UNIVERSIDAD
DE GRANADA

Department of Statistics and Operational Research

DISCRETE COMPLEX REDUNDANT SYSTEMS WITH LOSS OF UNITS AND AN INDETERMINATE NUMBER OF REPAIRPERSONS

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GRANADA, 2018

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This Dissertation was submitted in Partial Fulfillment of the Requirements for the Doctor of Philosophy Degree in *Statistics and Operations Research*

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2018

Editor: Universidad de Granada. Tesis Doctorales
Autor: Mohammed A. S. Dawabsha
ISBN: 978-84-1306-002-6
URI: http://hdl.handle.net/10481/53795

Preface

The approach of a research work is motivated by different studies and problems, raised previously, and by an objective to be achieved. Firstly, we would like to show the main aspects which led to the desirable development and objective. Several points of a reliability system have been taken into account in this work to answer a final question.

- Serious economic and human damage can be provoked when poor system reliability causes an unscheduled interruption or system failure. Redundant systems and preventive maintenance, which is employed to avoid this outcome, or at least to improve system reliability, involves regular, routine maintenance to help keep equipment up and running.
- Nowadays, it is well known that the classical binary system of 'failure vs. operational state' has been extended by multi-state systems (MMS), the efficiency of a system may vary according to the performance level of interest.
- Different problems appear when complex reliability systems are modeled. The modelling process and the measures associated with the model have intractable expressions of highly complex applicability and interpretation. Furthermore, a reliability system can be subject to several types of events that can produce failure or degradation.
- In reliability literature is usual to consider that when a non-repairable failure occurs, the unit is replaced in a negligible time. Sometimes this assumption is not real enough and loss of units can be considered whereas the systems continue to work.
- In a repairable system the number of repairpersons and the embedded times and costs do not change over time. It seems logic to consider a variable number of repairpersons depending on the number of units in the system.

- Most studies of reliability focus on dynamic reliability systems in a continuous-time setting, while very few take into account the discrete-time case. However, not all systems can be continuously monitored, and some must be observed at certain times, for reasons such as the internal structure of the system, the need for periodic inspections, etc.

For these reasons the following question arises then, why not study a discrete complex multi-state system subject to different events and loss of units, with preventive maintenance, in a well-structured algorithmic form, optimizing the number of repairpersons?

Multi state systems

A system that has a finite number of performance levels and different failure modes with different effects on the entire system performance is called a multi-state system (MSS). Traditional reliability theory considers systems in which the units perform in terms of binary models composed of up state (performing) and down state (failure). Multi-state reliability systems have been developed and applied intensively. So, any system consisting of different binary-state units that have a cumulative effect on the entire system performance has to be considered a MSS. Multi-state systems were first introduced by Murchland (1975) [38], since that time MSS reliability began intensive development. Essential achievements that were attained up to the mid 1980's were reflected in Natvig (1985) [40] and in El-Neweihi and Proschan (1984) [12]. MSS can have a finite number of performance/degradation stages. This approach has been studied using methods such as Markov and semi-Markov models, generating functions, Lz-transform and Monte Carlo simulations. Markov models have been considered to analyses the behaviour of multi-state systems (Li et al. (2017) [28], Peng et al. (2017) [48], Xie et al. (2018) [68]). Yeh and Fiondella (2017) [69] determined the optimal redundancy allocation such that computer networks reliability is maximized. In this respect, too, Yi and Cui (2017) [70] have used Z-transform to analyse repairable aggregated semi-Markov ternary systems experiencing degradation and internal shocks, while Lisnianski et al. (2017) [32] proposed a method based on an Lz-transform of the discrete-state continuous time Markov process, and also on Ushakov's Universal Generating Operator, to evaluate the sensitivity of an aging MSS under minimal repair. Finally, in this area, Levitin et al. (2017b) [26] presented a novel Markov model of standby systems composed of multi-state elements in which, when an operating element fails, the standby element with the best technical state is chosen.

Many real-life systems contain multiple components with different performance levels. Power and computer systems are among many examples of multi-state systems. In recent years, Lisnianski et al. (2010) [31] performed a comprehensive analysis of multi-state systems, and Eryilmaz (2010) [13] studied measures for single-unit multi-state systems and multi-state *k*-out-of-*n*: G systems. In this field, too, Ruiz-Castro (2016b) [55] analysed a complex multi-state system by considering Markov counting and reward processes. Lisnianski and Frenkel (2012) [30] considered Markov processes in the analysis of multi-state systems, highlighting the benefits of their application. However, when multiple states interact within a system, problems of a complex mathematical nature may arise.

Redundant Systems and Preventive maintenance

Redundant systems and preventive maintenance are of considerable research interest to improve overall reliability, prevent system failures and reduce costs. Serious damage and considerable financial losses are caused when a system failure occurs due to poor reliability. To avoid it, several reliability methodologies are commonly employed such as redundancy and maintenance policies. Redundant systems have been proposed in the reliability literature to solve different problems. The literature related to cold, warm and *k*-out-of-*n* systems is extensive. For example, an optimisation problem was addressed by Levitin et al. (2014) [27], who considered cold and warm standby groups. Wells (2014) [67] extended known analytic results to a case with repairable and non-repairable failures, while Levitin et al. (2017a) [25] presented a method for evaluating the probability of mission success for an arbitrary redundancy level in several 1-out-of-*n* subsystems where the environment is modelled by the Poisson process of shocks, by increasing the failure rate. Kim and Kim (2017) [22] suggested the exact reliability function for a cold standby redundant subsystem with an imperfect detector/switch. Recently, the reliability of a parallel system with active multicomponents and a single cold-standby unit has been investigated by Yongjin et al. (2018) [71].

A *k*-out-of-*n*: *G* system is an *n*-system which works if at least *k* units are operational. This system, introduced by Birnbaum et al. (1961) [4], is a redundant system that is applied in various fields, such as electronic, industrial and military systems. A generalised *k*-out-of-*n* with parallel modules was developed by Cui and Xie (2005) [7]. Recently, Kamalja (2017) [19] modelled a generalised *k*-out-of-*n*: *F* system with parallel modules.

Preventive maintenance is also essential to avoid total failures that can produce great damage. It is intended to improve system reliability and to increase profits. The provision of optimal maintenance is widely recommended as an effective way of minimising system downtime and hence maintenance costs. Effective system maintenance improves overall reliability, prevents system failures and increases the benefit derived from the system. Preventive maintenance has been discussed by Osaki and Asakura (1970) [47], who studied the behaviour of a two-unit system. Mahfoud et al. (2016) [35] Reviewed this question and conducted a careful examination of the status of application-oriented research into the preventive maintenance and optimisation of medical devices. In the field of survival analysis, Huilin et al. (2015) [17] introduced models of condition-based maintenance (CBM), and Laggounea et al. (2010) [24] developed a preventive maintenance model to coordinate component replacements in a multi-component system. A standard study and advance problems of maintenance policies for reliability systems can be seen in Nakagawa (2005) [39]. Zhong and Jin (2014) [73] included preventive maintenance in a cold standby two-component system, using semi-Markovian processes. In order to keep components running properly, the working component receives periodic preventive maintenance. An optimal replacement policy was developed by Zhang and Wang (2011) [72] to cope with a deteriorating system with multiple types of failures. Under this approach, the application of an optimal replacement policy ensures that the long-run expected reward per unit of time is maximised. Qiu et al. (2017) [49] recently studied optimal maintenance policies for a competing-risk repairable system with a working state and a general number of failure modes undergoing periodic inspections, and Daneshkhah et al. (2017) [9] has developed probabilistic sensitivity analysis methods to study the sensitivity of optimised preventive maintenance decisions. Many models have been proposed to evaluate the reliability of fault-tolerant systems subject to external shocks and internal degradation. For example, a generalised reliability system subject to degradation processes and to cumulative damage from external shocks was developed by Li and Pham (2005) [28]. In a related field, , Liu et al. (2016) [33] analysed the reliability of memory chips subject to a single-event upset and to a total ionising dose effect. Preventive maintenance has also been described by Ruiz-Castro (2013, 2014) [53, 56] for use in complex systems, with either a multi-state unit or with a general set of cold standby multi-state units.

One interesting aspect in this work is the following. In reliability studies, it is usually assumed that when a system unit undergoes a non-repairable failure it is replaced by a new one within a negligible time. This assumption, however, is not always realistic. Another, perhaps more practical option, is that of redundant systems, in which a unit that undergoes a non-repairable failure will not be replaced while the system is operational. This situation has been analysed for different redundant multi-state systems (Ruiz-Castro, (2018) [52]; Ruiz-Castro and Fernández-Villodre, (2012) [63]).

Phase type distribution and Markovian arrival processes

When complex multi-state systems are modeled, the development and measures associated have intractable expressions of highly complex applicability and interpretation. A good way to analyze mathematically complex systems is through phase-type distributions and Markovian Arrival Processes thanks to the good properties and the algorithmic matrix form.

In reliability, in the continuous case, there are several distributions that are frequently used in practice, such as the exponential, Erlang and Weibull distributions. The latter involves calculations that are, in fact, unmanageable, due to the analytic expression. Phase-type (PH) distributions play an important role in this way. This class of distributions was introduced by Neuts (1975) [44], describe in detail by Neuts (1981) [42], and has been applied in fields such as reliability and queuing theory. One important property is that the set of PH-distributions is dense in the set of probability distributions on the nonnegative half-line. Then, when general distributions are present in the system, they can be approximated through phase-type distributions. Some well-known discrete probability distributions, such as geometric distribution, negative binomial distribution, …, are phase type and in the discrete case the approximation is equivalence. All discrete distributions with finite support can be represented by discrete-phase (PH) distributions. Thus, any discrete distribution with finite support is phase-type distributed. Multiple redundant multi-state systems have been modelled by considering phase-type distributions. Ruiz-Castro and Li (2011) [62] modelled a multi-state *k*-out-of-*k*: *G* system where the embedded lifetimes are PH-distributed.

A MAP is a well-structured counting process that enables reliability modelling to be developed in an algorithmic and computational form. This class of process, which is related to PH distributions, was introduced by Neuts (1979) [41] and comprehensively reviewed by Artalejo et al. (2010) [2]. The MAP has attractive properties from the viewpoint of stochastic point processes. It is one of the most general classes of stochastic counting processes and contains many commonly-used arrival processes such as the Poisson process, the PH renewal process and the Markov-modulated Poisson process (MMPP). Moreover, the MAP is dense, meaning it can approximate an arbitrary stochastic point process to a given degree of accuracy. It has been applied in fields such as telecommunication and traffic queuing systems, reliability and industrial engineering.

Two special cases of this process are Batch MAP (BMAP) and Marked MAP (MMAP). In the first case, arrivals in batch are allowed, and in the second, several types of arrivals are counted. In all cases, the arrival rates of events can be customised for different situations, which highlight the inherent versatility of this class of processes. In recent studies, He (2014) [16] and Alfa (2016) [1] presented the main results associated with MAP. In that work he developed the theory of MAPs in an intuitive way, observing that MAPs with marked arrivals, or MMAPs, are an extension of MAPs when marked arrivals occur. A disadvantage of using MMAPs is the parameterisation effort required. This problem was analysed by Buchholz et al. (2014) [6], who made several proposals on how to estimate the parameters encountered in real problems. The problem for the identifiability for the two-state Markovian arrival process is analysed by Ramírez-Cobo et al. (2010) [50] and the non-stationary version of MAPs is considered by Rodríguez (2015) [51]. An MMAP enables us to model complex multi-state systems in a well-structured way, and to obtain results in an algorithmic and computational form. This approach is of interest in telecommunication, where different types of events are counted. In this respect, approaches based on PH and the MAP have been extensively considered in reliability studies.

This class of counting processes makes it possible to model complex systems with well-structured results, thanks to their matrix-algebraic form. Many reliability systems have inputs to the system over time, such as a repairable failure, a non-repairable failure, preventive maintenance or an external shock. When a multi-state system is considered, the number of events over time can be modelled through a Markovian arrival process (MAP). A warm standby system, considering a MMAP, was recently analyzed by Ruiz-Castro (2016a) [54]. Ruiz-Castro (2016b) [55] modelled redundant complex MSS with different types of events, considering PH and MAPs, while Okamura et al. (2009) [46] addressed a parameter estimation problem of the MAP by proposing a numerical procedure for fitting a MAP and a MMPP in order to group data with an algorithm based on the expectation-maximisation (EM) approach.

Discrete time

Most queueing and reliability models in the literature before the early 1990s were developed in continuous time. Only the models such as the M/G/1 and GI/M/1 that were based on the embedded Markov chains studied the systems in discrete times. Such models were well studied in the 1950s (Kendall (1951) Kendall (1953) [20, 21]). The discrete time models developed before then were few and far between. The other discrete time models that were later studied are those by Galliher and Wheeler (1958) [14], Dafermos and Neuts (1971) [8], Neuts and Klimko (1973) [45], and Minh (1978) [37], just to name a few. However, researchers then did not see any major reasons to study queues in discrete time, except when it was felt that by doing so made difficult models easier to analyze. Examples of such cases include the embedded systems and the queues with time-varying parameters. Modelling of a communication system is an area that uses queueing models significantly and continuous time models were seen as adequate for the purpose. However now that communication systems are more digital than analogue, and we work in time slots, discrete time modelling has become more appropriate. Hence new results for discrete time analysis and books specializing on this approach are required. Time is a continuous quantity, however, for practical measurement related purposes, this quantity is sometimes considered as discrete, especially in modern communication systems where we work in time slots. If for example we observe a system every minute, how do we relate the event occurrence times with those events that occur in between our two consecutive observation time points? This is what makes discrete time systems slightly different from continuous time systems. Besides most models, for practical large systems, end up getting solved using numerical methods. Most numerical methods usually require some form of discretization. Therefore, reliability modelling in discrete time is necessary. In this respect, it is important to note that discrete time is not an immediate consequence of continuous time, and that relatively little research has focused on this question. In fact, discrete case is more difficult in the most of the cases given that events can occur simultaneously.

Reliability systems that evolve in discrete time have been proposed to analyse the behaviour of devices in fields such as civil and aeronautical engineering. Thus, Warrington and Jones (2003) [66] proposed a method that integrates discrete event simulation with path sets to achieve a dynamic system. This method was applied to the analysis of Tornado aircraft movements. In the software reliability engineering literature, studies of the fault debugging environment have been made using discrete-time modelling. A discrete-time model suitable for a periodic debugging schedule, describing maximum likelihood estimation for the model parameters, was presented by Dewanji et al. (2011) [11]. Another discrete-time model of software reliability for such a scenario of periodic debugging has been developed by Das et al. (2016) [10]. Discrete-time non-homogeneous Poisson process-based software reliability models must be developed and formulated taking into account the diversity of debugging scenarios. In this respect, Shatnawi (2016) [65] provides a new insight into the development of discrete-time modelling in software reliability engineering. Semi-Markov processes have also been considered to model discrete-time reliability systems (Barbu and Limnios (2008) [3], Georgiadis and Limnios (2014) [15]). Also, redundant Markovian multi-state systems have been studied in discrete time (Li et al. (2017) [29], Ruiz-Castro and Li (2011) [64]). Therefore, reliability modelling in discrete time is necessary. In this respect, it is important to note that discrete time is not an immediate consequence of continuous time, and that relatively little research has focused on this question.

Project aims

The overall aim of this project is to model complex systems that evolve in discrete time through Markovian Arrival Processes with marked arrivals (D-MMAP) in an algorithmic and computational form. These systems are subject to several types of failure, repairable and/or non-repairable, as a consequence of internal wear or external shocks. Random inspections are included in the models and preventive maintenance is carried out as a consequence of this. Loss of units is introduced; i.e. each time that a non-repairable occurs, it is removed and no replaced. Variable numbers of repairpersons are considered; i.e. each time that a non-repairable occurs, (the number of repairpersons changes and depends on the number of units in the system). The system will be optimised by considering two different standpoints: the profitability of preventive maintenance and the number of repairpersons present according to the number of units in the system.

In particular, the following complex multi-state systems are developed; complex one-unit system, complex cold standby systems, complex warm standby systems and k-out-of-n: G system, in a well structured and algorithmic form. The following aspects are analyzed for each system proposed.

- The system is subject to multiple failure factors (internal and accidental external failures, repairable or non-repairable).
- Preventive maintenance is included as a consequence of random inspections.
- We build new models with loss of units and with a non-fixed number of repairpersons. The number of repairpersons will vary according to the number of units in the system.
- Phase type distributions and D-MMAPs are considered in the modelling. Thus, the results are given in an algorithmic and computational way.
- A transient analysis is carried out and the stationary distribution is worked out by considering matrix analytic methods.

- For both cases, transient and stationary regime, several reliability measures of interest such as the availability, reliability, conditional probability of different types of failures, etcetera are calculated in a well-structured way.
- Rewards and costs are included in the models to optimize the behavior of the system according to preventive maintenance and number of repairpersons.
- All results are expressed in algorithmic and computational form and they have been implemented computationally with *Matlab*.

Structure of the work

This work has been performed in a sequential form, from a multi-state complex one-unit system to complex redundant systems.

- *Chapter* 1 presents the basic theory that is going to play an important role throughout this work. Phase type distributions, Markovian Arrival Processes, BMAP, MMAP, Costs, etc will be introduced in this chapter.
- *Chapter* 2 analyzes the behavior of one unit multi-state dynamic system subject to multiple events through a Markovian arrival process with marked arrivals (MMAP). This study considers if preventive maintenance is profitable or not, and also shows how the system can be optimised according to its internal performance and the external cumulative damage states revealed by inspection. A numerical example, optimising the system by determining the optimum states from an economic standpoint, illustrates the versatility of the model proposed. The results achieved in this chapter were presented at the conference.

    - SEIO 2018 (J.E. Ruiz-Castro and Mohammed Dawabsha (2018). [60]). *A Markovian arrival process with marked transitions to model multi-state complex system subject to multiple events*.

    This chapter is submitted to be published to the journal (2nd revision),

  - *Discrete Events Dynamic Systems* (Ruiz-Castro and Mohammed Dawabsha, 2017). *A discrete MMAP for analysing the behaviour of a multi-state complex dynamic system subject to multiple events.*

- *Chapter* 3 describes cold standby systems with multiple variable repairpersons, evolving in discrete time. The online unit works as the one of chapter 2. This complex system is modelled by a MMAP in an algorithmic and computational form. Two interesting contributions are made in the present study. The number of repairpersons is indeterminate and variable depending on the number of units in the system. This chapter has been published in the journal,

  - *Reliability Engineering and System Safety* (Ruiz-Castro ; Mohammed Dawabsha, and Francisco Javier Alonso (2018). [62]). *Discrete-time markovian arrival processes to model multi-state complex systems with loss of units and an indeterminate variable number of repairpersons*. DOI: 10.1016/j.ejor.2018.02.019

  And the contents of the chapter were presented at the international conferences,

  - MMR 2017 (Ruiz-Castro and Mohammed Dawabsha (2017). [59]). *Modeling a redundant multi-state system with loss of units through a MMAP*.
  - EMS 2017 (Ruiz-Castro and Mohammed Dawabsha (2017). [57]). *A Markovian arrival process with marked transitions to model a complex system with loss of units.*

- *Chapter* 4 shows complex multi-state warm standby systems subject to different types of failures with loss of units. In this study we extend chapter 3 to the warm standby case. We model general reliability systems and associated measures to analyse the behaviour and effectiveness of preventive maintenance depending on the number of repairpersons and net rewards. The content of this chapter was presented at the international conference,

  - CMStatistics2017 (Mohammed Dawabsha and Ruiz-Castro (2017). [58]. *Modeling a complex multi-state warm standby system with loss of units through a D-MMAP*.

- *Chapter* 5 describes a multi-state complex *k*-out-of-*n*: *G* system with loss of units. Due to the complexity of this model, this chapter is a first step to model a complex system where each unit has similar features than the online units described in previous chapters. Several interesting reliability measures are obtained, for both transient and stationary regime. A numerical example is given to show the versatility of the model. The contents of this chapter has been published as chapter of book in,

  - *Reliability Engineering. Theory and Applications. Taylor and Francis* (Ruiz-Castro and Mohammed Dawabsha (2018) [61]). *Modelling a multi-state k-out-of-n: G system with loss of units*.

ACKNOWLEDGEMENTS

There are many people in my life that I can't forget them, these people supported me during the last few years to complete my education, and without them this success would not have been possible.

I would like to thank my supervisor, Dr. Juan Eloy, for encouraging and supporting me during my doctoral studies. I have learned so much from him, not only on research related to the study, but also in the life experience. I am so very grateful for the opportunity to work under your guidance. I would also like to express my thanks to, Dr. Jose Miguel Angulo, for providing professional advice and guidance.

I would also like to express my appreciation and gratitude to my family, friends and brothers, especially I want to thank my mother and father, and I hope I will make them proud in the coming years. I love you all so much!

Finally, I would like to say to my parents, as well as my brothers, that they have been a big source of inspiration throughout different stations of my life, without their support and encouragement, it would have been more difficult and perhaps impossible for me to complete my doctoral studies.

Mohammed A.S. Dawabsha

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Chapter 1

Preliminaries

1.1 Introduction

In this chapter we first provide the basic definitions for discrete-time Markov-chains and several results are given for absorbing Markov chains. Costs and rewards are introduced in an algorithmic matrix form. Phase type distributions and an introduction to the Markovian Arrival Processes are given. We discuss the basic concepts of Batch Markovian arrival processes (BMAPs) and marked Markovian arrival Process (MMAP).

1.2. Discrete-time Markov chains

This section concerns discrete-time Markov chains (DTMC) defined on a finite state space with order $m+a$. These Markov chains are assumed to be homogeneous.

1.2.1 Definitions

The basic concepts about a discrete-time Markov chain are given in this section.

Definition 1.1. (Discrete-time Markov chain) A stochastic process $\{X_n; n \geq 0\}$ on state space E is said to be a Dicrete-Time Markov Chain if, for any integer n and for all i_{n+1}, i_n, …, i_0 in E,

$$P\left(X_{n+1}=i_{n+1} \mid X_n=i_n, X_{n-1}=i_{n-1}, \ldots, X_0=i_0\right)=P\left(X_{n+1}=i_{n+1} \mid X_n=i_n\right).$$

This Markov chain is said to be homogeneous if for any n = 0, 1,… and for any i and j in E, then

$$P\left(X_{n+1}=j \mid X_n=i\right)=P\left(X_1=j \mid X_0=i\right)=p_{ij}.$$

Sometimes the Markov property is described in words as "given the present state of the system, the future state is independent of its past". The probability p_{ij} is called *one-step transition probability* of the DTMC and it can be expressed as an element of the *one-step transition probability matrix*. If the finite state space is given by $E=\{1, 2, \ldots, m+a\}$, this matrix is defined as a $(m+a)$ x $(m+a)$ matrix as follows,

$$\mathbf{P}=\begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1,m+a} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2,m+a} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3,m+a} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{m+a,1} & p_{m+a,2} & p_{m+a,3} & \cdots & p_{m+a,m+a} \end{pmatrix}.$$

Clearly, each element of this matrix is greater or equal than zero and the addition of each row is equal to one (stochastic matrix). And reciprocally, any square matrix, with both properties mentioned above, can be thought of as a transition probability matrix of a DTMC.

It is of interest to work out the probability of occupying a determinate state after n steps given the initial state. It is named as the *n-step transition probability*. This probability is,

$$P\left(X_n=j \mid X_0=i\right)=p_{ij}^{(n)}.$$

The *n-step transition probability matrix* is the matrix whose elements are $p_{ij}^{(n)}$. Then,

$$\mathbf{P}^{(n)}=\begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} & p_{13}^{(n)} & \cdots & p_{1,m+a}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & p_{23}^{(n)} & \cdots & p_{2,m+a}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & p_{33}^{(n)} & \cdots & p_{3,m+a}^{(n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{m+a,1}^{(n)} & p_{m+a,2}^{(n)} & p_{m+a,3}^{(n)} & \cdots & p_{m+a,m+a}^{(n)} \end{pmatrix}.$$

The n-step transition probability matrix can be worked out from the well-known Chapman-Kolmogorov equation, $\mathbf{P}^{(n)}=\mathbf{P}^{(\nu)}\mathbf{P}^{(n-\nu)}$ for any integer ν less or equal to n.

Given that $\mathbf{P}^{(2)} = \mathbf{P}^2$, by a simple induction, we will know $\mathbf{P}^{(n)} = \mathbf{P}^n$ holds for any non-negative integer n.

If the initial distribution of the process is known, $\mathbf{b} = (b_1, \ldots, b_{m+a})$ where $b_i = P(X_0 = i)$ for i = 1,…, m+a, then the probability $b_j^{(n)} = P(X_n = j)$ for all j in E and $n \geq 0$ (*transient distribution of the process*) can be calculated as the j-th element of the following vector,

$$\mathbf{b}^{(n)} = \mathbf{b}^{(0)}\mathbf{P}^{(n)} = \mathbf{b}^{(0)}\mathbf{P}^n .$$

1.2.2 Occupancy times

The occupancy time is the expected amount of time the DTMC spends in a given state during a given interval of time. Let $N_j(n)$ be the number of times the DTMC visits state j over the time {0, 1, 2, … , n} and the expected value when the DTMC was in state i initially, $m_{ij}(n) = E\left[N_j(n) \mid X_0 = i\right]$. Then, the *occupancy times matrix*, whose elements are $m_{ij}(n)$, is given by

$$\mathbf{M}(n) = \sum_{r=0}^{n} \mathbf{P}^r \ .$$

1.2.3 Limiting, stationary and occupancy distributions

We assume a DTMC with finite state space E. In this section we study the limiting behavior of X_n as n tends to infinity, the stationary distribution and the occupancy distribution and when they exist. Definitions are given and then existence and uniqueness results are shown.

Definition 1.2. The *limiting or steady-state distribution* (if it exists) is defined as $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m, \pi_{m+a})$ where $\pi_j = \lim_{n\to\infty} P(X_n = j)$ for j in E.

Definition 1.3. A distribution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m, \pi_{m+a})$ is called *stationary distribution* if

$$P(X_0 = i) = \pi_i \text{ for all } 1 \leq i \leq m+a \ \Rightarrow\ P(X_n = i) = \pi_i \text{ for all } 1 \leq i \leq m+a \text{ and } n \geq 0 \ .$$

Definition 1.4. A distribution $\boldsymbol{\pi} = \left(\pi_1, \ldots, \pi_m, \pi_{m+a}\right)$ is called *occupancy distribution* if

$$\pi_j = \lim_{n\to\infty} \frac{E\left(N_j\left(n\right)\right)}{n+1}.$$

This value can be interpreted as the long-run fraction of the time the DTMC spends in state j.

The following property plays an important role in the analysis of the existence and uniqueness of these distributions.

A DTMC is called irreducible if, for every i and j in E, there is a $k > 0$ such that $P\left\{X_k = j \mid X_0 = i\right\} > 0$. This condition holds if and only if it is possible to go from any state i to any state j in one or more steps.

A DTMC is *periodic* with period d if d is the largest integer for all i in E such that

$$P\left(X_n = i \mid X_0 = i\right) > 0 \Rightarrow n \text{ is an integer multiple d}.$$

If $d = 1$ then the DTMC is called *aperiodic*.

The following result shows the existence and uniqueness for finite DTMC. A finite state irreducible aperiodic DTMC has a unique limiting distribution and it coincides with the stationary and occupancy distribution. This one is the solution of the equation system $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ and $\sum_{i=1}^{m+a} \pi_i = 1$.

1.2.4. Costs

Throughout this work, costs are included in the models to study the effectiveness from an economic standpoint. It has been performed by using the following results for a DTMC. Let X_n be the state of a system at time n. We assume that $\left\{X_n; n \geq 0\right\}$ is a discrete time markov chain on state space with transition probability matrix $\mathbf{P}$. Assume the system incurs a random cost of $C(i)$ monetary units every time it visits the state i. Let $c(i) = E(C(i))$ be the expected cost incurred at every visit to state i. Despite our thinking of $C(i)$ as a cost per visit, it may be any other quantity, like reward per visit, loss per visit, profit per visit, etc. (Kulkarni, 1999, [23]).

1.2.4.1 Expected Total Cost over a finite Horizon

In this subsection a way to develop methods of computing expected total cost (ETC) up to a given finite time *n*, called the horizon. The cost incurred up to time *n* is given by

$$\sum_{r=0}^{n} C\left(X_r\right),$$

and the expected value up to time *n* when $X_0 = i$ for $1 \le i \le m+a$ can be expressed as

$$g(i,n) = E\left(\sum_{r=0}^{n} C\left(X_r\right) \middle| X_0 = i\right) = \sum_{j=1}^{m+a} m_{ij}\left(n\right).c(j),$$

where $c\,(\,j\,)$ is the cost produced each time that the state *j* is visited.

This result can be expressed in a matrix form as

$$\mathbf{g}\left(n\right) = \mathbf{M}\left(n\right) \cdot \mathbf{c} = \sum_{r=0}^{n} \mathbf{P}^{r} \cdot \mathbf{c},$$

where $\mathbf{g}\left(n\right) = \begin{pmatrix} g\left(1,n\right) \\ g\left(2,n\right) \\ \vdots \\ g\left(m+a,n\right) \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} c\left(1\right) \\ c\left(2\right) \\ \vdots \\ c\left(m+a\right) \end{pmatrix}$.

1.2.4.2 Long-Run Expected Cost per unit time

In reliability is important to study the net reward or cost per unit of time. Thus, the measure worked out in section above can be expressed per unit of time as $\dfrac{g\left(i,n\right)}{n+1}$. When *n* tends to infinity this measure is the long-run cost rate. It can be proved that this value is given by

$$g = g(i) = \lim_{n\to\infty} \frac{g(i,n)}{n+1} = \sum_{j=1}^{m+a} \pi_j \, c(j) = \boldsymbol{\pi} \cdot \mathbf{c},$$

being $\boldsymbol{\pi}$ the stationary distribution of the DTMC.

1.2.5 Absorbing Markov Chains

A Markov chain with at least one absorbing state is called an *absorbing Markov chain*. If the state space is finite all the states are absorbing or transient. We assume that there are a absorbing states and m transient states. Then, the transition probability matrix associated to the Markov chain can be expressed by considering matrix blocks in the following way

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{T} & \mathbf{T}^0 \\ \hline \mathbf{0} & \mathbf{I} \end{array} \right),$$

where $\mathbf{T}$ is a square matrix with order m whose elements are the transition probabilities between any two transient states, $\mathbf{I}$ is the identity matrix with order a, and $\mathbf{T}^0$ is a matrix with order m x a that contains the one-step transition probabilities from a transient state up to an absorbing one. The matrix $\mathbf{T}$ is a sub-stochastic matrix given the structure of the matrix $\mathbf{P}$ and given that the states related to $\mathbf{T}$ are transient, the matrix $\mathbf{I}-\mathbf{T}$ is non-singular.

Several characteristics of an absorbing Markov chain are shown next.

- **Time to absorption and state of absorption**. In an absorbing Markov chain the absorption occurs with a probability equal to one. Let $\mathbf{B}^{(n)}$ a matrix whose element (i, j) is the probability that the process is absorbed into state j at the n-th step, given that initially the process started in state i. Then,

$$\mathbf{B}^{(n)} = \mathbf{T}^{n-1}\mathbf{T}^0 \;\; ; \;\; n \geq 1 .$$

  Let $\mathbf{B}$ a matrix whose element (i, j) is the probability that the process eventually gets absorbed in state j given that initially the process started in the transient state i. Then,

$$\mathbf{B} = \sum_{n=1}^{\infty} \mathbf{B}^{(n)} = \sum_{n=1}^{\infty} \mathbf{T}^{n-1}\mathbf{T}^0 = (\mathbf{I}-\mathbf{T})^{-1}\mathbf{T}^0 .$$

- **Mean time to absorption**. Let $\mathbf{D}$ be the matrix whose element (i, j) is the mean time to absorption in state j given that initially the process started in the transient state i. This measure is given by

$$\mathbf{D} = \sum_{n=1}^{\infty} n\mathbf{B}^{(n)} = \sum_{n=1}^{\infty} n\mathbf{T}^{n-1}\mathbf{T}^{0} = \left(\mathbf{I}-\mathbf{T}\right)^{-2}\mathbf{T}^{0}.$$

Given the initial distribution, **b**, the mean time to absorption by any absorbing state is given by

$$\mathbf{bDe} = \mathbf{b}\left(\mathbf{I}-\mathbf{T}\right)^{-2}\mathbf{T}^{0}\mathbf{e}.$$ [1]

Given that the matrix **P** is a stochastic matrix, then it is clear that $\mathbf{T}^{0}\mathbf{e} = \left(\mathbf{I}-\mathbf{T}\right)\mathbf{e}$.

Therefore,

$$\mathbf{bDe} = \mathbf{b}\left(\mathbf{I}-\mathbf{T}\right)^{-1}\mathbf{e}.$$

1.3 Discrete phase-type (DPH) distributions

Phase-type distributions were introduced by Neuts (1975) [44] as a generalization of the exponential distribution, and are getting to be very commonly used these days after Neuts (1981) [42] made them very popular and easily accessible. They are often referred to as the PH distribution. The PH distribution class is a highly versatile class of probability distributions. It is well known that the main barrier to the explicit solution of even very simple stochastic models is the increasing complexity of the conditional probability distributions that arise in their analysis. The pervasiveness in stochastic modeling of the exponential distribution and of the related Poisson process is rarely due to persuasive empirical evidence in support of their assumption, but, far more so, to the ease of conditioning which results from the lack-of-memory property. Many well known distributions are PH. In fact, Johnson and Taaffe (1989) [18] have shown that most of the commonly occurring distributions can be approximated by the phase type distributions using moment matching approach based on three moments. The approach is based on using mixtures of two Erlang distributions - not necessarily of common order. They seem to obtain very good fit for most of the cases which they studied. There are other works by Telek and his team for the fitting discrete PH (Bobbio et al (2004) [5]).

[1] Throughout this work the vector **e** and $\mathbf{e}_a$ denotes a column vector of ones's with appropriate order and a column vector of one's with order a, respectively.

1.3.1 Definition

We consider a Markov process on the states {1,…, m+1} with transition probability matrix

$$\mathbf{P} = \left(\begin{array}{c|c} \mathbf{T} & \mathbf{T}^0 \\ \hline \mathbf{0} & 1 \end{array} \right),$$

where $\mathbf{T}$ is a square matrix with order m and $\mathbf{T}^0$ is a matrix with order m x 1 that contains the one-step transition probabilities from a transient state up to the absorbing one[2]. Let $\boldsymbol{\alpha}$ be the initial distribution for the transient states (we assume that $\boldsymbol{\alpha}\mathbf{e}$=1).

Definition 1.5. A probability density $\{p_k\}$ on the set of nonnegative integers is called a discrete phase type (DPH) distribution if it is the density of the time until absorption in an absorbing finite state Markov chain.

The pair $(\boldsymbol{\alpha}, \mathbf{T})$ is called the representation of DPH and m is the order of the phase type distribution. The probability mass function of the time until absorption is given by for $p_k = \boldsymbol{\alpha}\mathbf{T}^{k-1}\mathbf{T}^0, \quad \text{for} \quad k \geq 1$.

The value p_0 is the probability that the process is initially in the absorbent state (in this work we suppose that it is equal to zero) while p_k is the probability of absorption at time k.

Its probability generating function and the factorial moments are given by

$$H(z) = z\boldsymbol{\alpha}(\mathbf{I} - z\mathbf{T})^{-1}\mathbf{T}^0, \text{ for} |z| \leq 1 \text{ and } H^{(k)}(1) = k!\boldsymbol{\alpha}\mathbf{T}^{k-1}(\mathbf{I} - \mathbf{T})^{-k}\mathbf{e},$$

respectively.

It follows that the mean time to absorption is $\mu = \boldsymbol{\alpha}\,(\mathbf{I} - \mathbf{T})^{-1}\mathbf{e}$.

1.3.2 Some properties of Phase Type distributions

Some well-known discrete distributions are phase type. Some of them are degenerate distribution, geometric distribution, negative binomial distribution and mixed geometric distribution. But, not only some well-known distributions are phase type, but also any general discrete distribution with finite support is a PH distribution.

[2] Throughout this work, given a matrix $\mathbf{A}$ the column vector $\mathbf{A}^0$ is defined as $\mathbf{A}^0$=$\mathbf{Ae}$

A number of operations on PH lead again to distributions of phase type. Let X and Y two PH distributions with representation $(\boldsymbol{\alpha},\mathbf{T})$ and $(\boldsymbol{\beta},\mathbf{S})$ respectively. Then,

1. The sum is a PH distribution with representation $(\phi,\mathbf{U})$ where $\phi=(\boldsymbol{\alpha},\mathbf{0})$ and

$$\mathbf{U}=\begin{pmatrix}\mathbf{T} & \mathbf{T}^0\otimes\boldsymbol{\beta}\\ \mathbf{0} & \mathbf{S}\end{pmatrix}.$$ [3]

2. The mixture with $0\le\theta_i\le1$ for i = 1, 2 and $\theta_1+\theta_2=1$, is PH distributed with representation $(\phi,\mathbf{U})$ where $\phi=(\theta_1\boldsymbol{\alpha},\theta_2\boldsymbol{\beta})$ and

$$\mathbf{U}=\begin{pmatrix}\mathbf{T} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}\end{pmatrix}.$$

3. The minimum is a PH distribution with representation $(\phi,\mathbf{U})$ where $\phi=(\boldsymbol{\alpha}\otimes\boldsymbol{\beta})$ and $\mathbf{U}=\mathbf{T}\otimes\mathbf{S}$.

4. The maximum is a PH distribution with representation $(\phi,\mathbf{U})$ where $\phi=(\boldsymbol{\alpha}\otimes\boldsymbol{\beta},\mathbf{0})$ and

$$\mathbf{U}=\begin{pmatrix}\mathbf{T}\otimes\mathbf{S} & \mathbf{T}^0\otimes\mathbf{S} & \mathbf{T}\otimes\mathbf{S}^0\\ \mathbf{0} & \mathbf{S} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{T}\end{pmatrix}.$$

These results can be extended to the case of any finite number of PH distributions.

1.4 Phase type renewal process

Renewal processes have been analyzed extensively in the literature.

Deninition 1.6. A counting process $\{N_\nu;\nu\ge0\}$ is called a renewal process if the inter-event times $\{X_\nu;\nu\ge1\}$ are independent and identically distributed.

Phase type distributions are distributions of the time until the absorption in an absorbing Markov chain. If after absorption the chain is restarted, then it represents the distribution of a renewal process. Let N_ν be the number of renewals in the interval (0, ν) and J_ν the phase of the PH distribution at time ν. Then, if we define

$$p_{ij}(k,\nu)=P(N_\nu=k,J_\nu=j\mid N_0=0,J_0=i),$$

[3] Given the matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}$, with order m x n and p x q respectively, the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ is a matrix with order mp x nq defined as $\mathbf{A}\otimes\mathbf{B}=(a_{ij}\mathbf{B})$.

as the element (i, j) of the matrix $\mathbf{P}(k,\ \nu)$, then we have

$$\mathbf{P}(0,0)=\mathbf{I},$$

$$\mathbf{P}(0,\nu+1)=\mathbf{T}\cdot\mathbf{P}(0,\nu) \quad ; \quad \nu\geq 0$$

$$\mathbf{P}(k,\nu+1)=\mathbf{T}\cdot\mathbf{P}(k,\nu)+\mathbf{T}^0\boldsymbol{\alpha}\mathbf{P}(k-1,\nu) \quad ; \quad k=1,2,... \quad ; \quad \nu\geq 0.$$

The probability generating matrix function is defined as

$$\mathbf{P}^*(z,\nu)=\sum_{k=0}^{\nu} z^k\mathbf{P}(k,\nu) \quad ; \quad \nu\geq 0 \ \text{ and } |z|<1$$

then

$$\mathbf{P}^*(z,\nu)=\left(\mathbf{T}+z\mathbf{T}^0\boldsymbol{\alpha}\right)^{\nu} \quad ; \quad \nu\geq 0 \ \text{ and } |z|<1.$$

The mean number of events up to time ν is given by

$$\boldsymbol{\alpha}\sum_{k=1}^{\nu} k\mathbf{P}(k,\nu)\cdot\mathbf{e}=\boldsymbol{\alpha}\left.\frac{\partial\mathbf{P}^*(z,\nu)}{\partial z}\right|_{z=1}\mathbf{e}=\boldsymbol{\alpha}\left.\frac{\partial\left(\mathbf{T}+z\mathbf{T}^0\boldsymbol{\alpha}\right)^{\nu}}{\partial z}\right|_{z=1}\mathbf{e}=\boldsymbol{\alpha}\sum_{k=0}^{\nu-1}\left(\mathbf{T}+\mathbf{T}^0\boldsymbol{\alpha}\right)^k\mathbf{T}^0\boldsymbol{\alpha}\mathbf{e} \quad ; \quad \nu\geq 1.$$

If initially the process begins in a transient state with probability equal to one then this mean number is $\boldsymbol{\alpha}\sum_{k=0}^{\nu-1}\left(\mathbf{T}+\mathbf{T}^0\boldsymbol{\alpha}\right)^k\mathbf{T}^0 \quad ; \quad \nu\geq 1$.

