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Personalized individual semantics in computing with words for supporting linguistic group decision making

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Personalized individual semantics in computing with words for supporting linguistic group decision making

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1 Introduction

Decision making is a process that plays an important role in our daily lives. The decisions whether they are important or not, are influencing our lives. In some decision making problems, numerical values are used to express experts' preferences. However, due to the complexity and uncertainty of decision making environment, some problems cannot be dealt with by precise and exact numbers, and the experts may prefer to use some linguistic information to express their judgments, such as, good, bad, or medium. Thus, the use of linguistic approaches is necessary in decision making with linguistic information. The use of linguistic information often implies the use of computing with words (CW). CW is a methodology in which the objects of computation are words and propositions drawn from a natural language [Zad75b, Zad75c, Zad75a].

For linguistic approaches are as quantitative as any standard number-crunching method, the successful use of linguistic terms is highly dependent on the determination of a valid membership function [Dub11]. It is an important issue on the choice of the monotonic mapping to encode linguistic information with numerical values in CW.

In recent years, some functions and computational models to represent the linguistic information with numerical meaning and to deal with linguistic inputs in CW process are proposed. The 2-tuple linguistic model [HM00] is well suited to deal with linguistic terms that are uniformly and symmetrically distributed without loss of information, where the 2-tuples are composed of a linguistic term and a numeric value assessed in (-0.5, 0.5). Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set, the mapping function of the model is between 2-tuple and a numerical value lying between 0 and g. In some decision making situations, the linguistic terms that are not uniformly and symmetrically distributed are often used to express the preference. To deal with this kind of linguistic terms, some extensions of 2-tuple linguistic models are proposed,

(1) Proportional 2-tuple linguistic model [WH06]. It provides a computational model to deal with the proportional 2-tuples, which is composed by two linguistic terms with the symbolic proportions. This model provides the transformations between proportional 2-tuple and the numerical value between 0 and 1.

(2) The model based on a linguistic hierarchy [HHVM08]. This model presents a methodology to deal with the unbalanced linguistic terms with the linguistic hierarchy structure. For presenting the numerical meaning of linguistic information, each linguistic value has a corresponding

numerical value between 0 and g in an unbalanced way.

(3) Numerical scale model [DXY09]. It defines a function that establishes a one to one mapping between the linguistic information and numerical scales.

The above models have greatly contributed to the research and development of CW in dealing with linguistic information with its corresponding numerical values. However, in CW, there is a fact that words mean different things for different people, that is, in decision making there should be different numerical meanings of words for decision makers, while the above models do not study this issue. In order to deal with the previous fact, two models are introduced:

- The use of type-2 fuzzy sets based on lower and upper possibility distributions with a third dimension nature [MW10], which groups all meanings from people in just one representation function. It allows handling linguistic uncertaines with upper and lower membership functions.

- The multi-granular linguistic term set [HM01] was proposed for experts to choose different granularity linguistic term sets to express their preferences, because it takes into account the different backgrounds and knowledge of experts, so that the use of linguistic term set in expressing preferences is not same.

Although type-2 fuzzy sets and multi-granular can reflect the different understanding of words, they cannot represent the specific semantics or meaning of words. i.e., for each expert the exact meaning for each word should be different. For example, when reviewing an article, two referees may think the reviewed article is interesting, but the term interesting often has different numerical meaning for both referees.

In this thesis, we focus on the idea that words have different meanings for different people, we call it personalized individual semantics (PISs). To show the PIS, we study some decision making models and approaches with linguistic preference relations to compute the personalized numerical scales for linguistic terms. Under different decision making contexts, the constructed models and the designed computation processes are different. We consider three decision making contexts to study the PIS of each expert, i.e., classical decision making, hesitant linguistic decision making and large-scale group decision making (LSGDM). The characteristics of the three decision making environments are as follows,

(1) Classical decision making with linguistic information involves a group of experts, who express their preference using single linguistic terms, and make a decision choice among a set of alternatives.

(2) Considering that in some situations the use of single linguistic term is not enough to represent experts' knowledge and opinions, experts prefer to use several linguistic terms in expressing their preference. Under this context, the hesitant linguistic decision making is used to deal with experts' hesitations.

(3) In LSGDM, there is a large number of decision makers participated in decision making problems. The LSGDM is more complex than the traditional group decision making (GDM), because of the relatively large group size and the complexity of the decision making problems as well as the decision makers themselves, who have different knowledge and backgrounds.

In constructing the PIS models in decision making problems, the consistency of the provided preferences by experts plays an important role [ACH+08, DHV15]. The consistency ensures that the decision maker is being neither random nor illogical in their pairwise comparisons, and the unacceptable consistency may lead to inconsistent and unreliable results. In this thesis, the consistency status of linguistic expressions provided by decision makers is used as a basic idea in building the PIS models. We provide a natural premise to construct the connection of consistency between linguistic preference relations and numerical preference relations. Besides, to obtain the personalized numerical scales of linguistic terms, throughout this thesis, we utilize the numerical scale model, which established a one to one mapping between linguistic information and numerical values and has some desired properties in connecting it with 2-tuple linguistic models.

This thesis consists of two main parts: the first one illustrates the statement of the problems addressed and the results obtained from the proposed models. The second part is a compilation of the main publications that are associated with this thesis.

This thesis is structured as: Section 2 provides some related preliminaries used throughout this contribution. In Section 3, the basic ideas and the challenges that justify the development of this thesis are discussed. Section 4 proposes the objective of this thesis. Section 5 presents the methodology used in the thesis. A summary of the consistency-driven models proposed to show PIS is made in Section 6. Section 7 presents a discussion of the results obtained in the thesis. Finally, Section 8 draws the conclusion of this thesis and in Section 9 the future works are discussed.

Introducción

La toma de decisiones es un proceso que juega un papel importante en nuestra vida diaria. Las decisiones, ya sean importantes o no, están influyendo en nuestras vidas. En algunos problemas de toma de decisiones, los valores numéricos se utilizan para expresar las preferencias de los expertos. Sin embargo, debido a la complejidad e incertidumbre del entorno de toma de decisiones, algunos problemas no se pueden resolver con números precisos y exactos, y los expertos pueden preferir utilizar algún tipo de información lingüística para expresar sus juicios, como pueden ser los términos bueno, malo o medio. Por lo tanto, el uso de enfoques lingüísticos es necesario en la toma de decisiones. El uso de información lingüística a menudo implica el uso de la computación con palabras (CW). CW es una metodología en la que los objetos de cómputo son palabras y proposiciones extraídas de un lenguaje natural [Zad75b, Zad75c, Zad75a].

Para que los enfoques lingüísticos sean tan cuantitativos como cualquier método estándar de cálculo numérico, el uso exitoso de los términos lingüísticos depende en gran medida de la determinación de una función de pertenecia válida [Dub11]. Es una cuestión importante en la elección del mapeo monotónico codificar el valor lingüístico con valores numéricos en CW.

En los últimos años, se proponen algunas funciones y modelos computacionales para representar la información lingüística con un significado numérico y para manejar las entradas lingüísticas en el proceso de CW. El modelo lingüístico de 2-tuplas [HM00] es muy adecuado para tratar con términos lingüísticos que se distribuyen uniforme y simétricamente sin pérdida de información, siendo las 2-tuplas compuestas por un término lingüístico y un valor numérico comprendido en el intervalo (-0.5, 0.5). Sea $S = \{s_0, s_1, ..., s_g\}$ un conjunto de términos lingüísticos, la función de mapeo del modelo se establece entre 2-tuplas y un valor numérico comprendido entre 0 y g. En algunas situaciones de toma de decisiones, los términos lingüísticos que no están distribuidos uniforme y simétricamente se usan a menudo para expresar la preferencia. Para lidiar con este tipo de términos lingüísticos, se han propuesto algunas extensiones de los modelos lingüísticos de 2-tuplas,

(1) Modelo lingüístico proporcional de 2-tuplas [WH06]. Proporciona un modelo computacional para tratar las 2-tuplas proporcionales, que está compuesto por dos términos lingüísticos con las proporciones simbólicas. Este modelo proporciona las transformaciones entre 2-tuplas proporcionales y el valor numérico entre 0 y 1.

(2) El modelo basado en una jerarquía lingüística [HHVM08]. Este modelo presenta una metodología para tratar los términos lingüísticos desbalanceados mediante la estructura jerárquica lingüística. Para representar el significado numérico de la información lingüística, cada valor lingüístico tiene un valor numérico correspondiente entre 0 y g de una manera desbalanceada.

(3) Modelo de escala numérica [DXY09]. Define una función que establece un mapeo uno a uno entre la información lingüística y las escalas numéricas.

Los modelos anteriores han contribuido en gran medida a la investigación y desarrollo en el tratamiento de información lingüística mediante sus correspondientes valores numéricos. Sin embargo, en CW, se presenta el hecho de que las palabras significan cosas diferentes para diferentes personas, es decir, debe haber diferentes significados numéricos para palabras diferentes, mientras que los modelos anteriores no tiene en cuenta este problema. Para tratar el problema anterior, se presentan dos modelos:

- El uso de conjuntos difusos tipo 2 se basa en distribuciones de posibilidades bajas y altas con una naturaleza de tercera dimensión [MW10], que agrupa todos los significados de las personas en una sola función de representación. Permite manejar incertidumbres lingüísticas con funciones de pertenencia superior e inferior.

- El conjunto de términos lingüísticos multi-granular [HM01] se propuso para que los expertos elijan conjuntos de términos lingüísticos con distinta granularidad para expresar sus preferencias, teniendo en cuenta los diferentes antecedentes y conocimientos de los expertos.

Aunque los conjuntos difusos tipo 2 y multi-granular pueden reflejar las diferentes interpretaciones de las palabras, no pueden representar la semántica específica o el significado de las palabras, es decir, para cada experto no se trata el significado exacto de cada palabra. Por ejemplo, al revisar un artículo, dos revisores pueden pensar que el artículo revisado es interesante, pero el término interesante a menudo tiene un significado numérico diferente para ambos revisores.

En esta tesis, nos centramos en la idea de que las palabras tienen un significado diferente para diferentes personas, lo llamamos semántica individual personalizada (PIS). Para exponer el PIS, estudiamos algunos modelos y enfoques de toma de decisiones con relaciones de preferencia lingüística para calcular las escalas numéricas personalizadas para términos lingüísticos. Bajo diferentes contextos de toma de decisiones, los modelos construidos y los procesos de cálculo diseñados son diferentes. Consideramos tres contextos de toma de decisiones para estudiar el PIS de cada experto, es decir, toma de decisiones clásica, toma de decisiones lingüísticas vacilantes y toma de decisiones en grupo a gran escala (LSGDM). Las características de los tres entornos de toma de decisiones son las siguientes,

(1) La toma de decisiones clásica con información lingüística involucra a un grupo de expertos que expresan su preferencia utilizando términos lingüísticos únicos y toman una decisión entre un conjunto de alternativas.

(2) Considerando que en algunas situaciones el uso de términos lingüísticos únicos no es suficiente para representar el conocimiento y las opiniones de los expertos, los expertos prefieren usar varios términos lingüísticos para expresar sus preferencias. Bajo este contexto, la decisión lingüística vacilante se utiliza para tratar las dudas de los expertos.

(3) En LSGDM, hay una gran cantidad de personas que toman decisiones participando en los problemas de toma de decisiones. El LSGDM es más complejo que la toma de decisiones de grupo tradicional (GDM), debido al tamaño relativamente grande del grupo y la complejidad de los problemas de toma de decisiones, así como a los propios responsables de la toma de decisiones, que tienen diferentes conocimientos y antecedentes.

Al construir los modelos de PIS en los problemas de toma de decisiones, la consistencia de las preferencias proporcionadas por los expertos juega un papel importante [ACH + 08, DHV15]. La coherencia garantiza que el responsable de la toma de decisiones no sea ni aleatorio ni ilógico en sus comparaciones por pares, y la coherencia inaceptable puede dar lugar a resultados inconsistentes y poco confiables. En esta tesis, el estado de consistencia de las expresiones lingüísticas proporcionadas por los expertos se usa como una idea básica en la construcción los modelos PIS. Además, para obtener las escalas numéricas personalizadas de términos lingüísticos, a lo largo de esta tesis, utilizamos el modelo de escala numérica, que estableció un mapeo uno a uno entre la información lingüística y los valores numéricos y tiene algunas propiedades deseadas al conectarlo con modelos lingüísticos de 2-tuplas.

Esta tesis consta de dos partes principales: la primera ilustra la exposición de los problemas abordados y los resultados obtenidos para los modelos propuestos. La segunda parte es una recopilación de las principales publicaciones asociadas a esta tesis.

Esta tesis está estructurada de la siguiente manera: la Sección 2 proporciona algunos preliminares relacionados utilizados a lo largo de esta contribución. En la Sección 3, se discuten las ideas básicas y los desafíos que justifican el desarrollo de esta tesis. La Sección 4 propone el objetivo de esta tesis. La Sección 5 presenta la metodología utilizada en la tesis. En la Sección 6 se presenta un resumen de los modelos impulsados por la consistencia propuestos para mostrar el PIS. La Sección 7 presenta una discusión de los resultados obtenidos en la tesis. Finalmente, la Sección 8 extrae la conclusión de esta tesis y en la Sección 9 se discuten los trabajos futuros.

2 Preliminaries

Some basic knowledge about the basis of the linguistic computation models and linguistic group decision making are introduced in this section.

2.1 2-tuple linguistic models

In the following, the 2-tuple linguistic model and its extensions (proportional 2-tuple linguistic model, the model based on a linguistic hierarchy and numerical scale model) are introduced.

2.1.1 2-tuple linguistic representation model

The 2-tuple linguistic representation model, presented by Herrera and Martínez [HM00] represents the linguistic information by a 2-tuple $(s_i, \alpha) \in \overline{S} = S \times [-0.5, 0.5)$, where $s_i \in S$ and $\alpha \in [-0.5, 0.5)$. Formally, let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation. The 2-tuple that expresses the equivalent information to β is then obtained as:

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5), \tag{I.1}$$

where

$$\Delta(\beta) = (s_i, \alpha), \quad with \begin{cases} s_i, \ i = round(\beta) \\ \alpha = \beta - i, \ \alpha \in [-0.5, 0.5) \end{cases}. \tag{I.2}$$

Function Δ is a one to one mapping function whose inverse function $\Delta^{-1} : \overline{S} \rightarrow [0, g]$ is defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha$. When $\alpha = 0$ in (s_i, α) it is then called a simple term.

In [HM00] a computational model was also defined for the 2-tuple linguistic model in which different operations were introduced:

(1) A 2-tuple comparison operator: Let (s_k, α) and (s_l, γ) be two 2-tuples. Then:

(i) if $k < l$, then (s_k, α) is smaller than (s_l, γ).

(ii) if $k = l$, then

(a) if $\alpha = \gamma$, then (s_k, α), (s_l, γ) represents the same information.

(b) if $\alpha < \gamma$, then (s_k, α) is smaller than (s_l, γ).

(2) A 2-tuple negation operator:

$Neg((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha)))$.

(3) Several 2-tuple aggregation operators have been developed (see [HM00, MH12]).

2.1.2 Proportional 2-tuple linguistic model

Wang and Hao [WH06] proposed the proportional 2-tuple linguistic model, which is an extension of the 2-tuple linguistic model for CW. This proportional 2-tuple model can help make a more accurate expression of the results, such as $(0.2s_i, 0.8s_{i+1})$ for the case when someone's grades in the answerscripts of a whole course are distributed as $20\%s_i$ and $80\%s_{i+1}$.

Definition 1. *(Proportional 2-tuple [WH06]). Let $S = \{s_0, s_1, ..., s_g\}$ be an ordinal term set, and $IS = I \times S = \{(\alpha, s_i)\}$, $\alpha \in [0, 1]$, $i = 0, 1, ..., n$. Given a pair (s_i, s_{i+1}) of two successive ordinal* *terms of* S*, any two elements* (α, s_i) (β, s_{i+1}) *of* IS *called a symbolic proportion pair and* α, β *are called a pair of symbolic proportions of pair* (s_i, s_{i+1}) *if* $\alpha + \beta = 1$*. Let* $\overline{S} = \{(\alpha s_i, (1-\alpha)s_{i+1})\}, \alpha \in [0,1], i = 0, 1, ..., g$*, then* $\overline{S}$ *is called ordinal proportional 2-tuple set.*

In general, the element semantics in a linguistic term set are given by fuzzy numbers (defined in the [0,1] interval), which are described by linear triangular membership functions or linear trapezoidal membership functions. For instance, the linear trapezoidal membership function is achieved by a 4-tuple (a, b, c, d), b and c indicate the interval in which the membership value is 1, and a and d are the left and right limits of the definition domain of a trapezoidal membership function. Wang and Hao [WH06] proposed an interesting generalized version of the 2-tuple fuzzy linguistic representation model. The semantics of linguistic terms used in the Wang and Hao's model are defined by symmetrical trapezoidal fuzzy numbers. If the semantics of s_i is defined by $T[b_i - \sigma_i, b_i, c_i, c_i + \sigma_i]$ in the Wang and Hao model the canonical characteristic value (CCV) of s_i is $\frac{b_i+c_i}{2}$, i.e., $CCV(s_i) = \frac{b_i+c_i}{2}$.

Definition 2. *(Canonical characteristic value [WH06]). Let* $S = \{s_0, s_1, ..., s_g\}$ *be an ordinal term set,* $\alpha \in [0,1]$*,* $c_i \in [0,1]$ *, and* $c_0 < c_1, ..., < c_g$*, for* $CCV(s_i) = c_i$*,* $(\alpha s_i, (1-\alpha)s_{i+1}) \in \overline{S}$*, define the function* CCV *on* S *by*

$$CCV(\alpha s_i, (1-\alpha)s_{i+1}) = \alpha CCV(s_i) + (1-\alpha)CCV(s_{i+1}) = \alpha c_i + (1-\alpha)c_{i+1}. \tag{I.3}$$

The function CCV^{-1} *is defined as*

$$CCV^{-1}(\beta) = CCV^{-1}(\alpha c_i + (1-\alpha)c_{i+1}) = (\alpha s_i, (1-\alpha)s_{i+1}). \tag{I.4}$$

2.1.3 The model based on a linguistic hierarchy

A linguistic hierarchy is a set of levels where each level is a linguistic term set with a different granularity from the remaining levels of the hierarchy. Each level belonging to a linguistic hierarchy is denoted as $l(t, n(t))$, with t being a number that indicates the level of the hierarchy and $n(t)$ being the granularity of the linguistic term set of t. Generally, the linguistic term set $S^{n(t+1)}$ of level $t+1$ is obtained from its predecessor $S^{n(t)}$ as $l(t, n(t)) \to l(t+1, 2 \cdot n(t) - 1)$.

In linguistic hierarchies LH, the transformation function between terms from different levels to represent 2-tuple linguistic representations is defined as follows [HHVM08]: for any linguistic levels t and t', $TF^t_{t'} : l(t, n(t)) \to l(t', n(t'))$, such that

$$TF^t_{t'}(s_i^{n(t)}, \alpha^{n(t)}) = \Delta(\frac{\Delta^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1}) \tag{I.5}$$

Generally, in the computational model defined for the linguistic hierarchy LH, any level in the LH may be selected to unify the multigranular linguistic information in the computational model defined for the linguistic hierarchy LH. In this study, the maximum level t_m in the LH is used, i.e., $l(t_m, n(t_m)) = S^{n(t_m)} = \{s_0^{n(t_m)}, ..., s_{n(t_m)-1}^{n(t_m)}\}$. Any 2-tuple linguistic representation (s_i, α) can be transformed by the unbalanced linguistic transformation process into the term in $LH = \cup_t l(t, n(t))$ and vice versa. The detailed transformation process is given as follows:

(1) Representation in the linguistic hierarchy: the representation algorithm uses the linguistic hierarchy LH to model the unbalanced terms in S. Therefore, the first step towards accomplishing the process of CW is to transform the unbalanced terms in S into their corresponding terms in the LH, by means of the transformation function ψ associating each unbalanced linguistic 2-tuple (s_i, α) with its respective linguistic 2-tuple in $LH(\overline{S})$, i.e.,

$$\psi : \overline{S} \to LH(\overline{S}) \tag{I.6}$$

such that $\psi(s_i, \alpha) = (s_{I(i)}^{G(i)}, \alpha)$, for $\forall (s_i, \alpha) \in \overline{S}$.

(2) Computational phase: it computes the linguistic information based on 2-tuple linguistic model and the linguistic hierarchy. First, it uses Eq. (I.6) to transform $(s_{I(i)}^{G(i)}, \alpha)$ into linguistic 2-tuples in $\overline{S^{n(t_m)}}$, denoted as $(s_{I'(i)}^{n(t_m)}, \lambda)$, i.e.,

$$(s_{I'(i)}^{n(t_m)}, \lambda) = \Delta(\frac{\Delta^{-1}(s_{I(i)}^{G(i)}, \alpha)(n(t_m) - 1)}{G(i) - 1}) \tag{I.7}$$

Then, the computational model developed for the 2-tuple linguistic representation model is used over $\overline{S^{n(t_m)}}$ with a result denoted as $(s_r^{n(t_m)}, \lambda_r) \in \overline{S^{n(t_m)}}$.

(3) Retranslation process: A retranslation process is used to transform the result $(s_r^{n(t_m)}, \lambda_r) \in \overline{S^{n(t_m)}}$ into the unbalanced term in $\overline{S}$, by using the transformation function ψ^{-1}, i.e.,

$$\psi^{-1} : LH(\overline{S}) \to \overline{S} \tag{I.8}$$

such that $\psi^{-1}(s_r^{n(t_m)}, \lambda_r) = (s_{result}, \lambda_{result}) \in \overline{S}$. The details of the methodology to address unbalanced linguistic terms are described in Herrera et al. [HHVM08].

2.1.4 Numerical scale model

Dong et al. [DXY09, DZHY13] extended the 2-tuple linguistic model by the numerical scale and the interval numerical scale for integrating different linguistic models and increasing the accuracy of the 2-tuple linguistic model. The concept of the numerical scale was introduced in [DXY09] for transforming linguistic terms into real numbers:

Definition 3. *(Numerical scale [DXY09]). Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set, and R be the set of real numbers. The function: $NS : S \to R$ is defined as a numerical scale of S, and $NS(s_i)$ is called the numerical index of s_i. If the function NS is strictly monotone increasing, then NS is called an ordered numerical scale.*

Let S be defined as before. The numerical scale NS for (s_i, α), is defined by

$$NS(s_i, \alpha) = \begin{cases} NS(s_i) + \alpha \times (NS(s_{i+1}) - NS(s_i)) & \alpha \geq 0 \\ NS(s_i) + \alpha \times (NS(s_i) - NS(s_{i-1})) & \alpha < 0 \end{cases}$$

If $NS(s_i) < NS(s_{i+1})$, for $i = 0, 1, ..., g-1$, the numerical scale NS on S is ordered. The concept of the interval numerical scale is defined as Definition 4.

Definition 4. *(Interval numerical scale [DZHY13]). Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set, and $M = \{[A_L, A_R] | A_L, A_R \in R, A_L \leq A_R\}$ be a set of interval numbers. The function, $INS : S \to M$ is defined as an interval numerical scale of S, and $INS(s_i)$ is called the interval numerical index of s_i.*

Let $INS(s_i) = [A_L^i, A_R^i]$, where $INS_L(s_i) = A_L^i$ and $INS_R(s_i) = A_R^i$. The numerical scale INS for (s_i, α), is defined by

$$INS(s_i, \alpha) = [A_L^i, A_R^i] \tag{I.9}$$

where

$$A_L^i = \begin{cases} INS_L(s_i) + \alpha \times (INS_L(s_{i+1}) - INS_L(s_i)), \ \alpha \geq 0 \\ INS_L(s_i) + \alpha \times (INS_L(s_i) - INS_L(s_{i-1})), \ \alpha < 0 \end{cases} \tag{I.10}$$

and

$$A_R^i = \begin{cases} INS_R(s_i) + \alpha \times (INS_R(s_{i+1}) - INS_R(s_i)), \ \alpha \geq 0 \\ INS_R(s_i) + \alpha \times (INS_R(s_i) - INS_R(s_{i-1})), \ \alpha < 0 \end{cases} \tag{I.11}$$

The interval numerical scale INS on S is ordered, if $INS(s_i) \leq INS(s_{i+1})$.

2.2 Linguistic group decision making

A linguistic group decision making situation arises when a group of experts are asked to express their preferences about a set of alternatives using linguistic information. In GDM problems with preference relations, there are two measures that have been considered before obtaining a final solution [CMM+08]:

(1) Individual consistency. The individual consistency is applied to ensure that expert is being neither random nor illogical in pairwise comparisons.

(2) Consensus. Consensus means that the group of decision makers agreed to their preferences to some extent.

In this subsection, the linguistic preference relations and the basic measures of consistency and consensus are introduced.

2.2.1 Preference relations in linguistic decision making

In linguistic decision making problems, once the set of alternatives and a linguistic scale are established, experts provide their preferences using linguistic terms on the scale to construct a preference relation. The preference relations can be constructed by single linguistic terms or complex linguistic expressions under different decision making situations. The most commonly used preference relation structures under linguistic context are linguistic preference relation and hesitant fuzzy linguistic preference relation (HFLPR).

Let $X = \{x_1, x_2, ..., x_n\}(n \geq 2)$ be a finite set of alternatives. When a decision maker makes pairwise comparisons using the linguistic term set S, he/she can construct a linguistic preference relation $L = (l_{ij})_{n\times n}$, whose element l_{ij} estimates the preference degree of alternative x_i over x_j. Linguistic preference relations based on linguistic 2-tuples can be formally defined as Definition 5.

Definition 5. *(Linguistic preference relation [ACH+08, ACC+09]). The matrix $L = (l_{ij})_{n\times n}$, where $l_{ij} \in S$, is called a simple linguistic preference relation. The matrix $L = (l_{ij})_{n\times n}$, where $l_{ij} \in \overline{S}$, is called a 2-tuple linguistic preference relation. If $l_{ij} = Neg(l_{ji})$ for $i, j = 1, 2, ..., n$, then L is considered reciprocal.*

Torra [Tor10] introduced the hesitant fuzzy set. Similar to the situations that are described and managed by hesitant fuzzy sets in [Tor10], decision makers may hesitate between several linguistic terms before assessing an alternative in linguistic decision making. Bearing this idea in mind, Rodríguez et al. [RMH12] gave a concept of hesitant fuzzy linguistic term set (HFLTS) as follows:

Definition 6. *(Hesitant fuzzy linguistic term set [RMH12]). Let* $S = \{s_0, s_1, ..., s_g\}$ *be a linguistic term set, where* $g+1$ *is odd. A HFLTS,* M_S*, is an ordered finite subset of consecutive linguistic terms of* S.

Based on the use of HFLTSs, Rodríguez et al. [RMH13] proposed the concept of HFLPR as Definition 7.

Definition 7. *(Hesitant fuzzy linguistic preference relation [RMH13]). Let* H_S *be a set of HFLTSs based on* S*. A HFLPR based on* S *is presented by a matrix* $H = (H_{ij})_{n\times n}$*, where* $H_{ij} \in H_S$ *and* $Neg(H_{ij}) = H_{ji}$.

2.2.2 Consistency and consensus in linguistic group decision making

As mentioned before, consistency is an important issue in decision making with preference relations. Regarding the consistency measures of preference relations, generally, it includes additive consistency, multiplicative consistency and ordinal consistency. In this thesis, we use the additive transitivity to characterize the consistency of linguistic preference relations under numerical scale, see Definition 8.

Definition 8. *(Consistent linguistic preference relation [DXY09]). Let* $L = (l_{ij})_{n\times n}$ *be a linguistic preference relation based on* S*.* L *is considered consistent if* $NS(l_{ij}) + NS(l_{jk}) - NS(l_{ik}) = 0.5$ *for* $i, j, k = 1, 2, ..., n$.

Then, the consistency index (CI) of a linguistic preference relation L is defined as,

$$CI(L) = 1 - \frac{2}{3gn(n-1)(n-2)} \sum_{i,j,k=1}^{n} |NS(l_{ij}) + NS(l_{jk}) - NS(l_{ik}) - 0.5| \quad \text{(I.12)}$$

The larger the value of $CI(L)$ the more consistent L is. If $CI(L) = 1$, then L is a consistent linguistic preference relation.

The consensus process in decision making is defined as a dynamic and iterative group discussion. By computing the consensus degree, the consensus among the decision makers is detected. If the consensus level is not acceptable, then a feedback recommendation is applied to improve the consensus. Otherwise, the consensus is achieved. Generally, the computation of consensus measure is done by measuring the difference between individual preference and collective preference. A general consensus process is shown in Fig.1.

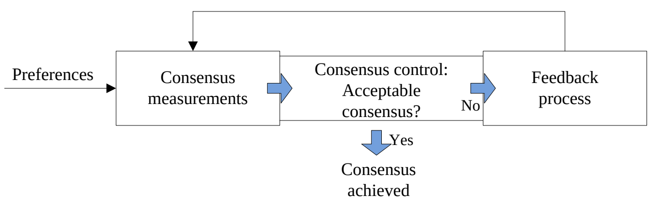


Fig.1 General consensus reaching process scheme

3 Justification

After the presentation of all the main concepts related to the topic, we present the basic ideas of proposing the PIS with CW in the following.

Yager points out the importance of the translation and retranslation processes in CW [Yag04]. Fig. 2 provides a schematic view of all elements involved in a process of CW.

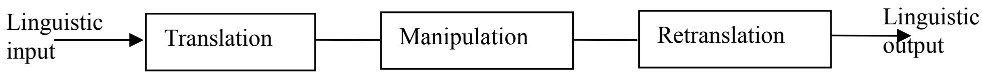


Fig. 2. Yager's CW scheme

As mentioned before, it is an important issue in the choice of the monotonic mapping encoding the linguistic issues in CW process. The linguistic computational model discussed before all use the same membership function or indexes to represent linguistic information among different experts. In this thesis, for the fact that words mean different things to different people, under the linguistic decision making context, the fact can be expressed as for different experts, the numerical meaning of linguistic variables should be different.

To show the individual differences in CW process, it is necessary to introduce a framework that fulfills the phases of the classical CW scheme and also fulfills the previous fact. Therefore, a PIS model to reflect individual differences in understanding the meaning of words is proposed (see Fig. 3).

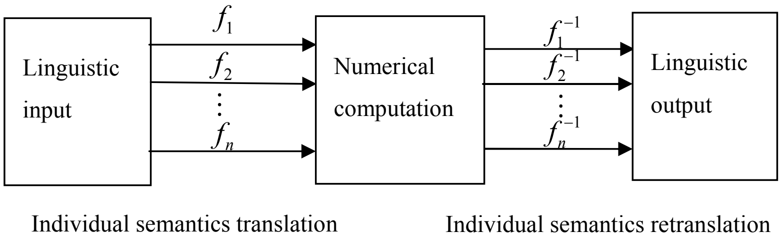


Fig. 3. The framework for PIS model

Fig. 3 provides a basic framework for the CW process with PIS. It is composed by the following three processes:

(1) Individual semantics translation. This process translates linguistic terms of a linguistic term set S, into the individual semantics defined by different numbers in the established numerical domain Φ. Formally, it can be expressed as the mapping $f_k : S \to \Phi$, where f_k is called the individual semantics translation, associated with the decision maker e_k.

(2) Numerical computation. It computes the output of individual semantics translation , which provides the numerical values for each linguistic term. Generally, the aggregation operators may be used in the computation process to aggregate the individual information, and from which the output is a numerical value.

(3) Individual semantics retranslation. Individual semantics retranslation is the inverse operation of individual semantics translation, and it is applied to retranslate the output of numerical computation into linguistic terms in S understandable to individuals. The individual semantics retranslation can be expressed as the inverse of f_k, denoted as f_k^{-1}.

It can be seen that the PIS model provides a novel way to represent the specific semantics of each expert. By using the PIS model, we investigate how to reflect and show the different individual semantics in different decision making contexts based on numerical scale model, and also study the application of the PIS in GDM problems.

(1) The 2-tuple linguistic models discussed in the thesis include the 2-tuple linguistic model, proportional 2-tuple linguistic model and the model based on a linguistic hierarchy. The 2-tuple linguistic model is suited to deal with the linguistic terms that are uniformly and symmetrically distributed, while other two models are developed to deal with the terms that are not uniformly and symmetrically distributed. Therefore, it would be interesting to provide a universal model to connect the above three models so that the universal model can have both desired properties, then the universal model can be applied to more decision making problems to deal with the linguistic preferences in an easier way.

(2) In classic decision making, the previous fact that words mean different things for different people is highlighted. To deal with this fact, a fundamental PIS model to deal with linguistic expressions and to obtain the different individual semantics among experts is necessary. The construction of the fundamental PIS model provides a basis for the models built in other decision making contexts.

(3) In hesitant linguistic decision making, instead the use of single linguistic term, the complex expression HFLTS, which is created by a set of linguistic terms, is used to express experts' preferences. The problem about how to obtain the individual semantics of HFLTSs provided by experts in hesitant linguistic GDM needs to be discussed.

(4) The LSGDM, which deals with a large number of experts, is more complex than the usual GDM. Generally, the consensus-based decisions are necessary and required in decision making problems. It would be an interesting study to discuss how to apply the PIS model in LSGDM to solve the consensus problem.

Thus, the study of the PIS in CW process provides a key point to develop the computation in different GDM problems to implement the idea that words mean different things to different people.

4 Objectives

The PIS makes it facilitate to manage the personalized linguistic information in CW, and by solving the PIS model, the different semantics presented in different forms, such as single numerical values, interval numerical values or trapezoidal fuzzy membership values, are presented regarding difference decision making environments. Therefore, the aim of this thesis is to analyze the PIS among experts in CW process, to deal with the previous fact that words mean different things for different people. This thesis is organized in several objectives as follows,

- **To connect the numerical scale model with 2-tuple linguistic models.** Discuss the characteristics of the numerical scale model and analyze its connection with 2-tuple linguistic model, proportional 2-tuple linguistic model and the model based on a linguistic hierarchy, to show the numerical scale model can deal with both the balanced linguistic terms and also the unbalanced linguistic terms.

- **To establish a fundamental PIS model to deal with the previous fact.** Construct a consistency-driven optimization model to obtain the PIS of linguistic terms for experts. Before constructing the model, the consistency measure of linguistic preference relations is discussed. Then based on the consistency measurement, a fundamental PIS model to implement the individual semantics translation process to represent the different individual semantics of linguistic terms is proposed. The proposed model is based on the numerical scale model because of its desired features to deal with different linguistic representations in a general way.

- **To extend the PIS model in hesitant linguistic GDM problems.** In hesitant linguistic decision making, the HFLTSs are used to model the hesitation and to express the preferences of experts. First, to show the consistency of HFLPR comprehensively, an average consistency measurement of HFLPR is proposed. Then we construct the average consistency-driven optimization model to reflect the different numerical meaning of linguistic terms with HFLPR and to investigate the method to show the individual semantics of HFLTS. The representation of the PIS for HFLTSs is in the form of trapezoidal fuzzy numbers, in order to capture more the uncertainties and hesitation in decision making problems.

- **To work on the use of PIS model in LSGDM problems.** Integrate the idea of personalizing individual semantics into LSGDM problems to observe the consensus formation, based on a natural premise that decision makers having similar semantics and preferences are easier to communicate with each other. By solving the PIS model, the personalized numerical meaning of each linguistic term is obtained. Then with the previous premise and the obtained personalized numerical scale, the consensus reaching process is constructed, with the aim of helping experts be more willing to change their preferences in LSGDM problems, so that a better consensus can be obtained in a more natural way.

5 Methodology

This section introduces the methodology used in the thesis. Considering that the main idea of this study is to obtain and analyze the different individual semantics for different experts by building PIS models based on numerical scale model in linguistic decision making problems, the related methods are provided as follows,

1. **Hypothesis formulation.** The hypothesis proposed in the study should be reasonable and suitable for the discussed decision making problems. The hypothesis can provide a good and useful tool to guide and influence the decision making process. In constructing the consistency-driven optimization model for PIS, we propose a premise about the consistency relation between the linguistic preference relation and its transformed numerical preference relation. In building the consensus model in LSGDM problem, we propose a natural premise to show a general situation when the experts are more willing to follow the adjustment suggestions of moderators.

2. **Establishment of optimization model.** In this thesis, the PIS of linguistic information for experts are obtained by solving consistency-driven optimization-based models, which are established based on consistency measurements of preference relations to show the different individual semantics or numerical meanings of linguistic expressions for different decision makers under different decision making environments, such as, in classical GDM, in hesitant linguistic GDM and in LSGDM.

3. **Simulation analysis.** It provides an effective and easier way to describe the results obtained from the proposed models. With the simulation experiment, the feasibility and validity of the proposed model in decision making problems are discussed. For example, in studying the consensus process with PIS, the simulation analysis is applied to explore the variation of the consensus level, to detect whether the desired result can be achieved through the proposed model.

4. **Comparative study.** It is the act of comparing two or more studies with a view to discovering something about the similarities and differences between studies. In this study, we summarize and analyze the existing studies regarding the topic, and by comparing the computation process or results of the proposed model with the existing studies, to further discuss the characteristics of the proposed model.

6 Summary

In this section, a summary of the proposals included in this thesis is presented, describing the main contents along with the obtained results associated with the journal publication are provided. The research carried out for this thesis and the results obtained in each case are collected into the following published papers:

- Y.C. Dong, C.C. Li, F. Herrera, Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. Information Sciences, 367 (2016) 259-278.
- C.C. Li, Y.C. Dong, F. Herrera, E. Herrera-Viedma, L. Martínez, Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. Information Fusion, 33 (2017) 29-40.
- C.C. Li, R.M. Rodríguez, L. Martínez, Y. Dong, F. Herrera, Personalized individual semantics based on consistency in hesitant linguistic GDM with comparative linguistic expressions. Knowledge-Based Systems, 145 (2018) 156-165.
- C.C. Li, Y.C. Dong, F. Herrera, A consensus model for large-scale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups. Submitted to IEEE Transaction on Fuzzy Systems.

The remainder of this section is organized into the four objectives defined in Section 4. First, Section 6.1 provides the connection between numerical scale model and 2-tuple linguistic models. Then, Section 6.2 illustrates PIS models to obtain the personalized numerical scales in linguistic decision making contexts. Specifically, Section 6.2.1 shows the fundamental PIS model for CW, Section 6.2.2 discusses the PIS in hesitant linguistic decision making and Section 6.2.3 describes the use of PIS in consensus problems under LSGDM.

6.1 Numerical scale to connect the 2-tuple linguistic models

The numerical scale provides a concept to show the general transformation process between linguistic terms and numerical values. A connection between the numerical scale model and the 2-tuple linguistic models (2-tuple linguistic model, proportional 2-tuple linguistic model and unbalanced linguistic model) is proposed by providing the equivalence among these models in some sense. From the connection, it is shown that the numerical scale model can deal with both balanced linguistic terms and unbalanced linguistic terms.

(1) Connection between numerical scale model and 2-tuple linguistic model

The 2-tuple linguistic model is suitable to deal with uniformly and symmetrically distributed linguistic term sets. It shows a map to map function between linguistic terms and numerical values, i.e., $\Delta^{-1}(s_i) = i$. By connecting the transformation function Δ^{-1} in 2-tuple linguistic model and numerical scale NS in numerical scale model, the following proposition shows the equivalence between these two models.

Proposition 1. *When setting $NS(s_i) = i$ for $i = 0, 1, ..., g$, we have $NS((s_\alpha, x_\alpha)) = \Delta^{-1}((s_\alpha, x_\alpha))$, for any $(s_\alpha, x_\alpha) \in \overline{S}$.*

Proposition 1 shows that the computation between numerical scale model and the 2-tuple linguistic model is equal when setting $NS(s_i) = i$.

(2) Connection between numerical scale and proportional 2-tuple linguistic model

As a generalization of 2-tuple linguistic model, the proportional 2-tuple linguistic model provides a way to represent the experts' preferences using two linguistic terms with proportions. In proportional 2-tuple linguistic model, the function canonical characteristic values CCV is used to transform proportional 2-tuples into numerical values. Proposition 2 shows the connection between the use of numerical scale NS and the function CCV in the transformation between linguistic terms and numerical values.

Proposition 2. *When setting $NS(s_i) = CCV(i)$ for $i = 0, 1, ..., g$, we have $NS((s_\alpha, x_\alpha)) = CCV(h^{-1}(s_\alpha, x_\alpha))$, for any $(s_\alpha, x_\alpha) \in \overline{S}$.*

Proposition 2 shows the equivalence of computation between the numerical scale model and proportional 2-tuple linguistic model when setting $NS(s_i) = CCV(i)$.

(3) Connection between numerical scale and the model based on a linguistic hierarchy

This connection is achieved by proving the equivalence between the numerical scale NS and the function Δ^{-1} developed on the linguistic hierarchy. Before making the connection between the numerical scale model and a model based on a linguistic hierarchy, we first present a revised retranslation process, which does not change the essence of the original retranslation process and provides a basis for its connection with the numerical scale model.

The revised retranslation process, where the input is the linguistic 2-tuple based on a linguistic hierarchy and the output is the unbalanced term, is designed as follows,

$$(s_{result*}, \lambda_{result*}) = \psi^{-1\prime}(s_x^{n(t)}, \alpha) \tag{I.13}$$

where

$$s_{result*} = \begin{cases} s_k, & d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) < d(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) \\ s_{k+1}, & d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) \geq d(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) \end{cases} \tag{I.14}$$

and

$$\lambda_{result*} = \begin{cases} \frac{d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))}{d(s_{I'(k)}^{n(t_m)}, s_{I'(k+1)}^{n(t_m)})}, & s_{result*} = s_k \\ -\frac{d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))}{d(s_{I'(k)}^{n(t_m)}, s_{I'(k+1)}^{n(t_m)})}, & s_{result*} = s_{k+1} \end{cases} \tag{I.15}$$

Then we prove that the results obtained through the revised retranslation process are the same as the ones obtained by the original retranslation process. Based on Eqs. (I.13)-(I.15), we set the numerical scale for linguistic term s_i as follows

$$NS(s_i) = \Delta^{-1}(s_{I'(i)}^{n(t_m)}), \quad i = 0, 1, ..., g \tag{I.16}$$

And it is proved that if the numerical scale is set as Eq. (I.16), the linguistic computational models between both models are equivalent.

The journal paper associated to this part is:

- Y.C. Dong, C.C. Li, F. Herrera. Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. Information Sciences, 367 (2016) 259-278.

6.2 Personalized numerical scales in linguistic decision making problems

As mentioned before, in CW process, there is a fact that words mean different things for different people. The type-2 fuzzy sets and multi-granular linguistic models [MW10, HM01] do show the different or multiple meanings of words for different people, but they do not represent the specific semantics of each individual. Therefore, the main work of this thesis is to personalize individual semantics in CW process to show the individual differences in understanding the meaning of words. By solving the PIS model, the different numerical meanings for linguistic information regarding different experts are obtained. The following sections propose the building and application of PIS models under three different decision making contexts.

6.2.1 PIS model for CW

Fig.3 in Section 3 shows the CW framework with PIS, which includes the individual semantics translation, computation process and individual semantics retranslation. In the following, we illustrate the first process individual semantics translation, which is the most important part in the CW process.

The consistency is an important issue in decision making using preference relations. The lack of consistency can lead to inconsistent results, which makes the unreliability of the decision and the inaccuracy of the results. To represent the individual semantics, we propose a consistency-driven optimization-based model to obtain the personalized individual interval numerical scales of the 2-tupe linguistic terms. This model is based on the following premise:

Premise 1. *If linguistic preference relations provided by individuals are consistent, then the interval fuzzy preference relations, transformed by the established interval numerical scales, should be as much consistent as possible.*

The proposed model is built based on the consistency property; we illustrate the construction of the PIS model from two parts:

(1) Consistency basics. Based on Premise 1, it shows that with the transformation between linguistic terms and numerical values based on numerical scale model, the consistency control of linguistic preference relation can be achieved by that of interval fuzzy preference relation. To measuring the consistency of interval fuzzy preference relation, we propose the optimistic consistency and pessimistic consistency of interval fuzzy preference relation, which shows the best and worst consistency indexes of all fuzzy preference relations associated to the interval fuzzy preference relation.

