Design based research on students’ reasoning evolution about randomness and decision-making

Investigación basada en el diseño de la evolución del razonamiento del alumnado sobre la aleatoriedad y la toma de decisiones

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Abstract

We present a design-based research on students’ reasoning evolution about the randomness and the decision making in a chance game. The research is supported theoretically in the analysis of how the complex and multifaceted notion of decision emerged and evolved simultaneously to the notions of randomness and probability. This theoretical framework has been used to retrospectively analyse the theoretical foundations of students taking part in the research (ages 15 and 16) on the decisions to be made to win in the game and how these decisions were based on the different conceptions about the randomness of the generator and the generated sequences of events. We conclude that the nature granted to the decisions made to win and the randomness granted to the game have evolved simultaneously.

Keywords: Randomness, decision-making, design based research, secondary school

Resumen

Presentamos una investigación basada en el diseño sobre evolución del razonamiento del alumnado sobre la aleatoriedad y la toma de decisiones en un juego de azar. La investigación se sustenta teóricamente en el análisis de cómo la compleja y multifacética noción de decisión ha emergido y evolucionado a la par de las nociones de aleatoriedad y probabilidad. Este marco teórico ha sido la base para el análisis retrospectivo para identificar los sustentos teóricos del alumnado participante en la investigación (15 y 16 años) sobre las decisiones a tomar para ganar en el juego y cómo dichas decisiones se basaban en las diferentes concepciones sobre la aleatoriedad del generador y de las secuencias generadas. Se concluye que la naturaleza otorgada a las decisiones tomadas para ganar y la aleatoriedad otorgada al juego han evolucionado a la par.

Keywords: Aleatoriedad, riesgo, toma de decisiones, investigaciones basadas en el diseño

1. Introduction

Educational design based research (DBR) has been a substantial educational research framework to provide theoretical and empirical grounded products for stochastic education. In this relatively new research approach, the design of educational materials is a crucial part of the research (Bakker & van Eerde, 2014). DBR has different approaches that differ if they are grounded on theory or empirical products (Engeström, 2011). Two examples of DBR are substantial to ground the theoretical approach of the DBR presented in this paper.

On the one hand, Abrahamson (2012) presented a DBR that investigated the cognition and instruction of probability. The construct of an epistemic resource emerged with his attempt to respond to empirical findings about the research assumption that, when analysing compound-events random generators, students typically do not appreciate the relevance of order among singleton events (Batanero, Navarro-Pelayo, & Godino,

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1997). He concluded that a learner could make appropriate mathematical analysis of compound-events by objectifying pre-symbolic notions of probability using a customized event space (Abrahamson, 2012). In this paper, this mathematical analysis of compound-events by objectifying pre-symbolic notions of probability would be a point of depart of a trajectory that aims to identify students' reasoning on decision making under situations of uncertainty and of risk.

On the other hand, Bakker and van Eerde (2014) presented an example of a DBR to answer the question of how can we promote coherent reasoning about distribution in relation to data, variability and sampling in a way that is meaningful for students with little statistical background. In particular, Bakker and Gravemejer (2004) presented a theoretical framework to analyse the relation between data and distribution. The structure of this theoretical framework can be read upward and downward. In the upward perspective novice students perceive the individual values of the data to construct the notion of frequency distribution of a data set. In the downward perspective students use probability distributions to model data. Moreover, they conjecture that experts in statistics can easily combine the upward and downward perspectives.

The combination between this upward/downward perspective about the relationship between the frequency distribution of a data set and the probability is not unique. From a probabilistic perspective, Nilsson, Eckert and Pratt (2018) examine the challenges and opportunities of experimentation-based instruction in the learning of the bi-directional relationship between a classical a priori and a frequentist model of probability (Borovcnik & Kapadia, 2014). Nilsson et al. (2018) concluded about the need to take a fine-grained account of the social and situational nature of how students express and develop an understanding of randomness and probability in particular learning environments.