The matrix $\mathbf{T}^*=\mathbf{T}+\mathbf{T}^0\boldsymbol{\alpha}$ is a stochastic matrix that represents the transition matrix of the phase process associate with this process.

1.5 Discrete Time Markovian Arrival Processes (DMAP)

The phase type renewal process can be extended to the case when inter-events times are correlated with tractable mathematically results. The resulting model is denoted as Markovian Arrival Processes (MAPs). Earlier Neuts (1979, 1992) [41, 43] and Lucantoni (1991) [34] presented the Markovian arrival process (MAP). MAPs are a very flexible with interesting computational and algorithmic properties.

Define two sub-stochastic matrices $\mathbf{D}_0$ and $\mathbf{D}_1$, both of the dimensions n. The elements $(\mathbf{D}_0)_{ij}$ refer to transition from state i to state j without an (event) arrival because the transitions are all within transient states. The elements $(\mathbf{D}_1)_{ij}$ refer to transition from state i into the absorbing state 0 with an instantaneous restart from the transient state j with an (event) arrival during the absorption. The matrix $\mathbf{D}_0+\mathbf{D}_1$ is a stochastic matrix and we assume that it is irreducible. Let N_ν be the number of arrivals up to time ν, and I_ν the state of the Markov process at time ν.

Definition 1.7. Define $\{N_\nu, I_\nu; \nu \geq 0\}$ as a discrete time Markov chain with state space $\{0,1,2,\ldots\}$ x $\{1,2,\ldots,n\}$, where n is a positive integer, and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{D}_1 & & & \\ & \mathbf{D}_0 & \mathbf{D}_1 & & \\ & & \mathbf{D}_0 & \mathbf{D}_1 & \\ & & & & \\ & & & & \end{pmatrix}.$$

Assume that $N_0 = 0$. Then $\{N_\nu, I_\nu; \nu \geq 0\}$ is called a Markovian arrival process (MAP) with representation $(\boldsymbol{\alpha}, \mathbf{D}_0, \mathbf{D}_1)$, being $\boldsymbol{\alpha}$ the initial distribution for $\{I_\nu;\ \nu \geq 0\}$.

If we define

$$p_{ij}(k,\nu) = P(N_\nu = k, J_\nu = j \mid N_0 = 0, J_0 = i) ,$$

as the element (i, j) of the matrix $\mathbf{P}(k,\ \nu)$, then we have

$$\mathbf{P}(0,0) = \mathbf{I},$$

$$\mathbf{P}(0,\nu+1) = \mathbf{P}(0,\nu)\cdot\mathbf{D}_0 \quad ; \quad \nu \geq 0$$

$$\mathbf{P}(k,\nu+1) = \mathbf{P}(k,\nu)\cdot\mathbf{D}_0 + \mathbf{P}(k-1,\nu)\cdot\mathbf{D}_1 \quad ; \quad k = 1,2,\ldots \quad ; \quad \nu \geq 0,$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{P}(k,\nu) = \mathbf{0}$ for $k \geq \nu + 1$.

The matrix probability generating function is

$$\mathbf{P}^*(z,\nu) = (\mathbf{D}_0 + z\mathbf{D}_1)^\nu \quad ; \quad \nu \geq 0,\ |z| \leq 1.$$

The mean number of events up to time ν is given by

$$\boldsymbol{\alpha}\sum_{k=1}^{\nu} k\mathbf{P}(k,\nu)\cdot\mathbf{e} = \boldsymbol{\alpha}\left.\frac{\partial \mathbf{P}^*(z,\nu)}{\partial z}\right|_{z=1}\mathbf{e} = \boldsymbol{\alpha}\left.\frac{\partial (\mathbf{D}_0 + z\mathbf{D}_1)^\nu}{\partial z}\right|_{z=1}\mathbf{e} = \boldsymbol{\alpha}\sum_{k=0}^{\nu-1}\mathbf{D}^k\mathbf{D}_1\mathbf{e} \quad ; \quad \nu \geq 1,$$

where $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1$.

1.6 Discrete Time Batch Markovian Arrival Processes (BMAP)

The Markovian arrival processes are extended by considering batch arrivals. Define sub-stochastic matrices $\mathbf{D}_k$, k = 0, 1, 2, …, m, such that $\mathbf{D} = \sum_{k=0}^{m} \mathbf{D}_k$ is stochastic and irreducible. The elements $(\mathbf{D}_k)_{ij}$ refer to transition from state i into the absorbing state 0 with an instantaneous restart from the transient state j with k (event) arrival during the absorption. Let N_ν be the number of arrivals up to time ν, and I_ν the state of the Markov process at time ν.

Definition 1.8. Define $\{(N_\nu, I_\nu); \nu \geq 0\}$ as a discrete time Markov chain with state space $\{0,1,2,\ldots\} \text{ x } \{1,2,\ldots,n\}$, where n is a positive integer, and transition probability matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \cdots & \mathbf{D}_m & & \\ & \mathbf{D}_0 & \mathbf{D}_1 & \cdots & \mathbf{D}_m & \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Assume that $N_0 = 0$. Then $\{(N_\nu, I_\nu); \nu \geq 0\}$ is called a Batch Markovian Arrival Process (BMAP) with representation $(\boldsymbol{\alpha}, \mathbf{D}_0, \mathbf{D}_1, ..., \mathbf{D}_m)$, being $\boldsymbol{\alpha}$ the initial distribution for $\{I_\nu;\ \nu \geq 0\}$.

If we define

$$p_{ij}(a, \nu) = P(N_\nu = a, J_\nu = j \mid N_0 = 0, J_0 = i) ,$$

as the element (i, j) of the matrix $\mathbf{P}(k, \nu)$, then we have

$$\mathbf{P}(0,0) = \mathbf{I} ,$$

$$\mathbf{P}(0, \nu+1) = \mathbf{P}(0, \nu) \cdot \mathbf{D}_0 \quad ; \quad \nu \geq 0$$

$$\mathbf{P}(a, \nu+1) = \sum_{k=0}^{\min\{m,a\}} \mathbf{P}(a-k, \nu) \cdot \mathbf{D}_k .$$

The matrix probability generating function is

$$\mathbf{P}^*(z,\nu)=\left(\mathbf{D}_0+\sum_{k=1}^{m}z^k\mathbf{D}_k\right)^{\nu} \quad ; \quad \nu\geq 0\,,\ |z|\leq 1\,.$$

The mean number of events up to time ν is given by

$$\boldsymbol{\alpha}\sum_{k=1}^{\nu}k\mathbf{P}(k,\nu)\cdot\mathbf{e}=\boldsymbol{\alpha}\left.\frac{\partial\mathbf{P}^*(z,\nu)}{\partial z}\right|_{z=1}\mathbf{e}=\boldsymbol{\alpha}\left.\frac{\partial\left(\mathbf{D}_0+\sum_{k=1}^{m}z^k\mathbf{D}_k\right)^{\nu}}{\partial z}\right|_{z=1}\mathbf{e}=\boldsymbol{\alpha}\sum_{k=0}^{\nu-1}\mathbf{D}^k\left(\sum_{l=1}^{m}l\mathbf{D}_l\right)\mathbf{e} \quad ;\nu\geq 1.$$

If we define $\boldsymbol{\pi}$ such that

$$\boldsymbol{\pi}\mathbf{D}=\boldsymbol{\pi} \quad , \ \boldsymbol{\pi}\mathbf{e}=\mathbf{e}\,.$$

then the mean number of events up to time ν in stationary regime is given by

$$\nu\sum_{l=1}^{m}l\boldsymbol{\pi}\mathbf{D}_l\mathbf{e} \quad ;\nu\geq 1.$$

1.7 Marked Markovian Arrival Processes (MMAP)

Markovian Arrival processes with marked arrivals (MMAP) are generalizations of BMAPs that accommodate processes with different types of events. Define sub-stochastic matrices $\left\{\mathbf{D}_h;h\in C^0\right\}$ such that $\mathbf{D}=\sum_{h\in C_0}\mathbf{D}_h$ is stochastic and irreducible. The elements $\left(\mathbf{D}_h\right)_{ij}$ refer to transition from state i into the absorbing state 0 with an instantaneous restart from the transient state j with one (event) arrival type h during the absorption. Let $\{N_h(\nu);\nu\geq 0\}$ and $\{X(\nu);\nu\geq 0\}$ be the number of events of type $h\in C^0$ (C^0 being the set composed of all types of events) and the underlying Markov process associated with the MMAP respectively.

Definition 1.9. The process $\left\{\left(N_h(\nu),h\in C^0,X(\nu)\right)\ ;\ \nu\geq 0\right\}$ is called MMAP.

To analyse the number of events in [0, ν], several functions must be defined. The probability distribution of the MMAP is given by the matrix

$$\mathbf{P}(\{n_h,h\in C^0\},\nu)=\left(P\{X(\nu)=j,N_h(\nu)=n_h,h\in C^0\mid X(0)=i\}\right).$$

The joint probability generating function for the number of arrivals in [0, ν] is defined as

$$\mathbf{P}^*(\{z_h, h \in C^0\}, \nu) = \sum_{\{n_h \geq 0, h \in C^0\}} \mathbf{P}(\{n_h, h \in C^0\}, \nu) \prod_{h \in C^0} z_h^{n_h} \ .$$

The MMAPs are used throughout this work and the main results have been developed for discrete case in Section 2.5.

Chapter 2

A multi-state dynamic one-unit system subject to multiple events

2.1 Introduction

The present chapter focuses on modelling a complex multi-state system that evolves in discrete time through a Markovian arrival process with marked arrivals (MMAP). This system is subject to several types of failure, repairable and/or non-repairable, as a consequence of internal wear or external shocks. Random events occur over time and if they impact on the system, diverse consequences can occur, including deterioration of internal performance, extreme failure or cumulative external damage. The internal performance state and that of cumulative external damage are partitioned according to the risk of failure: minor or major. Preventive maintenance is introduced, in conjunction with random inspection. If major internal or external damage is observed, the unit is sent to the repair facility for preventive maintenance. If a repairable failure occurs the unit is sent to the repair facility for corrective repair. The corrective repair and the preventive maintenance times have different distributions depending on the system state at which it failed or was observed. When a non-repairable failure occurs the device is replaced by an identical one. In this study, measures such as reliability, availability and expected number of events over time are obtained, and the correlation coefficient between two different types of events is determined and applied. Rewards and costs depending on the system state at which it failed or was inspected are included in the model. The model determines when preventive maintenance should be applied to optimise the behaviour of the system, from different standpoints. The modelling and the results obtained are presented in a matrix-algorithmic computational form, and are implemented computationally with Matlab.

2.1.1 Motivation and contribution

The model we present can be applied in fields such as civil, industrial and computer engineering. For instance, in computer engineering, the hard drive attached to a computer server is periodically inspected by an installed monitoring program that analyses logic and physics parameters to detect possible errors caused by internal and external events. In industrial engineering, any facility that requires a reliable electrical supply must have available generating sets capable of generating electricity in case of need. A genset is a diesel motor with a generator subject to repairable or total failures, for which preventive maintenance is necessary.

An interesting situation that can arise regarding preventive maintenance in the context of complex systems in which different types of failure may occur is when inspection reveals major damage to the system, which must then go to the repair facility for preventive maintenance, where different cost and time distributions may be present. We analyse this question by considering optimisation from the standpoints of cost and reliability.

This study extends previous research in this area in the following ways:

- The system passes through an indeterminate level of degradation, associated with its performance status. The unit is subject to failures that may be repairable or non-repairable, internal or external shocks.
- External shocks can produce consequences such as extreme failure, cumulative external damage (a non-repairable failure if a threshold is reached), and aggravation of the internal degradation or internal failure.
- Preventive maintenance is performed in response to random inspections.
- The major and minor states for internal performance and cumulative external damage can vary in number.

- The repair time distribution depends on the internal degradation status of the system.
- The preventive maintenance time distribution depends on the internal degradation and external cumulative damage observed by inspection.
- All results are expressed in algorithmic form, with PH distributions and Markovian Arrival Processes, with marked arrivals in discrete time (D-MMAP).
- Transient and long-term algorithmic analyses are performed. The stationary distribution is constructed using matrix analytic methods.
- Rewards and costs of repair and preventive maintenance, depending on the system state at which the unit failed or was inspected, are included.
- In addition to the optimisation analysis performed with respect to preventive maintenance, we also calculated the optimum internal status and the optimum level of external cumulative damage when preventive maintenance should be carried out.

This chapter is organised as follows. The system and its modelling are described in Section 2.2. The MMAP that governs the system is given in Section 2.3. The following section presents the transient and stationary distributions in an algorithmic form. Measures such as reliability, availability and the analysis of the time between events are addressed in Section 2.4. Section 2.5 then focuses on the mean number of events and correlations. Rewards are considered in Section 2.6, after which in Section 2.7 we analyse the optimisation process. A numerical example to illustrate the versatility of the model is given in Section 2.8. Finally, an Appendix is given in Section 2.9.

2.2 The system and the model

In this section, the assumptions underlying the system are described in detail. To model the system, the state-space must be well structured and so the system behaviour is modelled in a matrix-algorithmic form.

2.2.1 Assumptions of the system

We assume a multi-state complex system subject to repairable and/or non-repairable internal failures, external shocks and inspections. The internal performance of the system is composed of several states which are partitioned into two well-differentiated groups: minor and major damage states, which reflect a low and high risk of failure, respectively. From each of these operational states a repairable or non-repairable failure may occur. The unit is also exposed to external shocks. When a shock occurs and the system is operational, it may undergo one of three possible consequences: internal deterioration, cumulative external damage or extreme failure. Each time an external shock takes place, the cumulative external damage increases by passing through an external damage state. When the cumulative external damage reaches a given threshold, the unit undergoes a non-repairable failure. In addition, when the unit undergoes an internal repairable failure, the system is sent to the repair facility for corrective repair. Analogously to the internal case, the cumulative external damage states are partitioned into minor and major damage states. Finally, an external shock may produce an extreme non-repairable failure. After a non-repairable failure, whether internal or the consequence of an external shock, the unit is removed and replaced by an identical one. Preventive maintenance is introduced into the system in response to random inspections, of which periodic inspection is a particular case. When an inspection takes place, the internal and the cumulative external damage states are observed. If a major internal or cumulative external damage state is observed, the unit is sent to the repair facility for preventive maintenance. The time distributions for repairs and preventive maintenance depend on the system state when inspection was performed. The repair facility is staffed by one repairperson. The system is based on the following assumptions.

***Assumption* 2.1.** The internal operational time of the unit system is *PH*-distributed with representation (α, $\mathbf{T}$) with order n. The n operational states are partitioned into minor damage states (the first n_1 states) and major damage states (states n_1+1,…, n). State 1 indicates that the system does not present significant damage.

***Assumption* 2.2.** When an internal failure occurs, it may be repairable or non-repairable. The probability of the system undergoing a repairable or non-repairable failure from a transient state is given by the column vectors $\mathbf{T}_r^0$ and $\mathbf{T}_{nr}^0$, respectively. The probability of failure from the internal state i at one step is given by the i-th element of the column vector $\mathbf{T}^0 = \mathbf{T}_r^0 + \mathbf{T}_{nr}^0$.

***Assumption* 2.3**. Events that may produce failures of the system due to external shocks occur according to a phase-type renewal process. If the system is operational, the unit undergoes the effect of this shock. The time between two consecutive events is *PH* distributed with representation (γ, $\mathbf{L}$). The order of the matrix $\mathbf{L}$ is equal to t.

***Assumption* 2.4**. If the unit is operational, an external shock produces one of three different effects: extreme failure (non-repairable), external cumulative damage or aggravation of internal degradation.

***Assumption* 2.5.** An extreme (non-repairable) failure occurs with a probability equal to ω^0 after an external shock.

***Assumption* 2.6**. External damage may pass through an indeterminate number of external degradation states, d, which are partitioned into minor damage states (the first d_1 states) and major damage states (states d_1+1,..., d). If the external degradation state is i, then the external shock changes to state j with probability q_{ij}. These probabilities are contained in the matrix $\mathbf{Q}$. A cumulative external damage threshold is reached from the external damage states after an external shock, which is reflected in the probability column vector $\mathbf{Q}^0$. If this threshold is reached, the unit undergoes a non-repairable failure. Prior to such an external shock, the unit is in external degradation state 1 (no damage due to external shock). The initial distribution for external damage when a unit is at its initial online situation $\omega = (1,0,...,0)_{1xd}$.

***Assumption* 2.7.** An external shock modifies the internal degradation state while the unit is operational. If the internal degradation state is i, then the external shock changes it to state j with probability w_{ij}. These probabilities are included in matrix $\mathbf{W}$. An internal repairable failure may occur for this reason from any performance state with a probability column vector $\mathbf{W}^0$.

***Assumption* 2.8**. When a repairable failure occurs from operational state i, the unit system is sent to the repair facility. The repair time required depends on the state i and it is *PH* distributed with representation ($\boldsymbol{\beta}^{c,i}$, $\mathbf{S}_{c,i}$) with order $z_{c,i}$ for $i = 1,\ldots,n$.

***Assumption* 2.9.** While the unit is operational, random inspections may be made. The time between two consecutive inspections is *PH* distributed with representation ($\boldsymbol{\eta}$, $\mathbf{M}$) with order ε.

***Assumption* 2.10**. If an inspection observes major internal damage (state i) or major external cumulative damage (state j) then the unit goes to the repair facility for preventive maintenance. The preventive maintenance time depends on these states and it is *PH* distributed with representation ($\boldsymbol{\beta}^{p,i,j}$, $\mathbf{S}_{p,i,j}$) with order $z_{p,i,j}$ for $i = n_1+1, \ldots, n$ and $j = d_1+1, \ldots, d$. We assume $i = 0$ or $j = 0$ if minor internal or external damage is observed respectively.

***Assumption* 2.11**. When the online unit undergoes a non-repairable failure, it is replaced by an identical unit.

Te operation of the system is given in Figure 2.1.

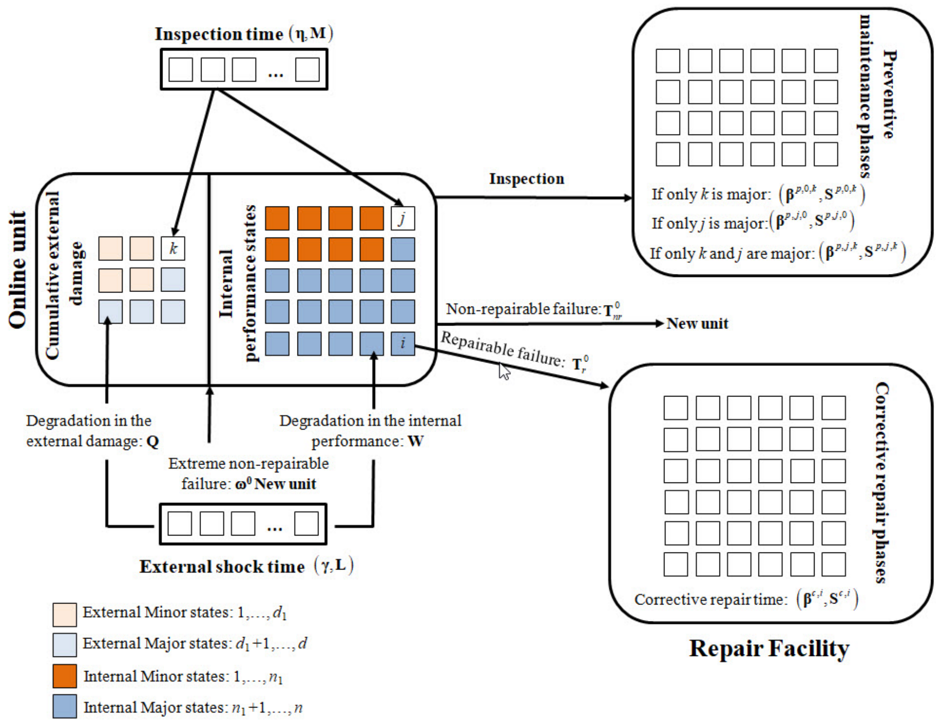

Figure 2.1. Diagram of the system

2.2.2 The Model

The system described above is governed by a vector Markov process. The state space E is composed of the macro-states $E=\{E^1, E^2, E^3\}$, where E^k contains the phases when the unit is operational (k = 1), the unit is in corrective repair (k = 2) and the unit is in preventive maintenance (k = 3). The macro-states E^2 and E^3 are composed of macro-states $E^{2,i}$ and $E^{3,i,j}$ respectively depending on the state at which the system failed or was inspected respectively. The phases of these macro-states are given by

$$E^1 = \{(i, j, u, m); 1 \le i \le n, 1 \le j \le t, 1 \le u \le d, 1 \le m \le \varepsilon\},$$
$$E^2 = \{E^{2,i}; 1 \le i \le n\},$$
$$E^{2,i} = \{(j,a); 1 \le j \le t, 1 \le a \le z_{c,i}\}, \text{ for } i = 1, \ldots, n,$$
$$E^3 = \{E^{3,i,j}; \{n_1+1 \le i \le n, j=0\}, \{d_1+1 \le j \le d, i=0\}, \{n_1+1 \le i \le n, d_1+1 \le j \le d\}\},$$
$$E^{3,i,j} = \{(j,a); 1 \le j \le t, 1 \le a \le z_{p,i,j}\},$$

where i denotes the phase of the internal operational time, j the phase of the external shock time, u the cumulative external damage, m the phase of the inspection time and a the phase of the corrective repair or the preventive maintenance time.

When a complex system is subject to several types of events, it is important to analyse their behaviour in order to avoid or delay economic or catastrophic failures. The unit is subject to several types of events which may cause failures. Three different impacts on the online unit are considered; repairable internal failure (A), inspection revealing major internal and/or external damage (B) and non-repairable failure (C).

The transition probabilities associated with these events are modelled in a well-structured way. Before discussing these probabilities, some auxiliary matrices are introduced.

The matrices $\mathbf{U}_1, \mathbf{U}_2^i$ and $\mathbf{V}_1, \mathbf{V}_2^i$ are square matrices of order n and d respectively, whose elements (s, t) are given by,

$$\mathbf{U}_1(s,t) = \begin{cases} 1 & ; \quad 1 \le s = t \le n_1 \\ 0 & ; \quad \text{otherwise} \end{cases}, \quad \mathbf{U}_2^i(s,t) = \begin{cases} 1 & ; \quad s = t = i \\ 0 & ; \quad \text{otherwise} \end{cases},$$

$$\mathbf{V}_1(s,t) = \begin{cases} 1 & ; \quad 1 \le s = t \le d_1 \\ 0 & ; \quad \text{otherwise} \end{cases}, \quad \mathbf{V}_2^i(s,t) = \begin{cases} 1 & ; \quad s = t = i \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

These matrices are applied when minor internal or cumulative external damage is observed by inspection ($\mathbf{U}_1$ and $\mathbf{V}_1$, respectively), or when an event occurs specifically and exclusively during state i ($\mathbf{U}_2^i$ and $\mathbf{V}_2^i$).

The transition probabilities for the complex system, depending on the type of event (repairable failure, major damage revealed by inspection, non-repairable failure), are given as follows.

No events

These transitions take place when no events occur, with or without inspection (inspection only reveals minor damage). There are four possible outcomes:

a. No inspection is made ($\mathbf{M}$), there is no external shock ($\mathbf{L}$) and the internal performance can modify its state without failure ($\mathbf{T}$).
b. No inspection is made ($\mathbf{M}$) but an external shock takes place ($\mathbf{L}^0\gamma$) producing external damage without non-repairable failure ($\mathbf{Q}\left(1-\omega^0\right)$) and the internal performance can be modified ($\mathbf{TW}$).
c. An inspection is made ($\mathbf{M}^0\eta$), no external shock takes place and the internal performance and external cumulative damage are in a minor damage state that can be modified without failure ($\mathbf{U}_1\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{V}_1$).
d. Inspection ($\mathbf{M}^0\eta$) and external shock both take place. There is no failure. The external shock may provoke internal and/or external cumulative damage, but in either/both cases, inspection reveals the damage to be minor ($\mathbf{U}_1\mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_1\mathbf{Q}\left(1-\omega^0\right)$).

The transition probability matrix is

$$\begin{aligned}\mathbf{H}_0 = &\left[\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{I}+\mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{Q}\left(1-\omega^0\right)\right]\otimes\mathbf{M}\\ &+\left[\mathbf{U}_1\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{V}_1+\mathbf{U}_1\mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_1\mathbf{Q}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0\eta.\end{aligned}$$

Internal Repairable failure (A)

The unit may undergo a repairable failure from the operational internal state i due to wear or external shock. In the first case, this occurs because the repairable internal failure is produced from state i ($\mathbf{U}_2^i\mathbf{T}_r^0$). In the second case, an internal failure may take place because a shock modifies the internal behaviour ($\mathbf{U}_2^i\mathbf{TW}^0$) without producing an extreme failure or causing the external threshold damage state to be reached ($\mathbf{Qe\omega}\left(1-\omega^0\right)$).

The matrix that governs this transition is given by

$$\mathbf{H}_1^i=\left[\mathbf{U}_2^i\mathbf{T}_r^0\otimes\mathbf{L}\otimes\mathbf{e}_d+\left(\mathbf{U}_2^i\mathbf{T}_r^0+\mathbf{U}_2^i\mathbf{TW}^0\right)\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{Qe}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon,\text{ for } i=1,\dots,n.$$

Inspection reveals major internal and/or external damage (B)

While the online unit is working, an inspection may take place. If it reveals any of the following situations, the unit must go the repair facility:

a. Major internal damage from state i without external shock, $\mathbf{U}_2^i\left(\mathbf{e}-\mathbf{T}^0\right)$ and $\mathbf{V}_1\mathbf{e}$, or with external shock, $\mathbf{U}_2^i\mathbf{TWe}$ and $\mathbf{V}_1\mathbf{Qe}\left(1-\omega^0\right)$.
b. Major cumulative external damage from state j without external shock, $\mathbf{U}_1\left(\mathbf{e}-\mathbf{T}^0\right)$ or $\mathbf{V}_2^j\mathbf{e}$, and with external shock, $\mathbf{U}_1\mathbf{TWe}$ and $\mathbf{V}_2^j\mathbf{Qe}\left(1-\omega^0\right)$.
c. Major internal and external cumulative damage from state i and j, respectively, without external shock $\mathbf{U}_2^i\left(\mathbf{e}-\mathbf{T}^0\right)$ and $\mathbf{V}_2^j\mathbf{e}$, or with external shock $\mathbf{U}_2^i\mathbf{TWe}$ and $\mathbf{V}_2^j\mathbf{Qe}\left(1-\omega^0\right)$.

Therefore,

$$\mathbf{H}_2^{i,0}=\left[\mathbf{U}_2^i\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{L}\otimes\mathbf{V}_1\mathbf{e}+\mathbf{U}_2^i\mathbf{TWe}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_1\mathbf{Qe}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0,$$

$$\mathbf{H}_2^{0,j}=\left[\mathbf{U}_1\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{L}\otimes\mathbf{V}_2^j\mathbf{e}+\mathbf{U}_1\mathbf{TWe}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_2^j\mathbf{Qe}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0,$$

$$\mathbf{H}_2^{i,j}=\left[\mathbf{U}_2^i\left(\mathbf{e}-\mathbf{T}^0\right)\otimes\mathbf{L}\otimes\mathbf{V}_2^j\mathbf{e}+\mathbf{U}_2^i\mathbf{TWe}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_2^j\mathbf{Qe}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0,$$

for $i = n_1+1,\dots, n$ and $j = d_1+1,\dots, d$.

Non-repairable failure (C)

While the unit is working, a non-repairable failure may occur, due to wear from any internal operational state ($\mathbf{T}_{nr}^0\boldsymbol{\alpha}$) or as a consequence of an external shock. This situation is arises when an external shock causes an extreme failure (ω^0) or when the cumulative external threshold is reached ($\mathbf{Q}^0$). In either case, the operational time of the online unit, the cumulative external damage and the inspection time are all reinitialised ($\boldsymbol{\alpha}$, $\boldsymbol{\omega}$, $\boldsymbol{\eta}$). The matrix is given by

$$\mathbf{H}_3 = \left[\mathbf{T}_{nr}^0\boldsymbol{\alpha} \otimes \left[\mathbf{L} \otimes \mathbf{e}\boldsymbol{\omega} + \mathbf{L}^0\boldsymbol{\gamma} \otimes \mathbf{Qe}\boldsymbol{\omega}\left(1-\omega^0\right)\right] + \mathbf{e}\boldsymbol{\alpha} \otimes \mathbf{L}^0\boldsymbol{\gamma} \otimes \left(\mathbf{e}\boldsymbol{\omega}\omega^0 + \mathbf{Q}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right)\right] \otimes \mathbf{e}\boldsymbol{\eta}.$$

The state space and the model when preventive maintenance is not considered (case for $n_1 = n$ and $d_1 = d$) are described in the Appendix 2A.

2.2.3 The Markovian Arrival Process with marked arrivals

The system for the model with preventive maintenance is governed by the Markovian arrival process with marked arrivals (MMAP) with representation $(\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3)$, where $\mathbf{D}_1$ denotes the matrix associated with the repairable failure event, $\mathbf{D}_2$ denotes a major positive inspection with preventive maintenance and $\mathbf{D}_3$ denotes a non-repairable failure. The transition probability matrix of the Markov chain is given by $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3$. This matrix is built by considering the macro-states E^1, E^2 and E^3 given in Section 2.2.2.

Matrix $\mathbf{D}_0$

The matrix $\mathbf{D}_0$ contains the transitions when no failure or preventive maintenance take place. This matrix is given by

$$\mathbf{D}_0 =$$

$$\begin{array}{cc} & \begin{array}{ccc} E^1 & E^2 & E^3 \end{array} \\ \begin{array}{c} E^1 \\ E^2 \\ E^3 \end{array} & \begin{pmatrix} \mathbf{H}_0 & \mathbf{0} & \mathbf{0} \\ \begin{pmatrix} \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{c,1} \\ \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{c,2} \\ \vdots \\ \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{c,n} \end{pmatrix} & diag((\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\mathbf{S}_{c,1},\ldots,(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\mathbf{S}_{c,n}) & \mathbf{0} \\ \begin{pmatrix} \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{p,n_1+1,0} \\ \vdots \\ \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{p,n,0} \\ \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{p,0,d_1+1} \\ \vdots \\ \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{p,0,d} \\ \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{p,n_1+1,d_1+1} \\ \vdots \\ \boldsymbol{\alpha}\otimes(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\boldsymbol{\omega}\otimes\boldsymbol{\eta}\otimes\mathbf{S}^0_{p,n,d} \end{pmatrix} & \mathbf{0} & diag((\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\mathbf{S}_{p,n_1+1,0},\ldots,(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma})\otimes\mathbf{S}_{p,n,d}) \end{pmatrix} \end{array}$$

Matrix $\mathbf{D}_1$

The matrix $\mathbf{D}_1$ contains the transitions when a repairable internal failure occurs. This matrix is given by

$$\mathbf{D}_1 = \begin{array}{cc} & \begin{array}{ccc} E^1 & E^2 & E^3 \end{array} \\ \begin{array}{c} E^1 \\ E^2 \\ E^3 \end{array} & \begin{pmatrix} \mathbf{0} & (\mathbf{H}_1^1\otimes\boldsymbol{\beta}^{c,1},\ldots,\mathbf{H}_1^n\otimes\boldsymbol{\beta}^{c,n}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}.$$

Matrix $\mathbf{D}_2$

The matrix $\mathbf{D}_2$ contains the transitions when preventive maintenance takes place. This matrix is given by

$$\mathbf{D}_2 = \begin{array}{c} \\ E^1 \\ E^2 \\ E^3 \end{array} \begin{array}{c} \begin{array}{ccc} E^1 & E^2 & E^3 \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathbf{0} & (\mathbf{H}_2^{n_1+1,0} \otimes \boldsymbol{\beta}^{p,n_1+1,0},\dots,\mathbf{H}_2^{n,d} \otimes \boldsymbol{\beta}^{p,n,d}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}.$$

Matrix $\mathbf{D}_3$

The matrix $\mathbf{D}_3$ contains the transitions when a non-repairable failure occurs. This matrix is given by

$$\mathbf{D}_3 = \begin{array}{c} \\ E^1 \\ E^2 \\ E^3 \end{array} \begin{array}{c} \begin{array}{ccc} E^1 & E^2 & E^3 \end{array} \\ \begin{pmatrix} \mathbf{H}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}.$$

2.3 Transient and stationary distributions

The transition probabilities are presented in a computational, algorithmic way by considering matrix blocks. The transition probability matrix is given by

$$\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3 ,$$

and can be expressed by considering the blocks of the macro-states E^k, for k = 1,2,3 as

$$\mathbf{D} = \begin{array}{c} \\ E^1 \\ E^2 \\ E^3 \end{array} \begin{array}{c} \begin{array}{ccc} E^1 & E^2 & E^3 \end{array} \\ \begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} = \mathbf{0} \\ \mathbf{D}_{31} & \mathbf{D}_{32} = \mathbf{0} & \mathbf{D}_{33} \end{pmatrix} \end{array}.$$

External shocks may occur independently of whether the system is operational or not. For this reason, the initial distribution for the time of the external shock is the stationary distribution of the process with transition probability matrix $\mathbf{L} + \mathbf{L}^0\boldsymbol{\gamma}$. This stationary distribution is equal to

$$\boldsymbol{\gamma}^* = [1, \mathbf{0}](\mathbf{e}_t \,|\, (\mathbf{I} - \mathbf{L} - \mathbf{L}^0\boldsymbol{\gamma})^*)^{-1},$$

where $(\mathbf{I} - \mathbf{L} - \mathbf{L}^0\boldsymbol{\gamma})^*$ is the matrix $\mathbf{I} - \mathbf{L} - \mathbf{L}^0\boldsymbol{\gamma}$ without the first column.

The initial distribution of the system is given by

$$\phi = (\boldsymbol{\alpha} \otimes \boldsymbol{\gamma}^* \otimes \boldsymbol{\omega} \otimes \boldsymbol{\eta}, \mathbf{0}),$$

and so the transient distribution is given by $\mathbf{a}(\nu) = \phi \mathbf{D}^{\nu}$.

The transition probability matrix in n steps has been calculated by matrix blocks to minimise the computational cost in a recursive form. Then,

$$\mathbf{D}^n = \begin{pmatrix} \mathbf{D}_{11}^{(n)} & \mathbf{D}_{12}^{(n)} & \mathbf{D}_{13}^{(n)} \\ \mathbf{D}_{21}^{(n)} & \mathbf{D}_{22}^{(n)} & \mathbf{D}_{23}^{(n)} \\ \mathbf{D}_{31}^{(n)} & \mathbf{D}_{32}^{(n)} & \mathbf{D}_{33}^{(n)} \end{pmatrix},$$

where

$$\mathbf{D}_{ij}^{(1)} = \mathbf{D}_{ij}$$

$$\mathbf{D}_{ij}^{(n)} = \sum_{\substack{k=1 \\ (k,j)\neq\{(2,3),(3,2)\}}}^{3} D_{ik}^{(n-1)} \mathbf{D}_{kj}.$$

The stationary distribution vector of $\mathbf{D}$ is denoted by $\boldsymbol{\theta}$ and verifies $\boldsymbol{\theta}\mathbf{D} = \boldsymbol{\theta}$ and $\boldsymbol{\theta}\mathbf{e} = 1$. This stationary distribution is partitioned according to the macro-states, $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3)$, and it is calculated by considering matrix-analytic methods. The balance equations are expressed by blocks as

$$\boldsymbol{\theta}_1 \mathbf{D}_{11} + \boldsymbol{\theta}_2 \mathbf{D}_{21} + \boldsymbol{\theta}_3 \mathbf{D}_{31} = \boldsymbol{\theta}_1 \tag{2.1}$$

$$\boldsymbol{\theta}_1 \mathbf{D}_{12} + \boldsymbol{\theta}_2 \mathbf{D}_{22} = \boldsymbol{\theta}_2 \tag{2.2}$$

$$\boldsymbol{\theta}_1 \mathbf{D}_{13} + \boldsymbol{\theta}_3 \mathbf{D}_{33} = \boldsymbol{\theta}_3. \tag{2.3}$$

From (2.2) and (2.3)

$$\boldsymbol{\theta}_2 = \boldsymbol{\theta}_1 \mathbf{D}_{12} (\mathbf{I} - \mathbf{D}_{22})^{-1} \tag{2.4}$$

$$\boldsymbol{\theta}_3 = \boldsymbol{\theta}_1 \mathbf{D}_{13} (\mathbf{I} - \mathbf{D}_{33})^{-1}. \tag{2.5}$$

From (2.1), (2.2), (2.3) and from the normalization equation, $\boldsymbol{\theta}_1\mathbf{e} + \boldsymbol{\theta}_2\mathbf{e} + \boldsymbol{\theta}_3\mathbf{e} = 1$, we have that

$$\boldsymbol{\theta}_1 \left[\mathbf{D}_{11} + \mathbf{D}_{12} (\mathbf{I} - \mathbf{D}_{22})^{-1} \mathbf{D}_{21} + \mathbf{D}_{13} (\mathbf{I} - \mathbf{D}_{33})^{-1} \mathbf{D}_{31} - \mathbf{I} \right] = \mathbf{0},$$

$$\boldsymbol{\theta}_1 \left[\mathbf{I} + \mathbf{D}_{12} (\mathbf{I} - \mathbf{D}_{22})^{-1} + \mathbf{D}_{13} (\mathbf{I} - \mathbf{D}_{33})^{-1} \right] \mathbf{e} = 1.$$

If we denote as $\mathbf{R}_1 = \mathbf{D}_{11} + \mathbf{D}_{12}(\mathbf{I}-\mathbf{D}_{22})^{-1}\mathbf{D}_{21} + \mathbf{D}_{13}(\mathbf{I}-\mathbf{D}_{33})^{-1}\mathbf{D}_{31} - \mathbf{I}$ and $\mathbf{R}_2 = \mathbf{e}_{ntd\varepsilon} + \mathbf{D}_{12}(\mathbf{I}-\mathbf{D}_{22})^{-1}\mathbf{e}_{tz_1} + \mathbf{D}_{13}(\mathbf{I}-\mathbf{D}_{33})^{-1}\mathbf{e}_{tz_2}$ then

$$\boldsymbol{\theta}_1 = [1,0]\left[\mathbf{R}_2 \mid \mathbf{R}_1^*\right]^{-1},$$

where $\mathbf{R}_1^*$ is the matrix $\mathbf{R}_1$ without the first column vectors $\boldsymbol{\theta}_2$ and $\boldsymbol{\theta}_3$ are obtained from (2.4) and (2.5) respectively.

2.4 Measures: Availability, reliability and distribution of time between events

The following measures associated with this system were determined.

Availability

The availability is the probability that the unit will be operational (macro-state E^1) at a certain time. If initially the system is operational then

$$A(\nu) = \left(\boldsymbol{\alpha} \otimes \boldsymbol{\gamma}^* \otimes \boldsymbol{\omega} \otimes \boldsymbol{\eta}\right)\mathbf{D}_{11}^{(\nu)}\mathbf{e}.$$

The availability in the stationary regime is given by $A = \boldsymbol{\theta}_1\mathbf{e}$.

Reliability

Regarding system reliability, various situations can be considered. The first is that of the time elapsed to first failure or preventive maintenance. This time is *PH* distributed and is represented as $(\boldsymbol{\alpha} \otimes \boldsymbol{\gamma}^* \otimes \boldsymbol{\omega} \otimes \boldsymbol{\eta}, \mathbf{H}_0)$.

On the other hand, we may be interested in the time elapsed until the first time that the unit stops working. As non-repairable failures do not interrupt system performance, this time is *PH* distributed, and is described as $(\boldsymbol{\alpha} \otimes \boldsymbol{\gamma}^* \otimes \boldsymbol{\omega} \otimes \boldsymbol{\eta}, \mathbf{D}_{11})$. In both cases the reliability function is given by $R(\nu) = (\boldsymbol{\alpha} \otimes \boldsymbol{\gamma}^* \otimes \boldsymbol{\omega} \otimes \boldsymbol{\eta})\mathbf{A}^{\nu}\mathbf{e}$, being $\mathbf{A}$ the matrix $\mathbf{H}_0$ or $\mathbf{D}_{11}$ respectively.