Definition 9. *(Fuzzy preference relation associated to $\widetilde{V}$). Let $\widetilde{V} = (\widetilde{v_{ij}})_{n\times n}$, where $\widetilde{v_{ij}} = [v_{ij}^-, v_{ij}^+]$, be an interval fuzzy preference relation. $F = (f_{ij})_{n\times n}$ is a fuzzy preference relation associated to $\widetilde{V}$ if $v_{ij}^- \leq f_{ij} \leq v_{ij}^+$ and $f_{ij} + f_{ji} = 1$.*

Being $N_{\widetilde{V}}$ the set of the fuzzy preference relation associated to $\widetilde{V}$.

Definition 10. *(Consistency of interval fuzzy preference relation). Let* $\widetilde{V} = (\widetilde{v_{ij}})_{n\times n}$ *be an interval fuzzy preference relation, let* $F = (f_{ij})_{n\times n}$ *be a fuzzy preference relation associated to* $\widetilde{V}$*, and let* $N_{\widetilde{V}}$ *be the set of the fuzzy preference relation associated to* $\widetilde{V}$*. The optimistic consistency index (OCI) of* $\widetilde{V}$ *is then defined as follows,*

$$OCI(\widetilde{V}) = \max_{F\in N_{\widetilde{V}}} CI(F) \tag{I.17}$$

i.e., $OCI(\widetilde{V}) = \max_{F\in N_{\widetilde{V}}} (1 - \frac{2}{3n(n-1)(n-2)} \times \sum_{i,j,z=1;i\neq j\neq z}^{n} |f_{ij} + f_{jz} - f_{iz} - 0.5|)$.
And the pessimistic consistency index of $\widetilde{V}$ *is,*

$$PCI(\widetilde{V}) = \min_{F\in N_{\widetilde{V}}} CI(F) \tag{I.18}$$

i.e., $PCI(\widetilde{V}) = \min_{F\in N_{\widetilde{V}}} (1 - \frac{2}{3n(n-1)(n-2)} \times \sum_{i,j,z=1;i\neq j\neq z}^{n} |f_{ij} + f_{jz} - f_{iz} - 0.5|)$.

(2) Building the consistency-driven optimization model to show PIS. Based on Definition 10, construct a consistency-driven optimization model to personalize individual semantics with linguistic preference relation. The objective of the proposed model is to maximize the optimistic and pessimistic consistency of the transformed interval fuzzy preference relations from linguistic preference relations. By solving the model using the software tools, such as Lingo, the personalized individual interval numerical scales of linguistic terms are obtained, and for different decision makers, the obtained numerical scales are different, which reflects the different understating of words for different people.

The journal paper associated to this part is:

- C.C. Li, Y.C. Dong, F. Herrera, E. Herrera-Viedma, L. Martínez, Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. Information Fusion, 33 (2017) 29-40.

6.2.2 PIS in hesitant linguistic decision making

In hesitant linguistic decision making, decision makers use several linguistic terms to express their preferences instead of using single linguistic terms. Considering the fact that words mean different things for different people in CW, we provide an approach with HFLTSs in decision making problems to personalize individual semantics. The main aim of the proposed approach is to show the different numerical meanings of HFLTSs for different decision makers, which can reflect the individual differences in understanding the meaning of words. It consists of a two-step procedure:

(1) PIS approach to obtain personalized numerical scales. An average consistency-driven model is proposed to set personalized numerical scales for linguistic terms with comparative linguistic expressions.

Before constructing the average consistency-driven model, we first show the definition of the average consistency measure for HFLPR.

Definition 11. *(Linguistic preference relation associated to HFLPR). Let* $H = (H_{ij})_{n\times n}$ *be a HFLPR.* $L = (l_{ij})_{n\times n}$ *is a linguistic preference relation associated to* H*, if* $l_{ij} = H_{ij}^{t}$*,* $t = \{1, ..., \#H_{ij}\}$ *and* $l_{ij} = Neg(l_{ji})$*.*

Being N_H the set of the linguistic preference relations associated to H.

Definition 12. *(Average consistency of HFLPR). Let H be a HFLPR. The value of average consistency index $ACI(H)$ is determined by the average consistency of all linguistic preference relations associated to the HFLPR, i.e.,*

$$\begin{aligned} ACI(H) &= \frac{1}{\#N_H} \times \sum\nolimits_{L \in N_H} CI(L) \\ &= \frac{1}{\#N_H} \times \sum_{h=1}^{\#N_H} (1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,j,z=1}^{n} \left| NS(l_{ij}^h) + NS(l_{jz}^h) - NS(l_{iz}^h) - 0.5 \right|) \end{aligned} \tag{I.19}$$

where $\#N_H$ is the number of of linguistic preference relations in H, i.e., $\#N_H = \prod_{i=1}^{n} \prod_{j=i+1}^{n} \#H_{ij}$.

To obtain the personalized numerical scales of the linguistic terms in HFLTS, an average consistency-driven optimization-based model is proposed. The objective of the model is to maximize the ACI of HFLPR, and the constraints are to set reasonable and natural range for numerical scales of linguistic terms. Solving the model with the software packages, it is obtained the personalized numerical scales for each linguistic term in linguistic term set associated with each decision makers.

(2) Fuzzy envelope for HFLTSs. Based on the personalized numerical scales obtained from the average consistency-driven model, a process to personalize individual semantics with comparative linguistic expressions via the fuzzy envelope for HFLTSs represented by fuzzy membership function is proposed.

- Representing the personalized individual semantics of linguistic terms

Based on the personalized numerical scale, each linguistic term s_i can be represented by the triangular fuzzy membership functions as follows,

$$A^k(s_i) = \begin{cases} T(NS^k(s_0), NS^k(s_0), NS^k(s_1)) & i = 0 \\ T(NS^k(s_{i-1}), NS^k(s_i), NS^k(s_{i+1})) & i = 1, ..., g-1 \\ T(NS^k(s_0), NS^k(s_0), NS^k(s_1)) & i = g \end{cases} \tag{I.20}$$

- Representing the personalized individual semantics of HFLTSs via fuzzy envelope

The fuzzy envelope for HFLTSs provided by decision makers is presented by means of triangular membership function based on Eq. (I.20). Let H^k be a HFLTS provided by decision maker e^k, its envelope is defined by trapezoidal fuzzy members as $env(H^k) = T(a^k, b^k, c^k, d^k)$.

HFLTSs can be easily converted from comparative linguistic expression generated by the context-free grammar. From the perspective of comparative linguistic expressions [RMH12], the hesitant linguistic expressions can be divided into three types: between s_i and s_j $(i \neq 0, i \neq g)$; at least s_i and at most s_i, then we consider these three cases to compute the fuzzy envelope for HFLTSs. In computing the fuzzy envelope, some aggregation operators are used to aggregate the numerical scales of linguistic terms in HFLTSs, such as min and max operators, OWA operators.

Fig. 4 shows the fuzzy envelopes for HFLTSs between s_i and s_j, $\{s_i, s_{i+1}, ..., s_j\}$ $(i \neq 0; i \neq g)$, at least s_i, $\{s_i, s_{i+1}, ..., s_g\}$ and at most s_i, $\{s_0, s_1, ..., s_i\}$.

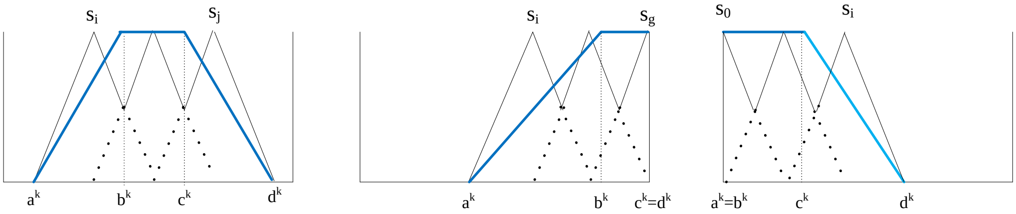


Fig.4 Fuzzy envelopes for HFLTSs

For the HFLPRs provided by different decision makers, the obtained fuzzy envelopes for HFLTSs are different. This shows the different understanding of decision makers for hesitant linguistic information.

The journal article associated to this part is:

- C.C. Li, R. M. Rodríguez, L. Martínez, Y.C. Dong, F. Herrera, Personalized individual semantics based on consistency in hesitant linguistic GDM with comparative linguistic expressions. Knowledge-Based Systems, 145 (2018) 156-165.

6.2.3 LSGDM with PIS

The LSGDM is a decision making process with a large number of decision makers, it provides a more complex situation than the usual GDM problems. The aim of this subsection is to provide the method to obtain the PIS of experts in LSGDM problems and also study the use of PIS in achieving a consensus. Therefore, we propose a consensus model based on PIS in LSGDM, which includes two processes: the PIS process and consensus process. In psychology it is highlighted that individuals rely on the opinions from their close friends or people with similar interests [HK02, WCFHV17], so that in this paper we provide a premise that decision makers having similar semantics and preferences are easier to communicate with each other. Based on this premise, we design a consensus process with the aim of helping decision makers more willing to change their preferences. While in the existing studies regarding consensus in LSGDM, the willingness of decision makers to change their judgments according to the suggestions provided by the moderator is ignored. We conduct the consensus model in LSGDM from two processes:

(1) PIS process: In the PIS process, a consistency-driven model aiming at maximizing the consistency of linguistic preference relation is proposed to obtain the individual semantics. By solving the model, the personalized numerical scales of linguistic terms for each expert are obtained.

(2) Consensus process: The consensus process with PIS is to help decision makers adjust their judgments by analyzing the similarities of preferences and semantics among decision makers. As mentioned earlier a noticeable drawback usually found in large groups is that decision makers do not want to modify their preferences in the feedback process, to overcome the drawback, the proposed consensus process consists of a consensus measure phase and a feedback recommendation phase as follows:

- **Consensus measure phase.** This phase is provided to obtain the consensus degree of the large group. First, by measuring the individual consensus degree, two opposing consensus groups are obtained: one group is the decision makers with acceptable consensus, denoted as G_A, and the other is the decision maker with unacceptable consensus, denoted as G_U.

The more the decision makers in the first group, the better the consensus is. Let $E = \{e_1, e_2, ..., e_m\}$ be the set of decision makers, then the consensus level is computed as follows,
$CL = \frac{\#G_A}{m}$
with $CL \in [0, 1]$, where $\#G_A$ means the number of decision makers in G_A.

- **Feedback recommendation phase.** This phase generates the recommendation for decision makers in group G_U and adjusts their preferences based on consensus rules to improve the consensus. First, considering the difference among individual semantics, a PIS based clustering method to classify decision makers with similar individual semantics is proposed, the obtained clusters are called semantic-based clusters. Then based on the opposing consensus groups and semantic-based clusters, recommendation rules are proposed for designing a feedback for decision makers with unacceptable consensus, to help them reach a higher consensus. Generally, the recommendation rule includes the identification rule and the direction rule. The identification rule is used to find out the decision maker in the group with unacceptable consensus which is needed to change his/her preferences. The decision maker e_κ, whose consensus level satisfies $CL_\kappa = \max_{e_k \in G_U} CL_k$, should change his/her preferences. The direction rule finds out the direction to change the preferences of decision maker e_κ according to his/her corresponding decision maker with similar semantics in group G_A.

The journal article associated to this part is:

- C.C. Li, Y.C. Dong, F. Herrera, A consensus model for large-scale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups. Submitted to IEEE Transaction on Fuzzy Systems.

7 Discussion of results

In this section, we make a discussion about the results obtained in each stage of the thesis.

7.1 Numerical scale to connect the 2-tuple linguistic models

The numerical scale model is an extension of the 2-tuple linguistic model to address unbalanced linguistic terms. From Propositions 1 and 2, the link between numerical scale model and 2-tuple linguistic model and proportional linguistic 2-tuple model is constructed. It is shown that by setting certain values of numerical scale, the numerical scale model is equal to other two models in some sense.

For the link between numerical scale model and the model based on a linguistic hierarchy, a revised retranslation process, providing a basis for the connection with numerical scale model, is presented. It simplifies the computations of original process, and does not change the essence of it. Besides, it is proved that the results obtained are same as the ones obtained by the original retranslation process. By setting certain numerical scale of linguistic terms, the equivalence of the linguistic computational models between these two models is proved.

From the above analysis, the advantages of numerical scale model are that it can deal with both balanced and unbalanced linguistic term sets. With this desired feature, the application of the numerical scale model can be extended into a more flexible decision making context.

7.2 Personalized numerical scales in linguistic decision making problems

The personalization of individual semantics under different decision making environments is discussed.

7.2.1 PIS model for CW

The constructed CW framework with PIS provides a process about dealing with decision making problems that words mean different things for different people; it includes three parts: individual semantics translation, computation process and individual semantics retranslation. Compared with the classic CW framework [Yag04], it emphasizes the different individual semantics for different experts. Based on the premise that the consistency should be still hold if the linguistic preference relation provided by decision makers is transformed into interval fuzzy preference relation with numerical scales, a consistency-driven optimization model to show the numerical meaning of linguistic terms for each expert is proposed, it is a fundamental PIS model. This model provides a basis for the construction of PIS model in other linguistic decision making contexts to obtain individual semantics, such as hesitant linguistic decision making and LSGDM.

7.2.2 PIS in hesitant linguistic decision making

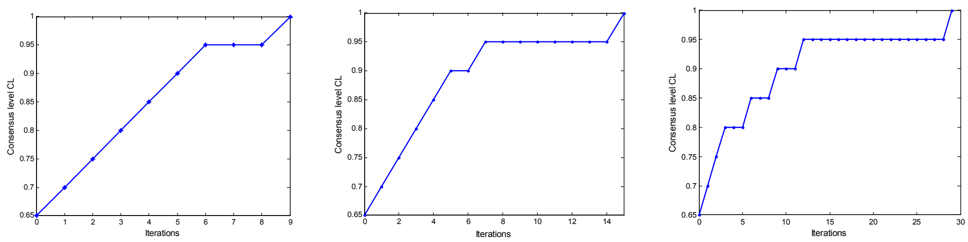
The process to derive the individual semantics of HFLTSs is divided into two parts: obtain the personalized individual numerical scale of linguistic terms and obtain the fuzzy envelope for HFLTSs to show PIS.

According to the fundamental PIS model, a consistency-driven optimization model in hesitant linguistic decision making is presented to obtain the individual numerical scales of linguistic terms in HFLTSs. The model is constructed based on an average consistency measure for HFLPRs, which is determined by the average consistency degree of all linguistic preference relations associated to the HFLPR.

Based on the individual numerical scales of linguistic terms, the PIS of HFLTSs is constructed and expressed as a trapezoidal fuzzy membership function. The use of the PIS provides a novel way to show decision makers' numerical meaning individually, and presents a method to deal with the fact that words mean different things for different people in hesitant linguistic decision making. But in the paper, the application of the PIS of HFLTS in solving decision making problems is not discussed.

7.2.3 LSGDM with PIS

In LSGDM problems with PIS, we discuss a method for improving the consensus considering the characteristics of LSGDM. The proposed consensus model is provided based on a natural premise: decision makers having similar semantics and preferences are easier to communicate with each other. Theoretically, we propose the fuzzy clustering algorithm with individual semantics and recommendation rules to conduct the consensus reaching process. To show the feasibility of the proposed consensus model, we make a simulation analysis to analyze the relation between the consensus level and the effect of the changing extent of the decision makers' preferences in the feedback process. It is observed that the consensus level can be improved through the consensus model, which demonstrates that our proposal provides an effective way to build consensus in LSGDM with PIS. Set α be the changing extent of decision makers' preference, and the number of the iterations depends on the value of α. Fig. 5 provides the variation trend of the consensus level.



$\alpha = 2$ $\alpha = 5$ $\alpha = 8$

Fig.5 The process to improve the consensus

But, the influence of the previous premise in achieving a consensus in LSGDM is not discussed. If without the previous premise, what the variation of the consensus levels is in the consensus reaching process in the LSGDM, is not studied in the paper.

8 Concluding remarks

In this section we present the results obtained from the research carried out during this PhD dissertation. These results follow the common goal of personalizing individual semantics for experts in decision making problems with linguistic information. This study focuses on the idea of personalizing individual semantics of experts and discusses its use in decision making problems with CW. We have constructed consistency-driven optimization-based models to obtain the PIS of linguistic information for different experts under three different decision making contexts, and utilized the theoretical analysis and simulation analysis to show the feasibility and reliability of the proposed models.

As mentioned before, the first objective is to show the connection between the numerical scale model and the 2-tuple linguistic models. With the connection, the desired property of numerical scale model that can deal with both the balanced and unbalanced linguistic terms is proved. The connection between the numerical scale model and 2-tuple linguistic model and proportional 2-tuple linguistic model has been stated in [WH06]. The model based on a linguistic hierarchy is well suited to deal with unbalanced linguistic terms, we propose a connection between the model based on a linguistic hierarchy and the numerical scale model, and prove the equivalence of the linguistic computational models between these two models.

Then, based on the numerical scale model, in order to fulfill the previous fact that words mean different things for different people, we propose consistency-driven approaches to investigate the PIS in classical decision making, hesitant linguistic decision making and LSGDM problems. The proposed approaches in deriving the PIS of experts are based on consistency measurements of linguistic preference relations.

In classical linguistic decision making, a consistency-driven optimization-based model, which provides a fundamental PIS approach, is proposed to personalize and represent the different individual semantics of linguistic terms with linguistic preference relations. The outputs of the proposed model in the form of interval numerical values show the different numerical meaning of linguistic terms for different experts.

In hesitant linguistic decision making, an average consistency measure of HFLPRs is proposed. It is determined by the average consistency of all linguistic preference relations associated to the HFLPR. Based on the average consistency, an average consistency-driven optimization model to obtain the personalized numerical scales of linguistic terms for different experts is proposed, which provides a basis for constructing the PIS for HFLTSs. The PISs of HFLTSs are represented by the fuzzy envelope of HFLTSs expressed by the trapezoidal fuzzy membership functions, which provides a new way to compute and represent the numerical meaning of HFLTSs.

In LSGDM, we incorporate the idea that different experts have different understanding of words in the consensus process with the aim of improving the willingness of decision makers in adjusting their preference in feedback process. First, a natural premise that individuals are more likely to communicate with people that have similar semantics and preference with them, is proposed. To make easier the computation with decision makers' preference in the LSGDM, we consider two types of groups for classifying the large group: consensus-based groups and semantics-based groups. By proposing the consensus measure, two opposing consensus-based groups are obtained based on the individual consensus degrees. Considering the previous premise, for obtaining the semantics-based groups, a fuzzy clustering method is provided to classify the decision makers into several groups based on the PIS. Then, a consensus rule under the LSGDM context, which is constructed based on two opposing consensus groups and semantics-based groups, is proposed to help experts more willing to change their preferences in the feedback process, so that the consensus in the LSGDM can be improved.

Conclusiones

En esta sección presentamos los resultados obtenidos en la investigación llevada a cabo durante esta tesis doctoral. Estos resultados persiguen el objetivo de personalizar la semántica individual para los expertos en problemas de toma de decisiones utilizando información lingüística. Este estudio se centra en la idea de personalizar la semántica individual de los expertos y analiza su uso en los problemas de toma de decisiones con CW. Hemos construido modelos de optimización basados en la coherencia para obtener el PIS de información lingüística para diferentes expertos en tres contextos diferentes de toma de decisiones, y utilizamos el análisis teórico y el análisis de simulación para mostrar la viabilidad y fiabilidad de los modelos propuestos.

Como se mencionó anteriormente, el primer objetivo es mostrar la conexión entre el modelo de escala numérica y los modelos lingüísticos de 2-tuplas. Con dicha conexión, se prueba la propiedad deseada del modelo de escala numérica que puede tratar tanto con los términos lingüísticos balanceados como desbalanceados. La conexión entre el modelo de escala numérica, el modelo lingüístico de 2-tuplas y el modelo lingüístico proporcional de 2-tuplas se ha establecido en [WH06]. El modelo basado en una jerarquía lingüística es muy adecuado para tratar los términos lingüísticos desbalanceados, proponemos una conexión entre el modelo basado en una jerarquía lingüística y el modelo de escala numérica, y mostramos la equivalencia de los modelos computacionales lingüísticos entre estos dos modelos.

A continuación, basándonos en el modelo de escala numérica, para cumplir con el hecho anterior de que las palabras significan cosas diferentes para diferentes personas, proponemos enfoques basados en la coherencia para investigar el PIS en la toma de decisiones clásica, toma de decisiones lingüísticas vacilantes y problemas LSGDM. Los enfoques propuestos para derivar el PIS de expertos se basan en mediciones de coherencia de las relaciones de preferencia lingüística.

En la toma de decisiones lingüística clásica, se propone un modelo de optimización basado en la coherencia, que proporciona un enfoque PIS fundamental, para personalizar y representar las diferentes semánticas individuales de los términos lingüísticos con las relaciones de preferencia lingüística. Los resultados del modelo propuesto en forma de valores numéricos de intervalo muestran el diferente significado numérico de los términos lingüísticos para diferentes expertos.

En decisiones lingüísticas vacilantes, se propone una medida de consistencia promedio de HFLPRs. Está determinado por la consistencia promedio de todas las relaciones de preferencia lingüística asociadas al HFLPR. Con base en la consistencia promedio, se propone un modelo de optimización impulsado por la consistencia promedio para obtener escalas numéricas personalizadas de términos lingüísticos para diferentes expertos, lo que proporciona una base para construir el PIS para HFLTS. Los PIS de HFLTS están representados por la envolvente difusa de HFLTS expresada por las funciones de pertenencia difusa trapezoidal, que proporciona una nueva forma de calcular y representar el significado numérico de los HFLTS.

En LSGDM, incorporamos la idea de que los diferentes expertos tienen una comprensión diferente de las palabras en el proceso de consenso con el objetivo de mejorar la disposición de quienes toman las decisiones para ajustar sus preferencias en el proceso de retroalimentación. En primer lugar, se propone la premisa natural de que los individuos se comunican mejor con personas que tienen una semántica y preferencias similares con ellos. Para facilitar el cálculo con la preferencia de los responsables de la toma de decisiones en el LSGDM, consideramos dos tipos de grupos para clasificar al grupo grande: grupos basados en consenso y grupos semánticos. Al proponer la medida de consenso, se obtienen dos grupos de consenso opuestos basados en los grados de consenso individuales. Teniendo en cuenta la premisa anterior, para obtener los grupos basados en la semántica, se proporciona un método de agrupamiento difuso para clasificar a los que toman las decisiones en varios grupos basados en el PIS. Luego, se propone una regla de consenso bajo el contexto LSGDM, que se basa en dos grupos de consenso opuestos y grupos semánticos, para ayudar a los expertos a cambiar sus preferencias en el proceso de retroalimentación, para que el consenso en el LSGDM pueda ser mejorado.

9 Future works

In presenting the researches in this thesis, new further topics to extend the studies come out. In what follows, some interesting research lines are introduced which are worth to be explored in the near future.

9.1 Development of more novel approaches to show PISs of experts

More novel ideas and approaches to reflect the different individual semantics of words for different experts should be studied. In this thesis, considering the idea that words mean different things for different people, we propose some approaches for personalizing individual semantics for decision makers under three different decision making environments. The proposed methods to show PIS are all focusing on solving the consistency-driven optimization-based models with the aim of maximizing the consistency of linguistic preference relations. By solving the models, the personalized numerical scales of linguistic terms are obtained, which can reflect the different individual semantics among decision makers.

But except the approaches based on the consistency property, it is considered that there should be more approaches to show the different understanding of words for difference people, such as the methods with sentimental analysis. Therefore, in the future, more approaches for reflecting the different individual semantics of experts would be studied.

9.2 The further use of PIS for HFLTSs in hesitant linguistic decision making

There should be more discussions about the use of fuzzy envelope of HFLTSs in decision making problems. In the thesis, we only study the approaches to represent the individual semantics of HFLTS for different experts, and the use of PIS for HFLTS in hesitant decision making is not studied. In future works, we will discuss more about the use of the individual semantics for HFLTSs in decision making,

(1) The use in GDM problems. In GDM with HFLPR where each expert has PIS for linguistic information, the computation process with the fuzzy envelope of HFLTSs, which are expressed as trapezoidal fuzzy membership functions, will be discussed to obtain the best alternative(s).

(2) Study the consensus in GDM. Consensus is always an important issue in the GDM with preference relation, with the aim of obtaining a consistent and most satisfactory result. Achieving a consensus among experts in hesitant linguistic GDM with PIS is an interesting topic to be investigated.

(3) Comparative study. It is clear that there are many different decision making approaches to tackle hesitant linguistic information. However, it is also necessary to provide a comparison framework to evaluate and compare the performance of the proposed model with PIS with the existing studies.

9.3 The application of PIS approaches

The PIS provides a novel way to show the individual difference in understanding the meaning of words, it extends the classical way of providing same semantics of linguistic information for different people in decision making problems. Investigating the PIS of words for different people is a new topic in decision making field; it would be an interesting idea to study its application in future works to increase the feasibility of the approaches proposed in the thesis.

A social network is the graph of relationships and interactions within a group of individuals and plays a fundamental role as a medium for the spread of information, ideas, and influence among its members. The decision making problem in social network often has a large number of experts, the application of PIS in LSGDM with social network would be a good topic for future studies. It is sure that under the social network context, more desired properties about the PIS and the LSGDM can be found.

Opinion dynamics [DeG74] is closely related to the management of public opinions and a research tool widely used to investigate the opinion evolution in many collective phenomena. With the idea of opinion dynamics, the process to control and manipulate the experts' opinion in GDM so that a satisfied or consensus result can be obtained would be a good idea to discuss.

To show more the flexibility and feasibility of the proposed PIS approaches, except in the situation under social network and opinion dynamics, the application of the PIS in more real decision making problems needs to be studied and discussed in the future.

Chapter II

Publications: Published Papers

1 Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information

- Y.C. Dong, C.C. Li, F. Herrera, Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. Information Sciences, 367 (2016) 259-278.
  - Status: **Published**.
  - Impact Factor (JCR 2016): 4.832
  - Subject Category: Computer Science, Artificial Intelligence, Ranking 16 / 123 (Q1).

Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information

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Abstract: The 2-tuple linguistic representation model is widely used as a basis for computing with words (CW) in linguistic decision making problems. Two different models based on linguistic 2-tuples (i.e., the model of the use of a linguistic hierarchy and the numerical scale model) have been developed to address term sets that are not uniformly and symmetrically distributed, i.e., unbalanced linguistic term sets (ULTSs). In this study, we provide a connection between these two different models and prove the equivalence of the linguistic computational models to handle ULTSs. Further, we propose a novel CW methodology where the hesitant fuzzy linguistic term sets (HFLTSs) can be constructed based on ULTSs using a numerical scale. In the proposed CW methodology, we present several novel possibility degree formulas for comparing HFLTSs, and define novel operators based on the mixed 0-1 linear programming model to aggregate the hesitant unbalanced linguistic information.

Keywords: Unbalanced linguistic term set, hesitant linguistic term set, numerical scale model, computing with words.

1. Introduction

Using linguistic information in decision making problems implies the need for computing with words (CW) [18, 19, 25, 27, 28, 29, 36, 37, 38, 59, 60, 61]. Several different linguistic computational models for CW have been presented in [4, 7, 8, 21, 26, 28, 41, 42, 56, 57]. In particular, Herrera and Martínez [21] initiated the 2-tuple linguistic representation model. This model is well suited for dealing with linguistic term sets that are uniformly and symmetrically distributed and results in matching of the elements in the initial linguistic terms. The 2-tuple linguistic model has been successfully used in a wide range of applications

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(e.g., [2, 13, 31, 33, 50, 52]). An overview on the advances with this representation model can be studied in [34, 35]. In recent years, there exist two notable progresses of the 2-tuple linguistic model: unbalanced linguistic term set (ULTS) and hesitant fuzzy linguistic term set (HFLTS). First, the 2-tuple linguistic model presented in Herrera and Martínez [21] only guarantees accuracy in dealing with the linguistic term sets that are uniformly and symmetrically distributed. In some cases, the term sets that are not uniformly and symmetrically distributed, i.e., ULTSs (see Fig. 1) [20], are often used in decision making problems.

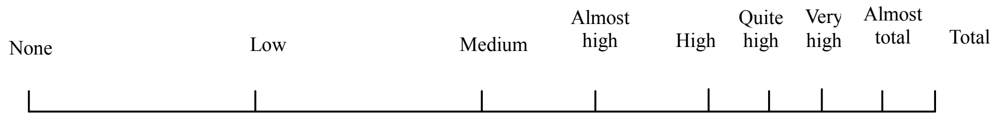

Fig. 1. Example of an ULTS of 8 labels [20]

Two different models based on linguistic 2-tuples have been developed to address ULTSs.

- The first model was presented in Herrera et al. [20], based on a linguistic hierarchy [9, 17, 22, 55] (applied to information retrieval system [24], consensus models [5, 6], aggregation operators [23, 39], and olive oil sensory evaluation [32], among others).
- The second model was presented and developed in Wang and Hao [51, 53], and Dong et al. [15, 16], which is referred to in this study as the numerical scale model. Wang and Hao [51, 53] proposed a generalized version (i.e., the proportional 2-tuple linguistic representation model) of the 2-tuple linguistic representation model to deal with ULTSs. The Wang and Hao model is based on the concepts of symbolic proportion and the canonical characteristic values (CCVs) of linguistic terms.

Traditional linguistic 2-tuples and proportional 2-tuples are used in the Herrera and Martínez model [21] and the Wang and Hao model [51, 53], respectively. By defining the concept of numerical scale, Dong et al. [11, 15] proposed the numerical scale model based on traditional linguistic 2-tuples to deal with ULTSs, and proved that setting a certain numerical scale in the numerical scale model yields the Wang and Hao model. Truck and Malenfant [47] proposed a unified model based on a vectorial approach to integrate both models. Also, Abchir and Truck [1] proposed an extension of the 2-tuple linguistic model to address ULTSs in an elegant and concise way.

Meanwhile, when using the 2-tuple linguistic model [21] in decision making problems, experts provide a single term as an expression of their knowledge (or preferences). However, in some situations, the experts cannot easily provide a single term as an expression of their knowledge; they may prefer to think of several terms at the same time to provide their preferences instead of a single linguistic term. Several methodologies that are involved in using linguistic expressions instead of single terms have been proposed in [3, 30, 45]. In particular, Rodríguez et al. [42] introduced the concept of a HFLTS, which provides a linguistic and computational basis for increasing the richness of linguistic elicitation based on the use of context-free grammar through the use of comparative terms. Rodríguez et al. [43] further developed a group decision making (GDM) model that address comparative linguistic expressions based on HFLTSs. Based on the concept of HFLTS described by Rodríguez et al., Zhu and Xu [62] developed consistency measures for hesitant fuzzy linguistic preference relations, and Wei et al. [54] developed comparison methods and studied the aggregation theory for HFLTSs, among others. Rodríguez et al. [44] presented a complete review of hesitant fuzzy sets and recent results on HFLTS. Prior studies have significantly advanced decision making analysis with HFLTSs, but the HFLTS model presented in Rodríguez et al. [42] is based on the 2-tuple linguistic model [21] without ULTSs. It is natural that both the balanced linguistic term sets and ULTSs can be used as a basis to construct HFLTSs.

Despite these 2-tuple linguistic models are quite useful, there still exist the following two gaps that should be filled:

- As mentioned above, both the model based on a linguistic hierarchy and the numerical scale model can address ULTSs. Therefore, the challenge naturally becomes how to connect these two different models to address ULTSs. So, the first aim of this study is to provide a connection between these two models. The analytical results in Section 3 will show the equivalence of these two linguistic computational models.
- It is natural that both the balanced linguistic term sets and ULTSs could be used as a basis for constructing HFLTSs. So, a second aim of this paper is to propose a CW methodology based on the numerical scale model and ULTSs to handle hesitant unbalanced linguistic information. In Section 4 we propose the novel CW methodology based on the numerical scale model, in which ULTSs can be used to construct HFLTSs.

The remainder of this paper is organized as follows: Section 2 introduces basic knowledge regarding linguistic 2-tuples setting numerical scales, as well as the model to address ULTSs. This is followed by Section 3, which shows the equivalence between the model based on a linguistic hierarchy and the numerical scale model to address ULTSs. Section 4 proposes the novel CW methodology for hesitant unbalanced linguistic information. In Section 5 we define two hesitant linguistic aggregation operators based on numerical scale model, HLWA and HLOWA operators, together with an algorithm for obtaining the aggregation results. Section 6 concludes this paper with final remarks.

2. Preliminaries

This section introduces the basic knowledge regarding the linguistic 2-tuples (subsection 2.1), the numerical scales (subsection 2.2), as well as the model to address ULTSs (subsection 2.3).

2.1. The 2-tuple linguistic representation model

The 2-tuple linguistic representation model, presented in Herrera and Martínez [21], represents the linguistic information by a 2-tuple (s_i, α), where $s_i \in S$ and $\alpha \in [-0.5, 0.5)$. This linguistic model defines a function with the purpose of making transformations between linguistic 2-tuples and numerical values. Formally, let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and $\beta \in [0, g]$ a value representing the result of a symbolic aggregation operation. Then, the 2-tuple that expresses the equivalent information to β is obtained by means of the following function:

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5), \tag{1}$$

where

$$\Delta(\beta) = (s_i, \alpha),\ with \begin{cases} s_i, i = round(\beta) \\ \alpha = \beta - i,\ \alpha \in [-0.5, 0.5) \end{cases}. \tag{2}$$

Clearly, Δ is a one to one mapping function. For convenience, its range is denoted as $\overline{S}$. Accordingly, Δ has an inverse function with $\Delta^{-1} : \overline{S} \rightarrow [0, g]$ and $\Delta^{-1}((s_i, \alpha)) = i + a$.

A computational model has been developed for the Herrera and Martínez model, in which the following exists:

(1) A 2-tuple comparison operator: let (s_k, α) and (s_l, r) be two 2-tuples. Then:

(i) If $k < l$,then (s_k, α) is smaller than (s_l, r).

(ii) If $k = l$, then

(a) If $\alpha = r$, then (s_k, α), (s_l, r) represents the same information.

(b) If $\alpha < r$, then (s_k, α) is smaller than (s_l, r).

(2) A 2-tuple negation operator:

$$Neg((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha))) \tag{3}$$

(3) Several 2-tuple aggregation operators have been developed (see [21, 34]). For example, let $L = \{l_1, ..., l_m\}$, where $l_i \in \overline{S}$ is a set of terms to aggregate, and let $w = \{w_1, ..., w_m\}$ be an

associated weighting vector that satisfies $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$. Then, based on the ordered weighted averaging (OWA) operator [58], the 2-tuple ordered weighted averaging (TOWA) operator in the Herrera and Martínez model is computed as

$$TOWA_w(l_1,...,l_m) = \Delta(\sum_{i=1}^{m} w_i y_i^*), \tag{4}$$

where y_i^* is the i th largest of the y_i values, and $y_i = \Delta^{-1}(l_i)$.

Let $s_i, s_j \in S$ be two simple terms. Xu [57] defined the deviation measure between s_i and s_j as follows: $d(s_i,s_j) = \frac{|i-j|}{g+1}$. For linguistic 2-tuples $(s_i,\alpha),(s_j,r) \in \overline{S}$, Dong et al. [12] similarly defined the deviation measure between (s_i,α) and (s_j,r) as follows, $d((s_i,\alpha),(s_j,\gamma)) = \frac{|\Delta^{-1}((s_i,\alpha)) - \Delta^{-1}((s_j,\gamma))|}{g+1}$. If only one pre-established linguistic label set is used in a decision making model, Dong et al. [12] simply considered

$$d((s_i,\alpha),(s_j,\gamma)) = |\Delta^{-1}((s_i,\alpha)) - \Delta^{-1}((s_j,\gamma))| \tag{5}$$

2.2. Numerical scale model

By defining the concept of *numerical scale*, Dong et al. [15] proposed an extension of the 2-tuple linguistic representation model to address ULTSs.

Definition 1: [15] Let $S = \{s_i | i = 0,1,2,...,g\}$ be a linguistic term set, and R be the set of real numbers. The function: $NS: S \rightarrow R$ is defined as a numerical scale of S, and $NS(s_i)$ is called the numerical index of s_i. If the function NS is strictly monotone increasing, then NS is called an ordered numerical scale.

Definition 2: [15] Let S, $\overline{S}$, NS on S be as before. For $(s_i,\alpha) \in \overline{S}$, the numerical scale $\overline{NS}$ on $\overline{S}$ is defined by

$$\overline{NS}((s_i,\alpha)) = \begin{cases} NS(s_i) + \alpha \times (NS(s_{i+1}) - NS(s_i)), \alpha \geq 0 \\ NS(s_i) + \alpha \times (NS(s_i) - NS(s_{i-1})), \alpha < 0 \end{cases} \tag{6}$$

To simplify the notation, $\overline{NS}$ will also be denoted as NS in this study.

Proposition 1: [15] Setting $NS(s_i) = i \, (i = 0,1,...,g)$ yields the Herrera and Martínez model.

The semantics of linguistic terms used in the Wang and Hao model are defined by symmetrical trapezoidal fuzzy numbers. If the semantics of s_i is defined by $T[b_i - \sigma_i, b_i, c_i, c_i + \sigma_i]$, in the Wang and Hao model, the CCV of s_i is $\frac{b_i + c_i}{2}$, i.e.,

$CCV(s_i) = \frac{b_i + c_i}{2}$.

Proposition 2: [15] Setting $NS(s_i) = CCV(s_i)(i = 0,1,...,g)$ yields the Wang and Hao model.

Propositions 1 and 2 provide a good linkage of the numerical scale to the Herrera and Martínez model and the Wang and Hao model.

Note 1: By setting the numerical scale $NS(s_i) = CCV(s_i)(i = 0,1,...,g)$, Dong et al. [15] showed that the Wang and Hao model can be redescribed as a linguistic model based on traditional 2-tuples. For notational simplicity, we use traditional 2-tuples throughout this study.

2.3. The model to address ULTSs

The model of Herrera et al. [20] to deal with ULTSs is based on a linguistic hierarchy and the 2-tuple linguistic representation model. Linguistic hierarchy has been presented and developed in [9, 17, 22, 55]. Over the linguistic hierarchy, a computational symbolic model based on the 2-tuple is defined to accomplish processes of CW [22, 48, 49].

A linguistic hierarchy is a set of levels where each level is a linguistic term set with a different granularity from the remaining levels of the hierarchy. Each level belonging to a linguistic hierarchy is denoted as $l(t, n(t))$, with t being a number that indicates the level of the hierarchy and $n(t)$ being the granularity of the linguistic term set of t. Generally, the linguistic term set $S^{n(t+1)}$ of level $t+1$ is obtained from its predecessor $S^{n(t)}$ as $l(t, n(t)) \rightarrow l(t+1, 2 \cdot n(t) - 1)$.

In linguistic hierarchies LH, the transformation function between terms from different levels to represent 2-tuple linguistic representations is defined as follows [22]: for any linguistic levels t and t', $TF_{t'}^{t} : l(t, n(t)) \rightarrow l(t', n(t'))$, such that

$$TF_{t'}^{t}(s_i^{n(t)}, \alpha^{n(t)}) = \Delta(\frac{\Delta^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1}). \quad (7)$$

Generally, in the computational model defined for the linguistic hierarchy LH, any level in the LH may be selected to unify the multigranular linguistic information in the computational model defined for the linguistic hierarchy LH. In this study, the maximum level t_m in the LH is used, i.e., $l(t_m, n(t_m)) = S^{n(t_m)} = \{s_0^{n(t_m)}, ..., s_{n(t_m)-1}^{n(t_m)}\}$.

Any 2-tuple linguistic representation (s_i, α) can be transformed by the unbalanced linguistic transformation process into the term in $LH = \bigcup_t l(t, n(t))$ and vice versa. The detailed transformation process is given as follows (see Fig. 2):

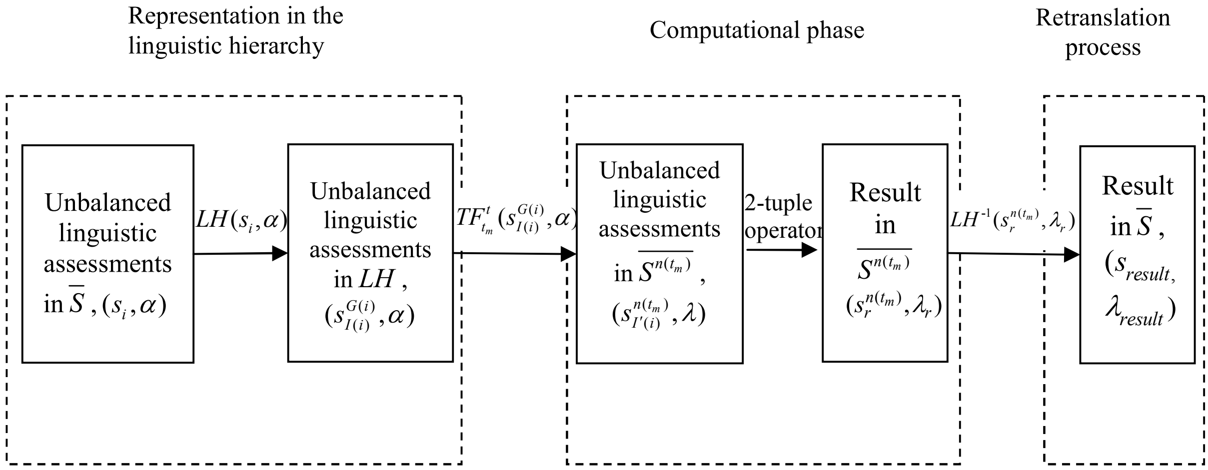

Fig.2. The unbalanced linguistic transformation process [20]

(1) Representation in the linguistic hierarchy: the representation algorithm uses the linguistic hierarchy LH to model the unbalanced terms in $\overline{S}$. Therefore, the first step towards accomplishing the process of CW is to transform the unbalanced terms in $\overline{S}$ into their corresponding terms in the LH, by means of the transformation function ψ associating each unbalanced linguistic 2-tuple (s_i, α) with its respective linguistic 2-tuple in $LH(\overline{S})$, i.e.,

$$\psi : \overline{S} \to LH(\overline{S}), \tag{8}$$

such that $\psi(s_i, \alpha) = (s_{I(i)}^{G(i)}, \alpha)$, for $\forall (s_i, \alpha) \in \overline{S}$.

(2) Computational phase: it accomplishes the process of CW by using the computation model defined for the linguistic hierarchy. First, it uses Eq.(7) to transform $(s_{I(i)}^{G(i)}, \alpha)$ into linguistic 2-tuples in $\overline{S^{n(t_m)}}$, denoted as $(s_{I'(i)}^{n(t_m)}, \lambda)$, i.e.,

$$(s_{I'(i)}^{n(t_m)}, \lambda) = \Delta\left(\frac{\Delta^{-1}(s_{I(i)}^{G(i)}, \alpha) \cdot (n(t_m) - 1)}{G(i) - 1}\right). \tag{9}$$

Then, the computational model developed for the 2-tuple linguistic representation model is used over $\overline{S^{n(t_m)}}$ with a result denoted as $(s_r^{n(t_m)}, \lambda_r) \in \overline{S^{n(t_m)}}$.

(3) Retranslation process: A retranslation process is used to transform the result $(s_r^{n(t_m)}, \lambda_r) \in \overline{S^{n(t_m)}}$ into the unbalanced term in $\overline{S}$, by using the transformation function ψ^{-1}, i.e.,

$$\psi^{-1} : LH(\overline{S}) \to \overline{S}, \tag{10}$$

such that $\psi^{-1}(s_r^{n(t_m)}, \lambda_r) = (s_{result,} \lambda_{result}) \in \overline{S}$.

The details of the methodology to address ULTSs are described in Herrera et al. [20].