In 2015 when initiating a DBR, the social and situational nature of game of chance enviroment was taken into account to design the Integer Addition Bingo (IAB) task (Serradó 2018). In general, this DBR had the purpose of enhancing the stochastic reasoning of 48 Spanish students (Grade 7, age 12) when making decisions in situations involving uncertainty and risk. In particular, the task was designed with the aim of improving students' learning and reasoning about risk management through a process of understanding the random nature of the game. The retrospective analysis of the DBR, presented in Serradó (2018), lead us to identify four mental levels of reasoning: (a) *pre-structural*, where decisions are based on personal preference; (b) *uni-structural*, where decisions are rationally bound in situations of uncertainty; (c) *multi-structural*, where decisions are rationally bound either in situations of uncertainty or in situations of risk; and, (d) *relational*, when decisions are rationally bound either in situations of uncertainty and of risk. Moreover, six learning trajectories were identified. Three of these trajectories are interesting for this paper because they informed about a restricted progression from pre-structural to uni-structural reasoning. The evolution to a higher level of reasoning was constricted by a deterministic view of the chance game, the difficulties of discerning between the randomness of the generator, the randomness of the events and the sequences of events, and a lack of previous knowledge about measures of centrer for frequency distributions.

In order to surpass these difficulties, Serradó (2018) suggested to initiate a second cycle of DBR through improving the IAB task design. Three improvements were suggested: (a) engaging students in a dialogue about the uncertainty of the randomly generated numbers; (b) reasoning about the differences between the numbers randomly generated by the IAB random generator and the random events obtained by the addition of the randomly-generated numbers; and, (c) facilitating students' ability to discriminate between events and sequences of events.

In this paper, we present a second retrospective analysis considering the cyclic nature of DBR (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). This retrospective analysis aims to analyse the evolution of students's reasoning about the random nature granted to the IAB generator, the events and sequences of events generated, and the decisions made.

2. Decision making in situations of uncertainty and of risk

Over the past decade several studies have advanced in the understanding of what decisions are, how decision-making skill is acquired and its acquisition measured (e.g. Baron, 2008; Chen, Moskowitz & Shue, 2016; Cokely et al., 2018; Gigerenzer, 2002). In particular, Cokely et al. (2018) integrative review of skilled human decision making describe four factors for assessing adults' expertise. These are: domain general-skill, statistical numeracy (i.e. practical probabilistic reasoning), specialized knowledge, and cognitive abilities. From these four factors, we are interested in the relationship between the development of decision-making and statistical numeracy.

This relationship has been largely documented from an epistemological, psychological and ontological points of view, concluding that the complex multifaceted notion of decision has emerged simultaneously to the concept of randomness and probability (Cokely et al., 2018). The ideas of randomness and probability can be seen to range along an ontological and epistemological spectrum (Saldanha &Liu 2014). On one end of the spectrum is a deterministic stance, in which randomness and probability are regarded not as inherent features of objetive nature, but rather as residing wholly in the mind as an expression of our ignorance of the causes of actions, and therefore of the true deterministic course of events. On the other end of the spectrum, randomness and probability are seen as inherent features of nature in which genuinely statistical sequences are thought to exist. Coherently with this framework, we also envision the ideas of decision to range along an ontological and epistemological spectrum. On one end of the spectrum are personal decisions based in the ignorance of the random nature of the event. On the other end of the spectrum, decisions under situations of risk in which the probabilistic distribution of the random events is genuinely recognised. From an ontological point of view, the different views on this spectrum emerge from questioning what decision, randomness and probability are.

Roughtly speaking the meanings of decision, randomness and probability are connected though the concept of judgement. In the stochastic field, there is much research on random sequences and the individual judgement of probability of sequences that have to be compared (e.g. Chernoff, 2013) that would help individuals to make decisions; however, there is less research about the influence of judgements on decision making. When conceptualizing the notion of decision, historically researchers on the field have distinguished between judgements (e.g. estimates) and decisions (e.g. choices).

Assimilating the concept of decision with "a choice of action –of what to do or not to do" (Baron, 2008, pp. 6) is basing it on the beliefs about what actions will achieve personal goals. In concordance, we are going to speak about *decisions based on personal preference*. In decisions based on personal preference, individuals do not need to look up the data to see if they decisions would work. Individuals could ignore totally the data on hand or might try a holistic estimate first (Baron, 2008).