2.5 Mean number of events and correlations

In this section, we consider the mean number of events associated with the Markovian arrival process with marked arrivals and correlations.

2.5.1 Mean and variance of the number of events at a certain time

Let $\{N_h(\nu); \nu \geq 0\}$ and $\{X(\nu); \nu \geq 0\}$ be the number of events of type $h \in C^0$ (C^0 being the set composed of all types of events) and the underlying Markov process associated with the MMAP respectively.

To analyse the number of events in [0, ν], several functions must be defined. The probability distribution of the MMAP is given by the matrix

$$P(\{n_h, h \in C^0\}, \nu) = (P\{X(\nu) = j, N_h(\nu) = n_h, h \in C^0 \mid X(0) = i\}).$$

The joint probability generating function for the number of arrivals in [0, ν] is defined as

$$P^*(\{z_h, h \in C^0\}, \nu) = \sum_{\{n_h \geq 0, h \in C^0\}} P(\{n_h, h \in C^0\}, \nu) \prod_{h \in C^0} z_h^{n_h}$$

and it is equal to

$$\mathbf{P}^*(\mathbf{z}, \nu) = \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu},$$

where $\mathbf{D}^*(\mathbf{z}) = \mathbf{D}^*(\{z_h, h \in C^0\}) = \mathbf{D}_0 + \sum_{h \in C^0} z_h \mathbf{D}_h$.

It is well known that

$$\left.\frac{\partial \mathbf{P}^*(\mathbf{z},\nu)}{\partial z_h}\right|_{\mathbf{z}=(1,\ldots,1)} \cdot \mathbf{e} = E[N_h(\nu)] \text{ and } \left.\frac{\partial^2 \mathbf{P}^*(\mathbf{z},\nu)}{\partial z_h^2}\right|_{\mathbf{z}=(1,\ldots,1)} \cdot \mathbf{e} = E[N_h(\nu)[N_h(\nu)-1]] \text{ for } h \in C^0.$$

Mean number of events

The mean number of events type $h \in C^0$ up to time $\nu \geq 1$ is given by

$$E[N_h(\nu)] = \left.\frac{\partial \mathbf{P}^*(\mathbf{z},\nu)}{\partial z_h}\right|_{\mathbf{z}=(1,\ldots,1)} \mathbf{e} = \sum_{i=0}^{\nu-1} \left.\left[\mathbf{D}^*(\mathbf{z})\right]^i \frac{\partial \mathbf{D}^*(\mathbf{z},\nu)}{\partial z_h} \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-i-1}\right|_{\mathbf{z}=(1,\ldots,1)} \mathbf{e}$$

$$= \sum_{i=0}^{\nu-1} \mathbf{D}^i \mathbf{D}_h \mathbf{D}^{\nu-i-1} \mathbf{e} = \sum_{i=0}^{\nu-1} \mathbf{D}^i \mathbf{D}_h \mathbf{e},$$

given that $\mathbf{D}$ is a stochastic matrix. If the initial distribution is given by ϕ then

$$E_{\phi}[N_h(\nu)] = \phi \sum_{i=0}^{\nu-1} \mathbf{D}^i \mathbf{D}_h \mathbf{e}.$$

If the model is in the stationary regime then

$$E_{\boldsymbol{\theta}}[N_h(\nu)] = \boldsymbol{\theta} \sum_{i=0}^{\nu-1} \mathbf{D}^i \mathbf{D}_h \mathbf{e} = \nu \boldsymbol{\theta} \mathbf{D}_h \mathbf{e}, \tag{2.6}$$

as $\boldsymbol{\theta}$ verifies $\boldsymbol{\theta}\mathbf{D} = \boldsymbol{\theta}$.

Variance

The variance is obtained from the second partial derivative of the joint probability generating function with respect to z_h.

Thus, for $\nu \geq 2$

$$\begin{aligned}
\frac{\partial^2 \mathbf{P}^*(\mathbf{z},\nu)}{\partial z_h^2} &= \frac{\partial^2 \mathbf{P}^*(\mathbf{z},\nu)}{\partial z_h^2} \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-1} + \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} \sum_{i=0}^{\nu-2} \left[\mathbf{D}^*(\mathbf{z})\right]^i \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-i-2} \\
&+ \sum_{i=1}^{\nu-2} \left[\sum_{j=0}^{i-1} \left[\mathbf{D}^*(\mathbf{z})\right]^j \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} \left[\mathbf{D}^*(\mathbf{z})\right]^{i-j-1} \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-i-1} \right. \\
&+ \left[\mathbf{D}^*(\mathbf{z})\right]^i \frac{\partial^2 \mathbf{P}^*(\mathbf{z},\nu)}{\partial z_h^2} \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-i-1} \\
&+ \left. \left[\mathbf{D}^*(\mathbf{z})\right]^i \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} \sum_{j=0}^{\nu-i-2} \left[\mathbf{D}^*(\mathbf{z})\right]^j \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-i-j-2} \right] \\
&+ \sum_{j=0}^{\nu-2} \left[\mathbf{D}^*(\mathbf{z})\right]^j \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-j-2} \frac{\partial \mathbf{D}^*(\mathbf{z})}{\partial z_h} + \left[\mathbf{D}^*(\mathbf{z})\right]^{\nu-1} \frac{\partial^2 P^*(\mathbf{z},\nu)}{\partial z_h^2}.
\end{aligned}$$

This function evaluated in $\mathbf{z} = (1,\ldots,1)$ is equal to

$$\begin{aligned}
\left.\frac{\partial^2 P^*(\mathbf{z},\nu)}{\partial^2 z_h}\right|_{\mathbf{z}=(1,\ldots,1)} \mathbf{e} &= E[N_h(\nu)[N_h(\nu)-1]] = \\
&= \left[\mathbf{D}_h \sum_{i=0}^{\nu-2} \mathbf{D}^i + \sum_{i=1}^{\nu-2} \left[\sum_{j=0}^{i-1} \mathbf{D}^j \mathbf{D}_h \mathbf{D}^{i-j-1} + \mathbf{D}^i \mathbf{D}_h \sum_{j=0}^{\nu-i-2} \mathbf{D}^j \right] \right. \\
&\left. + \sum_{j=0}^{\nu-2} \mathbf{D}^j \mathbf{D}_h \mathbf{D}^{\nu-j-2} \right] \mathbf{D}_h \mathbf{e}.
\end{aligned}$$

Given the initial distribution ϕ, the variance is equal to

$$
\begin{aligned}
Var_{\phi}[N_h(\nu)] &= E_{\phi}[N_h(\nu)[N_h(\nu)-1]] + E_{\phi}[N_h(\nu)] - E_{\phi}[N_h(\nu)]^2 \\
&= \phi\left[\mathbf{D}_h\sum_{i=0}^{\nu-2}\mathbf{D}^i + \sum_{i=1}^{\nu-2}\left[\sum_{j=0}^{i-1}\mathbf{D}^j\mathbf{D}_h\mathbf{D}^{i-j-1} + \mathbf{D}^i\mathbf{D}_h\sum_{j=0}^{\nu-i-2}\mathbf{D}^j\right] + \sum_{j=0}^{\nu-2}\mathbf{D}^j\mathbf{D}_h\mathbf{D}^{\nu-j-2}\right]\mathbf{D}_h\mathbf{e} \\
&\quad + E_{\phi}[N_h(\nu)] - E_{\phi}[N_h(\nu)]^2
\end{aligned}
$$

(2.7)

If initially the system is in the stationary regime then,

$$
E_{\boldsymbol{\theta}}[N_h(\nu)[N_h(\nu)-1]] = 2\boldsymbol{\theta}\mathbf{D}_h\sum_{i=1}^{\nu-1}\sum_{j=0}^{i-1}\mathbf{D}^j\mathbf{D}_h\mathbf{e}.
$$

The variance for $N_h(\nu)$ is then given by

$$
\begin{aligned}
Var_{\boldsymbol{\theta}}[N_h(\nu)] &= E_{\boldsymbol{\theta}}[N_h(\nu)[N_h(\nu)-1]] + E_{\boldsymbol{\theta}}[N_h(\nu)] - E_{\boldsymbol{\theta}}[N_h(\nu)]^2 \\
&= \boldsymbol{\theta}\left[\nu I + 2\mathbf{D}_h\sum_{i=1}^{\nu-1}\sum_{j=0}^{i-1}\mathbf{D}^j\right]\mathbf{D}_h\mathbf{e} - (\nu\boldsymbol{\theta}\mathbf{D}_h\mathbf{e})^2.
\end{aligned}
\tag{2.8}
$$

Covariance

The covariance between $N_h(\nu)$ and $N_k(\nu)$ at time $\nu \geq 2$ for $k \neq h$ can be described as follows,

$$
Cov_{\phi}\left(N_h(\nu), N_k(\nu)\right) = \frac{1}{2}\left\{Var_{\phi}\left(N_h(\nu)+N_k(\nu)\right) - Var_{\phi}\left(N_h(\nu)\right) - Var_{\phi}\left(N_k(\nu)\right)\right\}. \tag{2.9}
$$

The process $N_h(\nu)+N_k(\nu)$ is the number of events type h or k in [0, ν], therefore from (2.7)

$$
\begin{aligned}
Var_{\phi}\left(N_h(\nu)+N_k(\nu)\right) &= \phi\left[\left(\mathbf{D}_h+\mathbf{D}_k\right)\sum_{i=0}^{\nu-2}\mathbf{D}^i + \sum_{i=1}^{\nu-2}\left[\sum_{j=0}^{i-1}\mathbf{D}^j\left(\mathbf{D}_h+\mathbf{D}_k\right)\mathbf{D}^{i-j-1}\right.\right. \\
&\quad \left.\left. + \mathbf{D}^i\left(\mathbf{D}_h+\mathbf{D}_k\right)\sum_{j=0}^{\nu-i-2}\mathbf{D}^j\right] + \sum_{j=0}^{\nu-2}\mathbf{D}^j\left(\mathbf{D}_h+\mathbf{D}_k\right)\mathbf{D}^{\nu-j-2}\right]\left(\mathbf{D}_h+\mathbf{D}_k\right)\mathbf{e} \\
&\quad + E_{\phi}\left[N_h(\nu)\right] + E_{\phi}\left[N_k(\nu)\right] - \left(E_{\phi}\left[N_h(\nu)\right] + E_{\phi}\left[N_k(\nu)\right]\right)^2.
\end{aligned}
$$

For the stationary version of the MMAP we have from (2.8) that

$$
Var_{\boldsymbol{\theta}}\left(N_h(\nu)+N_k(\nu)\right) = \boldsymbol{\theta}\left[\nu\mathbf{I} + 2\left[\mathbf{D}_h+\mathbf{D}_k\right]\sum_{i=1}^{\nu-1}\sum_{j=0}^{i-1}\mathbf{D}^j\right]\left[\mathbf{D}_h+\mathbf{D}_k\right]\mathbf{e} - \left(\nu\boldsymbol{\theta}\left[\mathbf{D}_h+\mathbf{D}_k\right]\mathbf{e}\right)^2. \tag{2.10}
$$

From (2.8), (2.9) and (2.10), this yields

$$
\begin{aligned}
Cov_{\boldsymbol{\theta}}\left(N_h(\nu), N_k(\nu)\right) &= \boldsymbol{\theta}\left[\mathbf{D}_h \sum_{i=1}^{\nu-1}\sum_{j=0}^{i-1}\mathbf{D}^j\mathbf{D}_k + \mathbf{D}_k \sum_{i=1}^{\nu-1}\sum_{j=0}^{i-1}\mathbf{D}^j\mathbf{D}_h\right]\mathbf{e} \\
&+\frac{1}{2}\left[\left(\nu\boldsymbol{\theta}\mathbf{D}_k\mathbf{e}\right)^2 - \left(\nu\boldsymbol{\theta}\left[\mathbf{D}_h + \mathbf{D}_k\right]\mathbf{e}\right)^2 + \left(\nu\boldsymbol{\theta}\mathbf{D}_h\mathbf{e}\right)^2\right] \\
&= \boldsymbol{\theta}\left[\mathbf{D}_h \sum_{i=1}^{\nu-1}\sum_{j=0}^{i-1}\mathbf{D}^j\mathbf{D}_k + \mathbf{D}_k \sum_{i=1}^{\nu-1}\sum_{j=0}^{i-1}\mathbf{D}^j\mathbf{D}_h\right]\mathbf{e} \\
&-\nu^2\boldsymbol{\theta}\mathbf{D}_h\mathbf{e}\boldsymbol{\theta}\mathbf{D}_k\mathbf{e}.
\end{aligned}
$$

Correlation coefficient functions

The correlation coefficient at time ν between $N_h(\nu)$ and $N_k(\nu)$ is given by

$$
{}_{\boldsymbol{\varphi}}r_{h,k}(\nu) = \frac{Cov_{\boldsymbol{\varphi}}(N_h(\nu), N_k(\nu))}{\sqrt{Var_{\boldsymbol{\varphi}}(N_h(\nu))Var_{\boldsymbol{\varphi}}(N_k(\nu))}}, \tag{2.11}
$$

where $\boldsymbol{\varphi} = \phi$ or $\boldsymbol{\varphi} = \boldsymbol{\theta}$ for the transient or stationary regime respectively.

The square of this value is the determination coefficient function.

2.5.2 Covariance between the numbers of events in non-overlapping intervals

Let $\{N_h(\nu, \nu+\tau); \nu \geq 0\}$ be the number of events type h in the interval $\left]\nu, \nu+\tau\right]$, i.e. $N_h(\nu, \nu+\tau) = N_h(\nu+\tau) - N_h(\nu)$. The numbers of events in non-overlapping intervals are correlated for a MMAP, but the numbers are conditionally independent, i.e. if $X(\nu)$ is known, then the number events process in $\left]\nu, \nu+\tau\right]$ is independent of that in [0, ν]. Let the probability matrix be

$$
\begin{aligned}
&\mathbf{P}(\{n_{1h}, h \in C^0\}, \nu, \{n_{2k}, k \in C^0\}, \tau) \\
&= \left(P\{N_h(\nu) = n_{1h}, N_k(\nu, \nu+\tau) = n_{2k}; h, k \in C^0, X(\nu+\tau) = j \mid X(0) = i\}\right),
\end{aligned}
$$

then, the joint probability generating function for the number of arrivals in [0, ν] and in $\left]\nu, \nu+\tau\right]$ is defined as

$$
\begin{aligned}
&P^*(\{z_{1h}, h \in C^0\}, \nu, \{z_{2k}, h \in C^0\}, \tau) \\
&= \sum_{\substack{\{n_{1h} \geq 0, h \in C^0\} \\ \{n_{2k} \geq 0, k \in C^0\}}} P(\{n_{1h}, h \in C^0\}, \nu, \{n_{2k}, k \in C^0\}, \tau) \prod_{\substack{h \in C^0 \\ k \in C^0}} z_{1h}^{n_{1h}} z_{2h}^{n_{2h}},
\end{aligned}
$$

and from the Markov property it is equal to

$$\mathbf{P}^*(\{z_{1h}, h \in C^0\}, \nu, \{z_{2h}, h \in C^0\}, \tau) = \mathbf{P}^*(\mathbf{z}_1, \nu, \mathbf{z}_2, \tau) = [\mathbf{D}^*(\mathbf{z}_1)]^\nu [\mathbf{D}^*(\mathbf{z}_2)]^\tau .$$

From this result,

$$E_\phi[N_h(\nu)N_k(\nu, \nu+\tau)] = \phi \frac{\partial^2 P^*(\mathbf{z}_1, \nu, \mathbf{z}_2, \tau)}{\partial z_{1h} \partial z_{2k}}\Bigg|_{\mathbf{z}_1, \mathbf{z}_2 = (1,\dots,1)} \mathbf{e} = \phi \sum_{i=0}^{\nu-1} \mathbf{D}^i \mathbf{D}_h \sum_{j=0}^{\tau-1} \mathbf{D}^{\nu+j-i-1} \mathbf{D}_k \mathbf{e},$$

and then

$$Cov_\phi[N_h(\nu)N_k(\nu, \nu+\tau)] = \phi \left[\sum_{i=0}^{\nu-1} \mathbf{D}^i \mathbf{D}_h \left(\sum_{j=0}^{\tau-1} \mathbf{D}^{\nu+j-i-1} - \mathbf{e}\phi \sum_{j=\nu}^{\nu+\tau-1} \mathbf{D}^j \right) \right] \mathbf{D}_k \mathbf{e}.$$

If the system is initially in the stationary regime then

$$Cov_{\boldsymbol{\theta}}[N_h(\nu)N_k(\nu, \nu+\tau)] = \boldsymbol{\theta} \sum_{i=0}^{\nu-1} \mathbf{D}_h \sum_{j=0}^{\tau-1} \mathbf{D}^{\nu+j-i-1} \mathbf{D}_k \mathbf{e} - \tau\nu \boldsymbol{\theta} \mathbf{D}_h \mathbf{e} \boldsymbol{\theta} \mathbf{D}_k \mathbf{e}.$$

2.6 Rewards

To optimise the performance of a system, it is useful to examine its work times, repair times, corrective and preventive maintenance actions, rewards and costs. The system described considers repairable and non-repairable failures and preventive maintenance. It is also interesting to examine whether preventive maintenance is economically profitable and what are the optimum states at which preventive maintenance should be applied after observation.

In this analysis, we assume that the expected reward is equal to *b* while the system is operational and that the expected costs while the unit is operational, per unit of time if the system is in a minor or major damage internal state, are equal to c_1 and c_2 respectively. While the system is in the repair facility, a cost depending on the state from where the unit has come is produced. We assume a cost equal to cr_i^a per unit of time when it is in corrective repair phase *a* from operational state *i* and $pr_{i,j}^a$ per unit of time when it is in preventive maintenance phase *a* from internal operational state *i* and external cumulative damage *j*. These costs are arranged in the column vectors $\mathbf{cr}_i = \left(cr_i^1, \dots, cr_i^{z_{c,i}}\right)'$ and $\mathbf{pr}_{i,j} = \left(pr_{i,j}^1, \dots, pr_{i,j}^{z_{p,i,j}}\right)'$ respectively. A fixed expected cost, for one or more of various possible causes, is introduced for each time that a corrective repair or a preventive maintenance takes place, equal to *CR* and *PM* respectively.

While the unit is not operational the system experiences a loss equal to A per unit of time. Finally, each new unit has a cost of C. Figure 2.2 shows the complexity of costs and rewards.

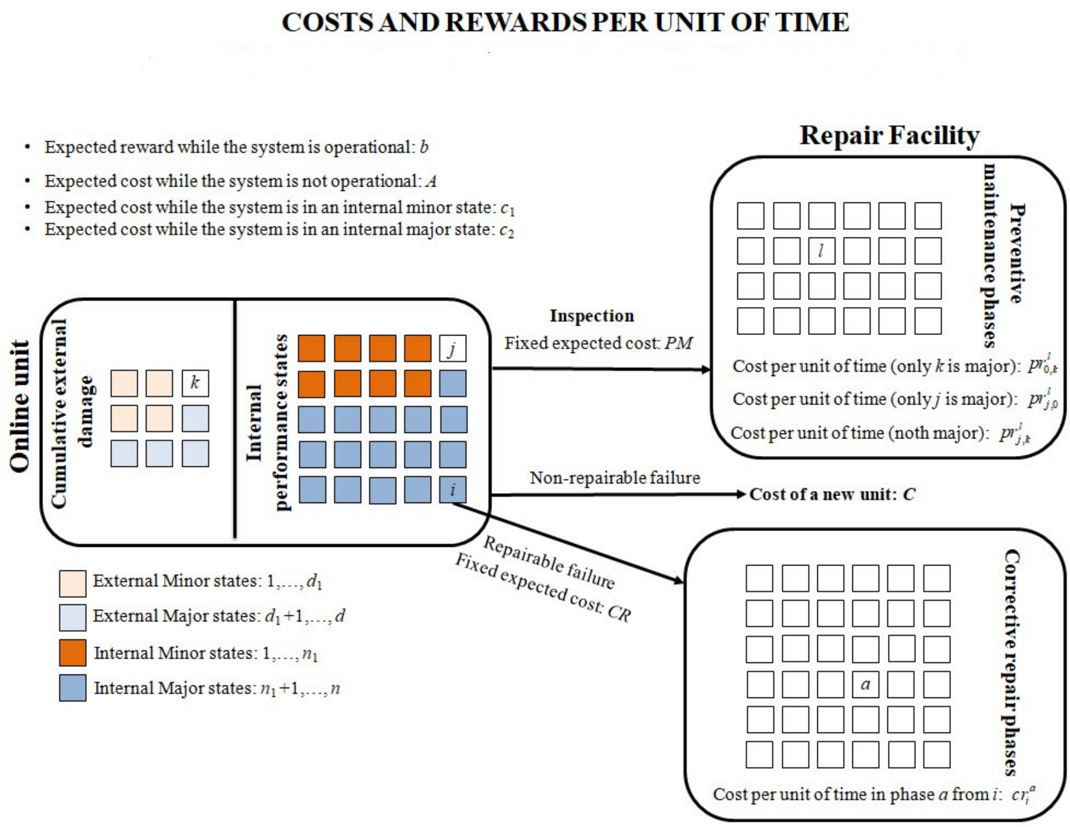

Figure 2.2. Diagram of costs and rewards

2.6.1 The net reward vector

A net reward vector is built according to the state space described in Section 2.2.2. The net reward vector when the system is in the macro-state E^1 is given by

$$\mathbf{nr}_1 = b\mathbf{e}_{ntd\varepsilon} - \begin{pmatrix} c_1\mathbf{e}_{n_1} \\ c_2\mathbf{e}_{n-n_1} \end{pmatrix} \otimes e_{td\varepsilon} \ .$$

The cost vectors for the macro-states E^2 and E^3 (when the system is in corrective repair and in preventive maintenance respectively) are $\mathbf{nr}_2 = -A\mathbf{e}_{t\sum_{i=1}^{n} z_{c,i}} - \begin{pmatrix} \mathbf{e}_t \otimes \mathbf{cr}_1 \\ \vdots \\ \mathbf{e}_t \otimes \mathbf{cr}_n \end{pmatrix}$ and

$$\mathbf{nr}_3 = -A\mathbf{e}_{t\left(\sum_{j=n_1+1}^{n} z_{p,i,0} + \sum_{j=d_1+1}^{d} z_{p,0,j} + \sum_{i=n_1+1}^{n}\sum_{j=d_1+1}^{d} z_{p,i,j}\right)} - \begin{pmatrix} \mathbf{e}_t \otimes \mathbf{pr}_{n_1+1,0} \\ \vdots \\ \mathbf{e}_t \otimes \mathbf{pr}_{n,0} \\ \mathbf{e}_t \otimes \mathbf{pr}_{0,d_1+1} \\ \vdots \\ \mathbf{e}_t \otimes \mathbf{pr}_{0,d} \\ \mathbf{e}_t \otimes \mathbf{pr}_{n_1+1,d_1+1} \\ \vdots \\ \mathbf{e}_t \otimes \mathbf{pr}_{n,d} \end{pmatrix},$$

respectively.

Finally, the net reward vector associated to the state space is given by

$$\mathbf{c} = \begin{pmatrix} \mathbf{nr}_1 \\ \mathbf{nr}_2 \\ \mathbf{nr}_3 \end{pmatrix}.$$

2.6.2 Expected net rewards

An interesting aspect associated with the performance of a reliability system is that of the net reward per unit of time up to a certain time. This measure is composed of the expected net reward per unit of time minus the fixed cost for corrective repair, preventive maintenance and new units (the cost of the initial unit is included in this measure).

The cumulative expected net reward from the beginning up to time ν is given by

$$ENR_{\phi}(\nu) = \phi\sum_{n=0}^{\nu} \mathbf{D}^n\mathbf{c} - CR \cdot E_{\phi}[N_1(\nu)] - PM \cdot E_{\phi}[N_2(\nu)] - C \cdot \left(1 + E_{\phi}[N_3(\nu)]\right).$$

Per unit of time this is equal to

$$\Psi_{\phi}(\nu) = \frac{ENR_{\phi}(\nu)}{\nu + 1}.$$

If initially the system is in stationary regime then from (9) and (15),

$$ENR_{\boldsymbol{\theta}}(\nu) = (\nu+1)\boldsymbol{\theta}\mathbf{c} - \nu\cdot CR\cdot\boldsymbol{\theta}\mathbf{D}_1\mathbf{e} - \nu\cdot PM\cdot\boldsymbol{\theta}\mathbf{D}_2\mathbf{e} - C\cdot(1+\nu\cdot\boldsymbol{\theta}\mathbf{D}_3\mathbf{e})$$

and

$$\Psi_{\boldsymbol{\theta}}(\nu) = \boldsymbol{\theta}\mathbf{c} - \frac{\nu\cdot CR}{\nu+1}\boldsymbol{\theta}\mathbf{D}_1\mathbf{e} - \frac{\nu\cdot PM}{\nu+1}\boldsymbol{\theta}\mathbf{D}_2\mathbf{e} - \frac{C}{\nu+1}(1+\nu\boldsymbol{\theta}\mathbf{D}_3\mathbf{e}).$$

Finally, independently of the initial distribution, the net reward per unit of time (steady state) is given by

$$\Psi = \boldsymbol{\theta}\mathbf{c} - CR\cdot\boldsymbol{\theta}\mathbf{D}_1\mathbf{e} - PM\cdot\boldsymbol{\theta}\mathbf{D}_2\mathbf{e} - C\cdot(\boldsymbol{\theta}\mathbf{D}_3\mathbf{e}) \quad . \tag{2.12}$$

2.7 Optimization

When preventive maintenance is introduced, the lifetime of the unit is of course extended, but at what price? Is this maintenance profitable? Multiple rewards and costs and corrective repair and preventive maintenance time distributions depend on the system state at which the unit failed and on the major damage state observed by inspection as it is shown in Figures 2.1 and 2.2.

An important question is that of when preventive maintenance should be carried out, i.e. what is the threshold between minor and major damage? To answer this question, the expected net reward in the stationary regime should be taken into account. This measure is developed in (16). In fact, this function depends on the structure of the matrices from n_1, d_1. When an inspection takes place, the unit goes to the repair facility for preventive maintenance if the internal performance state and/or external cumulative damage observed are greater than n_1 and d_1 respectively.

Accordingly, a preventive maintenance policy is carried out adjusting n_1 and d_1 such that the net reward per unit of time, $\Psi(n_1, d_1)$ is maximum, for $n_1 = 1, \ldots, n$ and $d_1 = 1,\ldots, d$. Figure 2.3 shows a diagram of the maintenance policy.

Other measures could be taken into account such as reliability and availability, both of them depend on the barrier between minor and major internal and external damage, n_1, d_1, respectively.

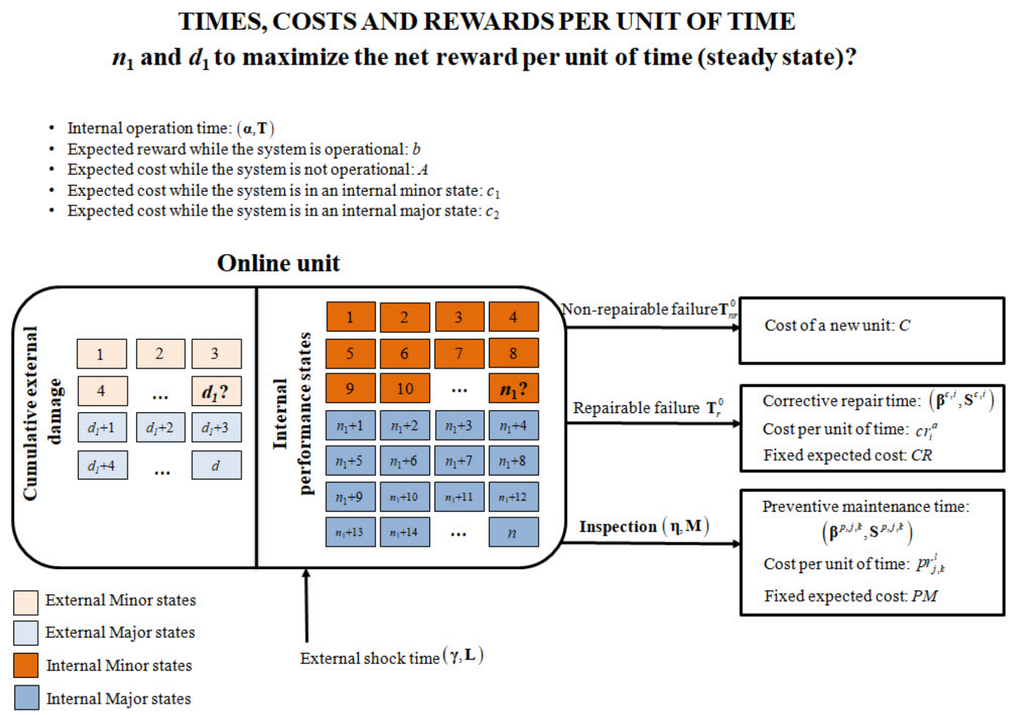

Figure 2.3. Diagram of the preventive maintenance policy

2.8 A numerical example

This section highlights the value of preventive maintenance by comparing similar systems with and without preventive maintenance and with different maintenance policies, depending on the states at which inspection reveals major internal (n_1) and external cumulative damage (d_1). We assume a system composed of a generating set in a facility that requires a reliable electrical supply. This generating set is subject to degradation and may fail for the same reasons as any motor, provoking either a total or a repairable failure. The generating set passes through various degradation stages, and an internal repairable or non-repairable failure may occur from any of the different states. In addition, this device is subject to external failures which can modify its internal behaviour or even produce a non-repairable failure. Random inspections take place and the level of internal degradation and of cumulative external damage is observed. Lifetime distributions for the repair and preventive maintenance times depend, logically, on the level of degradation at which the system failed or was inspected. We assume a system with five internal states. The phase-type distributions embedded in the system – internal failure time, time between two consecutive external shocks, inspection time, corrective repair and preventive maintenance times depending on the previous state – are shown in Tables 2.1, 2.2 and 2.3.

Internal failure time	**External shock time**	**Inspection time**
$\boldsymbol{\alpha}=(1,0,0,0,0)$ $\mathbf{T}=\begin{pmatrix} 0.99 & 0.001 & 0 & 0 & 0 \\ 0 & 0.99 & 0.001 & 0 & 0 \\ 0 & 0 & 0.8 & 0.003 & 0 \\ 0 & 0 & 0 & 0.8 & 0.003 \\ 0 & 0 & 0 & 0 & 0.8 \end{pmatrix}$ Mean time: 110.0508	$\boldsymbol{\gamma}=(1,0)$ $\mathbf{L}=\begin{pmatrix} 0.88 & 0.08 \\ 0.98 & 0.008 \end{pmatrix}$ Mean time: 26.3780	$\boldsymbol{\eta}=(1,0,0)$ $\mathbf{M}=\begin{pmatrix} 0.86 & 0.01 & 0.05 \\ 0.8 & 0.04 & 0 \\ 0.8 & 0.1 & 0.04 \end{pmatrix}$ Mean time: 12.4671

Table 2.1. Internal failure, external shock and inspection phase-type distributions

Corrective repair time from state 1	Corrective repair time from state 2	Corrective repair time from state 3
$\boldsymbol{\beta}^{c,1}=(1,0)$ $\mathbf{S}_{c,1}=\begin{pmatrix} 0.6 & 0.25 \\ 0.03 & 0.8 \end{pmatrix}$ Mean time: 6.2069	$\boldsymbol{\beta}^{c,2}=(1,0)$ $\mathbf{S}_{c,2}=\begin{pmatrix} 0.68 & 0.15 \\ 0.03 & 0.9 \end{pmatrix}$ Mean time: 9.0909	$\boldsymbol{\beta}^{c,3}=(1,0)$ $\mathbf{S}_{c,3}=\begin{pmatrix} 0.19 & 0.05 \\ 0.05 & 0.19 \end{pmatrix}$ Mean time: 10.2703
Corrective repair time from state 4	**Corrective repair time from state 5**	
$\boldsymbol{\beta}^{c,4}=(1,0)$ $\mathbf{S}_{c,4}=\begin{pmatrix} 0.85 & 0.1 \\ 0.04 & 0.8 \end{pmatrix}$ Mean time: 11.5385	$\boldsymbol{\beta}^{c,5}=(1,0)$ $\mathbf{S}_{c,5}=\begin{pmatrix} 0.87 & 0.1 \\ 0.04 & 0.75 \end{pmatrix}$ Mean time: 12.2807	

Table 2.2. Corrective repair phase-type distributions

Two types of internal failures are considered, repairable and non-repairable. The probability of either case occurring, from a transient state, is given by the column vectors $\mathbf{T}_r^0=(0.008,0.008,0.195,0.195,0.1)'$ and $\mathbf{T}_{nr}^0=(0.001,0.001,0.002,0.002,0.1)'$ respectively. We assume that an external shock can produce an extreme non-repairable failure with probability 0.4 and that the matrix governing the transitions between cumulative episodes of external damage is

$$\mathbf{Q} = \begin{pmatrix} 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 \end{pmatrix}.$$

When an external shock is produced, the cumulative threshold damage is reached from any transient state according to the column vector $\mathbf{Q}^0 = (0, 0, 0.5, 0.7)'$. In this case a non-repairable failure occurs.

While the unit is operational and an external shock occurs, the internal state may be modified. The matrix governing this transition after an external shock is

$$\mathbf{W} = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 & 0 \\ 0 & 0.6 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0.6 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0 & 0.4 \end{pmatrix}.$$

From the matrix **W**, it can be seen that if an external shock occurs then an internal repairable failure will take place only if the system is in state 4 or 5 ($\mathbf{W}^0$=(0,0,0,0.2,0.6)’).

Rewards

An interesting aspect regarding a complex reliability model subject to different types of failure and of repair (corrective and preventive), is to analyse the economic profit obtainable.

It is assumed that while the system is operational a reward equal to b=10 per unit of time is produced. However, while the system is active, a cost is also incurred. This cost per unit of time varies according to whether the system is working in a minor or a major state of internal damage. If the system is working in a minor damage state, a cost equal to

2 monetary units is incurred and this cost is 4 per unit of time if the system is in a major damage state.

If the system is in the repair facility, different costs arise. Each time the system undergoes a repairable failure, a fixed cost equal to CR=10 is incurred and a cost of 1, 2, 4, 7 or 10 monetary units per unit of time is incurred while the system is in corrective repair and if it failed from internal operational state 1, 2, 3, 4 or 5 respectively. An analogous outcome is obtained for preventive maintenance. Each positive inspection provokes a fixed cost of 2 monetary units and the costs per unit of time in preventive maintenance are as shown in Table 2.4.

Finally, while the system is in the repair facility, economic losses of 10 monetary units per unit of time are incurred. Each new unit installed has a value equal to 200 monetary units.

The proportional time spent in each macro-state and the net reward per unit of time in the stationary regime from (2.12) for the different systems according to d_1 and n_1 are shown in Table 2.5.

The proportional number of failures and major inspections per unit of time in the stationary regime were analysed for different systems according to d_1 and n_1. The values obtained are given in Table 2.6.

Table 2.5 shows that the maximum net reward per unit of time is reached for $n_1 = 3$ and $d_1 = 2$. Then, given the operational time, the corrective repair times, the preventive maintenance times and the costs and rewards, the most profitable policy is to undertake preventive maintenance when the internal performance is in state 4 or 5 and when the cumulative external damage is in state 3 or 4.

Next, we focus on the behaviour of the optimum model $n_1 = 3$ and $d_1 = 2$. If a new system, with an initial distribution of ϕ is considered, then a comparison can be drawn between the net reward per unit of time for this optimum model and one without preventive maintenance. Figure 2.4 shows the net reward up to a certain time and per unit of time for both models (optimum and without preventive maintenance).

Preventive maintenance time from only internal state 2	Preventive maintenance time from only internal states 3 or 4	Preventive maintenance time from only internal state 5
$\boldsymbol{\beta}^{p,2,0}=(1,0)$ $\mathbf{S}_{p,2,0}=\begin{pmatrix}0.02 & 0.02\\0.01 & 0.01\end{pmatrix}$ Mean time: 1.0412	$\boldsymbol{\beta}^{p,i,0}=(1,0)$ $\mathbf{S}_{p,i,0}=\begin{pmatrix}0.3 & 0.2\\0.1 & 0.2\end{pmatrix}$ Mean time: 1.8519	$\boldsymbol{\beta}^{p,5,0}=(1,0)$ $\mathbf{S}_{p,5,0}=\begin{pmatrix}0.5 & 0.2\\0.1 & 0.4\end{pmatrix}$ Mean time: 2.8571
Preventive maintenance time from only external cumulative damage 2	Preventive maintenance time from only external cumulative damage 3	Preventive maintenance time from only external cumulative damage 4
$\boldsymbol{\beta}^{p,0,2}=(1,0)$ $\mathbf{S}_{p,0,2}=\begin{pmatrix}0.01 & 0\\0 & 0.01\end{pmatrix}$ Mean time: 1.0101	$\boldsymbol{\beta}^{p,0,3}=(1,0)$ $\mathbf{S}_{p,0,3}=\begin{pmatrix}0.02 & 0\\0 & 0.02\end{pmatrix}$ Mean time: 1.0204	$\boldsymbol{\beta}^{p,0,4}=(1,0)$ $\mathbf{S}_{p,0,4}=\begin{pmatrix}0.2 & 0.01\\0.02 & 0.1\end{pmatrix}$ Mean time: 1.2642
Preventive maintenance time from internal state 2 and external cumulative damage 2	Preventive maintenance time from internal state 2 and external cumulative damage 3	Preventive maintenance time from internal state 2 and external cumulative damage 4
$\boldsymbol{\beta}^{p,2,2}=(1,0)$ $\mathbf{S}_{p,2,2}=\begin{pmatrix}0.4 & 0.15\\0.05 & 0.02\end{pmatrix}$ Mean time: 1.9466	$\boldsymbol{\beta}^{p,2,3}=(1,0)$ $\mathbf{S}_{p,2,3}=\begin{pmatrix}0.4 & 0.2\\0.01 & 0.01\end{pmatrix}$ Mean time: 2.0101	$\boldsymbol{\beta}^{p,2,4}=(1,0)$ $\mathbf{S}_{p,2,4}=\begin{pmatrix}0.5 & 0.05\\0.05 & 0.02\end{pmatrix}$ Mean time: 2.1128
Preventive maintenance time from internal state 3 or 4 and external cumulative damage 2	Preventive maintenance time from internal state 3 or 4 and external cumulative damage 3	Preventive maintenance time from internal state 3 or 4 and external cumulative damage 4
$\boldsymbol{\beta}^{p,i,2}=(1,0)$ $\mathbf{S}_{p,i,2}=\begin{pmatrix}0.3 & 0.2\\0.4 & 0.5\end{pmatrix}$ Mean time: 2.5926	$\boldsymbol{\beta}^{p,i,3}=(1,0)$ $\mathbf{S}_{p,i,3}=\begin{pmatrix}0.56 & 0.1\\0.3 & 0.1\end{pmatrix}$ Mean time: 2.7322	$\boldsymbol{\beta}^{p,i,4}=(1,0)$ $\mathbf{S}_{p,i,4}=\begin{pmatrix}0.3 & 0.5\\0.25 & 0.2\end{pmatrix}$ Mean Mean time: 2.9885
Preventive maintenance time from internal state 5 and external cumulative damage 2	Preventive maintenance time from internal state 5 and external cumulative damage 3	Preventive maintenance time from internal state 5 and external cumulative damage 4
$\boldsymbol{\beta}^{p,5,2}=(1,0)$ $\mathbf{S}_{p,5,2}=\begin{pmatrix}0.56 & 0.2\\0.2 & 0.4\end{pmatrix}$ Mean time: 3.5714	$\boldsymbol{\beta}^{p,5,3}=(1,0)$ $\mathbf{S}_{p,5,3}=\begin{pmatrix}0.58 & 0.2\\0.15 & 0.45\end{pmatrix}$ Mean Mean time: 3.7313	$\boldsymbol{\beta}^{p,5,4}=(1,0)$ $\mathbf{S}_{p,5,4}=\begin{pmatrix}0.62 & 0.2\\0.2 & 0.45\end{pmatrix}$ Mean time: 4.4379

Table 2.3. Preventive maintenance phase-type distributions

States observed by inspection (i, j)	(2, 0)	(3, 0) (4, 0)	(5, 0)	(0, 2)	(0, 3) (0, 4)	(2, 2)	(2, 3) (2, 4)
Monetary units per unit of time if the unit goes to repair facility	0.2	0.5	1	0.1	1	0.2	1.1
States observed by inspection (i, j)	(3, 2)	(3, 3)	(3, 4)	(4, 2)	(4, 3) (4, 4)	(5, 2)	(5, 3) (5, 4)
Monetary units per unit of time if the unit goes to repair facility	0.6	1.5	1.2	0.5	1	0.9	2

Table 2.4. Cost per unit of time when the unit goes to the repair facility after inspection has revealed major internal damage state *i* and major external cumulative damage *j* (0 indicates minor damage on inspection)

	$\boldsymbol{\theta}_1 \cdot \mathbf{e}$	$\boldsymbol{\theta}_2 \cdot \mathbf{e}$	$\boldsymbol{\theta}_3 \cdot \mathbf{e}$	$\Psi(n_1, d_1)$
d_1=1 ; n_1=1	0.9222	0.0625	0.0153	3.0492
d_1=1 ; n_1=2	0.9246	0.0630	0.0124	3.0830
d_1=1 ; n_1=3	0.9253	0.0630	0.0116	3.0940
d_1=1 ; n_1=4	0.9260	0.0631	0.0109	3.1041
d_1=1 ; n_1=5	0.9261	0.0631	0.0108	3.1054
d_1=2 ; n_1=1	0.9239	0.0626	0.0136	3.0709
d_1=2 ; n_1=2	0.9260	0.0634	0.0106	3.0965
d_1=2 ; n_1=3	**0.9266**	**0.0636**	**0.0098**	**3.1056**
d_1=2 ; n_1=4	0.9266	0.0643	0.0091	3.1004
d_1=2 ; n_1=5	0.9267	0.0643	0.0090	3.1014
d_1=3 ; n_1=1	0.9316	0.0621	0.0063	2.9890
d_1=3 ; n_1=2	0.9328	0.0635	0.0036	2.9329
d_1=3 ; n_1=3	0.9332	0.0640	0.0028	2.9322
d_1=3 ; n_1=4	0.9308	0.0670	0.0022	2.8701
d_1=3 ; n_1=5	0.9310	0.0670	0.0020	2.8726
d_1=4 ; n_1=1	0.9334	0.0620	0.0046	2.9690
d_1=4 ; n_1=2	0.9347	0.0636	0.0017	2.8836
d_1=4 ; n_1=3	0.9351	0.0641	0.0008	2.8797
d_1=4 ; n_1=4	0.9322	0.0676	0.0001	2.8078
d_1=4 ; n_1=5	0.9321	0.0679	0	2.8003

Table 2.5. Proportional time spent in each macro-state and net reward per unit of time in the stationary regime

For both models, the expected net reward up to a certain time increases with time, and both models are loss-making up to a certain time. The optimum system with (without) preventive maintenance incurs losses up to time 60 (63). The expected profit from the start until this time is 2.4332 (2.2585) and, from then on, the expected profit per unit of time from the start is equal to 0.0406 (0.0358). Taking into account the transient analysis, the expected net reward up to time 1000 for the optimum model with preventive maintenance is equal to 2921.7, in contrast to the 2626.5 for the model without preventive maintenance.