3. Connecting the numerical scale model to the model based on a linguistic hierarchy

In this section, we provide a connection between the model based on a linguistic hierarchy [20] and the numerical scale model [15]. Specifically, ULTSs are redefined in Section 3.1, a revised retranslation process in the model based on a linguistic hierarchy is proposed in Section 3.2, the equivalence between both models is analysed in Section 3.3, and an illustrative example is provided in Section 3.4.

Fig. 3 illustrates the proof process of the connection between the model based on a linguistic hierarchy and the numerical scale model. Specifically, Proposition 3 proves the equivalence of the revised and original retranslation processes in the model based on a linguistic hierarchy. Propositions 4 and 5 guarantee the equivalence of the aggregation operators and negation operators used in these two models, respectively. Finally, using the Propositions 3 and 4 in [15], we find that the numerical scale model can provide a unified framework to integrate the Herrera and Martínez model [21], the Wang and Hao model [51, 53], and the model based on a linguistic hierarchy [20].

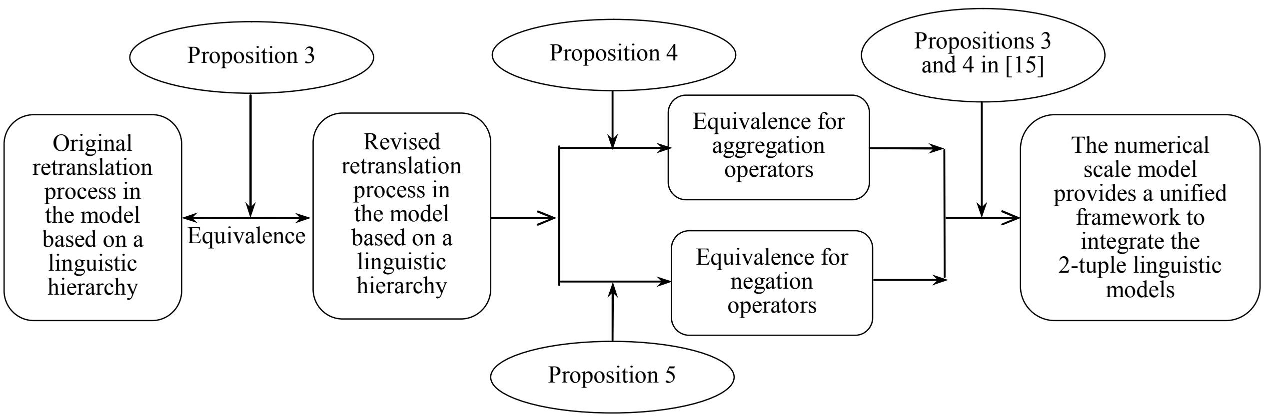

Fig.3. The proof process of the connection between the model based on a linguistic hierarchy and the numerical scale model

Note 2: In our conference paper [14], we provide some preliminary results regarding the equivalence between the numerical scale model and the model based on a linguistic hierarchy. In this study, we validate this equivalence by providing the proof of Proposition 3. Besides, to fully explain the content for readers, we will briefly introduce the results presented in the conference paper [14].

3.1. Definition of ULTSs

In the model of Herrera et al. [20] and the Wang and Hao model [51], ULTSs are defined in different ways. Specifically, the concept of the midterm and the concept of equally informative CCVs are used in these two models, respectively. For unified notation, inspired by the midterm used in Herrera et al. [20] and equally informative CCVs presented in Wang and Hao [51], this paper redefines ULTSs based on numerical scale (see Definition 3).

Definition 3: Let $S=\{s_0,s_1,...,s_g\}$ and NS on S be as before, and let s^* be the middle term in S. Then, S is a uniformly and symmetrically distributed linguistic term set if the following two conditions are satisfied:

(1) There exists a unique constant $\lambda>0$ such that $NS(s_i)-NS(s_j)=\lambda(i-j)$ for $i,j=0,1,...,g$.

(2) Let $\overline{\overline{S}}=\{s|s\in S, s>s^*\}$ and $\underline{\underline{S}}=\{s|s\in S, s<s^*\}$. Let $\#(\overline{\overline{S}})$ and $\#(\underline{\underline{S}})$ be cardinality of $\overline{\overline{S}}$ and $\underline{\underline{S}}$, respectively. Then, $\#(\overline{\overline{S}})=\#(\underline{\underline{S}})$.

If S is a uniformly and symmetrically distributed term set, then S is called a balanced linguistic term set (with respect to NS). Otherwise, S is called an ULTS.

Clearly, the ULTSs in both the model based on a linguistic hierarchy [20] and the Wang and Hao model [51] satisfy this new definition.

3.2. The revised retranslation process in the model based on a linguistic hierarchy

In the model based on a linguistic hierarchy [20], a retranslation process is used to transform the terms in *LH* into the terms in the ULTS *S*. Here, we present a revised retranslation process, providing a basis for connecting it with the numerical scale model. Meanwhile, we show that the results, obtained through the revised retranslation process, are the same as the ones obtained by the original retranslation process.

Let S, $\overline{S}$ and $s_{I'(i)}^{n(t_m)}$ be as in Section 2.3. Let $(s_x^{n(t)},\alpha)$ be any 2-tuple term in LH, and let $(s_r^{n(t_m)},\lambda)=TF_{t_m}^{t}(s_x^{n(t)},\alpha)$. The main idea of the revised retranslation process is based on the use of the deviation measure (i.e., Eq. (5)). Without the loss of generality, if $s_{I'(k)}^{n(t_m)}\le(s_r^{n(t_m)},\lambda)\le s_{I'(k+1)}^{n(t_m)}$, then the revised retranslation process $\psi^{-1\prime}$ can be described as Eqs. (11)-(13):

$$(s_{result*},\lambda_{result*})=\psi^{-1\prime}(s_x^{n(t)},\alpha), \tag{11}$$

where,

$$s_{result*}=\begin{cases} s_k, & d(s_{I'(k)}^{n(t_m)},(s_r^{n(t_m)},\lambda))<d(s_{I'(k+1)}^{n(t_m)},(s_r^{n(t_m)},\lambda)) \\ s_{k+1}, & d(s_{I'(k)}^{n(t_m)},(s_r^{n(t_m)},\lambda))\geq d(s_{I'(k+1)}^{n(t_m)},(s_r^{n(t_m)},\lambda)) \end{cases}, \tag{12}$$

and

$$\lambda_{result*}=\begin{cases} \dfrac{d(s_{I'(k)}^{n(t_m)},(s_r^{n(t_m)},\lambda))}{d(s_{I'(k)}^{n(t_m)},s_{I'(k+1)}^{n(t_m)})}, & s_{result*}=s_k \\ -\dfrac{d(s_{I'(k+1)}^{n(t_m)},(s_r^{n(t_m)},\lambda))}{d(s_{I'(k)}^{n(t_m)},s_{I'(k+1)}^{n(t_m)})}, & s_{result*}=s_{k+1} \end{cases}. \tag{13}$$

Let $(s_{result},\lambda_{result})\in\overline{S}$ be the transformed unbalanced term based on the original retranslation process ψ^{-1}, associated with $(s_x^{n(t)},\alpha)$. Then, we have Proposition 3.

Proposition 3: For any 2-tuple term $(s_x^{n(t)},\alpha)$ in the LH, $\psi^{-1}(s_x^{n(t)},\alpha)=\psi^{-1\prime}(s_x^{n(t)},\alpha)$, i.e., $(s_{result},\lambda_{result})=(s_{result*},\lambda_{result*})$.

The proof of Proposition 3 is provided in the Appendix.

Proposition 3 guarantees that the result, obtained by the revised retranslation process, is the same as the one obtained by the original retranslation process.

Note 3: The revised retranslation process will be more convenient for connecting the model based on a linguistic hierarchy to the numerical scale model, which is discussed in Section 3.3. As such, in the rest of this study the revised retranslation process is adopted. To simplify the notation, the revised retranslation process $\psi^{-1\prime}$ is still denoted as ψ^{-1}.

3.3. Equivalence between the numerical scale model and the model based on a linguistic hierarchy

Before connecting both models, we propose an approach to set the numerical scale. Let S, $\overline{S}$ and $s_{I'(i)}^{n(t_m)}$ be as before. This approach to set the numerical scale is described as Eq. (14):

$$NS(s_i)=\Delta^{-1}(s_{I'(i)}^{n(t_m)}),\ i=1,2,...,g \tag{14}$$

If the numerical scale is set as Eq. (14), this section shows the equivalence of the linguistic computational models between both models. Because the comparison operators defined in the model based on a linguistic hierarchy and the numerical scale model are the same, we only analyse the equivalence for the aggregation operators and negation operators in the rest of this section.

3.3.1. Equivalence for aggregation operators

When analysing the equivalence of the aggregation operators, we only consider the OWA operator. The results for the other aggregation operators are similar. In the model based on a linguistic hierarchy and the numerical scale model, the OWA operators can be defined as Definitions 4 and 5, respectively.

Definition 4: Let S, $\overline{S}$ and $S^{n(t_m)}$ and $\overline{S^{n(t_m)}}$ be as before. Let $L=\{l_1,...,l_m\}$, where $l_i \in \overline{S}$ is a set of terms to aggregate. Let $(s_{i'}^{n(t_m)},\lambda_{i'})$ be the corresponding 2- tuple term in $\overline{S^{n(t_m)}}$, associated with l_i. Let $w=\{w_1,w_2,...,w_m\}$ be an associated weighting vector that satisfies $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$. The 2-tuple ordered weighted averaging (TOWA) operator in the model based on a linguistic hierarchy is computed as

$$TOWA_w^{LH}(l_1,l_2,...,l_m)=\psi^{-1}(\Delta(\sum_{i=1}^{m} w_i y_i^*)) \tag{15}$$

where y_i^* is the i th largest of the y_i values, and $y_i=\Delta^{-1}(s_{i'}^{n(t_m)},\lambda_{i'})$.

Definition 5: Let S and $\overline{S}$ be as before. Let $L=\{l_1,...,l_m\}$, where $l_i \in \overline{S}$ is a set of terms to aggregate. Let NS be an ordered numerical scale over $\overline{S}$ and $w=\{w_1,w_2,...,w_m\}$ be an associated weighting vector that satisfies $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$. The TOWA operator under NS is computed as

$$TOWA_w^{NS}(l_1,l_2,...,l_m)=NS^{-1}(\sum_{i=1}^{m} w_i y_i^*) \tag{16}$$

where y_i^* is the i th largest of the y_i values, and $y_i = NS(l_i)$.

Before analysing the equivalence of $TOWA_w^{LH}$ and $TOWA_w^{NS}$, we provide Lemmas 1-3.

Lemma 1: Let NS be an ordered numerical scale, i.e., $NS(s_i) < NS(s_{i+1})$. For $\forall y \in [NS(s_0), NS(s_g)]$, if $NS(s_i) \leq y \leq NS(s_{i+1})$, then the inverse operation of NS is

$$NS^{-1}(y)=\begin{cases}(s_i,\dfrac{y-NS(s_i)}{NS(s_{i+1})-NS(s_i)}), & y<\dfrac{NS(s_{i+1})+NS(s_i)}{2}\\ (s_{i+1},\dfrac{y-NS(s_{i+1})}{NS(s_{i+1})-NS(s_i)}), & y \geq \dfrac{NS(s_{i+1})+NS(s_i)}{2}\end{cases} \tag{17}$$

The proof of Lemma 1 is provided in [14].

Lemma 2: Let $(s_J^{n(t_m)},\lambda)$ be the corresponding 2-tuple term in $\overline{S^{n(t_m)}}$, associated with $(s_i,\alpha)\in \overline{S}$. If the numerical scale is set as Eq. (14), i.e., $NS(s_i)=\Delta^{-1}(s_{I'(i)}^{n(t_m)})$ $(i=1,2,..,g)$. Then,

$$NS(s_i,\alpha)=\Delta^{-1}(s_J^{n(t_m)},\lambda). \tag{18}$$

The proof of Lemma 2 is provided in [14].

Lemma 3: For any $s \in \overline{S^{n(t_m)}}$, $NS^{-1}(\Delta^{-1}(s)) = \psi^{-1}(s)$ if $NS(s_i) = \Delta^{-1}(s_{I'(i)}^{n(t_m)})$ $(i = 1, 2, ..., g)$.

The proof of Lemma 3 is provided in [14].

Using Lemmas 1-3 yields Proposition 4.

Proposition 4: Let S and $\overline{S}$ be as before. Let $L = \{l_1, ..., l_m\}$, where $l_i \in \overline{S}$ is a set of terms to aggregate and $w = \{w_1, w_2, ..., w_m\}$ is an associated weighting vector. Then,

$$TOWA_w^{LH}(l_1, l_2, ..., l_m) = TOWA_w^{NS}(l_1, l_2, ..., l_m) \tag{19}$$

under the condition that the numerical scale is set as Eq. (14), i.e., $NS(s_i) = \Delta^{-1}(s_{I'(i)}^{n(t_m)})$, $i = 1, 2, ..., g$.

The proof of Proposition 4 is provided in [14].

Proposition 4 guarantees the equivalence of the OWA operators, used in the model based on a linguistic hierarchy and the numerical scale model, if the numerical scale is set as Eq. (14).

3.3.2. Equivalence for negation operators

The negation operators can be defined as Definitions 6 and 7, respectively.

Definition 6: Let S, $\overline{S}$, $S^{(t(m))}$ and $\overline{S^{(t(m))}}$ be as before. Let $s \in \overline{S}$, and let s' be the corresponding 2-tuple term in $\overline{S^{(t(m))}}$, associated with s. Then, the negation operator in the model based on a linguistic hierarchy is defined as

$$Neg^{LH}(s) = \psi^{-1}(Neg(s')). \tag{20}$$

Definition 7: Let S and $\overline{S}$ be as before and NS be an ordered numerical scale over $\overline{S}$. For any $s \in \overline{S}$, the negation operator under NS is defined as

$$Neg^{NS}(s) = \begin{cases} NS^{-1}(2NS(s^*) - NS(s)), & NS(s_0) \le 2NS(s^*) - NS(s) < NS(s_g) \\ null, others \end{cases}, \tag{21}$$

where $null$ denotes undefined elements.

Lemma 4: If the numerical scale is set as Eq. (14), i.e., $NS(s_i) = \Delta^{-1}(s_{I'(i)}^{n(t_m)})\,(i = 1, 2, ..., g)$, then NS is an ordered numerical scale, such that $NS(s_0) = 0, NS(s_{\frac{g}{2}}) = \frac{n(t_m) - 1}{2}$, and $NS(s_g) = n(t_m) - 1$.

The proof of Lemma 4 is provided in [14].

Proposition 5: Let S and $\overline{S}$ be as before. For any $s \in \overline{S}$, $Neg^{LH}(s) = Neg^{NS}(s)$.

The proof of Proposition 5 is provided in [14].

Proposition 5 guarantees the equivalence of the negation operators, used in the model based on a linguistic hierarchy and the numerical scale model.

Note 4: Setting the numerical scale of linguistic term sets is a key task in the numerical scale CW framework. In the existing studies, Dong et al. [15] defined the concepts of the transitive calibration matrix and its consistent index. By maximizing the consistency level, Dong et al. [15] developed an optimization-based approach to compute the numerical scale of a linguistic term set. Meanwhile, Wand and Hao [51] and Dong et al. [16] developed the CCV approach to set a numerical scale. The results in this section show the model based on a linguistic hierarchy and provide a novel numerical scale approach (i.e., Eq. (14)). If the numerical scale is set as Eq. (14), we analytically prove the equivalence of the linguistic computational models by equating the model based on a linguistic hierarchy and the numerical scale model.

3.4. Illustrative example

Herrera et al. [20] proposed an example to evaluate students' knowledge using different tests to obtain a global evaluation. In this example, an ULTS is used,

$$S=\{s_0=F, s_1=D, s_2=C, s_3=B, s_4=A\}$$

A student, Martina Grant, has completed six different tests to demonstrate her knowledge. The evaluations of tests are assessed using the ULTS S. The unbalanced linguistic assessments are listed in Table 1.

In this example, we set $t_m=3$. According to the model of Herrera et al., the values for $s_{I(i)}^{G(i)}$, $Brid(s_i)$ and $s_{I'(i)}^{n(t_m)}$ are listed in Table 2.

First, we illustrate the equivalence of the OWA operators, used in the model of Herrera et al. and the numerical scale model. In this example, the tests are equally important, i.e., $w=(1/6,1/6,1/6,1/6,1/6,1/6)^T$. We set the numerical scale: $NS(s_0)=0$, $NS(s_1)=4$, $NS(s_2)=6$, $NS(s_3)=7$, $NS(s_4)=8$. According to Definition 5, we have

$$TOWA_w^{NS}(s_4,s_1,s_1,s_2,s_3,s_4)=NS^{-1}(\frac{8+4+4+6+7+8}{6})=NS^{-1}(\frac{37}{6}).$$

Based on Eq. (17), we have $NS^{-1}(\frac{37}{6})=(s_2,0.16)=(C,0.16)$. i.e., $TOWA_w^{NS}(s_4,s_1,s_1,s_2,s_3,s_4)=(C,0.16)$.

Table 1 Unbalanced linguistic assessments in each exam

	T_1	T_2	T_3	T_4	T_5	T_6
M. Grant	s_4	s_1	s_1	s_2	s_3	s_4

Table 2 The values for $s_{I(i)}^{G(i)}$, $Brid(s_i)$ and $s_{I'(i)}^{n(t_m)}$

s_i	$s_{I(i)}^{G(i)}$	$Brid(s_i)$	$s_{I'(i)}^{n(t_m)}$
$s_0 = F$	$s_{I(0)}^{G(0)} = s_0^3$	False	s_0^9
$s_1 = D$	$s_{I(1)}^{G(1)} = s_1^3$	True	s_4^9
$s_2 = C$	$s_{I(2)}^{G(2)} = s_3^5$	True	s_6^9
$s_3 = B$	$s_{I(3)}^{G(3)} = s_7^9$	False	s_7^9
$s_4 = A$	$s_{I(4)}^{G(4)} = s_8^9$	False	s_8^9

In [20], Herrera et al. have shown $TOWA_w^{LH}(s_4, s_1, s_1, s_2, s_3, s_4) = (C, 0.16)$. So $TOWA_w^{LH}(s_4, s_1, s_1, s_2, s_3, s_4) = TOWA_w^{NS}(s_4, s_1, s_1, s_2, s_3, s_4) = (C, 0.16)$.

Next, we illustrate the equivalence of the negation operators, used in the model based on a linguistic hierarchy and the numerical scale model. For example, based on Eq. (20), $Neg^{LH}(s_2) = \psi^{-1}(Neg(s_6^9)) = \psi^{-1}(s_2^9)$.

Furthermore, because $s_0^9 = s_{I'(0)}^{n(t_3)} \le s_2^9 \le s_{I'(1)}^{n(t_3)} = s_4^9$ and $d(s_0^9, s_2^9) = d(s_2^9, s_4^9)$, using Eqs. (11)-(13) obtains $\Psi^{-1}(s_2^9) = (s_1, -0.5)$, i.e., $Neg^{LH}(s_2) = (s_1, -0.5) = (D, -0.5)$.

Based on Eq. (21), $Neg^{NS}(s_2) = NS^{-1}(8 - NS(s_2)) = NS^{-1}(2)$. According to Eq. (17), $NS^{-1}(2) = (s_1, -0.5)$, i.e., $Neg^{NS}(s_2) = (s_1, -0.5) = (D, -0.5)$. So, $Neg^{LH}(s_2) = Neg^{NS}(s_2)$ $= (D, -0.5)$.

4. Hesitant unbalanced linguistic information

In this section, we first propose the novel CW methodology based on the numerical scale model, in which ULTSs can be used to construct HFLTSs. Then, we define several possibility degree formulas for comparing HFLTSs.

4.1. Setting numerical scales for HFLTSs using ULTSs

In this subsection, for the use of ULTSs in HFLTSs, we set the numerical scales of HFLTSs. Torra [46] proposed the hesitant fuzzy sets. Based on the hesitant fuzzy sets, the concept of an HFLTS is introduced in [42], as Definition 8.

Definition 8: [42] Let $S = \{s_0, ..., s_g\}$ be a linguistic term set. An HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.

Definition 9: [42] Let $S = \{s_0, ..., s_g\}$ be a linguistic term set. Let H_S^1 and H_S^2 be two HFLTSs on S,

(1) The intersection $H_S^1 \cap H_S^2$ of H_S^1 and H_S^2 is defined by,

$H_s^1 \cap H_s^2 = \{s_k | s_k \in H_s^1 \ and \ s_k \in H_s^2\}$;

(2) The union $H_S^1 \cup H_S^2$ of H_S^1 and H_S^2 is defined by,

$H_S^1 \cup H_S^2 = \{s_k | s_k \in H_S^1 \ or \ s_k \in H_S^2\}$.

Definition 10: [42] Let H_S be an HFLTS of S. Let $H_S^- = \min_{s_i \in H_S}(s_i)$, $H_S^+ = \max_{s_i \in H_S}(s_i)$, and $env(H_S) = [H_S^-, H_S^+]$. Then, H_S^-, H_S^+ and $env(H_S)$ are called the lower bound, the upper bound and the envelope of H_S.

Definition 11: Let H_S be an HFLTS of S, and let NS be an ordered numerical scale over S. Then, the numerical scale of H_S is defined by,

$$HNS(H_S) = \{NS(s_i) | s_i \in H_S\} \tag{22}$$

Definition 12: Let H_S and NS be defined as before, then the negation operator of H_S is defined as follows,

$$Neg^{HNS}(H_S) = \{s | s = Neg^{NS}(h), h \in H_S\} \tag{23}$$

4.2. Possibility degree formulas for comparing HFLTSs

In this subsection, we define several possibility degree formulas for comparing HFLTSs.

Let $S = \{s_0, ..., s_g\}$ be a linguistic term set, let NS be an ordered numerical scale over S, and let $s_i, s_j \in S$. Then, if $s_i > s_j$, we define the degree that s_i is greater than s_j based on numerical scale as follows:

$$d(s_i > s_j) = NS(s_i) - NS(s_j). \tag{24}$$

If $s_i < s_j$, we define the degree that s_i is less than s_j based on numerical scale as follows:

$$d(s_i < s_j) = NS(s_j) - NS(s_i). \tag{25}$$

Let H_S^1 and H_S^2 be two HFLTSs on S, based on Eq. (24), the degree that H_S^1 is greater than H_S^2 is defined as follows:

$$d(H_S^1 > H_S^2) = \sum_{s_i \in H_S^1, s_j \in H_S^2, s_i > s_j} (NS(s_i) - NS(s_j)). \tag{26}$$

Similarly, based on Eq. (25), the degree that H_S^1 is less than H_S^2 is defined as follows:

$$d(H_S^1 < H_S^2) = \sum_{s_i \in H_S^1, s_j \in H_S^2, s_i < s_j} (NS(s_j) - NS(s_i)). \tag{27}$$

Example 1: Herrera et al. [20] proposed an example to evaluate students knowledge using different tests to obtain a global evaluation. Using the ULTS S in [20], i.e., $S = \{s_0 = F, \ s_1 = D, s_2 = C, \ s_3 = B, \ s_4 = A\}$. Based on Eq. (14), we set the numerical scale:

$NS(s_0)=0$, $NS(s_1)=4$, $NS(s_2)=6$, $NS(s_3)=7$ and $NS(s_4)=8$.

Let $H_S^1=\{s_0,s_1,s_2\}$ and $H_S^2=\{s_1,s_2,s_3\}$ be two HFLTSs on S. According to Eqs. (26) and (27), we have $d(H_S^1>H_S^2)=NS(s_2)-NS(s_1)=6-4=2$ and $d(H_S^1<H_S^2)=(NS(s_1)-NS(s_0))+(NS(s_2)-NS(s_0))+(NS(s_2)-NS(s_1))+(NS(s_3)-NS(s_0))$ $+(NS(s_3)-NS(s_1))+(NS(s_3)-NS(s_2))=23$.

Based on Eqs. (26) and (27), we propose the possibility degree formulas for comparing HFLTSs.

Definition 13: Let H_S^1 and H_S^2 be defined as before. The possibility degree that H_S^1 is equal to H_S^2 is defined by

$$P(H_S^1=H_S^2)=\frac{\#(H_S^1\cap H_S^2)}{\#(H_S^1\cup H_S^2)} \quad . \tag{28}$$

The possibility degree that H_S^1 is greater than H_S^2 is defined by

$$P(H_S^1>H_S^2)=(1-P(H_S^1=H_S^2))\cdot\frac{d(H_S^1>H_S^2)}{d(H_S^1>H_S^2)+d(H_S^1<H_S^2)}. \tag{29}$$

The possibility degree that H_S^1 is less than H_S^2 is defined by

$$P(H_S^1<H_S^2)=(1-P(H_S^1=H_S^2))\cdot\frac{d(H_S^1<H_S^2)}{d(H_S^1>H_S^2)+d(H_S^1<H_S^2)}. \tag{30}$$

Here, we introduce an algorithm to get $P(H_S^1=H_S^2)$, $P(H_S^1>H_S^2)$ and $P(H_S^1<H_S^2)$ (see Table 3). The time complexity of this algorithm is $O(n^2)$.

Table 3 An algorithm for obtianing the possibility degree between two HFLTSs

Input: Two HFLTSs based on S, H_S^1 and H_S^2
Output: $P(H_S^1=H_S^2)$, $P(H_S^1>H_S^2)$ and $P(H_S^1<H_S^2)$
Begin:
Step 1: Compute $\#(H_S^1\cap H_S^2)$ and $\#(H_S^1\cup H_S^2)$.
Let $P(H_S^1=H_S^2)=\frac{\#(H_S^1\cap H_S^2)}{\#(H_S^1\cup H_S^2)}$.
Step 2: Let $k=0$, $d_k(H_S^1>H_S^2)=0$ and $d_k(H_S^1<H_S^2)=0$.
Step 3: For each element $s_i\in H_S^1$
For each element $s_j\in H_S^2$
If $s_i\ge s_j$
Do $d_{k+1}(H_S^1>H_S^2)=d_k(H_S^1>H_S^2)+NS(s_i)-NS(s_j)$
$d_{k+1}(H_S^1<H_S^2)=d_k(H_S^1<H_S^2)$.

Else

Do $d_{k+1}(H_S^1 < H_S^2) = d_k(H_S^1 < H_S^2) + NS(s_j) - NS(s_i)$

$d_{k+1}(H_S^1 > H_S^2) = d_k(H_S^1 > H_S^2)$.

Let $k = k+1$.

Step 4: Let $d(H_S^1 > H_S^2) = d_k(H_S^1 > H_S^2)$ and $d(H_S^1 < H_S^2) = d_k(H_S^1 < H_S^2)$. Then,

$$P(H_S^1 > H_S^2) = (1 - P(H_S^1 = H_S^2)) \cdot \frac{d(H_S^1 > H_S^2)}{d(H_S^1 > H_S^2) + d(H_S^1 < H_S^2)},$$

$$P(H_S^1 < H_S^2) = (1 - P(H_S^1 = H_S^2)) \cdot \frac{d(H_S^1 < H_S^2)}{d(H_S^1 > H_S^2) + d(H_S^1 < H_S^2)}.$$

End

According to Eqs. (28)-(30), we provide the following comparison operators for HFLTSs as Definition 14.

Definition 14: If $P(H_S^1 > H_S^2) > \frac{1 - P(H_S^1 = H_S^2)}{2}$, then H_S^1 is superior to H_S^2, denoted by $H_S^1 \succ H_S^2$.

If $P(H_S^1 > H_S^2) = \frac{1 - P(H_S^1 = H_S^2)}{2}$, then H_S^1 is indifferent from H_S^2, denoted by $H_S^1 \sim H_S^2$.

If $P(H_S^1 > H_S^2) < \frac{1 - P(H_S^1 = H_S^2)}{2}$, then H_S^1 is inferior to H_S^2, denoted by $H_S^1 \prec H_S^2$.

Example 2: Let S and NS be as in Example 1. Let $H_S^1 = \{s_2, s_3\}$ and $H_S^2 = \{s_1, s_2, s_3, s_4\}$. Next, we compute the possibility degree between H_S^1 and H_S^2 in the following.

Clearly, $H_S^1 \cap H_S^2 = \{s_2, s_3\}$ and $H_S^1 \cup H_S^2 = \{s_1, s_2, s_3, s_4\}$, so $P(H_S^1 = H_S^2) = \frac{\#(H_S^1 \cap H_S^2)}{\#(H_S^1 \cup H_S^2)} = \frac{1}{2}$.

Because $d(H_S^1 > H_S^2) = 6$ and $d(H_S^2 > H_S^1) = 4$, thus,

$$P(H_S^1 > H_S^2) = (1 - P(H_S^1 = H_S^2)) \cdot \frac{d(H_S^1 > H_S^2)}{d(H_S^1 > H_S^2) + d(H_S^1 < H_S^2)} = \frac{1}{2} \cdot \frac{3}{5} = 0.3$$

and $P(H_S^1 < H_S^2) = (1 - P(H_S^1 = H_S^2)) \cdot \frac{d(H_S^1 < H_S^2)}{d(H_S^1 > H_S^2) + d(H_S^1 < H_S^2)} = \frac{1}{2} \cdot \frac{2}{5} = 0.2$.

It is obvious that $P(H_S^1 > H_S^2) = 0.3 > 0.2 = P(H_S^2 > H_S^1)$, i.e., $H_S^1 \succ H_S^2$.

Next, we present some desired properties of the comparison operators.

Property 1: $P(H_S^1 > H_S^2) + P(H_S^1 = H_S^2) + P(H_S^1 < H_S^2) = 1$.

The proof of Property 1 is provided in the Appendix.

Property 2: For two HFLTSs H_S^1 and H_S^2, the following statements are equivalent in showing that H_S^1 is superior to H_S^2, i.e., $H_S^1 \succ H_S^2$.

(1) $P(H_S^1 > H_S^2) > \dfrac{1 - P(H_S^1 = H_S^2)}{2}$

(2) $P(H_S^1 \geq H_S^2) > P(H_S^2 \geq H_S^1)$

(3) $P(H_S^1 > H_S^2) > P(H_S^2 > H_S^1)$

(4) $d(H_S^1 > H_S^2) > d(H_S^2 > H_S^1)$

The proof of Property 2 is provided in the Appendix.

Property 3: For any two HFLTSs H_S^1 and H_S^2,

(1) $H_S^1 \succ H_S^2$ if H_S^1 and H_S^2 satisfy one of the following relationships,

(i) $H_S^{1+} \geq H_S^{2+}$ and $H_S^{1-} > H_S^{2-}$, or $H_S^{1+} > H_S^{2+}$ and $H_S^{1-} \geq H_S^{2-}$

(ii) $H_S^{1-} < H_S^{2-} \leq H_S^{2+} < H_S^{1+}$ and

$$\sum_{H_S^{2+} < s_i \leq H_S^{1+}} \sum_{H_S^{2-} \leq s_j \leq H_S^{2+}} (NS(s_i) - NS(s_j)) > \sum_{H_S^{1-} \leq s_i < H_S^{2-}} \sum_{H_S^{2-} \leq s_j \leq H_S^{2+}} (NS(s_j) - NS(s_i))$$

(iii) $H_S^{2-} < H_S^{1-} \leq H_S^{1+} < H_S^{2+}$ and

$$\sum_{H_S^{1-} \leq s_i \leq H_S^{1+}} \sum_{H_S^{2-} \leq s_j < H_S^{1-}} (NS(s_i) - NS(s_j)) > \sum_{H_S^{1-} \leq s_i \leq H_S^{1+}} \sum_{H_S^{1+} < s_j \leq H_S^{2+}} (NS(s_j) - NS(s_i))$$

(2) $H_S^1 \sim H_S^2$ if H_S^1 and H_S^2 satisfy one of the following relationships,

(i) $H_S^{1-} = H_S^{2-}$ and $H_S^{1+} = H_S^{2+}$

(ii) $H_S^{1-} < H_S^{2-} \leq H_S^{2+} < H_S^{1+}$ and

$$\sum_{H_S^{2+} < s_i \leq H_S^{1+}} \sum_{H_S^{2-} \leq s_j \leq H_S^{2+}} (NS(s_i) - NS(s_j)) = \sum_{H_S^{1-} \leq s_i < H_S^{2-}} \sum_{H_S^{2-} \leq s_j \leq H_S^{2+}} (NS(s_j) - NS(s_i))$$

(iii) $H_S^{2-} < H_S^{1-} \leq H_S^{1+} < H_S^{2+}$ and

$$\sum_{H_S^{1-} \leq s_i \leq H_S^{1+}} \sum_{H_S^{2-} \leq s_j < H_S^{1-}} (NS(s_i) - NS(s_j)) = \sum_{H_S^{1-} \leq s_i \leq H_S^{1+}} \sum_{H_S^{1+} < s_j \leq H_S^{2+}} (NS(s_j) - NS(s_i))$$

The proof of Property 3 is provided in the Appendix.

Properties 2 and 3 provide several equivalent statements to implement the comparison operators between two HFLTSs.

Property 4: For any two HFLTSs H_S^1 and H_S^2 based on S, if S is a balanced linguistic term set, then

(1) $H_S^1 \succ H_S^2$ if and only if $NS(H_S^{1-}) + NS(H_S^{1+}) > NS(H_S^{2-}) + NS(H_S^{2+})$;

(2) $H_S^1 \sim H_S^2$ if and only if $NS(H_S^{1-}) + NS(H_S^{1+}) = NS(H_S^{2-}) + NS(H_S^{2+})$.

The proof of Property 4 is provided in the Appendix.

Wei et al. [54] provided a comparison method for HFLTSs under the condition that S is a balanced linguistic term set. Property 4 shows that the comparison results in our study are the same as that in Wei et al. [54], if S is a balanced linguistic term set.

5. Hesitant linguistic aggregation operators based on numerical scale model

In this section, we define the aggregation operators HLWA and HLOWA to aggregate the hesitant unbalanced linguistic information. Moreover, we provide an algorithm based on a mixed 0-1 linear programming model to obtain the aggregation results.

5.1. Hesitant unbalanced linguistic aggregation operators

In this subsection we define the novel HLWA and HLOWA operators. First, we introduce the definition of the convex combination of two linguistic terms, presented in [10].

Definition 15: [10] Let $S=\{s_0,s_1,...,s_g\}$ be a linguistic term set. For two linguistic terms s_i and s_j, the convex combination of s_i and s_j is defined as,

$$C^2(w_1,s_i,w_2,s_j)=w_1\odot s_i\oplus w_2\odot s_j=s_k\,, \tag{31}$$

where $w_1+w_2=1$, $w_1\geq 0$ and $w_2\geq 0$, $k=\min\{g,round\left(w_1\times i+w_2\times j\right)\}$, and "$round$" is the usual $round$ operation.

Based on Definition 15, we provide the novel convex combination of s_i and s_j based on numerical scale.

Definition 16: Let S be defined as before. For two linguistic terms s_i and s_j, the novel convex combination of s_i and s_j based on numerical scale is defined as,

$$C^2_{NS}(w_1,s_i,w_2,s_j)=round'\left(NS^{-1}\left(w_1\times NS(s_i)+w_2\times NS(s_j)\right)\right), \tag{32}$$

where $w_1+w_2=1$, $w_1\geq 0$, $w_2\geq 0$, and $round'$ is a novel $round$ operation over $\overline{S}$, i.e., $round'(s_k,\alpha)=s_k$.

Example 3: Let S and NS be as in Example 1. According to Eq. (32), if $w_1=w_2=0.5$, the novel convex combination of s_1 and s_2 based on numerical scale is

$$\begin{aligned}C^2_{NS}(0.5,s_1,0.5,s_2)&=round'(NS^{-1}(0.5\times NS(s_1)+0.5\times NS(s_2)))=round'(NS^{-1}(0.5\times 4+0.5\times 6))\\&=round'(NS^{-1}(5))=round'(s_2,-0.5)=s_2\end{aligned}$$

Similarly, if $w_1=0.2$ and $w_2=0.8$, the novel convex combination of s_3 and s_4 based on numerical scale is

$$\begin{aligned}C^2_{NS}(0.2,s_3,0.8,s_4)&=round'(NS^{-1}(0.2\times NS(s_3)+0.8\times NS(s_4)))=round'(NS^{-1}(0.2\times 7+0.8\times 8))\\&=round'(NS^{-1}(7.8))=round'(s_4,-0.2)=s_4\end{aligned}$$

Next, based on the novel convex combination of linguistic terms, we introduce the novel convex combination of HFLTSs.

Definition 17: Let S and NS be defined as before. Let H_S^1 and H_S^2 be two HFLTSs on S, then the novel convex combination of H_S^1 and H_S^2 is defined as follows:

$$C_{NS}^2(w_1, H_S^1, w_2, H_S^2) = \{round'(NS^{-1}(w_1 \times NS(s_i) + w_2 \times NS(s_j))) | s_i \in H_S^1, s_j \in H_S^2\}, \quad (33)$$

where $w_1 + w_2 = 1$, $w_1 \geq 0$ and $w_2 \geq 0$.

Let $\{H_S^1, H_S^2, ..., H_S^m\}$ be m HFLTSs on S, then the convex combination of m HFLTSs is defined as follows:

$$C_{NS}^m(w_1, H_S^1, w_2, H_S^2, ..., w_m, H_S^m)$$
$$= \{round'(NS^{-1}(w_1 \times NS(r_1) + w_2 \times NS(r_2) + ... + w_m \times NS(r_m))) | r_k \in H_S^k\}, \quad (34)$$

where $\sum_{k=1}^{m} w_k = 1$ and $w_k \geq 0$.

Example 4: Let $S = \{s_0, s_1, ..., s_6\}$ be a linguistic term set, and the numerical scale of S is defined as follows:

$NS(s_0) = 0$, $NS(s_1) = 5$, $NS(s_2) = 6$, $NS(s_3) = 7$, $NS(s_4) = 8$, $NS(s_5) = 10$, and $NS(s_6) = 16$. Let $H_S^1 = \{s_0, s_1\}$ and $H_S^2 = \{s_5, s_6\}$ be two HFLTSs on S, and let $w_1 = 0.5$ and $w_2 = 0.5$. Then,

$$C_{NS}^2(0.5, H_S^1, 0.5, H_S^2) = \{round'(NS^{-1}(0.5 \times 0 + 0.5 \times 10)), round'(NS^{-1}(0.5 \times 5 + 0.5 \times 10)),$$
$$round'(NS^{-1}(0.5 \times 0 + 0.5 \times 16)), round'(NS^{-1}(0.5 \times 5 + 0.5 \times 16))\}$$
$$= \{round'(NS^{-1}(5)), round'(NS^{-1}(7.5)), round'(NS^{-1}(8)), round'(NS^{-1}(10.5))\}$$
$$= \{round'(s_1), round'(s_4, -0.5), round'(s_4), round'(s_5, 0.08)\} = \{s_1, s_4, s_5\}.$$

Based on Example 4, we find that the novel convex combination result of H_S^1 and H_S^2 is not an ordered finite subset of the consecutive linguistic terms of S, i.e., it is not an HFLTS. So here, we define the extended HFLTS. The novel convex combination result of H_S^1 and H_S^2 and the result of the following novel operators are all extended HFLTSs.

Definition 18: Let S be defined as before. Any subset of S is called an extended HFLTS of S, denoted by EH_S.

Note 5: Clearly, the comparison operator in Section 3 can be used to compare extended HFLTSs. To simplify the notation, the EH_S will be denoted as H_S in this paper.

Based on Definition 17, we define the novel HLWA and HLOWA operators based on numerical scale as Definitions 19 and 20.

Definition 19: Let $\{H_S^1, H_S^2, ..., H_S^m\}$ be a set of HFLTSs on S, and let $\{w_1, w_2, ..., w_m\}$ be an associated weighting vector that satisfies $w_k \geq 0$ and $\sum_{k=1}^{m} w_k = 1$; then, the novel $HLWA$ operator based on numerical scale is defined as,

$$HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^1, w_2, H_S^2, ..., w_m, H_S^m) \quad (35)$$

Definition 20: Let $\{H_S^1, H_S^2, ..., H_S^m\}$ be a set of HFLTSs on S, and let $\{w_1, w_2, ..., w_m\}$ be an associated weighting vector that satisfies $w_k \geq 0$ and $\sum_{k=1}^{m} w_k = 1$; then, the novel $HLOWA$ operator based on numerical scale is computed as,

$$HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^{\sigma(1)}, w_2, H_S^{\sigma(2)}, ..., w_m, H_S^{\sigma(m)}) \quad (36)$$

where $(\sigma(1), \sigma(2), ..., \sigma(m))$ is the permutation of $(1, 2, ..., m)$ such that $H_S^{\sigma(k-1)} \succ H_S^{\sigma(k)}$ for $k = 2, 3, ..., m$.

Note 6: Compared with the HLWA and HLOWA operators provided in Wei et al. [54], our proposed HLWA and HLOWA operators are based on the numerical scale model and can not only address HFLTSs in the balanced linguistic context but also in the unbalanced linguistic context. Meanwhile, when using the novel HLWA and HLOWA operators to aggregate HFLTSs, the *round* function is only used one time, while in Wei et al. [54], the *round* function is used $m-1$ times. Thus, our proposed HLWA and HLOWA operators can provide a more accurate result.

Property 5: Let $\{H_S^1, H_S^2, ..., H_S^m\}$ be a set of HFLTSs on S. Then, the novel HLWA and HLOWA operators based on numerical scale satisfy the following properties,

(1) Boundary,

$\min\{H_S^k \mid k = 1, 2, ..., m\} \leq HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \leq \max\{H_S^k \mid k = 1, 2, ..., m\}$,

and $\min\{H_S^k \mid k = 1, 2, ..., m\} \leq HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \leq \max\{H_S^k \mid k = 1, 2, ..., m\}$.

(2) Idempotency,

$HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = H_S^z$, when $H_S^k = H_S^z$ for $k = 1, 2, ..., m$.

(3) Commutativity,

If $(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m})$ is any permutation of $(H_S^1, H_S^2, ..., H_S^m)$, then we have

$HLOWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) = HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$.

(4) Monotonicity,

$HLWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) > HLWA_w^{NS}(H_S^{\beta 1}, H_S^{\beta 2}, ..., H_S^{\beta m})$ if $H_S^{\alpha k} > H_S^{\beta k}$,

$HLOWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) > HLOWA_w^{NS}(H_S^{\beta 1}, H_S^{\beta 2}, ..., H_S^{\beta m})$ if $H_S^{\alpha k} > H_S^{\beta k}$.

(5) $HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = H_S^k$, when $w_k = 1$. $HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = H_S^k$, when $w_i = 1$ and $k = \sigma(i)$.

The proof of Property 5 is provided in the Appendix.

5.2. An algorithm for obtaining the aggregation results of the novel HLWA and HLOWA operators

In this subsection, we provide an algorithm based on a mixed 0-1 linear programming model to obtain the aggregation results of the novel HLWA and HLOWA operators.

Property 6: Let $\{H_S^1, H_S^2, ..., H_S^m\}$ be a set of HFLTSs based on S, and let $\{w_1, w_2, ..., w_m\}$ be an associated weighting vector that satisfies $w_k \geq 0$ and $\sum_{k=1}^{m} w_k = 1$. Let

$$s_L = round'(NS^{-1}(w_1 \times NS(H_S^{1-}) + w_2 \times NS(H_S^{2-}) + ... + w_m \times NS(H_S^{m-}))), \quad (37)$$

$$s_R = round'(NS^{-1}(w_1 \times NS(H_S^{1+}) + w_2 \times NS(H_S^{2+}) + ... + w_m \times NS(H_S^{m+}))), \quad (38)$$

then $HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \subseteq \{s_i | s_i \in S, s_L \leq s_i \leq s_R\}$.