2.1. Decisions based on personal preference and the ignorance of chance

In the case that individuals totally ignore the data on hand, we presume that in their decisions three primitive ideas of chance will arose, such as: believing in a destiny predetermined by God or spirits, assuming personal chance factor, unequal for different individuals or accepting natural necessity (Batanero, Henry, & Parzysz, 2005). The first idea of chance, believing in a destinity predetermined by God, is prescribing to randomness a divine nature. Randomness, in this case, is the expression of God's will (Borovcnik, 2018). The second one, assuming personal chance factor, means opossing reasons to causality and accepting that randomness has not specific causes independently to the specific conditions in an experiment.

In Ancient times, craftmanships were used as randomisers for decision making. Those objects exhibited perfect symmetries and helped to understand the structural homogeneity of the shapes as a relationship between symmetry and randomness (Gandhi, 2018). Finally, the acceptance of a natural necessity of the object emerge gradually as a consequence of neglecting the deterministic nature (Saldanha & Liu, 2014). This epistemological point of view proposed by Saldanha and Liu, (2014) is coherent with the ontological proposal of Piaget and Inheler (1975) of the development of the idea of necessity.

2.2.Decisions based on personal preference relying on probabilistic heuristics and biases

Piaget and Inhelder (1975) confirmed the existence of three stages in the development of the idea of necessity for the complex understanding of chance and probability. In the first one, children do not distinguish possible from necessary events. In the second stage, children begin to differentiate between the necessary and the possible. Finally, children translate unpredictable and incomprenhensible chance into a system of operations that are still incomplete and effected unsystematically.

If there is incomplete information, people would have to rely on heuristics to bridge the lack of information which is not available by assumptions biased by heuristic strategies (Borovcnik & Kapadia, 2011). Five heuristics and biases in probability have been studied for its importance when making decisions (Baron, 2008). Those are: the representativeness heuristic (Kahneman & Tversky, 1973), the availability heuristic (Tversky & Kahneman, 1973), the subadditivity in frequency judgements (Mumford & Dawes, 1999), the hindsight bias in replications (Slovic & Fischloff, 1977) and the averaging (Birnbaum & Mellers, 1983).

People often rely on simple heuristics to empower effective decision making when they have limited time, knowledge (e.g. numerical or stochastical) and cognitive resources (Cokely et al. 2018). This empowerment would provide people with rational decision standards.

2.3. Decisions rationally bounded and the emergence of the mathematical conceptualization of randomness

Consequently, a new paradigm for understanding decision making will emerge in this spectrum that goes from the decisions based on personal preference to decisions of risk. This paradigm defines decision as *"the outcome of an inference problem using both a prediction and investigation of the current case's merits"* (Chen, Moskowitz, & Shue, 2016, p. 1185). These are rationally, normatively superior decisions that can be defined by optimization analysis, which coherently integrate values, goals, preferences, and constrains. This integration is made according with *standards* of logic, probability and statistics (Cokely et al. 2018, Baron, 2008). One of the most influential standards emerged in 1654 when Blaise Pascal and Pierre Fermat wrote about the glambling problem. Their letters became the founding documents of the logical system at the heart of modern science: decision and probability theory (Hacking, 2006).

On those initial developments of the decision theory, randomness was related to equiprobability; because they were closely linked to games of chance, for which the principle of equal probabilities is reasonable (Batanero & Serrano, 1999). In spite of the existence of games of chance (such as, IAB presented in this paper) in which the principle of equal probabilities is not reasonable; the equiprobability bias emerges (Lecoutre, 1992), and remains remarkably stable across all ages (Pratt, 2000). The emergence of this equiprobability bias constricts the possible decisions between the different members of the class, because subjects only consider randomness when all the possible results are equally probably (Batanero, 2016).

Meanwhile, when it is accepted the existence of multiple possibilities in the same conditions, a new paradigm for randomness as uncertainty, emerges (Batanero, 2016). Coherently, we can talk about decisions made under situations of uncertainty. The critizism, about the definition of randomness as equiprobability made by Kyburg (1974), applies for randomness as uncertainty since imposes that the object is a random member of a class if the class is finite. In consequence, under this paradigm for randomness, decisions made under situations of uncertainty can only be carried out if the class is finite.

