What is an ordered set of ordered subsets in combinatorics? A paper in honour of Carmen Batanero in her 70th birthday

¿Qué es un conjunto ordenado de subconjuntos ordenados en combinatoria? Un artículo en honor de Carmen Batanero en su 70º cumpleaños

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Abstract

This paper analyzes the multiple results and the reasons given for their divergent answers by subjects of different age, gender and training to an apparently very simple problem in combinatorics, called "The Pearl" or "The Four-Toy Problem". The unexpected consequences of this semiotic analysis lead to questioning the usual definition of order relation, and to detecting two different kinds of order for the results of school tasks about combinations and permutations, here named "internal" and "external order". At the end, some suggestions for classroom communication, for the design of didactic situations, for assessment, and for Didactics of Mathematics or Mathematics Education research derived from the semiotic analysis of the Four-Toy Problem are reviewed and motivated.

Resumen

El presente trabajo analiza los múltiples resultados y las razones que dan distintos sujetos de edad, género y preparación diferentes a un problema aparentemente muy sencillo de combinatoria, llamado "La Perla" o "El Problema de los Cuatro Juguetes". Las consecuencias inesperadas de este análisis semiótico conducen a poner en cuestión la definición usual de las relaciones de orden, y a detectar dos tipos de orden diferentes para los resultados de tareas escolares sobre combinaciones y permutaciones, que se llaman aquí "orden interno" y "externo". Finalmente, se proponen y motivan una serie de sugerencias para la comunicación en el aula, para el diseño de situaciones didácticas, para la evaluación, y para la investigación en Didáctica de las Matemáticas o Educación Matemática que se derivan del análisis del análisis semiótico del Problema de los Cuatro Juguetes.

1. Introduction

I vividly recall those three crammed days of hard work with Carmen Batanero and one of her doctoral students, Luis Serrano, when we worked on the last refinements and corrections to his doctoral dissertation in Granada, preparing for the public defence of his thesis. It was over 20 years ago, in the Spring of the year 1996, when the three of us first had a glimpse of some unexpected problems in elementary Combinatorics. The following was the announcement of the defence of Luis Serrano Romero's dissertation in Granada:

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The Title: "Significados institucionales y personales de objetos matemáticos ligados a la aproximación frecuencial de la enseñanza de la probabilidad".[i]

The Jury: Carlos Vasco Uribe Universidad Nacional de Colombia; Mikel Shaughnessy, Portland State University; Ángel Gutiérrez Rodríguez, Universidad de Valencia.

All three of us liked the thesis very much, but I was particularly puzzled by the stubborn difficulties young and old students repeatedly showed in Serrano's dissertation, some of them reported in earlier works by Carmen and her students, when dealing with apparently simple instructions about order of symbols, ordered, well-ordered and partially ordered sets and subsets, permutations, combinations, and other seemingly easy and universally shared terminology of set theory and combinatorics. I was even more puzzled by the mutual misunderstandings and misinterpretations by all of us, Luis, Carmen and the three reviewers, about the same sentences and paragraphs of the dissertation and of the apparently clear comments and suggestions for corrections we had sent in writing or proposed verbally during the preparation for the defence and in the defence itself. That puzzlement has not abandoned me for over 20 years. I am pleased to report on my reflections about these topics.

After two centuries of permutation groups and Galois theory and after a century of Set Theory, Combinatorics, Probability and Statistics, it was unthinkable that there still were some hidden ambiguities and misleading definitions of such a clear concept as *order* in the ubiquitous expression "ordered sets and subsets" and in the treatment of combinations and permutations in elementary Combinatorics, the unavoidable introduction to Statistics and Probability Theory.

As it often happens in Mathematics Education research, a stubborn opinion of a sharp student becomes the crack that allows a glimpse of a pearl inside the hard shell. The pearl shines fleetingly, but the young person's puzzling conjecture, question or deviant statement is easily dismissed by us teachers and researchers as soon as it comes up: we just throw it into the trash can labelled "well-known student errors", or "pupils' misunderstandings" or—more refined but just as pejorative—"misconceptions". The shiny pearl fades into oblivion.

Many questions surfaced in our conversations in 1996 and kept surging and fading for the years to come: What is the deep difference between permutations and combinations? Is it just order vs. disorder? Is there a hidden problem in the definitions of ordered sets and subsets? Are there different conceptions of order even among experts? Is there a hidden problem in defining ordered pairs, triplets and quadruplets? Are there non-obvious classes of combinations and permutations not yet clearly identified? What is Combinatorics really about?

I still puzzle about those questions, and therefore the aim of this paper is *not* to "solve" this or that specific problem of Combinatorics, not even the one I consider "the Pearl", but rather to "dis-solve" some of our certainties about order and disorder, combinations and permutations, letters, numbers and other symbols, even about sets and systems and their subsets and subsystems.

As a result of those musings in Granada over 20 years ago, it turned out that the study of just one apparently very simple problem in Set Theory and Combinatorics has led me in a long journey to a critique of the traditional Problem-Solving approach to Didactics of Mathematics and to direct my attention to other kinds and models of word problems, exercises, puzzles, conjectures and paradoxes, not only in Set Theory and

Combinatorics but in all of Mathematics and Logic. Even deeper, to a questioning of teaching, assessing, writing tests, text-books, test items and word problems in School Mathematics in general, not only in Set Theory and Combinatorics.

Here is “the Pearl”. Let us call it “The Four-Toy Problem”. You can hardly think of a simpler word problem, but—as it is now politically incorrect to start with the word problem—let us start with the problem situation (or problematic situation, didactic situation—Guy Brousseau— or learning situation—Roger Shank).