Similarly, let $(\sigma(1), \sigma(2), ..., \sigma(m))$ be the permutation of $(1, 2, ..., m)$ such that $H_S^{\sigma(k-1)} \succ H_S^{\sigma(k)}$. Let

$$s_L = round'(NS^{-1}(w_1 \times NS(H_S^{\sigma(1)-}) + w_2 \times NS(H_S^{\sigma(2)-}) + ... + w_m \times NS(H_S^{\sigma(m)-}))), \quad (39)$$

$$s_R = round'(NS^{-1}(w_1 \times NS(H_S^{\sigma(1)+}) + w_2 \times NS(H_S^{\sigma(2)+}) + ... + w_m \times NS(H_S^{\sigma(m)+}))), \quad (40)$$

then $HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \subseteq \{s_i | s_i \in S, s_L \leq s_i \leq s_R\}$.

The proof of Property 6 is provided in the Appendix.

Let $\{H_S^1, H_S^2, ..., H_S^m\}$ be a set of HFLTSs based on S. In the following, we propose a mixed 0-1 linear programming model to obtain $HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$. The process to obatin $HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$ is similar.

Let s_L and s_R be as in Eqs. (37) and (38), and let s_h be any term in the set $\{s_i | s_i \in S, s_L \leq s_i \leq s_R\}$. The main idea of the mixed 0-1 linear programming model is to justify whether $s_h \in HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$ or not.

Let $U = \{w_1 \times NS(r_1) + w_2 \times NS(r_2) + ... + w_m \times NS(r_m) | r_k \in H_S^k\}$. We hope to find out $z \in U$ which is closest to $NS(s_h)$, i.e.,

$$\begin{cases} \min |z - NS(s_h)| \\ s.t.\ \ z \in U \end{cases} \tag{41}$$

Solving Eq. (41) obtains the optimal solution to z, denoted by z^*. In the following, we provide Proposition 6 to justify whether $s_h \in HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$ or not.

Proposition 6: Let $Q=(z^* - \frac{NS(s_h)+NS(s_{h-1})}{2})/(\frac{NS(s_h)+NS(s_{h+1})}{2} - z^*)$. $s_h \in HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$ if and only if $Q \geq 0$.

The proof of Proposition 6 is provided in the Appendix.

Proposition 7: Let $env(H_S^k) = [s_{I^{k-}}, s_{I^{k+}}]$. Then, model (41) can be equivalently transformed into model (42).

$$\begin{cases} \min |z - NS(s_h)| \\ s.t. \begin{cases} x_i^k \in \{0,1\},\ k = 1,2,...,m, i = I^{k-}, I^{k-}+1, ..., I^{k+}-1, I^{k+} \\ \sum_{i=I^{k-}}^{I^{k+}} x_i^k = 1,\ k = 1,2,...,m, \\ z = \sum_{k=1}^{m} \sum_{i=I^{k-}}^{I^{k+}} w_k \times x_i^k \times NS(s_i) \end{cases} \end{cases} \tag{42}$$

The proof of Proposition 7 is provided in the Appendix.

According to models (41) and (42) and Proposition 6, the algorithm for obtaining $HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$ is proposed (see Table 4).

Table 4 An algorithm for obtaining the aggregation result of novel HLWA operator

Input: A set of HFLTSs, $\{H_S^1, H_S^2, ..., H_S^m\}$, the associated weighting vector $\{w_1, w_2, ..., w_m\}$

Output: $H_S = HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$

Begin:

Step 1: Let

$s_L = round'(NS^{-1}(w_1 \times NS(H_S^{1-}) + w_2 \times NS(H_S^{2-}) + ... + w_m \times NS(H_S^{m-})))$,

$s_R = round'(NS^{-1}(w_1 \times NS(H_S^{1+}) + w_2 \times NS(H_S^{2+}) + ... + w_m \times NS(H_S^{m+})))$,

$H_S = \{s_L, s_R\}$.

Step 2: For each element $s_h \in \{s_{L+1}, s_{L+2}, ..., s_{R-2}, s_{R-1}\}$, we solve

$$\begin{cases}\min\left|z-NS(s_h)\right| \\ s.t.\begin{cases} x_i^k\in\{0,1\},\ \ k=1,2,...,m, i=I^{k-},I^{k-}+1,...,I^{k+}-1,I^{k+} \\ \sum_{i=I^{k-}}^{I^{k+}} x_i^k=1,\ \ k=1,2,...,m, \\ z=\sum_{k=1}^{m}\sum_{i=I^{k-}}^{I^{k+}} w_k\times x_i^k\times NS(s_i)\end{cases}\end{cases},$$

and obtain the optimal solution z^*.

Let $Q=(z^*-\frac{NS(s_h)+NS(s_{h-1})}{2})/(\frac{NS(s_h)+NS(s_{h+1})}{2}-z^*)$. If $Q\geq 0$, then $H_S=H_S\cup s_h$.

Step 3: Output H_S.

End

Note 7: According to Miller [40], an individual cannot simultaneously compare more than 7 ± 2 objects without confusion. Thus, the granularity of ULTSs must be less than 9. As a result, the proposed mixed 0-1 linear programming is not a large-scale optimization problem. Generally, a mixed 0-1 linear programming model with a few hundred binary variables can be effectively and rapidly solved by several software packages (e.g., Lingo).

Example 5: Let S be defined as in Example 4. Let $\{H_S^1,H_S^2,H_S^3,H_S^4\}$ be a set of HFLTSs on S. The HFLTSs $\{H_S^1,H_S^2,H_S^3,H_S^4\}$ and the associated weighting vector w are given as follows:

$H_S^1=\{s_0,s_1\}$, $H_S^2=\{s_5,s_6\}$, $H_S^3=\{s_5,s_6\}$, $H_S^4=\{s_0\}$, and $w=\{0.375,0.25,0.25,0.125\}$.

Next, we use the above algorithm to get the result of $HLWA_w^{NS}(H_S^1,H_S^2,H_S^3,H_S^4)$.

Step 1: Calculate s_L and s_R,

$$s_L=round'(NS^{-1}(0.375\times NS(s_0)+0.25\times NS(s_5)+0.25\times NS(s_5)+0.125\times NS(s_0)))=s_1$$
$$s_R=round'(NS^{-1}(0.375\times NS(s_1)+0.25\times NS(s_6)+0.25\times NS(s_6)+0.125\times NS(s_0)))=s_5$$

Step 2: For $s_h\in\{s_2,s_3,s_4\}$, we use the following model (43) to get optimal solution of z, z^*, respectively.

$$\begin{cases} \min |z - NS(s_h)| \\ s.t. \begin{cases} x_i^1 \in \{0,1\},\ i=0,1 \\ x_i^2 \in \{0,1\},\ i=5,6 \\ x_i^3 \in \{0,1\},\ i=5,6 \\ x_0^1 + x_1^1 = 1 \\ x_5^2 + x_6^2 = 1 \\ x_5^3 + x_6^3 = 1 \\ x_0^4 = 1 \\ 0.375 \times (0 \times x_0^1 + 5x_1^1) + 0.25 \times (10x_5^2 + 16x_6^2) + 0.25 \times (10x_5^3 + 16x_6^3) + 0 \times x_0^4 = z \end{cases} \end{cases} \tag{43}$$

By solving model (43), we obtain $z^* = \begin{cases} 6.5 & s_h = s_2 \\ 6.875 & s_h = s_3 \\ 8 & s_h = s_4 \end{cases}$.

Because $Q = (6.5 - \frac{NS(s_1) + NS(s_2)}{2}) / (\frac{NS(s_2) + NS(s_3)}{2} - 6.5)$ is non-existent, we have $s_2 \notin H_S$.

Because $Q = (6.875 - \frac{NS(s_2) + NS(s_3)}{2}) / (\frac{NS(s_3) + NS(s_4)}{2} - 6.875) > 0$, we have $s_3 \in H_S$.

Because $Q = (8 - \frac{NS(s_3) + NS(s_4)}{2}) / (\frac{NS(s_4) + NS(s_5)}{2} - 8) > 0$, we have $s_4 \in H_S$.

Thus, $HLWA_w^{NS}(H_S^1, H_S^2, H_S^3, H_S^4, H_S^5) = \{s_1, s_3, s_4, s_5\}$.

6. Conclusions

In this study, we propose a connection between the model based on a linguistic hierarchy and the numerical scale model, and prove the equivalence of the linguistic computational models by equating the model based on a linguistic hierarchy and the numerical scale model to address ULTSs. Furthermore, we propose the novel CW methodology to address HFLTSs based on the numerical scale model, where the HFLTSs can be constructed based on ULTSs.

The results in this paper show that the numerical scale model provides a unified framework to connect the traditional linguistic 2-tuples, the proportional 2-tuples and the model based on a linguistic hierarchy. Meanwhile, compared with the existing studies that address HFLTSs, the novel CW methodology can not only address HFLTSs in the balanced linguistic context but also in the unbalanced linguistic context.

It could be an interesting future research topic to discuss the use of the numerical scale model in hesitant unbalanced linguistic GDM problems.

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Appendix: Proofs

The proof of Proposition 3.

Let $S = S_L \cup S_C \cup S_R$, $S_L = S_{LE} \cup S_{LC}$, $S_R = S_{RE} \cup S_{RC}$, and $T_{LH} = \{t_{LE}, t_{LC}, t_{RE}, t_{RC}\}$. The definitions for S_L, S_C, S_R, S_{LE}, S_{LC}, S_{RE}, S_{RC}, t_{LE}, t_{LC}, t_{RE}, and t_{RC} are same as the ones used in the model of Herrera et al. [20].

For any 2-tuple $(s_x^{n(t)}, \alpha) \in LH(\overline{S})$, let $(s_r^{n(t_m)}, \lambda) = TF_{t_m}^{t}(s_x^{n(t)}, \alpha)$. Without loss of generality, $s_{I'(k)}^{n(t_m)} \le (s_r^{n(t_m)}, \lambda) \le s_{I'(k+1)}^{n(t_m)}$. Here, we only consider the case of $d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) < d(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))$. The proof for the case of $d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) \ge d(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))$ is similar.

When $d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) < d(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))$, according to the revised retranslation process (i.e., Eqs. (11)-(13)), we have $s_{result*} = s_k$, and $\lambda_{result*} = \dfrac{d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))}{d(s_{I'(k)}^{n(t_m)}, s_{I'(k+1)}^{n(t_m)})}$

In order to analyze the values for s_{result} and λ_{result} we consider two cases: Case A and Case B.

Case A: $G(k) = n(t)$ and $I(k) = x$. In Case A, according to the original retranslation process LH^{-1}, presented in Herrera et al. [20], we have $s_{result} = s_k = s_{result*}$.

Meanwhile, because $s_{I'(k)}^{n(t_m)} \le (s_r^{n(t_m)}, \lambda)$, $\alpha \ge 0$.

We continue to consider two subcases of Case A: Case A.1 and Case A.2.

Case A.1: $Brid(s_k) = False$. In Case A.1, since $LH^{-1}(s_x^{n(t)}, \alpha) = (s_k, \alpha)$,

$$\lambda_{result} = \alpha = \frac{d(s_x^{n(t)}, (s_x^{n(t)}, \alpha))}{d(s_x^{n(t)}, s_{x+1}^{n(t)})} = \frac{d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))}{d(s_{I'(k)}^{n(t_m)}, s_{I'(k+1)}^{n(t_m)})} = \lambda_{result*} \tag{44}$$

Case A.2: $Brid(s_k) = True$. Without loss of generality, we assume that $s_k \in S_R$ and $t_{RC} = t_{RE} - 1$ (the proofs for the other cases are similar). We continue to consider two subcases of Case A.2: Case A.2.1 and Case A.2.2.

Case A.2.1: $s_k \in S_{RE}$. In Case A.2.1, since $\alpha \ge 0$ and $s_k \in S_{RE}$, we have $LH^{-1}(s_x^{n(t)}, \alpha) = (s_k, \alpha)$. Similar to Eq. (44), we have $\lambda_{result} = \alpha = \dfrac{d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda))}{d(s_{I'(k)}^{n(t_m)}, s_{I'(k+1)}^{n(t_m)})} = \lambda_{result*}$.

Case A.2.2: $s_k \in S_{RC}$. In Case A.2.2, since $\alpha \ge 0$ and $s_k \in S_{RC}$, according to the original retranslation process LH^{-1},

$$\lambda_{result} = \left(\frac{\Delta^{-1}(s_k^{n(t)}, \alpha) \times (n(t+1) - 1)}{n(t) - 1}\right) - round\left(\frac{\Delta^{-1}(s_k^{n(t)}, \alpha) \times (n(t+1) - 1)}{n(t) - 1}\right) = 2\alpha$$

Meanwhile,

$$2\alpha=\frac{d(s_{2x}^{n(t)},(s_{2x}^{n(t)},2\alpha))}{d(s_{2x}^{n(t)},s_{2x+1}^{n(t)})}=\frac{d(s_{I'(k)}^{n(t_m)},(s_r^{n(t_m)},\lambda))}{d(s_{I'(k)}^{n(t_m)},s_{I'(k+1)}^{n(t_m)})}=\lambda_{result*}$$. So $\lambda_{result}=2\alpha=\lambda_{result*}$.

Case B: $G(k)\neq n(t)$ or $I(k)\neq x$. Then, $LH^{-1}(s_x^{n(t)},\alpha)=LH^{-1}(TF_{t'}^{t}(s_x^{n(t)},\alpha))$, with $t'\in\{t_{LE},t_{LC},t_{RC},t_{RE}\}$ being a level such that if $TF_{t'}^{t}(s_x^{n(t)},\alpha)=(s_{x'}^{n(t')},\alpha')$, then $G(k)=n(t')$ or $I(k)=x'$. By the proof of Case A, we also obtain $s_{result}=s_{result*}$ and $\lambda_{result}=\lambda_{result*}$. Based on Case A and Case B, we have $s_{result}=s_{result*}$ and $\lambda_{result}=\lambda_{result*}$.

This completes the proof of Proposition 3.

The proof of Property1.

Proof: By Definition 13, we have,

$$P(H_S^1>H_S^2)+P(H_S^1=H_S^2)+P(H_S^1<H_S^2)$$

$$=(1-P(H_S^1=H_S^2))\cdot(\frac{d(H_S^1>H_S^2)}{d(H_S^1>H_S^2)+d(H_S^1<H_S^2)}+\frac{d(H_S^1<H_S^2)}{d(H_S^1>H_S^2)+d(H_S^1<H_S^2)})+P(H_S^1=H_S^2)$$

$$=1-P(H_S^1=H_S^2)+P(H_S^1=H_S^2)=1$$

This completes the proof of Property 1.

The proof of Property 2.

Proof: From Property 1, we know $P(H_S^1>H_S^2)+P(H_S^1=H_S^2)+P(H_S^1<H_S^2)=1$, i.e., $P(H_S^1>H_S^2)+P(H_S^1<H_S^2)=1-P(H_S^1=H_S^2)$

If $P(H_S^1>H_S^2)>\frac{1-P(H_S^1=H_S^2)}{2}$, then $P(H_S^1<H_S^2)<\frac{1-P(H_S^1=H_S^2)}{2}$, it is obvious that $P(H_S^1>H_S^2)>P(H_S^2>H_S^1)$.

For $P(H_S^1\geq H_S^2)=P(H_S^1>H_S^2)+P(H_S^1=H_S^2)$, $P(H_S^2\geq H_S^1)=P(H_S^2>H_S^1)+P(H_S^1=H_S^2)$

If $P(H_S^1>H_S^2)>P(H_S^2>H_S^1)$, then $P(H_S^1\geq H_S^2)>P(H_S^2\geq H_S^1)$.

And, from Eqs. (29) and (30), if $P(H_S^1>H_S^2)>P(H_S^2>H_S^1)$, it is obvious that $d(H_S^1>H_S^2)>d(H_S^2>H_S^1)$.

This completes the proof of Property 2.

The proof of Property 3.

Proof: Since the proof of (1) and (2) is similar, here we only give the proof of (1).

Next we prove, if H_S^1 and H_S^2 satisfy the relationships (i), (ii) or (iii), then $H_S^1\succ H_S^2$.

(i) if $H_S^{1+}\geq H_S^{2+}$ and $H_S^{1-}>H_S^{2-}$, or $H_S^{1+}>H_S^{2+}$ and $H_S^{1-}\geq H_S^{2-}$, then we get three cases: $H_S^{2+}<H_S^{1-}$, $H_S^{2-}\leq H_S^{1-}<H_S^{2+}<H_S^{1+}$ or $H_S^{2-}<H_S^{1-}<H_S^{2+}\leq H_S^{1+}$.

If $H_S^{2+}<H_S^{1-}$, it is obvious that $d(H_S^2>H_S^1)=0$. According to Definitions 13 and 14, we get $P(H_S^1 > H_S^2) = 1 - P(H_S^1 = H_S^2) > \frac{1 - P(H_S^1 = H_S^2)}{2}$, thus $H_S^1 \succ H_S^2$.

If $H_S^{2-} \le H_S^{1-} < H_S^{2+} < H_S^{1+}$, for

$d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2) = \sum_{H_S^{2+} < s_i \le H_S^{1+}} \sum_{H_S^{2-} \le s_j \le H_S^{2+}} (NS(s_i) - NS(s_j)) > 0$.

If $H_S^{2-} < H_S^{1-} < H_S^{2+} \le H_S^{1+}$, for

$d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2) = \sum_{H_S^{1-} \le s_i \le H_S^{1+}} \sum_{H_S^{2-} \le s_j < H_S^{1-}} (NS(s_i) - NS(s_j)) > 0$.

Thus, according to Property 2, if $d(H_S^1 > H_S^2) > d(H_S^2 > H_S^1)$, then $H_S^1 \succ H_S^2$.

(ii) If $H_S^{1-} < H_S^{2-} \le H_S^{2+} < H_S^{1+}$, for

$$d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2) = \sum_{s_i \in H_S^1, s_j \in H_S^2, s_i > s_j} (NS(s_i) - NS(s_j)) - \sum_{s_i \in H_S^1, s_j \in H_S^2, s_j > s_i} (NS(s_j) - NS(s_i))$$

$$= \sum_{H_S^{2+} < s_i \le H_S^{1+}} \sum_{H_S^{2-} \le s_j \le H_S^{2+}} (NS(s_i) - NS(s_j)) - \sum_{H_S^{1-} \le s_i < H_S^{2-}} \sum_{H_S^{2-} \le s_j \le H_S^{2+}} (NS(s_j) - NS(s_i)) > 0$$, so $H_S^1 \succ H_S^2$.

(iii) If $H_S^{2-} < H_S^{1-} \le H_S^{1+} < H_S^{2+}$, for

$$d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2) = \sum_{s_i \in H_S^1, s_j \in H_S^2, s_i > s_j} (NS(s_i) - NS(s_j)) - \sum_{s_i \in H_S^1, s_j \in H_S^2, s_j > s_i} (NS(s_j) - NS(s_i))$$

$$= \sum_{H_S^{1-} \le s_i \le H_S^{1+}} \sum_{H_S^{2-} \le s_j < H_S^{1-}} (NS(s_i) - NS(s_j)) - \sum_{H_S^{1-} \le s_i \le H_S^{1+}} \sum_{H_S^{1+} < s_j \le H_S^{2+}} (NS(s_j) - NS(s_i)) > 0$$, so $H_S^1 \succ H_S^2$.

This completes the proof of Property 3.

The proof of Property 4.

Proof: Since the proof of (1) and (2) is similar, so here we only prove (1).

We prove the sufficient condition and the necessary condition. First, we prove the sufficient condition, i.e., if $NS(H_S^{1-}) + NS(H_S^{1+}) > NS(H_S^{2-}) + NS(H_S^{2+})$, then $H_S^1 \succ H_S^2$.

There are three possible relationships between H_S^1 and H_S^2 under the condition of $NS(H_S^{1-}) + NS(H_S^{1+}) > NS(H_S^{2-}) + NS(H_S^{2+})$. They are,

(i) $H_S^{1+} \ge H_S^{2+}$ and $H_S^{1-} > H_S^{2-}$, or $H_S^{1+} > H_S^{2+}$ and $H_S^{1-} \ge H_S^{2-}$

(ii) $H_S^{1-} < H_S^{2-} \le H_S^{2+} < H_S^{1+}$ and $NS(H_S^{1+}) - NS(H_S^{2+}) > NS(H_S^{2-}) - NS(H_S^{1-})$

(iii) $H_S^{2-} < H_S^{1-} \le H_S^{1+} < H_S^{2+}$ and $NS(H_S^{1-}) - NS(H_S^{2-}) > NS(H_S^{2+}) - NS(H_S^{1+})$

Next, we prove that $H_S^1 \succ H_S^2$ if H_S^1 and H_S^2 satisfy one of the above relationships.

(i) The proof of (i) is similar with the proof of (i) in Property 3, so we omit this proof here.

(ii) If $H_S^{1-} < H_S^{2-} \le H_S^{2+} < H_S^{1+}$ and $NS(H_S^{1+}) - NS(H_S^{2+}) > NS(H_S^{2-}) - NS(H_S^{1-})$, let $H_S^{1+\prime} = NS^{-1}(NS(H_S^{2-}) - NS(H_S^{1-}) + NS(H_S^{2+}))$, i.e.,

$NS(H_S^{1+\prime})-NS(H_S^{2+})=NS(H_S^{2-})-NS(H_S^{1-})$. Then

$d(H_S^1>H_S^2)-d(H_S^1<H_S^2)=\sum_{H_S^{1-}\le s_i\le H_S^{1+\prime}}\sum_{H_S^{2-}\le s_j\le H_S^{2+}}(NS(s_i)-NS(s_j))+\sum_{H_S^{1+\prime}<s_i\le H_S^{1+}}\sum_{H_S^{2-}\le s_j\le H_S^{2+}}(NS(s_i)-NS(s_j))$,

since
$$\sum_{H_S^{1-}\le s_i\le H_S^{1+\prime}}\sum_{H_S^{2-}\le s_j\le H_S^{2+}}(NS(s_i)-NS(s_j))=\sum_{H_S^{1-}\le s_i<H_S^{2-}}\sum_{H_S^{2-}\le s_j\le H_S^{2+}}(NS(s_i)-NS(s_j))+\sum_{H_S^{2+}<s_i\le H_S^{1+\prime}}\sum_{H_S^{2-}\le s_j\le H_S^{2+}}(NS(s_i)-NS(s_j))$$

$=[(NS(H_S^{2+})-NS(H_S^{2-})+1)\cdot(NS(H_S^{1+\prime})-NS(H_S^{2+}))]\cdot[NS(H_S^{1+\prime})-NS(H_S^{2+})-(NS(H_S^{2-})-NS(H_S^{1-}))]=0$,

and

$\sum_{H_S^{1+\prime}<s_i\le H_S^{1+}}\sum_{H_S^{2-}\le s_j\le H_S^{2+}}(NS(s_i)-NS(s_j))>0$, thus $d(H_S^1>H_S^2)-d(H_S^1<H_S^2)>0$, according to property 2, we arrive at that $H_S^1\succ H_S^2$.

(iii) If $H_S^{2-}<H_S^{1-}\le H_S^{1+}<H_S^{2+}$ and $NS(H_S^{1-})-NS(H_S^{2-})>NS(H_S^{2+})-NS(H_S^{1+})$, let $H_S^{2-\prime}=NS^{-1}(NS(H_S^{1-})-(NS(H_S^{2+})-NS(H_S^{1+})))$, i.e.,

$NS(H_S^{1-})-NS(H_S^{2-\prime})=NS(H_S^{2+})-NS(H_S^{1+})$, then

$d(H_S^1>H_S^2)-d(H_S^1<H_S^2)=\sum_{H_S^{1+}\le s_i\le H_S^{1+}}\sum_{H_S^{2-\prime}\le s_j<H_S^{2+}}(NS(s_i)-NS(s_j))+\sum_{H_S^{1+}\le s_i\le H_S^{1+}}\sum_{H_S^{2-}\le s_j<H_S^{2-\prime}}(NS(s_i)-NS(s_j))$

since $\sum_{H_S^{1+}\le s_i\le H_S^{1+}}\sum_{H_S^{2-\prime}\le s_j<H_S^{2+}}(NS(s_i)-NS(s_j))=$

$=[(NS(H_S^{1+})-NS(H_S^{1-})+1)\cdot(NS(H_S^{2+})-NS(H_S^{1+}))]\cdot[NS(H_S^{1-})-NS(H_S^{2-\prime})-(NS(H_S^{2+})-NS(H_S^{1+}))]=0$

and

$\sum_{H_S^{1+}\le s_i\le H_S^{1+}}\sum_{H_S^{2-}\le s_j<H_S^{2-\prime}}(NS(s_i)-NS(s_j))>0$, thus $d(H_S^1>H_S^2)-d(H_S^1<H_S^2)>0$, according to property 2, we arrive at that $H_S^1\succ H_S^2$.

Second, we prove the necessary condition, i.e., if $H_S^1\succ H_S^2$, then $NS(H_S^{1-})+NS(H_S^{1+})>NS(H_S^{2-})+NS(H_S^{2+})$.

According to Property 3, three cases may satisfy the condition of $H_S^1\succ H_S^2$.

(iv) $H_S^{1+}\ge H_S^{2+}$ and $H_S^{1-}>H_S^{2-}$, or $H_S^{1+}>H_S^{2+}$ and $H_S^{1-}\ge H_S^{2-}$

(v) $H_S^{1-}<H_S^{2-}\le H_S^{2+}<H_S^{1+}$

(vi) $H_S^{2-}<H_S^{1-}\le H_S^{1+}<H_S^{2+}$

We prove that the conclusion follows for each aforementioned case.

(iv) $H_S^{1+}\ge H_S^{2+}$ and $H_S^{1-}>H_S^{2-}$, or $H_S^{1+}>H_S^{2+}$ and $H_S^{1-}\ge H_S^{2-}$, it is easy to obtain that

$NS(H_S^{1-})+NS(H_S^{1+})>NS(H_S^{2-})+NS(H_S^{2+})$.

(v) We use "reduction to absurdity" to prove (v).

Assuming that if $H_S^{1-} < H_S^{2-} \le H_S^{2+} < H_S^{1+}$, and $NS(H_S^{1-}) + NS(H_S^{1+}) = NS(H_S^{2-}) + NS(H_S^{2+})$, then $H_S^1 \succ H_S^2$.

Because $d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2)$

$= [(NS(H_S^{2+}) - NS(H_S^{2-}) + 1) \cdot (NS(H_S^{1+}) - NS(H_S^{2+}))] \cdot [NS(H_S^{1+}) - NS(H_S^{2+}) - (NS(H_S^{2-}) - NS(H_S^{1-}))] = 0$,

which contradicts to the hypothesis of $d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2) > 0$. Thus

$NS(H_S^{1-}) + NS(H_S^{1+}) \ne NS(H_S^{2-}) + NS(H_S^{2+})$.

Assuming that if $H_S^{1-} < H_S^{2-} \le H_S^{2+} < H_S^{1+}$, and $NS(H_S^{1-}) + NS(H_S^{1+}) < NS(H_S^{2-}) + NS(H_S^{2+})$, then $H_S^1 \succ H_S^2$.

Let $H_S^{1-'} = NS^{-1}(NS(H_S^{2-}) - (NS(H_S^{1+}) - NS(H_S^{2+})))$, then similar to cases (ii) and (iii), we arrive at that $d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2) = \sum_{H_S^{1-} \le s_i < H_S^{1-'}} \sum_{H_S^{2-} \le s_j < H_S^{2+}} (NS(s_i) - NS(s_j)) < 0$, which contradicts to the hypothesis of $d(H_S^1 > H_S^2) - d(H_S^1 < H_S^2) > 0$.

Thus, if $H_S^1 \succ H_S^2$, then $NS(H_S^{1-}) + NS(H_S^{1+}) > NS(H_S^{2-}) + NS(H_S^{2+})$.

(vi) The proof of (vi) is similar with the proof of (v), so we omit this proof here.

This completes the proof of Property 4.

The proof of Property 5.

Proof: (1) Let $\min\{H_S^k | k = 1, 2, ..., m\} = H_S^\alpha$ and $\max\{H_S^k | k = 1, 2, ..., m\} = H_S^\beta$, then

$HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^1, w_2, H_S^2, ..., w_m, H_S^m) \le C_{NS}^m(w_1, H_S^\beta, w_2, H_S^\beta, ..., w_m, H_S^\beta)$

$= \{round'(NS^{-1}(w_1 \times NS(r_1) + w_2 \times NS(r_2) + ... + w_m \times NS(r_m))) | r_k \in H_S^\beta\} = H_S^\beta$

$HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^1, w_2, H_S^2, ..., w_m, H_S^m) \ge C_{NS}^m(w_1, H_S^\alpha, w_2, H_S^\alpha, ..., w_m, H_S^\alpha)$

$= \{round'(NS^{-1}(w_1 \times NS(r_1) + w_2 \times NS(r_2) + ... + w_m \times NS(r_m))) | r_k \in H_S^\alpha\} = H_S^\alpha$.

Hence, $\min\{H_S^k | k = 1, 2, ..., m\} \le HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \le \max\{H_S^k | k = 1, 2, ..., m\}$.

Similarly, we can get

$\min\{H_S^k | k = 1, 2, ..., m\} \le HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \le \max\{H_S^k | k = 1, 2, ..., m\}$.

(2) Since $H_S^k = H_S^z$ for $k = 1, 2, ..., m$, it follows that

$$HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^1, w_2, H_S^2, ..., w_m, H_S^m)$$
$$= C_{NS}^m(w_1, H_S^z, w_2, H_S^z, ..., w_m, H_S^z) = H_S^z$$

$$HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^{\sigma(1)}, w_2, H_S^{\sigma(2)}, ..., w_m, H_S^{\sigma(m)})$$
$$= C_{NS}^m(w_1, H_S^z, w_2, H_S^z, ..., w_m, H_S^z) = H_S^z$$

hence, $HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = H_S^z$, when $H_S^k = H_S^z$ for $k = 1, 2, ..., m$.

(3) According to Definition 20, let

$HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^{\sigma(1)}, w_2, H_S^{\sigma(2)}, ..., w_m, H_S^{\sigma(m)})$,

$HLOWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) = C_{NS}^m(w_1, H_S^{\sigma(\alpha 1)}, w_2, H_S^{\sigma(\alpha 2)}, ..., w_m, H_S^{\sigma(\alpha m)})$.

Because $(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m})$ is any permutation of $(H_S^1, H_S^2, ..., H_S^m)$, so $HLOWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) = HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$.

(4) Let $HLWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) = C_{NS}^m(w_1, H_S^{\alpha 1}, w_2, H_S^{\alpha 2}, ..., w_m, H_S^{\alpha m})$

$= \{round'(NS^{-1}(w_1 \times NS(r_{\alpha 1}) + w_2 \times NS(r_{\alpha 2}) + ... + w_m \times NS(r_{\alpha m}))) | r_{\alpha k} \in H_S^{\alpha k}\}$

and $HLWA_w^{NS}(H_S^{\beta 1}, H_S^{\beta 2}, ..., H_S^{\beta m}) = C_{NS}^m(w_1, H_S^{\beta 1}, w_2, H_S^{\beta 2}, ..., w_m, H_S^{\beta m})$

$= \{round'(NS^{-1}(w_1 \times NS(r_{\beta 1}) + w_2 \times NS(r_{\beta 2}) + ... + w_m \times NS(r_{\beta m}))) | r_{\beta k} \in H_S^{\beta k}\}$

Since $H_S^{\alpha k} > H_S^{\beta k}$, there must exist the possibility of $r_{\alpha k} > r_{\beta k}$, then $HLWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) > HLWA_w^{NS}(H_S^{\beta 1}, H_S^{\beta 2}, ..., H_S^{\beta m})$ if $H_S^{\alpha k} > H_S^{\beta k}$.

Similarly, we can get $HLOWA_w^{NS}(H_S^{\alpha 1}, H_S^{\alpha 2}, ..., H_S^{\alpha m}) > HLOWA_w^{NS}(H_S^{\beta 1}, H_S^{\beta 2}, ..., H_S^{\beta m})$ if $H_S^{\alpha k} > H_S^{\beta k}$.

(5) If $w_k = 1$, then

$HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^1, w_2, H_S^2, ..., w_k, H_S^k, ..., w_m, H_S^m)$

$= \{round'(NS^{-1}(1 \times NS(r_k) | r_k \in H_S^k\} = H_S^k$.

Similarly, if $w_i = 1$ and $k = \sigma(i)$, then

$HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) = C_{NS}^m(w_1, H_S^{\sigma(1)}, w_2, H_S^{\sigma(2)}, ..., w_i, H_S^{\sigma(i)}, ..., w_m, H_S^{\sigma(m)})$

$= \{round'(NS^{-1}(1 \times NS(r_{\sigma(i)}) | r_{\sigma(i)} \in H_S^{\sigma(i)}\} = H_S^k = \{round'(NS^{-1}(1 \times NS(r_k) | r_k \in H_S^k\} = H_S^k$.

This completes the proof Property 5.

The proof of Property 6.

Proof: Because NS is an ordered numerical scale over S, and the novel HLWA and HLOWA operators satisfy monotonicity according to Property 5, we can obtain $HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \subseteq \{s_i | s_i \in S, s_L \le s_i \le s_R\}$ and $HLOWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m) \subseteq \{s_i | s_i \in S, s_L \le s_i \le s_R\}$.

This completes the proof Property 6.

The proof of Proposition 6.

Proof: We prove the sufficient condition and the necessary condition.

(i) Sufficiency. If $s_h \in HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$, then based on Eq. (41), we know there must exist a z^*, which satisfies $round'(NS^{-1}(z^*)) = s_h$, and $NS^{-1}(z^*) \in [(s_h, -0.5), (s_h, 0.5))$. Thus $\frac{NS(s_h) + NS(s_{h-1})}{2} \le z^* < \frac{NS(s_h) + NS(s_{h+1})}{2}$ and $Q \ge 0$.

(ii) Necessity. If $Q \ge 0$, it means that $\frac{NS(s_h) + NS(s_{h-1})}{2} \le z^* < \frac{NS(s_h) + NS(s_{h+1})}{2}$, so $round'(NS^{-1}(z^*)) = s_h$. According to Eqs. (34) and (35), we arrive at that $s_h \in HLWA_w^{NS}(H_S^1, H_S^2, ..., H_S^m)$.

This completes the proof of Proposition 6.

The proof of Proposition 7.

Proof: From model (41), we have that $z = w_1 \times NS(r_1) + w_2 \times NS(r_2) + ... + w_m \times NS(r_m) = \sum_{k=1}^{m} (w_k \times NS(r_k))$, $r_k \in H_S^k$. From model (42), we have that $z = \sum_{k=1}^{m} \sum_{i=I^{k-}}^{I^{k+}} w_k \times x_i^k \times NS(s_i)$, $x_i^k \in \{0,1\}$ and $\sum_{i=I^{k-}}^{I^{k+}} x_i^k = 1$. It is obvious that the z in model (41) and the z in model (42) have the same meaning (they all mean choosing an element from each HFLTS, and then aggregate these elements into overall results using the associated weights), and the z in model (41) can be equivalently expressed by the z in model (42). Thus, model (41) can be equivalently transformed into model (42).

This completes the proof of Proposition 7.

REFERENCES

[1] M.-A. Abchir, I. Truck, Towards an extension of the 2-tuple linguistic model to deal with unbalanced linguistic term sets, Kybernetika 49 (1) (2013) 164-180.

[2] S. Alonso, F.J. Cabrerizo, F. Chiclana, F. Herrera, E. Herrera-Viedma, Group decision-making with incomplete fuzzy linguistic preference relations, Int. J. Intell. Syst. 24 (2) (2009) 201-222.

[3] I. Beg, T. Rashid, TOPSIS for hesitant fuzzy linguistic term sets, Int. J. Intell. Syst. 28 (12) (2013) 1162–1171.

[4] P.P. Bonissone, A fuzzy sets based linguistic approach: Theory and applications, in Approximate Reasoning in Decision Analysis. Amsterdam, The Netherlands: North-Holland, (1982) 329-339.

[5] F.J. Cabrerizo, S. Alonso, E. Herrera-Viedma, A consensus model for group decision making problems with unbalanced fuzzy linguistic information, Int. J. Inf. Tech. Decis.

Making 8 (1) (2009) 109-131.
[6] F.J. Cabrerizo, I.J. Pérez, E. Herrera-Viedma, Managing the consensus in group decision making in an unbalanced fuzzy linguistic context with incomplete information, Knowl.-Based Syst. 23 (2010) 169-181.
[7] Y.Z. Cao, M.S. Ying, G.Q. Chen, Retraction and generalized extension of computing with words, IEEE Trans. Fuzzy Syst. 15 (6) (2007) 1238-1250.
[8] Y.Z. Cao, G.Q. Chen, A fuzzy petri-nets model for computing with words, IEEE Trans. Fuzzy Syst. 18 (3) (2010) 486-499.
[9] O. Cordón, F. Herrera, I. Zwir, Linguistic modeling by hierarchical systems of linguistic rules, IEEE Trans. Fuzzy Syst. 10 (1) (2001) 2-20.
[10] M. Delgado, J.L. Verdegay, M.A. Vila, On aggregation operations of linguistic labels, Int. J. Intell. Syst. 8 (3) (1993) 351-370.
[11] Y.C. Dong, E. Herrera-Viedma, Consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic GDM with preference relation, IEEE Trans. Cybern. 45 (4) (2015) 780-792.
[12] Y.C. Dong, W.C. Hong, Y.F. Xu, S. Yu, Selecting the individual numerical scale and prioritization method in the analytic hierarchy process: A 2-tuple fuzzy linguistic approach, IEEE Trans. Fuzzy Syst. 19 (1) (2011) 13-25.
[13] Y.C. Dong, C.C. Li, Y.F. Xu, X. Gu, Consensus-based group decision making under multi-granular unbalanced 2-tuple linguistic preference relations, Group Decis. Negot. 24 (2) (2015) 217-242.
[14] Y.C. Dong, C.C. Li, F. Herrera, Connecting the numerical scale model to the unbalanced linguistic term sets, IEEE Int. Conf. Fuzzy Syst. (FUZZ-IEEE) (2014) 455-462.
[15] Y.C. Dong, Y.F. Xu, S. Yu, Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model, IEEE Trans. Fuzzy Syst. 17 (6) (2009) 1366-1378.
[16] Y.C. Dong, G.Q. Zhang, W.C. Hong, Y.F. Xu, Linguistic computational model based on 2-tuples and intervals, IEEE Trans. Fuzzy Syst. 21 (2013)1006-1018.
[17] M. Espinilla, J. Liu, L. Martínez, An extended hierarchical linguistic model for decision-making problems, Comput. Intell. 27 (3) (2011) 489-512.
[18] F. Herrera, S. Alonso, F. Chiclana, E. Herrera-Viedma, Computing with words in decision making: foundations, trends and prospects, Fuzzy Optim. Decis. Making 8 (4) (2009) 337-364.
[19] F. Herrera, E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision

problems under linguistic information, Fuzzy Sets Syst. 115 (1) (2000) 67-82.
[20] F. Herrera, E. Herrera-Viedma, L. Martínez, A fuzzy linguistic methodology to deal with unbalanced linguistic term sets, IEEE Trans. Fuzzy Syst. 16 (2) (2008) 354-370.
[21] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (6) (2000) 746-752.
[22] F. Herrera, L. Martínez, A model based on linguistic 2-tuples for dealing with multigranularity hierarchical linguistic contexts in multiexpert decision-making, IEEE Trans. Syst., Man Cybern. B, Cybern. 31 (2) (2001) 227-234.
[23] E. Herrera-Viedma, F.J. Cabrerizo, I.J. Pérez, M.J. Cobo, S. Alonso, F. Herrera, Applying linguistic OWA operators in consensus models under unbalanced linguistic information, Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice Studies in Fuzziness and Soft Computing 265 (2011) 167-186.
[24] E. Herrera-Viedma, A.G. López-Herrera, A model of information retrieval system with unbalanced fuzzy linguistic information, Int. J. Intell. Syst. 22 (11) (2007) 1197-1214.
[25] J. Kacprzyk, S. Zadrozny, Computing with words in decision making: Through individual and collective linguistic choice rules, Int. J. Uncertain., Fuzz. Knowl.-Based Syst. 9 (2001) 89-102.
[26] J. Kacprzyk, S. Zadrozny, Computing with words is an implementable paradigm: fuzzy queries, linguistic data summaries, and natural-language generation, IEEE Trans. Fuzzy Syst. 18 (3) (2010) 461-472.
[27] E.S. Khorasani, S. Rahimi, W. Calvert, Formalization of generalized constraint language: A crucial prelude to computing with words, IEEE Trans. Syst., Man Cybern. B, Cybern. 43 (1) (2013) 246-258.
[28] J. Lawry, An alternative approach to computing with words, Int. J. Uncertain., Fuzz. Knowl.-Based Syst. 9 (2001) 3-16.
[29] C.C. Li, Y.C. Dong, F. Herrera, E. Herrera-Viedma, L. Martínez, Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching, Inf. Fusion 33 (2017) 29-40.
[30] J. Ma, D. Ruan, Y. Xu, G. Zhang, A fuzzy-set approach to treat determinacy and consistency of linguistic terms in multi-criteria decision making, Int. J. Approx. Reason. 44 (2) (2007) 165-181.
[31] L. Martínez, Sensory evaluation based on linguistic decision analysis, Int. J. Approx. Reason. 44 (2) (2007)148-164.
[32] L. Martínez, M. Espinilla, J. Liu, L.G. Pérez, P.J. Sánchez, An evaluation model with

unbalanced linguistic information applied to olive oil sensory evaluation, J. Multi.-Valued Logic and Soft Comp. 15 (2009) 229-250.

[33] L. Martínez, M. Espinilla, L.G. Pérez, A linguistic multigranular sensory evaluation model for olive oil, Int. J. Comp. Intell. Syst. 1 (2) (2008) 148-158.

[34] L. Martínez, F. Herrera, An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges, Inf. Sci. 207 (2012) 1-18.

[35] L. Martínez, R.M. Rodríguez, F. Herrera, The 2-tuple linguistic model: computing with words in decision making, Springer, 2015.

[36] L. Martínez, D. Ruan, F. Herrera, Computing with words in decision support systems: An overview on models and applications, Int. J. Comp. Intell. Syst. 3 (4) (2010) 382-395.

[37] J.M. Mendel, D. Wu, Perceptual computing: Aiding people in making subjective judgments. New Jersey: IEEE Press and John Wiley, 2010.

[38] J.M. Mendel, L.A. Zadeh, R. Yager, J. Lawry, H. Hagras, S. Guadarrama, What computing with words means to me, IEEE Comput. Intell. Mag. 5 (1) (2010) 20-26.

[39] D. Meng, Z. Pei, On weighted unbalanced linguistic aggregation operators in group decision making, Inf. Sci. 223 (2013) 31-41.

[40] G.A. Miller, The magical number seven plus or minus two: Some limits on our capacity of processing information, Psychol. Rev. 63 (1956) 81-97.

[41] M.R. Rajati, J.M. Mendel, On advanced computing with words using the generalized extension principle for type-1 fuzzy sets, IEEE Trans. Fuzzy Syst. 22 (5) (2014) 1245-1261.

[42] R.M. Rodríguez, L. Martínez, F. Herrera, Hesitant fuzzy linguistic term sets for decision making, IEEE Trans. Fuzzy Syst. 20 (1) (2012) 109-119.

[43] R.M. Rodríguez, L. Martínez, F. Herrera, A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets, Inf. Sci. 241 (2013) 28-42.

[44] R.M. Rodríguez, L. Martínez, V. Torra, Z.S. Xu, F. Herrera, Hesitant fuzzy sets: State of the art and future directions, Int. J. Intell. Syst. 29 (2014) 495-524.

[45] Y. Tang, J. Zheng, Linguistic modelling based on semantic similarity relation among linguistic labels, Fuzzy Sets Syst. 157 (12) (2006) 1662-1673.

[46] V. Torra, Hesitant fuzzy sets, Int. J. Intell. Syst. 25 (6) (2010) 529-539.

[47] I. Truck, J. Malenfant, Towards a unification of some linguistic representation models: A vectorial approach, The 9th International Flins Conference on Computational Intelligence in Decision and Control, Chengdu, China, (2010) 610-615.