Even though the class would be finite, discerning between favourable and possible outcomes can be paradoxical (Borovcnik & Kapadia, 2014). This paradoxe has consequences in understanding the compound events, because students do not appreciate the relevance of order among singleton events (Batanero, et al. , 1997). In such cases, the line between favourable, unfavourable and possible judgements for decision making has to be drawn by the theoretical winning probability. That means understanding the proportional relationship of the quantities that are involved (Saldanha & Liu, 2014)

2.4. Local and global decisions under uncertainty

In consequence, a second paradox may emerge, when those probabilities are linked to the relative frequencies, because the expected value may differ from the theoretical one. Deepening in the meaning of this last paradox means understanding that the relative frequencies are based on independently repeating a random experiment. There are many idiosyncratic perceptions about how randomness manifest in repeated trials (Borovcnik & Kapadia, 2011). Those subject who base their perceptions in short-term behaviour of the frequencies may wrongly and intuitively think that an experiment which is random has a unique formulation. And, consequently, rationally bounded their decisions under uncertainty on the local judgement of the randomness on this short-term behaviour of the frequencies. We have used the term local, for describing the local decisions under uncertainty in coherence with the closed descriptions of the local perception of randomness made by Toohey (1995) and Pratt (2000).

Contrarily, a global perception of randomness involves the students' understanding of patterns in the long run and in the distributions (Pratt, 2000). Meanwhile, Toohey (1995) speaks exclusively that the global perspective is reliant on the frequency distribution of different outcomes; Pratt (2000) goes further and talks about global resources. Three global resources would provide an aggregated overall view of the stochastic nature of the situation, which are: (a) *probability*, the proportion of outcomes for each possibility is predictable; (b) *large numbers*, the proportion of prior results for each possibility in the sample space will stabilize as an increasing number of results is considered; and (c) *distribution*, the observer is able to exert control over these proportions through manipulation of the sample space.

If the decisions are based on the use of these three global resources, we are going to talk about *global rationally bounded decisions under uncertainty*. Despite the advance that using these three global resources would mean for decision making, Nilsson, Eckert and Pratt (2018) inform about the students' difficulties in understanding the difference between drawing conclusions based on global (long-term) and local (short-term) behaviour of a data. They claim that it would be critical for understanding this difference, the discussion of the relationship of sample size for the bi-directional relationship between the proportions of a sample space and the relative frequencies in a sample. Under this global understanding relies the relationship between a classical a priori and a frequentist model of probability (Borovcnik & Kapadia, 2014). Acknowledging this relationship means giving an accurate meaning to the sample distribution and the stable frequency distribution; and, in coherence basing these global rationally bounded decisions in the distribution known. According to Knight (1921), in this situation where the distribution is know it will involve risk. And, the decions made globally bounded recognising the complex meaning of the distribution will be *decisions made under a situation of risk.*

Summing up, we have presented in this section a theoretical background to gain insights about the random nature granted to the experiment, the events and sequences of events and the decisions made. The schema presented informs about the evolution on the construction of the notion of decision coherently to the historical grown of the notion of randomness and probability. On the one end of this schema, there are the decisions based on personal preference; on the other end, there are the global rationally bounded decisions mader under a situation of risk.

3. Methodology

Integer Addition Bingo [IAB] is a game of chance based on the game of bingo (Serradó, 2018). In bingo, each player has a card arrangement filled with different numbers. Numbers are called out randomly, and players mark the numbers on their cards if they are called. Players complete against one another to be the first to have a winning arrangement. The IAB cards have ten numbers from -10 to +10. Unlikely in traditional bingo, the numbers are not called out. Rather, students watch as an applet randomly generates two numbers from -5 to +6. Students mentally add the two numbers, and if the number result of the addition is on their card, they mark the result on it. The first student to have marked all of the numbers on their card wins the game.

In 2015, an IAB task was designed and implemented to 48 Grade 7 students (ages 12-14) in a Spanish middle school located in a low socio-economic coastal city. In the retrospective analysis presented in Serradó (2018), we argued that students reasoning was constricted by a deterministic view of the game of chance (IAB generator). And, students' had difficulties in discerning between the randomness of the generator, the randomness of the events and the sequences of events. To surpass these difficulties, Serradó (2018) suggested to initiate a second cycle of DBR through improving the IAB task design.