I begin by spreading four toys on the floor or on a low table for all students of any age to see. A word of caution: In this learning situation I do not use mathematical symbols. If necessary, I would only use, and only much later, the initials of the names my students would give on-the-spot to those toys; but in order to comply with scientific standards for refereed journals, I will write here four symbols to stand for the four toys, like ‘a, b, c, d’ for the case of a plastic airplane, a ball, a car and a doll.

Then, I write in large lettering the following very simple question or instruction on the board (or I take good care to assign somebody to write it down, preferably to record it in a tape recorder or smart phone, you will soon see why):

Question: In how many ways can I split these four toys into two small groups?

Please try the Pearl yourself with young and old students, elementary, middle and high-school pupils, pre-service and in-service teachers, with undergraduate and even with your graduate students. Make sure the Four-Toy question stands out in large lettering throughout the learning situation. While writing and rewriting this paper during the present year 2018, I tried the Pearl again with a group of doctoral students in Mathematics Education in Bogotá. If you have not tried it yourself, you might not believe what I am going to report. Please stop your automatic-rejection system for a while, until you are able to replicate by yourself this problem situation I call “the Pearl” and until you have read this paper at least twice. Thank you.

One puzzling first fact: as soon as I finish writing the Four-Toy question, children jump out of their seats and start touching and pushing the four toys all over the floor or the table, separating them in small batches with their hands and arms. On the other hand, adults sit still, grasp their notebooks firmly or open them to jot down scribbles, usually single letters or numbers in different arrays. They hardly look around, concentrating only in getting the right answer. This behavioural contrast deserves reflection.

Children come up with an answer in a few seconds; usually, they cry out only a single number-word, like “Four!”, looking directly at your eyes. If you do a little pragmatic analysis and pay attention to the illocutionary point, you will notice that the way the number-word is pronounced by children does not express a certainty of having gotten the right answer, but just a good guess, accompanied with an honest request for the expert’s approval.

A puzzling second fact: sadly enough, in the case of teachers or graduate students, you will hear the same nuance that hints to teacher- or expert-dependency, not to creative, autonomous, authentic mathematical mental activity. Not “Magister dixit!” but “Magister, dic!”

A puzzling third fact: children and adults in my experience seem equally puzzled if I do not immediately approve or reject their answer. If I say: “Are you sure?” with a slight wrinkle of my left eyebrow, they will often either withdraw or change their answer. Try it. Then, I start my research session by waiting for a few conjectures from the floor, without approval or rejection on my part, with a “doubting Tom” look in my face—“dead-pan face” they call it. Then, I go back to the proponent of a particularly puzzling conjecture and start by asking him or her to repeat the given answer, and, after everybody has heard it again, then I follow up by honestly using expressions like: “What are your reasons to support your answer?” or “How would you convince me that your answer is correct?” Again, after hearing a few different solutions and the arguments that back them, I follow up with a generic question to all present: “Is there another answer anyone of you could also argue that it is correct from your viewpoint?”

A puzzling fourth fact: every subject, child, student or teacher in my experience seems to have the expectation that the initial question has one and only one correct answer. In Guy Brousseau’s terminology in his Theory of Didactic Situations TDS (Brousseau, 2002/2007), there seems to be an implicit clause in the didactic contract forbidding teachers, text-books and examination papers to ask questions with multiple correct answers. Even after a 15-to-20-minute session of lively discussion about divergent plausible answers and the reasons behind them, many teachers and experts stay after the end of the session and approach me in private to ask: “What was the correct answer?”

That attitude merits a statement as a puzzling fifth fact. In my experience, I have up to now always wasted my time trying to change the attitude of teachers and experts towards multiple-correct-answer questions by showing them my Pearl. My hope in writing this paper is that you give me enough time and credit to slightly change your attitude after you try the Pearl yourself and finish reading thid paper (twice, please).

Of course, I have a preferred answer. When I designed the Pearl, I tried to review all possible answers to list those with good mathematical reasons behind them, and I must now confess that my preferred answer was… [~~DELETED~~]. I am sorry, but I feel unable to write my favourite answer here, because it would forever ruin the future experiences with the Pearl. I can produce what I consider very good reasons to back up my favourite answer, but that does not make it *the* correct answer. That is the whole point of this paper: the Pearl is one of few simple word problems in elementary Mathematics that has multiple correct answers, depending on the mental models in which the Four-Toy question is interpreted by the participants. Because of the expert-dependency mentioned in the second fact above, my favourite answer would become the only correct answer and it would ruin every future learning situation built around the Four-Toy question.

Nevertheless, perhaps because my answer is not the result of any direct substitution in any of the usual formulas for permutations or combinations, it is perhaps the least common answer I usually get from any group of teachers or experts I have worked with. I should now say that the most common answers are the even numbers between 4 and 16.

In dozens of trials, I had collected five different answers with good reasons to support them, ranging from 3 to 16. Mostly, the reasons given for a specific list of splits or combinations were either simply showing some arrangements of the initials with an indication like “and there are no more splits”; sometimes that “it was the result of this formula”, or at times very elaborated explanations of why these specific splits were possible, why they were different, and why there could be no more groups. I was initially surprised that five different answers could be supported with good reasons coming from different interpretations of the three words “ways”, “split” and “group”. I thought I had heard enough results, and that there would be no more reasonable ones.

Thus, I was very surprised the day I first heard the answer “Only two!”. The ten-year-old boy was firm and sure of his answer “Two!” When I asked for the reasons behind it, the boy said: “Just two-and-two and three-and-one!” (In Spanish, using a colloquial way of saying “two-and-two” or “two-by-two”—see endnote ii—he said: “No más diá dos-y-dos y diá tres-y-uno!”). Indeed, can you think of more ways? What is “ways”, anyway?