[48] I. Truck, H. Akdag, Manipulation of qualitative degrees to handle uncertainty: formal

models and applications, Knowl. Inf. Syst. 9 (4) (2006) 385- 411.

[49] I. Truck, Comparison and links between two 2-tuple linguistic models for decision making, Knowl.-Based Syst. 87 (2015) 61-68.

[50] S.-Y. Wang, Applying 2-tuple multigranularity linguistic variables to determine the supply performance in dynamic environment based on product-oriented strategy, IEEE Trans. Fuzzy Syst. 16 (1) (2008) 29-39.

[51] J.H. Wang, J.Y. Hao, A new version of 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 14 (3) (2006) 435-445.

[52] J.H. Wang, J.Y. Hao, Fuzzy linguistic PERT, IEEE Trans. Fuzzy Syst. 15 (2) (2007) 133-144.

[53] J.H. Wang, J.Y. Hao, An approach to computing with words based on canonical characteristic values of linguistic labels, IEEE Trans. Fuzzy Syst. 15 (4) (2007) 593-604.

[54] C.P. Wei, N. Zhao, X.J. Tang, Operators and comparisons of hesitant fuzzy linguistic term sets, IEEE Trans. Fuzzy Syst. 22 (3) (2014) 575-585.

[55] D. Wu, J.M. Mendel, Computing with words for hierarchical decision making applied to evaluating a weapon system, IEEE Trans. Fuzzy Syst. 18 (3) (2010) 441-460.

[56] Z. Xu, Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment, Inf. Sci. 168 (2004)171-184.

[57] Z. Xu, Deviation measures of linguistic preference relations in group decision making, Omega. 33 (2005) 249-254.

[58] R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst., Man, Cybern. 18 (1988) 183-190.

[59] R. Yager, An approach to ordinal decision making, Int. J. Approx. Reason. 12 (1995) 237-261.

[60] L.A. Zadeh, The concept of a linguistic variable and its applications to approximate reasoning, Part I, Inf. Sci. 8 (1975) 199-249.

[61] G.Q. Zhang, Y.C. Dong, Y.F Xu, Consistency and consensus measures for linguistic preference relations based on distribution assessments, Inf. Fusion 17 (2014) 46-55.

[62] B. Zhu, Z. Xu, Consistency measures for hesitant fuzzy linguistic preference relations, IEEE Trans. Fuzzy Syst. 22 (1) (2014) 35-45.

2 Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching

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Personalized individual semantics in Computing with Words for supporting linguistic Group Decision Making. An Application on Consensus reaching

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Abstract

In group decision making (GDM) dealing with Computing with Words (CW) has been highlighted the importance of the statement, *words mean different things for different people*, because of its influence in the final decision. Different proposals that either grouping such different meanings (uncertainty) to provide one representation for all people or use multi-granular linguistic term sets with the semantics of each granularity, have been developed and applied in the specialized literature. Despite these models are quite useful they do not model individually yet the different meanings of each person when he/she elicits linguistic information. Hence, in this paper a personalized individual semantics (PIS) model is proposed to personalize individual semantics by means of an interval numerical scale and the 2-tuple linguistic model. Specifically, a consistency-driven optimization-based model to obtain and represent the PIS is introduced. A new CW framework based on the 2-tuple linguistic model is then defined, such a CW framework allows us to

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deal with PIS to facilitate CW keeping the idea that words mean different things to different people. In order to justify the feasibility and validity of the PIS model, it is applied to solve linguistic GDM problems with a consensus reaching process.

Keywords: computing with words, 2-tuple linguistic model, semantics, group decision making, preference relations

1. Introduction

Human beings usually employ words in most of their computing and reasoning processes without the necessity of any precise number. Computing with words (CW) is a methodology in which the objects of computation are words and propositions drawn from a natural language [49, 50] that arises to emulate such human behaviors. Hence a crucial feature of CW is that its processes deal with linguistic inputs to obtain linguistic outputs easy to understand by human beings. Different computing schemes have been proposed for CW that could be summarized in Fig. 1. Yager [48] points out the importance of the translation and retranslation processes to achieve the aims of the CW.

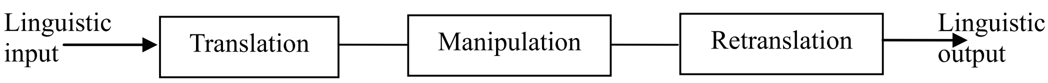

Fig. 1. Yager's CW scheme

It is important to remark that CW involves a wide-ranging ramifications and applications from learning to decision making passing by many others [23, 13, 15, 39, 40]. Our interest in this paper is focused on the use of CW in decision making [28]. Specifically on group decision making (GDM) because its use implies another key and controversial point about CW, that it is the fact that *words mean different things for different people* [1, 16, 29, 30]. In order to deal with previous fact that increases the difficulty of managing the uncertainty of linguistic information, two mainstreams have been developed in the literature:

1. The use of type-2 fuzzy sets based on low and upper possibility distributions with a third dimension in between [29], that group all meanings from people in just one representation function and,

2. The use of multi-granular linguistic models [14, 19, 33] in which multiple linguistic term sets can be used by experts according to either their degree of knowledge or their comfort or their similarity with the semantics of each granularity.

In spite of both previous methods are quite useful to deal with the multiple meanings of words and have been also widely used for CW in multiple different problems, they do not represent yet the specific semantics of each individual. For example, when reviewing an article, two referees both think this article is "*Good*", but the term "*Good*" often has different numerical meaning for these two referees. Hence, in this paper a personalized individual semantics (PIS) model is proposed to customize individual semantics by means of an interval numerical scale [6, 12] and the 2-tuple linguistic model [18]. In order to do so, this paper develops two main proposals:

a) A new model to represent PIS, such that it will be based on the interval numerical scale because of its features to deal with different linguistic representations in a precise way [6, 12].

b) A framework for CW dealing with PIS, based on the 2-tuple linguistic model [27], including personalized 2-tuple linguistic operators are proposed, because of its good features for managing linguistic information in CW processes [38]. This framework will cope with PIS and redesign the CW phases pointed out in Fig. 1 to obtain customized accurate linguistic results easy to interpret and understand by individuals.

There are a lot of researches regarding GDM problems using linguistic preference relations, such as aggregation operators [3], consistency measures [8, 10], consensus models [9, 20, 34] and so on. In order to justify the feasibility and validity of the PIS model, it will be applied to a linguistic GDM problem with a consensus reaching process, by defining the concept of the *individual linguistic understanding*.

The remainder of this paper is arranged as follows. Section 2 introduces a basic description of the 2-tuple linguistic model, the numerical scale and preference relations. Section 3 introduces a consistency-driven optimization-based model to obtain the interval numerical scale of PIS for decision makers in linguistic GDM problems. Section 4 proposes a new CW framework based on the 2-tuple linguistic model for dealing with PIS. Section 5 presents a consensus reaching process for linguistic GDM problems with PIS. Section 6 then concludes this paper.

2. Preliminaries

This section introduces the basic necessary knowledge to understand our proposals, regarding the 2-tuple linguistic model, the numerical scale and preference relations.

2.1. The 2-tuple linguistic model

The 2-tuple linguistic representation model, presented in Herrera and Martínez [18] represents the linguistic information by a 2-tuple $(s_i, \alpha) \in \overline{S} = S \times [-0.5, 0.5)$, where $s_i \in S$ and $\alpha \in [-0.5, 0.5)$. Formally, let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation. The 2-tuple that expresses the equivalent information to β is then obtained as:

$$\Delta : [0, g] \rightarrow S \times [-0.5, 0.5), \tag{1}$$

where

$$\Delta(\beta) = (s_i, \alpha), with \begin{cases} s_i, i = round(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5) \end{cases} . \tag{2}$$

Function Δ, it is a one to one mapping whose inverse function $\Delta^{-1} : \overline{S} \rightarrow [0, g]$ is defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha$. When $\alpha = 0$ in (s_i, α) is then called simple term.

In [18] it was also defined a computational model for linguistic 2-tuples in which different operations were introduced:

(1) A 2-tuple comparison operator: Let (s_k, α) and (s_l, γ) be two 2-tuples. Then:

(i) if $k < l$, then (s_k, α) is smaller than (s_l, γ).

(ii) if $k = l$, then

(a) if $\alpha = \gamma$, then (s_k, α), (s_l, γ) represents the same information.

(b) if $\alpha < \gamma$, then (s_k, α) is smaller than (s_l, γ).

(2) A 2-tuple negation operator:

$Neg((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha)))$.

(3) Several 2-tuple aggregation operators have been developed (see [18, 31]).

2.2. Numerical scale to extend the 2-tuple linguistic model

Dong et al. [11, 12] extended the 2-tuple linguistic model by the numerical scale and the interval numerical scale for integrating different linguistic models and increasing the accuracy of the 2-tuple linguistic computational model.

(1) Numerical scale

The concept of the numerical scale was introduced in [11] for transforming linguistic terms into real numbers:

Definition 1. *[11] Let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be a linguistic term set, and R be the set of real numbers. The function: $NS : S \to R$ is defined as a numerical scale of S, and $NS(s_i)$ is called the numerical index of s_i. If the function NS is strictly monotone increasing, then NS is called an ordered numerical scale.*

Definition 2. *[11] Let $S, \overline{S}$ and NS be as before. The numerical scale $\overline{NS}$ on $\overline{S}$ for $(s_i, \alpha) \in \overline{S}$, is defined by*

$$\overline{NS}((s_i, \alpha)) = \begin{cases} NS(s_i) + \alpha \times (NS(s_{i+1}) - NS(s_i)), & \alpha \geq 0 \\ NS(s_i) + \alpha \times (NS(s_i) - NS(s_{i-1})), & \alpha < 0 \end{cases}. \quad (3)$$

To simplify the notation, $\overline{NS}$ will also be denoted as NS in this paper.

In [11] NS was introduced as a a family of functions, that usually are ordered functions, if so it was proved that its inverse NS^{-1} exists. For example, setting $NS(s_i) = \Delta^{-1}(s_i)$ (i.e., $NS(s_0) = 0, NS(s_1) = 1, ..., NS(s_g) = g$) yields the 2-tuple linguistic model [18].

(2) Interval numerical scale

The concept of the interval numerical scale [12] extends the numerical scale model to transform linguistic terms into numerical interval values:

Definition 3. *[12] Let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be a linguistic term set, and let $M = \{[A_L, A_R] | A_L, A_R \in [0, 1], A_L \leq A_R\}$ be a set of interval values in [0, 1]. The function $INS : S \to M$ is defined as an interval numerical scale of S, and $INS(s_i)$ is called the interval numerical index of s_i.*

If $INS(s_i) = [A_L^i, A_R^i]$, then the functions INS_L and INS_R are defined as follows: $INS_L(s_i) = A_L^i$ and $INS_R(s_i) = A_R^i$. The interval numerical scale INS is ordered if $INS_L(s_i) < INS_L(s_{i+1})$ and $INS_R(s_i) < INS_R(s_{i+1})$ for $i = 0, 1, ..., g - 1$.

Definition 4. *[12] Let $S, \overline{S}$ and NS be as before. For $(s_i, \alpha) \in \overline{S}$, the interval numerical scale INS on $\overline{S}$ is defined by*

$$INS((s_i, \alpha)) = [A_L, A_R], \tag{4}$$

where

$$A_L = \begin{cases} INS_L(s_i) + \alpha \times (INS_L(s_{i+1}) - INS_L(s_i)), & \alpha \geq 0 \\ INS_L(s_i) + \alpha \times (INS_L(s_i) - INS_L(s_{i-1})), & \alpha < 0 \end{cases} \tag{5}$$

$$A_R = \begin{cases} INS_R(s_i) + \alpha \times (INS_R(s_{i+1}) - INS_R(s_i)), & \alpha \geq 0 \\ INS_R(s_i) + \alpha \times (INS_R(s_i) - INS_R(s_{i-1})), & \alpha < 0 \end{cases}. \tag{6}$$

Dong et al. [12] introduced the inverse operation of INS noted as INS^{-1} and its generalization, a simplified inverse operation INS^{-1} is defined as:

Definition 5. *Let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be a linguistic term set, INS be an ordered interval numerical scale on $\overline{S}$, and $M = \{[A_L, A_R] | A_L, A_R \in [0, 1], A_L \leq A_R\}$ a set of interval values in $[0, 1]$. The inverse operation INS^{-1} is defined as:*

$$INS^{-1} : M \longrightarrow \overline{S}, \tag{7}$$

where for any $A \in M$, $INS^{-1}(A) = s$ and

$$d(A, INS(s)) = \min_{x \in \overline{S}} d(A, INS(x)). \tag{8}$$

In Eq.(8), d is a distance function for interval values. Different distance functions might be applied to computing INS^{-1}, and in this paper it is used the Euclidean distance, i.e., $d([a, b], [c, d]) = \sqrt{(a - c)^2 + (b - d)^2}$, because it provides correct results and is generally utilized in the retranslation process in CW [48].

In [12] was also introduced a way to compute INS and INS^{-1}.

Example 1. Let $S = \{s_0, s_1, ..., s_4\}$. Let $INS(s_0) = [0, 0.1]$, $INS(s_1) = [0.2, 0.25]$, $INS(s_2) = 0.5$, $INS(s_3) = [0.75, 0.8]$, and $INS(s_4) = [0.751, 1]$.

(1) Hence the interval numerical index of a linguistic 2-tuple $(s_1, 0.3)$ is $INS((s_1, 0.3)) = [A_L, A_R]$ and according to Eq.(5):
$A_L = INS_L(s_1)+0.3\times(INS_L(s_2)-INS_L(s_1)) = 0.2+0.3\times(0.5-0.2) = 0.29$.

Moreover, according to Eq.(6):
$A_R = INS_R(s_1)+0.3\times(INS_R(s_2)-INS_R(s_1)) = 0.25+0.3\times(0.5-0.25) = 0.325$.

Consequently, $INS((s_1, 0.3)) = [A_L, A_R] = [0.29, 0.325]$.

(2) To illustrate how to obtain the value of $INS^{-1}([0.6, 0.8])$ it must be used Eqs. (7) and (8):
$\min_{x\in\overline{S}} d([0.6, 0.8], INS(s)) = d([0.6, 0.8], INS(s_3, -0.246)) = 0.013$.
Therefore, $INS^{-1}([0.6, 0.8]) = (s_3, -0.246)$.

In the linguistic computational model with the interval numerical scale, the input are linguistic terms, and the output are 2-tuple linguistic intervals to avoid the loss of information. Further detail regarding the operations with the interval numerical scale can be found in [12].

Remark 1[7, 11]. The numerical scale can provide a connection among the 2-tuple linguistic model and its variants, additionally can set different numerical scales for the 2-tuple linguistic model [18], the Wang and Hao model [41] and the unbalanced linguistic model based on a linguistic hierarchy [17].

Remark 2[11]. The interval numerical scale can be reduced to the numerical scale. So the interval numerical scale will be used as the basis to develop the 2-tuple linguistic model with PIS in this paper.

2.3. Linguistic and numerical preference relations. Consistency

Let $X = \{X_1, X_2, ..., X_m\}(n \geq 2)$ be a finite set of alternatives. When a decision maker provides pairwise comparisons using the linguistic term set S, he/she can construct a linguistic preference relation $L = (l_{ij})_{n\times n}$, whose element l_{ij} estimates the preference degree of alternative X_i over X_j. Linguistic preference relations based on linguistic 2-tuples can be formally defined as:

Definition 6. *[2] The matrix $L = (l_{ij})_{n\times n}$, where $l_{ij} \in S$, is called a simple linguistic preference relation. The matrix $L = (l_{ij})_{n\times n}$, where $l_{ij} \in \overline{S}$, is called a 2-tuple linguistic preference relation. If $l_{ij} = Neg(l_{ji})$ for $i, j = 1, 2, ..., n$, then L is considered reciprocal.*

In addition, the numerical preference relations are often used in decision making. A kind of numerical preference relations, i.e., fuzzy preference relations were also introduced.

Definition 7. *[22, 36] The matrix $F = (f_{ij})_{n\times n}$, where $f_{ij} \in [0,1]$ and $f_{ij} + f_{ji} = 1$ for $i, j = 1, 2, ..., n$, is called a fuzzy preference relation.*

The study of consistency in a preference relation is very important, because it ensures that preferences are neither random nor illogical. Generally, *ordinal* [45] and *cardinal* [4] consistency are two common types of consistency for a preference relation. The former is closely related to the transitivity of the corresponding preference relation meanwhile the latter is a stronger concept because it not only implies the transitivity of preferences, but also the intensity of preference expressed by comparisons. Here, it is revised the cardinal consistency index (CI) based on additive transitivity [21] for fuzzy preference relations, F, because it will be extended in our proposal:

$$CI(F) = 1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,j,k=1; i\neq j\neq k}^{n} |f_{ij} + f_{jk} - f_{ik} - 0.5|. \quad (9)$$

Due to the complexity and uncertainty involved in real-world decision problems, sometimes it is unrealistic to acquire exact judgments. Thus, fuzzy preference relations are extended to interval fuzzy preference relations.

Definition 8. *[46] The matrix $\widetilde{V} = (\widetilde{v_{ij}})_{n\times n}$, where $\widetilde{v_{ij}} = [v_{ij}^-, v_{ij}^+] \subseteq [0,1]$ and $v_{ij}^- + v_{ji}^+ = 1$ for $i, j = 1, 2, ..., n$, is called an interval fuzzy preference relation.*

Definition 9. *[5] Let $\widetilde{V} = (\widetilde{v_{ij}})_{n\times n}$, where $\widetilde{v_{ij}} = [v_{ij}^-, v_{ij}^+]$, be an interval fuzzy preference relation. $F = (f_{ij})_{n\times n}$ is a fuzzy preference relation associated to $\widetilde{V}$ if $v_{ij}^- \leq f_{ij} \leq v_{ij}^+$ and $f_{ij} + f_{ji} = 1$.*

Being $N_{\widetilde{V}}$ the set of the fuzzy preference relations associated to $\widetilde{V}$.
Remark 3. Reciprocity is an important property of preference relations. However, when S is not uniformly and symmetrically distributed, the reciprocity of linguistic preference relations cannot be guaranteed. In this situation, it is assumed that the decision maker only provides his/her preferences for the upper/lower triangular entries of L.

3. Personalized individual semantics based on interval numerical scales

As aforementioned, the difficulty of carrying out CW processes with the issue *words mean different things for different people* that naturally arises in problems with multiple experts like GDM problems still remains open. Even though, different proposals have been introduced in the literature based on type-1 [14] and type-2 [29] fuzzy sets, dealing with multiple linguistic term sets and grouping individual representations respectively. In fact, neither of them represents specifically the PIS of each expert involved in the GDM problem.

Therefore, the first objective of this paper is to introduce an interval numerical scale based method to personalize individual semantics represented by interval values from the linguistic preference relations elicited by the experts taking part in the GDM problem. This representation will be managed in the CW framework presented later in Section 4.

The method to obtain the PIS consists of a consistency-driven optimization model. Before introducing this model it is necessary to fix some notations, premises and a consistency measure for interval fuzzy preference relations introduced in the coming subsections.

3.1. Basics

Let $S = \{s_i | i = 0, 1, ..., g\}$ be a linguistic term set, INS^k be an ordered interval numerical scale on S associated with the individual $e_k (k = 1, 2, ..., m)$, and $L^k = (l_{ij}^k)_{n \times n}$ be the linguistic preference relation based on S associated with e_k. The matrix $\widetilde{V^k} = (\widetilde{v_{ij}^k})_{n \times n}$, in which $v_{ij}^k = [v_{ij}^{k-}, v_{ij}^{k+}] = INS^k(l_{ij}^k)$, is called the numerical preference relation transformed by INS^k, associated with L^k.

Remark 3 pointed out that when an individual only provides his/her preference information for the upper/lower triangular entries of the linguistic preference relations based on S, the reciprocity of the numerical preference relation $\widetilde{V^k}$ will not be violated. Besides, in the 2-tuple linguistic model with the interval numerical scale, the support of the INS^k of S is the interval $[0, 1]$. As a result, $\widetilde{V^k}$ is an interval fuzzy preference relation. Hence, L^k and $\widetilde{V^k}$ represent the same preference, associated with e_k. So, $\widetilde{V^k}$ should be consistent if L^k is consistent. From this reasoning in [6] was provided the following premise:

Premise 1 [6]. If linguistic preference relations provided by individuals are consistent, then the interval fuzzy preference relations, transformed by the established interval numerical scales, should be as much consistent as possible.

The Premise 1 implies the need of consistency in interval fuzzy preference relations, and the ordinal consistency can be guaranteed by the transformation from linguistic to interval fuzzy preference relations. However, the cardinal consistency should be still studied with a specific measure, and here we propose the cardinal interval consistency index based on Eq. (9):

Definition 10. *Let $\widetilde{V} = (\widetilde{v_{ij}})_{n\times n}$ be an interval fuzzy preference relation, let $F = (f_{ij})_{n\times n}$ be a fuzzy preference relation associated to $\widetilde{V}$, and let $N_{\widetilde{V}}$ be the set of the fuzzy preference relation associated to $\widetilde{V}$. The optimistic consistency index (OCI) of $\widetilde{V}$ is then defined as follows,*

$$OCI(\widetilde{V}) = \max_{F\in N_{\widetilde{V}}} CI(F), \tag{10}$$

i.e.,

$$OCI(\widetilde{V}) = \max_{F\in N_{\widetilde{V}}} (1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,j,z=1;i\neq j\neq z}^{n} |f_{ij} + f_{jz} - f_{iz} - 0.5|), \tag{11}$$

and the pessimistic consistency index (PCI) of $\widetilde{V}$ is,

$$PCI(\widetilde{V}) = \min_{F\in N_{\widetilde{V}}} CI(F), \tag{12}$$

i.e.,

$$PCI(\widetilde{V}) = \min_{F\in N_{\widetilde{V}}} (1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,j,z=1;i\neq j\neq z}^{n} |f_{ij} + f_{jz} - f_{iz} - 0.5|). \tag{13}$$

In the proposed interval consistency index, $OCI(\widetilde{V})$ and $PCI(\widetilde{V})$ reflect the best and worst consistency indexes of all fuzzy preference relations associated to $\widetilde{V}$, respectively.

In previous studies regarding the consistency measure of $\widetilde{V}$ (e.g., [6, 42]), $OCI(\widetilde{V})$ was considered as the consistency degree of $\widetilde{V}$. However, $OCI(\widetilde{V})$

cannot accurately measure the consistency degree of $\widetilde{V}$ such as it is illustrated in Example 2.

Example 2. Consider the following interval fuzzy preference relation:

$$\widetilde{V} = \begin{pmatrix} [0.5, 0.5] & [0.2, 1] & [0.1, 0.3] \\ [0, 0.8] & [0.5, 0.5] & [0.3, 0.9] \\ [0.7, 0.9] & [0.1, 0.7] & [0.5, 0.5] \end{pmatrix}.$$

Solving Eq. (10) obtains $\max_{F \in N_{\widetilde{V}}} CI(F) = CI(F^1) = 1$, where

$$F^1 = \begin{pmatrix} 0.5 & 0.5 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.7 & 0.7 & 0.5 \end{pmatrix},$$

so, $OCI(\widetilde{V}) = CI(F^1) = 1$. Solving Eq. (12) obtains $\min_{F \in N_{\widetilde{V}}} CI(F) = CI(F^2) = 0.133$, in which

$$F^2 = \begin{pmatrix} 0.5 & 1 & 0.1 \\ 0 & 0.5 & 0.9 \\ 0.9 & 0.1 & 0.5 \end{pmatrix},$$

so, $PCI(\widetilde{V}) = CI(F^2) = 0.133$.

In Example 2, F^1 reflects the best consistency degree of $\widetilde{V}$, and F^2 reflects the worst consistency degree of $\widetilde{V}$. Besides, $OCI(\widetilde{V}) = 1$, i.e., $\widetilde{V}$ is fully consistent based on $OCI(\widetilde{V})$. But, $PCI(\widetilde{V}) = 0.133$ is very low. Hence, $OCI(\widetilde{V})$ cannot accurately measure the consistency degree of $\widetilde{V}$.

Remark 4. Clearly, the consistency index CI of any fuzzy preference relation F associated to $\widetilde{V}$ is in the interval $[PCI(\widetilde{V}), OCI(\widetilde{V})]$, i.e., $CI(F) \in [PCI(\widetilde{V}), OCI(\widetilde{V})]$ for any $F \in N_{\widetilde{V}}$.

Therefore our proposal will use $OCI(\widetilde{V})$ and $PCI(\widetilde{V})$ because they reflect better the consistency degree of $\widetilde{V}$ than just the use of $OCI(\widetilde{V})$.

3.2. A consistency-driven optimization-based model to obtain personalized individual semantics

From our view the personal own meaning (semantics) that each individual provides to words when eliciting linguistic preferences are closely related to her/his consistency. Therefore this section introduces a consistency-driven optimization-based model to obtain the personalized individual interval numerical scales of the 2-tuple linguistic terms.

Let $INS^k(s_i) = [A_L^{i,k}, A_R^{i,k}]$ be the interval numerical index of s_i, associated with the decision maker e_k. According to Premise 1, if L^k is consistent, then $\widetilde{V^k}$ should be as much consistent as possible. It is then necessary to maximize $PCI(\widetilde{V^k})$ by,

$$\max \min CI(F^{2k}), \tag{14}$$

where

$$F^{2k} \in N_{\widetilde{V^k}},\ k = 1, 2, ..., m. \tag{15}$$

In the previous studies [6, 42] it was required that $OCI(\widetilde{V^k}) = 1$, so

$$CI(F^{1k}) = 1,\ k = 1, 2, ..., m, \tag{16}$$

where

$$F^{1k} \in N_{\widetilde{V^k}},\ k = 1, 2, ..., m. \tag{17}$$

Based on the existing several 2-tuple linguistic models (e.g., the Herrera and Martínez model [18], the Wang and Hao model [41], and the unbalanced linguistic model [17]), the ordered initial numerical index a_i of s_i can be provided by different functions that computes NS. For example, in the Herrera and Martínez model, $a_i = NS(s_i) = \Delta^{-1}(s_i)/g$; in the Wang and Hao model, $NS(s_i)$ is determined by canonical characteristic values; in the unbalanced linguistic model, $NS(s_i)$ is determined by a linguistic hierarchy. This paper assumes that $a_i \in INS^k(s_i)$, i.e.,

$$0 \le A_L^{i,k} \le a_i \le A_R^{i,k} \le 1, i = 0, 1, ..., g; k = 1, 2, ..., m. \tag{18}$$

Moreover, INS^k is ordered, then:

$$INS_L^k(s_i) < INS_L^k(s_{i+1}), i = 0, 1, ..., g-1; k = 1, 2, ..., m, \tag{19}$$

and

$$INS_R^k(s_i) < INS_R^k(s_{i+1}), i = 0, 1, ..., g-1; k = 1, 2, ..., m. \tag{20}$$

Based on Eqs. (14)-(20), an optimization model to set individual interval numerical scales of linguistic terms $INS^k(s_i) = [A_L^{i,k}, A_R^{i,k}]$ can be constructed as follows,

$$\begin{cases} \max\min CI(F^{2k}) \\ s.t.\ F^{2k} \in N_{\widetilde{V^k}}, \quad k = 1,2,...,m \\ CI(F^{1k}) = 1, \quad k = 1,2,...,m \\ F^{1k} \in N_{\widetilde{V^k}}, \quad k = 1,2,...,m \\ 0 \le A_L^{i,k} \le a_i \le A_R^{i,k} \le 1 \qquad i = 0,1,...,g; k = 1,2,...,m \\ INS_L^k(s_i) < INS_L^k(s_{i+1}) \qquad i = 0,1,...,g-1; k = 1,2,...,m \\ INS_R^k(s_i) < INS_R^k(s_{i+1}) \qquad i = 0,1,...,g-1; k = 1,2,...,m \end{cases} \tag{21}$$

Remark 5. Using $[PCI(\widetilde{V}), OCI(\widetilde{V})]$ for measuring the consistency degree of $\widetilde{V}$, model (21) sets $OCI(\widetilde{V^k}) = 1$ based on the previous studies[6, 42], and the objective function is set to maximize $PCI(\widetilde{V^k})$. In this way, $\widetilde{V^k}$ can be as much consistent as possible.

Let $INS^k(s_i) = [A_L^{i,k}, A_R^{i,k}]$, and let $p(s)$, where $s \in S$, be the position index of s. For example, if $s = s_i$, then $p(s) = i$. Thus $INS^k(l_{ij}^k) = [A_L^{p(l_{ij}^k),k}, A_R^{p(l_{ij}^k),k}](l_{ij}^k \ne null)$.

Proposition 1. *Model (21) can be equivalently transformed into model (22)-(30), denoted as P.*

$$\max\ \min\ 1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,j,z=1; i\ne j\ne z}^{n} \left| f_{ij}^{2k} + f_{jz}^{2k} - f_{iz}^{2k} - 0.5 \right| \tag{22}$$

s.t.

$$A_L^{p(l_{ij}^k),k} \le f_{ij}^{2k} \le A_R^{p(l_{ij}^k),k}, i,j = 1,2,...,n, i \ne j; k = 1,2,...,m; l_{ij}^k \ne null \tag{23}$$

$$f_{ij}^{2k} + f_{ji}^{2k} = 1, i,j = 1,2,...,n, i \ne j; k = 1,2,...,m \tag{24}$$

$$f_{ij}^{1k} + f_{jz}^{1k} - f_{iz}^{1k} = 0.5, i,j = 1,2,...,n, i \ne j; k = 1,2,...,m \tag{25}$$

$$A_L^{p(l_{ij}^k),k} \le f_{ij}^{1k} \le A_R^{p(l_{ij}^k),k}, i,j = 1,2,...,n, i \ne j; k = 1,2,...,m; l_{ij}^k \ne null \tag{26}$$

$$f_{ij}^{1k} + f_{ji}^{1k} = 1, i,j = 1,2,...,n, i \ne j; k = 1,2,...,m \tag{27}$$

$$0 \le A_L^{i,k} \le a_i \le A_R^{i,k} \le 1, i = 0,1,...,g \tag{28}$$

$$A_L^{i,k} < A_L^{i+1,k}, i = 0,1,...,g-1; k = 1,2,...,m \tag{29}$$

$$A_R^{i,k} < A_R^{i+1,k}, i = 0,1,...,g-1; k = 1,2,...,m \tag{30}$$

The proof of Proposition 1 is provided in Appendix.

Model P can be easily transformed into a max-min linear programming model. By solving P, it is obtained the individual interval numerical indexes $INS^k(s_i) = [A_L^{i,k}, A_R^{i,k}]$ that reflect in the best possible way the individual meaning of words because it reflects the best consistency in their preferences. According to Miller [32], an individual cannot simultaneously compare more than 7 ± 2 objects without producing confusion. So, the size of matrices, i.e., n, should be smaller than 9. As a result, the proposed model P is a small-scale optimization problem, and can be effectively and rapidly solved by several software packages (e.g., Matlab and Lingo).

3.3. Illustration of the consistency-driven optimization-based model

The following example illustrates the consistency-driven optimization-based model.

Example 3. Let's suppose a set of five decision makers, $E = \{e_1, e_2, ..., e_5\}$ and a set of five alternatives, $X = \{X_1, X_2, ..., X_5\}$. Let $S = \{s_0 = \textit{extremely poorer}, s_1 = \textit{much poorer}, s_2 = \textit{fair}, s_3 = \textit{better}, s_4 = \textit{extremely better}\}$ be an established linguistic term set. The decision maker e_k supplies the linguistic preference relation based on S, L^k, to express his/her opinions over X. These preference relations $L^k (k = 1, 2, ..., 5)$ are listed as follows.

$$L^1 = \begin{pmatrix} null & s_3 & s_4 & s_1 & s_1 \\ null & null & s_3 & s_0 & s_1 \\ null & null & null & s_0 & s_0 \\ null & null & null & null & s_3 \\ null & null & null & null & null \end{pmatrix}, \quad L^2 = \begin{pmatrix} null & s_2 & s_0 & s_0 & s_0 \\ null & null & s_1 & s_1 & s_1 \\ null & null & null & s_2 & s_1 \\ null & null & null & null & s_1 \\ null & null & null & null & null \end{pmatrix},$$

$$L^3 = \begin{pmatrix} null & s_3 & s_0 & s_1 & s_1 \\ null & null & s_0 & s_1 & s_1 \\ null & null & null & s_3 & s_3 \\ null & null & null & null & s_1 \\ null & null & null & null & null \end{pmatrix}, \quad L^4 = \begin{pmatrix} null & s_2 & s_1 & s_0 & s_0 \\ null & null & s_1 & s_1 & s_1 \\ null & null & null & s_2 & s_2 \\ null & null & null & null & s_2 \\ null & null & null & null & null \end{pmatrix},$$

$$L^5 = \begin{pmatrix} null & null & null & null & null \\ s_2 & null & null & null & null \\ s_3 & s_3 & null & null & null \\ s_4 & s_3 & s_2 & null & null \\ s_4 & s_4 & s_3 & s_3 & null \end{pmatrix}.$$

Without loss of generality, let the initial numerical index a_i of s_i be $\{0, 0.25, 0.5, 0.75, 1\}$. Solving model P obtains the interval numerical indexes that represent the PIS of each decision maker $INS^k(s_i) = [A_L^i, A_R^i]$, which are listed in Table 1.

Table 1. Values of $INS^k(s_i)(k = 1, 2, ..., 5; i = 0, 1, ..., 4)$

	$INS^1(s_i)$	$INS^2(s_i)$	$INS^3(s_i)$	$INS^4(s_i)$	$INS^5(s_i)$
$i=0$	$[0, 0.125]$	$[0, 0.375]$	$[0, 0.249]$	$[0, 0.249]$	0
$i=1$	$[0.25, 0.375]$	$[0.25, 0.376]$	$[0.25, 0.499]$	$[0.249, 0.25]$	0.25
$i=2$	0.5	0.5	0.5	0.5	0.5
$i=3$	$[0.625, 0.75]$	$[0.625, 0.75]$	$[0.748, 0.75]$	0.75	$[0.75, 0.751]$
$i=4$	$[0.75, 1]$	$[0.626, 1]$	1	1	$[0.751, 1]$

Using the obtained interval numerical scale INS^k transforms L^k into the interval fuzzy preference relation $\widetilde{V^k} = (\widetilde{v_{ij}^k})_{5\times5}$, where $\widetilde{v_{ij}^k} = [v_{ij}^{k-}, v_{ij}^{k+}](k = 1, 2, ..., 5)$.

$$\widetilde{V^1} = \begin{pmatrix} null & [0.625, 0.75] & [0.75, 1] & [0.25, 0.375] & [0.25, 0.375] \\ null & null & [0.625, 0.75] & [0, 0.125] & [0.25, 0.375] \\ null & null & null & [0, 0.125] & [0, 0.125] \\ null & null & null & null & [0.625, 0.75] \\ null & null & null & null & null \end{pmatrix},$$

$$\widetilde{V^2} = \begin{pmatrix} null & 0.5 & [0, 0.375] & [0, 0.375] & [0, 0.375] \\ null & null & [0.25, 0.376] & [0.25, 0.376] & [0.25, 0.376] \\ null & null & null & 0.5 & [0.25, 0.376] \\ null & null & null & null & [0.25, 0.376] \\ null & null & null & null & null \end{pmatrix},$$

$$\widetilde{V^3} = \begin{pmatrix} null & [0.748, 0.75] & [0, 0.249] & [0.25, 0.499] & [0.25, 0.499] \\ null & null & [0, 0.249] & [0.25, 0.499] & [0.25, 0.499] \\ null & null & null & [0.748, 0.75] & [0.748, 0.75] \\ null & null & null & null & [0.25, 0.499] \\ null & null & null & null & null \end{pmatrix},$$

$$\widetilde{V^4} = \begin{pmatrix} null & 0.5 & [0.249, 0.25] & [0, 0.249] & [0, 0.249] \\ null & null & [0.249, 0.25] & [0.249, 0.25] & [0.249, 0.25] \\ null & null & null & 0.5 & 0.5 \\ null & null & null & null & 0.5 \\ null & null & null & null & null \end{pmatrix},$$

$$\widetilde{V^5} = \begin{pmatrix} null & null & null & null & null \\ 0.5 & null & null & null & null \\ [0.75, 0.751] & [0.75, 0.751] & null & null & null \\ [0.751, 1] & [0.75, 0.751] & 0.5 & null & null \\ [0.751, 1] & [0.751, 1] & [0.75, 0.751] & [0.75, 0.751] & null \end{pmatrix}.$$

The optimistic consistency index of $\widetilde{V^k}$, $OCI(\widetilde{V^k})$, and the pessimistic consistency index of $\widetilde{V^k}$, $PCI(\widetilde{V^k})(k = 1, 2, ..., 5)$, are listed in Table 2 showing high values of consistency according to *Remark* 5.

Table 2. Values of $OCI(\widetilde{V^k})$ and $PCI(\widetilde{V^k})(k = 1, 2, ..., 5)$

	$\widetilde{V^1}$	$\widetilde{V^2}$	$\widetilde{V^3}$	$\widetilde{V^4}$	$\widetilde{V^5}$
$PCI(\widetilde{V^k})$	0.825	0.825	0.816	0.933	0.883
$OCI(\widetilde{V^k})$	1	1	1	1	1

Remark 6. Despite the representation of PIS is a very challenging and complex task in *Proposition 1* has been introduced an interval based representation of PIS. This solution is valid but still improvable. It seems relevant for future research to study models that provide fuzzy representations for PIS, but it is not the aim of the current research in this paper.

4. A CW framework with PIS based on the 2-tuple linguistic model

his section, a framework for CW dealing with PIS based on 2-tuple linguistic model is proposed.

4.1. A 2-tuple linguistic framework based on Yager's CW scheme

This subsection introduces a CW linguistic framework to manage the linguistic information with PIS in real-world problems, which fulfils the phases of the CW scheme showed in Fig. 1, such that it will be able:

- To obtain linguistic inputs
- To represent the personalized individual semantics
- To carry out the CW processes
- Finally, to return linguistic outputs taking into account PIS

The numerical interval individual semantics obtained from the consistency-driven optimization-based model allows reflecting individual differences in understanding the meaning of words. Thus, the new CW framework to deal with individual semantics, in which $S = \{s_0, s_1, ..., s_g\}$ is the established linguistic term set, $\Phi = \{A|A \subseteq [0,1]\}$ the established numerical domain, and $E = \{e_1, e_2, ..., e_n\}$ the set of n individuals, should extend the scheme of Fig. 1. Therefore, our proposal consists of the scheme depicted in Fig. 2, composed by the following three processes:

- *Individual semantics translation.* This process translates linguistic terms in S into the individual semantics defined by interval values in the established numerical domain Φ. The individual semantics translation process can be carried out by the consistency-driven optimization-based model introduced in Section 3. Formally, it can be expressed as the mapping $INS^k : S \to \Phi$, where INS^k is called the individual semantics translation, associated with e_k.

- *Numerical computation.* The output of individual semantics translation activates numerical computation over Φ, whose output is an interval numerical value.

- *Individual semantics retranslation.* It is the inverse operation of individual semantics translation, and it is applied to retranslate the output of numerical computation into linguistic 2-tuples in $\overline{S}$ easy to understand for individuals. The individual semantics retranslation can be expressed as the inverse of INS^k, denoted as $INS^{k,-1}$.

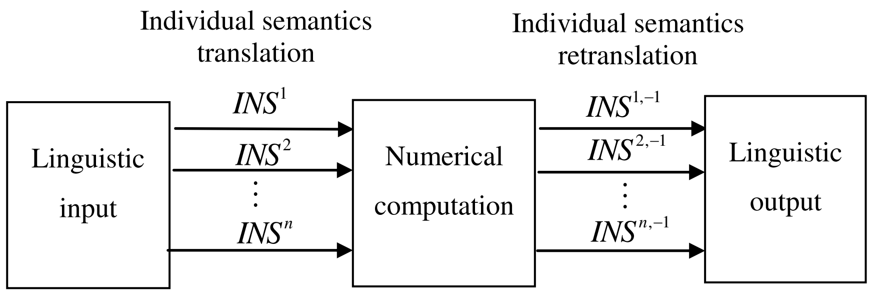

Fig. 2. The framework for the 2-tuple linguistic model with individual semantics

Following, different operators for *numerical computation* based on the linguistic 2-tuple are further detailed, the other two processes are based on results presented in previous sections and not further detailed here.

4.2. Comparison and aggregation: The personalized 2-tuple linguistic operators

The comparison and aggregation operators in the computational model of the 2-tuple linguistic model have been investigated extensively. However, the existing 2-tuple linguistic models only can be suitable to deal with decision problems in the context that a word has the same numerical meaning for different people.

In this subsection, following the CW framework in Fig. 2, it is proposed the personalized 2-tuple linguistic comparison and aggregation operators for the numerical computation phase to deal with the problem that words mean different things to different people.

Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set, and let $E = \{e_1, e_2, ..., e_m\}$ be the set of decision makers. Let INS^k be the interval numerical scale over S, associated with the decision maker e_k. In the following personalized 2-tuple linguistic comparison and aggregation operators are presented.

(1) Personalized 2-tuple linguistic comparison operator

Let r_κ and r_ρ be two linguistic terms provided by decision makers e_κ and e_ρ, then
(i) $r_\kappa \succ r_\rho$ if and only if $INS^\kappa(r_\kappa) > INS^\rho(r_\rho)$;
(ii) $r_\kappa \sim r_\rho$ if and only if $INS^\kappa(r_\kappa) = INS^\rho(r_\rho)$;
(iii) $r_\kappa \prec r_\rho$ if and only if $INS^\kappa(r_\kappa) < INS^\rho(r_\rho)$.

Remark 7. There are many proposals for comparing interval values. Without loss of generality, in this paper it is used the comparison operator introduced in [43] to compare interval values.

Example 4. Let $E = \{e_1, e_2, e_3\}$ be three decision makers, and let $S = \{s_0, s_1, ..., s_4\}$ be the linguistic term set. As shown in Section 3, different decision makers have different interval numerical scales over S. Without loss of generality, the individual interval numerical scale INS^k over S, associated with e_k, is set as follows,

$INS^1(s_0) = [0, 0.25]$, $INS^1(s_1) = [0.3, 0.45]$, $INS^1(s_2) = 0.5$, $INS^1(s_3) = [0.6, 0.7]$, and $INS^1(s_4) = [0.75, 1]$;

$INS^2(s_0) = 0$, $INS^2(s_1) = [0.1, 0.25]$, $INS^2(s_2) = 0.5$, $INS^2(s_3) = [0.8, 0.9]$, and $INS^2(s_4) = [0.9, 1]$;

$INS^3(s_0) = [0, 0.1]$, $INS^3(s_1) = [0.2, 0.4]$, $INS^3(s_2) = 0.5$, $INS^3(s_3) = [0.75, 0.8]$, and $INS^3(s_4) = [0.8, 1]$.

Let r_1, r_2 and r_3 be the linguistic terms provided by decision makers e_1, e_2 and e_3, respectively.

If $r_1 = s_1$, $r_2 = s_1$, and $r_3 = s_3$, using the personalized 2-tuple linguistic comparison operator it can be obtained $r_2 \prec r_1 \prec r_3$ because of $INS^1(r_1) < INS^3(r_3) < INS^2(r_2)$.