	Repairable failure ratio	**Major inspection ratio**	**Non-repairable failure ratio**
d_1=1 ; n_1=1	0.0098	0.0109	0.0163
d_1=1 ; n_1=2	0.0099	0.0105	0.0163
d_1=1 ; n_1=3	0.0099	0.0105	0.0163
d_1=1 ; n_1=4	0.0099	0.0105	0.0163
d_1=1 ; n_1=5	0.0099	0.0105	0.0163
d_1=2 ; n_1=1	0.0098	0.0097	0.0163
d_1=2 ; n_1=2	0.0099	0.0088	0.0164
d_1=2 ; n_1=3	0.0100	0.0087	0.0164
d_1=2 ; n_1=4	0.0100	0.0086	0.0164
d_1=2 ; n_1=5	0.0101	0.0086	0.0164
d_1=3 ; n_1=1	0.0098	0.0046	0.0175
d_1=3 ; n_1=2	0.0098	0.0023	0.0179
d_1=3 ; n_1=3	0.0101	0.0019	0.0179
d_1=3 ; n_1=4	0.0104	0.0016	0.0179
d_1=3 ; n_1=5	0.0104	0.0016	0.0179
d_1=4 ; n_1=1	0.0097	0.0038	0.0178
d_1=4 ; n_1=2	0.0098	0.0009	0.0184
d_1=4 ; n_1=3	0.0102	0.0004	0.0184
d_1=4 ; n_1=4	0.0105	0.0000	0.0184
d_1=4 ; n_1=5	0.0105	****	0.0184

Table 2.6. Proportional number of repairable and non-repairable failures and preventive maintenance per unit of time in stationary regime

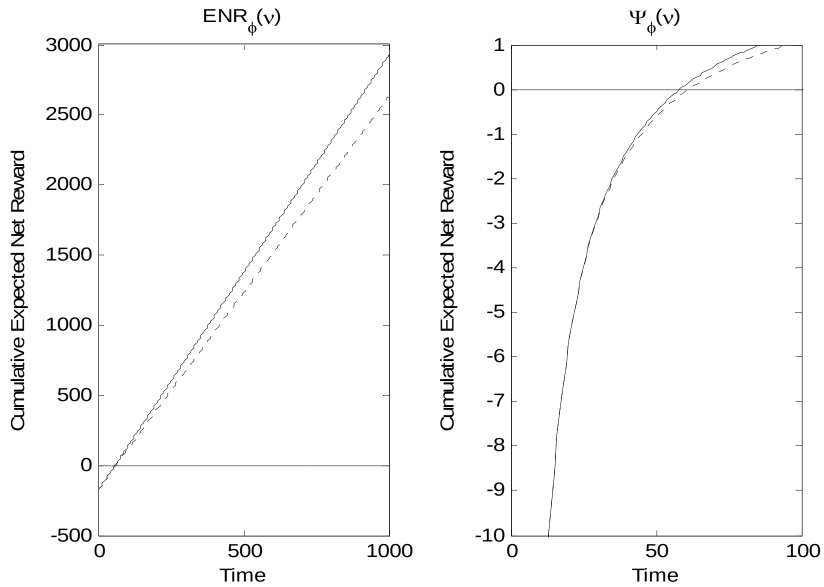

Figure 2.4. Expected net reward up to a certain time and per unit of time (with preventive maintenance, continuous line; without preventive maintenance, dashed line)

After including the times, several measures were calculated. Table 2.7 shows the probability of the system being in any of the macro-states (operational, corrective repair or preventive maintenance) at various times for each model: optimum and without preventive maintenance.

Time (ν)	E_1	E_2	E_3
10	0.9440 (0.9500)	0.0485 (0.0500)	0.0075
20	0.9313 (0.9371)	0.0594 (0.0629)	0.0093
50	0.9270 (0.9324)	0.0632 (0.0676)	0.0098
100	0.9266 (0.9322)	0.635 (0.0678)	0.0098
∞	0.9266 (0.9321)	0.0636 (0.0679)	0.0098

Table 2.7. Probability of the system being in each macro-state (in parentheses, the optimum model without preventive maintenance)

The mean number of events is described in Section 2.5. If we assume that the model is in a stationary regime, then the mean number of repairable and non-repairable failures and the mean number of major inspection events for both models can be calculated from (2.6). These details are shown in Table 2.8.

Time (ν)	Repairable failure $E_\theta\left[N_1(\nu)\right]$	Preventive maintenance $E_\theta\left[N_2(\nu)\right]$	Non-repairable failure $E_\theta\left[N_3(\nu)\right]$
50	0.4954 (0.5220)	0.3757	0.8180 (0.8716)
100	0.9947 (1.0470)	0.8095	1.6370 (1.7858)
200	1.9934 (2.0957)	1.6771	3.2749 (3.6244)
500	4.9895 (5.2419)	4.2799	8.1888 (9.1379)
1000	9.9830 (10.4854)	8.6180	16.3786 (18.3270)

Table 2.8. Mean times of events up to a certain time (in parentheses the model without preventive maintenance)

The difference between the number of repairable and non-repairable failures is considerable. Thus, up to time 1000 the mean number of non-repairable failures decreases by almost two units and the mean number of repairable failures decreases by half a unit.

The correlation between the numbers of events is also described. Figure 2.5 shows the correlation coefficient function obtained from (2.11).

All correlations between events decrease with time and are negative. The events repairable failure, preventive maintenance and non-repairable failure are denoted by 1, 2 and 3 respectively. In the optimum model with preventive maintenance, the maximum negative correlation occurs between repairable failure and preventive maintenance. Comparison of the models with and without preventive maintenance shows that the negative correlation between repairable and non-repairable failures decreases in both cases, but is considerably larger for the second model. The correlation between the number of repairable failures and non-repairable failures at time 1000 is equal to -0.1083 for the model with preventive maintenance and -0.1649 for the model without preventive maintenance.

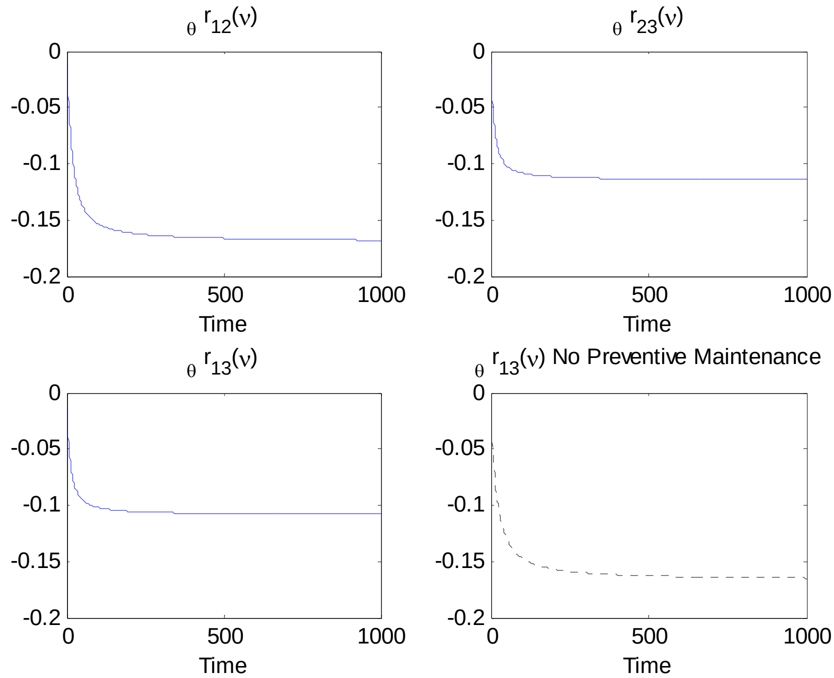

Figure 2.5. Correlation Coefficient function (model with preventive maintenance, continuous line; model without preventive maintenance, dashed line)

2.9 Appendix 2A

In a similar way to the procedure described in Section 2.2, the state space and events are built for the model without preventive maintenance, n_1=n and d_1=d. In this case, the state space E is composed of the macro-states $E=\left\{E^1,E^2\right\}$, where E^k contains the phases when the unit is operational (k = 1) and the unit is in corrective repair (k = 2). The phases are given by

$$E^1=\{(i,j,u,m);1\le i\le n,1\le j\le t,1\le u\le d,1\le m\le \varepsilon\},$$

$$E^2=\{E^{2,i};1\le i\le n\},$$

$$E^{2,i}=\{(j,a);1\le j\le t,1\le a\le z_{c,i}\},\ \text{for } i = 1,\ldots, n,$$

For this new situation the matrices are given by

$$\mathbf{H}_0=\mathbf{T}\otimes\mathbf{L}\otimes\mathbf{I}+\mathbf{TW}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{Q}\left(1-\omega^0\right),$$

$$\mathbf{H}_1^i=\mathbf{U}_2^i\mathbf{T}_r^0\otimes\mathbf{L}\otimes\mathbf{e}_d+\left(\mathbf{U}_2^i\mathbf{T}_r^0+\mathbf{U}_2^i\mathbf{TW}^0\right)\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{Qe}\left(1-\omega^0\right);\ i = 1,\ldots, n.$$

$$\mathbf{H}_3=\mathbf{T}_{nr}^0\boldsymbol{\alpha}\otimes\left[\mathbf{L}\otimes\mathbf{e}\boldsymbol{\omega}+\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{Qe}\boldsymbol{\omega}\left(1-\omega^0\right)\right]+\mathbf{e}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\left(\mathbf{e}\boldsymbol{\omega}\omega^0+\mathbf{Q}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right).$$

Chapter 3

Multi-state complex cold standby systems subject to multiple events with loss of units

3.1 Introduction

This chapter describes the algorithmic procedure used to model three multi-state complex cold standby systems with loss of units and an indeterminate variable number of repairpersons, using MMAPs. Two main contributions are making: on the one hand, we consider the loss of units with variable numbers of repairpersons; on the other, complex MMAPs are used to model complex systems with multiple events, after which the stationary distribution is determined. In the first of these systems, the online unit is only subject to failure by wear; the second extends this by including external shocks with diverse consequences, and the third includes inspections, so that the effects of preventive maintenance and of the variable number of repairpersons, depending on the number of units present in the system, are analysed. An optimal maintenance policy enables policymakers to decide what level of degradation should be taken into account for preventive maintenance in response to an inspection, whether preventive maintenance is profitable and the optimum number of repairpersons at a given time. This study extends previous research in this area in the following ways: the online multi-state unit passes through an indeterminate level of degradation, external shocks can produce several consequences (extreme failure, cumulative external damage, aggravation of the internal degradation or internal failure), preventive maintenance is performed in response to random inspections, the loss of units is considered (when a non-repairable failure occurs, the unit is not replaced while the system is operational), variable numbers of repairpersons are considered (the number of repairpersons depends on the number of units in the system), rewards and costs are included in the system and an optimising example is shown and all results are expressed in algorithmic form, with PH distributions and Markovian Arrival Processes, with marked arrivals in discrete time (D-MMAP).

The applications considered range from performance and reliability/availability analyses of different configurations of non-repairable and repairable systems, to the development of maintenance strategies providing the desired system functioning, to the optimisation of system structure, performance and maintenance schedules. In this respect, Markopoulos and Platis (2017) [36] considered MSS and semi-Markov modelling to restructure an IEEE 6 BUS RBTS energy system in order to enhance its reliability. Real-life systems are modelled in the present chapter. The model presented can be applied in fields such as civil, industrial and computer engineering. For instance, in computer engineering, a computer server with three hard drives, two of which are available in cold standby, might be assumed. The online hard drive is periodically inspected by an installed monitoring program that analyses logic and physics parameters to detect possible errors. In civil engineering a fundamental element in well machinery is the drill bit. This is essential to advance the construction and it is subject to wear and/or breakage. Drill bits are very expensive and so they are regularly inspected and preventive maintenance is considered. New drill bits are kept in cold standby.

The rest of this chapter is organised as follows. The systems and the state-spaces are detailed in Section 3.2. In Section 3.3 the online unit and the repair facility are modelled. The MMAP for each system are developed in section 3.4. Measures of the transient and the stationary distributions are obtained by considering matrix-analytics methods in Section 3.5, after which costs and rewards are introduced in Section 3.6. A numerical application illustrating the versatility of the model is presented in Section 3.7, section 3.8 presents the main conclusions drawn and finally an Appendix with the main expressions is given in Section 3.9.

3.2 The systems

Three complex systems are described and modelled. The first is the most basic and the last, the most complex. The systems are available in cold standby, and the online unit is multi-state and subject to different types of events.

SYSTEM I

The online unit is multi-state and subject to internal repairable or non-repairable failure. The internal performance of the system is composed of several states which are partitioned into two well-differentiated groups: minor and major damage states, which reflect a low and high risk of failure, respectively.

SYSTEM II

The online unit is multi-state and subject to internal repairable or non-repairable failure and external shocks with different consequences, such as extreme failure of the online unit (non-repairable), degradation of the internal performance of the online unit, caused by a repairable internal failure, and cumulative external damage where if a threshold is reached a non-repairable failure occurs. Each time an external shock takes place, the cumulative external damage increases by passing through an external damage state. These cumulative external damage states are also well-differentiated in two groups: minor and major cumulative external damage states.

SYSTEM III

Random inspections are added to system II. If a major internal stage is reached and/or major external cumulative damage is observed by the inspection, the unit is sent to the repair facility for preventive maintenance. The time distributions for repairable failures and for preventive maintenance may be different.

Three main contributions are incorporated in these systems. The initial number of units in the system is general, K, each time that a unit undergoes a non-repairable failure is removed and the number of repairpersons in the repair facility is general and varies each time that a non-repairable failure occurs. The system continues working while there are units in the system. The number of repairpersons when there k units in the system is denoted by R_k where $1 \leq R_k \leq k$.

The systems are modelled and presented sequentially; the state-space, the modelling of the online unit, that of the repair facility, the associated MMAPs (from the online unit and the repair facility), the measures used and the costs produced. Examples are given in the modelling of the repair facility to illustrate the algorithmic approach used.

3.2.1 Assumptions

The cumulative assumptions for the systems are the following.

SYSTEM I

The behavior of the operational time of the online unit works as assumptions 2.1 and 2.2, given in Section 2.2.1. New assumptions are supposed for this new system.

***Assumption* 3.1**. When the online unit undergoes a non-repairable failure then it is removed and the number of the repairpersons is modified.

***Assumption* 3.2**. The corrective repair time when the online unit fails is *PH* distributed with representation ($\boldsymbol{\beta}_1$, $\mathbf{S}_1$). The order of this matrix is equal to z_1 (number of corrective repair phases).

Assumption 3.3. When the system is composed of only one unit and this one undergoes a non-repairable failure, the system is replaced by new and identical K-units system.

SYSTEM II

The system described above is extended by including external shocks over the online unit. Then, the assumptions 2.3, 2.4, 2.5, 2.6 and 2.7, given in Section 2.2.1, are added to the assumptions described for *System I*. Figure 3.1 shows a diagram for *systems I* and *II*.

SYSTEM III

Preventive maintenance as response to random inspections is considered in *System III*. The assumptions for *System II* are extended with the following new assumptions.

***Assumption* 3.4.** While the online place is busy, random inspections can occur. The time between two consecutive inspections is *PH* distributed with representation ($\boldsymbol{\eta}$, $\mathbf{M}$). The order of the matrix $\mathbf{M}$ is equal to ε.

***Assumption* 3.5**. When all standby units in repair facility, none unit is repaired and one inspection occurs then the online unit is continuous working.

***Assumption* 3.6**. When major internal or/and cumulative external damage is observed then the unit goes to repair facility for preventive maintenance. Preventive maintenance time is *PH* distributed with representation ($\boldsymbol{\beta}_2$, $\mathbf{S}_2$). The order of this matrix is equal to z_2 (number of preventive maintenance states).

Figure 3.2 shows a diagram of *system III*.

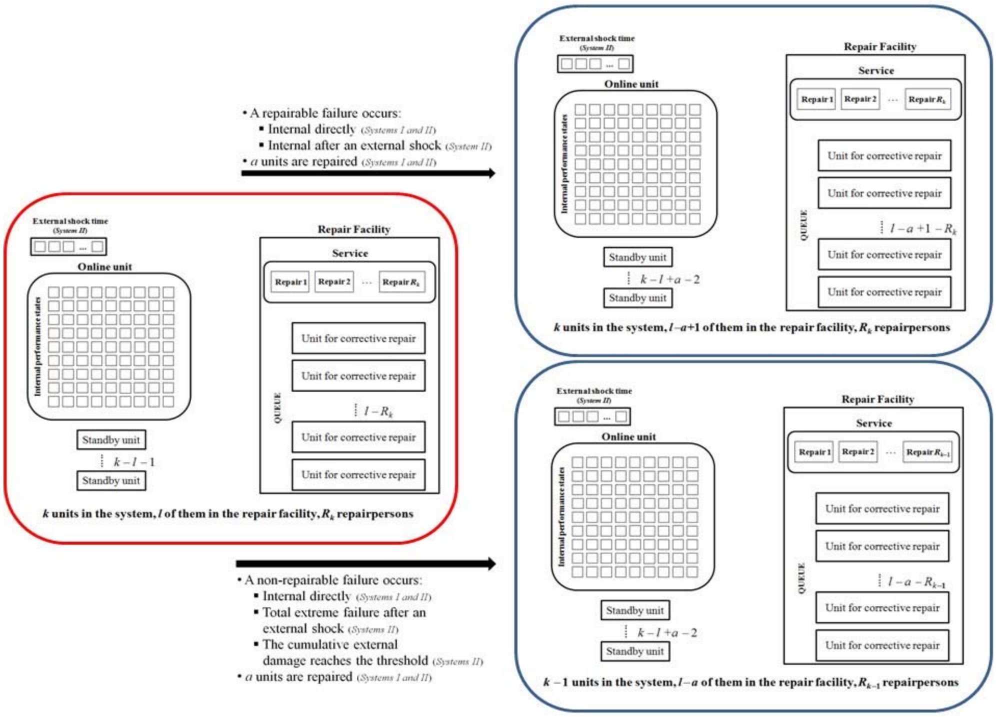

Figure 3.1. Diagram of *systems I* and *II*

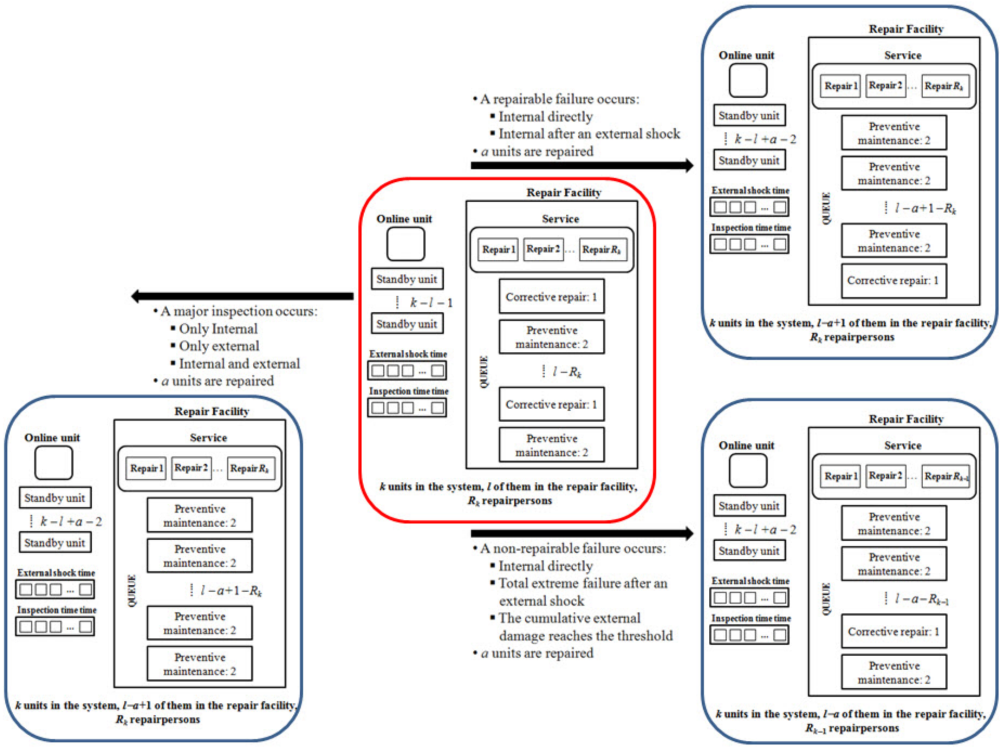

Figure 3.2. Diagram of *system III*

3.2.2 The state-space

The state-space of the system is composed of macro-states. This state-space is different according to the systems.

SYSTEM I

The state space of the system is composed of two levels of macro-states. This state space is denoted by $S=\{\mathbf{U}^K,\mathbf{U}^{K-1},\ldots,\mathbf{U}^1\}$, where $\mathbf{U}^k$ is the second level, containing the phases when there are k units in the system. These macro-states are composed of the macro-states of the first level, $\mathbf{U}^k=\{\mathbf{E}_0^k,\mathbf{E}_1^k,\ldots,\mathbf{E}_k^k\}$ where $\mathbf{E}_s^k$ contains the phases when there are k units in the system and s units are in the repair facility. The phases of the system if the online unit is in state i and the units in corrective repair, if any, are in states $r_1,\ldots,r_{\min\{s,R_k\}}$ are for $k = 1,\ldots, K$ and $s = 1,\ldots, k-1$,

$$\mathbf{E}_0^k = \left\{ \left(k;i\right); i = 1,...,n \right\};$$

$$\mathbf{E}_s^k = \left\{ \left(k, s; i, r_1, \ldots, r_{\min\{s,R_k\}}\right); i = 1,...,n,\ \mathrm{r}_h = 1,..., z_1, h = 1, \ldots, \min\left\{s, R_k\right\} \right\}$$

$$\mathbf{E}_k^k = \left\{ \left(k, k; r_1, \ldots, r_{\min\{s,R_k\}}\right); \mathrm{r}_h = 1,..., z_1, h = 1, \ldots, \min\left\{s, R_k\right\} \right\}.$$

SYSTEM II

The state-space of system II is again composed of two levels, but in this case the states of the inspection time, *j*, and the external cumulative damage, *u*, are included. Then, for $k = 1,\ldots, K$ and $s = 1,\ldots, k-1$,

$$\mathbf{E}_0^k = \left\{ \left(k, 0; i, j, u\right); i = 1,...,n,\ j = 1,...,t,\ u = 1,...,d \right\};$$

$$\mathbf{E}_k^k = \left\{ \left(k, s; j, r_1, \ldots, r_{\min\{s,R_k\}}\right); j = 1,...,t,\ \mathrm{r}_h = 1,..., z_1, h = 1, \ldots, \min\left\{s, R_k\right\} \right\}$$

$$\mathbf{E}_s^k = \left\{ \left(k, s; i, j, u, r_1, \ldots, r_{\min\{s,R_k\}}\right); i = 1,...,n,\ j = 1,...,t,\ u = 1,...,d, \right.$$
$$\left. \mathrm{r}_h = 1,..., z_1, h = 1, \ldots, \min\left\{s, R_k\right\} \right\}.$$

SYSTEM III

The state space of System III is composed of three levels of macro-states. In this case the order of the units in the repair facility has to be saved in memory, as there are two types of repair, corrective and preventive maintenance. For this reason, the macro-state $\mathbf{E}_s^k$ is composed of the first level of macro-states $\mathbf{E}_{i_1,\ldots,i_s}^k$. These macro-states contain the phases when there are *k* units in the system, with *s* of them in the repair facility, and the type of repair is given by the ordered sequence $i_{1,\ \ldots\ ,}\ i_S$. The values of i_l are equal to 1 or 2 if the unit is in corrective repair or preventive maintenance, respectively. Then, for $k = 1,\ldots, K$,

$$\mathbf{E}_0^k = \left\{ \left(k, 0; i, j, u, m\right); i = 1,...,n,\ j = 1,...,t,\ u = 1,...,d,\ m = 1,...,\varepsilon \right\}$$

$$\mathbf{E}_s^k = \left\{ \mathbf{E}_{i_1,\ldots,i_s}^k; i_l = 1,2; l = 1,...,s,\ j = 1,...,s \right\} \text{ for } s = 1,\ldots,k \text{ where}$$

$$\mathbf{E}_{i_1,\ldots,i_s}^k = \left\{ \left(k, s; i, j, u, m, r_1, \ldots, r_{\min\{s,R_k\}}\right); i = 1,...,n,\ j = 1,...,t, \right.$$
$$\left. u = 1,...,d,\ m = 1,...,\varepsilon,\ \mathrm{r}_h = 1,..., z_{i_h}, h = 1, \ldots, \min\left\{s, R_k\right\} \right\}$$

for $s = 1,\ldots,k-1$ and $\mathbf{E}^k_{i_1,\ldots,i_k} = \left\{\left(k, s; j, r_1,\ldots, r_{R_k}\right); j = 1,...,t, r_h = 1,..., z_{i_h}, h = 1,\ldots, R_k\right\}$.

The phase $\left(k, s; i, j, u, m, r_1,\ldots, r_{\min\{s,R_k\}}\right)$ indicates that there are k units in the system of which s of them are in the repair facility, the internal performance is in state i, the external shock time is in state j, the cumulative damage undergone by external shocks is given by u, m is the phase of the inspection time and r is the corrective repair/preventive maintenance phase for the units that are being repaired in the repair facility.

3.3 Modelling the systems

The systems are governed by a Markov process vector in discrete time with the state space described in Section 3.2.2. To model any proposed complex system, the behaviour of the online unit and of the repair facility must be described separately. This section shows the case of System III but analogous reasoning can be performed for Systems I and II. The corresponding matrices for all systems are given in Appendices 3A and 3B.

3.3.1 Modelling the online unit

The online unit of *system III* can undergo different types of events. These ones are partitioned as:

A_1: Internal repairable failure due to internal degradation
A_2: Internal repairable failure due to external shocks
B_1: Major revision for only major internal degradation after inspection
B_2: Major revision for only major external cumulative damage after inspection
B_3: Major revision in both cases (internal and external cumulative damage)
C_1: Non-repairable failure due to internal degradation
C_2: Non-repairable failure due to one external shock
O: No events

The transition for each event affecting the online unit is obtained as follows. The repairable case (A_1, A_2) is discussed below, and the remaining cases are shown in Appendix 3A. An internal repairable failure (A_1) can occur due to internal degradation or after an external shock. In the first case, the online unit undergoes an internal repairable failure and another unit occupies the online place ($\mathbf{T}_r^0\boldsymbol{\alpha}$); an external shock occurs or does not ($\mathbf{L}^0\boldsymbol{\gamma}, \mathbf{L}$ respectively); if it does, cumulative damage occurs but there is no non-repairable failure, $\mathbf{De\omega}\left(1-\omega^0\right)$. If an inspection takes place at the same time, the unit undergoes a repairable failure and the inspection time begins for the new online unit $(\mathbf{e}_\varepsilon\eta)$. This transitions is governed by

$$\mathbf{H}_{\text{rep}}^1 = \left[\mathbf{T}_r^0\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{e\omega}+\mathbf{T}_r^0\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon\eta.$$

If the online unit is the only operational unit and a repair does not occur, then none unit will occupy the online place at the next time. In this case,

$$\mathbf{H}_{\text{rep}}^{'1} = \left[\mathbf{T}_r^0\otimes\mathbf{L}\otimes\mathbf{e}_d+\mathbf{T}_r^0\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon.$$

A similar reasoning can be applied when an external shock provokes an internal repairable failure (A_2). In this case, an external shock occurs ($\mathbf{L}^0\boldsymbol{\gamma}$) and the internal behaviour is modified to address the internal failure ($\mathbf{TW}^0\boldsymbol{\alpha}$). This shock does not provoke a non-repairable failure ($\mathbf{De\omega}\left(1-\omega^0\right)$). The transition matrix is governed by

$$\mathbf{H}_{\text{rep}}^2 = \left[\mathbf{TW}^0\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon\eta.$$

Analogously, if at the next time the online place is empty then

$$\mathbf{H}_{\text{rep}}^{'2} = \left[\mathbf{TW}^0\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De}\left(1-\omega^0\right)\right]\otimes\mathbf{e}_\varepsilon.$$

The rest of the matrices are given in Appendix 3A.

3.3.2 Modeling the repair facility

As mentioned above, the modelling is developed for System III, but the method described is valid for Systems I and II if only corrective repair and non-repairable failures are considered. The transition matrix for the repair facility depends on the number of repairpersons, the number of units in the repair facility, the number of units that are successfully repaired and the type of failure (if any) of the online unit. The number of repairpersons when there are k units in the system, with $k \leq K$, is given by $R_k \leq k$ and the number of units in the repair facility is denoted as l. Let a be the number of units which finish the repair. Let k_h be the ordinal of the repairpersons who concluded the repair, and let i_h and j_h be the type of repair (corrective, 1, preventive maintenance, 2) for the ordered units, after and before the transition, respectively. The online unit can undergo two types of events that can require the unit to be sent to the repair facility: repairable failure or major inspection. This fact is included in the modelling through the variable *mr*, which is equal to 0 if the unit does not undergo an event and 1 if a repairable failure or a major inspection occurs. The online unit is also subject to non-repairable failure. If this occurs, it is denoted by $nr = 1$, otherwise it is equal to 0.When a non-repairable failure occurs, the number of repairpersons can be modified. In this case, if there are fewer repairpersons after a transition than remaining units being repaired, some of these units will be returned to the queue in the repair facility. The number of units to be returned is denoted by *b*. This value is given by

$$b = \max\left\{\min\left\{l, R_k\right\} - a - I_{\{nr=1\}} R_{k-1} - I_{\{nr=0\}} R_k, 0\right\}.$$

To model the behavior of the repair facility we define the following matrix function that governs the behavior in one transition of the units that are being repaired where the order of the units repaired are specified. This function is given by

$$C\left(k,l,a,b;k_1,\ldots,k_a;i_1,\ldots,i_{l-a+mr};j_1,\ldots,j_l\right) = \begin{cases} \mathbf{S}(1)\otimes\cdots\otimes\mathbf{S}\left(\min\left\{l,R_k\right\}\right) & ; \quad i_{s-\sum_{z=1}^{a} I_{\{k_z<s\}}} = j_s; s=1,\ldots,l; s\neq k_z, \forall z \\ \mathbf{0} & ; \quad \text{otherwise} \end{cases}$$

for $k \le K$, $l \ge 1$, $a \ge 1, b \ge 0$, where

$$S(h) = \begin{cases} \mathbf{S}^0_{j_h} & ; \quad \exists z \in \{1,\ldots,a\} \mid h = k_z \\ \mathbf{e} - \mathbf{S}^0_{j_h} & ; \quad h \text{ is the ordinal of the last } b \text{ units being repaired without ending} \\ \mathbf{S}_{j_h} & ; \quad \text{otherwise} \end{cases}.$$

If $a = 0$, then the definition is analogous but we will consider the following notation

$$C\left(k,l,a=0,b;i_1,\ldots,i_{l+mr};j_1,\ldots,j_l\right) = \begin{cases} S_{j_1}\otimes\ldots\otimes S_{j_{\min\{l,R_k\}-b}}\otimes\left(\mathbf{e}-S^0_{j_{\min\{l,R_k\}-b+1}}\right)\otimes\ldots\otimes\left(\mathbf{e}-S^0_{j_{\min\{l,R_k\}}}\right) & ; \quad i_s = j_s; 1\le s\le l \\ \mathbf{0} & ; \quad \text{otherwise} \end{cases}$$

If $a=\min\{l,R_k\}$, then the definition is analogous but we will consider the following notation

$$C\left(k,l,a=\min\{l,R_k\},0;j_1,\ldots,j_l\right)=S_{j_1}^0\otimes\ldots\otimes S_{j_{\min\{l,R_k\}}}^0 .$$

Example 3.1. For instance, we assume a system composed of 4 repairpersons and 6 units (k=6, R_6=4), 5 of them in the repair facility (l = 5; preventive maintenance, corrective repair, preventive maintenance, corrective repair and preventive maintenance respectively). At the next time three unit that are being repaired finishes the repair, and the online unit undergoes a non-repairable failure. The number of repairpersons is only two when the system is composed of 5 units (R_5=2) and the units in the repair facility after non-repairable failure are types corrective repair and preventive maintenance respectively.

In this case the number of units that are devolved to the queue in the repair facility is $b=\max\{\min\{5,4\}-3-2,0\}=0$. If the first three units are repaired then this transition for the units that was being repaired with the established order is

$$C\left(k=6,l=5,a=3,b=0;k_1=1,k_2=2,k_3=3;i_1=1,i_2=2;j_1=2,j_2=1,j_3=2,j_4=1,j_5=2\right)$$
$$=S_2^0\otimes S_1^0\otimes S_2^0\otimes S_1.$$

From this matrix function the transition probability, if only a is known and the order is not specified, is given by

$$B\left(k,l,a,b;i_1,\ldots,i_{l-a+mr};j_1,\ldots,j_l\right)$$
$$=\begin{cases} C\left(k,l,0,b;i_1,\ldots,i_{l+mr};j_1,\ldots,j_l\right) & ; \quad a=0,b\geq 0,l>0 \\ \displaystyle\sum_{k_1=1}^{\min\{l,R_k\}-a+1}\sum_{k_2=k_1+1}^{\min\{l,R_k\}-a+2}\cdots\sum_{k_a=k_{a-1}+1}^{\min\{l,R_k\}} C\left(k,l,a,b;k_1,\ldots,k_a;i_1,\ldots,i_{l-a+mr};j_1,\ldots,j_l\right) & ; \quad a>0,a\neq\min\{l,R_k\} \\ C\left(k,l,a,0;j_1,\ldots,j_l\right) & ; \quad a=\min\{l,R_k\}, \\ 1 & ; \quad a=0,b=0,l=0 \end{cases}$$

Example 3.2. If the example 3.1 is considered, then transition probability matrix when the order of the units repaired are not specified is given by

$$B\left(k=6,l=5,a=3,b=0;i_1=1,i_2=2;j_1=2,j_2=1,j_3=2,j_4=1,j_5=2\right)$$
$$=S_2^0\otimes S_1^0\otimes S_2^0\otimes S_1+\mathbf{0}+S_2^0\otimes S_1\otimes S_2^0\otimes S_1^0+\mathbf{0} .$$

After one transition, new units that were in queue or not can entry in repair. The number of units that will begin the repair at the next time is given by

$$\varepsilon = \min\left\{\max\left\{0, l - R_k\right\} + mr, I_{\{nr=1\}}R_{k-1} + I_{\{nr=0\}}R_k \right.$$
$$\left. - \min\left\{\min\left\{R_k, l\right\} - a, I_{\{nr=1\}}R_{k-1} + I_{\{nr=0\}}R_k\right\}\right\}.$$

The matrix function that governs the transition probability of the repair facility when a of l units are repaired for $l > 0$ and $a \neq l$ is given by

$$E\left(k,l,a,b;i_1,\ldots,i_{l-a+mr};j_1,\ldots,j_l;mr,nr\right) =$$
$$\begin{cases} B\left(k,l,a,0;i_1,\ldots,i_{l-a+mr};j_1,\ldots,j_l\right) \otimes \boldsymbol{\beta}^{i_{\min(l,R_k)-a+1}} \otimes \ldots \otimes \boldsymbol{\beta}^{i_{\min(l,R_k)-a+\varepsilon}} & ; \quad \varepsilon > 0 \\ B\left(k,l,a,b;i_1,\ldots,i_{l-a+mr};j_1,\ldots,j_l\right) & ; \quad \varepsilon = 0 \\ \mathbf{0} & ; \quad \text{otherwise} \end{cases}$$

If $l = 0$ or $a = l$ with $l \leq R_k$ then this function is denoted as

$$E\left(k,l=0,a=0,b=0;mr=0,nr=0,1\right) = 1, E\left(k,l=0,a=0,b=0;i_{mr};mr=1,nr=0\right) = \boldsymbol{\beta}^{i_{mr}}$$
$$, E\left(k,l,a=l,b=0;j_1,\ldots,j_l;mr=0,nr=0,1\right) = B\left(k,l,l,0;j_1,\ldots,j_l\right),$$
$$E\left(k,l,a=l,b=0;i_{mr};j_1,\ldots,j_l;mr=1,nr=0\right) = B\left(k,l,l,0;j_1,\ldots,j_l\right) \otimes \boldsymbol{\beta}^{i_{mr}}.$$

Example 3.3. If the example 3.1 and 3.2 are considered, then the number of units that entry in repair is given by $\varepsilon = \min\left\{\max\left\{0,1\right\} + 0, 2 - \min\left\{4-3,2\right\}\right\} = 1$. Therefore, the transition probability for the repair facility is given by

$$E\left(k=6,l=5,a=3,b=0;i_1=1,i_2=2;j_1=2,j_2=1,j_3=2,j_4=1,j_5=2;mr=0,nr=1\right)$$
$$= \left(S_2^0 \otimes S_1^0 \otimes S_2^0 \otimes S_1 + S_2^0 \otimes S_1 \otimes S_2^0 \otimes S_1^0\right) \otimes \boldsymbol{\beta}^{i_2=2}.$$

3.4 The Markovian Arrival Processes with marked arrivals

The systems *I*, *II* and *III*, are modeled by different MMAPs by considering the different types of events described in section 3.3.1. The MMAPs for the different systems have the following representations,

Model I: $\left(\mathbf{D}^{O}, \mathbf{D}^{A_1}, \mathbf{D}^{C_1}, \mathbf{D}^{FC_1}\right)$

Model II: $\left(\mathbf{D}^{O}, \mathbf{D}^{A_1}, \mathbf{D}^{A_2}, \mathbf{D}^{C_1}, \mathbf{D}^{C_2}, \mathbf{D}^{FC_1}, \mathbf{D}^{FC_2}\right)$

Model III: $\left(\mathbf{D}^{O}, \mathbf{D}^{A_1}, \mathbf{D}^{A_2}, \mathbf{D}^{B_1}, \mathbf{D}^{B_2}, \mathbf{D}^{B_3}, \mathbf{D}^{C_1}, \mathbf{D}^{C_2}, \mathbf{D}^{FC_1}, \mathbf{D}^{FC_2}\right)$.