Then, I thought for a year or two that I knew all six ‘good answers” from 2 to 16, until an 8-year-old boy came out with the answer “Only one!”. I was floored and could not keep my dead-pan face as I manage to do most of the time. His strong challenge was immediate:

> —How else if not two-and-two? (In Spanish, “No hay más sino diá dos!”).[ii]

My rebuttal:

> —How about three and one?
>
> —Only one toy is not a little group!—he countered. (In Spanish, “Un solo juguete no es grupito”).
>
> —Touché!

The problem of empty sets and singletons surfaced again at that moment. For mathematicians, “2^n” is a good answer to “How many subsets are there in a set of cardinality *n*”, because the cardinality of the set of parts, counting the whole set as a part and the empty set as a part, mind you, turns out to be the n^{th} power of two. No matter what mathematicians think, I can attest that no child in the many sessions about the Pearl has proposed or accepted that “zero-and-four” or “four-and-zero” are good ways of splitting the four toys into two small groups.

Some cautious mathematicians would agree that subsets of cardinality *0* and *n* are somehow “improper parts” of the totality, but even they would still insist in calling them “sets”, “subsets” and “parts”. How about calling them “little groups”? What is “a group” or “a little group”, anyway?

Most mathematicians in my experience would even reject the use of “group” for a set or a subset (or a part). They have reserved that sacred word for the most important binary systems in all of Mathematics, and a very restricted subset of those: they must be associative (in the sense of being semigroups), modular (in the sense of being monoids with a two-sided identity element), closed and invertible. Not even very respectable binary systems like Moufang loops (or most loops for that matter) would qualify as groups. What could children say? Nothing, I suppose, but that is not the problem: the problem is that we do not care about what they might say.

Going back to the problem of counting the empty set, the singleton sets and the full set as “parts” or “subsets” of a given set, I do ask children sincerely what they think about that, not using the words “sets” and “subsets”, “empty sets” or “singletons”, of course, but letting them use “group”, “little group”, “batch”, or any of their preferred terms for splitting larger batches into smaller. The polysemic character of “part” was obvious. All children in my experience reject the so-called “empty set” as a little group or a group or part, as attested by the absence of acceptance of a “four-and-zero” split. But most of them accepted that “three-and-one” was a good way of splitting the four toys, which is good evidence that they accept singletons as subsets. But the child who came up with the answer “Only one!” was also clear about rejecting singletons as subsets, because he was sure that only one toy is not a group, not even “a little group”. For him, “Three-and-one" was not a good way of splitting the four toys into two little groups. Was he right? Is semantics enough to analyse these answers, or do we need all the power of semiotics?

2. Two kinds of orders

So much for the answers in low numbers, like one, two, three, maybe up to seven and eight. Eight is hard to support as an answer, though it is a frequent answer. When ask for reasons to support it, most subjects change it back to seven, six or five. The research question that came up in Granada over 20 years ago was about why so many children and adults would come up with numbers *larger* than eight, and even more than that, up to 16, 24 and 64. Can you guess the sources of those large answers and the possible arguments to support them?

The reasons some participants give for the larger numbers they conjecture for the different ways of splitting the toys always have to do with lists of symbols and with appeals to order and disorder, which are usually considered as akin to "permutations and combinations", whether the proponents use those words or not. The proponents show you lists of strings of symbolos taken from the initial four symbols, like 'abc—d' and 'd—cba', and insist in taking them as "different ways" of splitting the four toys, as well as 'bac—d' and 'cab—d'. Are they different or not? In some sense, yes, in some sense, not.

The problem is in what sense yes, and in what sense not. The conjecture that came out in Granada over 20 years ago was that probably they were writing out the lists of symbol strings, not only taking into account the micro- or internal order of symbols in the string, especially the lexical order or alphabetical order a-b-c-d. In our terminology, but also to the macro- or external list order. In the first sense, they were taking into account the different permutations of the symbols in the resulting strings, and the different order of those permutations, while the usual questions in Combinatorics about "How many ways" without mentioning order usually mean only that we expect the subjects to search for combinations, not permutations.

I realized that the way the Four-Toy question was stated did not exclude the attention to order. Of course, children do not use the words "permutations", but some of them talked about combinations without the explicit rejection of internal order implied by text-book and teachers questions. Even many high-school and college students who also failed to use the word "permutation" itself, but the way they list the "different ways" clearly showed they were considering pairs or triplets with the same letters in different order as "different ways" of splitting the larger "group" into smaller ones, not attending to the customary implicit exclusion of order in the questions about how many combinations of symbols are there in a restricted initial set.

Typically, in our case, but also in Luis Serrano's cases, the subjects claimed that longer-than-expected lists of ways of grouping objects into subsets ("combinations and permutations") were "good" because "they are all different". As in the 'abc—d' and 'd—cba' cases above, they would point to a list of two pairs (or a triplet and a singleton), and then show another pair or triplet, like '(a, b)—(c, d)' and '(c, d)—(a, b)', and claim that they were different "because of the order" (we might say "in the external order" in the list) or point to two triplets, like '(a, b, c), (a, c, b)', and claim that they were different "because of the order of the letters inside" (we might say "in the internal order in the string"). Their favourite reason—if you call that a reason—was simply to point to the symbol strings and say: "This is not the same as this".

In many sessions around the Pearl, this double interpretation of order came out often; but who had heard of "external order" or "internal order" in the teaching and learning of combinations and permutations? As I recall, every text-book of probability or statistics would speak about order only for permutations, not for combinations, and, in the case of permutations, the order was always the internal order of the symbols in the string, like the forward order (a, b, c) or the backward order (c, b, a); never, to my recall, the external order in which the strings of symbols were listed.