In 2016, the IAB task was revised with the aim of: (a) reducing the number of sessions; (b) include questions to individually and cooperatively discuss about the randomness of the generator and sequences of events; and (c) promoting deliberate dialogue about students' decisions about the election of the card or its construction. In 2017, this revised version of the IAB task was implemented during four sessions of one hour to 30 students grade 10 (ages 15-17). Twenty-eight of these students participated also in the 2015 implementation of the task and two students did not have previous knowledge of the task or of probability. Students were involved in a sequential process of playing IAB, doing mathematics, and dialoguing about the decisions made. We have retrospectively analysed the recordings and videotapes of the deliberate dialogue about the decisions made by the students to elect and/or contruct the cards to play with. On those deliberate dialogues, we have identified the random nature granted to the IAB generator, the events and sequences of events generated. In the next section, we present the results of this retrospective analysis.

4. Results and discussion

After playing all the students with the same cards, they initiated a deliberate dialogue about which numbers would appear in the next game.

16	J	I have written that it is possible that we get the same numbers and that it is possible that we don't get them. Because it is a chance game and it has not a pre-established order. Because it has a different order if it is between some limits.
17	T	Do you agree? Do you want to add something else?
18	N	Are these multiple possibilities?
19	T	There exist multiple possibilities.
20	All	Yes?
21	T	Why?
22	M	Because they have the same probability of appearing.
23	V	Because they are equally likely outcomes

In this initial dialogue, we observe the successive emergence of different conceptualizations of randomness. Firstly, students assumed a personal chance factor accepting that randomness has not specific causes due to the inexistence of a pattern (Batanero, et al., 2005). There was a student that questioned the existence of multiple possibilities in the same conditions, as a new paradigm for randomness as uncertainty (Batanero, 2016). For one of the students was reasonable to argue based on the principle of equal probabilities (Batanero & Serrano,1999); although this principle is not reasonable in this case. We consider that the equiprobability bias emerged (Lecoutre, 1992). And, comparing the data of this student V, with the previous data obtained in 2015, we can confirm that this bias has remained remarkably stable (Pratt, 2000).

Under this understanding of the random nature of the IAB game, students made their first decisions by electing between four cards (card C1: +0, +0, +0, +0, +0, +0, +0, +0, +0, +0; card C2: -10, -9, -8, -7, -6, -5, -4, -3, -2, -1; card C3: -10, -8, -6, -4, -2, +2, +2, +4, +6, +8, +10; and card C4: -9, -8, -5, -5, +1, +3, +4, +10, +10, +10). The deliberate dialogue describing their elections is related with the propensity of appearing some numbers.

70 Ta I have chosen the second card, because this card has positive and negative [numbers], big and small. It has more possibilities of winning that other that had only positive or negative numbers.
71 T Does somebody want to argue differently?
72 C I do not agree with T, because I think that the second card do not have zero and it is the number with highest frequency [of appearance]. So, I selected the fourth card that the numbers are repeated twice and the numbers are close to zero. And, I did not chose the first, because all the numbers are zero and the fact that there are so many repeated. You can lose.
74 MG. I have elected the zero, because the zero is the number to be repeated more times. If it appears more times, ok?
75 Á I agree with C. I have elected the fourth card because it has the same number of positive numbers and negative ones. It has not pattern, like the second one that has even numbers. It has twice the cero, that it is the number with higher possibility of appearance. In some sense, I agree with MG, but not completely, because sometimes the zero it is not going to appear. And, finally there is variability in the order of the numbers.

Student Ta reasoned about her decisions based in the representativeness that positive and negative numbers have in the whole class of possible outcomes. We consider that her decision is based on her personal preference, relying in her decisions the representativeness heuristic (Kahneman &Tversky, 1973). Meanwhile, when the student C does not agree with student Ta, she describes her rationally bounded decision based on a model of randomness as uncertainty (Batanero, 2016). On expressing her rationally bounded decision, the ongoing dialogue of the students advanced making judgements based on their understanding of the proportional relationship of the quantities that are involved (Saldanha & Liu, 2014).

On this deliberate dialogue, the student Á confronts the other students with the fact that: *"the zero is the number with higher possibility of appearance […] sometimes the zero is not going to appear"* (student Á). In words of Borovcnik and Kapadia (2014), students are confronted with one of the paradoxes that can emerge when discerning between favourable and possible outcomes.

The student Á also points out the inexistence of patterns in the cards. In coherence with the theoretical framework presented by Pratt (2000) and Toohey (1995), this remark could be an expression of the global perception of the resource. It means that he had the intuition about the aggregate overall view of the stochastic nature of the pseudo-randomizer. The dialogue continues deepening in their understanding of the random nature of the IAB generator, their outcomes and events.