FC_1 and FC_2 denote the same that C_1 and C_2 respectively when only one unit is present in the system. These events will be used to count the number of new systems by time.

The matrix $\mathbf{D}^Y$ contains the transition probabilities when the event Y has occurred for $Y = O, A_1, A_2, B_1, B_2, B_3, C_1, C_2, FC_1, FC_2$.

The matrix $\mathbf{D}^Y$ is built following three matrix block levels, always when the event Y occurs. The third level corresponds to the transitions from the macro-state $\mathbf{U}^k$ to $\mathbf{U}^k$ or $\mathbf{U}^{k-1}$. These matrix blocks are composed of the matrices $\mathbf{D}_{lh}^{Y,k}$ which correspond to the transitions between the macro-states from $\mathbf{E}_l^k$ to either $\mathbf{E}_h^k$ or $\mathbf{E}_h^{k-1}$ (level 2).

Finally, when preventive maintenance is introduced (*system III*) several types of repairing can be produced. Therefore, the type of failure of the units in the repair facility has to be saved in memory. The matrices $\mathbf{D}_{lh}^{Y,k}$ are composed of matrix blocks corresponding to the transition from the macro-states $\mathbf{E}_{i_1,\ldots,i_l}^k$ to $\mathbf{E}_{i_1,\ldots,i_h}^k$ or $\mathbf{E}_{i_1,\ldots,i_h}^{k-1}$. The matrix block $\mathbf{D}_{lh}^{Y,k}\left(i_1,\ldots,i_h; j_1,\ldots,j_l\right)$ contains the transition probabilities described above where the type of repair in the repair facility is ordered for the case before and after transition. These blocks are built by considering the matrices $\mathbf{H}$ defined in section 3.1 and developed in Appendix 3.A (level 1). Next, the case A_1 for the Model III, a repairable internal failure occurs, is described in detail. The rest is given in an algorithmic form in Appendix 3B.

Building the matrix $\mathbf{D}^{A_1}$

This matrix $\mathbf{D}^{A_1}$ is a matrix block that governs the transitions when an internal repairable failure occurs. Then this matrix is a diagonal matrix block $\mathbf{D}^{A_1} = diag\left(\mathbf{D}^{A_1,K}, \mathbf{D}^{A_1,K-1}, \mathbf{D}^{A_1,K-2}, \ldots, \mathbf{D}^{A_1,1}\right)$ given that a non-repairable failure does not occur. The matrix block $\mathbf{D}^{A_1,k}$ contains the transitions when this fact occurs with k units in the system. The elements of the matrix $\mathbf{D}^{A_1,k}$ for k = 1,..., K are matrix blocks by considering the number of units in the system, l and h before and after the transition respectively. It is given by

$$\mathbf{D}^{A_1,k} = \left(\mathbf{D}_{lh}^{A_1,k}\right)_{l,h=0,\ldots,k}$$

where $\mathbf{D}_{lh}^{A_1,k} = \mathbf{0}$ if $h > l+1$ or $h < l+1-\min\{l, R(k)\}$ or $l = k$.

Finally, these matrices are again composed of matrix blocks by taking into account the order of the units in the repair facility. Thus, if there are k units in the system, l of them in the repair facility with ordered type of failure $\left(j_1,\ldots,j_l\right)$, a units are repaired and one repairable failure occurs then the transition matrix is given by

$$\mathbf{D}_{l,l+1-a}^{A_1,k}\left(i_1,\ldots,i_{l+1-a-1},1;j_1,\ldots,j_l\right)=\left(\mathbf{H}_{\text{rep}}^{1}I_{\{l<k-1\text{ or }a>0\}}+\mathbf{H}_{\text{rep}}^{'1}I_{\{l=k-1\text{ and }a=0\}}\right)$$
$$\otimes E\left(k,l,a,0;i_1,\ldots,i_{l+1-a-1},1;j_1,\ldots,j_l;1,0\right),$$

for l=1,…,k–1; a=0,…,min{l, R_k} and $k > 1$.

The matrix is obtained according to the following algorithm.

1. $\mathbf{D}^{A_1}=diag\left(\mathbf{D}^{A_1,K},\mathbf{D}^{A_1,K-1},\mathbf{D}^{A_1,K-2},\ldots,\mathbf{D}^{A_1,1}\right)$
2. $\mathbf{D}^{A_1,k}=\left(\mathbf{D}_{lh}^{A_1,k}\right)_{l,h=0,\ldots,k}$ for k =1,..., K
3. Building blocks $\mathbf{D}_{lh}^{A_1,k}$ for $l+1-\min\{l,R(k)\}\le h\le l+1$ and $l\neq k$

 3.1. If $l = 0$

  a. Calculating $\mathbf{D}_{01}^{A_1,k}(1)=\left(\mathbf{H}_{\text{rep}}^{1}I_{\{k>1\}}+\mathbf{H}_{\text{rep}}^{'1}I_{\{k=1\}}\right)\otimes E\left(k,0,0,0;1;1,0\right)$

   a.1. If $k>1\rightarrow\mathbf{H}_{\text{rep}}^{1}$ else $\mathbf{H}_{\text{rep}}^{'1}$

   a.2. Function $E\left(k,0,0,0;1;1,0\right)=B\left(k,0,0,0;1\right)\otimes\boldsymbol{\beta}^{1}$

    a.2.1. Calculating ε and b (in this case it is equal to one and zero respectively)

    a.2.2. Calculating function $B\left(k,0,0,0;1\right)=1$

 3.2. For l=1,…,k–1; a=0,…,min{l, R_k} with $k > 1$

  a. Calculating

$$\mathbf{D}_{l,l+1-a}^{A_1,k}\left(i_1,\ldots,i_{l+1-a-1},1;j_1,\ldots,j_l\right)=\left(\mathbf{H}_{\text{rep}}^{1}I_{\{l<k-1\text{ or }a>0\}}+\mathbf{H}_{\text{rep}}^{'1}I_{\{l=k-1\text{ and }a=0\}}\right)$$
$$\otimes E\left(k,l,a,0;i_1,\ldots,i_{l+1-a-1},1;j_1,\ldots,j_l;1,0\right)$$

   a.1. If $l<k-1$ or $a>0\rightarrow\mathbf{H}_{\text{rep}}^{1}$ else $\mathbf{H}_{\text{rep}}^{'1}$

   a.2. Function $E\left(k,l,a,0;i_1,\ldots,i_{l+1-a-1},1;j_1,\ldots,j_l;1,0\right)$

    a.2.1. Calculating ε and b (in this case b is equal to zero)

    a.2.2. Calculating function

$$B\left(k,l,a,0;i_1,\ldots,i_{l-a+1-1},1;j_1,\ldots,j_l\right)$$

The rest of matrices are given in Appendix 3B.

3.5 The transient and stationary distribution and Measures

The transient and stationary distributions have been built so as several measures of interest. These measures are developed for the *system III*. The other models can be achieved in a similar way.

3.5.1 The transient distribution

Once built the D-MMAP, the transition probability matrix that governs the discrete Markov chain associated to the system III and it is given by $\mathbf{D} = \mathbf{D}^0 + \mathbf{D}^{A_1} + \mathbf{D}^{A_2} + \mathbf{D}^{B_1} + \mathbf{D}^{B_2} + \mathbf{D}^{B_3} + \mathbf{D}^{C_1} + \mathbf{D}^{C_2} + \mathbf{D}^{FC_1} + \mathbf{D}^{FC_2}$. Given the initial distribution of the system $\boldsymbol{\theta}$, the transient distribution is worked out as $\mathbf{p}^{\nu} = \boldsymbol{\theta}\mathbf{D}^{\nu}$. Therefore, the probability of being in the macro-state $\mathbf{E}_s^k$ at time ν is the corresponding part of $\mathbf{p}^{\nu}$ and it is denoted by $\mathbf{p}^{\nu}_{\mathbf{E}_s^k}$.

3.5.2 Stationary distribution

The stationary distribution, $\boldsymbol{\pi}$, has been built solving the balance equations by applying matrix analytic methods. It is well known that the stationary distribution verifies $\boldsymbol{\pi}\mathbf{D} = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\mathbf{e} = 1$. This system has been solved for the macro-state $\mathbf{E}^k$, k units in the system. The stationary distribution for this macro-state is denoted by $\boldsymbol{\pi}_{\mathbf{E}^k}$. Then, the stationary distribution is $\boldsymbol{\pi} = \left(\boldsymbol{\pi}_{\mathbf{E}^K}, \boldsymbol{\pi}_{\mathbf{E}^{K-1}}, \ldots, \boldsymbol{\pi}_{\mathbf{E}^1}\right)$. From the MMAP the transition probability matrices for the transition from $\mathbf{E}^k$ to $\mathbf{E}^k$ or $\mathbf{E}^{k-1}$ are denoted by

$$
\begin{aligned}
&\mathbf{D}_{k,k} = \mathbf{D}^{O,k} + \mathbf{D}^{A_1,k} + \mathbf{D}^{A_2,k} + \mathbf{D}^{B_1,k} + \mathbf{D}^{B_2,k} + \mathbf{D}^{B_3,k} \quad ; \quad k = 1,\ldots,K \\
&\mathbf{D}_{k,k-1} = \mathbf{D}^{C_1,k} + \mathbf{D}^{C_2,k} \quad ; \quad k = 2,\ldots,K \\
&\mathbf{D}_{1,K} = \mathbf{D}^{FC_1,1} + \mathbf{D}^{FC_2,1}
\end{aligned}
$$

The stationary distribution has been worked out from the balance equations. These probabilities are equal to

$$\boldsymbol{\pi}_{\mathbf{E}^k} = \boldsymbol{\pi}_{\mathbf{E}^1}\mathbf{R}_{1,k}\text{, for } k = 2,\ldots,K,$$

Being

$$\mathbf{R}_{1,K} = \mathbf{D}_{1,K}\left(\mathbf{I} - \mathbf{D}_{K,K}\right)^{-1}$$

$$\mathbf{R}_{1,k} = \mathbf{R}_{1,k+1}\mathbf{D}_{k+1,k}\left(\mathbf{I}-\mathbf{D}_{k,k}\right)^{-1} \quad \text{for } k=2,\ldots,K-1.$$

The vector $\boldsymbol{\pi}_{\mathbf{E}^1}$ can be expressed as

$$\boldsymbol{\pi}_{\mathbf{E}^1} = \left(1,\mathbf{0}\right)\left(\left(\mathbf{I}+\sum_{k=2}^{K}\mathbf{R}_{1,k}\right)\mathbf{e}\middle|\left[\mathbf{D}_{1,1}+\mathbf{R}_{1,2}\mathbf{D}_{2,1}-\mathbf{I}\right]^{*}\right)^{-1},$$

Where the matrix $\mathbf{A}^*$ is the matrix $\mathbf{A}$ without the first column.

The stationary distribution associated to the macro-state $\mathbf{E}_s^k$ is given by the corresponding part of $\boldsymbol{\pi}_{\mathbf{E}^k}$ and it is denoted by $\boldsymbol{\pi}_{\mathbf{E}_s^k}$.

3.5.3 Measures

Several interesting reliability measures such as availability, reliability, mean times and mean number of events are calculated in this section for the transient and stationary regime.

3.5.3.1 Availability

The availability is the probability that the system is operational at time ν. It is given by

$$A(\nu) = 1-\sum_{k=1}^{K}\mathbf{p}_{E_k^k}^{\nu}\cdot\mathbf{e}.$$

This measure is also calculated in the stationary case and it is equal to $A = 1-\sum_{k=1}^{K}\boldsymbol{\pi}_{E_k^k}\cdot\mathbf{e}$.

3.5.3.2 Reliability

Two different reliability functions have been built: the time up to the first time that the system in non-operational (all units in the repair facility) and the time up to the first time that the system is replaced (all units have undergone a non-repairable failure).

In the first case, the probability distribution is given by the phase-type distribution with representation $(\boldsymbol{\theta}',\mathbf{D}')$ where the vector and the matrix are equal to $(\boldsymbol{\theta},\mathbf{D})$ restricted to the macro-states E_s^k for $k = 1,\ldots, K$ and $s = 0,\ldots,k-1$.

In the second case, the time up to the first time that the system is replaced by another an identical one is phase-type distributed with representation $(\boldsymbol{\theta}, \mathbf{D}^*)$ where the matrix is given by $\mathbf{D}$ with the blocks $\mathbf{D}^{FC1,1} = \mathbf{D}^{FC2,1} = \mathbf{0}$.

3.5.3.3 Mean time in each macro-state

The mean time that the system is in macro-state $\mathbf{E}_s^k$ (k units in the system and s of them in the repair facility) up to time ν is given by

$$\psi_{k,s}(\nu) = \sum_{m=0}^{\nu} \mathbf{p}_{E_s^k}^{m} \cdot \mathbf{e}.$$

From this expression, the mean time in macro state $\mathbf{E}^k$ (k units in the system) is given by

$$\psi_k(\nu) = \sum_{s=0}^{k} \psi_{k,s}(\nu).$$

The corresponding stationary values are $\psi_{k,s} = \sum_{m=0}^{\nu} \boldsymbol{\pi}_{E_s^k} \cdot \mathbf{e}$ and $\psi_k = \sum_{s=0}^{k} \psi_{k,s}$.

3.5.3.4 Mean operational time up to time ν

From the measures described above in section 3.5.3, the mean time that the system is operational up to time ν can be calculated. It is given by

$$\mu_{op}(\nu) = \sum_{k=1}^{K} \sum_{s=0}^{k-1} \psi_{k,s}(\nu).$$

This mean time in stationary regime is the operational time ratio and it is $\mu_{op} = \sum_{k=1}^{K} \sum_{s=0}^{k-1} \psi_{k,s}$.

3.5.3.5 Mean time that the repairpersons are idle and busy

The systems proposed in this work have different number of repairpersons depending on the number of units in the system. One interesting aspect is to analyse the mean cumulative time that the repairpersons are idle up to a certain time. This measure is given by

$$\mu_{idle}(\nu) = \sum_{k=1}^{K} \sum_{s=0}^{k-1} \left(R_k - \min\{R_k, s\}\right) \cdot \psi_{k,s}(\nu).$$

In the stationary regime this measure is the mean number of idle repairpersons per unit of time,

$$\mu_{idle\_s} = \sum_{k=1}^{K}\sum_{s=0}^{k-1}\left(R_k - \min\left\{R_k, s\right\}\right)\cdot \psi_{k,s} .$$

Following a similar reasoning to analyse the number of repairpersons that are busy

$$\mu_{busy}\left(\nu\right) = \sum_{k=1}^{K}\sum_{s=1}^{k}\min\left\{R_k, s\right\}\cdot \psi_{k,s}\left(\nu\right),$$

and in the stationary regime $\mu_{busy\_s} = \sum_{k=1}^{K}\sum_{s=1}^{k}\min\left\{R_k, s\right\}\cdot \psi_{k,s}$.

3.5.3.6 Mean time working on corrective and preventive repair

The repairpersons can be working on corrective repair or preventive maintenance. The mean time that the repairpersons are working on corrective repair and preventive maintenance up to time ν is given respectively by

$$\mu_{corr}\left(\nu\right) = \sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}^{\nu}_{E_s^k}\cdot \mathbf{q}_s^k\left(1\right) \text{ And } \mu_{pm}\left(\nu\right) = \sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}^{\nu}_{E_s^k}\cdot \mathbf{q}_s^k\left(2\right),$$

Where $\mathbf{q}_s^k\left(1\right)$ and $\mathbf{q}_s^k\left(2\right)$ are column vectors that contains the number of repairpersons that are working on corrective repair and preventive maintenance respectively according to the macro-state E_s^k. These column vectors are given by

$$\mathbf{q}_s^k\left(1\right) = \begin{pmatrix} d_s^k\left(1\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}} z_1^{d_s^k(1)} z_2^{\min\{s,R_k\}-d_s^k(1)}} \\ d_s^k\left(2\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}} z_1^{d_s^k(2)} z_2^{\min\{s,R_k\}-d_s^k(2)}} \\ \vdots \\ d_s^k\left(I_{\{s=K\}}2^{K-1} + I_{\{s\neq K\}}2^s\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}} z_1^{d_s^k(i)} z_2^{\min\{s,R_k\}-d_s^k(i)}} \end{pmatrix},$$

for

$$\mathbf{q}_s^k\left(2\right) = \begin{pmatrix} g_s^k\left(1\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}} z_1^{\min\{s,R_k\}-g_s^k(1)} z_2^{g_s^k(1)}} \\ g_s^k\left(2\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}} z_1^{\min\{s,R_k\}-g_s^k(2)} z_2^{g_s^k(2)}} \\ \vdots \\ g_s^k\left(I_{\{s=K\}}2^{K-1} + I_{\{s\neq K\}}2^s\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}} z_1^{\min\{s,R_k\}-g_s^k(2)} z_2^{g_s^k(2)}} \end{pmatrix}$$

being $d_s^k\left(i\right)$ the *i*-th element of the vector $\mathbf{d}_s^k = \begin{pmatrix}1\\0\end{pmatrix} \odot \overset{\min\{s,R_k\}}{\cdots} \odot \begin{pmatrix}1\\0\end{pmatrix} \otimes \mathbf{e}_{2^{\max\{s-R_k,0\}}}$ with

$\mathbf{d}_k^k = I_{\{k=R_k\}}\left[\begin{pmatrix}1\\0\end{pmatrix}\odot\overset{k-1}{\cdots}\odot\begin{pmatrix}1\\0\end{pmatrix}+\mathbf{e}_{2^{k-1}}\right]+I_{\{k>R_k\}}\begin{pmatrix}1\\0\end{pmatrix}\odot\overset{R_k}{\cdots}\odot\begin{pmatrix}1\\0\end{pmatrix}\otimes\mathbf{e}_{2^{k-R_k-1}}$ and $g_s^k(i)$ the i-th element of the vector $\mathbf{g}_s^k = \begin{pmatrix}0\\1\end{pmatrix}\odot\overset{\min\{s,R_k\}}{\cdots}\odot\begin{pmatrix}0\\1\end{pmatrix}\otimes\mathbf{e}_{2^{\max\{s-R_k,0\}}}$ and $\mathbf{g}_k^k = I_{\{k=R_k\}}\begin{pmatrix}0\\1\end{pmatrix}\odot\overset{k-1}{\cdots}\odot\begin{pmatrix}0\\1\end{pmatrix}+I_{\{k>R_k\}}\begin{pmatrix}0\\1\end{pmatrix}\odot\overset{R_k}{\cdots}\odot\begin{pmatrix}0\\1\end{pmatrix}\otimes\mathbf{e}_{2^{k-R_k-1}}$ for k = 1,…, K and s = 1,…,k where $\mathbf{a}\odot\mathbf{b}=\mathbf{a}\otimes\mathbf{e}_m+\mathbf{e}_n\otimes\mathbf{a}$ being $\mathbf{a}$ and $\mathbf{b}$ column vectors with order n and m respectively.

These measures in stationary regime are $\mu_{corr\_s}=\sum_{k=1}^{K}\sum_{s=1}^{k}\boldsymbol{\pi}_{E_s^k}\cdot\mathbf{q}_s^k(1)$ and $\mu_{pm\_s}=\sum_{k=1}^{K}\sum_{s=1}^{k}\boldsymbol{\pi}_{E_s^k}\cdot\mathbf{q}_s^k(2)$ respectively.

3.5.3.7 Mean number of events

Thanks to the structure built, the expected number of events up to a certain time ν is worked out. It is given by $\Lambda^Y(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\mathbf{D}^Y\mathbf{e}$, for $Y = A_1, A_2, B_1, B_2, B_3, C_1, C_2\ FC_1, FC_2$. In stationary regime, the mean number of events per unit of time is $\Lambda^Y=\boldsymbol{\pi}\mathbf{D}^Y\mathbf{e}$.

3.6 Costs and Rewards

Several costs and rewards have been included in the model to study the effectiveness of the model from an economic standpoint. Thus, we assume that there is a gross profit per unit of time while the system is operational equal to B. While the system is operational a mean cost per unit of time depending on the operational phase occurs. This cost is given by the vector $\boldsymbol{c}_0$. There are two different types of repair, corrective repair and preventive maintenance. The mean cost per unit of time when a unit is in corrective repair or preventive maintenance depending on the repair phase is given by the vectors $\boldsymbol{cr}_1$ and $\boldsymbol{cr}_2$ respectively. Also, we assume a fixed cost per unit of time for each repairperson equal to H and a loss per unit of time while the system is not operational equal to C. Finally, each time that the online unit undergoes a repairable failure or a major inspection a fixed cost is produced equal to fcr or fpm respectively. The mean cost per one new unit is fnu (the cost of a new system is $K{\cdot}fnu$).

To calculate the total net profit up to time ν is necessary to build the vector cost for the macro-state E_s^k and several rewards and costs functions.

3.6.1 Net profit vector associated to the phases

When the systems is composed of *k* units and *s* of them are in the repair facility, then the online unit provokes a net reward for the different phases of the system given by

$$\mathbf{nr}_s^k = \begin{cases} B\mathbf{e}_{ntd\varepsilon} - \mathbf{c_0} \otimes \mathbf{e}_{td\varepsilon} & ; \quad s = 0 \\ B\mathbf{e}_{ntd\varepsilon \cdot 2^{s-\min\{s,R_k\}} \sum_{i=1}^{2^s} z_1^{d_s^k(i)} z_2^{g_s^k(i)}} - \begin{pmatrix} \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon 2^{s-\min\{s,R_k\}} z_1^{d_s^k(1)} z_2^{g_s^k(1)}} \\ \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon 2^{s-\min\{s,R_k\}} z_1^{d_s^k(2)} z_2^{g_s^k(2)}} \\ \vdots \\ \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon 2^{s-\min\{s,R_k\}} z_1^{d_s^k(2^s)} z_2^{g_s^k(2^s)}} \end{pmatrix} & ; \quad s = 1,\ldots,k-1 \\ -C \cdot \mathbf{e}_{t \cdot 2^{k-R_k-I_{\{s=K,R_K<K\}}} \sum_{i=1}^{2^{s-I_{\{s=K\}}}} z_1^{d_s^k(i)} z_2^{g_s^k(i)}} & ; \quad s = k \end{cases}$$

Then, for the state space it is

$$\mathbf{nr} = \left(\mathbf{nr}_0^K{}', \mathbf{nr}_1^K{}', \ldots, \mathbf{nr}_K^K{}', \mathbf{nr}_0^K{}', \mathbf{nr}_1^{K-1}{}', \ldots, \mathbf{nr}_{K-1}^{K-1}{}', \ldots, \mathbf{nr}_0^1{}', \mathbf{nr}_1^1{}'\right)'.$$

If the repair facility is considered, the cost vector per unit of time depending on the type of repair associated to the macro-state E_s^k for *s* = 1,…, *k*, is

$$\mathbf{nc}_s^k = \begin{pmatrix} \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}2^{s-\min\{s,R_k\}-I_{\{s=K,R_K\neq K\}}}} \otimes \overbrace{\mathbf{cr}_1 \odot \mathbf{cr}_1 \odot \cdots \odot \mathbf{cr}_1}^{\min\{s,R_k\}} \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}2^{s-\min\{s,R_k\}-I_{\{s=K,R_K\neq K\}}}} \otimes \mathbf{cr}_1 \odot \mathbf{cr}_1 \odot \cdots \odot \mathbf{cr}_2 \\ \vdots \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}2^{s-\min\{s,R_k\}-I_{\{s=K,R_K\neq K\}}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \cdots \odot \mathbf{cr}_1 \\ \mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}2^{s-\min\{s,R_k\}-I_{\{s=K,R_K\neq K\}}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \cdots \odot \mathbf{cr}_2 \end{pmatrix}, \ \mathbf{nc}_1^k = \mathbf{0}_{ntd\varepsilon}$$

$$\mathbf{nc}_K^K = \begin{pmatrix} \mathbf{e}_t \otimes \overbrace{\mathbf{cr}_1 \odot \mathbf{cr}_1 \odot \cdots \odot \mathbf{cr}_1 \odot \mathbf{cr}_1}^{K} \\ \mathbf{e}_t \otimes \mathbf{cr}_1 \odot \mathbf{cr}_1 \odot \cdots \odot \mathbf{cr}_2 \odot \mathbf{cr}_1 \\ \vdots \\ \mathbf{e}_t \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \cdots \odot \mathbf{cr}_1 \odot \mathbf{cr}_1 \\ \mathbf{e}_t \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \cdots \odot \mathbf{cr}_2 \odot \mathbf{cr}_1 \end{pmatrix}.$$

The total vector for the cost due to repair is given by

$$\mathbf{nc} = \left(\mathbf{nc}_0^K{}', \mathbf{nc}_1^K{}', \ldots, \mathbf{nc}_K^K{}', \mathbf{nc}_0^K{}', \mathbf{nc}_1^{K-1}{}', \ldots, \mathbf{nc}_{K-1}^{K-1}{}', \ldots, \mathbf{nc}_0^1{}', \mathbf{nc}_1^1{}'\right)'.$$

Thus, the net profit vector associated to the macro-state E_s^k is given by $\mathbf{c}_0^k = \mathbf{nr}_0^k$, $\mathbf{c}_s^k = \mathbf{nr}_s^k - \mathbf{nc}_s^k$ for s =1,…,k. Finally, the net column profit vector associated to the macro-state E^k is given by $\mathbf{c}^k = \left(\mathbf{c}_0^k, \ldots, \mathbf{c}_k^k\right)'$ thus the global net column profit vector associated to the macro-state E is given by

$$\mathbf{c} = \mathbf{nr} - \mathbf{nc} = \begin{pmatrix} \mathbf{c}^K \\ \mathbf{c}^{K-1} \\ \vdots \\ \mathbf{c}^1 \end{pmatrix}.$$

3.6.2 Rewards Measures

Several rewards measures have been built in transient and stationary regime.

Mean net profit up to time ν

The mean net profit by considering only the online unit up to time ν is given by

$$\Phi_w^{\nu} = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{nr},$$

and it is in stationary regime the meat neat profit per unit of time, $\Phi_{w\_s} = \boldsymbol{\pi} \cdot \mathbf{nr}$.

Mean cost due to corrective and preventive repair

The mean cost due to corrective repair and preventive maintenance up to time up ν is given respectively by

$\Phi_{cr}^{\nu} = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{cr}$ and $\Phi_{pm}^{\nu} = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{pm}$ where $\mathbf{mc}^{cr}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_2 = \mathbf{0}_{z_2}$ and $\mathbf{mc}^{pm}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_1 = \mathbf{0}_{z_1}$, being $\mathbf{0}_a$ a column vector of zeros with order a.

These measures in stationary regime, net cost per unit of time due to corrective or preventive maintenance, are $\Phi_{cr\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{cr}$ and $\Phi_{pm\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{pm}$ respectively.

Total net profit

The total net profit up to time ν is worked out by adding costs and profits produced by the events. If the fixed cost per event are included then it is equal to

$$\begin{aligned}\Phi^{\nu} = {} & \Phi_w^{\nu} - \Phi_{cr}^{\nu} - \Phi_{pm}^{\nu} - \left(1 + \Lambda^{FC1}(\nu) + \Lambda^{FC2}(\nu)\right) \cdot K \cdot fnu \\ & - \left(\Lambda^{A1}(\nu) + \Lambda^{A2}(\nu)\right) \cdot fcr - \left(\Lambda^{B1}(\nu) + \Lambda^{B2}(\nu) + \Lambda^{B3}(\nu)\right) \cdot fpm - \left(\mu_{idle} + \mu_{busy}\right) \cdot H.\end{aligned}$$

Finally, the total net profit per unit of time (stationary regime) is

$$\begin{aligned}\Phi_s = {} & \Phi_{w\_s} - \Phi_{cr\_s} - \Phi_{pm\_s} - \left(1 + \Lambda^{FC1} + \Lambda^{FC2}\right) \cdot K \cdot fnu \\ & - \left(\Lambda^{A1} + \Lambda^{A2}\right) \cdot fcr - \left(\Lambda^{B1} + \Lambda^{B2} + \Lambda^{B3}\right) \cdot fpm - \left(\mu_{idle} + \mu_{busy}\right) \cdot H.\end{aligned}$$

3.7 A numerical example

Any facility that requires a reliable electrical supply (such as department stores, hospitals, military installations and hydroelectric plants) must have additional generating resources available. When the ordinary electricity supply fails, a cold standby generating set comes into action. For a large dam, at least two such generating sets must be installed in cold standby. The generating set may fail for the same reasons as any motor, provoking either a total failure of the motor or a repairable failure, and preventive maintenance may be necessary. Therefore, we assume a cold standby system composed of three units, as Systems II and III. To optimise the system, two questions must be answered. Is preventive maintenance profitable? How many repairpersons, depending on the number of units in the system, would have to be deployed to optimise the profit? In this numerical example, the effectiveness of preventive maintenance is analysed and the optimum number of repairpersons is calculated.

System times

The internal behaviour of the online unit passes through five performance levels, where the degradation is minor in the first three stages and major in the last two. The online unit is also subject to external shocks and inspections. The operational time distribution of the online unit, the inspection time distribution and the external shock time are PH distributed with representation given in Table 3.1.

Internal operational time	External shock	Inspection time
$\boldsymbol{\alpha} = (1,0,0,0,0)$ $\mathbf{T} = \begin{pmatrix} 0.99 & 0.002 & 0 & 0 & 0 \\ 0 & 0.9 & 0.001 & 0 & 0 \\ 0 & 0 & 0.9 & 0.002 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0.6 \end{pmatrix}$ Mean time: 102.0201	$\boldsymbol{\gamma} = (1,0)$ $\mathbf{L} = \begin{pmatrix} 0.89 & 0.1 \\ 0.1 & 0.8 \end{pmatrix}$ Mean time: 25	$\boldsymbol{\eta} = (1,0)$ $\mathbf{M} = \begin{pmatrix} 0.85 & 0.1 \\ 0.45 & 0.4 \end{pmatrix}$ Mean time: 15.56

Table 3.1. Internal operational, external shock and inspection time distributions.

Each time that the online unit undergoes an external shock, a total non-repairable failure occurs with a probability equal to 0.05. If no such failure occurs, the internal performance may be degraded according to the following probability matrix

$$\mathbf{W} = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 & 0 \\ 0 & 0.6 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0.6 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0 & 0.4 \end{pmatrix}.$$

When an external shock takes place, cumulative external damage occurs. Four external degradation levels are assumed, the first two of which are minor and the last two, major. Changes in the external degradation levels are governed by the matrix

$$\mathbf{D} = \begin{pmatrix} 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 \end{pmatrix},$$

where initially the external degradation level is the stage 1 (without external damage).

Each time that a repairable failure or a major inspection occurs, the online units goes to the repair facility. The corrective repair time and the preventive maintenance time distributions are given in Table 3.2.

Performance of the systems according to the number of repairpersons

As mentioned above, the systems with and without preventive maintenance (Systems III and II, respectively) are compared by considering all possibilities for the number of repairpersons. Thus, the system *i_j_k* denotes a system with i, j, k repairpersons when there are 1, 2, 3 units in the system respectively for i =1, j =1, 2 , k = 1, 2, 3. In total there are 12 possible systems, six with preventive maintenance and six without. Several measures have been worked out and compared in transient and stationary regime.

Corrective repair time distribution	Preventive maintenance time distribution
$\boldsymbol{\beta}^1 = (1, 0)$ $\mathbf{S}_1 = \begin{pmatrix} 0.91 & 0.01 \\ 0 & 0.8 \end{pmatrix}$ Mean time: 11.67	$\boldsymbol{\beta}^2 = (1, 0)$ $\mathbf{S}_2 = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{pmatrix}$ Mean time: 1.23

Table 3.2. Corrective repair and preventive maintenance time distributions

Figure 3.3 shows the mean operational time and the mean number of idle repairpersons per unit of time for Systems II and III. The optimum mean operational time ratio is reached for System 1_2_3 when the operational time ratio in a stationary regime is equal to 0.9467 for the system with preventive maintenance and 0.9346 for the case without preventive maintenance. This outcome is to expected but as the repairpersons have a cost, it is interesting to analyse the mean number of idle repairpersons. In this case, the maximum is reached for System 1_2_3 with a mean number of idle repairpersons per unit of time in the stationary regime equal to 1.8847 and 1.7663 for the systems with and without preventive maintenance, respectively.

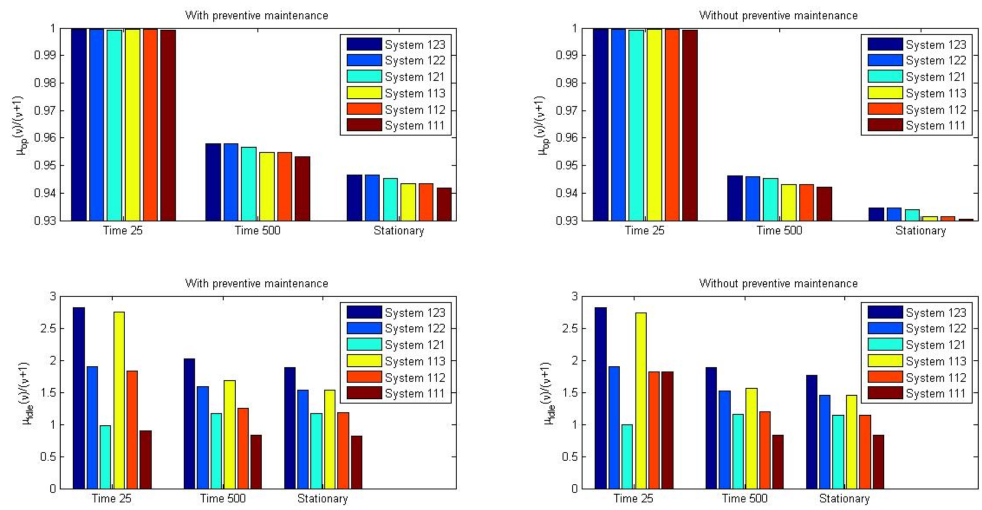

Figure 3.3. Mean operational ratio (first row) and mean number of idle repairpersons per unit of time (second row) with preventive maintenance (first column) and without preventive maintenance (second column)

Another interesting aspect to study is that of the mean number of events up to a certain time. This measure was calculated for every system and for several units of time. Table 3.3 shows the results obtained for 1500 units of time.

The number of new systems up to time 1500 is given by the last column of the Table 3.3. The minimum is reached when always one repairperson is assumed.

SYSTEM		$\Lambda^{A_1}(1500)$	$\Lambda^{A_2}(1500)$	$\Lambda^{B_1}(1500)$	$\Lambda^{B_2}(1500)$	$\Lambda^{B_3}(1500)$	$\Lambda^{C_1}(1500)$	$\Lambda^{C_2}(1500)$	$\Lambda^{FC_1}(1500)$ $+\Lambda^{FC_2}(1500)$
Time 1500	1_2_3	21.6051 (22.3598)	0.0160 (0.0158)	0.0741 (----)	12.1965 (----)	0.2521 (----)	3.5750 (3.1678)	5.6187 (8.5369)	4.1283 (5.3222)
	1_2_2	21.6038 (22.3585)	0.0160 (0.0158)	0.0741 (----)	12.1957 (----)	0.2521 (----)	3.5748 (3.1676)	5.6184 (8.5365)	4.1280 (5.3219)
	1_2_1	21.5840 (22.3403)	0.0160 (0.0157)	0.0733 (----)	12.1253 (----)	0.2499 (----)	3.5703 (3.1651)	5.6237 (8.5298)	4.1283 (5.3177)
	1_1_3	21.5328 (22.2864)	0.0159 (0.0157)	0.0738 (----)	12.1531 (----)	0.2512 (----)	3.5634 (3.1577)	5.6012 (8.5098)	4.1136 (5.3037)
	1_1_2	21.5315 (22.2850)	0.0159 (0.0157)	0.0738 (----)	12.1522 (----)	0.2512 (----)	3.5632 (3.1575)	5.6009 (8.5093)	4.1133 (5.3034)
	1_1_1	21.5053 (22.2638)	0.0159 (0.0157)	0.0730 (----)	12.0797 (----)	0.2489 (----)	3.5577 (3.1546)	5.6045 (8.5016)	4.1123 (5.2985)

Table 3.3. Mean number of events up to time 1500 (without preventive maintenance in parenthesis)

Analysis of systems when costs and rewards are included

Rewards and costs have been included in the analysis to optimize the model form an economical standpoint. Each time that the system is operational a reward equal to B=100 is produced and a lost with the same quantity is considered while the system is not operational. The operational cost per unit of time while the online unit is working depends on the internal degradation level according to the vector $\mathbf{c}_0 = (10, 20, 30, 40, 50)$. While the unit is being repaired two different costs per unit of time can be produced according if they are corrective repair or preventive maintenance. In the first case a cost equal to 5 is given and in the second case 0.5. Finally when a repairable failure occurs a fixed cost equal to 20 is produced and if it is a major inspection this cost is 1. A new unit of the system costs 200 and any repairperson has a cost of one per unit of time.

The mean net reward has been calculated for any system to achieve the more profitable system. Figure 3.4 shows them per unit of time for the cases with and without preventive maintenance.

If the mean net profit is observed in stationary regime the most profitable situation is for system 1_2_1 with preventive maintenance. Initially the number of repairpersons should be only one, when the first non-repairable failure occurs then one repairperson is added and finally only one repairperson should be when another non-repairable failure occurs. The optimum mean net profit in this case is equal to 74.0513 in stationary regime.

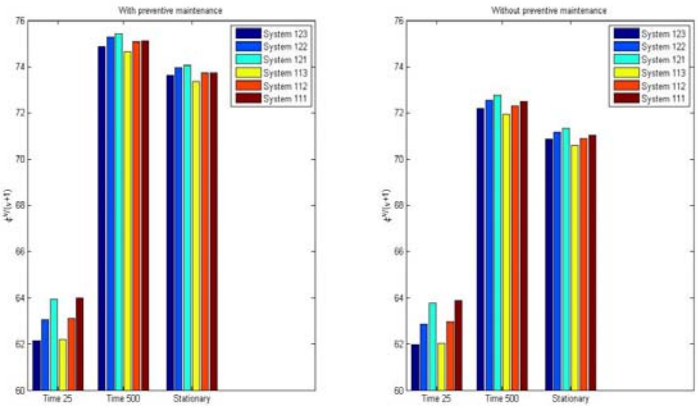

Figure 3.4. Mean net profit per unit of time up to time 25, 500 and the stationary case for system II (with preventive maintenance) and system III (without preventive maintenance)

3.8 Conclusions

In this study, three multi-state cold standby systems, evolving in discrete time, are modelled in an algorithmic and computational form using Markovian arrival processes with marked arrivals. The online unit is a multi-state device depending on degradation/performance levels. The three systems are modelled following similar methods, ranked from simplest to most complex. The latter includes multiple events: internal failure, external shocks with different consequences and inspections. Corrective repair and preventive maintenance are included as responses to a repairable failure and to major damage (internal or external) when the unit is inspected, respectively. Non-repairable failures, whether internal or due to an external shock, are possible and in this case the unit is removed.

Two interesting contributions are made in the present study. The number of repairpersons is indeterminate and variable depending on the number of units in the system. A system can be optimised by considering two different standpoints: the profitability of preventive maintenance and the number of repairpersons present according to the number of units in the system.

This complex system is modelled by a MMAP, which is shown to be useful for expressing the modelling and its associated measures in a well-structured form. Furthermore, this method makes it possible to determine the transient and stationary distributions and measures associated with the system in a matrix-algorithmic and computational form.

Other redundant systems such as warm standby systems and *k*–out-of-*n*: *G* systems can be modelled following this algorithmic methodology. Also, in a similar way and following this methodology, repairpersons could be replaced by repair sources, a situation in which costs and the associated repair times need not be the same.

Several measures, developed in an algorithmic form, are worked out in transient and stationary regime in an algorithmic and computational way. A numerical example illustrates the versatility of the modelling performed, and the optimum system is obtained.