Thus, after I had collected enough answers and their well-argued reasons, I had to start the data analysis from scratch. The new trends in Semiotics in Mathematics Education, coming out especially after two special issues of ESM and RELME in 2006, helped me much. I started the new round of data analysis by highlighting three very polysemic words in the initial question or instruction. You guessed right: "WAYS", "SPLIT", "GROUPS".

We had not realized that "the way" you arrange a singleton and doublet, or two or three doublets, or a singleton and a triplet, was also "a way of ordering" or "of splitting" the larger set into subsets. We were thinking about the results, the list of combinations or permutations obtained *after* the splitting in small groups was finished, not of the ways in the sense of strategies. Students seemed to be thinking about the ways to proceed *before* they would start constructing the list of symbol strings or "splittings". There were some young students and adults who considered that the format 'one-and-three' was different from the format 'three-and-one', thus getting twice as many combinations as the majority. Others were considering the format "two-and-two" as a single strategy, and then considering the results as split again into (ab)—(cd) and (cd)—(ab).

3. Permutation and combination formulas

We did recall at the time the universally accepted distinction between combinations and permutations, and the universally accepted Combinatorial Wisdom: If the required sets are *ordered*, use the permutation formulas; if the sets are *not* ordered, use the combination formulas. Is this also one "way of splitting" as a strategy?

In fact, no children in my experience with the Pearl, but many adults, especially College students, in-service teachers and graduate students, did try to use one of the conventional formulas for permutations or combinations to get the total number of "ways to split" or "splittings" of the four toys. You could easily see that happening if they would answer "Sixteen!" (2^4 = 16) or "Twenty-four!" (4! = 24) or even larger numbers, like 32 or 64. If you asked for the reasons, the subjects would recite the formula, and acknowledge that they had tried to list every combination in different orders (which we would have called them "permutations") until they got the same number that they had found by using the formula. Many of them quickly ran out of possibilities by listing different orders of the symbols (internal order), and if they needed more, they tried to appeal to different orders of the groups (external order) to get close to the predicted number.

4. Back to two kinds of orders (or more)

Some Combinatorics problems require to find—or at least to count how many—*subsets* of points, numbers or symbols taken from a larger set S are there under some specified conditions, and others require to find—or count—how many *ordered sets* of those elements there are. On purpose, the Four-Toy question does not hint to order or disorder, allowing the subjects to choose what they would call "different ways". What I had not even suspected when designing and reusing the Pearl was that there were two kinds of orders, internal and external, and much less, that even the notion of order in small sets was not as clear and distinct as mathematicians think it is.

Supposedly, from the institutionalized or epistemic viewpoint, *sets* are totally unstructured systems of any number of elements or components, i.e., having no specific binary relations between pairs of elements, and *ordered* sets are at least partially structured systems with one specific binary relation that satisfies reflexivity, anti-symmetry and transitivity.

Let us start with the simplest case beyond ordered pairs and ordered triplets, in which there seems to be full agreement among experts in Probability, Statistics, and Combinatorics. In case we have four symbols, letters, numbers or whatever, there could be *internal* orderings among the symbols inside the set or in each subset of a partition, and in the case of two or more subsets of a partition, there could also be *external* orderings among those subsets.

It seemed to us—and it still seems so—that internal orders are easy, like trying to order the sets {a, b, c, d} or {1, 2, 3, 4}. Everybody knows that we have 24 internal orders for permutations of four distinct symbols without repetition, starting with what we write as '(a, b, c, d)' or '(1, 2, 3, 4)' and then scrambling or permuting the symbols. It is also clear that we have six internal orders for triplets and just two for pairs; in case of a partition into two pairs, there are only two external orders: one of those pairs first, and the other second, or vice-versa.

So far, so good; but when we try to generalize beyond five or ten, where we can use our fingers ("digital computing"), and now use sequences like a(i), b(j), c(k), …, x(n) with indexes (or indices or subindices a_n or exponents a^n), we would be very surprised to find that the apparently exhaustive 24 orders of {1, 2, 3, 4} would yield 24^2 = 576 proto-sequences or ordered quadruplets we usually write '(a(1), b(2), c(3), d(4))' or '(a_1, b_2, c_3, d_4)'. And we were still inside internal orders!

We had forgotten that the letters hide a culturally invisible alphabetical order, which should rather be called "abecedarial order" in this case, because we are using Latin letters, not Greek. This order is definitely not the same as the one used by the Greeks with the letters of their alphabet: for them, the third letter of the alphabet was not 'c' but 'γ'.

Initially, we thought there were no problems in agreeing that there are n-factorial or n! ways of linearly ordering a subset of n different symbols, usually written '(a, b, c, …, x)' or '(1, 2, 3, …, n)'; but when we read that 'x' the first time, we did not notice anything. It took me years to realize what I now think is obvious: that in the abecedarial order, 'x' is not a variable but a constant: the "antepenultimate" or next-to-next-to-last letter of the abecedarium, just before 'y' and 'z'. Think about the sentence: "Let X be a letter of the alphabet". Is X a constant or a variable? How about X = 10?

What would 'x' be in the Greek alphabet? Capital 'X' does exist, and it would be called 'xi' but it could also be 'chi', now the next-to-last letter just before omega 'ω', but no lower-case 'X' exists that looks like our 'x': it's 'χ'!