(2) Personalized 2-tuple linguistic aggregation operators

Definition 11. *Let $S = \{s_0, s_1, ..., s_g\}$, $E = \{e_1, e_2, ..., e_m\}$ and INS^k be defined as before. Let $R = \{r_1, r_2, ..., r_m\}$ be a set of linguistic terms to aggregate, where $r_k \in S$ are the linguistic terms given by decision makers $e_k (k = 1, 2, ..., m)$, and let $W = \{w_1, w_2, ..., w_m\}$ be a weighting vector that satisfies $w_k \geq 0$ and $\sum_{k=1}^{m} w_k = 1$, then the personalized 2-tuple linguistic weighted averaging (PTLWA) operator is defined as*

$$PTLWA_W(r_1, r_2, ..., r_m) = (\widetilde{r_1}, \widetilde{r_2}, ..., \widetilde{r_m})^T, \tag{31}$$

where $\widetilde{r_k} = INS^{k,-1}(q)$ and $q = w_1 \times INS^1(r_1) + w_2 \times INS^2(r_2) + ... + w_m \times INS^m(r_m)$. The personalized 2-tuple linguistic ordered weighted averaging (PTLOWA) operator is computed as

$$PTLOWA_W(r_1, r_2, ..., r_m) = (\widetilde{r_1}, \widetilde{r_2}, ..., \widetilde{r_m})^T, \tag{32}$$

where $\widetilde{r_k} = INS^{k,-1}(q)$, $q = w_1 \times INS^{\sigma(1)}(r_{\sigma(1)}) + w_2 \times INS^{\sigma(2)}(r_{\sigma(2)}) + ... + w_m \times INS^{\sigma(m)}(r_{\sigma(m)})$, and $(\sigma(1), \sigma(2), ..., \sigma(m))$ is the permutation of $(1, 2, ..., m)$ such that $INS^{\sigma(k-1)}(r_{\sigma(k-1)}) \succ INS^{\sigma(k)}(r_{\sigma(k)})$ for $k = 2, 3, ..., m$.

In Definition 11, q is the numerical computation result over the linguistic terms $\{r_1, r_2, ..., r_m\}$, and $\widetilde{r_k}$ $(k = 1, 2, ..., m)$ are the linguistic 2-tuples, which show the different understanding of the decision makers e_k to the numerical computation result q.

Below, Example 5 illustrates the calculation of the PTLOWA operator. The calculation of the PTLWA operator is similar.

Example 5. Let $E = \{e_1, e_2, e_3\}$, $S = \{s_0, s_1, ..., s_4\}$ and INS^k $(k = 1, 2, 3)$ be as Example 4. Let r_1, r_2, and r_3 be the linguistic terms provided by decision makers e_1, e_2, and e_3, respectively. Without loss of generality, let $r_1 = s_1$, $r_2 = s_1$, and $r_3 = s_3$.

(1) Individual semantics translation. According to INS^k $(k = 1, 2, 3)$ in Example 4, we have $INS^1(s_1) = [0.3, 0.45]$, $INS^2(s_1) = [0.1, 0.25]$ and $INS^3(s_3) = [0.75, 0.8]$.

(2) Numerical computation. Without loss of generality, let the weighting vector $W = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, then $q = \frac{1}{3} \times INS^1(s_1) + \frac{1}{3} \times INS^2(s_1) + \frac{1}{3} \times INS^3(s_3) = [0.383, 0.5]$.

(3) Individual semantics retranslation. Since $\widetilde{r_1} = INS^{1,-1}(q) = (s_1, 0.449)$,

$\widetilde{r_2} = INS^{2,-1}(q) = (s_2, -0.207)$ and $\widetilde{r_3} = INS^{3,-1}(q) = (s_2, -0.345)$, we have $PTLOWA_W(s_1, s_1, s_3) = (\widetilde{r_1}, \widetilde{r_2}, \widetilde{r_3})^T = ((s_1, 0.449), (s_2, -0.207), (s_2, -0.345))^T$.

Some desired properties of the PTLOWA operator are introduced. The properties of the PTLWA operator can be analyzed similarly.

Proposition 2. *Let $S = \{s_0, s_1, ..., s_g\}$, $E = \{e_1, e_2, ..., e_m\}$, $R = \{r_1, r_2, ..., r_m\}$, $W = \{w_1, w_2, ..., w_m\}$, and INS^k be defined as before. Then the PTLOWA operator satisfies the following properties,*

(1) Boundary. Let $q_1 = \min\limits_{\alpha \in \{1,...,m\}} \sum\limits_{k=1}^{m} (w_k \times INS^k(r_\alpha))$ and $q_2 = \max\limits_{\alpha \in \{1,...,m\}} \sum\limits_{k=1}^{m} (w_k \times INS^k(r_\alpha))$. Then $(INS^{1,-1}(q_1), INS^{2,-1}(q_1), ..., INS^{m,-1}(q_1))^T \leq PTLOWA_W (r_1, r_2, ..., r_m) \leq (INS^{1,-1}(q_2), INS^{2,-1}(q_2), ..., INS^{m,-1}(q_2))^T$.

(2) Idempotency. $PTLOWA_W(r_1, r_2, ..., r_m) = (r_1, r_2, ..., r_m)^T$ if $r_k \sim r_t$ for any $k, t \in \{1, 2, ..., m\}$.

(3) Commutativity. If $(r'_1, r'_2, ..., r'_m)$ is any permutation of $(r_1, r_2, ..., r_m)$, then we have $PTLOWA_W(r'_1, r'_2, ..., r'_m) = PTLOWA_W(r_1, r_2, ..., r_m)$.

(4) Monotonicity. $PTLOWA_W(r_1, r_2, ..., r_m) > PTLOWA_W(r'_1, r'_2, ..., r'_m)$ if $r_k \succ r'_k$ for $k = 1, 2, ..., m$.

The proof of Proposition 2 is provided in Appendix.

Once it has been introduced different operators for carrying out the *Numerical Computation* process of the CW framework with PIS (see Fig. 2), it is convenient to show the differences between CW processes carried out by previous models in the literature and our proposal to clarify the differences and advantages of using PIS in those problems in which can be necessary. To do so, below it is proposed a comparison among different functions to compute the numerical indexes according to the linguistic modelling and using the PTLOWA operator.

Let S, E, W, and r_k be defined as Examples 4 and 5., then consider five different cases:

Case A. The numerical index is computed by the 2-tuple linguistic model [18]:

$INS^1 = INS^2 = INS^3 = \Delta^{-1}$

Case B. The numerical index is computed by:
$INS^1 = INS^2 = INS^3 = \{[0, 0.25], [0.3, 0.45], 0.5, [0.6, 0.7], [0.75, 1]\}$
Case C. The numerical index is computed by:
$INS^1 = INS^2 = INS^3 = \{0, [0.1, 0.25], 0.5, [0.8, 0.9], [0.9, 1]\}$
Case D. The numerical index is computed by:
$INS^1 = INS^2 = INS^3 = \{[0, 0.1], [0.2, 0.4], 0.5, [0.75, 0.8], [0.8, 1]\}$
Case E. The numerical index is computed by using the personalized 2-tuple linguistic operators, in which:
$INS^1 = \{[0, 0.25], [0.3, 0.45], 0.5, [0.6, 0.7], [0.75, 1]\}$,
$INS^2 = \{0, [0.1, 0.25], 0.5, [0.8, 0.9], [0.9, 1]\}$,
$INS^3 = \{[0, 0.1], [0.2, 0.4], 0.5, [0.75, 0.8], [0.8, 1]\}$.

Comparing the results obtained, among the numerical indexes in different 2-tuple linguistic modelling showed in Table 3, can be found out that the personalized 2-tuple linguistic operators provide not only obvious different results because of computations but also different rankings due to the consideration of different meaning of linguistic information by each expert.

Table 3. Results for different numerical indexes

	Comparison	Weighted averaging operator
Case A	$r_1 \sim r_2 \prec r_3$	$(s_2, -0.333)$
Case B	$r_1 \sim r_2 \prec r_3$	$(s_2, -0.432)$
Case C	$r_1 \sim r_2 \prec r_3$	$(s_2, -0.337)$
Case D	$r_1 \sim r_2 \prec r_3$	$(s_2, -0.317)$
Case E	$r_3 \succ r_1 \succ r_2$	$((s_1, 0.449), (s_2, -0.21), (s_2, -0.351))^T$

5. Solving a linguistic GDM problem with PIS: A consensus based model

This section presents the application of the PIS model to deal with the consensus-based linguistic GDM with individual semantics. Specifically, it is introduced the notation for GDM problems with individual semantics together a resolution framework and finally a consensus reaching process is provided and developed.

5.1. A GDM framework with PIS

Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set, $X = \{X_1, X_2, ..., X_n\}$ be a set of alternatives, and $E = \{e_1, e_2, ..., e_m\}$ be a set of decision makers. In the GDM with individual semantics, each decision maker provides his/her preferences over X by a linguistic preference relation $L^k = (l_{ij}^k)_{n\times n}$ ($k = 1, 2, ..., m$), where $l_{ij}^k \in S$ estimates the preference degree of decision maker e_k for alternative X_i over X_j. Meanwhile, decision makers have their individual semantics over S, namely, they use different interval numerical scales of S. Consequently, it is necessary to support decision makers, who have individual semantics described by individual interval numerical scales INS^k over S, to reach an agreed solution for the linguistic GDM problem.

Therefore, a new framework to deal with the consensus-based linguistic GDM with individual semantics is introduced. It includes three processes depicted in Fig. 3: individual semantics translation process, selection process and consensus reaching process.

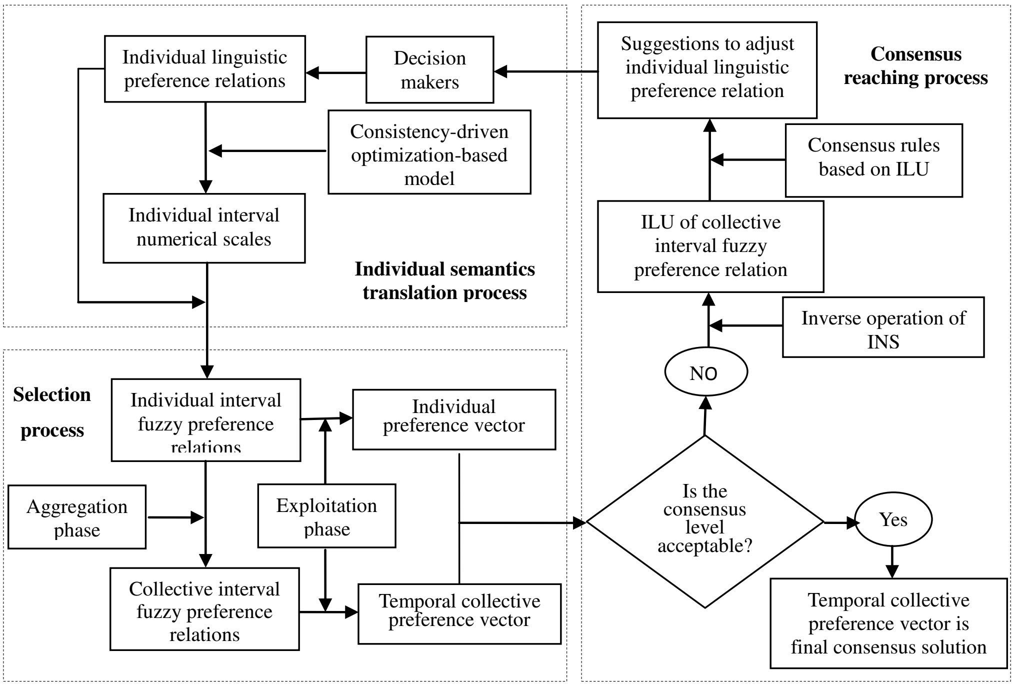

Fig. 3. The framework for the consensus-based linguistic GDM with PIS

(1) Individual semantics translation process

The individual semantics translation process uses the consistency-driven optimization-based model proposed in Section 3.2 to generate the individual interval numerical scales of S, INS^k, by applying the individual linguistic preference relation L^k as data resource. Using the individual interval numerical scale INS^k to quantify L^k obtains the individual interval fuzzy preference relation $\widetilde{V^k} = (\widetilde{v_{ij}^k})_{n\times n}$, where

$$\widetilde{v_{ij}^k} = [v_{ij}^{k-}, v_{ij}^{k+}] = INS^k(l_{ij}^k)\ (k = 1, 2, ..., m). \tag{33}$$

(2) Selection process

It aims at obtaining the collective ranking of alternatives by applying two phases: aggregation phase and exploitation phase.

The aggregation phase aggregates individual interval fuzzy preference relations $\{\widetilde{V^1}, \widetilde{V^2}, ..., \widetilde{V^m}\}$ into a collective preference relation $\widetilde{V^c} = (\widetilde{v_{ij}^c})_{n\times n}$. The aggregation operation can be carried out by means of either the weighted average (WA) operator or ordered weighted average (OWA) operator [47]. In this paper, the WA operator is used, i.e.,

$$\widetilde{v_{ij}^c} = [v_{ij}^{c-}, v_{ij}^{c+}] = [\sum_{k=1}^{m} \lambda_k \cdot v_{ij}^{k-}, \sum_{k=1}^{m} \lambda_k \cdot v_{ij}^{k+}], \tag{34}$$

where $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ is the weighting vector of decision makers $\{e_1, e_2, ..., e_m\}$ that satisfies $\lambda_k \in [0, 1]$ and $\sum_{k=1}^{m} \lambda_k = 1$.

In the exploitation phase, the collective preference vector $Z^c = (z_1^c, z_2^c, ..., z_n^c)^T$ is obtained from $\widetilde{V^c}$ to order alternatives, where

$$z_i^c = [\sum_{j=1}^{n} w_j \cdot v_{ij}^{c-}, \sum_{j=1}^{n} w_j \cdot v_{ij}^{c+}], \tag{35}$$

and $W = \{w_1, w_2, ..., w_n\}$ is an associated weighting vector that satisfies $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

From the values z_i^c, the ranking of alternatives $\{X_1, X_2, ..., X_n\}$ is obtained. The larger the value of z_i^c, the better the alternative.

(3) Consensus reaching process

It aims at reaching a higher agreement level among decision makers. The details of the consensus reaching process are introduced in Section 5.2.

5.2. Consensus reaching process

A consensus reaching process can be viewed as an iterative process with several consensus rounds, in which the decision makers adjust their preferences following the consensus rules until the maximum possible consensus level is achieved. Generally consensus reaching process includes two parts [34]: (i) A consensus measure process computes the level of agreement among experts and, (ii) A feedback mechanism guides the process to improve the agreement among them.

(1) Consensus measure

Let $Z^k = (z_1^k, z_2^k, ..., z_n^k)^T$ be the individual preference vector obtained from $\widetilde{V^k}$ to rank alternatives, where

$$z_i^k = [\sum_{j=1}^{n} w_j \cdot v_{ij}^{k-}, \sum_{j=1}^{n} w_j \cdot v_{ij}^{k+}], for\ i = 1, 2, ..., n. \tag{36}$$

Let $O^c = (o_1^c, o_2^c, ..., o_n^c)^T$, where o_i^c is the position of alternative X_i in Z^c. For example, if $Z^c = ([0.1, 0.2], [0.6, 0.7], [0.3, 0.4], [0.8, 0.9])$, then $O^c = (4, 2, 3, 1)^T$. Similarly, we get $O^k = (o_1^k, o_2^k, ..., o_n^k)^T$, where o_i^k is the position of alternative X_i in Z^k.

The consensus measure used in our proposal for consensus reaching process is defined as:

Definition 12. *The consensus level associated with decision maker e_k, $CL_k \in [0, 1]$, is given by*

$$CL_k = 1 - 2\sum_{i=1}^{n} \frac{|o_i^c - o_i^k|}{n^2}. \tag{37}$$

The consensus level of all decision makers $\{e_1, e_2, ..., e_m\}$, $CL \in [0, 1]$, is given by

$$CL = 1 - 2\sum_{k=1}^{m}\sum_{i=1}^{n} \frac{|o_i^c - o_i^k|}{mn^2}. \tag{38}$$

A larger CL value indicates a higher consensus degree among the decision makers $\{e_1, e_2, ..., e_m\}$.

(2) Feedback mechanism

In our proposal the feedback mechanism is based on different consensus rules that help decision makers to make their opinions closer across the consensus reaching process. Before introducing the consensus rules, it is proposed the concept of the individual linguistic understanding of the collective interval fuzzy preference relation, associated with each decision maker (see Definition 13), which provides the basis of the consensus rules.

Definition 13. *Let INS^k be an ordered interval numerical scale on S, associated with the decision maker e_k, and let $\widetilde{V^c} = (\widetilde{v^c_{ij}})_{n\times n}$ be a collective interval fuzzy preference relation. Then $L^{k*} = (l^{k*}_{ij})_{n\times n}$, where*

$$l^{k*}_{ij} = INS^{k,-1}(\widetilde{v^c_{ij}}), \tag{39}$$

is called the individual linguistic understanding of the collective interval fuzzy preference relation $\widetilde{V^c}$, associated with the decision maker e_k.

The individual linguistic understanding reflects the linguistic meaning of the collective interval fuzzy preference relation $\widetilde{V^c}$, associated with individual decision makers. According to Eqs. (33), (34) and (39), the individual linguistic understanding of $\widetilde{V^c}$ can be expressed by a PTLWA operator, i.e.,

$$(l^{1*}_{ij}, l^{2*}_{ij}, ..., l^{m*}_{ij}) = PTLWA_\lambda(l^1_{ij}, l^2_{ij}, ..., l^m_{ij}). \tag{40}$$

Naturally, different decision makers have different linguistic understanding over $\widetilde{V^c}$. The individual linguistic understanding of $\widetilde{V^c}$ is illustrated in Example 6.

Example 6. Let $E = \{e_1, e_2\}$ and let $S = \{s_0, s_1, ..., s_4\}$. According to Section 3, different decision makers set different interval numerical scales over S. Without loss of generality, the individual interval numerical scales INS^1 and INS^2 over S, associated with e_1 and e_2, respectively, is defined as follows:
$INS^1(s_0) = [0, 0.25], INS^1(s_1) = [0.3, 0.45], INS^1(s_2) = 0.5, INS^1(s_3) = [0.6, 0.7]$ *and* $INS^1(s_4) = [0.75, 1]$.
$INS^2(s_0) = 0, INS^2(s_1) = [0.1, 0.25], INS^2(s_2) = 0.5, INS^2(s_3) = [0.8, 0.9]$ *and* $INS^2(s_4) = [0.9, 1]$.

Let the collective interval fuzzy preference relation $\widetilde{V^c}$ be as follows,

$$\widetilde{V^c} = \begin{pmatrix} null & [0.3, 0.4] & [0.4, 0.6] & [0.75, 1] \\ null & null & [0.2, 0.6] & [0, 0.3] \\ null & null & null & [0.7, 0.8] \\ null & null & null & null \end{pmatrix}.$$

Then, the individual linguistic understandings of $\widetilde{V^c}$, associated with decision makers e_1 and e_2, are L^{1*} and L^{2*}, respectively, i.e.,

$$L^{1*} = \begin{pmatrix} null & (s_1, -0.25) & (s_1, -0.235) & s_4 \\ null & null & (s_2, -0.4) & (s_0, 0.46) \\ null & null & null & (s_4, -0.333) \\ null & null & null & null \end{pmatrix},$$

and

$$L^{2*} = \begin{pmatrix} null & (s_2, -0.472) & (s_2, -0.345) & (s_3, 0.25) \\ null & null & (s_2, 0.16) & (s_1, 0.18) \\ null & null & null & (s_3, -0.12) \\ null & null & null & null \end{pmatrix}.$$

According to L^{1*} and L^{2*}, decision makers e_1 and e_2 have different linguistic understanding over $\widetilde{V^c}$. Based on the individual linguistic understanding of the collective interval fuzzy preference relation and the consensus level CL_k associated with e_k, two consensus rules namely, identification rule and direction rule to guide the feedback process are introduced:

(1) *Identification rule.* The identification rule identifies the decision makers contributing less to reach a high degree of consensus.

From the ranking position of each decision maker e_k according to CL_k, the larger the CL_k, the higher position of decision maker e_k. If the decision maker's position is high, then the decision maker does not need to change his/her preferences, but if it is low then the decision maker has to change his/her preferences. A satisfaction consensus threshold $\overline{CL}$ is computed to calculate how many decision makers need to change their preferences. If $CL_k < \overline{CL}$, $\overline{CL} \in [0, 1]$, the decision maker e_k needs to change his/her preferences. Generally, the decision maker e_τ, whose consensus level $CL_\tau = \min_k CL_k \, (k = 1, 2, ..., m)$, needs to change his/her preferences.

(2) *Direction rule.* The direction rule finds out the direction to change the preferences of decision makers.

Let $L^{k*} = (l_{ij}^{k*})_{n \times n}$ be the individual linguistic understanding of the collective interval fuzzy preference relation $\widetilde{V^c}$, associated with e_k. Let $\overline{L^k} = (\overline{l_{ij}^k})_{n \times n}$ be the adjusted linguistic preference relation associated with e_k. Then the direction rules are as follows:

(i) If l_{ij}^k is smaller than l_{ij}^{k*}, then decision maker e_k should increase the evaluation associated with the pairwise (X_i, X_j). Specifically, the adjusted preference value should be $\overline{l_{ij}^k} \in \{s \,|s \in S, s \in (l_{ij}^k, l_{ij}^{k*}]\}$.

(ii) If $l_{ij}^k = l_{ij}^{k*}$, then the decision maker e_k should not change the evaluations associated with the pairwise (X_i, X_j).

(iii) If l_{ij}^k is larger than l_{ij}^{k*}, then the decision maker e_k should decrease the evaluation associated with the pairwise (X_i, X_j). Specifically, the adjusted preference value should be $\overline{l_{ij}^k} \in \{s \,|s \in S, s \in [l_{ij}^{k*}, l_{ij}^k)\}$.

The following Algorithm 1 provides a formal description of the consensus reaching process.

Algorithm 1

Input: The individual linguistic preference relation based on S, $L^k = (l_{ij}^k)_{n\times n}$ $(k = 1, 2, ..., m)$, the weighting vectors λ=$\{\lambda_1, \lambda_2, ..., \lambda_m\}$ and $W = \{w_1, w_2, ..., w_n\}$, the established consensus threshold $\overline{CL}$, and the established maximum number of iterations h_{max}.

Output: Adjusted linguistic preference relation $\overline{L^k} = (\overline{l_{ij}^k})_{n\times n}$ $(k = 1, 2, ..., m)$.

Step 1: Let $h = 0$ and $L_h^k = (l_{ij,h}^k)_{n\times n} = (l_{ij}^k)_{n\times n}$ $(k = 1, 2, ..., m)$

Step 2: The consistency-driven optimization-based model presented in Section 3.2 is used to set the individual interval numerical scales INS^k.

Step 3: Using INS^k to quantify L_h^k obtains the individual interval fuzzy preference relation $\widetilde{V_h^k}$. Then, using Eq. (34) obtains the collective interval fuzzy preference relation $\widetilde{V_h^c}$. Next, using Eqs. (35) and (36) obtains the collective preference vector $Z^c = (z_1^c, z_2^c, ..., z_n^c)^T$ and the individual preference vector $Z^k = (z_1^k, z_2^k, ..., z_n^k)^T$. Finally, based on Definition 12, the consensus level CL_h is calculated. If $CL_h > \overline{CL}$ or $h > h_{max}$, then go to Step 6; otherwise, continue with the next step.

Step 4: Using Eq. (39) obtains the individual linguistic understanding of the collective interval fuzzy preference relation $\widetilde{V_h^c}$, associated with e_k, $L_h^{k*} = (l_{ij,h}^{k*})_{n\times n}$ $(k = 1, 2, ..., m)$.

Step 5: Based on the identification rule, the decision maker e_τ, who has the lowest consensus level, needs to change his/her preferences. Then, according to the direction rule, the adjusted suggestions associated with decision maker e_τ and the pairwise (X_i, X_j) are obtained, i.e.,

$$l_{ij,h+1}^\tau \in \begin{cases} \{s \,|s \in S, s \in (l_{ij,h}^\tau, l_{ij,h}^{\tau*}]\} & if\ l_{ij,h}^\tau \prec l_{ij,h}^{\tau*} \\ l_{ij,h}^\tau & if\ l_{ij,h}^\tau = l_{ij,h}^{\tau*} \\ \{s \,|s \in S, s \in [l_{ij,h}^{\tau*}, l_{ij,h}^\tau)\} & if\ l_{ij,h}^\tau \succ l_{ij,h}^{\tau*} \end{cases} \tag{41}$$

Based on Eq. (41), construct the new individual linguistic preference relation $L_{h+1}^\tau = (l_{ij,h+1}^\tau)_{n\times n}$. Let $h = h + 1$. Then, go to Step 2.

Step 6: Let $\overline{L^k} = L_h^k$. Output the adjusted linguistic preference relation $\overline{L^k} = (\overline{l_{ij}^k})_{n\times n}$ $(k = 1, 2, ..., m)$.

Next, we provide Example 7 to illustrate the selection process and the consensus reaching process.

Example 7. Once finished the individual semantics translation process in Example 4, we keep solving the problem (Example 3) to apply the selection process and the consensus reaching process to it.

(1) Selection process

Without loss of generality, let the weighting vectors $\lambda = W = \{\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\}$. Using Eq. (34) obtains the collective interval fuzzy preference relation, $\widetilde{V^c} = (\widetilde{v_{ij}^c})_{5\times 5}$, i.e.,

$$\widetilde{v_{ij}^c} = [v_{ij}^{c-}, v_{ij}^{c+}] = [(1/5)\sum_{k=1}^{5} v_{ij}^{k-}, (1/5)\sum_{k=1}^{5} v_{ij}^{k+}].$$

Matrix $\widetilde{V^c}$ is listed as follows,

$$\widetilde{V^c} = \begin{pmatrix} [0.5, 0.5] & [0.575, 0.6] & [0.25, 0.425] & [0.1, 0.349] & [0.1, 0.349] \\ [0.4, 0.425] & [0.5, 0.5] & [0.275, 0.375] & [0.2, 0.3] & [0.2, 0.35] \\ [0.575, 0.75] & [0.625, 0.725] & [0.5, 0.5] & [0.45, 0.475] & [0.349, 0.4] \\ [0.651, 0.9] & [0.7, 0.8] & [0.525, 0.55] & [0.5, 0.5] & [0.375, 0.475] \\ [0.651, 0.9] & [0.65, 0.85] & [0.6, 0.651] & [0.525, 0.625] & [0.5, 0.5] \end{pmatrix}.$$

Then using Eq. (35) yields $z_i^c = [(1/5)\sum_{j=1}^{5} v_{ij}^{c-}, (1/5)\sum_{j=1}^{5} v_{ij}^{c+}]$. The values of $z_i^c (i = 1, 2, ..., 5)$ are listed below,
$z_1^c = [0.305, 0.445]$, $z_2^c = [0.315, 0.39]$, $z_3^c = [0.5, 0.57]$, $z_4^c = [0.55, 0.645]$, and $z_5^c = [0.585, 0.705]$.

The larger the value of z_i^c, the better the alternative. Based on the comparison operations of interval numbers [43], the collective ranking of alternatives is $X_5 \succ X_4 \succ X_3 \succ X_1 \succ X_2$.

Similarly, we can get the individual rankings of alternatives, they are as follows,

$e_1 : X_4 \succ X_5 \succ X_1 \succ X_2 \succ X_3$

$e_2 : X_5 \succ X_4 \sim X_3 \succ X_2 \succ X_1$

$e_3 : X_3 \succ X_5 \succ X_4 \succ X_1 \succ X_2$

$e_4 : X_5 \succ X_4 \succ X_3 \succ X_2 \succ X_1$

$e_5 : X_5 \succ X_4 \sim X_3 \succ X_2 \succ X_1$

(2) Consensus reaching process

According to Eq. (37), $CL_1 = 0.52$, $CL_2 = 0.76$, $CL_3 = 0.68$, $CL_4 = 0.84$, and $CL_5 = 0.76$. Then, based on Eq. (38), the consensus level of all decision makers is $CL = 0.712$.

The consensus rules are then applied to help decision makers reach a high consensus. The consensus rules are carried out in the following two steps:

(i) Identification rule

From the position ranking of each decision maker e_k according to CL_k, it is found that the position of the decision maker e_1 in the ranking is the lowest. Clearly, the decision maker e_1 needs to change his/her preferences.

(ii) Direction rule

Firstly, the individual linguistic understanding of collective interval fuzzy preference relation $\widetilde{V^c}$, associated with the decision maker e_1, L^{1*}, is obtained:

$$L^{1*} = \begin{pmatrix} null & (s_2, 0.01) & (s_1, 0.08) & (s_1, -0.352) & (s_1, -0.352) \\ null & null & (s_1, 0.08) & (s_1, -0.25) & (s_1, -0.15) \\ null & null & null & (s_1, -0.2) & (s_1, 0.357) \\ null & null & null & null & (s_2, -0.44) \\ null & null & null & null & null \end{pmatrix}.$$

Then, let the decision maker e_1 change his/her preference values according to the direction rule, the new preference relation L^1 is obtained as follows:

$$L^1 = \begin{pmatrix} null & s_2 & s_1 & s_0 & s_1 \\ null & null & s_1 & s_0 & s_1) \\ null & null & null & s_1 & s_1 \\ null & null & null & null & s_3 \\ null & null & null & null & null \end{pmatrix}.$$

Applying the selection process again,the individual ranking of alternatives is obtained, associated with decision maker e_1, that is $X_4 \succ X_5 \succ X_3 \succ X_1 \sim X_2$, and the collective ranking of alternatives, $X_5 \succ X_4 \succ X_3 \succ X_2 \succ X_1$. Then, applying the consensus reaching process again, the consensus level of all decision makers is obtained: $CL = 0.824$.

6. Conclusions

In this paper it has been introduced a Personalized Individual Semantics (PIS) approach to model and solve linguistic GDM problems with preference relations to improve the management of different meanings of words for different people.

First a consistency-driven optimization-based model to personalize and represent the individual semantics based on the interval numerical scale is introduced. Second a new CW framework based on the 2-tuple linguistic model for dealing with personalized individual semantics is developed and eventually both are applied to linguistic GDM problem with a consensus reaching process.

In the future, we plan to work on the potential use of PIS for large scale decision making [24, 25, 26, 35, 37, 44] to handle large groups with different PIS according to their preferences.

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References

[1] M.A. Abchir and I. Truck. Towards an extension of the 2-tuple linguistic model to deal with unbalanced linguistic term sets. *Kybernetika*, 49(1):164–180, 2013.

[2] S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, J. Alcalá-Fdez, and C. Porcel. A consistency-based procedure to estimate missing pairwise preference values. *International Journal of Intelligent Systems*, 23(2):155–175, 2008.

[3] H. Bustince, F. Herrera, and J. Montero. *Fuzzy Sets and Their Extensions: Representation, Aggregation and Models:Intelligent Systems from Decision Making to Data Mining, Web Intelligence and Computer Vision*, volume 220 of *Studies in Fuzziness and Soft Computing*. Springer-Verlag, 2008.

[4] F. Chiclana, E. Herrera-Viedma, S. Alonso, and F. Herrera. Cardinal consistency of reciprocal preference relations: A characterization of multiplicative transitivity. *IEEE Transactions on Fuzzy Systems*, 17(1):14–23, Feb 2009.

[5] Y. Dong, X. Chen, C.C. Li, W.C. Hong, and Y.F. Xu. Consistency issues of interval pairwise comparison matrices. *Soft Computing*, 19(8):2321–2335, 2015.

[6] Y. Dong and E. Herrera-Viedma. Consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic GDM with preference relation. *IEEE Transactions on Cybernetics*, 45(4):780–792, 2015.

[7] Y. Dong, C.C. Li, and F. Herrera. Connecting the numerical scale model to the unbalanced linguistic term sets. In *IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, pages 455–462, 2014.

[8] Y. Dong, C.C. Li, and F. Herrera. An optimization-based approach to adjusting unbalanced linguistic preference relations to obtain a required consistency level. *Information Sciences*, 292:27–38, 2015.

[9] Y. Dong, C.C. Li, Y.F. Xu, and X. Gu. Consensus-based group decision making under multi-granular unbalanced 2-tuple linguistic preference relations. *Group Decision and Negotiation*, 24(2):217–242, 2015.

[10] Y. Dong, Y. Xu, and H. Li. On consistency measures of linguistic preference relations. *European Journal of Operational Research*, 189(2):430–444, 2008.

[11] Y. Dong, Y. Xu, and S. Yu. Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model. *IEEE Transactions on Fuzzy Systems*, 17(6):1366–1378, 2009.

[12] Y. Dong, G. Zhang, W.C. Hong, and S. Yu. Linguistic computational model based on 2-tuples and intervals. *IEEE Transactions on Fuzzy Systems*, 21(6):1006–1018, 2013.

[13] H. Doukas, C. Karakosta, and J. Psarras. Computing with words to assess the sustainability of renewable energy options. *Experts Systems with Applications*, 37(7):5491–5497, 2010.

[14] M. Espinilla, J. Liu, and L. Martínez. An extended hierarchical linguistic model for decision-making problems. *Computational Intelligence*, 27(3):489–512, 2011.

[15] F. Herrera, S. Alonso, F. Chiclana, and E. Herrera-Viedma. Computing with words in decision making: Foundations, trends and prospects. *Fuzzy Optimization and Decision Making*, 8(4):337–364, 2009.

[16] F. Herrera, E. Herrera-Viedma, and L. Martínez. A fusion approach for managing multi-granularity linguistic terms sets in decision making. *Fuzzy Sets and Systems*, 114(1):43–58, 2000.

[17] F. Herrera, E. Herrera-Viedma, and L. Martínez. A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on Fuzzy Systems*, 16(2):354–370, 2008.

[18] F. Herrera and L. Martínez. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 8(6):746–752, 2000.

[19] F. Herrera and L. Martínez. A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic context in multi-expert decision making. *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, 31(2):227–234, 2001.

[20] E. Herrera-Viedma, F.J. Cabrerizo, J. Kacprzyk, and W. Pedrycz. A review of soft consensus models in a fuzzy environment. *Information Fusion*, 17(4):4–13, 2014.

[21] E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso. Group decision-making model with incomplete fuzzy preference relations based on additive consistency. *IEEE Transactions on Systems, Man and Cybernetics, Part B, Cybernetics*, 37(1):176–189, 2007.

[22] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque. Some issues on consistency of fuzzy preference relations. *European Journal of Operational Research*, 154(1):98–109, 2004.

[23] C-C. Li and Y. Dong. Multi-attribute group decision making methods with proportional 2-tuple linguistic assessments and weights. *International Journal of Computational Intelligence Systems*, 7(4):758–770, 2014.

[24] B. Liu, Y. Shen, X. Chen, Y. Chen, and X. Wang. A partial binary tree dea-da cyclic classification model for decision makers in complex

multi-attribute large-group interval-valued intuitionistic fuzzy decision-making problems. *Information Fusion*, 18(3):119–130, 2014.

[25] B. Liu, Y. Shen, Y. Chen, X. Chen, and Y. Wang. A two-layer weight determination method for complex multi-attribute large-group decision-making experts in a linguistic environment. *Information Fusion*, 23:156–165, 2015.

[26] Y. Liu, Z.P. Fan, and X. Zhang. A method for large group decision-making based on evaluation information provided by participators from multiple groups. *Information Fusion*, in press, 2015.

[27] L. Martínez and F. Herrera. An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences*, 207(1):1–18, 2012.

[28] L. Martínez, D. Ruan, and F. Herrera. Computing with words in decision support systems: An overview on models and applications. *International Journal of Computational Intelligence Systems*, 3(4):382–395, 2010.

[29] J.M. Mendel and D. Wu. *Perceptual Computing: Aiding People in Making Subjective Judgments*. IEEE-Wiley, 2010.

[30] J.M. Mendel, L.A. Zadeh, E. Trillas, R.R. Yager, J. Lawry, H. Hagras, and S. Guadarrama. What computing with words means to me: Discussion forum. *IEEE Computational Intelligence Magazine*, 5(1):20–26, 2010.

[31] J.M. Merigó, M. Casanovas, and L. Martínez. Linguistic aggregation operators for linguistic decision making based on the dempster-shafer theory of evidence. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 18(3):287–304, 2010.

[32] G.A. Miller. The magical number seven plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, 63:81–97, 1956.

[33] J.A. Morente-Molinera, I.J. Pérez, M.R. Ureña, and E. Herrera-Viedma. On multi-granular fuzzy linguistic modelling in group decision making problems: a systematic review and future trends. *Knowledge-Based Systems*, 74:49–60, 2015.

[34] I. Palomares, F.J. Estrella, L. Martínez, and F. Herrera. Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study. *Information Fusion*, 20:252–271, 2014.

[35] I. Palomares, L. Martínez, and F. Herrera. A consensus model to detect and manage noncooperative behaviors in large-scale group decision making. *IEEE Transactions on fuzzy systems*, 22(3):516–530, 2014.

[36] W. Pedrycz, P. Ekel, and R. Parreiras. *Fuzzy Multicriteria Decision-Making: Models, Methods and Applications.* John Wiley & Sons, Ltd. Chichester, UK, 2010.

[37] F.J. Quesada, I. Palomares, and L. Martínez. Managing experts behavior in large-scale consensus reaching processes with uninorm aggregation operators. *Applied Soft Computing*, 35:873–887, 2015.

[38] R.M. Rodríguez and L. Martínez. An analysis of symbolic linguistic computing models in decision making. *International Journal of General Systems*, 42(1):121–136, 2013.

[39] K.S. Schmucker. *Fuzzy Sets, Natural Language Computations, and Risk Analysis.* Computer Science Press, Rockville, MD, 1984.

[40] M. Tong and P.P. Bonissone. A linguistic approach to decision making with fuzzy sets. *IEEE Transactions on Systems, Man and Cybernetics*, 10(11):716–723, 1980.

[41] J.H. Wang and J. Hao. A new version of 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 14(3):435–445, 2006.

[42] Y.M. Wang, J.B. Yang, and D.L. Xu. Interval weight generation approaches based on consistency test and interval comparison matrices. *Applied Mathematics and Computation*, 167(1):252–273, 2005.

[43] Y.M. Wang, J.B. Yang, and D.L. Xu. A preference aggregation method through the estimation of utility intervals. *Computers and Operations Research*, 32:2027–2049, 2005.

[44] X.H. Xu, Z.J. Du, and X.H. Chen. Consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions. *Decision Support Systems*, 79:150–160, 2015.

[45] Y.J. Xu, R. Patnayakuni, and H.M. Wang. The ordinal consistency of a fuzzy preference relation. *Information Sciences*, 224:152–164, 2013.

[46] Z.S. Xu and J. Chen. Some models for deriving the priority weights from interval fuzzy preference relations. *European Journal of Operational Research*, 184(1):266–280, 2008.

[47] R.R. Yager. On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Transactions on Systems, Man, and Cybernetics*, 18:183–190, 1988.

[48] R.R. Yager. On the retranslation process in Zadeh's paradigm of computing with words. *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, 34(2):1184–1195, 2004.

[49] L.A. Zadeh. Fuzzy logic = computing with words. *IEEE Transactions on Fuzzy Systems*, 4(2):103–111, 1996.

[50] L.A. Zadeh. *From Computing with Numbers to Computing with Words-From Manipulation of Measurements to Manipulation of Perceptions*, pages 35–68. In: Computing with Words, P.P. Wang (Ed.). Wiley Series on Intelligent Systems. John Wiley and Sons, 2001.

Appendix

The proof of Proposition 1

In model P, constraints (23)and (24)guarantee that $F^{2k} \in N_{\widetilde{V^k}}$, constraint (25) guarantees that $CL(F^{1k}) = 1$, constraints (26) and (27) guarantee that$F^{1k} \in N_{\widetilde{V^k}}$, constraint (28) guarantees that $a_i \in INS^k(s_i)$, and constraints(29) and (30) guarantee that INS^k is ordered.

This completes the proof of Proposition 1.

The proof of Proposition 2

The properties of the PTLOWA operator (1)-(4) are proved as follows,

(1) Let $q_1 = \min\limits_{\alpha\in\{1,...,m\}} \sum\limits_{k=1}^{m} w_k INS^k(r_\alpha)$, $q_2 = \max\limits_{\alpha\in\{1,...,m\}} \sum\limits_{k=1}^{m} w_k INS^k(r_\alpha)$, and $q = \sum\limits_{k=1}^{m} w_k INS^k(r_k)$.

Since INS is ordered, it is clear that $q_1 \le q \le q_2$. So, we can get $INS^{k-}(q_1) \le INS^{k-}(q) \le INS^{k-}(q_2)$ for $k = 1, 2, ..., m$, and $(INS^{1-}(q_1), INS^{2-}(q_1), ..., INS^{m-}(q_1))^T \le PTLOWA_W(r_1, r_2, ..., r_m) \le (INS^{1-}(q_2), INS^{2-}(q_2), ..., INS^{m-}(q_2))^T$, which completes the proof of property (1).

(2) Since $r_k \sim r_t$ for $k = 1, 2, ..., m$, it follows that $INS^k(r_k) = INS^t(r_t)$, and $z_t = w_1 \times INS^1(r_1) + w_2 \times INS^2(r_2) + ... + w_m \times INS^m(r_m) = w_1 \times INS^{\sigma(1)}(r_{\sigma(1)}) + w_2 \times INS^{\sigma(2)}(r_{\sigma(2)}) + ... + w_m \times INS^m(r_{\sigma(m)}) = INS^t(r_t)$. So, $PTLOWA_W(r_1, r_2, ..., r_m) = (INS^{1-}(z_t), INS^{2-}(z_t), ..., INS^{m-}(z_t))^T = (r_1, r_2, ..., r_m)^T$, which completes the proof of property (2).

(3) Let $PTLOWA_W(r_1, r_2, ..., r_m) = (INS^{1-}(z), INS^{2-}(z), ..., INS^{m-}(z))^T$, where $z = w_1 \times INS^{\sigma(1)}(r_{\sigma(1)}) + w_2 \times INS^{\sigma(2)}(r_{\sigma(2)}) + ... + w_m \times INS^{\sigma(m)}(r_{\sigma(m)})$. Let $PTLOWA_W(r'_1, r'_2, ..., r'_m) = (INS^{1-}(z'), INS^{2-}(z'), ..., INS^{m-}(z'))^T$, where $z' = w_1 \times INS^{\sigma(1)}(r'_{\sigma(1)}) + w_2 \times INS^{\sigma(2)}(r'_{\sigma(2)}) + ... + w_m \times INS^{\sigma(m)}(r'_{\sigma(m)})$. Because $(r'_1, r'_2, ..., r'_m)$ is any permutation of $(r_1, r_2, ..., r_m)$, so we have $PTLOWA_W(r'_1, r'_2, ..., r'_m) = PTLOWA_W(r_1, r_2, ..., r_m)$, which completes the proof of property (3).

(4) Let $PTLOWA_W(r_1, r_2, ..., r_m) = (INS^{1-}(z), INS^{2-}(z), ..., INS^{m-}(z))^T$, where $z = w_1 \times INS^{\sigma(1)}(r_{\sigma(1)}) + w_2 \times INS^{\sigma(2)}(r_{\sigma(2)}) + ... + w_m \times INS^m(r_{\sigma(m)})$.

Let $PTLOWA_W(r'_1, r'_2, ..., r'_m) = (INS^{1-}(z'), INS^{2-}(z'), ..., INS^{m-}(z'))^T$, where $z' = w_1 \times INS^1(r'_{\sigma(1)}) + w_2 \times INS^2(r'_{\sigma(2)}) + ... + w_m \times INS^m(r'_{\sigma(m)})$.

Since $r_k \succ r'_k$ and INS is ordered, it follows that $r_{\sigma(k)} \succ r'_{\sigma(k)}$ and $z > z'$, then we can get

$PTLOWA_W(r_1, r_2, ..., r_m) = (INS^{1-}(z), INS^{2-}(z), ..., INS^{m-}(z))^T > PTLOWA_W(r'_1, r'_2, ..., r'_m) = (INS^{1-}(z'), INS^{2-}(z'), ..., INS^{m-}(z'))^T$.

This completes the proof of Proposition 2.

3 Personalized individual semantics based on consistency in hesitant linguistic GDM with comparative linguistic expressions

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Personalized individual semantics based on consistency in hesitant linguistic group decision making with comparative linguistic expressions

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Abstract

In decision making problems, decision makers may prefer to use more flexible linguistic expressions instead of using only one linguistic term to express their preferences. The recent proposals of hesitant fuzzy linguistic terms sets (HFLTSs) are developed to support the elicitation of comparative linguistic expressions in hesitant decision situations. In group decision making (GDM), the statement that words mean different things for different people has been highlighted and it is natural that a word should be defined by individual semantics described by different numerical values. Considering this statement in hesitant linguistic decision making, the aim of this paper is to personalize individual semantics in the hesitant GDM with comparative linguistic expressions to show the individual difference in understanding the meaning of words. In our study, the personalized individual semantics are carried out by the fuzzy envelopes of HFLTSs based on the personalized numerical scales of linguistic term set.