3.9 Appendices

3.9.1 Appendix 3A

MODEL I

The transition matrices for the online unit for the *system* I case are,

O: No events: $\mathbf{H}_0 = \mathbf{T}$

A_1: Repairable internal failure: $\mathbf{H}_{\text{rep}} = \mathbf{T}_r^0 \boldsymbol{\alpha}$; $\mathbf{H}'_{\text{rep}} = \mathbf{T}_r^0$

C_1: Non-repairable failure: $\mathbf{H}_{\text{nrep}} = \mathbf{T}_{nr}^0 \boldsymbol{\alpha}$; $\mathbf{H}'_{\text{nrep}} = \mathbf{T}_{nr}^0$

MODEL II

The transition matrices for the online unit for the *system* II case are,

O: No events: $\mathbf{H}_0 = \mathbf{T} \otimes \mathbf{L} \otimes \mathbf{I} + \mathbf{TW} \otimes \mathbf{L}^0 \boldsymbol{\gamma} \otimes \mathbf{D}\left(1-\omega^0\right)$

A: Repairable internal failure

A_1: Repairable internal failure not due to shock:

$\mathbf{H}^{1}_{\text{rep}} = \mathbf{T}^{0}_{r}\boldsymbol{\alpha} \otimes \mathbf{L} \otimes \mathbf{e}\boldsymbol{\omega} + \mathbf{T}^{0}_{r}\boldsymbol{\alpha} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\mathbf{e}\boldsymbol{\omega}\left(1-\omega^{0}\right)$

$\mathbf{H}^{'1}_{\text{rep}} = \mathbf{T}^{0}_{r} \otimes \mathbf{L} \otimes \mathbf{e}_{d} + \mathbf{T}^{0}_{r} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\mathbf{e}\left(1-\omega^{0}\right)$

A_2: Repairable internal failure due to shock: $\mathbf{H}^{2}_{\text{rep}} = \mathbf{T}\mathbf{W}^{0}\boldsymbol{\alpha} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\mathbf{e}\boldsymbol{\omega}\left(1-\omega^{0}\right)$

$\mathbf{H}^{'2}_{\text{rep}} = \mathbf{T}\mathbf{W}^{0} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\mathbf{e}\left(1-\omega^{0}\right)$

C: Non-repairable failure

C_1: Non-repairable internal failure: $\mathbf{H}^{1}_{\text{nrep}} = \mathbf{T}^{0}_{nr}\boldsymbol{\alpha} \otimes \mathbf{L} \otimes \mathbf{e}\boldsymbol{\omega} + \mathbf{T}^{0}_{nr}\boldsymbol{\alpha} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\mathbf{e}\boldsymbol{\omega}\left(1-\omega^{0}\right)$

$\mathbf{H}^{'1}_{\text{nrep}} = \mathbf{T}^{0}_{nr} \otimes \mathbf{L} \otimes \mathbf{e} + \mathbf{T}^{0}_{nr} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\mathbf{e}\left(1-\omega^{0}\right)$

C_2: Non-repairable failure due to shock: $\mathbf{H}^{2}_{\text{nrep}} = \mathbf{e}\boldsymbol{\alpha} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \left(\mathbf{e}\boldsymbol{\omega}\omega^{0} + \mathbf{D}^{0}\boldsymbol{\omega}\left(1-\omega^{0}\right)\right)$

$\mathbf{H}^{'2}_{\text{nrep}} = \mathbf{e} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \left(\mathbf{e}\omega^{0} + \mathbf{D}^{0}\left(1-\omega^{0}\right)\right).$

MODEL III

Auxiliary matrices for minor/major inspection

The matrix $\mathbf{U}_l$ and $\mathbf{V}_l$, for l =1,2, are square matrices of order n and d respectively, whose element (s, t) is given by,

$$U_1(s,t) = \begin{cases} 1 & ; \quad 1 \le s = t \le n_1 \\ 0 & ; \quad \text{otherwise} \end{cases}, \quad U_2(s,t) = \begin{cases} 1 & ; \quad s = t > n_1 \\ 0 & ; \quad \text{otherwise} \end{cases},$$

$$V_1(s,t) = \begin{cases} 1 & ; \quad 1 \le s = t \le d_1 \\ 0 & ; \quad \text{otherwise} \end{cases}, \quad V_2(s,t) = \begin{cases} 1 & ; \quad s = t > d_1 \\ 0 & ; \quad \text{otherwise} \end{cases}.$$

The matrices $\mathbf{U}$ and $\mathbf{V}$ will be taken into account when one inspection occurs and the internal degradation level and cumulative external damage are observed respectively. The subscripts 1 and 2 will be considered when the damage observed is minor or major respectively.

O: No events:

$$\begin{aligned}\mathbf{H}_0 &= \left[\mathbf{T} \otimes \mathbf{L} \otimes \mathbf{I} + \mathbf{T}\mathbf{W} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\left(1-\omega^{0}\right)\right] \otimes \mathbf{M} \\ &+ \left[\mathbf{U}_1\mathbf{T} \otimes \mathbf{L} \otimes \mathbf{V}_1 + \mathbf{U}_1\mathbf{T}\mathbf{W} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{V}_1\mathbf{D}\left(1-\omega^{0}\right)\right] \otimes \mathbf{M}^{0}\eta\end{aligned}$$

$$\begin{aligned}\mathbf{H}^{'}_{\text{mr}} &= \left[\mathbf{U}_1\mathbf{T} \otimes \mathbf{L} \otimes \mathbf{V}_2\mathbf{I} + \mathbf{U}_2\mathbf{T} \otimes \mathbf{L} \otimes \mathbf{I}\right. \\ &\left.+ \mathbf{U}_1\mathbf{T}\mathbf{W} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{V}_2\mathbf{D}\left(1-\omega^{0}\right) + \mathbf{U}_2\mathbf{T}\mathbf{W} \otimes \mathbf{L}^{0}\boldsymbol{\gamma} \otimes \mathbf{D}\left(1-\omega^{0}\right)\right] \otimes \mathbf{M}^{0}\eta\end{aligned}$$

B: Major revision

B_1: Major revision for only internal major damage

$$\mathbf{H}_{\mathrm{mr}}^{1} = \left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{V}_1\mathbf{e}\boldsymbol{\omega}+\mathbf{U}_2\mathbf{TWe}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_1\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0\eta$$

B_2: Major revision for only external cumulative damage

$$\mathbf{H}_{\mathrm{mr}}^{2} = \left[\mathbf{U}_1\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{V}_2\mathbf{e}\boldsymbol{\omega}+\mathbf{U}_1\mathbf{TWe}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_2\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0\eta$$

B_3: Major revision for internal and external cumulative damage

$$\mathbf{H}_{\mathrm{mr}}^{3} = \left[\mathbf{U}_2\left(\mathbf{e}-\mathbf{T}^0\right)\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{V}_2\mathbf{e}\boldsymbol{\omega}+\mathbf{U}_2\mathbf{TWe}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{V}_2\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\right]\otimes\mathbf{M}^0\eta$$

C: Non-repairable failure

C_1: Non-repairable internal failure:

$$\mathbf{H}_{\mathrm{nrep}}^{1} = \mathbf{T}_{nr}^{0}\boldsymbol{\alpha}\otimes\mathbf{L}\otimes\mathbf{e}\boldsymbol{\omega}\otimes\mathbf{e}\boldsymbol{\eta}+\mathbf{T}_{nr}^{0}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De}\boldsymbol{\omega}\left(1-\omega^0\right)\otimes\mathbf{e}\eta$$

$$\mathbf{H}_{\mathrm{nrep}}^{'1} = \mathbf{T}_{nr}^{0}\otimes\mathbf{L}\otimes\mathbf{e}\otimes\mathbf{e}+\mathbf{T}_{nr}^{0}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\mathbf{De}\left(1-\omega^0\right)\otimes\mathbf{e}$$

C_2: Non-repairable failure due to shock: $\mathbf{H}_{\mathrm{nrep}}^{2} = \mathbf{e}\boldsymbol{\alpha}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\left(\mathbf{e}\boldsymbol{\omega}\omega^0+\mathbf{D}^0\boldsymbol{\omega}\left(1-\omega^0\right)\right)\otimes\mathbf{e}\eta$

$$\mathbf{H}_{\mathrm{nrep}}^{'2} = \mathbf{e}\otimes\mathbf{L}^0\boldsymbol{\gamma}\otimes\left(\mathbf{e}\omega^0+\mathbf{D}^0\left(1-\omega^0\right)\right)\otimes\mathbf{e}\,.$$

3.9.2 Appendix 3B

The matrices for the Markovian arrival processes have been developed in the following way.

$\mathbf{D}^{Y} = diag\left(\mathbf{D}^{Y,K},\mathbf{D}^{Y,K-1},\mathbf{D}^{Y,K-2},\ldots,\mathbf{D}^{Y,1}\right)$ for $Y = O, A_1, A_2, B_1, B_2$ and

$$\mathbf{D}^{Y} = \begin{pmatrix} \mathbf{0} & \mathbf{D}^{Y,K} & & & \\ & \mathbf{0} & \mathbf{D}^{Y,K-1} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \mathbf{D}^{Y,2} \\ \mathbf{0} & & & & \mathbf{0} \end{pmatrix},$$ for $Y = C_1, C_2$ and $$\mathbf{D}^{Y} = \begin{pmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & & \vdots \\ \mathbf{0} & & \ddots & \vdots \\ \mathbf{D}^{Y,1} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}$$

for $Y = FC_1, FC_2$.

Matrix $\mathbf{D}^{Y,k}$ for $Y = O, C_1, C_2, FC_1, FC_2$

The elements of the matrix $\mathbf{D}^{Y,k}$ for $k=1,\ldots,K$ and $Y=O, C_1, C_2, FC_1, FC_2$, are given by

$$\mathbf{D}^{Y,k}=\begin{cases}\left(\mathbf{D}_{lh}^{Y,k}\right)_{l,h=0,\dots,k} & ; \quad Y=O\\ \left(\mathbf{D}_{lh}^{Y,k}\right)_{\substack{l=0,\dots,k\\ h=0,\dots,k-1}} & ; \quad Y=C_1,C_2\\ \left(\mathbf{D}_{lh}^{Y,1}\right)_{\substack{l=0,1\\ h=0,\dots,K}} & \quad Y=FC_1,FC_2;k=1\end{cases}$$

where $\mathbf{D}_{lh}^{O,k}=\mathbf{0}$ if $h>l$ or $h<l-\min\{l,R_k\}$, $\mathbf{D}_{lh}^{C_i,k}=\mathbf{0}$ if $h>l$ or $h<l-\min\{l,R_k\}$ or $l=k$ and $\mathbf{D}_{lh}^{FC_i,1}=\mathbf{0}$ for all l and h excepting for the case $l=h=0$.

For $k=1,\dots,K$,

$$\mathbf{D}_{00}^{Y,k}=\begin{cases}\mathbf{H}_0+I_{\{k=1\}}\mathbf{H}_{\mathrm{mr}}^{'} & ; \quad Y=O\\ \mathbf{H}_{\mathrm{nrep}}^{type} & ; \quad Y=C_{type} \text{ or } Y=FC_{type}\end{cases},$$

For $l=1,\dots,R_k$

$$\mathbf{D}_{l,0}^{Y,k}\left(j_1,\dots,j_l\right)=\begin{cases}\mathbf{H}_0\otimes E\left(k,l,l,0;j_1,\dots,j_l;0,0\right) & ; \quad Y=O \text{ and } l<k\\ \mathbf{H}_{\mathrm{nrep}}^{type}\otimes E\left(k,l,l,0;j_1,\dots,j_l,0,1\right) & ; \quad Y=C_{type} \text{ and } 1<k\,,\\ \boldsymbol{\zeta}\otimes E\left(k,l,l,0;j_1,\dots,j_l;0,0\right) & ; \quad Y=O \text{ and } l=k\end{cases}$$

with $\boldsymbol{\zeta}=\boldsymbol{\alpha}$ for *system I*, $\boldsymbol{\zeta}=\boldsymbol{\alpha}\otimes\left(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma}\right)$ for *system II* and $\boldsymbol{\zeta}=\boldsymbol{\alpha}\otimes\left(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma}\right)\otimes\boldsymbol{\eta}\otimes\boldsymbol{\omega}$ for *system III*.

For $l=1,\dots,k-1$; $a=0,\dots,\min\{R_k,l-1\}$ with $k>1$,

$$\mathbf{D}_{l,l-a}^{Y,k}\left(i_1,\dots,i_{l-a};j_1,\dots,j_l\right)=$$
$$\begin{cases}\left[\mathbf{H}_0+I_{\{l=k-1 \text{ and } a=0\}}\mathbf{H}_{\mathrm{mr}}^{'}\right]\otimes E\left(k,l,a,0;i_1,\dots,i_{l-a};j_1,\dots,j_l;0,0\right) & ; \quad Y=O\\ \left(\mathbf{H}_{\mathrm{nrep}}^{type}I_{\{l<k-1 \text{ or } a>0\}}+\mathbf{H}_{\mathrm{nrep}}^{'type}I_{\{l=k-1 \text{ and } a=0\}}\right)\otimes E\left(k,l,a,b;i_1,\dots,i_{l-a};j_1,\dots,j_l;0,1\right) & ; \quad Y=C_{type}\end{cases}.$$

For a = $1,\dots,\min\{R_k,k-1\}$,
$\mathbf{D}_{k,k-a}^{O,k}\left(i_1,\dots,i_{k-a};j_1,\dots,j_k\right)=\boldsymbol{\zeta}\otimes E\left(k,k,a,0;i_1,\dots,i_{k-a};j_1,\dots,j_k;0,0\right)$, with $k>1$.

For $k=1,\dots,K$,

$$\mathbf{D}_{k,k}^{O,k}\left(i_1,\dots,i_k;j_1,\dots,j_k\right)=\begin{cases}E\left(k,k,0,0;i_1,\dots,i_k;j_1,\dots,j_k;0,0\right) & ; \quad \textit{system I}\\ \left(\mathbf{L}+\mathbf{L}^0\boldsymbol{\gamma}\right)\otimes E\left(k,k,0,0;i_1,\dots,i_k;j_1,\dots,j_k;0,0\right) & ; \quad \textit{systems II, III}.\end{cases}$$

Matrix $\mathbf{D}^{A_i,k}$

The elements of the matrix $\mathbf{D}^{A_i,k}$ for i=1, 2 and for$k$ = 1,...,K are given by

$$\mathbf{D}^{A_i,k} = \left(\mathbf{D}_{lh}^{A_i,k}\right)_{l,h=0,\ldots,k}$$

where $\mathbf{D}_{lh}^{A_i,k} = \mathbf{0}$ if $h > l+1$ or $h < l+1-\min\{l, R(k)\}$ or $l = k$.

For $type$ = 1, 2, then

$$\mathbf{D}_{01}^{A_{type},k}\left(1\right) = \left(\mathbf{H}_{\text{rep}}^{type} I_{\{k>1\}} + \mathbf{H}_{\text{rep}}^{'type} I_{\{k=1\}}\right) \otimes E\left(k,0,0,0;1;1,0\right).$$

For l=1,…,k–1; a=0,…,min$\{l, R_k\}$ with $k > 1$

$$\mathbf{D}_{l,l+1-a}^{A_{type},k}\left(i_1,\ldots,i_{l+1-a-1},1; j_1,\ldots,j_l\right) = \left(\mathbf{H}_{\text{rep}}^{type} I_{\{l<k-1 \text{ or } a>0\}} + \mathbf{H}_{\text{rep}}^{'type} I_{\{l=k-1 \text{ and } a=0\}}\right)$$
$$\otimes E\left(k,l,a,0;i_1,\ldots,i_{l+1-a-1},1; j_1,\ldots,j_l;1,0\right)$$

Matrix $\mathbf{D}^{B_i,k}$

The elements of the matrix $\mathbf{D}^{B_i,k}$ for i=1, 2, 3 and for k = 1,..., K are given by

$$\mathbf{D}^{B_i,k} = \left(\mathbf{D}_{lh}^{B_i,k}\right)_{l,h=0,\ldots,k}$$

where $\mathbf{D}_{lh}^{A_i,k} = \mathbf{0}$ if $h > l+1$ or $h < l+1-\min\{l, R_k\}$ or $l \geq k-1$.

For $type$ = 1, 2, 3 then

$$\mathbf{D}_{01}^{B_{type},k}\left(2\right) = \mathbf{H}_{\text{mr}}^{type} I_{\{k>1\}} \otimes E\left(k,0,0,0;2;1,0\right).$$

For l=1,…,k–2; a=0,…,min$\{l, R_k\}$ with $k > 1$

$$\mathbf{D}_{l,l+1-a}^{B_{type},k}\left(i_1,\ldots,i_{l+1-a-1},2; j_1,\ldots,j_l\right) = \mathbf{H}_{\text{mr}}^{type} \otimes E\left(k,l,a,0;i_1,\ldots,i_{l+1-a-1},2; j_1,\ldots,j_l;1,0\right).$$

Chapter 4

Complex multi-state warm standby systems subject to multiple events and repairpersons with loss of units

4.1 Introduction

In chapter 3 different complex redundant multi-state systems have been developed in a well-structured way by using Markovian Arrival Processes with Marked arrivals. These three systems are extended in this chapter by considering warm standby. Again, in the first of these systems, the online unit is only subject to failure by wear; the second extends this by including external shocks with diverse consequences, and the third includes inspections, so that the effects of preventive maintenance and of the variable number of repairpersons, depending on the number of units present in the system, are analysed. Also, in this chapter, each warm standby unit can undergo a repairable failure at any time with a probability equal to *p*. Each time that a repairable failure occurs, the unit goes to the repair facility. Two different repair time distributions can be carried out, for the online unit and for the warm standby units. Loss of units and a variable number of repairpersons are included in these systems. The model is developed in an algorithmic way and some attractive measures have been built. Rewards and costs are introduced according to the operational phases and the different types of repair. An optimal maintenance policy enables policymakers to decide what level of degradation should be taken into account for preventive maintenance in response to an inspection, whether preventive maintenance is profitable and the optimum number of repairpersons at a given time. Phase type distributions and Marked Markovian Arrival processes play again an important role in this chapter.

4.2 The systems

Similar systems to those described in Section 3.2 are considered in this chapter. We are going to focus on the most complex system, external shocks and random inspections (*System III*) but with standby units subject to repairable failures.

4.2.1 Assumptions and the state space of the system

The system is subject to the following assumptions. Regarding the online unit we assume the assumptions 2.1–2.7 and 2.9 given in Section 2.2.1. Also, assumptions 3.1–3.6 from Section 3.2.1 are considered. These assumptions are referred to the loss of units and the variable number of repairpersons, the repair time distribution of the online unit, the renewal of the system and to the preventive maintenance time distribution respectively. The following specific assumptions are introduced for the warm standby system.

***Assumption* 4.1**. Each warm standby unit can undergo a repairable failure at any time with a probability equal to p.

***Assumption* 4.2**. The corrective repair time when a warm standby unit fails is PH distributed with representation $\left(\boldsymbol{\beta}_0, \mathbf{S}_0\right)$. The order of this matrix is equal to z_0 (number of corrective repair phases for the repair of a warm standby unit that failed).

***Assumption* 4.3**. When the online unit undergoes either a repairable failure or a major inspection and at the same time a warm standby system fails, the online unit has priority in the repair facility.

The state-space of the system is composed of macro-states. This state-space can be expressed as it is given in Section 3.2.2.

4.3 Modeling the system

The systems are governed by a Markov process vector in discrete time in a similar way as Section 3.3. The behaviour of the online unit is as the cold standby system given in Section 3.3.1 but it is more complex the modelling of the repair facility. This section shows the case of System III but analogous reasoning can be performed for Systems I and II. The corresponding matrices for all systems are given in Appendices 4*A* and 4*B*.

The system that is being analyzed can be modelled through a Discrete Marked Markov Arrival Process (MMAP).

4.3.2 Modeling the repair facility

The number of repairpersons when there are k units in the system, with $k \leq K$, is given by $R_k \leq k$ and the number of units in the repair facility is denoted as l. Let a be the number of units which finish the repair and r the number of warm standby repairable failures. Let k_h be the ordinal of the repairpersons who concluded the repair, and let i_h and j_h be the type of repair (corrective warm standby, 0, corrective online unit, 1, preventive maintenance, 2) for the ordered units, after and before the transition, respectively. Online and warm-standby units can undergo different types of events. Let wr be the number of warm standby units that undergo a repairable failure at one step. Let mr be the variable indicatos that is equal to 1 if the online unit goes to the repair facility for any circumstances (repairable failure or major inspection) and 0 otherwise. Let nr an indicator which is equal to 1 if a non-repairable failure occurs and 0 otherwise. When the online unit undergoes a non-repairable failure, the number of repairpersons is modified. If there are fewer repairpersons after a transition than remaining units being repaired, some of these units will be returned to the queue in the repair facility. The number of units to be returned is denoted by b. This value is given by

$$b = \max\left\{\min\left\{l, R_k\right\} - a - I_{\{nr=1\}} R_{k-1} - I_{\{nr=0\}} R_k, 0\right\}.$$

For instance, if there are 4 units in a system ($k = 4$), 3 of them in the repair facility with 3 repairpersons ($R_k = 3$), none of them is repaired and one non-repairable occurs then the number of units to be returned to the queue if $R_{k-1} = 1$ is given by

$$b = \max\left\{\min\left\{3,3\right\} - 0 - 1, 0\right\} = \max\left\{2,0\right\} = 2.$$

To model the behavior of the repair facility we define the following matrix function that governs the behavior in one transition of the units that are being repaired where the order of the units repaired are specified. This function is given by

$$C\left(k,l,a,b;k_1,\ldots,k_a;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right)=$$

$$\begin{cases} \mathbf{S}(1)\otimes\cdots\otimes\mathbf{S}\left(\min\{l,R_k\}\right) & ; \begin{array}{l} i_{s-\sum_{z=1}^{a} I_{\{k_z<s\}}}=j_s;s=1,\ldots,l;s\neq k_z,\forall z \\ i_{l-a+1}=1,2;\ \text{if } mr=1 \\ i_s=0;s=l-a+mr+1,\ldots,l-a+mr+wr \end{array} \\ \mathbf{0} & ; \quad \text{otherwise} \end{cases}$$

for $k\leq K$, $l\geq 1$, $a\geq 1, b\geq 0$, where

$$S(h)=\begin{cases} \mathbf{S}_{j_h}^0 & ; \quad \exists z\in\{1,\ldots,a\}\mid h=k_z \\ \mathbf{e}-\mathbf{S}_{j_h}^0 & ; \quad h \text{ is the ordinal of the last } b \text{ units being repaired without ending} \\ \mathbf{S}_{j_h} & ; \quad \text{otherwise} \end{cases}.$$

If a = 0, then the definition is analogous but we will consider the following notation

$$C\left(k,l,a=0,b;i_1,\ldots,i_{l+mr+wr};j_1,\ldots,j_l\right)=$$

$$\begin{cases} S_{j_1}\otimes\ldots\otimes S_{j_{\min\{l,R_k\}-b}}\otimes\left(\mathbf{e}-S^0_{j_{\min\{l,R_k\}-b+1}}\right)\otimes\ldots\otimes\left(\mathbf{e}-S^0_{j_{\min\{l,R_k\}}}\right) & ; \begin{array}{l} i_s=j_s;1\leq s\leq l \\ i_{l+1}=1,2 \text{ if } mr=1 \\ i_s=0;s=l+mr+1,\ldots,l+mr+wr \end{array} \\ \mathbf{0} & ; \quad \text{otherwise} \end{cases}$$

If $a=\min\{l,R_k\}$, then the definition is analogous but we will consider the following notation

$$C\left(k,l,a=\min\{l,R_k\},0;j_1,\ldots,j_l\right)=S^0_{j_1}\otimes\ldots\otimes S^0_{j_{\min\{l,R_k\}}}.$$

From this matrix function the transition probability, if only a is known and the order is not specified, is given by

$$B\left(k,l,a,b;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right)$$
$$=\begin{cases} C\left(k,l,0,b;i_1,\ldots,i_{l+mr+wr};j_1,\ldots,j_l\right) & ; \quad a=0,b\geq 0,l>0 \\ \sum_{k_1=1}^{\min\{l,R_k\}-a+1}\sum_{k_2=k_1+1}^{\min\{l,R_k\}-a+2}\cdots\sum_{k_a=k_{a-1}+1}^{\min\{l,R_k\}} C\left(k,l,a,b;k_1,\ldots,k_a;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right) & ; \quad a>0, a\neq \min\{l,R_k\} \\ C\left(k,l,a,0;j_1,\ldots,j_l\right) & ; \quad a=\min\{l,R_k\}, \\ 1 & ; \quad a=0,b=0,l=0 \end{cases}$$

After one transition, new units that were in queue or not can entry in repair. The number of units that will begin the repair at the next time is given by

$$\varepsilon=\min\left\{\max\{l-R_k,0\}+mr+wr,\max\left\{I_{\{nr=1\}}R_{k-1}+I_{\{nr=0\}}R_k-\min\{R_k,l\}+a,0\right\}\right\}.$$

The matrix function that governs the transition probability of the repair facility when a of l units are repaired for $l>0$ and $a\neq l$ is given by

$$E\left(k,l,a,b;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l;mr,wr,nr\right)=$$
$$\begin{cases} B\left(k,l,a,0;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right)\otimes \boldsymbol{\beta}^{i_{\min(l,R_k)-a+1}}\otimes\ldots\otimes\boldsymbol{\beta}^{i_{\min(l,R_k)-a+\varepsilon}} & ; \quad \varepsilon>0 \\ B\left(k,l,a,b;i_1,\ldots,i_{l-a+mr+wr};j_1,\ldots,j_l\right) & ; \quad \varepsilon=0 \\ \mathbf{0} & ; \quad \text{otherwise} \end{cases}$$

If $l=0$ or $a=l$ with $l\leq R_k$ then this function is denoted as

$$E\left(k,l=0,a=0,b=0;0,\overset{s}{\ldots},0;mr=0,wr=s,nr=0,1\right)$$
$$=I_{\{s>0\}}\left(\boldsymbol{\beta}^0\otimes\overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\cdots}\otimes\boldsymbol{\beta}^0\right)+I_{\{s=0\}}$$
$$E\left(k,l=0,a=0,b=0;i_{mr},0,\overset{s}{\ldots},0;mr=1,wr=s,nr=0\right)$$
$$=\boldsymbol{\beta}^{i_{mr}}\otimes\left(I_{\{s>0\}}\left(\boldsymbol{\beta}^0\otimes\overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\cdots}\otimes\boldsymbol{\beta}^0\right)+I_{\{s=0\}}\right),$$

$$
\begin{aligned}
&E\left(k,l,a=l,b=0;0,\overset{s}{\ldots},0;j_1,\ldots,j_l;mr=0,wr=s,nr=0,1\right)\\
&=B\left(k,l,l,0;j_1,\ldots,j_l\right)\otimes\boldsymbol{\beta}^0\otimes\overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\ldots}\otimes\boldsymbol{\beta}^0,\\
&E\left(k,l,a=l,b=0;i_{mr},0,\overset{s}{\ldots},0;j_1,\ldots,j_l;mr=1,wr=s,nr=0\right)\\
&=B\left(k,l,l,0;j_1,\ldots,j_l\right)\otimes\boldsymbol{\beta}^{i_{mr}}\otimes\boldsymbol{\beta}^0\otimes\overset{\min\left\{s,I_{\{nr=0\}}R_k+I_{\{nr=1\}}R_{k-1}\right\}}{\ldots}\otimes\boldsymbol{\beta}^0.
\end{aligned}
$$

Example 4.1. For instance, we assume a system composed of 4 repairpersons and 8 units (k=8, R_8=4), 4 of them in the repair facility (l = 4; corrective repair online unit, corrective repair warm standby unit, preventive maintenance, corrective repair warm standby unit and preventive maintenance, respectively). At the next time three units that are being repaired finishes the repair, one warm standby unit fails and the online unit undergoes a non-repairable failure. The number of repairpersons is only two when the system is composed of 7 units (R_7=2) and the units in the repair facility after non-repairable failure are types corrective repair warm standby unit, preventive maintenance and corrective repair warm standby unit respectively.

In this case $b=\max\left\{\min\left\{5,4\right\}-3-2,0\right\}=0$ and

$$\varepsilon=\min\left\{\max\left\{5-1,0\right\}+0+1,\max\left\{2-\min\left\{4,5\right\}+3,0\right\}=1\right..$$

$$B\left(8,5,3,0;0,2,0;1,0,2,0,2\right)=\sum_{k_1=1}^{2}\sum_{k_2=k_1+1}^{3}\sum_{k_3=k_2+1}^{4}C\left(8,4,3,0;k_1,\ldots,k_a;0,2,0;1,0,2,0,2\right)$$

$$
\begin{aligned}
&=C\left(8,4,3,0;1,2,3;0,2,0;1,0,2,0,2\right)+C\left(8,4,3,0;1,2,4;0,2,0;1,0,2,0,2\right)\\
&+C\left(8,4,3,0;1,3,4;0,2,0;1,0,2,0,2\right)+C\left(8,4,3,0;2,3,4;0,2,0;1,0,2,0,2\right)
\end{aligned}
$$

$$=\mathbf{S}_1^0\otimes\mathbf{S}_0^0\otimes\mathbf{S}_2^0\otimes\mathbf{S}_0+\mathbf{0}+\mathbf{S}_1^0\otimes\mathbf{S}_0\otimes\mathbf{S}_2^0\otimes\mathbf{S}_0^0+\mathbf{0}.$$

Therefore,

$$
\begin{aligned}
E\left(8,5,3,0;0,2,0;1,0,2,0,2;0,1,1\right)&=B\left(8,5,3,0;0,2,0;1,0,2,0,2\right)\otimes\boldsymbol{\beta}^2\\
&=\left(\mathbf{S}_1^0\otimes\mathbf{S}_0^0\otimes\mathbf{S}_2^0\otimes\mathbf{S}_0+\mathbf{S}_1^0\otimes\mathbf{S}_0\otimes\mathbf{S}_2^0\otimes\mathbf{S}_0^0\right)\otimes\boldsymbol{\beta}^2.
\end{aligned}
$$

4.3.3 Modeling the online unit and the warm standby units

The online unit has already modeled and it is going to be extended when warm standby units are introduced in the analysis.

If wr indicates the number of standby units which are broken at a certain time and l the number of units in the repair facility before that time, then we define the matrix

$$\mathbf{H}_{k,c,l,wr} = \binom{k-l-1}{wr} p^{r} \left(1-p\right)^{k-l-1-wr} \mathbf{H}_{c},$$

$l = 0,\ldots, k-1$; $wr \le k-l-1$,

where $c = \left\{0, \binom{1}{rep}, \binom{2}{rep}, \binom{1}{mr}, \binom{2}{mr}, \binom{3}{mr}, \binom{1}{nrep}, \binom{2}{nrep}\right\}.$

This matrix $\mathbf{H}_{k,c,l,wr}$ contains the transition probabilities when there are k units, l of them non-operational, and at next time wr warm standby units break down and the online unit passes to the situation c in that time; where c is equal to 0 when the online unit keeps on working the next time, 1 when this one undergoes a repairable failure, 2 when it undergoes a major inspection and 3 when the online unit undergoes a non-repairable failure.

When all operational units fail and a repair does not occur then

$$\mathbf{H}^{'}_{k,c,l,k-l-1} = p^{k-l-1}\mathbf{H}^{'}_{c}, \quad \text{for } c = \left\{0, \binom{1}{rep}, \binom{2}{rep}, mr, \binom{1}{nrep}, \binom{2}{nrep}\right\}.$$

4.4 The Markovian Arrival Processes with marked arrivals

The behavior of the system; when the online unit, the warm standby units, and the repair facility are considered, is modelled through a D-MMAP.

When the events A, B and C have place at a certain time, the number of warm standby units that fail can vary from 0 to the total units in standby at the previous moment. The MMAPs for system III is given by the following representation,

$$\left(\mathbf{D}^{O},\mathbf{D}^{A_1},\mathbf{D}^{A_2},\mathbf{D}^{B_1},\mathbf{D}^{B_2},\mathbf{D}^{B_3},\mathbf{D}^{C_1},\mathbf{D}^{C_2},\mathbf{D}^{FC_1},\mathbf{D}^{FC_2}\right).$$

The matrix $\mathbf{D}^{Y}$ contains the transition probabilities when the event Y has occurred for $Y = O, A_1, A_2, B_1, B_2, B_3, C_1, C_2, FC_1, FC_2$. A similar reasoning given in Section 3.4 can be done to interpret these matrices but the internal structure is different in this case. The third level corresponds to the transitions from the macro-state $\mathbf{U}^{k}$ to $\mathbf{U}^{k}$ or $\mathbf{U}^{k-1}$. These matrix blocks are composed of the matrices $\mathbf{D}_{lh}^{Y,k}$ which correspond to the transitions between the macro-states from $\mathbf{E}_{l}^{k}$ to either $\mathbf{E}_{h}^{k}$ or $\mathbf{E}_{h}^{k-1}$ (level 2). Finally, several types of repairing can be produced, repairable failure for the online unit, repairable failure for one warm standby unit and preventive maintenance. Therefore, the type of failure of the units in the repair facility has to be saved in memory. The matrices $\mathbf{D}_{lh}^{Y,k}$ are composed of matrix blocks corresponding to the transition from the macro-states $\mathbf{E}_{i_1,\ldots,i_l}^{k}$ to $\mathbf{E}_{i_1,\ldots,i_h}^{k}$ or $\mathbf{E}_{i_1,\ldots,i_h}^{k-1}$. The matrix block $\mathbf{D}_{lh}^{Y,k}\left(i_1,\ldots,i_h; j_1,\ldots,j_l\right)$ contains the transition probabilities described above where the type of repair in the repair facility is ordered for the case before and after transition. These blocks are built by considering the matrices $\mathbf{H}$ defined in Section 3.3.1 and developed in *Appendix* 3*A*, the functions defined in Section 4.3.2 and 4.3.3. They are developed in Appendix 4*A*. The case A_1 for the Model III, a repairable internal failure occurs, is described in detail.

4.4.1 Building the matrix $\mathbf{D}^{A_1}$

The elements of the matrix $\mathbf{D}^{A_1}$ are given by $\mathbf{D}^{A_1} = diag\left(\mathbf{D}^{A_1,K},\mathbf{D}^{A_1,K-1},\mathbf{D}^{A_1,K-2},\ldots,\mathbf{D}^{A_1,1}\right)$, where $\mathbf{D}^{A_1,k}$ contains the transitions when one internal repairable failure of the online unit occurs and there k units in the system. This matrix composed of matrix blocks according to the number of units in the repair facility. This matrix block (from l units in the repair facility up to h) is $\mathbf{D}_{lh}^{A_1,k}$ and it is equal to $\mathbf{0}$ if $h < l + 1 - \min\{l, R_k\}$ or $l = k$.

For $h = 1, \ldots, k$ then

$$\mathbf{D}_{0h}^{A_1,k}\left(1,0,\overset{h-1}{\ldots},0\right) = \left(\mathbf{H}_{k,\binom{1}{rep},0,h-1} I_{\{h<k\}} + \mathbf{H}'_{k,\binom{1}{rep},0,h-1} I_{\{h=k\}}\right) \otimes E\left(k,0,0,0;1,0,\overset{h-1}{\ldots},0;1,h-1,0\right).$$

When there are l units in the repair facility the matrix $\mathbf{D}_{lh}^{A_1,k}$ depends on the number of warm standby units fail, wr. The matrix $\mathbf{D}_{lh}^{A_1,k,wr}$ contains the transitions when there are k units in the system, l of them in the repair facility; the online unit undergoes an internal repairable failure, wr warm standby unit fail and h units are in the repair facility after the transition.

Therefore, for $l=1,\ldots,k-1$; $a=0,\ldots,\min\{l, R_k\}$; $wr = 0,\ldots,k-l-1$ with $k>1$,

$$\mathbf{D}_{l,l+wr+1-a}^{A_1,k,wr}\left(i_1,\ldots,i_{l-a},1,0,\overset{wr}{\ldots},0;j_1,\ldots,j_l\right)=\left(\mathbf{H}_{k,\binom{1}{rep},l,wr}I_{\{wr<k-l-1\text{ or }a>0\}}+\mathbf{H}'_{k,\binom{1}{rep},l,wr}I_{\{wr=k-l-1\text{ and }a=0\}}\right)$$
$$\otimes E\left(k,l,a,0;i_1,\ldots,i_{l-a},1,0,\overset{wr}{\ldots},0;j_1,\ldots,j_l;1,wr,0\right).$$

Then,

$$\mathbf{D}_{lh}^{A_1,k}\left(i_1,\ldots,i_h;j_1,\ldots,j_l\right)=\sum_{a=\max\{0,l-h+1\}}^{\min\{k-h,\min\{l,R_k\}\}}\mathbf{D}_{lh}^{A_1,k,h-l-1+a}\left(i_1,\ldots,i_h;j_1,\ldots,j_l\right).$$

The rest blocks are given in the *Appendix* 4*A*.

4.5 The transient and stationary distribution and measures

The transient and stationary distributions can be built in a similar way as cold standby system shown in Section 3.5.1 and 3.5.2. Several measures can be also worked out for the warm standby system. Some of them have similar structure as cold standby system. Next, some of them are referenced.

Availability: Section 3.5.3.1

Reliability: Section 3.5.3.2

Mean time in each macro-state: Section 3.5.3.3

Mean operational time up to time ν: Section 3.5.3.4

Mean time that the repairpersons are idle and busy: Section 3.5.3.5

4.5.1. Mean time working on corrective repair for the online unit, corrective repair for warm standby units and preventive maintenance

The repairpersons can be working on corrective repair for the online unit, warm standby units that have fail and on preventive maintenance. The mean time that the repairpersons are working on these different situations up to time ν is given respectively by

$$\mu_{onlinecorr}(\nu)=\sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}_{E_s^k}^{m}\cdot\mathbf{q}_s^k(1),\ \mu_{warmcorr}(\nu)=\sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}_{E_s^k}^{m}\cdot\mathbf{q}_s^k(0),\quad \text{and}$$

$$\mu_{pm}(\nu)=\sum_{m=0}^{\nu}\sum_{k=1}^{K}\sum_{s=1}^{k}\mathbf{p}_{E_s^k}^{m}\cdot\mathbf{q}_s^k(2),$$

where $\mathbf{q}_s^k(0)$, $\mathbf{q}_s^k(1)$ and $\mathbf{q}_s^k(2)$ are column vectors that contains the number of repairpersons that are working on corrective repair for warm standby units, corrective repair for online units and preventive maintenance respectively according to the macro-state E_s^k. These column vectors are given by

$$\mathbf{q}_s^k(0)=\begin{pmatrix} h_s^k(1)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(1)}z_1^{d_s^k(1)}z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(1)}} \\ h_s^k(2)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(2)}z_1^{d_s^k(2)}z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(2)}} \\ \vdots \\ h_s^k\left(i=I_{\{s=K\}}2\cdot 3^{k-1}+I_{\{s\neq K\}}3^s\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(i)}z_1^{d_s^k(i)}z_2^{\min\{s,R_k\}-h_s^k(i)-d_s^k(i)}} \end{pmatrix},$$

$$\mathbf{q}_s^k(1)=\begin{pmatrix} d_s^k(1)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(1)}z_1^{d_s^k(1)}z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(1)}} \\ d_s^k(2)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(2)}z_1^{d_s^k(2)}z_2^{\min\{s,R_k\}-h_s^k(2)-d_s^k(2)}} \\ \vdots \\ d_s^k\left(i=I_{\{s=K\}}2\cdot 3^{k-1}+I_{\{s\neq K\}}3^s\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(i)}z_1^{d_s^k(i)}z_2^{\min\{s,R_k\}-h_s^k(i)-d_s^k(i)}} \end{pmatrix},$$

for

$$\mathbf{q}_s^k(2)=\begin{pmatrix} g_s^k(1)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(1)}z_1^{d_s^k(1)}z_2^{\min\{s,R_k\}-h_s^k(1)-d_s^k(1)}} \\ g_s^k(2)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(2)}z_1^{d_s^k(2)}z_2^{\min\{s,R_k\}-h_s^k(2)-d_s^k(2)}} \\ \vdots \\ g_s^k\left(i=I_{\{s=K\}}2\cdot 3^{k-1}+I_{\{s\neq K\}}3^s\right)\mathbf{e}_{t(nd\varepsilon)^{I_{\{s\neq k\}}}z_0^{h_s^k(i)}z_1^{d_s^k(i)}z_2^{\min\{s,R_k\}-h_s^k(i)-d_s^k(i)}} \end{pmatrix},$$

being $h_s^k(i)$ the *i*-th element of the vector $1\leq s<k$;

$$\mathbf{h}_s^k = \begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{\min\{s,R_k\}}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \mathbf{e}_{3^{\max\{s-R_k,0\}}} \quad \text{with}$$

$$\mathbf{h}_k^k = I_{\{k=R_k\neq K\}}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{k}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix}\right] + I_{\{k=R_k=K\}}\left[\begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{k-1}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \odot \begin{pmatrix}1\\0\end{pmatrix}\right]$$
$$+ I_{\{K\neq k>R_k\}}\begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \mathbf{e}_{3^{k-R_k}} + I_{\{K=k>R_k\}}\begin{pmatrix}1\\0\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \mathbf{e}_{2\cdot 3^{k-R_k-1}},$$

and $d_s^k(i)$ the i-th element of the vector .. and

$$\mathbf{d}_k^k = I_{\{k=R_k\neq K\}}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{k}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix}\right] + I_{\{k=R_k=K\}}\left[\begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{k-1}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix} \odot \begin{pmatrix}0\\1\end{pmatrix}\right]$$
$$+ I_{\{K\neq k>R_k\}}\begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \mathbf{e}_{3^{k-R_k}} + I_{\{K=k>R_k\}}\begin{pmatrix}0\\1\\0\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \mathbf{e}_{2\cdot 3^{k-R_k-1}}$$

and $g_s^k(i)$ the i-th element of the vector $\mathbf{g}_s^k = \begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{\min\{s,R_k\}}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_{3^{\max\{s-R_k,0\}}}$ and

$$\mathbf{g}_k^k = I_{\{k=R_k\neq K\}}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{k}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix}\right] + I_{\{k=R_k=K\}}\left[\begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{k-1}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_2\right]$$
$$+ I_{\{K\neq k>R_k\}}\begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_{3^{k-R_k}} + I_{\{K=k>R_k\}}\begin{pmatrix}0\\0\\1\end{pmatrix} \odot \overset{R_k}{\cdots} \odot \begin{pmatrix}0\\0\\1\end{pmatrix} \otimes \mathbf{e}_{2\cdot 3^{k-R_k-1}}$$

for $k = 1,\ldots,K$ and $s = 1,\ldots,k$.