We did not notice that the dear old 'n' itself as last element of the string would also have an intractable problem, because if the given sequence had fourteen letters or more, then 'n' would be again a constant of the alphabet, not a variable symbol for as-yetundetermined numbers. We unthinkingly agreed then that in Spanish, English and many other languages we have only nine or ten digits ending at ‘9’—or, do they end at ‘0’?—but if we had been computer hackers working in Hexadecimal, we would have stumbled with the even more intractable problems that after ‘9’ goes ‘A’, and ‘N’ does not exist; but in ASCII-128 it does, but ‘n’ is 110 = HEX 6E and ‘N’ is 78 = HEX 4E. Back to square one, or zero, or in ASCII code, what is the 0 = HEX 0 symbol, or the space or blank symbol 32 = HEX 20, or the zero symbol 48 = HEX 30? We were fortunate not to know much about Hexadecimal or ASCII codes when we studied Set Theory, Combinatorics, Probability and Statistical. The problem is that many of our young students know a lot about ASCII codes, QR codes and Unicode.

5. Subsets, singletons and empty sets

Here we found the first stumbling block for us as mathematicians and as teachers of Set Theory: when we say “systems of any number of elements”, we mean: “including zero and one”. Students instinctively refuse to accept that a single element forms a set, a batch, a group, a system, much less none! Would you have a collection of modern airplanes if you had only one? (Older people might still understand what “a collection of stamps” means). For most people of all ages, in the semiotic register of everyday language, be it current English or Spanish, sets, batches, groups or systems, ordered or unordered, must have at least *two* elements or components. They are not alone in their claim: every printed book of arithmetic from 1450 to 1900 explicitly stated that one was not a number but the source, the beginning or “principle” of number.

Since the Middle Ages, in European text-books there was no mention of zero as a number until Cantor was solemnly acclaimed and formally accepted in the mathematical Paradise by David Hilbert in 1900. For those same old text-books, “zero” was a figure, a cypher, a numeral, used only to mark the absence of tens, hundreds or thousands, and it was considered as useless and meaningless if written to the left of a number symbol (in Spanish, we used to repeat: “Cero a la izquierda no vale”).

Students easily win a round of verbal fencing with us teachers about the acceptance of empty sets and singletons if we previously have made the slip of attempting to define a set as a union of elements, or a reunion of elements, or a collection of points, or a group of elements, or a system of components, or any similar expression built around the few synonyms of ‘set’, like ‘class’, ‘ensemble’, ‘group’, ‘pack’, etc. Students clearly hear the final ‘s’ in the second part of those ‘of’-expressions, which we do not notice, and they correctly infer that in any set S there must be at least two of those elements, points, components, bits or “whatever’s”.

We have to step back and defend ourselves, usually appealing to invalid (and immoral) arguments of authority, and students are not quite convinced of our defensive discourses. The stability of those conceptions is well attested, and even with the best of methods we seldom succeed in changing those, for us, strange, false or mis-conceptions in the long run. They reappear often, even after a period of time where the subjects seemed to have overcome those misconceptions. The question that arises from the good reasons to divergent answers to the Four-Toy question is: Are they really “misconceptions”? Let us be a little more respectful of our students’ clever conjectures, and from now on let us non-judgmentally call them rather “alternative conceptions”.

6. The definition of order relation

That ambiguity of any possible verbal definition of the word ‘set’ and its synonyms is not the only one present in the initial statement of word problems in Combinatorics. Neither students nor teachers seem to notice that the words ‘order’ and ‘ordered’ are just as slippery, and if we add fuzzy qualifiers like “partial order” or “well-ordered”, things might get worse.

It is not clear from the words themselves what orders are “partial” and what would be the opposite of “partial orders”. Do teachers mean that there are “total orders”? What would that mean? Totally defined? Well-ordered?

I must confess I still do not know how to translate into direct discourse the adverbial expression “well-ordered”, either in English or Spanish. May I (or my students) correctly infer that there are “good orders” and “bad orders”? Which ones are “bad”? Are “bad orders” real orders at all? I am not so sure.

Are we sure that the expression ‘order relation’ is as well-defined as it looks? Try a strict order, like strictly less than; then, I bet a strict order is not an order! Try a cyclic order, like the order of the twelve hours on a clock face; I bet a cyclic order is not an order! After those two blows, we are forced to reconstruct the genus-species order of the Porphyrian Tree. Let us be more precise:

Start by defining that an internal binary relation R = R(_, _) = (_)R(_) = (_, _)R (using prefix, infix or postfix notation) inside a set X is an *order* relation iff R is reflexive, antisymmetric and transitive. Do you agree?

This definition is usually checked by testing the properties of the relation “less than or equal”: $\leq$ = (_)$\leq$(_) in a few number systems, and those of the relation “is contained in”: $\subseteq$ = (_)$\subseteq$(_) in a set of subsets. Everything seems to follow nicely, but after a little thinking, the reader will agree with me that the following three propositions would also hold*:*

Prop. 1. A strict order relation < = (_)<(_) (or >) on any set X is *not* an order relation.

Proof: Reflexivity is not satisfied.

Prop. 2. A cyclic order relation R = < (or >) on a finite set X is *not* an order relation.

Proof: Transitivity trivializes < to the universal relation in X; thus, it cannot be anti-symmetric.

Prop. 3. A strict cyclic order relation R = < (or >) cannot be defined on a finite set X of cardinality 0, 1 or 2.

Proof: Try it and see what happens with the ordered pairs of the graph of R.

Note 3.1. There can be no consistent written definition of ordered pair or ordered couple. Let us start by using Bourbaki’s attempt to define the ordered pair or ordered duplet or ordered couple (a, b) by introducing the binary coupling operator C = C(_, _) = (_)C(_) = (_, _)C (using prefix, infix or postfix notation):

Def. (a, b) :=: Cab = aCb = abC.