91 Mt It has failed, because there is a high variability of numbers. There is the same amount of positive and negative numbers.
92 T Ok? Has she looked for some kind of symmetry?
93 S Yes,
94 T She has opted for the symmetry of the card. Does somebody want to refute the idea, improve it or integrate it?

95 R I want to add to the conclusion of Mt that, because it is a chance game, the outcomes are going to be different.
96 T Because it is a chance game, does the outcomes are going to change? I do not know: what does it means?
97 R Let's see. That the results are random. We are not going to obtain the same outcomes in the first game than in the second one. But, it is true, that there is more probable the appearance of the numbers closes to zero; however, despite this fact in each throw and each game, the events are going to be different.

The student Mt expressed "it has failed" referring to her election of the card C3 that has the same number of positive and negative values. She has made her decisions based on the representativeness heuristic (Kahneman & Tversky, 1973) looking for the existence of patterns based on the symmetry of the numbers in the real straight line. She used this card as an expression of the symetry of the pseudo-random generator, setting an unnatural relationship between the symmetry of the outcomes of the pseudo-randomizer and the possible distribution of data. The expression of this structural homogenity of the outcomes in the straight line could be an expression of the student understanding of the relationship between symetry and randomness (Gandhi, 2018).

The student R wanted to refute the idea of randomness as symmetry of the pseudo-randomizer arguing about the sampling distribution. Althought it is true that the numbers with higher probability are close to zero, it is still an intuition for the student. Her understanding of the data distribution is reduced to the analysis of the variability obtained of the outcomes of a particular game and her prediction of the possible diferences between samples. Her argumentation using the variability of the data and the possible differences between samples informs that the student is surely developing an aggregate view of the data that could help her to understand the complexities of the sampling distribution (Baker & Gravemeijer, 2004). We interpret that her reasoning has been constricted by the hindsight bias (Slovic & Fischloff, 1997), because she has little or no objective basis for predicting that the numers wiht higher probability are close to zero. Furthermore and according to Toohey (1995), the student R is still adopting a local perception of randomness when refuting the Mt decisions based personal preferences. Students continue their dialogue deliberating about the random nature of the pseudo-randomizer and of the outcomes. The student C concludes:

145 C I want to add, that the random models of the ball and the card are different. Because I think that the random model of the balls is multiple possibilities; meanwhile the one of the card is equal probabilities. Because you try [to construct the card] in order that you have the same number of positive and negative [numbers], a number of zeros, and that you would see that the frequency of the previous game it is going to give you a benefice. And, yes there is randomness in the sequence.
146 T In the sequences of numbers that will appear.
147 C I think that they are going to be random, associated to model of randomness with multiple possibilities.
148 MG. There is dependence, because the order of the balls influence. It is not the same if you have first the first ball or the second.
149 T No, no!! I say: a different sequence of appearance -2, 1, 7, … or -1, 5, 7, …
150 MG. Ah!! Could be this [model] a lack of information?
151 T Why?
152 MG. Because you don't know surely which number is going to appear. You are blind. You write the number that you want and you are blind.
153 R At chance
154 T Why at chance?
155 R Because it is the probability that this number has to appear, then when I add…[silence]

156 Ta Tell me
157 Ta I think that it is random, because you have the same possibilities that the first shot will be negative or in the second one, and the probability changes [she hesitates]. Yes, it is random. That's all.

The student C argued with two different models of randomness. On the one hand, she associated the random nature of the IAB generator with the uncertainty of the multiple possibilities of the addition of the values of the two balls. In words of Batanero (2016), she concieved randomness as uncertainty. On the other hand, when the student C affirmed the necessity of a card with the same number of positive and negative numbers or zeros, her decisions would be constricted by a equiprobability bias (Batanero, 2016; Lecoutre, 1992). Finally, the student expressed her intuitions about the random nature of the sequence of events based on the multiple possibilities of this uncertain situation.