These measures in stationary regime are

$$\mu_{warmcorr} = \sum_{k=1}^{K}\sum_{s=1}^{k} \boldsymbol{\pi}_{E_s^k} \cdot \mathbf{q}_s^k(0),\ \mu_{onlinecorr} = \sum_{k=1}^{K}\sum_{s=1}^{k} \boldsymbol{\pi}_{E_s^k} \cdot \mathbf{q}_s^k(1) \text{ and } \mu_{pm} = \sum_{k=1}^{K}\sum_{s=1}^{k} \boldsymbol{\pi}_{E_s^k} \cdot \mathbf{q}_s^k(2).$$

4.5.2 Mean number of events

Thanks to the structure built, the expected number of events up to a certain time ν is worked out. The expressions are the same than the ones given in Section 3.5.3.7 for the cold standby system. It is given by $\Lambda^{Y}(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\mathbf{D}^{Y}\mathbf{e}$, for $Y = A_1, A_2, B_1, B_2, B_3, C_1, C_2, FC_1, FC_2$. In stationary regime, the mean number of events per unit of time is $\Lambda^{Y}=\boldsymbol{\pi}\mathbf{D}^{Y}\mathbf{e}$.

The mean number of failure of warm standby units is also worked out for this new system.

Warm standby failures

The expected number of warm standby repairable failures is given by

$$\Lambda^{w}(\nu)=\sum_{u=1}^{\nu}\mathbf{p}^{u-1}\mathbf{V},$$

where $\mathbf{V}$ is the column vector $\mathbf{V}=\left(\mathbf{V}^{K},\mathbf{V}^{K-1},\mathbf{V}^{K-2},\ldots,\mathbf{V}^{2},\mathbf{V}^{1}\right)'$, where $\mathbf{V}^{k}=\left(\mathbf{V}_{0}^{k},\mathbf{V}_{1}^{k},\ldots,\mathbf{V}_{k-2}^{k},\mathbf{0},\mathbf{0}\right)'$ for k = 2,…, K and $\mathbf{V}^{1}=\left(\mathbf{0},\mathbf{0}\right)'$. The column vector $\mathbf{V}_{l}^{k}$ is given by

$$\mathbf{V}_{l}^{k}=\sum_{Y=A_1,A_2,B_1,B_2,B_3,C_1,C_2}\;\sum_{h=l+2-I_{\{Y=C_1,C_2\}}}^{k-I_{\{Y=C_1,C_2\}}} I_{\{h=k\text{ and }(Y=B_1,B_2,B_3)\}}\left(h-l-1+I_{\{Y=C_1,C_2\}}\right)\mathbf{D}_{lh}^{Y,k}\mathbf{e},$$

for l = 0, 1, …, k–2. This vector $\mathbf{V}_{l}^{k}$ contains the expected number of warm standby units that fail at one step according to the phases when there are k units in the system and l of them in the repair facility.

In the stationary case it is given by $\Lambda^{w}=\boldsymbol{\pi}\mathbf{V}$.

4.6 Costs and Rewards

Several costs and rewards have been included in the model to study the effectiveness of the model from an economic standpoint. Some of them are similar to the ones described in Section 3.6. That is,

B: gross profit per unit of time while the system is operational
$\mathbf{c_0}$: mean cost per unit of time depending on the operational phase
$\mathbf{cr_1}$: mean cost per unit of time when a unit is in corrective repair from online depending on the repair phase

cr₂: mean cost per unit of time when a unit is in corrective repair from major inspection depending on the repair phase
H: fixed cost per unit of time for each repairperson
C: loss per unit of time while the system is not operational
fcr: fixed cost each time that the online unit undergoes a repairable failure
fpm: fixed cost each time that the online unit undergoes a major inspection
fnu: cost per one new unit.

New costs have been included by considering the warm standby system. Thus, the mean cost per unit of time when a unit is in corrective repair from warm standby depending on the repair phase is given by the vector $\mathbf{cr_0}$. Also, we assume a fixed cost each time that a warm standby unit undergoes a repairable failure equal to *fwr*.

To calculate the total net profit up to time ν is necessary to build the vector cost for the macro-state E_s^k and several rewards and costs functions.

4.6.1 Net profit vector associated to the phases

When the systems is composed of *k* units and *s* of them are in the repair facility, then the online unit provokes a net reward for the different phases of the system given by

$$\mathbf{nr}_s^k = \begin{cases} B\mathbf{e}_{ntd\varepsilon} - \mathbf{c_0} \otimes \mathbf{e}_{td\varepsilon} & ; \quad s = 0 \\ B\mathbf{e}_{ntd\varepsilon \cdot \sum_{i=1}^{3^s} z_o^{h_s^k(i)} z_1^{d_s^k(i)} z_2^{g_s^k(i)}} - \begin{pmatrix} \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon z_0^{h_s^k(1)} z_1^{d_s^k(1)} z_2^{g_s^k(1)}} \\ \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon z_0^{h_s^k(2)} z_1^{d_s^k(2)} z_2^{g_s^k(2)}} \\ \vdots \\ \mathbf{c}_0 \otimes \mathbf{e}_{td\varepsilon z_0^{h_s^k(3^s)} z_1^{d_s^k(3^s)} z_2^{g_s^k(3^s)}} \end{pmatrix} & ; \quad s = 1, \ldots, k-1 \\ -C \cdot \mathbf{e}_{t \cdot \sum_{i=1}^{I_{\{s=K\}} 2 \cdot 3^{k-1} + I_{\{s \neq K\}} 3^k} z_0^{h_s^k(i)} z_1^{d_s^k(i)} z_2^{g_s^k(i)}} & ; \quad s = k \end{cases}$$

Then, for the state space it is

$$\mathbf{nr} = \left(\mathbf{nr}_0^K{}', \mathbf{nr}_1^K{}', \ldots, \mathbf{nr}_K^K{}', \mathbf{nr}_0^K{}', \mathbf{nr}_1^{K-1}{}', \ldots, \mathbf{nr}_{K-1}^{K-1}{}', \ldots, \mathbf{nr}_0^1{}', \mathbf{nr}_1^1{}' \right)'.$$

If the repair facility is considered, the cost vector per unit of time depending on the type of repair associated to the macro-state E_s^k for $s = 1,\ldots, k$, with $(k, s) \neq (K, K)$ is

$$\mathbf{nc}_s^k = \begin{pmatrix} \mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_0 \\ \mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_1 \\ \mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_2 \\ \vdots \\ \mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_0 \\ \mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_1 \\ \mathbf{e}_{t(nd\varepsilon)^{I\{s\neq k\}}3^{s-\min\{s,R_k\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{\min\{s,R_k\}}{\cdots} \odot \mathbf{cr}_2 \end{pmatrix}, s \geq 1,\ \mathbf{nc}_0^k = \mathbf{0},$$

$$\mathbf{nc}_K^K = \begin{pmatrix} \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I\{K>R_K\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_0 \odot \mathbf{cr}_0 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I\{K>R_K\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_0 \odot \mathbf{cr}_1 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I\{K>R_K\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_1 \odot \mathbf{cr}_0 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I\{K>R_K\}}} \otimes \mathbf{cr}_0 \odot \mathbf{cr}_0 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_1 \odot \mathbf{cr}_1 \\ \vdots \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I\{K>R_K\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{R_K}{\cdots} \odot \mathbf{cr}_2 \odot \mathbf{cr}_0 \\ \mathbf{e}_{t\left(2\cdot 3^{K-R_K-1}\right)^{I\{K>R_K\}}} \otimes \mathbf{cr}_2 \odot \mathbf{cr}_2 \odot \overset{K}{\cdots} \odot \mathbf{cr}_2 \odot \mathbf{cr}_1 \end{pmatrix}.$$

The total vector for the cost due to repair is given by

$$\mathbf{nc} = \left(\mathbf{nc}_0^K{}', \mathbf{nc}_1^K{}', \ldots, \mathbf{nc}_K^K{}', \mathbf{nc}_0^K{}', \mathbf{nc}_1^{K-1}{}', \ldots, \mathbf{nc}_{K-1}^{K-1}{}', \ldots, \mathbf{nc}_0^1{}', \mathbf{nc}_1^1{}'\right)'.$$

Thus, the net profit vector associated to the macro-state E_s^k is given by $\mathbf{c}_0^k = \mathbf{nr}_0^k$, $\mathbf{c}_s^k = \mathbf{nr}_s^k - \mathbf{nc}_s^k$ for $s = 1,\ldots, k$. Finally, the net column profit vector associated to the macro-state E^k is given by $\mathbf{c}^k = \left(\mathbf{c}_0^k, \ldots, \mathbf{c}_k^k\right)'$ thus the global net column profit vector associated to the macro-state E is given by

$$\mathbf{c} = \mathbf{nr} - \mathbf{nc} = \begin{pmatrix} \mathbf{c}^K \\ \mathbf{c}^{K-1} \\ \vdots \\ \mathbf{c}^1 \end{pmatrix}.$$

4.6.2 Rewards Measures

Several rewards measures have been built in transient and stationary regime.

Mean net profit up to time ν

The mean net profit by considering only the online unit up to time ν is given by

$$\Phi_w^{\nu} = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{nr},$$

and it is in stationary regime the meat net profit per unit of time, $\Phi_{w\_s} = \boldsymbol{\pi} \cdot \mathbf{nr}$.

Mean cost due to corrective (online and warm standby) and preventive repair

The mean cost due to corrective repair and preventive maintenance up to time up ν is given respectively by

$$\Phi_{online\_cr}^{\nu} = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{online\_cr}, \quad \Phi_{warm\_cr}^{\nu} = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{warm\_cr} \quad \text{and} \quad \Phi_{pm}^{\nu} = \sum_{m=0}^{\nu} \mathbf{p}^m \cdot \mathbf{mc}^{pm},$$

where $\mathbf{mc}^{warm\_cr}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_1 = \mathbf{0}_{z_1}$, $\mathbf{cr}_2 = \mathbf{0}_{z_2}$; $\mathbf{mc}^{online\_cr}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_0 = \mathbf{0}_{z_0}$, $\mathbf{cr}_2 = \mathbf{0}_{z_2}$ and $\mathbf{mc}^{pm}$ is the vector $\mathbf{nc}$ with $\mathbf{cr}_0 = \mathbf{0}_{z_0}$ and $\mathbf{cr}_1 = \mathbf{0}_{z_1}$, being $\mathbf{0}_a$ a column vector of zeros with order a.

These measures in stationary regime, net cost per unit of time due to corrective or preventive maintenance, are $\Phi_{online\_cr\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{online\_cr}$, $\Phi_{warm\_cr\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{warm\_cr}$ and $\Phi_{pm\_s} = \boldsymbol{\pi} \cdot \mathbf{mc}^{pm}$ respectively.

Total net profit

The total net profit up to time ν is worked out by adding costs and profits produced by the events. If the fixed costs per event are included then it is equal to

$$\begin{aligned}\Phi^{\nu} &= \Phi_{w}^{\nu} - \Phi_{online\_cr}^{\nu} - \Phi_{warm\_cr}^{\nu} - \Phi_{pm}^{\nu} - \left(1 + \Lambda^{FC1}(\nu) + \Lambda^{FC2}(\nu)\right) \cdot K \cdot fnu \\ &\quad - \Lambda^{w}(\nu) \cdot fwr - \left(\Lambda^{A1}(\nu) + \Lambda^{A2}(\nu)\right) \cdot fcr - \left(\Lambda^{B1}(\nu) + \Lambda^{B2}(\nu) + \Lambda^{B3}(\nu)\right) \cdot fpm \\ &\quad - \left(\mu_{idle} + \mu_{busy}\right) \cdot H.\end{aligned}$$

Finally, the total net profit per unit of time (stationary regime) is

$$\begin{aligned}\Phi_{s} &= \Phi_{w\_s} - \Phi_{cr\_s} - \Phi_{pm\_s} - \left(1 + \Lambda^{FC1} + \Lambda^{FC2}\right) \cdot K \cdot fnu \\ &\quad - \Lambda^{w} \cdot fwr - \left(\Lambda^{A1} + \Lambda^{A2}\right) \cdot fcr - \left(\Lambda^{B1} + \Lambda^{B2} + \Lambda^{B3}\right) \cdot fpm - \left(\mu_{idle} + \mu_{busy}\right) \cdot H.\end{aligned}$$

4.7 A numerical example

We assume a warm standby system composed of three units, like *Systems II* and *III* in Section 3.7. We assume identical behaviour of the embedded operational times but the standby unit can undergo a failure. Again two questions must be answered. Is preventive maintenance profitable? How many repairpersons, depending on the number of units in the system, would have to be deployed to optimise the profit? In this numerical example, the effectiveness of preventive maintenance is analysed and the optimum number of repairpersons is calculated.

System times

The internal behaviour of the online unit passes through five performance levels, where the degradation is minor in the first three stages and major in the last two. The operational time distribution of the online unit, the inspection time distribution and the external shock time are PH distributed with representation given in Table 3.1.

Each time that the online unit undergoes an external shock, a total non-repairable failure occurs with a probability equal to 0.05. If no such failure occurs, the internal performance may be degraded according to the following probability matrix

$$\mathbf{W}=\begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 & 0 \\ 0 & 0.6 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0.6 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0 & 0.4 \end{pmatrix}.$$

When an external shock takes place, cumulative external damage occurs. Four external degradation levels are assumed, the first two of which are minor and the last two, major. Changes in the external degradation levels are governed by the matrix

$$\mathbf{D}=\begin{pmatrix} 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 \end{pmatrix}.$$

The external degradation level is the stage 1 initially (without external damage). Each warm standby is also subject to failure. Each one can fail at any time with a probability equal to $p = 0.05$.

Each time that a repairable failure, either online unit, warm standby unit or a major inspection occurs, the online units goes to the repair facility. The corrective repair times for each situation and the preventive maintenance time distribution are given in Table 4.1

Corrective repair time distribution Warm standby unit	Corrective repair time distribution Online unit	Preventive maintenance time distribution
$\boldsymbol{\beta}^0=(1,0)$ $\mathbf{S}_0=\begin{pmatrix} 0.9 & 0.02 \\ 0 & 0.6 \end{pmatrix}$ Mean time: 10.5	$\boldsymbol{\beta}^1=(1,0)$ $\mathbf{S}_1=\begin{pmatrix} 0.91 & 0.01 \\ 0 & 0.8 \end{pmatrix}$ Mean time: 11.67	$\boldsymbol{\beta}^2=(1,0)$ $\mathbf{S}_2=\begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.1 \end{pmatrix}$ Mean time: 1.23

Table 4.1. Corrective repair and preventive maintenance time distributions

As it can be seen, the corrective repair time distribution and the preventive maintenance time distribution are similar to the ones given in Table 3.2 in Section 3.7.

Performance of the systems according to the number of repairpersons

As mentioned above, the systems with and without preventive maintenance (*Systems III* and *II*, respectively) are compared by considering all possibilities for the number of repairpersons. Again, the system *i_j_k* denotes a system with *i*, *j*, *k* repairpersons when there are 1, 2, 3 units in the system respectively for i =1, j =1, 2 , k = 1, 2, 3. In total there are 12 possible systems, six with preventive maintenance and six without. Several measures have been worked out and compared in transient and stationary regime for the warm standby system.

Figure 4.1 shows the mean operational time and the mean number of idle repairpersons per unit of time for *Systems II* and *III*. The optimum mean operational time ratio is reached for system 1_2_2 when the operational time ratio in a stationary regime is equal to 0.9221 for the system with preventive maintenance and 0.9122 for system 1_2_3 without preventive maintenance. This outcome is expected but as the repairpersons have a cost, it is interesting to analyse the mean number of idle repairpersons. In this case, the maximum is reached for system 1_2_3 with a mean number of idle repairpersons per unit of time in the stationary regime equal to 1.5196 and 1.4714 for the systems with and without preventive maintenance, respectively.

Another interesting aspect to study is that of the mean number of events up to a certain time. This measure was calculated for every system and for several units of time. Tables 4.2 and 4.3 shows the results obtained for 1500 units of time.

The number of new systems up to time 1500 is given by the last column of the Table 4.3. The minimum is reached when one repairperson is assumed when there are one and two units in the system and two repairpersons when there are three.

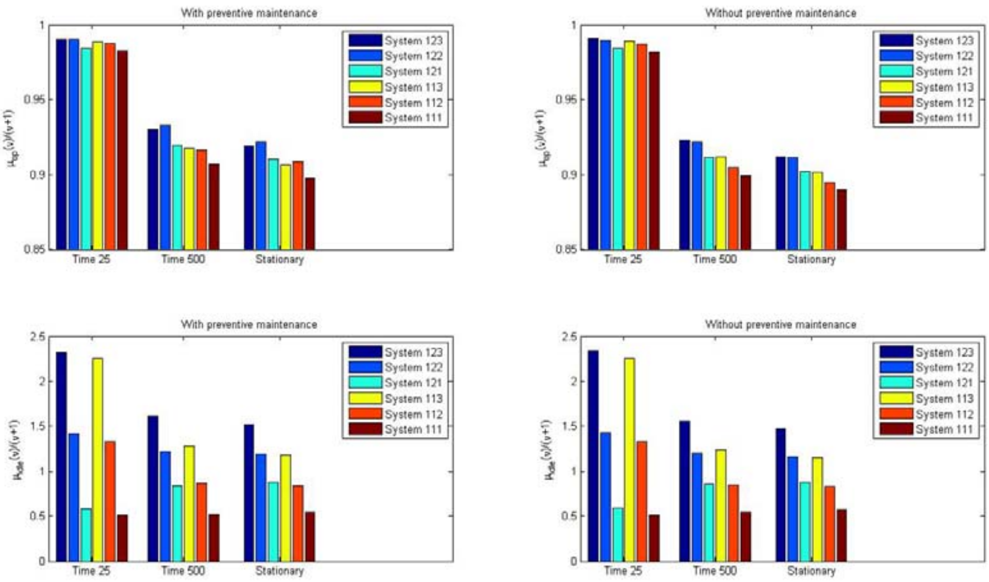

Figure 4.1. Mean operational ratio (first row) and mean number of idle repairpersons per unit of time (second row) with preventive maintenance (first column) and without preventive maintenance (second column)

	SYSTEM	$\Lambda^{A_1}(1500)$	$\Lambda^{A_2}(1500)$	$\Lambda^{B_1}(1500)$	$\Lambda^{B_2}(1500)$	$\Lambda^{B_3}(1500)$
Time 1500	1_2_3	21.2544 (21.8486)	0.0157 (0.0154)	0.0540	9.6627	0.1861
	1_2_2	21.3005 (21.8251)	0.0157 (0.0154)	0.0555	9.8917	0.1913
	1_2_1	21.0984 (21.6047)	0.0156 (0.0153)	0.0496	9.1592	0.1726
	1_1_3	20.9670 (21.5930)	0.0155 (0.0152)	0.0532	9.5296	0.1834
	1_1_2	20.9354 (21.4308)	0.0155 (0.0151)	0.0531	9.5139	0.1831
	1_1_1	20.8079 (21.3191)	0.0154 (0.0151)	0.0488	9.0283	0.1699

Table 4.2. Mean number of events up to time 1500 (without preventive maintenance in parenthesis)

SYSTEM		$\Lambda^{w}(1500)$	$\Lambda^{C_1}(1500)$	$\Lambda^{C_2}(1500)$	$\Lambda^{FC_1}(1500)$ $+\Lambda^{FC_2}(1500)$
Time 1500	1_2_3	1.3608 (1.2314)	3.4413 (3.1035)	5.9919 (8.3642)	4.2381 (5.2012)
	1_2_2	1.4167 (1.2294)	3.4562 (3.1002)	5.9599 (8.3552)	4.2316 (5.1956)
	1_2_1	1.2622 (1.1244)	3.4030 (3.0703)	6.0309 (8.2748)	4.2391 (5.1445)
	1_1_3	1.3422 (1.2156)	3.3965 (3.0686)	5.9151 (8.2701)	4.1770 (5.1370)
	1_1_2	1.3395 (1.1689)	3.3913 (3.0469)	5.9065 (8.2116)	4.1708 (5.0970)
	1_1_1	1.2444 (1.1079)	3.3579 (3.0314)	5.9534 (8.1700)	4.1774 (5.0730)

Table 4.3. Mean number of events up to time 1500 (without preventive maintenance in parenthesis)

Analysis of systems when costs and rewards are included

Rewards and costs have been included in the analysis to optimize the model form an economical standpoint in a similar way as given in Section 3.7. The following costs and rewards have been considered,

B=100	Reward per unit of time while the system is operational.
$\mathbf{c}_0 = (10, 20, 30, 40, 50)'$	Operational cost per unit of time while the online unit is working.
$\mathbf{cr_0}$=(3, 3)'	Cost per unit of time while a unit is being repaired from warm standby.
$\mathbf{cr_1}$ = (5, 5)'	Cost per unit of time while a unit is being repaired from online unit.
$\mathbf{cr_2}$=(0.5, 0.5)'	Cost per unit of time while a unit is in preventive maintenance.
fcr = 20	Fixed cost due to repairable failure of the online unit is produced.
fwr = 5	Fixed cost each time that a warm standby unit undergoes a repairable failure.
fpm = 1	Fixed cost due to one major inspection.
fnu = 200	Cost of one new unit.

The mean net reward has been calculated for any system to achieve the more profitable system. Figure 4.2 shows them per unit of time for the cases with and without preventive maintenance.

If the mean net profit is observed in stationary regime the most profitable situation is for system 1_2_2 with preventive maintenance. Initially the number of repairpersons should be only two, when the first non-repairable failure occurs then the number of repairpersons continues being two and finally only one repairperson should be when another non-repairable failure occurs. The optimum mean net profit in this case is equal to 68.3350 in stationary regime.

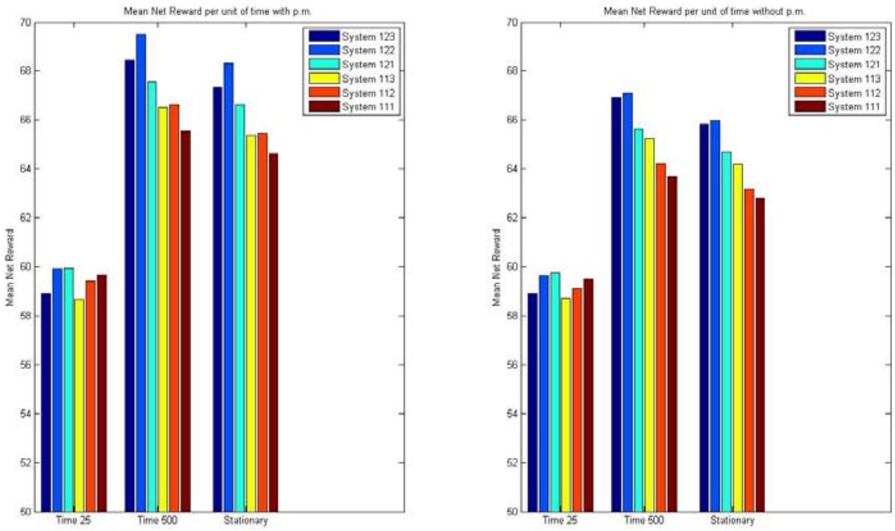

Figure 4.2. Mean net profit per unit of time up to time 25, 500 and the stationary case for *System II* (with preventive maintenance) and *System III* (without preventive maintenance)

Table 4.4 shows the mean net reward per unit of time for the cases with and without preventive maintenance. The most profitable situation of mean net profit is for system 1_2_2 with preventive maintenance.

	$\phi^{\nu}/(\nu+1)$					
System	1_2_3	1_2_2	1_2_1	1_1_3	1_1_2	1_1_1
$\nu = 25$	58.8789 (58.8882)	59.9001 (59.6256)	59.9360 (59.7518)	58.6865 (58.7034)	59.4086 (59.1121)	59.6707 (59.5156)
$\nu = 500$	68.4357 (66.9176)	69.4783 (67.0815)	67.5566 (65.6270)	66.4923 (65.2530)	66.6263 (64.2267)	65.5385 (63.6981)
Stationary	67.3379 (65.8385)	68.3350 (65.9736)	66.6226 (64.6795)	65.3496 (64.1894)	65.4558 (63.1707)	64.6127 (62.7953)

Table 4.4. Mean net profit per unit of time up to time 25, 500 and the stationary case for system II (with preventive maintenance) and system III (without preventive maintenance)

4.8 Appendix 4A

Blocks of the Markovian Arrival Process $\left(\mathbf{D}^{O},\mathbf{D}^{A_1},\mathbf{D}^{A_2},\mathbf{D}^{B_1},\mathbf{D}^{B_2},\mathbf{D}^{B_3},\mathbf{D}^{C_1},\mathbf{D}^{C_2},\mathbf{D}^{FC_1},\mathbf{D}^{FC_2}\right)$

$\mathbf{D}^{Y}=diag\left(\mathbf{D}^{Y,K},\mathbf{D}^{Y,K-1},\mathbf{D}^{Y,K-2},\ldots,\mathbf{D}^{Y,1}\right)$, for $Y = O, A_1, A_2, B_1, B_2, B_3$ and

$$\mathbf{D}^{Y}=\begin{pmatrix} \mathbf{0} & \mathbf{D}^{Y,K} & & & \\ & \mathbf{0} & \mathbf{D}^{Y,K-1} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \mathbf{D}^{Y,2} \\ \mathbf{0} & & & & \mathbf{0} \end{pmatrix}, \text{ for } Y=C_1, C_2, \text{ and } \mathbf{D}^{Y}=\begin{pmatrix} \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \ddots & & \vdots \\ \mathbf{0} & & \ddots & \vdots \\ \mathbf{D}^{Y,1} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix},$$

for $Y=FC_1, FC2$.

Matrix $\mathbf{D}^{Y,k}$ for $Y = O, C_1, C_2, FC_1, FC_2$

The elements of the matrix $\mathbf{D}^{Y,k}$ for $k = 1,...,K$ and $Y = O, C_1, C_2, FC_1, FC_2$, are given by

$$\mathbf{D}^{Y,k}=\begin{cases} \left(\mathbf{D}^{Y,k}_{lh}\right)_{l,h=0,\ldots,k} & ; \quad Y=O \\ \left(\mathbf{D}^{Y,k}_{lh}\right)_{\substack{l=0,\ldots,k\\ h=0,\ldots,k-1}} & ; \quad Y=C_1,C_2 \\ \left(\mathbf{D}^{Y,1}_{lh}\right)_{\substack{l=0,1\\ h=0,\ldots,K}} & \quad Y=FC_1,FC_2; k=1, \end{cases}$$

where $\mathbf{D}^{O,k}_{lh}=\mathbf{0}$ if $h = k$ for $l < k$ or $h <l - \min\{l, R_k\}$, $\mathbf{D}^{C_i,k}_{lh}=\mathbf{0}$ if $h <l - \min\{l, R_k\}$ or $l = k$ and $\mathbf{D}^{FC_i,1}_{lh}=\mathbf{0}$ for all l and h excepting for the case $l = h = 0$.

For $k = 1,\ldots, K$,

$$\mathbf{D}^{Y,k}_{00}=\begin{cases} \mathbf{H}_{k,0,0,0}+I_{\{k=1\}}\mathbf{H}'_{k,mr,0,0} & ; \quad Y=O \\ \mathbf{H}_{k,\binom{type}{nrep},0,0} & ; \quad Y=C_{type} \text{ or } Y=FC_{type} \end{cases}.$$

For $l =1,..., R_k$,

$$\mathbf{D}^{Y,k}_{l,0}\left(j_1,\ldots,j_l\right)=\begin{cases} \mathbf{H}_{k,0,l,0}\otimes E\left(k,l,l,0;j_1,\ldots,j_l;0,0,0\right) & ; \quad Y=O \text{ and } l<k \\ \mathbf{H}_{k,(nrep,type),l,0}\otimes E\left(k,l,l,0;j_1,\ldots,j_l;0,0,1\right) & ; \quad Y=C_{type} \text{ and } l<k \\ \boldsymbol{\zeta}\otimes E\left(k,l,l,0;j_1,\ldots,j_l;0,0,0\right) & ; \quad Y=O \text{ and } l=k, \end{cases}$$

with $\boldsymbol{\zeta} = \boldsymbol{\alpha}$ for *system I*, $\boldsymbol{\zeta} = \alpha \otimes \left(\mathbf{L} + \mathbf{L}^0 \boldsymbol{\gamma}\right)$ for *system II* and $\boldsymbol{\zeta} = \alpha \otimes \left(\mathbf{L} + \mathbf{L}^0 \boldsymbol{\gamma}\right) \otimes \boldsymbol{\eta} \otimes \boldsymbol{\omega}$ for *system III*.

For $wr = 1,\ldots, k-1$ with $k > 1$

$$\mathbf{D}_{0,wr}^{Y,k}\left(0,\overset{wr}{\ldots},0\right) =$$

$$\begin{cases} \left[\mathbf{H}_{k,0,0,wr} + I_{\{wr=k-1\}}\mathbf{H}'_{k,mr,0,wr}\right] \otimes E\left(k,0,0,0;0,\overset{wr}{\ldots},0;0,wr,0\right) & ; \quad Y = O \\ \left(\mathbf{H}_{k,\binom{type}{nrep},0,wr} I_{\{wr<k-1\}} + \mathbf{H}'_{k,\binom{type}{nrep},0,wr} I_{\{wr=k-1\}}\right) \otimes E\left(k,0,0,0;0,\overset{wr}{\ldots},0;0,wr,1\right) & ; \quad Y = C_{type}. \end{cases}$$

For $l = 1,\ldots, k-1$; $a = 0,\ldots, \min\{R_k, l\}$; $wr = 0,\ldots, k-l-1$ with $k > 1$ and $l + wr - a > 0$

$$\mathbf{D}_{l,l+wr-a}^{Y,k,wr}\left(i_1,\ldots,i_{l-a},0,\overset{wr}{\ldots},0; j_1,\ldots, j_l\right) =$$

$$\begin{cases} \left[\mathbf{H}_{k,0,l,wr} + I_{\{wr=k-l-1 \text{ and } a=0\}}\mathbf{H}'_{k,mr,l,wr}\right] \\ \otimes E\left(k,l,a,0;i_1,\ldots,i_{l-a},0,\overset{wr}{\ldots},0; j_1,\ldots, j_l;0,wr,0\right) & ; \quad Y = O \\ \left(\mathbf{H}_{k,\binom{type}{nrep},l,wr} I_{\{wr<k-l-1 \text{ or } a>0\}} + \mathbf{H}'_{k,\binom{type}{nrep},l,wr} I_{\{wr=k-l-1 \text{ and } a=0\}}\right) & ; \quad Y = C_{type}. \\ \otimes E\left(k,l,a,b;i_1,\ldots,i_{l-a}; j_1,\ldots, j_l;0,wr,1\right) \end{cases}$$

Then,

$$\mathbf{D}_{lh}^{Y,k}\left(i_1,\ldots,i_h; j_1,\ldots, j_l\right) = \sum_{a=\max\{0,l-h\}}^{\min\{k-h-1,\min\{l,R_k\}\}} \mathbf{D}_{lh}^{Y,k,h-l+a}\left(i_1,\ldots,i_h; j_1,\ldots, j_l\right) .$$

For $a = 1,\ldots, \min\{R_k, k-1\}$,

$$\mathbf{D}_{k,k-a}^{O,k}\left(i_1,\ldots,i_{k-a}; j_1,\ldots, j_k\right) = \boldsymbol{\zeta} \otimes E\left(k,k,a,0;i_1,\ldots,i_{k-a}; j_1,\ldots, j_k;0,0,0\right), \text{ with } k > 1.$$

For $k = 1,\ldots, K$,

$$\mathbf{D}_{k,k}^{O,k}\left(i_1,\ldots,i_k; j_1,\ldots, j_k\right) = \begin{cases} E\left(k,k,0,0;i_1,\ldots,i_k; j_1,\ldots, j_k;0,0,0\right) & ; \quad system\ I \\ \left(\mathbf{L} + \mathbf{L}^0\boldsymbol{\gamma}\right) \otimes E\left(k,k,0,0;i_1,\ldots,i_k; j_1,\ldots, j_k;0,0,0\right) & ; \quad systems\ II,\ III \end{cases}.$$

Matrix $\mathbf{D}^{A_i,k}$

The elements of the matrix $\mathbf{D}^{A_i,k}$ for i=1, 2 and for$k$ = 1,...,K are given by

$$\mathbf{D}^{A_i,k} = \left(\mathbf{D}^{A_i,k}_{lh}\right)_{l,h=0,\dots,k},$$

where $\mathbf{D}^{A_i,k}_{lh} = \mathbf{0}$ if $h < l+1-\min\{l, R_k\}$ or $l = k$.

For $type$ = 1, 2 and h=1, …, k then

$$\mathbf{D}^{A_{type},k}_{0h}\left(1,0,\overset{h-1}{\dots},0\right) = \left(\mathbf{H}_{k,\binom{type}{rep},0,h-1} I_{\{h<k\}} + \mathbf{H}'_{k,\binom{type}{rep},0,h-1} I_{\{h=k\}}\right).$$
$$\otimes E\left(k,0,0,0;1,0,\overset{h-1}{\dots},0;1,h-1,0\right)$$

For l=1,…,k–1; a=0,…,min{l, R_k}; wr = 0,…, k–l–1 with k > 1,

$$\mathbf{D}^{A_{type},k,wr}_{l,l+wr+1-a}\left(i_1,\dots,i_{l-a},1,0,\overset{wr}{\dots},0;j_1,\dots,j_l\right) = \left(\mathbf{H}_{k,\binom{type}{rep},l,wr} I_{\{wr<k-l-1 \text{ or } a>0\}} + \mathbf{H}'_{k,\binom{type}{rep},l,wr} I_{\{wr=k-l-1 \text{ and } a=0\}}\right)$$
$$\otimes E\left(k,l,a,0;i_1,\dots,i_{l-a},1,0,\overset{wr}{\dots},0;j_1,\dots,j_l;1,wr,0\right).$$

Then,

$$\mathbf{D}^{A_{type},k}_{lh}\left(i_1,\dots,i_h;j_1,\dots,j_l\right) = \sum_{a=\max\{0,l-h+1\}}^{\min\{k-h,\min\{l,R_k\}\}} \mathbf{D}^{A_{type},k,h-l-1+a}_{lh}\left(i_1,\dots,i_h;j_1,\dots,j_l\right).$$

Matrix $\mathbf{D}^{B_i,k}$

The elements of the matrix $\mathbf{D}^{B_i,k}$ for i = 1, 2, 3 and for k = 1,..., K are given by

$$\mathbf{D}^{B_i,k} = \left(\mathbf{D}^{B_i,k}_{lh}\right)_{l,h=0,\dots,k},$$

where $\mathbf{D}^{B_i,k}_{lh} = \mathbf{0}$ if $h < l+1-\min\{l, R_k\}$ or $h = k$ or $l = k$.

For $type$ = 1, 2, 3 then for h = 1, …, k–1

$$\mathbf{D}^{B_{type},k}_{0h}\left(2,0,\overset{h-1}{\dots},0\right) = \mathbf{H}_{k,\binom{type}{mr},0,h-1} I_{\{k>1\}} \otimes E\left(k,0,0,0;2,0,\overset{h-1}{\dots},0;1,h-1,0\right).$$

For l=1,…,k–1; a=0,…,min{l, R_k}; wr = 0,…, k–l–1 with k > 1,

$$\mathbf{D}_{l,l+wr+1-a}^{B_{type},k,wr}\left(i_1,...,i_{l-a},2,0,\overset{wr}{...},0;\ j_1,...,j_l\right)$$
$$=\mathbf{H}_{k,\binom{type}{mr},l,wr}\otimes E\left(k,l,a,0;i_1,...,i_{l-a},2,0,\overset{wr}{...},0;\ j_1,...,j_l;1,wr,0\right)$$

Then, $\mathbf{D}_{lh}^{B_{type},k}\left(i_1,...,i_h;\ j_1,...,j_l\right)=\sum_{a=\max\{0,l-h+1\}}^{\min\{k-h,\min\{l,R_k\}\}}\mathbf{D}_{lh}^{B_{type},k,h-l-1+a}\left(i_1,\ldots,i_h;\ j_1,\ldots,j_l\right)$.

Chapter 5

Modeling a multi-state k–out–of–n: G system with loss of units

5.1 Introduction

In this chapter, we model a discrete-time complex multi-state k-out-of-n: G system with loss of units, in a well-structured form. This one is a n–system that works if at least k units are operational. The lifetime of the units is governed by PH distributions in which different performance stages are introduced. The units of the system may undergo repairable and/or non-repairable failures. In the first case the unit goes to the repair facility and in the second, it is removed. Then, loss of units is introduced. The repair time is also PH distributed. When fewer than k units remain in the system, it is replaced by a new n-system.

In this chapter external shocks and preventive maintenance are not included. Furthermore, only one repairperson is considered. A natural extension will be to include different types of events on the units and to introduce a variable number of repairpersons depending on the number of units in the system. The system has been modelled in an algorithmic and computational form but without considering Markovian Arrival Processes with Marked arrivals. Even though it is a very complex model expressed in a well structured form, it can be extended by following the general methodology given in this work.

The chapter is organised as follows. In Section 5.2, the system is described in detail, setting out the assumptions made and the state space employed. The model is then described in Section 5.3 in algorithmic form. The transient and the stationary distributions are presented in Sections 5.4 and 5.5, respectively, where matrix analytic methods are used. Various measures are applied in Section 5.6 and a Markov counting process is developed in Section 5.7 to explain the mean number of new systems obtained. In Section 5.8, a numerical example is given to show the applicability of the model, and finally, in Section 5.9, the main conclusions drawn are presented.

5.2. Assumptions and state space

We assume a multi-state system composed initially of n independent units subject to different types of failure. Each multi-state unit may experience internal repairable and/or non-repairable failure. In the first case, the unit is transferred to the repair facility, where there is one repairperson (if this one is busy then the failed unit waits for in queue). However, if an operational unit undergoes a non-repairable failure, it is removed and not replaced. We thus propose a k-out-of-n: G system, which is operational while at least k units are operational. Therefore, when n–k+1 (or more) non-repairable failures occur, the system is replaced by a new, identical one (k–units system). The system satisfies the following assumptions.

***Assumption* 5.1**. The lifetime of each unit is discrete-time PH distributed with representation $(\boldsymbol{\alpha}, \mathbf{T})$. The order of the matrix $\mathbf{T}$ is the number of operational stages, m.

***Assumption* 5.2**. Each unit can undergo a repairable or non-repairable failure. We assume two absorbing states, one for each kind of failure. The probability of failure depends on the operational stage, thus the probability of repairable failure or non-repairable is given by the column vectors $\mathbf{T}_r^0$ and $\mathbf{T}_{nr}^0$, repectively. Clearly, the total absorbing vector produced by any transient state is given by $\mathbf{T}^0 = \mathbf{e} - \mathbf{T}\mathbf{e} = \mathbf{T}_r^0 + \mathbf{T}_{nr}^0$.

The repair time is PH distributed with representation $(\boldsymbol{\beta}, \mathbf{S})$ where the order of $\mathbf{S}$ is equal to t (number of repair stages).

***Assumption* 5.3**. When a non-repairable failure occurs, the unit is removed. The number of units in the system is always greater than or equal to k.

***Assumption* 5.4**. While the system is operational, several transitions may occur at the same time (operational unit and unit under repair).

***Assumption* 5.5**. If the number of units in the system is l, greater than or equal to k, then the system is operational only if at least k units are operational (the number of units under repair is less than or equal to $l–k$). Otherwise, the system is broken, in which case only the repairpersons continue operating, and all other operations cease.