My Italian mentor, Carlo Federici, noticed in the mid-seventies that Bourbaki’s definition of ordered pair (or couple, or duplet) would require a previous definition of ordered triple (or triplet) on the set

Y = {‘a’, ‘b’, ‘C’} of cardinality 3.

Note 3.2. Carlo Federici also noticed that Wiener-Kuratowski’s attempt to define the ordered pair or couple (a, b) by

Def. (a, b) :=: {a, {a, b}}

would require a previous definition of an ordered nine-tuple or nine-tuplet on the set

Y = {‘a’, ‘b’, ‘{‘, ‘}’, ‘,’} of cardinality 5.

Note 3.3. Carlo Federici also noticed that attempting to define an unordered pair by {a, b} would require a previous definition of an ordered quintuplet on the set Y = {‘a’, ‘b’, ‘{‘, ‘}’, ‘,’} of cardinality 5.

Note 3.4. A strict cyclic order relation R = < (or >) defined on a finite set X of cardinality

n>2 (for instance, n=12 for the hours of the day) cannot be totally defined on X without trivializing the relation, thus destroying anti-symmetry.

Note 3.5. There is no way to define an ordered semigroup, group, ring or field on Z_2 = {0, 1}, usually identified with the Cantorial ordinal 2 or the Cantorial cardinal 2.

Are we now so sure we knew what ‘order relation’ means? Is strict order < an order relation? How about the cyclic ordering of the twelve stops in the clock face?

7. Operators or results?

Were Évariste Galois, Chevallier, Liouville, Betti and Camille Jordan missing something in the notations ‘(ab)’, ‘(abc)’ and ‘(abcd)’ for permutations? Were those strings meant to represent a specific permutation as a dynamic operator capable of *permuting* symbols inside the string, or the result of that permutation? Which is which? In this case, one thing is to permute symbols when we see the symbol ‘(ab)’, which would mean that we start pushing the first symbol to the second place and then, the second symbol to the first place, getting ‘(ba)’. The second place is certainly the one after the first, but which one is the second symbol? Is it the symbol to the right of the first, or is the first ‘a’ and the second ‘b’ as in our alphabetical order? As a result of the transposition, (ba) is a different permutation, but as a transposition operator, (ba) is the same as (ab). Or isn’t it? One thing is to see a string of symbols in some order, like (abcd), before or after permuting a previous string of symbols, and another thing is to actively scramble them until we bring the last one to the first place. Why do we use the same symbol (abcd) for the operator and for the result of the fourth application of the operator? How do we note the identity operator that transforms (abcd) into (abcd)? Was it a single identity operator acting once, or was it the fourth application of the cycle (abcd)?

In short, is (abcd) the *result* or the *operator*? Is it the *state* or the *transition*? The ambiguous ending in “-tion” might mean either or both, or neither, or something else. Let us distinguish the operator, the action—in this case, the active or dynamic permutation (abcd)—from the static or passive result (dabc) after the first application. Are they the same, or are they as different as “energeia kai dynamis” or “actio et passio” in philosophy?

I thought of using the arrow notation to get the *state* clearly recorded as ‘)b, c, d, a>’ after the first *transition* produced by the application of (abcd) to an assumed standard initial state conventionally recorded as ‘)a, b, c, d>’. This looks very reasonable, but how should I write that out symbolically without using that many words? Let us try:

(abcd)()a, b, c, d>) =)b, c, d, a>.

That looks good; but shouldn't I use the arrow notation also for the permutation as the action or the operator or the transition? There is no reason against, but if I do it, I lose the distinction:

)a, b, c, d>()a, b, c, d>) =)b, c, d, a>.

Now it looks like I have applied the operator (abcd) twice, which would not yield the answer)b, c, d, a> but)c, d, a, b>!

I took this problem to the semiotics subgroup of PME led by Adalira Sáenz-Ludlow and Norma Presmeg. See their special issue of EMS, and D'Amore and Radford's special issue of RELME, for me the golden year 2006, when we started taking semiotics seriously in Mathematics Education.

In Bergen, Norway, I tried my hand at listing some uses of juxtapositions, like 'ab, 2ab, abc' in algebra and analysis. The paper was deemed too long to be published in the PME proceedings and too short to be published in the special issue of ESM.[iii]

In short, neither the words "ways", "split" and "group" are well defined, perhaps not even definable; the number of ways can refer to the general strategies of splitting, dividing or distributing or to the specific results of applying that strategy; the traditional definition of order that we consider institutionalized (and epistemic) by the three properties of reflexivity, anti-symmetry and transitivity is not so good for some obvious orders like strict order and cyclic order, and the notations for the permutations and combinations confuse students and teachers alike with respect to the active (operator) state or the passive (result) status of a permutation.

8. Some corollaries from reflections on the Pearl

First, some hints for writing "good" word problems and test questions

True-false questions do not provide any useful information about the quality of the mathematical processes going on in our students' minds. Even a list of three or four options, of which we think there is one and only one correct choice is no longer a good format for an assessment item. Single-answer problems are not really "problems" in the sense of thought-provoking learning situations. At most, they become good exercises, limited to those who have not yet automatized an algorithm or learned a useful formula (or worse yet, for those who already solved them but have a very poor memory).

The search for "pearls" as deliberately ambiguous statements, questions and puzzles to design productive learning situations around the multiple reasonable answers can be very fruitful not only to engage attention and creative thinking in our students, but also to advance critical thinking in Mathematics, especially in Probability and Statistics.