Although the teacher wanted that all the students advance on deliberating about the randomness of the sequence of events; the students were unable to express their beliefs, because they still did not captured the essence of the compound event of the addition of the numbers of two balls. The student MG and Ta appreciate the relevance of order among singleton events (Batanero, et al. , 1997); however, they wanted to stablish some dependence between those singleton events that did not help them to understand the compound event. Borovcnik and Kapadia (2014) analysed this paradoxe between the behaviour of singleton and compound events essential to understand the randomness under situations of uncertainty if the class is finite. Furthermore, Abrahamson (2012) suggested that an appropiate mathematical analysis of compound events needs of the understanding of the pre-symbolic notions of probabilities using the event space on hand. Coherently with this research, we think that the teacher should, before advancing in the understanding the randomness of sequences of events, deepen on students understanding of the theoretical probability distribution of the singleton events and compare it with the ones of the compound events.

A second paradoxe, expressed by Borovcnik and Kapadia (2014), emerged when students deliberate between the link between the probability and the relative frequencies.

260 MG That now we have written the zero several times, thinking that it was the most probable? But, no, it has appeared only once.
261 T Theoretically, it has more possibilities of appearance. But?
262 S But, this does not determine that it is going to appear. It cannot appear although it has a higher probability.
263 T This does not determine that it is going to appear. Look carefully the word that she has used: "It is not determined". Is it deterministic?
264 A They state: "No. It is random".
265 T It is random. So, it is true that we know that the zero is the one that has more probabilities. But, is it sure that we can obtain a sequence in which the zero is the ones with higher frequency?
266 S No
267 T Why
268 Ta Because you are not sure that the zero would appear.
269 N Because, we have multiple possibilities. It can be deterministic, but it is not the case.
270 T And, in this case it is not deterministic

Students begin their dialogue discussing again about the expectance of appearance of the number zero. They linked the probabilities to the relative frequencies, observing that the expected value may differ from the theoretical one (Borovcnik & Kapadia, 2014).

Students were able to abstract the idiosincratic perceptions of the sequence of events of on short-runs (Borovcnik & Kapadia, 2011) to conclude that the short-term behaviour of the frequencies has not a unique determinated formulation. Although students had still a local perception of randomness (Pratt, 2000; Toohney, 1995) teacher made them imagine what could happen with long-runs.

271	T	Let's our mind to fly! Think about what could happen if instead of 66 throws, we would have 350. I know that Fran is the winner, but that the game would have needed 350. What do you think it could have happened?
272	JR	Could have the zero appeared more times?
273	T	That the zero could have appeared more times.
274	C	That the relative frequency would have been smaller, because if you make a quotient with more numbers. This must be smaller.
275	T	But, he says that it would appear more times.
276	N	Then, it would be bigger.
277	T	Would be it bigger or not?

We consider that the students began to capture the agregate view of the stochastic nature of the situation, although they had problems on discerning the proportional relationships of the quantities that were involved (Saldanha & Liu, 2014). In coherence with the theoretical framework of Pratt (2000) about the global perception of randomness, we consider that the students had difficulties in discerning that the proportion of outcomes for each possibility was predictable. Although the students had difficulties in discerning those proportions, they were able to capture the similarity between the frequencies of each outcomes in long-runs with the theoretical probability when they visualize the situation.

278	A	That the relative frequency could be similar to the probability
279	T	Do you know what A argued about?
280	S	Yes!
281	T	He has argued more to the …
282	N	Probability
285	MG	If you have more throws, the relative frequency increases and it looks like the possibility
286	T	I have not used the word possibility. I have referred to theoretical probability.
287	N	And, it is repeated many times, and then it tends to be stable.
288	T	If you repeat it many times, then it tends to stabilize. Does this argument works for you?
289	S	[Affirm] Yes

Furthermore, they discussed about the stabilization of the frequency distribution when increasing the number of runs, providing a first insight about the behaviour of long runs in relation to the law of large numbers. According with Pratt (2000) this was a second condition four understanding the aggregate overall view of the stochastic nature of the situation. Despite the advance on conceptualizing the random nature of the situation, we observe that the student MG still hesitates when distinguishing between probability and possibility. We acknowledge the importance of the remark of Nilsson et al. (2018) about the need to take fine-grained account of the social and situational nature of how students express and develop an understanding of randomness and probability in particular learning environments, which aim to examine the opportunities for stablishing the bi-directional relationship between a classical a priori and a frequentist model of probability. In consequence, the teacher continues interrogating students about the difference between the stable frequency distribution and the theoretical one when increasing the runs.