***Assumption* 5.6.** After a repair the unit quality is as good as new.

***Assumption* 5.7**. The times involved in the model are independent.

The discrete case is more complex than the continuous one from assumption 5.4. The state space of the system, denoted by E, is described as follows.

This state space is composed of macro-states such that $E = \{U^n, U^{n-1}, \ldots, U^k\}$ where U^l denotes the phases when there are l units in the system. In turn, these macro-states are composed of macro-states U^l_s , for $l = k,\ldots, n$ and $s = 0,\ldots,k$; l units in the system and s of them in the repair facility, the first being repaired and the rest in queue. The phases are given by

$U^l_0 = \{(i_1,\ldots,i_l); 1 \le i_s \le m, s = 1,\ldots,l\}$ for $l = k,\ldots, n$,

$U^l_a = \{(i_1,\ldots,i_{l-a}; j); 1 \le i_s \le m, s = 1,\ldots,l-a; 1 \le j \le t\}$ for $a = 1,\ldots,l–1$,

$U^l_l = \{j; 1 \le j \le t\}$,

where i_s indicates the state of the s-th operational unit of the system and j the repair stage.

5.3 The model

The k-out-of-n system with loss of units is modelled by a vector Markov process with state-space as described in the previous section. The transition probability matrix, **P**, is composed of two levels of matrix blocks. The first level has the matrices $\mathbf{R}^{l,h}$ and contains the transition probabilities between the macro-states from U^l to U^h (i.e. from l units in the system to h units in the system), where $l–h$ non-repairable failures occur. The matrix has the following structure,

$$\mathbf{P}=\begin{pmatrix} \mathbf{R}^{n,n} & \mathbf{R}^{n,n-1} & \mathbf{R}^{n,n-2} & \cdots & \mathbf{R}^{n,k+1} & \mathbf{R}^{n,k} \\ \mathbf{R}^{n-1,n} & \mathbf{R}^{n-1,n-1} & \mathbf{R}^{n-1,n-2} & \cdots & \mathbf{R}^{n-1,k+1} & \mathbf{R}^{n-1,k} \\ \mathbf{R}^{n-2,n} & \mathbf{0} & \mathbf{R}^{n-2,n-2} & \cdots & \mathbf{R}^{n-2,k+1} & \mathbf{R}^{n-2,k} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{R}^{k+1,n} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{R}^{k+1,k+1} & \mathbf{R}^{k+1,k} \\ \mathbf{R}^{k,n} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R}^{k,k} \end{pmatrix}.$$

These matrices, $\mathbf{R}^{l,h}$, are composed of matrix blocks (second level). These new matrix blocks, $\mathbf{B}_{i,j}^{l,h}$, contain the transition between the macro-states, from U_i^l (*l* units in the system of which *i* are in the repair facility) to U_j^h (*h* units in the system of which *j* are in the repair facility).

5.3.1 Auxiliary functions

To build these matrix blocks, the following auxiliary functions are incorporated, taking into account the phases of the operational units.

1. Transition matrix for the operational units when the system is composed of *l* units, of which *a* are in the repair facility, when *w*=*l*–*h* non-repairable failures and *u* repairable failures take place, in a specific failure order ($s_1,\ldots,s_u$ is the ordinal of the repairable failures and $k_1,\ldots,k_w$ is the ordinal of the non-repairable failures). This situation is described as $C\left(l,a,w,u;k_1,\ldots,k_w;s_1,\ldots,s_u\right)$.

2. Transition matrix for the operational units when the system is composed of *l* units, of which *a* are in the repair facility, when *w* non-repairable failures and *u* repairable failures take place where the failure order is not established. This situation is described as $b\left(l,a,w,u\right)$.

3. If there are *l* units is the system, *a* of which are in the repair facility, and the number of non-repairable failures at the next step is greater than or equal to *l*–*k*+1, then the system has to be replaced. The probability of this occurring during the phases of the system is denoted by $d\left(k,l,a\right)$.

These functions are further developed in *Appendix 5A*.

5.3.2 Transition probability matrix

The transition probability matrix, **P**, is composed as follows.

For $l = k, \dots , n$

$$\mathbf{R}^{l,l} =$$

$$\begin{pmatrix}
\mathbf{B}^{l,l}_{00} & \mathbf{B}^{l,l}_{01} & \mathbf{B}^{l,l}_{02} & \cdots & \mathbf{B}^{l,l}_{0,l-k-2} & \mathbf{B}^{l,l}_{0,l-k-1} & \mathbf{B}^{l,l}_{0,l-k} & \mathbf{B}^{l,l}_{0,l-k+1} & \dots & \dots & \mathbf{B}^{l,l}_{0,l-1} & \mathbf{B}^{l,l}_{0,l} \\
\mathbf{B}^{l,l}_{1,0} & \mathbf{B}^{l,l}_{1,1} & \mathbf{B}^{l,l}_{1,2} & \cdots & \mathbf{B}^{l,l}_{1,l-k-2} & \mathbf{B}^{l,l}_{1,l-k-1} & \mathbf{B}^{l,l}_{1,l-k} & \mathbf{B}^{l,l}_{1,l-k+1} & \dots & \dots & \mathbf{B}^{l,l}_{1,l-1} & \mathbf{B}^{l,l}_{1,l} \\
 & \mathbf{B}^{l,l}_{2,1} & \mathbf{B}^{l,l}_{2,2} & \cdots & \mathbf{B}^{l,l}_{2,1l-k-2} & \mathbf{B}^{l,l}_{2,l-k-1} & \mathbf{B}^{l,l}_{2,l-k} & \mathbf{B}^{l,l}_{2,l-k+1} & \dots & \dots & \mathbf{B}^{l,l}_{2,l-1} & \mathbf{B}^{l,l}_{2,l} \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & \vdots & \vdots \\
 & & & \ddots & \mathbf{B}^{l,l}_{l-k-1,l-k-2} & \mathbf{B}^{l,l}_{l-k-1,l-k-1} & \mathbf{B}^{l,l}_{l-k-1,l-k} & \mathbf{B}^{l,l}_{l-k-1,l-k+1} & \dots & \dots & \mathbf{B}^{l,l}_{l-k-1,l-1} & \mathbf{B}^{l,l}_{l-k-1,l} \\
\mathbf{B}^{l,l}_{l-k,0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{B}^{l,l}_{l-k,l-k-1} & \mathbf{B}^{l,l}_{l-k,l-k} & \mathbf{B}^{l,l}_{l-k,l-k+1} & \dots & \dots & \mathbf{B}^{l,l}_{l-k,l-1} & \mathbf{B}^{l,l}_{l-k,l} \\
\mathbf{B}^{l,l}_{l-k+1,0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{B}^{l,l}_{l-k+1,l-k} & \mathbf{B}^{l,l}_{l-k+1,l-k+1} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & & & \vdots & & \ddots & \ddots & & \vdots & \vdots \\
\mathbf{B}^{l,l}_{l-1,0} & \mathbf{0} & \mathbf{0} & & & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{B}^{l,l}_{l-1,l-2} & \mathbf{B}^{l,l}_{l-1,l-1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{0} & \mathbf{B}^{l,l}_{l,l-1} & \mathbf{B}^{l,l}_{l,l}
\end{pmatrix}.$$

For $l = k, \dots , n-1$; $\mathbf{R}^{l,n} = \begin{pmatrix}
\mathbf{B}^{l,n}_{00} & \mathbf{0} & & & & \mathbf{0} \\
\vdots & \mathbf{0} & & & & \mathbf{0} \\
\mathbf{B}^{l,n}_{\min\{l-k,k-1\},0} & \mathbf{0} & & & & \mathbf{0} \\
\mathbf{0} & \vdots & \ddots & \ddots & \ddots & \vdots \\
 & \vdots & \ddots & \ddots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}
\end{pmatrix}_{l \times n}$ and

for $l = k+1, \dots , n$ and $h = l-1, ... , k$

$$\mathbf{R}^{l,h} = \begin{pmatrix} \mathbf{B}_{00}^{l,h} & \mathbf{B}_{01}^{l,h} & \mathbf{B}_{02}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{0h-2}^{l,h} & \mathbf{B}_{0h-1}^{l,h} & \mathbf{B}_{0h}^{l,h} \\ \mathbf{B}_{10}^{l,h} & \mathbf{B}_{11}^{l,h} & \mathbf{B}_{12}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{1h-2}^{l,h} & \mathbf{B}_{1h-1}^{l,h} & \mathbf{B}_{1h}^{l,h} \\ \mathbf{0} & \mathbf{B}_{21}^{l,h} & \mathbf{B}_{22}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{2h-2}^{l,h} & \mathbf{B}_{2h-1}^{l,h} & \mathbf{B}_{2h}^{l,h} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{32}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{3h-2}^{l,h} & \mathbf{B}_{3h-1}^{l,h} & \mathbf{B}_{3h}^{l,h} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_{\min\{h,l-k\},\min\{h,l-k\}-1}^{l,h} & \cdots & \mathbf{B}_{\min\{h,l-k\},h-1}^{l,h} & \mathbf{B}_{\min\{h,l-k\},h}^{l,h} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & & & & & \ddots & & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}_{l+1 \text{ x } h+1}.$$

The matrix $\mathbf{B}_{ij}^{l,h}$ for $1 \le i \le h-2$ and $i+1 \le j \le h-1$ is described in detail, the rest matrices are given in *Appendix 5B.*

Matrix $\mathbf{B}_{ij}^{l,h}$

We assume that there are l units in the system ($l = k + 1, \ldots, n$) and that i of them are in the repair facility ($1 \le i \le h-2$). We also assume that the number of the units in the repair facility is less than or equal to $l-k$. Thus, the system is operational and failures may occur. The probability matrix of having h units in the system ($h = l-1, \ldots, k$) with j of them in the repair facility ($i+1 \le j \le h-1$) is given by the following two possibilities.

a. $l-h$ non-repairable and $j-i$ repairable failures occur. The unit in the repair facility is not repaired.

$$b(l,i,l-h,j-i) \otimes \mathbf{S}.$$

b. $l-h$ non-repairable and $j-i+1$ repairable failures occur. The unit in the repair facility is repaired and the repair of another one begins.

$$b(l,i,l-h,j-i+1) \otimes \mathbf{S}^0\boldsymbol{\beta}.$$

Thus, the matrix block is given by

$$\mathbf{B}_{ij}^{l,h} = I_{\{i \le l-k\}} \left\{ b(l,i,l-h,j-i) \otimes \mathbf{S} + b(l,i,l-h,j-i+1) \otimes \mathbf{S}^0\boldsymbol{\beta} \right\}.$$

5.4 Transient distribution

The transition probability at *m* steps can be calculated from the transition probability matrix described above. This is obtained by considering the matrix blocks associated with the macro-states U^l, when there are *l* units in the system. If the transition probability matrix from macro-state U^i to U^j is denoted by $\mathbf{p}^{(m)}_{U^iU^j}$ for $i, j = k,...,n$ then in a recursive way we can obtain

$$\mathbf{p}^{(m)}_{U_iU_j} = \sum_{l=j\cdot I_{\{j\neq n\}}+k\cdot I_{\{j=n\}}}^{n} \mathbf{p}^{(m-1)}_{U_iU_l}\mathbf{R}^{lj}\,.$$

If the system is initially composed of *n* new units, then the transient distribution is given by $\mathbf{p}^{(m)} = (1,0,\ldots,0)\sum_{j=k}^{n}\mathbf{p}^{(m)}_{U^nU^j}$. $\mathbf{p}^{(m)}_{U^l}$ then denotes the values corresponding to the macro-state derived from $\mathbf{p}^{(m)}$. In an analogous way, $\mathbf{p}^{(m)}_{U^l_a}$ denotes the corresponding value for the macro-state *l* units in the system, when *a* of them are in the repair facility.

5.5 The long-run distribution

The long-run distribution, $\boldsymbol{\pi}$, is obtained by matrix-analytic methods. This distribution has been calculated for the macro-states U^l, when there are *l* units in the system. The stationary probability of being in this macro-state is denoted by $\boldsymbol{\pi}^l$, thus $\boldsymbol{\pi} = \left(\boldsymbol{\pi}^n, \boldsymbol{\pi}^{n-1},\ldots,\boldsymbol{\pi}^k\right)$. This stationary distribution verifies $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$ where the stationary distribution verifies the normalization condition. These equations can be expressed as

$$\boldsymbol{\pi}^n = \sum_{s=k}^{n}\boldsymbol{\pi}^s\mathbf{R}^{s,n}\,,$$

$$\boldsymbol{\pi}^l = \sum_{s=l}^{n}\boldsymbol{\pi}^s\mathbf{R}^{s,l}\,;\; l = k, ..., n-1,$$

$$\sum_{s=k}^{n}\boldsymbol{\pi}^s\mathbf{e} = 1\,.$$

The solution to this system is given by

$$\boldsymbol{\pi}^l = \boldsymbol{\pi}^n\mathbf{R}^l \quad;\quad l = k,..., n-1$$

where

$$\mathbf{R}^{l} = \left(\mathbf{R}^{n,l} + I_{\{l \le n-2\}} \sum_{s=l+1}^{n-1} \mathbf{R}^{s} \mathbf{R}^{s,l} \right) \left(\mathbf{I} - \mathbf{R}^{l,l} \right)^{-1} \quad ; \quad l = k,\ldots,n-1.$$

The vector $\boldsymbol{\pi}^n$ is obtained from the first balance equation and the normalization condition. This is equal to

$$\boldsymbol{\pi}^{n} = (1, \mathbf{0}) \left[\left(\mathbf{e} + \sum_{s=k}^{n-1} \mathbf{R}^{s} \mathbf{e} \right) \middle| \left(\mathbf{I} - \sum_{s=k}^{n-1} \mathbf{R}^{s} \mathbf{R}^{s,n} - \mathbf{R}^{n,n} \right)^{*} \right]^{-1},$$

where the matrix $\mathbf{A}^*$ is a matrix $\mathbf{A}$ without the first column.

$\boldsymbol{\pi}_a^l$ denotes the corresponding stationary values for the macro-state of l units in the system of which a are in the repair facility.

5.6 Measures

Several measures associated with the system have been applied, such as the availability, reliability and the conditional probability of failure.

5.6.1 Availability

The probability that at a certain time ν the system is operational is named availability. In the stationary case, this measure is given by

$$A(\nu) = \sum_{l=k}^{n} \sum_{a=0}^{l-k} \mathbf{p}_{U_a^l}^{(\nu)} \mathbf{e} \ .$$

This measure in the stationary case is given by $A = \sum_{l=k}^{n} \sum_{a=0}^{l-k} \boldsymbol{\pi}_a^l \mathbf{e}$.

5.6.2 Reliability

The first time that the system is not operational is PH distributed, denoted by $\left((1, \mathbf{0}, \ldots, \mathbf{0}), \mathbf{P}^*\right)$, where $\mathbf{P}^*$ is the matrix $\mathbf{P}$ restricted to the macro-state $\bigcup_{l=k}^{n} \bigcup_{a=0}^{l-k} U_a^l$. The reliability function (defined as the probability that at time ν the system will be operational before any failure has occurred) is given by

$$R(\nu) = (1, 0\ldots, 0) \left(\mathbf{I} - \mathbf{P}^* \right)^{-1} \mathbf{P}^{*\nu} \mathbf{P}^{*0}.$$

5.6.3 Conditional probability of failure

Two different conditional probabilities of failure are defined; that of repairable failure and that of non-repairable failure. Both are defined in a transient and also in a stationary regime.

5.6.3.1 Conditional probability of *r* repairable failures

If the system is working at time $\nu-1$, the probability that at the next time the system will have undergone only *r* repairable failures is given by

$$CPF_r(\nu)=\sum_{l=k}^{n}\left[\mathbf{p}_{U_0^l}^{(\nu-1)}\cdot b(l,k,a=0,r,0)\cdot\mathbf{e}_{m^l}+\sum_{a=1}^{l-r}\left[\mathbf{p}_{U_a^l}^{(\nu-1)}\cdot b(l,k,a,r,0)\otimes\mathbf{e}_t\right]\cdot\mathbf{e}_{t\cdot m^{l-a}}\right].$$

5.6.3.2 Conditional probability of *nr* non-repairable failures

If the system is working at time $\nu-1$, the probability that at the next time the system will have undergone only *nr* non-repairable failures is given by

$$CPF_r(\nu)=\sum_{l=k}^{n}\left[\mathbf{p}_{U_0^l}^{(\nu-1)}\cdot b(k,l,a=0,0,nr)\cdot\mathbf{e}_{m^l}+\sum_{a=1}^{l-r}\left[\mathbf{p}_{U_a^l}^{(\nu-1)}\cdot b(l,k,a,0,nr)\otimes\mathbf{e}_t\right]\cdot\mathbf{e}_{t\cdot m^{l-a}}\right].$$

5.7 Markov counting process to calculate the mean number of new systems

In this section, a Markov counting process is developed to calculate the mean number of new systems up to a certain time. To achieve this objective, the transition probability matrix from a state to a new system is determined. This matrix is given by

$$\mathbf{D}_{ns}=\begin{pmatrix}\mathbf{D}^{n,n} & \mathbf{0} & & \mathbf{0}\\ \mathbf{D}^{n-1,n} & \mathbf{0} & & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{D}^{k,n} & \mathbf{0} & & \mathbf{0}\end{pmatrix},$$

where for $l = k,\ldots, n$

$$\mathbf{D}^{l,n} = \begin{pmatrix} \mathbf{D}_{00}^{l,n} & \mathbf{0} & \mathbf{0} \\ \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{D}_{\min\{l-k,k-1\},0}^{l,n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & & \\ \vdots & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}_{l \times n},$$

with $\mathbf{D}_{00}^{l,n} = d(k,l,0) \otimes \boldsymbol{\alpha} \otimes \overset{n}{\cdots} \otimes \boldsymbol{\alpha}$ and for i =1,…, min{l–k, k–1}

$$\mathbf{D}_{i0}^{l,n} = d(k,l,i) \otimes \boldsymbol{\alpha} \otimes \overset{n}{\cdots} \otimes \boldsymbol{\alpha} \otimes \mathbf{e}.$$

Then, the expected number of new systems up to a certain time ν is given by

$$\phi(\nu) = (1,0,...,0) \sum_{i=0}^{\nu-1} \mathbf{P}^i \mathbf{D}\mathbf{e} \ . \tag{5.1}$$

In the stationary case it is given by $\phi = \boldsymbol{\pi}\mathbf{D}\mathbf{e}$, the number of new systems per unit of time.

5.8 Numerical example

Let us assume a 2-out-of-4: G system where the time of each operational unit is PH distributed with representation (α, $\mathbf{T}$) and where

$$\boldsymbol{\alpha} = (1,0,0) \text{ and } \mathbf{T} = \begin{pmatrix} 0.98 & 0.01 & 0.002 \\ 0 & 0.98 & 0.01 \\ 0 & 0 & 0.99 \end{pmatrix}.$$

Each device is composed of three different degradation levels. Each operational unit may undergo a repairable or a non-repairable failure. These failures can occur from states 1, 2 or 3, respectively, in accordance with the following column probability vectors,

$$\mathbf{T}_r^0 = \begin{pmatrix} 0.005 \\ 0.0075 \\ 0.0075 \end{pmatrix}, \mathbf{T}_{nr}^0 = \begin{pmatrix} 0.003 \\ 0.0025 \\ 0.0025 \end{pmatrix}.$$

Then, for instance, a repairable failure from degradation state 1 occurs with a probability equal to 0.005 and a non-repairable failure with a probability of 0.003. The mean time until a failure occurs is 110 units of time (u.t.), with mean time to repairable failure of 114.2857 u.t. and mean time to non-repairable failure of 100 u.t.

When a repairable failure occurs, the unit is transferred to the repair facility. The repair time pass through two different phases and it is PH distributed with representation ($\boldsymbol{\beta}$, $\mathbf{S}$) where

$$\boldsymbol{\beta} = (1,0)\ ,\ \mathbf{S} = \begin{pmatrix} 0.2 & 0.4 \\ 0.1 & 0.5 \end{pmatrix}.$$

The mean repair time is equal to 2.5 u.t. various measures can be applied. The macro-state of the system is composed of three macro-states $E = \{U^4, U^3, U^2\}$ where U^l denotes the phases when there are l units in the system. These macro-states are composed of macro-states U_s^l, s of them are in the repair facility. For instance, the macro-state U_2^4 is composed of 3x3x2 phases (two operational units, 3x3 possible phases, one unit being repaired, 2 possible phases and one unit in queue). For instance, the phase (1,3,2) of this macro-state indicates that the first operational unit is in degradation state 1, the second in degradation state 3 and the unit that is being repaired is in phase 2 of repairing. The macro-states U^4, U^3 and U^2 are composed of 81+54+18+6+2, 27+18+6+2 and 9+6+2 phases respectively. Thus, in a stationary regime, the ratio in each macro-state is given by the stationary distribution, as shown in Table 5.1.

	Number of units in the system		
Number of units in the repair facility	**4**	**3**	**2**
0	0.2109	0.2913	0.4554
1	0.0125	0.0141	0.0149
2	0.0005	0.0004	0.0000
3	0.0000	0.0000	
4	0.0000		

Table 5.1. Proportional time in each macro-state

The availability function is shown in Figure 5.1.

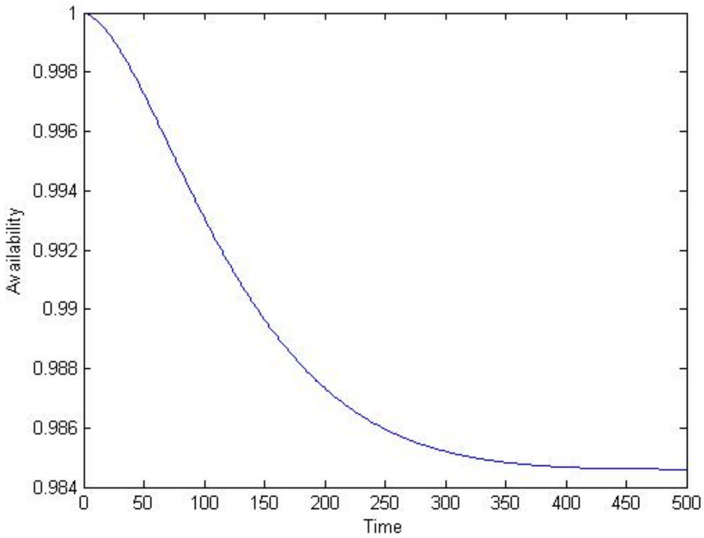

Figure 5.1. Availability of the system

The stationary availability is equal to 0.9846. This value is the operational time ratio and can be derived from Table 5.1. The probability of the system being operational at a certain time before the first system failure is given in Figure 5.2.

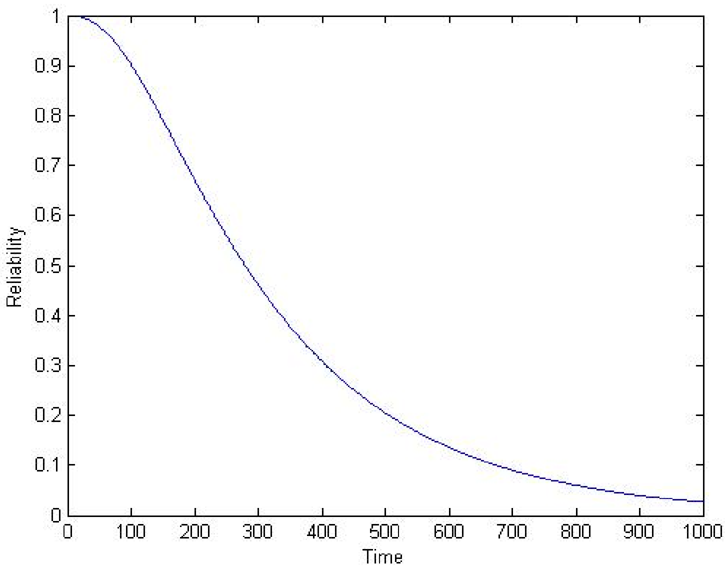

Figure 5.2. Reliability function

The mean time up to the first failure of the system is equal to 346.0609 u.t. Finally, Table 5.2 shows the mean number of new systems calculated for different times, from (5.1).

	Mean number of new systems									
ν	100	200	300	400	500	600	700	800	900	1000
$\phi(\nu)$	0.0468	0.2071	0.4286	0.6702	0.9164	1.1629	1.4091	1.6552	1.9012	2.1472

Table 5.2. Mean number of new system up to a certain time

The stationary value, proportional to the number of new systems per unit of time is 0.0025.

5.9 Conclusions

This chapter describes a multi-state complex k-out-of-n: G system, modelled in an algorithmic and computational form. The system is operational when at least k units are operational. Both repairable and non-repairable failures are included in the system. When a non-repairable failure occurs, the unit is removed. When the number of units in the system is less than or equal to n–k–1, it is considered to be non-operational and is replaced by a new one. At present, the system is modelled only for the case of internal failures, but more complex systems could also be modelled by the same approach.

The transient distribution and the long-run case have also been computed, and several interesting reliability measures obtained, for both transient and stationary regimes. The mean number of new systems per unit of time is calculated by means of a Markov counting process. A numerical example is given to show the versatility of the model.

The system modelled in this chapter can be extended by considering multiple types of failure with external shocks and random inspection followed by preventive maintenance. Also, this system can be generalized to the case of multiple and variable repairpersons where after each non-repairable failure the unit is removed.

5.10 Appendices

5.10.1 Appendix 5A

1. Transition matrix for the operational units when the system is composed of l units, a of them in the repair facility, l–h non-repairable failures occur and u repairable

failures take place when the failure order is determined ($s_1,\ldots,s_u$ the ordinal of the repairable failures and $k_1,\ldots,k_w$ the ordinal for the non-repairable failures).

For $h < l$ and $u > 0$,

$$C\left(l,a,w,u;k_1,\ldots,k_w;s_1,\ldots,s_u\right)=\mathbf{T}(1)\otimes\cdots\otimes\mathbf{T}(l-a)\text{, where}$$

$$\mathbf{T}(v)=\begin{cases}\mathbf{T}_r^0 & ;\quad v=s_1,\ldots,s_u\\ \mathbf{T}_{nr}^0 & ;\quad v=k_1,\ldots,k_w\\ \mathbf{T} & ;\quad \text{otherwise.}\end{cases}$$

This function is denoted as follows for the following cases,

- For $w = 0$ and $u > 0$, $C\left(l,a,w=0,u;s_1,\ldots,s_u\right)=\mathbf{T}(1)\otimes\cdots\otimes\mathbf{T}(l-a)$
- For $u = 0$ and $w > 0$, $C\left(l,a,w,u=0;k_1,\ldots,k_{l-h}\right)=\mathbf{T}(1)\otimes\cdots\otimes\mathbf{T}(l-a)$
- For $w = 0$ and $u = 0$, $C\left(l,a,w=0,u=0\right)=\mathbf{T}\overset{l-a}{\otimes\cdots\otimes}\mathbf{T}$
- For $w = l–a$, $C\left(l,a,w=l-a,u=0\right)=\mathbf{T}_{nr}^0\overset{l-a}{\otimes\cdots\otimes}\mathbf{T}_{nr}^0$
- For $u = l–a$, $C\left(l,a,w=0,u=l-a\right)=\mathbf{T}_r^0\overset{l-a}{\otimes\cdots\otimes}\mathbf{T}_r^0$.

2. Transition matrix for the operational units when the system is composed of *l* units, *a* of them are in the repair facility, *w* non-repairable failures occur and *u* repairable failures have place where the failure order is not determined.

   For $w > 0$ and $u > 0$,

$$b\left(l,a,w,u\right)=\sum_{k_1=1}^{l-a-w+1}\sum_{k_2=k_1+1}^{l-a-w+2}\cdots\sum_{k_w=k_{w-1}+1}^{l-a}\sum_{\substack{s_1=1\\ s_1\neq k_v\\ v=1,\ldots,w}}^{l-a-u+1}\sum_{\substack{s_2=s_1+1\\ s_2\neq k_v\\ v=1,\ldots,w}}^{l-a-u+2}\cdots\sum_{\substack{s_u=s_{u-1}+1\\ s_u\neq k_v\\ v=1,\ldots,w}}^{l-a}C\left(l,a,w,u;k_1,\ldots,k_w;s_1,\ldots,s_u\right)$$

According to the specific cases described above we have the following notation.

- For $w = 0$ and $l-a\neq u>0$,

$$b\left(l,a,w=0,u\right)=\sum_{s_1=1}^{l-a-u+1}\sum_{s_2=s_1+1}^{l-a-u+2}\cdots\sum_{s_u=s_{u-1}+1}^{l-a}C\left(l,a,0,u;s_1,\ldots,s_u\right)$$

- For $u = 0$ and $l-a\neq w>0$,

$$b\left(l,a,w,u=0\right)=\sum_{k_1=1}^{l-a-w+1}\sum_{k_2=k_1+1}^{l-a-w+2}\cdots\sum_{k_w=k_{w-1}+1}^{l-a}C\left(l,a,w,0;k_1,\ldots,k_w\right)$$

- For w = 0 and u = 0, $b\left(l,a,w=0,u=0\right)=c\left(l,a,0,0\right)=\mathbf{T}\otimes\overset{l-a}{\cdots}\otimes\mathbf{T}$ and if a = l $b\left(l,l,w=0,u=0\right)=1$
- For $w = l–a$, $b\left(l,a,w=l-a,u=0\right)=C\left(l,a,l-a,0\right)=\mathbf{T}_{nr}^{0}\otimes\overset{l-a}{\cdots}\otimes\mathbf{T}_{nr}^{0}$
- For $u = l–a$, $b\left(l,a,w=0,u=l-a\right)=C\left(l,a,0,l-a\right)=\mathbf{T}_{r}^{0}\otimes\overset{l-a}{\cdots}\otimes\mathbf{T}_{r}^{0}$.

3. If there are l units is the system, a of them in the repair facility, and the number of non-repairable failures is greater or equal than $l–k$+1, then the system has to be replaced. The probability of that it occurs by considering the phases of the system is

$$d\left(k,l,a\right)=\sum_{w=l-k+1}^{l-a} d\left(l,a,w\right) ,$$

where

$$d\left(l,a,w\right)=\sum_{k_1=1}^{l-a-w+1}\sum_{k_2=k_1+1}^{l-a-w+2}\cdots\sum_{k_w=k_{w-1}+1}^{l-a} D\left(l,a,w;k_1,\ldots,k_w\right)$$

and

$$D\left(l,a,w;k_1,\ldots,k_w\right)=\mathbf{T}\left(1\right)\otimes\cdots\otimes\mathbf{T}\left(l-a\right) \text{ with } \mathbf{T}\left(v\right)=\begin{cases}\mathbf{T}_{nr}^{0} & ; \quad v=k_1,\ldots,k_w\\ \mathbf{e}-\mathbf{T}_{nr}^{0} & ; \quad \text{otherwise.}\end{cases}$$

5.10.2 Appendix 5B

The transition probability matrix, **P**, is composed by the following block matrices.

For $l = k, \ldots, n$

$\mathbf{R}^{l,l}$ =

$$\begin{pmatrix}
\mathbf{B}_{00}^{l,l} & \mathbf{B}_{01}^{l,l} & \mathbf{B}_{02}^{l,l} & \cdots & \mathbf{B}_{0,l-k-2}^{l,l} & \mathbf{B}_{0,l-k-1}^{l,l} & \mathbf{B}_{0,l-k}^{l,l} & \mathbf{B}_{0,l-k+1}^{l,l} & \cdots & \cdots & \mathbf{B}_{0,l-1}^{l,l} & \mathbf{B}_{0,l}^{l,l}\\
\mathbf{B}_{1,0}^{l,l} & \mathbf{B}_{1,1}^{l,l} & \mathbf{B}_{1,2}^{l,l} & \cdots & \mathbf{B}_{1,l-k-2}^{l,l} & \mathbf{B}_{1,l-k-1}^{l,l} & \mathbf{B}_{1,l-k}^{l,l} & \mathbf{B}_{1,l-k+1}^{l,l} & \cdots & \cdots & \mathbf{B}_{1,l-1}^{l,l} & \mathbf{B}_{1,l}^{l,l}\\
 & \mathbf{B}_{2,1}^{l,l} & \mathbf{B}_{2,2}^{l,l} & \cdots & \mathbf{B}_{2,1l-k-2}^{l,l} & \mathbf{B}_{2,l-k-1}^{l,l} & \mathbf{B}_{2,l-k}^{l,l} & \mathbf{B}_{2,l-k+1}^{l,l} & \cdots & \cdots & \mathbf{B}_{2,l-1}^{l,l} & \mathbf{B}_{2,l}^{l,l}\\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots\\
 & & & \ddots & \mathbf{B}_{l-k-1,l-k-2}^{l,l} & \mathbf{B}_{l-k-1,l-k-1}^{l,l} & \mathbf{B}_{l-k-1,l-k}^{l,l} & \mathbf{B}_{l-k-1,l-k+1}^{l,l} & \cdots & \cdots & \mathbf{B}_{l-k-1,l-1}^{l,l} & \mathbf{B}_{l-k-1,l}^{l,l}\\
\mathbf{B}_{l-k,0}^{l,l} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_{l-k,l-k-1}^{l,l} & \mathbf{B}_{l-k,l-k}^{l,l} & \mathbf{B}_{l-k,l-k+1}^{l,l} & \cdots & \cdots & \mathbf{B}_{l-k,l-1}^{l,l} & \mathbf{B}_{l-k,l}^{l,l}\\
\mathbf{B}_{l-k+1,0}^{l,l} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{B}_{l-k+1,l-k}^{l,l} & \mathbf{B}_{l-k+1,l-k+1}^{l,l} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\
\vdots & \vdots & \vdots & & & \vdots & & \ddots & \ddots & & \vdots & \vdots\\
\mathbf{B}_{l-1,0}^{l,l} & \mathbf{0} & \mathbf{0} & & & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{B}_{l-1,l-2}^{l,l} & \mathbf{B}_{l-1,l-1}^{l,l} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{B}_{l,l-1}^{l,l} & \mathbf{B}_{l,l}^{l,l}
\end{pmatrix}$$

$$\mathbf{B}_{00}^{l,l}=b\left(l,0,0,0\right)+I_{\{l=n\}}d\left(k,l,0\right)\otimes\boldsymbol{\alpha}\otimes\overset{n}{\cdots}\otimes\boldsymbol{\alpha} .$$

For $1 \le j \le l$; $\mathbf{B}_{0j}^{l,l} = b(l,0,0,j) \otimes \boldsymbol{\beta}$,

$$\mathbf{B}_{10}^{l,l} = I_{\{1 \le l-k\}}\left[b(l,a=1,0,0) \otimes \boldsymbol{\alpha} \otimes \mathbf{S}^0 + I_{\{l=n,a \le k-1\}} d(k,l,1) \otimes \boldsymbol{\alpha} \otimes \overset{n}{\cdots} \otimes \boldsymbol{\alpha} \otimes \mathbf{e} \right]$$
$$+ I_{\{1>l-k\}}\left[\mathbf{I} \otimes \overset{l-1}{\ldots} \otimes \mathbf{I} \otimes \boldsymbol{\alpha} \otimes \mathbf{S}^0 \right].$$

For $2 \le i \le n-1$; $\mathbf{B}_{i0}^{l,l} = I_{\{i \le l-k; l=n; i \le k-1\}} d(k,l,i) \otimes \boldsymbol{\alpha} \otimes \overset{n}{\cdots} \otimes \boldsymbol{\alpha} \otimes \mathbf{e}$.

For $1 \le i \le l-1$; $\mathbf{B}_{ii}^{l,l} = I_{\{i \le l-k\}}\left[b(l,i,0,0) \otimes \mathbf{S} + b(l,i,0,1) \otimes \boldsymbol{\alpha} \otimes \mathbf{S}^0 \boldsymbol{\beta} \right] + I_{\{i>l-k\}} \mathbf{I} \otimes \overset{l-i}{\ldots} \otimes \mathbf{I} \otimes \mathbf{S}$,

$\mathbf{B}_{ll}^{l,l} = \mathbf{S}$.

For $2 \le i \le l$; $\mathbf{B}_{i,i-1}^{l,l} = \left[I_{\{i \le l-k\}} b(l,i,0,0) + I_{\{i>l-k\}} \left\{ I_{\{i \ne l\}} \left(\mathbf{I} \otimes \overset{l-i}{\ldots} \otimes \mathbf{I} \right) + I_{\{i=l\}} \right\} \right] \otimes \boldsymbol{\alpha} \otimes \mathbf{S}^0 \boldsymbol{\beta}$.

For $1 \le i \le l-2$; $i+1 \le j \le l-1$;

$\mathbf{B}_{ij}^{l,l} = I_{\{i \le l-k\}}\left[b(l,i,0,j-i) \otimes \mathbf{S} + b(l,i,0,j-i+1) \otimes \boldsymbol{\alpha} \otimes \mathbf{S}^0 \boldsymbol{\beta} \right]$,

and for $1 \le i \le l-1$; $\mathbf{B}_{i,l}^{l,l} = I_{\{i \le l-k\}} b(l,i,0,l-i) \otimes \mathbf{S}$

For $l = k, \ldots, n-1$; $\mathbf{R}^{l,n} = \begin{pmatrix} \mathbf{B}_{00}^{l,n} & \mathbf{0} & \mathbf{0} \\ \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{\min\{l-k,k-1\},0}^{l,n} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}_{l \times n}$

$\mathbf{B}_{00}^{l,n} = d(k,l,0) \otimes \boldsymbol{\alpha} \otimes \overset{n}{\cdots} \otimes \boldsymbol{\alpha}$.

For $i = 1, \ldots, l-1$; $\mathbf{B}_{i0}^{l,n} = I_{\{i \le k-1\}} d(k,l,i) \otimes \boldsymbol{\alpha} \otimes \overset{n}{\cdots} \otimes \boldsymbol{\alpha} \otimes \mathbf{e}$.

For $l = k+1, \ldots, n$ and $h = l-1, \ldots, k$

$$\mathbf{R}^{l,h} = \begin{pmatrix}
\mathbf{B}_{00}^{l,h} & \mathbf{B}_{01}^{l,h} & \mathbf{B}_{02}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{0h-2}^{l,h} & \mathbf{B}_{0h-1}^{l,h} & \mathbf{B}_{0h}^{l,h} \\
\mathbf{B}_{10}^{l,h} & \mathbf{B}_{11}^{l,h} & \mathbf{B}_{12}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{1h-2}^{l,h} & \mathbf{B}_{1h-1}^{l,h} & \mathbf{B}_{1h}^{l,h} \\
\mathbf{0} & \mathbf{B}_{21}^{l,h} & \mathbf{B}_{22}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{2h-2}^{l,h} & \mathbf{B}_{2h-1}^{l,h} & \mathbf{B}_{2h}^{l,h} \\
\mathbf{0} & \mathbf{0} & \mathbf{B}_{32}^{l,h} & \cdots & \cdots & \cdots & \mathbf{B}_{3h-2}^{l,h} & \mathbf{B}_{3h-1}^{l,h} & \mathbf{B}_{3h}^{l,h} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_{\min\{h,l-k\},\min\{h,l-k\}-1}^{l,h} & \cdots & \mathbf{B}_{\min\{h,l-k\},h-1}^{l,h} & \mathbf{B}_{\min\{h,l-k\},h}^{l,h} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix}_{l+1 \text{ x } h+1}$$

$\mathbf{B}_{00}^{l,h} = b(l,0,l-h,0)$.

For $1 \le i \le h-1$; $\mathbf{B}_{ii}^{l,h} = b(l,i,l-h,0) \otimes \mathbf{S} + b(l,i,l-h,1) \otimes \mathbf{S}^0 \boldsymbol{\beta}$.

$\mathbf{B}_{hh}^{l,h} = b(l,h,l-h,0) \otimes \mathbf{S}$.

$\mathbf{B}_{10}^{l,h} = b(l,1,l-h,0) \otimes \mathbf{S}^0$.

For $2 \le i \le h$; $\mathbf{B}_{i,i-1}^{l,h} = b(l,i,l-h,0) \otimes \mathbf{S}^0 \boldsymbol{\beta}$.

For $1 \le j \le h$; $\mathbf{B}_{0j}^{l,h} = b(l,0,l-h,j) \otimes \boldsymbol{\beta}$.

For $1 \le i \le h-2$; $i+1 \le j \le h-1$; $\mathbf{B}_{ij}^{l,h} = b(l,i,l-h,j-i) \otimes \mathbf{S} + b(l,i,l-h,j-i+1) \otimes \mathbf{S}^0 \boldsymbol{\beta}$.

For $1 \le i \le h-1$; $\mathbf{B}_{i,h}^{l,h} = b(l,i,l-h,h-i) \otimes \mathbf{S}$.

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