Second, some hints for "good teaching" of Probability and Statistics

It is progressively clear that no matter how "good" the verbal, symbolic, black or white board teaching of Probability and Statistics is, especially if text-book based, it will not help much in dissolving students' misconceptions but will rather induce them to keep their personal interpretations and their private mental models prudently repressed. They will fear externalizing them because of the risk to contradict traditional conceptions held as epistemic by teachers and textbooks and, hence, the risk of getting poor grades and even to infuriate their teachers.

If from the reflections on the Pearl we treat those personal, idiosyncratic interpretations of our students in their spontaneously activated models not as misconceptions or well-known students' errors, but as alternative conceptions, perhaps useful, powerful and creative, we will start patiently and sincerely asking our "deviant" students to produce strong reasons to support their interpretations of the question or problem, to develop those reasons and conceptions, to debate them with their peers, and even to attempt to externalize their mental models and interpretations on screen with graphic software (preferably 3D). In this fashion, we will start changing our teaching time into student-learning time. This change from teaching activities to learning situations is well aligned with the design of didactic situations in the sense of Guy Brousseau (2002/2007), where the 'adidactic phase' or moment is ideally extended as much as possible beyond the initial presentation of the situation and the moment of institutionalization, which is also necessary and must be planned according to the time constraints.

It is progressively clear that Combinatorics is not a part of Probability Theory or of Statistics. It is only a part of elementary Set Theory, usually restricted to counting the number of elements in a finite set and its subsets, as well as the number of selected subsets from a restricted set of objects, numbers or symbols, usually further restricted to permutations or combinations of them. The problem we have visualized with the ambiguities of students' interpretations of small sets and orders is that the deeper and more important, formative and interesting discussions and debates on sets and subsets, on alternative conceptions and interpretations of the standard technical words in Set Theory, on relations and functions, order relations and operators, and on alternative mental models and possible interpretations of the same well-formed formulas and apparently clear definitions are not dealt with; moreover: these discussions are suppressed or covered up by the repetition of ritual words, magic formulas and algorithmic routines to get well-known answers to single-answer word problems.[iv]

Third, some hints on Semiotics for researchers on Mathematical Education or Didactics of Mathematics

As I have learned from successive layers of analysis of the Pearl, the semiotic factors are paramount in any effort to analyse data from puzzling learning situations in which active teacher-student and student-student dialogues make semiotic conflicts appear, but even more so if divergent semiotic expressions and interpretation can go on for days, months or years without arousing explicit semiotic conflicts. In general, it is easily observable that most teachers are not conscious of those divergent interpretations and of the different mental models that they might activate in children's minds. Teachers and textbook writers feel sure they have expressed themselves clearly and correctly, and even today, most mathematicians and mathematics teachers think that the modern definitions of the relevant mathematical objects in school mathematics are now univocal and definitive, epistemic and institutionalized. In the 80's and early 90's, only a few researchers started to realize the problems of expression and interpretation of semiotic representations, including carefully crafted definitions and well-chosen diagram labels and characterizations; of the role of mental models and the difficulty to express them by gestures, diagrams or ordinary-language words; of the specifically semiotic difficulties of the transformations from one symbolic form to another. Still, most teachers and researchers do not seem to be very conscious of the need to guess the mental models and the semiotic representation registers available to children and youths in the classroom and in the reading of word-problem statements and accompanying drawings in tests and homework assignments.

In a sense, it is also true that the polysemic aspects of any sign, symbol, word, diagram or icon have been noticed and dealt with from the very early days of the discipline of Didactics of Mathematics or Mathematics Education in the 60's and 70's. In the 80's, this was the case with Régine Douady's theory of frames ("jeu de cadres") or Jim Kaput's theory of multiple representations: both are clearly semiotic, though not explicitly named as such.

Perhaps the revival of Peirce in the United States after 1945, J. L Austin's lectures at Harvard in 1955, later published as *How to do things with words* (1962), and then John Searle' pragmatics and the analysis of illocutionary points (Searle & Vandervecken, 1985) slowly turned the attention of Mathematics Education researchers to the use of these tools to refine their analysis of mathematical discourse in school textbooks, test items and classroom symbolic interactions.

In Europe, Umberto Eco and his learned treatises on semiotics—but especially his book "The Name of the Rose" in 1980—could be placed as the beginning of the progressive attention to semiotics and semiology in our discipline, but it took about 15 to 20 years to become generalized.

The most refined and explicitly semiotic studies in our discipline appeared perhaps in the works of Raymond Duval in the late 80's, with the powerful distinction between semiotic registers and semiotic representations and the general approach to analysis of diagrams, icons, words and other symbols we now call "the Noetic-Semiotic Approach NSA". These semiotic ideas, inspired in Benveniste, first appeared in book form in French in 1995, *Sémiosis et Pensée Humaine*, and—after dozens of papers in English and French, and translations into Spanish and Portuguese—only since 2017 are now widely available as a full book in English (see Duval, 1995; 2017).

In Spain, the explicit use of semiotics, but then inspired in Hjelmslev, gave rise about the same time to the Onto-Semiotic Approach OSA (in Spanish, Enfoque Onto-Semiótico EOS) of Godino, Batanero and Font. A long road has been travelled by Juan Díaz-Godino and Carmen Batanero since the first drafts of the OSA-EOS approach started to appear in local meetings and in the web page of the University of Granada in the early 90's, shortly afterwards followed by a seminal paper in *Recherches en Didactique des Mathématiques* in 1994, one year before Duval's book and 12 years before the ESM and RELIME compilations in 2006.

In the late 90's, two more explicitly semiotic approaches appeared: Luis Radford's Semiotic-Cultural Approach and his theory of objectivation, inspired in Bakhtin, Voloshinov and Vygotsky, and Adalira Sáenz-Ludlow's "semiotic interpretation games" (Sáenz-Ludlow, 2006), inspired in Charles Sanders Peirce. After the special issues of ESM and RELIME on semiotics in 2006, we can now say that explicit semiotic consciousness and semiotic tools in analysis are now established in our discipline, and we can find them more and more explicit in research projects, papers and dissertations.