Keywords: `group decision making`, hesitant fuzzy linguistic term set, comparative linguistic expressions, personalized individual semantics, numerical scale

1. Introduction

In real-world decision making, Computing with Words (CW) is often applied as a basis to solve the decision problems with linguistic information [14, 15, 16, 28, 29]. In recent years, different linguistic models are proposed for CW. Particularly, the 2-tuple linguistic

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representation model [8] provided a computation technique to deal with linguistic information without loss of information. Based on the 2-tuple linguistic representation model, the model based on a linguistic hierarchy [7] and the numerical scale model [2, 3] are developed to provide good methods to deal with the linguistic decision making problems with single linguistic term.

However, the complexity and time pressure of decision making problems nowadays make decision makers need more elaborated expressions than a simple linguistic label [20]. Hence, to overcome this limitation, Rodríguez et al. [21] introduced the concept of Hesitant Fuzzy Linguistic Term Set (HFLTS) to serve as the basis of increasing the flexibility of the elicitation of linguistic information by means of linguistic expressions.

To generate more elaborate linguistic expressions, Rodríguez et al. [21] provided a method to generate comparative linguistic expressions by using a context-free grammar and HFLTS. To deal with comparative linguistic expressions in Group Decision Making (GDM), a decision model was proposed in [22] to facilitate the elicitation of linguistic information in hesitant situation. Besides, to represent the semantics of comparative linguistic expressions, Liu and Rodríguez [11] proposed a representation way by means of a fuzzy envelope to carry out the CW processes and discussed its application in multicriteria decision making. Some further developments about the hesitant linguistic decision making can be found in [19, 23].

In GDM dealing with CW, there is a fact that words mean different things for different people [5, 15, 16]. For example, when evaluating the quality of a paper, three reviewers think the paper has "good" quality, but this term "good" has different semantics for these three reviewers. That makes the understanding and numerical meanings of "good" for different reviewers are different. The existing studies use the type-2 fuzzy sets [15] and multi-granular linguistic models [7, 17] for managing this issue. Although both methods deal with multiple meanings of words are quite useful, they do not represent yet the specific semantics of each individual. To overcome this problem, Li et al. [10] proposed a personalized individual semantics approach to model and solve linguistic GDM by means of numerical scales [1, 2, 3] and the 2-tuple linguistic model [8] to improve the management of different meanings of words for different people. This approach shows the good features for managing linguistic information in CW processes and can reflect individual personalized differences in understanding the meaning of words.

In hesitant linguistic decision making, although there are many studies (e.g., [2, 11, 15, 26]) to discuss the representations of HFLTSs, few studies consider the personalized individual semantics among decision makers when expressing the preferences using HFLTSs. Therefore, in this paper, we apply the idea of personalize individual semantics to reflect the different understanding of words for different decision makers in hesitant linguistic decision making. A new framework to personalize individual semantics in hesitant linguistic GDM with comparative linguistic expressions is proposed. This proposal consists of a two-step procedure:

- An average consistency-driven model is proposed to set personalized numerical scales for linguistic terms with comparative linguistic expressions. The proposed model is based on measuring the Average Consistency Index (ACI) of Hesitant Fuzzy Linguistic Preference Relations (HFLPRs) and provides a basis for developing the personalized individual semantics of HFLTSs.

- Based on the personalized numerical scales obtained from the average consistency-driven model, a process to personalize individual semantics with comparative linguistic expressions via the fuzzy envelope for HFLTSs represented by fuzzy membership function is proposed.

The proposed personalized individual semantics show the individual difference in understanding the meaning of comparative linguistic expressions. The use of the personalized individual semantics provides a new way to show decision makers' numerical meaning individually, and also provides a potential tool to obtain the optimal solution in hesitant linguistic GDM when dealing with the fact that words mean different things to different people.

The rest of this paper is arranged as follows. In Section 2, we present some basic knowledge. Then, in Section 3 the framework and models to personalize individual semantics with comparative linguistic expressions are proposed. Next, Section 4 provides numerical examples and analysis. Section 5 discusses the advantages and weakness of the proposed model. Section 6 concludes this paper with final remarks.

2. Preliminaries

In this section, we introduce the basic knowledge regarding the 2-tuple linguistic model, numerical scale, comparative linguistic expressions and HFLTSs.

2.1. The 2-tuple linguistic model and numerical scale

The 2-tuple linguistic representation model, presented by Herrera and Martínez [8], represents the linguistic information by a 2-tuple $(s_i, \alpha) \in \overline{S} = S \times [-0.5, 0.5)$, where $s_i \in S$ and $\alpha \in [-0.5, 0.5)$.

Definition 1. *[8] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation. The 2-tuple linguistic value that expresses the equivalent information to β is then obtained as:*

$$\Delta : [0, g] \to \overline{S},$$

being

$$\Delta(\beta) = (s_i, \alpha), \ \ with \begin{cases} s_i, \ \ i = round(\beta) \\ \alpha = \beta - i, \ \ \alpha \in [-0.5, 0.5) \end{cases}$$

Function Δ, it is a one to one mapping whose inverse function $\Delta^{-1} : \bar{S} \to [0, g]$ is defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha$. When $\alpha = 0$ in (s_i, α) is then called simple term.

A computational model for the 2-tuple linguistic model was defined in [8], in which different operations were introduced:

(1) A 2-tuple comparison operator: Let (s_k, α) and (s_l, γ) be two 2-tuples. Then:

(i) if $k < l$, then (s_k, α) is smaller than (s_l, γ).

(ii) if $k = l$, then

(a) if $\alpha = \gamma$, then (s_k, α), (s_l, γ) represents the same information.

(b) if $\alpha < \gamma$, then (s_k, α) is smaller than (s_l, γ).

(2) A 2-tuple negation operator:

$Neg((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha)))$.

(3) Several 2-tuple aggregation operators have been developed (see [8, 14]).

The concept of the numerical scale was defined to transform linguistic terms into real numbers:

Definition 2. *[3] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set, and R be the set of real numbers. The function: $NS : S \to R$ is defined as a numerical scale of S, and $NS(s_i)$ is called the numerical index of s_i. If the function NS is strictly monotone increasing, then NS is called an ordered numerical scale.*

Based on the concept of numerical scale, Dong et al. [2] proposed a connection of the numerical scale model with the 2-tuple linguistic model [8], the proportional 2-tuple linguistic model [25] and the model based on a linguistic hierarchy [6], respectively, by setting different certain values for $NS(s_i)$.

2.2. Comparative linguistic expressions and HFLTSs

To facilitate the elicitation of flexible and rich linguistic expressions, Rodríguez et al. [21] proposed an approach to generate comparative linguistic expressions by using a context-free grammar.

Definition 3. *[21] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and G_H be a context-free grammar. The elements of $G_H = \{V_N, V_T, I, P\}$ are defined as follows,*
$V_N = \{\langle primary\ term\rangle, \langle composite\ term\rangle, \langle unary\ relation\rangle, \langle binary\ relation\rangle, \langle conjunctiond\rangle\}$
$V_T = \{lower\ than, greater\ than,\ between, and,\ s_0, s_1, ..., s_g\}$
$I \in V_N$.

For the context-free grammar G_H, the production rules are as follows:
$P = \{I ::= \langle primary\ term\rangle\ |\langle composite\ term\rangle\ \langle composite\ term\rangle ::= \langle unary\ relation\rangle$
$\langle primary\ term\rangle\ |\ \langle binary\ relation\rangle\ \langle primary\ term\rangle\ \langle conjuction\rangle\ \langle primary\ term\rangle$
$\langle primary\ term\rangle ::= s_0|s_1|...|s_g$
$\langle unary\ relation\rangle ::= lower\ than|greater\ than$
$\langle binary\ relation\rangle ::= between$
$\langle conjuction\rangle ::= and\}$

By using the context-free grammar G_H, the comparative linguistic expressions are generated. Since they cannot be directly used for CW, Rodríguez et al. [21] provided a transformation function to transform them into HFLTSs.

Definition 4. *[21] Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set. A HFLTS, H_S, is an ordered finite subset of consecutive linguistic terms of S.*

Definition 5. *[21] Let H_S be a HFLTS of S. Let $H_S^- = \min_{s_i \in H_S}(s_i)$, $H_S^+ = \max_{s_i \in H_S}(s_i)$ and $env(H_S) = [H_S^-, H_S^+]$. Then, H_S^-, H_S^+ and $env(H_S)$ are called the lower bound, the upper bound and the envelope of H_S.*

Definition 6. *[21] Let S_{ll} be the expressions generated by G_H, and let E_{G_H} be a function that transforms linguistic expressions, $ll \in S_{ll}$, obtained by using G_H, into HFLTS, H_S.*

S is the linguistic term set used by G_H and S_{ll} is the expressions domain generated by G_H:

$E_{G_H} : S_{ll} \to H_S$

The comparative linguistic expressions generated by G_H using the production rules are converted into HFLTS by means of the following transformations:

$E_{G_H}(s_i) = \{s_i\}$;

$E_{G_H}(less\ than\ s_i) = \{s_j\,|s_j \in S \ and\ s_j \leq s_i\}$;

$E_{G_H}(greater\ than\ s_i) = \{s_j\,|s_j \in S \ and\ s_j > s_i\}$;

$E_{G_H}(between\ s_i\ and\ s_j) = \{s_k\,|s_k \in S \ and\ s_i \leq s_k \leq s_j\}$.

Based on the use of HFLTSs, Rodríguez et al. [22] proposed the concept of the HFLPR as Definition 7.

Definition 7. *[22] Let M_S be a set of HFLTSs based on S. A HFLPR based on S is presented by a matrix $H = (H_{ij})_{n\times n}$, where $H_{ij} \in M_S$ and $Neg(H_{ij}) = H_{ji}$.*

3. Personalizing individual semantics with comparative linguistic expressions in hesitant linguistic GDM

As aforementioned, there is a fact that words mean different things for different people. To represent the specifically personalized individual semantics of each decision maker in decision making, this section proposes the process to personalize individual semantics with comparative linguistic expressions in hesitant linguistic GDM.

3.1. Framework

GDM is defined as a decision situation where two or more experts, who have their own knowledge and preferences regarding the decision problem, take part and provide their preferences to reach a collective decision. Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set, $X = \{x_1, x_2, ..., x_n\}$ be a set of alternatives and $E = \{e^1, e^2, ..., e^m\}$ be a set of decision makers. Each decision maker provides his/her preferences with comparative linguistic expressions over X by a preference relation $P^k = (p^k_{ij})_{n\times n}(k = 1, 2, ..., m)$.

In order to carry out CW processes with comparative linguistic expressions p^k_{ij}, it is necessary to transform them into HFLTS H^k_{ij} by means of the transformation function E_{G_H}, i.e.,

$$E_{G_H}(p^k_{ij}) = H^k_{ij} \quad (1)$$

Therefore, in decision making, by using the transformation function E_{G_H}, the preference relation with comparative linguistic expressions P^k can be transformed into the HFLPR H^k [21, 22],

$$H^k = \left(H_{ij}^k\right)_{n\times n} = \left(E_{G_H}(p_{ij}^k)\right)_{n\times n} \tag{2}$$

Following the existing semantics definitions of linguistic terms and HFLTSs, in this paper it is assumed that:

(1) The semantics of linguistic terms $s_i \in S$ are represented by the trapezoidal (triangular) membership functions $A(s_i) = T(a_L^i, a_M^i, a_M^i, a_R^i)$. For simplicity we note $A(s_i) = T(a_L^i, a_M^i, a_R^i)$.

(2) The semantics of HFLTSs H_{ij}^k are defined by fuzzy envelopes using trapezoidal fuzzy membership functions, $env_F(H_{ij}^k) = T(a_{ij}^k, b_{ij}^k, c_{ij}^k, d_{ij}^k)$.

Considering the fact that words mean different things for different people in CW processes, in linguistic GDM, the representation way for the HFLTS should reflect the individual differences to understand the meaning of words. Keeping the previous fact in mind, an approach to express the personalized individual semantics of HFLTS in GDM is presented, which reflects the different meanings of HFLTS for different decision makers. It is implemented by a two-step procedure:

(1) The process to set the personalized numerical scales of linguistic terms over S. To achieve this process, we propose a consistency-driven approach based on the ACI of HFLPR in Section 3.2. The ACI is determined as the average consistency degree of all linguistic preference relations associated to a HFLPR.

(2) The process to represent the personalized individual semantics of HFLTSs. Based on the personalized numerical scales, we propose an approach to represent the personalized individual semantics by means of constructing the fuzzy envelope for HFLTSs in Section 3.3.

The framework to personalize individual semantics with comparative linguistic expressions in hesitant GDM is provided below (see Fig. 1).

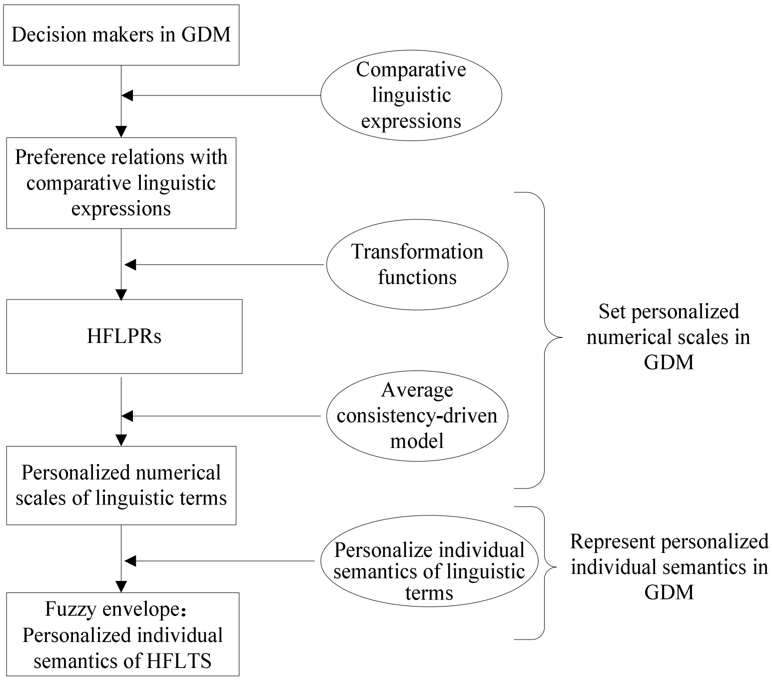

Fig.1 Framework to personalize individual semantics with comparative linguistic expressions

3.2. Setting personalized numerical scales of linguistic terms in GDM

According to Eqs. (1) and (2), the preference relations with comparative linguistic expressions are transformed into HFLPRs using the transformation function E_{G_H} to facilitate the CW processes in linguistic decision making. Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and $H = (H_{ij})_{n\times n}$ be a HFLPR based on S, where $H_{ij} = \{H_{ij}^t | t = 1,..., \#H_{ij}\}$ and $\#H_{ij}$ is the number of linguistic terms in H_{ij}.

Definition 8. *Let $H = (H_{ij})_{n\times n}$ be a HFLPR defined as before. $L = (l_{ij})_{n\times n}$ is a linguistic preference relation associated to H, if $l_{ij} = H_{ij}^t$, $t \in \{1, ..., \#H_{ij}\}$, and $l_{ij} = Neg(l_{ji})$.*

We denote N_H as the set of the linguistic preference relations associated to H.

Example 1. Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ be a linguistic term set. The HFLPR H is given as follows [26],

$$H = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_6, s_7\} & \{s_6, s_7\} \\ \{s_3\} & \{s_4\} & \{s_4, s_5\} & \{s_5, s_6\} \\ \{s_1, s_2\} & \{s_3, s_4\} & \{s_4\} & \{s_5\} \\ \{s_1, s_2\} & \{s_2, s_3\} & \{s_3\} & \{s_4\} \end{pmatrix}$$

For any $l_{ij} \in H_{ij}$ and $l_{ij} + l_{ji} = 1$, we have the linguistic preference relation $L =$

$(l_{ij})_{4\times4} \in N_H$, such as,

$$L = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_6\} & \{s_7\} \\ \{s_3\} & \{s_4\} & \{s_5\} & \{s_5\} \\ \{s_2\} & \{s_3\} & \{s_4\} & \{s_5\} \\ \{s_1\} & \{s_3\} & \{s_3\} & \{s_4\} \end{pmatrix}$$

Let NS be an ordered numerical scale on S, and in this paper we set the range of NS in the interval [0,1]. Additive transitivity is often used to character the consistency of linguistic preference relations [9, 27]. Following the additive transitivity, the consistency index (CI) of a linguistic preference relation L based on the numerical scales NS is defined as,

$$CI(L) = 1 - \frac{2}{3n(n-1)(n-2)} \sum_{i,j,z=1}^{n} |NS(l_{ij}) + NS(l_{jz}) - NS(l_{iz}) - 0.5| \qquad (3)$$

with $NS(l_{ij}) \in [0, 1]$.

To measure the consistency of HFLPRs, we propose the ACI based on Eq. (3) as follows.

Definition 9. *Let H be a HFLPR. The value of $ACI(H)$ is determined by the average consistency degree of all linguistic preference relations associated to the HFLPR, i.e.,*

$$ACI(H) = \frac{1}{\#N_H} \times \sum_{L \in N_H} CI(L) \qquad (4)$$

where $\#N_H$ is the number of linguistic preference relations in H, i.e., $\#N_H = \prod_{i=1}^{n} \prod_{j=i+1}^{n} \#H_{ij}$.

Example 2. Let S and H be as in Example 1, we have $\#N_H = \prod_{i=1}^{4} \prod_{j=i+1}^{4} \#H_{ij} = 16$. Using Eq. (3) to compute the consistency of the linguistic preference relation associated to H, such as, the consistency of the linguistic preference relation L provided in Example 1 is $CI(L) = 0.9583$. Then, by computing the average consistency of all the linguistic preference relations $L \in N_H$, the ACI of H are obtained, $ACI(H) = 0.9375$.

As mentioned before, it is possible to transform linguistic terms into the numerical scales, and both linguistic terms and numerical scales represent the same preference of decision maker. Considering this statement, we provide the following premise.

Premise 1: If HFLPRs are consistent, then the transformed preference relation based on the established numerical scale should be as much consistent as possible.

In the following, we construct an optimization-based model to set personalized numerical scales for linguistic terms with HFLPRs based on the average consistency measure.

Let $S = \{s_0, s_1, ..., s_g\}$ be a linguistic term set and $E = \{e^1, e^2, ..., e^m\}$ be a set of decision makers. Let $H^k = (H^k_{ij})_{n\times n}$ be a HFLPR based on S associated to decision maker e^k, where $H^k_{ij} = \{H^{t,k}_{ij} | t = 1,..., \#H^k_{ij}\}$, and let $L^{h,k} = (l^{h,k}_{ij})_{n\times n}\,(h = 1, 2, ..., \#N_{H^k})$ be the linguistic preference relations associated to H^k, i.e., $L^{h,k} \in N_{H^k}$. Let NS^k be the numerical scale associated with e^k.

Based on Premise 1, in order to guarantee that the HFLPR H^k is as consistent as possible, the objective function to maximize the ACI of H^k is as follows,

$$max \quad ACI(H^k) \tag{5}$$

where $ACI(H^k) = \frac{1}{\#N_{H^k}} \sum\limits_{h=1}^{\#N_{H^k}} (1 - \frac{2\sum\limits_{i,j,z=1}^{n} \left|NS(l^{h,k}_{ij})+NS(l^{h,k}_{jz})-NS(l^{h,k}_{iz})-0.5\right|}{3n(n-1)(n-2)})$ with $l^{h,k}_{ij} \in H^k_{ij}$ and $l^{h,k}_{ij} = Neg(l^{h,k}_{ji})$.

In this paper, without loss of generality, we set the range of numerical scales for linguistic terms $NS^k(s_i)$ as follows,

$$NS^k(s_i)\begin{cases} = 0 & i = 0 \\ = 0.5 & i = \frac{g}{2} \\ \in [(i-1)/g, (i+1)/g] & i = 1, 2, ..., g-1; i \neq \frac{g}{2} \\ = 1 & i = g \end{cases} \tag{6}$$

Besides, NS^k must be ordered. We introduce a constraint value $\lambda \in (0, 1)$ to restrict the distance between $NS^k(s_i)$ and $NS^k(s_{i+1})$, i.e.,

$$NS^k(s_{i+1}) - NS^k(s_i) \geq \lambda \tag{7}$$

Based on Eqs. (5)-(7), the consistency-driven optimization model P to obtain personalized numerical scales for linguistic terms with HFLPR H^k is constructed as follows,

$$\begin{cases} \max \quad ACI(H^k) \\ s.t.\ ACI(H^k) = \frac{1}{\#N_{H^k}} \sum\limits_{h=1}^{\#N_{H^k}} \left(1 - \frac{2\sum\limits_{i,j,z=1}^{n} \left| NS(l_{ij}^{h,k}) + NS(l_{jz}^{h,k}) - NS(l_{iz}^{h,k}) - 0.5 \right|}{3n(n-1)(n-2)}\right) \\ \quad l_{ij}^{h,k} \in H_{ij}^k \qquad i,j = 1,2,...,n \\ \quad l_{ij}^{h,k} = Neg(l_{ji}^{h,k}) \qquad i,j = 1,2,...,n \\ \quad NS^k(s_0) = 0 \\ \quad NS^k(s_{\frac{g}{2}}) = 0.5 \\ \quad NS^k(s_i) \in [(i-1)/g, (i+1)/g] \qquad i = 1,...,g-1; i \neq \frac{g}{2} \\ \quad NS^k(s_g) = 1 \\ \quad NS^k(s_{i+1}) - NS^k(s_i) \geq \lambda \qquad i = 0,1,...,g-1 \end{cases}$$

Solving model P uses the software packages Lingo or Matlab, we obtain the personalized numerical scales for each term in S associated with decision maker d_k, i.e., $NS^k(s_0)$, $NS^k(s_1)$, ..., $NS^k(s_g)$, and the optimal ACI of H^k. The personalized numerical scales will provide a basis to personalize individual semantics of HFLTSs in Section 3.3.

3.3. Personalizing individual semantics with HFLTSs in GDM

Based on the personalized numerical scales, we propose an approach to personalize individual semantics of HFLTSs by computing the fuzzy envelope expressed by trapezoidal fuzzy membership functions in GDM. The personalized individual semantics of HFLTSs reflect the decision makers' different understanding for HFLTSs.

The process to represent the personalized individual semantics of HFLTSs can be implemented by a two-step procedure: (1) Representing the personalized individual semantics of linguistic terms; and (2) Representing the personalized individual semantics of HFLTSs via fuzzy envelope. They are developed as follows:

(1) Representing the personalized individual semantics of linguistic terms

As stated above, in this paper we assume that the semantics of linguistic terms $s_i \in S$ are represented by the fuzzy membership functions $A(s_i) = (a_L^i, a_M^i, a_R^i)$. According to the fuzzy partitions [24], we have $a_R^{i-1} = a_M^i = a_L^{i+1}$, $i = 1,2,...,g-1$. Thus the set of points of all membership functions of the linguistic term set is given as

$$T = \{a_L^0, a_M^0, a_M^1, ..., a_M^g, a_R^g\} \tag{8}$$

From Section 3.2, the personalized numerical scales for the linguistic term set associated with decision maker e^k, $\{NS^k(s_0), NS^k(s_1), ..., NS^k(s_g)\}$, are obtained by the consistency-driven optimization model P. Based on the personalized numerical scale, each linguistic term s_i can be defined by triangular membership function as follows,

$$A^k(s_i) = \begin{cases} T(NS^k(s_0), NS^k(s_0), NS^k(s_1)) & i = 0 \\ T(NS^k(s_{i-1}), NS^k(s_i), NS^k(s_{i+1})) & i = 1, ..., g-1 \\ T(NS^k(s_{g-1}), NS^k(s_g), NS^k(s_g)) & i = g \end{cases} \tag{9}$$

Thus, Eq. (8) can be equivalently transformed into Eq. (10).

$$T = \{NS^k(s_0), NS^k(s_0), NS^k(s_1), ..., NS^k(s_{g-1}), NS^k(s_g), NS^k(s_g)\} \tag{10}$$

In this way, the personalized individual semantics of the linguistic terms s_i and the linguistic term set S, associated with each decision maker, can be represented by Eqs. (9) and (10).

(2) Representing the personalized individual semantics of HFLTSs via fuzzy envelope

Liu and Rodríguez [11] proposed a method to represent the semantics of the HFLTS via fuzzy envelope, using a trapezoidal fuzzy membership function $T(a, b, c, d)$ obtained by aggregating the fuzzy membership functions of the linguistic terms of the HFLTS. This method provides a basis for personalizing individual semantics of HFLTSs.

Based on the computation method proposed in [11], we propose the fuzzy envelopes for HFLTSs provided by decision makers in GDM to personalize individual semantics by means of trapezoidal membership functions $T(a^k, b^k, c^k, d^k)$. From the context-free grammar in Definition 3, the comparative linguistic expressions can be divided into three types: between s_i and s_j ($i \neq 0, j \neq g$), at least s_i and at most s_i. Based on the transformations between comparative linguistic expressions and HFLTSs in Definition 6, we consider the following three cases to compute the fuzzy envelope for HFLTSs:

Case A: Fuzzy envelope for the HFLTS $\{s_i, s_{i+1}, ..., s_j\}$ ($i \neq 0, j \neq g$)

In order to represent the personalized individual semantics of the HFLTS $\{s_i, s_{i+1}, ..., s_j\}$ by using the fuzzy envelope defined by the trapezoidal membership function $T(a^k, b^k, c^k, d^k)$, the min and the max operators to compute a^k and d^k are used, i.e.,

$a^k = \min\{NS^k(s_{i-1}), NS^k(s_i), ..., NS^k(s_j), NS^k(s_{j+1})\} = NS^k(s_{i-1})$

$d^k = \max\{NS^k(s_{i-1}), NS^k(s_i), ..., NS^k(s_j), NS^k(s_{j+1})\} = NS^k(s_{j+1})$

The way to obtain the parameters b^k and c^k is as follows,

(i) If $i+j$ is odd and $i+1=j$, then $b^k = NS^k(s_i)$ and $c^k = NS^k(s_{i+1})$.

(ii) If $i+j$ is odd and $i+1<j$, then

$b^k = OWA_{W^2}\{NS^k(s_i), NS^k(s_{i+1}), ..., NS^k(s_{\frac{i+j-1}{2}})\}$;

$c^k = OWA_{W^1}\{NS^k(s_j), NS^k(s_{j-1}), ..., NS^k(s_{\frac{i+j+1}{2}})\}$.

(iii) If $i+j$ is even, then

$b^k = OWA_{W^2}\{NS^k(s_i), NS^k(s_{i+1}), ..., NS^k(s_{\frac{i+j}{2}})\}$;

$c^k = OWA_{W^1}\{NS^k(s_j), NS^k(s_{j-1}), ..., NS^k(s_{\frac{i+j}{2}})\}$,

where $W^1 = (w_1^1, w_2^1, ..., w_n^1)^T$ with $w_1^1 = \alpha_1$, $w_2^1 = \alpha_1(1-\alpha_1)$, $w_3^1 = \alpha_1(1-\alpha_1)^2$,..., $w_{n-1}^1 = \alpha_1(1-\alpha_1)^{n-2}$, $w_n^1 = (1-\alpha_1)^{n-1}$ and $\alpha_1 = \frac{j-i-1}{g-1}$.
$W^2 = (w_1^2, w_2^2, ..., w_n^2)^T$ with $w_1^2 = {\alpha_2}^{n-1}$, $w_2^2 = (1-\alpha_2){\alpha_2}^{n-2}$, $w_3^2 = (1-\alpha_2){\alpha_2}^{n-3}$,..., $w_{n-1}^2 = (1-\alpha_2)\alpha_2$, $w_n^2 = 1-\alpha_2$ and $\alpha_2 = \frac{g-(j-i)}{g-1}$.

Case B: Fuzzy envelope for HFLTS $\{s_i, s_{i+1}, ..., s_g\}$

The fuzzy envelope for $\{s_i, s_{i+1}, ..., s_g\}$ defined by $T(a^k, b^k, c^k, d^k)$ is computed as follows,

$a^k = \min\{NS^k(s_{i-1}), NS^k(s_i), ..., NS^k(s_g)\} = NS^k(s_{i-1})$;

$b^k = OWA_{W^2}\{NS^k(s_i), NS^k(s_{i+1}) ...., NS^k(s_g)\}$;

$c^k = NS^k(s_g)$;

$d^k = \max\{NS^k(s_{i-1}), NS^k(s_i), ..., NS^k(s_g)\} = NS^k(s_g)$,

where $W^2 = ({\alpha_2}^{g-i}, (1-\alpha_2){\alpha_2}^{g-i-1}, (1-\alpha_2){\alpha_2}^{g-i-2}, ..., (1-\alpha_2)\alpha_2, 1-\alpha_2)^T$ with $\alpha_2 = \frac{i}{g}$.

Case C: Fuzzy envelope for HFLTS $\{s_0, s_1, ..., s_i\}$

The way to compute the fuzzy envelope for $\{s_0, s_1, ..., s_i\}$ defined by $T(a^k, b^k, c^k, d^k)$ is as follows,

$a^k = \min\{NS^k(s_0), NS^k(s_0), NS^k(s_1), ..., NS^k(s_i), NS^k(s_{i+1})\} = NS^k(s_0)$;

$b^k = NS^k(s_0)$;

$c^k = OWA_{W^1}\{NS^k(s_0), NS^k(s_1), NS^k(s_2), ..., NS^k(s_i)\}$;

$d^k = \max\{NS^k(s_0), NS^k(s_0), NS^k(s_1), ..., NS^k(s_i), NS^k(s_{i+1})\} = NS^k(s_{i+1})$,

where $W^1 = (\alpha_1, \alpha_1(1-\alpha_1), \alpha_1(1-\alpha_1)^2, ..., \alpha_1(1-\alpha_1)^{g-i-1}, (1-\alpha_1)^{g-i})^T$ with $\alpha_1 = \frac{i}{g}$.

Fig.2 shows the fuzzy envelopes for HFLTSs $\{s_i, s_{i+1}, ..., s_j\}$ $(i \neq 0; j \neq g)$, $\{s_i, s_{i+1}, ..., s_g\}$ and $\{s_0, s_1, ..., s_i\}$ in Cases A-C, respectively.

In Cases A-C, the OWA weights W^1 and W^2 presented in [4] are used for computing the fuzzy envelope for HFLTS. Besides, more details about the computation method to obtain the values α_1 and α_2 can be found in [11].

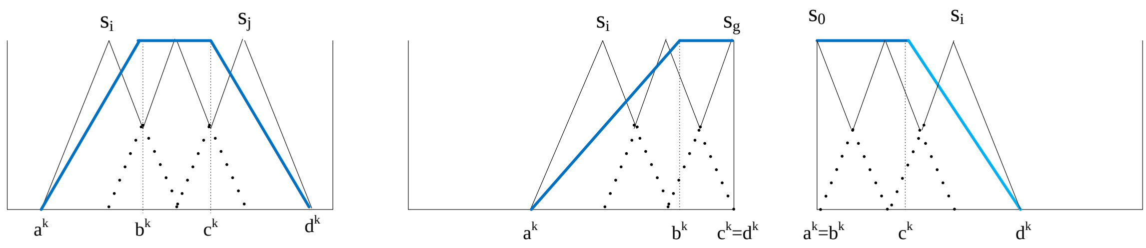

Fig.2 Fuzzy envelopes for the HFLTSs $\{s_i, s_{i+1}, ..., s_j\}$, $\{s_i, s_{i+1}, ..., s_g\}$ and $\{s_0, s_1, ..., s_i\}$

In this way, we generalize the use of fuzzy envelope to represent the personalized individual semantics of the HFLTS. Because semantics play a key role in CW, our proposal can provide a potential tool to help decision makers obtain the optimal solution in hesitant linguistic GDM when dealing with the idea that words mean different things to different people.

4. Numerical examples and analysis

In this section, numerical examples and a comparative study are provided to justify the feasibility of the proposed approach to personalize individual semantics in GDM with comparative linguistic expressions.

4.1. Numerical examples

In this example, there are five alternatives $X = \{x_1, x_2, x_3, x_4, x_5\}$ and four decision makers $E = \{e^1, e^2, e^3, e^4\}$. Each decision maker provides his/her preference relation with comparative linguistic expressions over X using the following linguistic term set,

$$S = \{s_0 = extremely\ poor, s_1 = very\ poor, s_2 = poor, s_3 = slightly\ poor, s_4 = Fair,$$
$$s_5 = slightly\ good\ , s_6 = good, s_7 = very\ good, s_8 = extremely\ good\}$$

The four HFLPRs transformed from preference relations with comparative linguistic expressions are provided as follows,

$$H^1 = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_0\} & \{s_2, s_3, s_4\} & \{s_5, s_6\} \\ \{s_3\} & \{s_4\} & \{s_4, s_5\} & \{s_1, s_2\} & \{s_2, s_3\} \\ \{s_8\} & \{s_3, s_4\} & \{s_4\} & \{s_6, s_7\} & \{s_1\} \\ \{s_4, s_5, s_6\} & \{s_6, s_7\} & \{s_1, s_2\} & \{s_4\} & \{s_0, s_1\} \\ \{s_2, s_3\} & \{s_5, s_6\} & \{s_7\} & \{s_7, s_8\} & \{s_4\} \end{pmatrix}$$

$$H^2 = \begin{pmatrix} \{s_4\} & \{s_3, s_4\} & \{s_5, s_6\} & \{s_1, s_2\} & \{s_0, s_1\} \\ \{s_4, s_5\} & \{s_4\} & \{s_6, s_7, s_8\} & \{s_2, s_3\} & \{s_0, s_1, s_2\} \\ \{s_2, s_3\} & \{s_0, s_1, s_2\} & \{s_4\} & \{s_5, s_6\} & \{s_4, s_5, s_6\} \\ \{s_6, s_7\} & \{s_5, s_6\} & \{s_2, s_3\} & \{s_4\} & \{s_4, s_5, s_6\} \\ \{s_7, s_8\} & \{s_6, s_7, s_8\} & \{s_2, s_3, s_4\} & \{s_2, s_3, s_4\} & \{s_4\} \end{pmatrix}$$

$$H^3 = \begin{pmatrix} \{s_4\} & \{s_1, s_2, s_3\} & \{s_4, s_5\} & \{s_6\} & \{s_7\} \\ \{s_5, s_6, s_7\} & \{s_4\} & \{s_2, s_3\} & \{s_0, s_1\} & \{s_7, s_8\} \\ \{s_3, s_4\} & \{s_5, s_6\} & \{s_4\} & \{s_6, s_7\} & \{s_5\} \\ \{s_2\} & \{s_7, s_8\} & \{s_1, s_2\} & \{s_4\} & \{s_5\} \\ \{s_1\} & \{s_0, s_1\} & \{s_3\} & \{s_3\} & \{s_4\} \end{pmatrix}$$

$$H^4 = \begin{pmatrix} \{s_4\} & \{s_1, s_2\} & \{s_2, s_3\} & \{s_3\} & \{s_3, s_4\} \\ \{s_6, s_7\} & \{s_4\} & \{s_4, s_5\} & \{s_6\} & \{s_7, s_8\} \\ \{s_5, s_6\} & \{s_3, s_4\} & \{s_4\} & \{s_8\} & \{s_0\} \\ \{s_5\} & \{s_2\} & \{s_0\} & \{s_4\} & \{s_1, s_2\} \\ \{s_4, s_5\} & \{s_0, s_1\} & \{s_8\} & \{s_6, s_7\} & \{s_4\} \end{pmatrix}$$

4.1.1. Illustration for setting personalized numerical scales

Based on the data H^1, H^2, H^3 and H^4, we illustrate the use of the consistency-driven optimization model to set the personalized numerical scale for linguistic term set.

Based on Eq. (6), the range of $NS^k(s_i)(k = 1, 2, 3, 4)$ is set as follows,

$$NS^k(s_i) \begin{cases} = 0 & i = 0 \\ = 0.5 & i = 4 \\ \in [(i-1)/8, (i+1)/8] & i = 1, 2, ..., 7; i \neq 4 \\ = 1 & i = 8 \end{cases}$$

Following, we show how to obtain the set of linguistic preference relations, N_{H^k}, associated to H^k. Here, we take the HFLPR H^1 as an example.

The linguistic preference relation set associated to H^1 is $N_{H^1} = \{L^{h,1} | h = 1, 2, ..., \#N_{H^1}\}$, where $\#N_{H^1} = \prod_{i=1}^{5} \prod_{j=2}^{5} \#H^1_{ij} = 192$. For $L^{h,1} = (l^{h,1}_{ij})_{n \times n}$, where $l^{h,1}_{ij} \in H^1_{ij}$ and $l^{h,1}_{ij} = Neg(l^{h,1}_{ji})$, it is obtained by the permutation and combination of the elements in H^1, such as

$$L^{1,1} = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_0\} & \{s_2\} & \{s_5\} \\ \{s_3\} & \{s_4\} & \{s_4\} & \{s_1\} & \{s_2\} \\ \{s_8\} & \{s_4\} & \{s_4\} & \{s_6\} & \{s_1\} \\ \{s_6\} & \{s_7\} & \{s_1\} & \{s_4\} & \{s_1\} \\ \{s_3\} & \{s_6\} & \{s_7\} & \{s_7\} & \{s_4\} \end{pmatrix}, L^{2,1} = \begin{pmatrix} \{s_4\} & \{s_5\} & \{s_0\} & \{s_3\} & \{s_5\} \\ \{s_3\} & \{s_4\} & \{s_5\} & \{s_1\} & \{s_2\} \\ \{s_8\} & \{s_3\} & \{s_4\} & \{s_7\} & \{s_1\} \\ \{s_5\} & \{s_7\} & \{s_1\} & \{s_4\} & \{s_0\} \\ \{s_3\} & \{s_6\} & \{s_7\} & \{s_8\} & \{s_4\} \end{pmatrix} \cdots$$

According to Eq. (7), without loss of generality, we set the constraint value $\lambda = 0.05$. Then, the optimization model to obtain the personalized numerical scale NS^k is as follows.

$$\begin{cases} max \quad ACI(H^k) \\ s.t. \ ACI(H^k) = \frac{1}{\#N_{H^k}} \times \sum_{h=1}^{\#N_{H^k}} (1 - \frac{1}{90} \sum_{i,j,z=1}^{n} \left| NS^k(l_{ij}^{h,k}) + NS^k(l_{jz}^{h,k}) - NS^k(l_{iz}^{h,k}) - 0.5 \right|) \\ \quad l_{ij}^{h,k} \in H_{ij}^k \quad i,j = 1,2,...,5 \\ \quad l_{ij}^{h,k} = Neg(l_{ji}^{h,k}) \quad i,j = 1,2,...,5 \\ \quad NS^k(s_0) = 0 \\ \quad NS^k(s_4) = 0.5 \\ \quad NS^k(s_8) = 1 \\ \quad NS^k(s_i) \in [(i-1)/8, (i+1)/8] \quad i = 1,...,7; i \neq 4 \\ \quad NS^k(s_{i+1}) - NS^k(s_i) \geq 0.05 \quad i = 0,1,...,7 \end{cases} \tag{11}$$

where $NS^k(s_i)$ $(k = 0, 1, ..., 8)$ are decision variables.

We solve the above model to obtain the personalized numerical scales $NS^k(s_i)$ $(k = 1, 2, 3, 4;\ i = 0, 1, ..., 8)$ (see Table 1). Besides, we provide Fig. 3 to show the difference of $NS^k(s_i)$ among different decision makers more clearly.

Table 1. Obtained values of $NS^k(s_i)$

	$NS^k(s_0)$	$NS^k(s_1)$	$NS^k(s_2)$	$NS^k(s_3)$	$NS^k(s_4)$	$NS^k(s_5)$	$NS^k(s_6)$	$NS^k(s_7)$	$NS^k(s_8)$
e_1	0	0.25	0.3	0.35	0.5	0.55	0.625	0.75	1
e_2	0	0.25	0.35	0.4	0.5	0.55	0.625	0.75	1
e_3	0	0.25	0.375	0.45	0.5	0.6	0.65	0.75	1
e_4	0	0.15	0.2	0.45	0.5	0.67	0.8	0.85	1

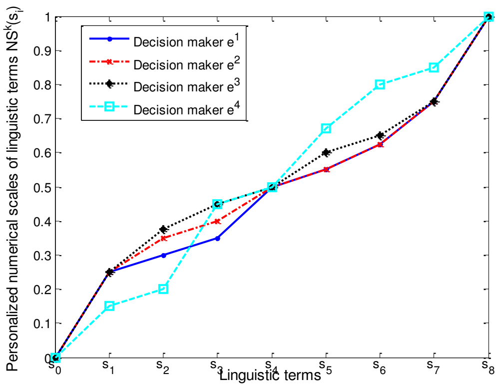

Fig. 3 Personalized numerical scales $NS^k(s_i)$

Moreover, we use four HFLPRs from [26] to compute the personalized individual semantics of linguistic terms by solving model (11). The obtained semantics are shown in Table 2 and Fig. 4.

Table 2. Obtained values of $NS^k(s_i)$ using the HFLPRs in [26]

	$NS^k(s_0)$	$NS^k(s_1)$	$NS^k(s_2)$	$NS^k(s_3)$	$NS^k(s_4)$	$NS^k(s_5)$	$NS^k(s_6)$	$NS^k(s_7)$	$NS^k(s_8)$
e_1	0	0.25	0.366	0.407	0.5	0.593	0.634	0.75	1
e_2	0	0.2375	0.375	0.49	0.5	0.5625	0.625	0.75	1
e_3	0	0.25	0.333	0.417	0.5	0.583	0.667	0.75	1
e_4	0	0.232	0.262	0.323	0.5	0.597	0.738	0.768	1

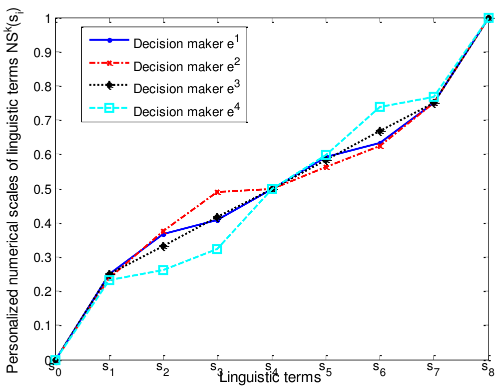

Fig. 4 Personalized numerical scales $NS^k(s_i)$ using the HFLPRs in [26]

According to Tables 1-2 and Figs. 3-4, the personalized numerical scales of linguistic term set associated with each decision maker are different, which provides a basis to personalize individual semantics to reflect the different meanings of linguistic terms for different decision makers.

4.1.2. Illustration for representing the personalized individual semantics

Next, we illustrate the process to represent the personalized individual semantics in GDM with HFLTSs based on the results shown in Table 1.

(1) Personalized individual semantics of linguistic terms

Following Eq. (9), we construct the personalized individual semantics for the linguistic terms $s_i \in S$ associated with decision makers $e^k (k = 1, 2, 3, 4)$ defined by the triangular membership functions as follows,

$$A^k(s_i) = \begin{cases} T(NS^k(s_0), NS^k(s_0), NS^k(s_1)) & i = 0 \\ T(NS^k(s_{i-1}), NS^k(s_i), NS^k(s_{i+1})) & i = 1, ..., 7 \\ T(NS^k(s_7), NS^k(s_8), NS^k(s_8)) & i = 8 \end{cases}$$

(i) Personalized individual semantics for linguistic terms associated with e^1 (see Fig.5)

$A^1(s_0) = T(0, 0, 0.25)$; $A^1(s_1) = T(0, 0.25, 0.3)$; $A^1(s_2) = T(0.25, 0.3, 0.35)$;

$A^1(s_3) = T(0.3, 0.35, 0.5)$; $A^1(s_4) = T(0.35, 0.5, 0.55)$; $A^1(s_5) = T(0.5, 0.55, 0.625)$;

$A^1(s_6) = T(0.55, 0.625, 0.75)$; $A^1(s_7) = T(0.625, 0.75, 1)$ and $A^1(s_8) = T(0.75, 1, 1)$.