292	S	The distribution of a sample of 500 throws is much more likely the theoretical, no?
293	T	And, Has it any relation with the concept use by N of smoothing?
294	S	Yes?
295	T	I have made the question! Is there any relationship with smoothing?
296	MG	Yes
297	T	Then, when the number of throws increases, that in this case is the sample size, what does it happen?
298	MC	It smoothens
299	T	Does it smoothen? One, one, because I get lost!
301	ML.	But, it never is going to be the same than the theoretical probability.
302	T	Never it is going to have the same value than the theoretical probability. But, what is it always going to happen?
303	Á	I will be close to it.
304	T	Tell me
305	C	And, every time the difference will be smaller.

The students discussed with the help of the teacher about the smotheness of the frequency distributions when the number of runs increases and compared these distributions with the sampling distribution. We understand that the expressions using *"close to it" or "the difference will be smaller"* refer to the description of the sample distribution and the stable frequency distribution. The students advanced in the understanding of the concept of sample and sampling distribution, although they did not capture all the stochastic nature that the aggregate view of this distributions (Pratt, 2000) could give them to understand the global resource of the situation. These argumentations helped students to recognise the risk involving the construction of the cards and, in coherence, they used the sample distribution to make decisions under the situation of risk (e.g. *"I will construct the card using the theoretical [one], because it is the best [distribution] that is closed to what it is going to happen" (student* A).

5. Final discussion and conclusions

We have presented a DBR with the aim of analysing the evolution of students reasoning during the social and situated deliberate dialogue about the relationship stablished between the random nature granted to the IAB game, the events and sequences of events generated and the decisions made.

Theoretically, we have presented a scheme that informs about the evolution of the construction of the notion of decision making coherent with the historical grown of the notion of randomness and probability. On the one end of this schema, there are the decisions based on personal preference linked with three primitive ideas of chance. Those are believing in a destinity predetermined by God or spirits, assuming personal chance factor, unequal for different individuals or accepting natural necessity (Batanero, Henry and Parzysz, 2005). On the other end, there are the global rationally bounded decisions made under a situation of risk (Knight, 1921) that means understanding the aggregate overall view of the stochastic nature of the situation, the probability, the large numbers and the distributions involved.

This theoreticall framework has helped to retrospectively analyse students' deliberate dialogue about the the random nature granted to the IAB situation. Students initially based their decisions on the necessity of understanding the random nature of the game generator. The lack of information made the emergence of some heuristics and bias, which were: the representativeness heuristic (Kahneman and Tversky, 1973), the hindsight bias (Slovic & Fischloff, 1997) and the equiprobability bias (Lecoutre, 1992).

Surpassing those heuristics and biases meant students' reasoning evolution on their understanding of the random nature of the IAB generator as the uncertainty of the situation (Batanero, 2016); and, in coherence, deliberating about the decisions made rationally bounded on the uncertainty of the situation. After understanding the random nature of the situation, students were confronted to deliberate about the decisions made and the random nature of the events obtained from the addition of two singleton events. Nevertheless, the students were unable to express their beliefs about the essence of the compound event. These results are coherent with the conclusions obtained by Batanero, et al. (1997) for students of the same age.

Students analysed the sequences of events obtained by short-runs of the IAB pseudo-randomizer. On those short-runs, they were able to abstract the idiosincratic percetion of the sequence of events (Borovnick & Kapadia, 2011). Students advanced on the understanding of the global perception of randomness when they began to capture an aggregate view of the stochastic nature of the situation in long-runs (Pratt, 2000). Describing the rationally bounded decisions made for constructing a card to win and comparing long-runs of different sizes helped students to gain insights about the IAB global resource. They reasoned about the stabilization of the distribution of frequencies and stablished a bi-directional relationship between the theoretical sample distribution and the stable frequentist distribution model. The students made their final decisions globally and rationally based on knowledge about the stabilized frequencies distribution and the theoretical one.

Summing up, students description of their decision made to elect and construct the cards to win in the IAB game have evolved simultaneously with the evolution of their conception about the randomness of the situation. Initally, students made decisions based on their personal preference based on the ignorance of the chance. A first, evolution of their decisions based on the uncertainty of the situation emerge when they were able to capture the multiple possibilities of the outcomes obtained in the random generator. Finally, students made globally rationally bounded decisions based in their understanding of stochastical nature of the IAB.

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