Although Godino, Batanero and Font explicitly acknowledge the polysemic plurality of every mathematical object, they often seem to confuse the polysemic character of the semiotic representations of the same object with a hard-to-pin-down polysemy of those objects themselves. It might the case that they still fail to recognize that the same objectively observable semiotic representation can play different roles and induce different meanings when produced by different semiotic registers.

In the onto-semiotic analysis of concrete classroom dialogues, it is easy to confuse the occurrence of a mathematical object with the occurrence of the use of a word or mathematical symbol related to it, with a verbal description, or with a definition of the object. The word “concept” is taken sometimes as meaning a mental concept, a mathematical object, and, at times, it also seems to be identified with its explicit definition. In the last case, it is not noticed that the given definition cannot be identified with the *concept* but would rather qualify as a *proposition* in the category “Languages”. But in some publications and diagrams, one might take propositions and arguments as separate categories from the overarching category “Languages”, and *definitions-concepts* as a separate category, although *definitions* might be better included into *propositions*, and these, as well as *arguments*, should be included in the category “Languages”.

It would then seem profitable to analyse in more detail this overarching category, perhaps separating more iconic-indexical languages from more symbolic-articulate languages that require reinterpretation (Vygotstky would have considered the latter as “second order” languages, requiring and promoting higher-order mental operations).

It would also be very profitable for onto-semiotic analysis to distinguish different semiotic representation registers and the ways of applying treatments to semiotic representations inside the same register and o conversions between registers. The local everyday-language would be profitably considered as a different register from the technical semiotic registers of school arithmetic; elementary algebra could be identified as a semiotic register for elementary arithmetic with variables and unknows, and Cartesian and Eulerian coordinates as a different semiotic register for analysis (see Neira, 2002).

Also, a refinement of the categories epistemic-cognitive and institutional-personal has to be developed from the former analysis of the Pearl. This will lead to Chevallard’s Anthropologic Theory of the Didactical (TAD: “du Didactique”, not “of Didactics”) and his observations about different layers of didactical transpositions, that allow no clear distinction of what is epistemic and what is “merely cognitive” and no identification of what is “institutional” or “institutionalized” with what is epistemic more solid and powerful.

As observed above, the observational category of Semiotic Conflicts in the classroom could also be extended to divergent semiotic interpretations that cause difficulties that cannot be classified as conflicts, because that are not detected as conflicts by teachers or students; they are only suggested to the researchers by hints and unintended signals, reappearing questions or unusual manoeuvres on the part of teachers. In the dissertations of my students Gloria Neira (2018) on Differential Calculus and Enrique Mateus (2018) on Integral Calculus, it was clear that the different standard and non-standard interpretations of the usual symbols in Calculus lead to silenced and repressed alternative conceptions that did not appear as conflicts but revealed that not all statements taught as *institutional* by the teacher are *epistemic*. In fact, they could not even be qualified as *institutional* in the same school or college, even without arousing any observable conflict with his students, and not all conflict-arousing statements are cognitively interpreted differently by students and teachers. This requires a new category of whatever is “taken for institutional” on the part of the teacher, but cognitively accepted by students without apparent conflict.

Finally, the consideration that the same clearly written proposition could be interpreted in very different directly-inaccessible mental models and, thus, give rise to mutually inconsistent correct answers ("correct", not "true", in the sense that each individual proponent might have very good reasons to express his or her answer from the viewpoint of an unconsciously activated interpretation model for which the answer holds) forces a rethinking of assessment, of teaching, and of doing research in Mathematics Education that leaves behind the classification of student errors or misconceptions and opens up avenues of research in the relationships between formal theories and mental models in learning, teaching and doing Mathematics, not only Probability and Statistics.

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[i] "Institutional and Personal Meanings of Mathematical Objects linked to the frequential approach to the teaching of Probability". This is the title of Luis Serrano Romero's dissertation, directed by Carmen Batanero, of which they later published a widely read paper in JRME: Batanero, C., & Serrano, L. (1999).

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[ii] The colloquial use of "diá" instead of the adult Spanish "de a" is the most common pronunciation heard in Colombian children (and in several other Latin-American countries) to refer to partitive division ("repartir diá dos" instead of "dividir entre dos", which is the sometimes called "quotitive division", both confused by adults when they say "dividir por dos" to mean either type of division) or, as in this case, to refer to distributive counting, similar to the distributive numbers in Classical Latin: *binis, ternis, quaternis, quinis…* These words would correspond to the Spanish expressions "de dos en dos", "de tres en tres", etc. Thus, for children in Colombia, "contar diá dos" goes "two, four, six, eight, …", and "contar diá tres" goes "tres, seis, nueve, …").

[iii] I also attempted to list the uses of '=', following early work by Merlyn Behr, later by Zalman Usiskin and more recently by Adalira Sáenz-Ludlow, and to apply the early pragmatic analysis à la Austin and Searle. See a Spanish version by my student Gloria Neira (2012) in a book chapter about the transition from Algebra to Calculus.

[iv] En la siguiente referencia hay un tratamiento sistemático del razonamiento combinatorio que podría enriquecer las reflexiones que se hacen en este trabajo: Batanero, Godino, Navarro-Pelayo (1994). *Razonamiento combinatorio*. Madrid: Síntesis

https://www.sintesis.com/educacion-matematica-en-secundaria-70/razonamiento-combinatorio-ebook-1309.html