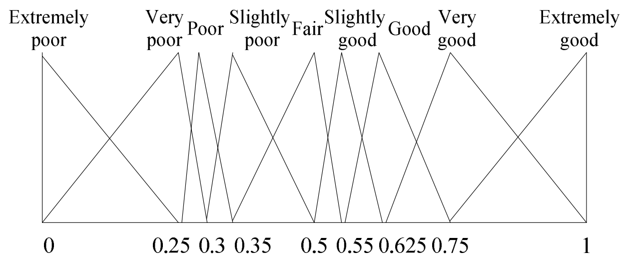

Fig.5 Personalized individual semantics for linguistic terms associated with e^1

(ii) Personalized individual semantics for linguistic terms associated with e^2 (see Fig.6)

$A^2(s_0) = T(0, 0, 0.25)$; $A^2(s_1) = T(0, 0.25, 0.35)$; $A^2(s_2) = T(0.25, 0.35, 0.4)$;

$A^2(s_3) = T(0.35, 0.4, 0.5)$; $A^2(s_4) = T(0.4, 0.5, 0.55)$; $A^2(s_5) = T(0.5, 0.55, 0.625)$;

$A^2(s_6) = T(0.55, 0.625, 0.75)$; $A^2(s_7) = T(0.625, 0.75, 1)$ and $A^2(s_8) = T(0.75, 1, 1)$.

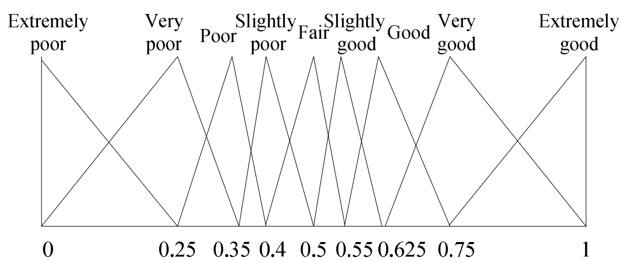

Fig.6 Personalized individual semantics for linguistic terms associated with e^2

(iii) Personalized individual semantics for linguistic terms associated with e^3 (see Fig.7)

$A^3(s_0) = T(0, 0, 0.25)$; $A^3(s_1) = T(0, 0.25, 0.375)$; $A^3(s_2) = T(0.25, 0.375, 0.45)$;

$A^3(s_3) = T(0.375, 0.45, 0.5)$; $A^3(s_4) = T(0.45, 0.5, 0.6)$; $A^3(s_5) = T(0.5, 0.6, 0.65)$;

$A^3(s_6) = T(0.6, 0.65, 0.75)$; $A^3(s_7) = T(0.65, 0.75, 1)$ and $A^3(s_8) = T(0.75, 1, 1)$.

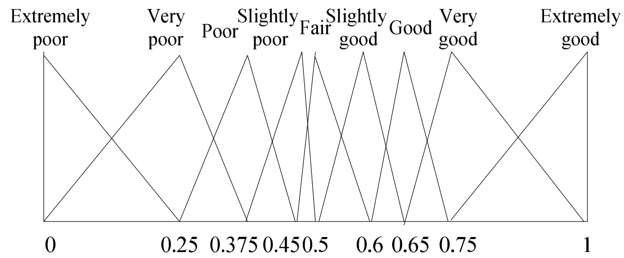

Fig.7 Personalized individual semantics for linguistic terms associated with e^3

(iv) Personalized individual semantics for linguistic terms associated with e^4 (see Fig.8)

$A^4(s_0) = T(0, 0, 0.15)$; $A^4(s_1) = T(0, 0.15, 0.2)$; $A^4(s_2) = T(0.15, 0.2, 0.45)$;

$A^4(s_3) = T(0.2, 0.45, 0.5)$; $A^4(s_4) = T(0.45, 0.5, 0.67)$; $A^4(s_5) = T(0.5, 0.67, 0.8)$;

$A^4(s_6) = T(0.67, 0.8, 0.85)$; $A^4(s_7) = T(0.8, 0.85, 1)$ and $A^4(s_8) = T(0.85, 1, 1)$.

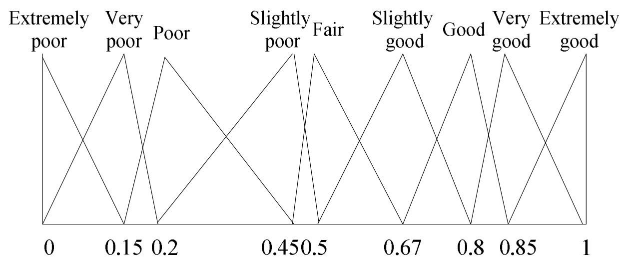

Fig.8 Personalized individual semantics for linguistic terms associated with e^4

(2) The personalized individual semantics for HFLTSs via fuzzy envelope

We take the HFLTSs $H^2_{12} = \{s_3, s_4\}$, $H^2_{23} = \{s_6, s_7, s_8\}$ and $H^2_{25} = \{s_0, s_1, s_2\}$ in the HFLPR H^2 as an example to illustrate the way to compute the personalized individual semantics of HFLTSs via fuzzy envelope defined by trapezoidal fuzzy membership functions.

(i) For the HFLTS $H^2_{12} = \{s_3, s_4\}$, according to Case A in Section 3.3, the set of elements to aggregate is

$T = \{NS^2(s_2), NS^2(s_3), NS^2(s_4), NS^2(s_5)\}$.

The parameters of the fuzzy envelope $env_F(H^2_{12}) = T\{a_{12}, b_{12}, c_{12}, d_{12}\}$ for H^2_{12} is computed as follows,

$a_{12} = \min\{NS^2(s_2), NS^2(s_3), NS^2(s_4), NS^2(s_5)\} = NS^2(s_2) = 0.35$

$d_{12} = \max\{NS^2(s_2), NS^2(s_3), NS^2(s_4), NS^2(s_5)\} = NS^2(s_5) = 0.55$

$b_{12} = NS^2(s_3) = 0.4$

$c_{12} = NS^2(s_4) = 0.5$

Thus, the personalized individual semantics for H_{12}^2 is $T\{0.35, 0.4, 0.5, 0.55\}$.

(ii) For the HFLTS $H_{23}^2 = \{s_6, s_7, s_8\}$, according to Case B in Section 3.3, the set of elements to aggregate is

$T = \{NS^2(s_5), NS^2(s_6), NS^2(s_7), NS^2(s_8), NS^2(s_8)\}$.

The parameters of the fuzzy envelope $env_F(H_{23}^2) = T\{a_{23}, b_{23}, c_{23}, d_{23}\}$ for H_{23}^2 is computed as follows,

$a_{23} = \min\{NS^2(s_5), NS^2(s_6), NS^2(s_7), NS^2(s_8), NS^2(s_8)\} = NS^2(s_5) = 0.55$

$d_{23} = \max\{NS^2(s_5), NS^2(s_6), NS^2(s_7), NS^2(s_8), NS^2(s_8)\} = NS^2(s_8) = 1$

and the parameter $c_{23} = NS^2(s_8) = 1$.

The point b_{23} is computed by the OWA operator with $\alpha_2 = \frac{3}{4}$ and the weighting vector $W^2 = ((\frac{3}{4})^2, (1 - \frac{3}{4}) \cdot \frac{3}{4}, (1 - \frac{3}{4}))^T$. Thus,

$$
\begin{aligned}
b_{23} &= (\tfrac{3}{4})^2 \cdot NS^2(s_8) + (1 - \tfrac{3}{4}) \cdot \tfrac{3}{4} \cdot NS^2(s_7) + (1 - \tfrac{3}{4}) \cdot NS^2(s_6) \\
&= \tfrac{9}{16} \cdot 1 + \tfrac{3}{16} \cdot 0.75 + \tfrac{1}{4} \cdot 0.625 \\
&= 0.859
\end{aligned}
$$

Thus, the personalized individual semantics for H_{23}^2 is $T\{0.55, 0.859, 1, 1\}$.

(iii) For the HFLTS $H_{25}^2 = \{s_0, s_1, s_2\}$, according to Case C in Section 3.3, the set of elements to aggregate is

$T = \{NS^2(s_0), NS^2(s_0), NS^2(s_1), NS^2(s_2), NS^2(s_3)\}$.

The parameters of the fuzzy envelope $env_F(H_{25}^2) = T\{a_{25}, b_{25}, c_{25}, d_{25}\}$ for H_{25}^2 is computed as follows,

$a_{25} = \min\{NS^2(s_0), NS^2(s_0), NS^2(s_1), NS^2(s_2), NS^2(s_3)\} = NS^2(s_0) = 0$

$d_{25} = \max\{NS^2(s_0), NS^2(s_0), NS^2(s_1), NS^2(s_2), NS^2(s_3)\} = NS^2(s_3) = 0.4$

and the parameter $b_{25} = NS^2(s_0) = 0$.

The point c_{25} is computed by the OWA operator with $\alpha_1 = \frac{1}{4}$ and the weighting vector $W^1 = (\frac{1}{4}, \frac{1}{4} \cdot (1 - \frac{1}{4}), (1 - \frac{1}{4})^2)^T$. Thus,

$$
\begin{aligned}
c_{25} &= \tfrac{1}{4} \cdot NS^2(s_2) + \tfrac{1}{4} \cdot (1 - \tfrac{1}{4}) \cdot NS^2(s_1) + (1 - \tfrac{1}{4})^2 \cdot NS^2(s_0) \\
&= \tfrac{1}{4} \cdot 0.35 + \tfrac{1}{4} \cdot (1 - \tfrac{1}{4}) \cdot 0.25 + (1 - \tfrac{1}{4})^2 \cdot 0 \\
&= 0.134
\end{aligned}
$$

Thus, the personalized individual semantics for H_{25}^2 is $T\{0, 0, 0.134, 0.4\}$.

Fig.9 shows the obtained personalized individual semantics for HFLTSs $H^2_{12} = \{s_3, s_4\}$, $H^2_{23} = \{s_6, s_7, s_8\}$ and $H^2_{25} = \{s_0, s_1, s_2\}$.

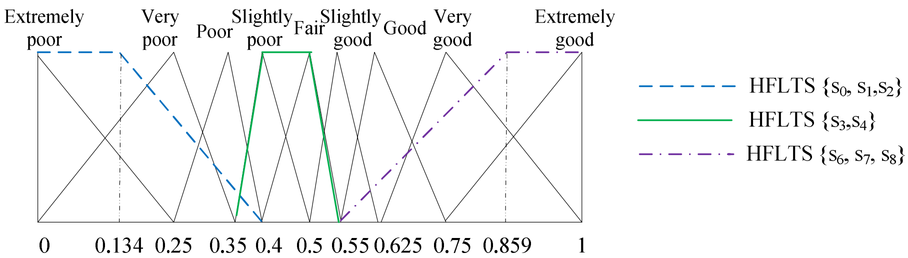

Fig.9 Personalized individual semantics for H^2_{12}, H^2_{23} and H^2_{25}

Based on our proposal to personalize individual semantics of HFLTSs, it is easy to obtain the personalized individual semantics of all HFLTSs in the HFLPRs H^1, H^2, H^3 and H^4. However, for saving space, we do not present them here.

4.2. Comparative study

In [11] and our proposal, the semantics of HFLTS are both expressed as trapezoidal membership functions. Next, we take some HFLTSs from HFLPRs provided in Section 4.1 to show the difference for the representations of HFLTSs in the following methods:

(1) The semantics of HFLTSs proposed in [11];

(2) The personalized individual semantics of HFLTSs in Section 3.3.

Table 3 shows the semantics of HFLTSs for different decision makers using the approach in [11] and our proposed approach.

Table 3. The semantics for several HFLTSs from Section 4.1 using the approach in [11] and our proposed approach

	In [11]	**Our proposal**
$H^1_{25} = \{s_2, s_3\}$	[0.125,0.25,0.375,0.5]	[0.25,0.3,0.35,0.5]
$H^2_{24} = \{s_2, s_3\}$	[0.125,0.25,0.375,0.5]	[0.25,0.35,0.4,0.5]
$H^3_{23} = \{s_2, s_3\}$	[0.125,0.25,0.375,0.5]	[0.25,0.375,0.45,0.5]
$H^4_{13} = \{s_2, s_3\}$	[0.125,0.25,0.375,0.5]	[0.15,0.2,0.45,0.5]
$H^1_{23} = \{s_4, s_5\}$	[0.375,0.5,0.625,0.75]	[0.35,0.5,0.55,0.625]
$H^3_{13} = \{s_4, s_5\}$	[0.375,0.5,0.625,0.75]	[0.45,0.5,0.6,0.65]
$H^4_{23} = \{s_4, s_5\}$	[0.375,0.5,0.625,0.75]	[0.45,0.5,0.67,0.8]
$H^1_{24} = \{s_1, s_2\}$	[0,0.167,0.333,0.5]	[0,0.25,0.3,0.35]
$H^2_{14} = \{s_1, s_2\}$	[0,0.167,0.333,0.5]	[0,0.25,0.35,0.4]
$H^4_{12} = \{s_1, s_2\}$	[0,0.167,0.333,0.5]	[0,0.15,0.2,0.45]

From Table 3, the following observations are highlighted:

- By applying the approach in Liu and Rodríguez [11], the semantics obtained for $\{s_2, s_3\}$, $\{s_4, s_5\}$ and $\{s_1, s_2\}$ of different decision makers are all expressed by the trapezoidal membership functions $[0.125, 0.25, 0.375, 0.5]$, $[0.375, 0.5, 0.625, 0.75]$ and $[0, 0.167, 0.333, 0.5]$, respectively.

- Using our proposed approach, the semantics obtained for $\{s_2, s_3\}$, $\{s_4, s_5\}$ and $\{s_1, s_2\}$ of different decision makers are different, such as, the semantics of $\{s_2, s_3\}$ for decision makers e^1, e^2, e^3 and e^4 are $[0.25, 0.3, 0.35, 0.5]$, $[0.25, 0.35, 0.4, 0.5]$, $[0.25, 0.375, 0.45, 0.5]$ and $[0.15, 0.2, 0.45, 0.5]$, respectively.

The above observations show that the approach in [11] provided the semantics rules, but it reflects the same semantics of HFLTSs for different decision makers. While our proposed approach reflects the different understanding of HFLTSs for different decision makers, it reflects the personalized individual semantics.

5. Discussion: advantages and weakness

In this section, we present some improvements and limitations of the proposed approach to personalize individual semantics in hesitant linguistic GDM.

1) Advantages. We find the following improvements of our proposal:

a) In Mendel and Wu [15], type-2 fuzzy set is used to deal with the multiple meanings of words, but it cannot represent the specific meaning of words. Comparing with the method in [15], our proposal is based on a different assumption to personalize individual semantics via numerical scale and consistency-driven methodology.

b) In recent years, HFLTSs are widely used in linguistic decision making (e.g., [2, 26]) based on the operation rules of 2-tuple linguistic model, but the semantics for HFLTSs are not discussed in most of these studies. In Liu and Rodríguez [11] the fuzzy envelope for HFLTSs has been proposed to describe the semantics of HFLTSs. Comparing with the approach in [11], the proposed approach provides a way to show the individual difference in understanding the meaning of HFLTSs.

c) Li et al. [10] proposed a model to personalize individual semantics of simple terms of a linguistic term set. Our proposal is a continuation of Li et al., and generalizes the work in [10] to personalize individual semantics of HFLTSs.

2) Weakness. We find the following limitations:

a) To our knowledge, there is not any framework to compare different CW methodologies in decision making. However, it is necessary to propose some criteria to compare our proposal with other CW methodologies (e.g., Mendel and Wu [15]).

b) Semantics should play an important role in linguistic GDM problems, but this paper mainly discusses how to personalize individual semantics in a hesitant linguistic and group context, and it is necessary to study how to use personalized individual semantics to improve the quality of hesitant linguistic GDM.

These limitations will be talked in the future research, to design CW comparison methodologies from different criteria, and to discuss the GDM improvements based on the personalized individual semantics.

6. Conclusion

In this paper, we propose the framework to personalize individual semantics in the hesitant linguistic GDM with comparative linguistic expressions to improve the management of different meanings of words for different people. An average consistency–driven approach to personalize numerical scales of the linguistic term set is first provided, then based on the personalized numerical scales, the fuzzy envelope for HFLTSs described by trapezoidal fuzzy membership function is proposed to personalize individual semantics with comparative linguistic expressions.

In the future, we plan to study the use of the personalized individual semantics in large scale GDM problems [12, 13, 18, 30].

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Reference

[1] Y.C. Dong and E. Herrera-Viedma. Consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic

GDM with preference relation. *IEEE Transactions on Cybernetics*, 45(4):780–792, 2015.

[2] Y.C. Dong, C.C. Li, and F. Herrera. Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. *Information Sciences*, 367–368:259–278, 2016.

[3] Y.C. Dong, Y.F. Xu, and S. Yu. Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model. *IEEE Transactions on Fuzzy Systems*, 17(6):1366–1378, 2009.

[4] D. Filev and R.R. Yager. On the issue of obtaining owa operator weights. *Fuzzy Sets and Systems*, 94(2):157–169, 1998.

[5] F. Herrera, E. Herrera-Viedma, and L. Martínez. A fusion approach for managing multi-granularity linguistic term sets in decision making. *Fuzzy Sets and Systems*, 114(1):43–58, 2000.

[6] F. Herrera, E. Herrera-Viedma, and L. Martínez. A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on Fuzzy Systems*, 16(2):354–370, 2008.

[7] F. Herrera and L. Martínez. A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making. *IEEE Transactions on Systems, Man, and Cybernetics, Part B:Cybernetics*, 31(2):227–234, 2001.

[8] F. Herrera and L. Martínez. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 8(6):746–752, 2000.

[9] E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso. Group decision-making model with incomplete fuzzy preference relations based on additive consistency. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 37:176–189, 2007.

[10] C.C. Li, Y.C. Dong, F. Herrera, E. Herrera-Viedma, and L. Martínez. Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching. *Information Fusion*, 33:29–40, 2017.

[11] H.B. Liu and R.M. Rodríguez. A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making. *Information Sciences*, 258:220–238, 2014.

[12] H.B. Liu, Y. Shen, Y. Chen, X. Chen, and Y. Wang. A two-layer weight determination method for complex multi-attribute large-group decision-making experts in a linguistic environment. *Information Fusion*, 23:156–165, 2015.

[13] Y. Liu, Z.P. Fan, and X. Zhang. A method for large group decision-making based on evaluation information provided by participators from multiple groups. *Information Fusion*, 29:132–141, 2016.

[14] L. Martínez and F. Herrera. An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences*, 207(1):1–18, 2012.

[15] J.M. Mendel and D. Wu. Perceptual computing: Aiding people in making subjective judgments. *Wiley and Sons*, 2010.

[16] J.M. Mendel, L.A. Zadeh, E. Trillas, R.R. Yager, J. Lawry, H. Hagras, and S. Guadarrama. What computing with words means to me. *IEEE Computational Intelligence Magazine*, 5(1):20–26, 2010.

[17] J.A. Morente-Molinera, I.J. Pérez, M.R. Ureña, and E. Herrera-Viedma. On multigranular fuzzy linguistic modeling in group decision making problems: A systematic review and future trends. *Knowledge-Based Systems*, 74(1):49–60, 2015.

[18] I. Palomares, L. Martínez, and F. Herrera. A consensus model to detect and manage noncooperative behaviors in large-scale group decision making. *IEEE Transactions on Fuzzy Systems*, 22(3):516–530, 2014.

[19] R.M. Rodríguez, B. Bedregal, H. Bustince, Y.C. Dong, B. Farhadinia, C. Kahraman, L. Martínez, V. Torra, Y.J. Xu, Z.S. Xu, and F. Herrera. A position and perspective analysis of hesitant fuzzy sets on information fusion in decision making. Towards high quality progress. *Information Fusion*, 29:89–97, 2016.

[20] R.M. Rodríguez, A. Labella, and L. Martínez. An overview on fuzzy modelling of complex linguistic preferences in decision making. *International Journal of Computational Intelligence Systems*, 9(sup1):81–94, 2016.

[21] R.M. Rodríguez, L. Martínez, and F. Herrera. Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions on Fuzzy Systems*, 20(1):109–119, 2012.

[22] R.M. Rodríguez, L. Martínez, and F. Herrera. A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. *Information Sciences*, 241:28–42, 2013.

[23] R.M. Rodríguez, L. Martínez, V. Torra, Z. Xu, and F. Herrera. Hesitant fuzzy sets: State of the art and future directions. *International Journal of Intelligent Systems*, 29(6):495–524, 2014.

[24] E.H. Ruspini. A new approach to clustering. *Information and Control*, 15(1):22–32, 1969.

[25] J.H. Wang and J.Y. Hao. A new version of 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on Fuzzy Systems*, 14(3):435–445, 2006.

[26] Z.B. Wu and J.P. Xu. Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. *Omega*, 65:28–40, 2016.

[27] Y.J. Xu, X. Liu, and H.M. Wang. The additive consistency measure of fuzzy reciprocal preference relations. *International Journal of Machine Learning and Cybernetics*, http://dx.doi.org/10.1007/s13042-017-0637-0.

[28] R.R. Yager. On the retranslation process in Zadeh's paradigm of computing with words. *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics*, 34(2):1184–1195, 2004.

[29] L.A. Zadeh. Fuzzy logic = computing with words. *IEEE Transactions on Fuzzy Systems*, 4(2):103–111, 1996.

[30] H.J. Zhang, Y.C. Dong, and E. Herrera-Viedma. Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions. *IEEE Transactions on Fuzzy Systems*, http://dx.doi.org/10.1109/TFUZZ.2017.2697403.

4 A consensus model for large-scale linguistic group decision making with a feedback recommendation based on clustered personalized individual semantics and opposing consensus groups

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A Consensus Model for Large-scale Linguistic Group Decision Making with a Feedback Recommendation based on Clustered Personalized Individual Semantics and Opposing Consensus Groups

Cong-Cong Li, Yucheng Dong, Francisco Herrera

Abstract: In linguistic large-scale group decision making (LSGDM), it is often necessary to achieve a consensus. Particularly, when computing with words and linguistic decision we must keep in mind that words mean different things to different people. Therefore, to represent the specific semantics of each individual, we need to consider the personalized individual semantics (PIS) model in linguistic LSGDM. In this paper, we propose a consensus model based on PIS for LSGDM. Specifically, a PIS process to obtain the individual semantics of linguistic terms with linguistic preference relations is introduced. Following, a consensus process based on PIS, including the consensus measure and feedback recommendation phases, is proposed to improve the willingness of decision makers who follow the suggestions to revise their preferences in order to achieve a consensus in linguistic LSGDM problems. The consensus measure defines two opposing consensus groups with respective acceptable and unacceptable consensus. In the feedback recommendation phase, a PIS based clustering method to get decision makers with similar individual semantics is proposed, and the recommendation rules design a feedback for decision makers with unacceptable consensus, finding suitable moderators from the decision makers with acceptable consensus based on cluster proximity.

Keywords: Large-scale group decision making, consensus, personalized individual semantics, preference relation

I. INTRODUCTION

Group decision making (GDM) consists in deriving a common solution from a group of decision makers over some set of alternatives. Generally, the consensus-based decisions are necessary and required in GDM problems. Thus, the studies of consensus processes [17, 18, 34, 38] are widely analyzed in some research so as to guide decision makers to reach a consensus before making a decision, so that the obtained solution is acceptable for the group.

Generally, the consensus process includes two parts [30]: (i) A consensus measure computes the level of agreement among decision makers and, (ii) A feedback recommendation phase improves the level of agreement among the decision makers. Usually, the consensus process is guided by a moderator [14, 20, 21], who is in charge of supervising and guiding decision makers to change their preference in the process.

In most consensus models, the decision making focuses on a small number of decision makers. However, with increasing social demand, in some situations (e.g., social networks and e-democracy), decisions need to be made by a large number of decision makers, referred to as the large-scale group decision making (LSGDM) [31].

The LSGDM problem is more complex than the usual GDM problems, because of the relatively large group size and the complexity of the decision making problems as well as the decision makers themselves, which have different knowledge and backgrounds. The existing studies regarding LSGDM can be divided into three categories, i.e., consensus processes in LSGDM [8, 31, 32, 33, 39, 42, 43, 45]; clustering approaches in LSGDM [27, 49]; the decision making method in LSGDM with different types of preferences [24, 26, 39, 44, 47]. These studies have greatly contributed to the research and development of LSGDM, but they are mostly proposed in a fuzzy context. In a linguistic decision making context, achieving a consensus result is also an important issue. Considering the uncertainty in real decision making problems, there are still some challenges regarding the consensus in LSGDM:

- In linguistic decision making, an important point to note about computing with words (CW) is the fact that words mean different things to different people [28, 29]. For example, when reviewing an article, three referees all think this article is "Good", but the term "Good" often has different numerical meanings for these three referees. In existing studies, the individual difference in understanding the meaning of words is usually ignored.
- In the consensus process in decision making, a noticeable drawback usually found in large groups is that decision makers do not want to modify their preferences according to the moderator's suggestions in the process of achieving a better consensus. For example, in many decision making issues regarding consensus, some recommendations for

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decision makers can be provided to update their preferences, but we often find that the consensus cannot be reached during the process, because of the unwillingness of decision makers to change their preferences, such as when selecting the best service companies or choosing the best candidate for a position in a university. The existing LSGDM studies regarding consensus do provide some good research in guiding decision makers to reach a higher consensus with a moderator, but they do not really consider the willingness of decision makers to follow the moderator's suggestions.

To represent the specific semantics of each individual, Li et al. [22] proposed a personalized individual semantics (PIS) model with linguistic preference relations by means of interval numerical scales and a 2-tuple linguistic model [13]. The PIS process is applied to obtain the personalized numerical scale of linguistic terms based on the consistency-driven methodology. In [23], a PIS model is proposed to set personalized numerical scales for hesitant fuzzy linguistic information.

In this paper, we propose a consensus model based on PIS in LSGDM, which includes two processes: the PIS process and consensus process. In psychology individuals relying on the opinions of their close friends or people with similar interests are highlighted [19, 25, 40], so in this paper we assume that decision makers having similar semantics and preferences find it easier to communicate with each other. Based on this assumption, we propose a consensus process with PIS to help decision makers become more willing to change their preferences. Incorporating the PIS process in the consensus model provides a good tool for analyzing the PISs of decision makers, with the aim of improving the consensus. The consensus process consists of a consensus measure phase and a feedback recommendation phase: the consensus measure computes the consensus level associated with each decision maker, based on which we classify decision makers into two opposing consensus groups: one group contains the decision makers with acceptable consensus and the other the decision makers with unacceptable consensus. The feedback recommendation includes a PIS based clustering method to get decision makers with similar individual semantics and recommendation rules to design a feedback for decision makers with unacceptable consensus based on the semantics similarity and distance among opposing decision makers.

Finally, we provide the numerical examples and simulation experiments to show the use of the proposed consensus model and to justify the validity of our proposal.

It should be noted that the existing clustering methods in LSGDM [31, 39, 42] classify decision makers with similar opinions with the aim of handling each cluster as a whole to decrease the management complexity, so that in the consensus process the consensus can be measured and improved by adjusting the clusters. But, in our proposal we propose that decision makers having similar semantics and preferences are more likely to communicate with each other. Then, the PIS based clustering method and the opposing consensus groups are employed to improve the willingness of decision makers, who follow their suggestions to revise their preferences in LSGDM.

The rest of this paper is arranged as follows. In Section II, we present some related preliminaries for the proposed model. In Section III a consensus model in LSGDM with PIS, which includes a consensus measure phase and a feedback recommendation phase, is proposed. Section IV shows an example and some simulations to illustrate the proposed consensus model. Section V concludes this paper with some final remarks.

II. Preliminaries

In this section, we introduce some basic knowledge regarding the 2-tuple linguistic models, linguistic preference relations and the PIS model.

A. 2-tuple linguistic model, numerical scale model and linguistic preference relations

The 2-tuple linguistic representation model [13] represents the linguistic information by a 2-tuple $(s_i,\alpha)\in\overline{S}=S\times[-0.5,0.5)$, where $s_i\in S$ and $\alpha\in[-0.5,0.5)$. Formally, let $S=\{s_0,s_1,...,s_g\}$ be a linguistic term set and $\beta\in[0,g]$ be a value representing the result of a symbolic aggregation operation. The 2-tuple that expresses the equivalent information to β is then obtained as:

$$\Delta:[0,g]\to\overline{S}, \tag{1}$$

being

$$\Delta(\beta)=(s_i,\alpha),\ with\begin{cases} s_i, i=round(\beta)\\ \alpha=\beta-i,\ \alpha\in[-0.5,0.5)\end{cases} \tag{2}$$

Function Δ, is a one to one mapping whose inverse function $\Delta^{-1}:\overline{S}\to[0,g]$ is defined as $\Delta^{-1}(s_i,\alpha)=i+\alpha.$ When $\alpha=0$ in (s_i,α) it is then called a simple term.

Dong et al. [6] proposed an extension of the 2-tuple based models [13, 15, 41] with the concept of numerical scale.

Definition 1 [6]. Let $S=\{s_0,s_1,...,s_g\}$ be a linguistic term set, and R be the set of real numbers. The function: $NS:S\to R$ is defined as a numerical scale of S, and $NS(s_i)$ is called the numerical index of s_i. If the function NS is strictly monotone increasing, then NS is called an ordered numerical scale.

Let S be defined as before. The numerical scale NS for (s_i,α), is defined by

$$NS(s_i,\alpha)=\begin{cases} NS(s_i)+\alpha\times(NS(s_{i+1})-NS(s_i)) & \alpha\ge 0\\ NS(s_i)+\alpha\times(NS(s_i)-NS(s_{i-1})) & \alpha<0\end{cases} \tag{3}$$

If $NS(s_i)<NS(s_{i+1})$, for $i=0,1,...,g-1$, the numerical scale NS on S is ordered. In particular, the numerical scale model provides a connection framework [9] among the 2-tuple linguistic model [13], the proportional 2-tuple linguistic model [41] and the unbalanced linguistic model based on a linguistic hierarchy [15].

Let $X=\{x_1,x_2,...,x_n\}$ be a finite set of alternatives. When a decision maker makes pairwise comparisons using the linguistic term set S, they can construct a linguistic preference relation $L=(l_{ij})_{n\times n}\subseteq X\times X$, with a membership function $u_L:X\times X\to S$, where $u_L(x_i,x_j)=l_{ij}$ denotes the linguistic preference degree of the alternative x_i over x_j.

The additive transitivity is often used to characterize the consistency of preference relations [1, 16, 46]. The consistency index (CI) of linguistic preference relations under numerical scale is defined as follows,

Definition 2 [6, 23]. Let $L=(l_{ij})_{n\times n}$ be a linguistic preference relation and NS be the numerical scale on S. Then, the CI of L under NS is defined as follows,

$$CI(L)=1-\frac{4}{n(n-1)(n-2)}\sum_{i,j,k=1;i<j<k}^{n}\left|NS(l_{ij})+NS(l_{jk})-NS(l_{ik})-0.5\right| \tag{4}$$

with $NS(l_{ij})\in[0,1]$.

The larger the value of $CI(L)$ the more consistent L is. If $CI(L)=1$, then L is a consistent linguistic preference relation.

B. Personalized individual semantics with linguistic preference relations

In linguistic decision making, the design of CW methodologies to enrich the linguistic vocabulary to close to the expression of human beings is important, and the successful use of linguistic labels is highly dependent on the determination of a valid membership function or the monotonic mapping encoding the linguistic values [5]. This is a crucial question that always appears in CW [6, 9, 12, 22, 35]. In recent years we have found different proposals regarding the choice of monotonic mapping encoding linguistic values, Yager [35] proposed using a generic ordering such as $G=\{g_0,g_1,...,g_n\}$ such that the only relationship on this scale is ordering $g_{i+1}>g_i$ for the operations in linguistic decision making; García-Lapresta and Pérez-Román [12] proposed ordered qualitative scales using proximity measures between consecutive labels and metrizable distances to make the comparisons of linguistic terms; Gou et al. [11] proposed a double linguistic hierarchy with the monotonic mapping to deal with the enriched vocabulary.

In GDM, the statement that words mean different things for different people [28, 29], has been highlighted because of its influence on the final decision. To represent the specific semantics of each individual via the monotonic mapping encoding the linguistic values, in [22] Li et al. proposed a PIS model to personalize individual semantics by means of an interval numerical scale and the 2-tuple linguistic model. Furthermore, in [23] a PIS model for hesitant fuzzy linguistic information via numerical scales is introduced.

In this paper, based on the approaches in [22] and [23], we introduce a consistency–driven optimization model to obtain the PISs of linguistic terms with linguistic preference relation.

Let $S=\{s_0,s_1,...,s_g\}$ be a set of linguistic terms and let $E=\{e_1,e_2,...,e_m\}$ be a set of decision makers. Let NS^k be an ordered numerical scale on S, associated with decision maker e_k, and let $L^k=(l^k_{ij})_{n\times n}$ be the linguistic preference relation based on S provided by e_k.

To guarantee the linguistic preference relation L^k is as consistent as possible, the objective function is to maximize the consistency index of L^k, i.e.,

$$\max CI(L^k) \tag{5}$$

where

$$CI(L^k)=1-\frac{4\sum_{i,j,z=1;i<j<z}^{n}\left|NS^k(l^k_{ij})+NS^k(l^k_{jz})-NS^k(l^k_{iz})-0.5\right|}{n(n-1)(n-2)}.$$

Without loss of generality, we set the range of numerical scales for linguistic terms as follows,

$$NS^k(s_i)\begin{cases}=0 & i=0\\ \in[(i-1)/g,(i+1)/g] & i=1,2,...,g-1;i\neq g/2\\ =0.5 & i=g/2\\ =1 & i=1\end{cases} \tag{6}$$

Besides, let λ be a small constraint value to guarantee that NS is ordered, i.e.,

$$NS(s_{i+1})-NS(s_i)\geq\lambda \tag{7}$$

In this paper, we set λ=0.01.

Thus, the consistency-driven optimization-based model to obtain the personalized numerical scales is introduced as follows,

$$\begin{cases}\max CI(L^k)=1-\dfrac{4\sum_{i,j,z=1;i<j<z}^{n}\left|NS^k(l^k_{ij})+NS^k(l^k_{jz})-NS^k(l^k_{iz})-0.5\right|}{n(n-1)(n-2)}\\ s.t.\begin{cases}NS^k(s_0)=0\\ NS^k(s_i)\in[(i-1)/g,(i+1)/g] & i=0,1,...,g-1;i\neq g/2\\ NS^k(s_{g/2})=0.5\\ NS^k(s_g)=1\\ NS^k(s_{i+1})-NS^k(s_i)\geq\lambda & i=0,1,...,g-1\end{cases}\end{cases} \tag{8}$$

Model (8) is a linear programming model, and it can be solved by using the some software tools, such as Lingo. By solving this model, we obtain the PISs for linguistic terms in linguistic term set S, i.e., $NS^k(s_0)$, $NS^k(s_1),...,NS^k(s_g)$. For different decision makers, the obtained PISs may be different, showing the different understanding of decision makers.

III. Consensus Model in LSGDM with PIS

In this section, we introduce two opposing consensus groups based on the consensus measure and a PIS based clustering method. Then we develop a novel recommendation rule to improve the willingness of decision makers who follow the suggestions to revise their preferences in order to achieve a consensus in linguistic LSGDM problems.

A. Framework

The problem described in this paper is how to reach a consensus with a large amount of decision makers who have PISs for the established linguistic term set. Here we present a consensus framework for LSGDM problems with PIS. The proposed framework includes two processes (see Fig. 1): PIS process and consensus process. The consensus process is developed in two phases: consensus measure phase for setting opposing consensus groups and feedback recommendation phase.

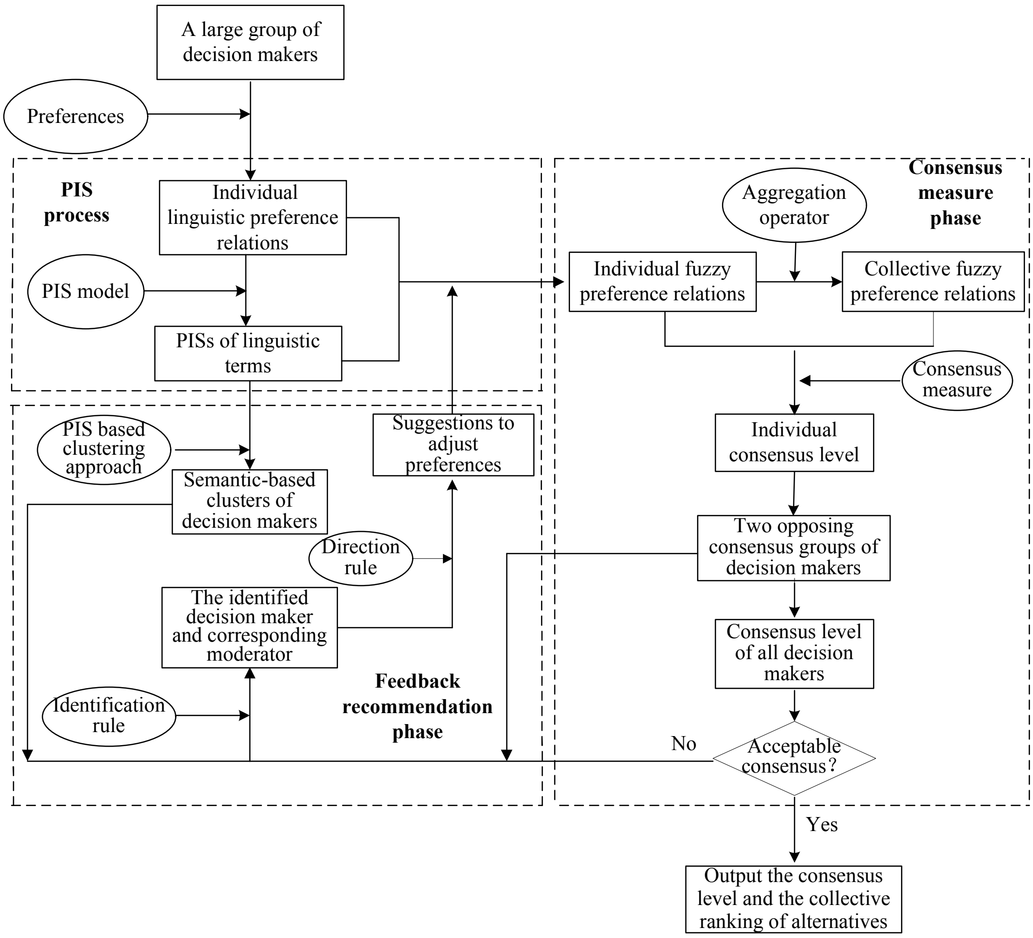


Fig.1. The large-scale consensus reaching framework

First, by applying the PIS method proposed in Section II.B, the PISs of linguistic terms are obtained, so that the individual linguistic preference relations are transformed into individual fuzzy preference relations. Then, a consensus process with PIS in LSGDM is provided to help decision makers reach a consensus. The consensus process consists of two phases: consensus measure and feedback recommendation.

(1) Consensus measure

Collective fuzzy preference relations are obtained by aggregating individual fuzzy preference relations with aggregation operators. Then, we can measure the consensus level associated with each decision maker by measuring the difference between the individual preferences and collective preference. If the obtained consensus level is higher than the established consensus threshold, then the decision maker is of acceptable consensus. Otherwise, the decision maker is of unacceptable consensus.

Next, the decision makers are divided into two opposing groups based on their consensus levels,

(a) Consensus group A, denoted as G_A : decision makers with acceptable consensus;

(b) Consensus group U, denoted as G_U : decision makers with unacceptable consensus.

The proposed consensus measure of all decision makers is based on these two opposing consensus groups, the more decision makers in G_A , the higher the consensus is. The consensus measure is formally provided in Section III.B.

(2) Feedback recommendation

To achieve a higher consensus, the decision makers in G_A should be as many as possible, and the decision makers in G_U should be as few as possible. Therefore, we propose a feedback recommendation based on PIS to guide the decision makers with unacceptable consensus in G_U to change their preferences. A PIS based clustering method is first proposed to obtain semantic-based clusters by grouping the decision makers with similar semantics. Then the recommendation rule, based on the two opposing consensus groups and the semantic-based clusters, provides a novel way to help decision makers with unacceptable consensus to change their preferences. In Section III.C we introduce the feedback recommendation phase.

B. *Consensus measure phase*

The consensus measure with PIS is provided to measure the consensus degree of each decision maker and to obtain two opposing consensus groups with respective acceptable and unacceptable consensus. If the obtained consensus level is larger than the established consensus threshold, then an acceptable consensus is reached. Otherwise, the feedback recommendation phase proposed in Section III.C is applied to provide suggestions to improve the consensus level among the decision makers.

Based on the PIS process (see Section II.B), the personalized numerical scales of linguistic terms NS^k are obtained. Using the individual numerical scale NS^k to quantify the individual linguistic preference relation $L^k = \left(l_{ij}^k\right)_{n\times n}$ $(k = 1,2,...,m)$ obtains the individual fuzzy preference relation $F^k = \left(f_{ij}^k\right)_{n\times n}$ $(k = 1,2,...,m)$, i.e.,

$$f_{ij}^k = NS^k\left(l_{ij}^k\right) \qquad i = 1,2,...,n; j = i+1,...,n \tag{9}$$

By aggregating the individual fuzzy preference relations, the collective fuzzy preference relation is obtained. Let $F^k = (f_{ij}^k)_{n\times n}$ $(k = 1,2,...,m)$ be the individual fuzzy preference relations and $F^c = (f_{ij}^c)_{n\times n}$ be the collective fuzzy preference relation.

Without lack of generality, the weighted average operator is used as the aggregation operator, i.e.,

$$f_{ij}^c = \sum_{k=1}^{m} w_k \cdot f_{ij}^k \tag{10}$$

where $W = (w_1, w_2, ..., w_m)$ is the weighting vector of decision makers $\{e_1, e_2, ..., e_m\}$ and $\sum_{k=1}^{m} w_k = 1$. It is noted that other aggregation operators, such as ordered weighted average operator, can also be applied to aggregate the individual preferences, which will not change the essence of the model.

In our proposal, the consensus associated with each decision maker is based on measuring the difference between the individual preference and the collective preference (see Definition 3).

Definition 3 [4]. The consensus level associated with decision maker e_k is defined as follows,

$$CL_k = 1 - \sum_{i,j=1;i\neq j}^{n} \frac{\left|f_{ij}^k - f_{ij}^c\right|}{n(n-1)} \tag{11}$$

Let ε be a parameter to justify whether the consensus associated with decision maker e_k is acceptable or not. If $CL_k \geq \varepsilon$, then decision maker e_k is of acceptable consensus. Otherwise, decision maker e_k is of unacceptable consensus. In this way, we provide two opposing groups of decision makers based on their consensus levels (see Table 1).

Table 1. Two opposing consensus groups

Consensus group G_A (decision makers with acceptable consensus)	Consensus group G_U (decision makers with unacceptable consensus)
$\{e_k \mid CL_k \geq \varepsilon; k \in \{1,2,...,m\}\}$	$\{e_k \mid CL_k < \varepsilon; k \in \{1,2,...,m\}\}$

Based on the two opposing consensus groups, we define the consensus of all decision makers as follows,

Definition 4. The consensus level CL of all decision makers $\{e_1, e_2, ..., e_m\}$ is computed as follows,

$$CL = \frac{\#G_A}{m} = \frac{\#(\{e_k \mid CL_k \geq \varepsilon\})}{m} \tag{12}$$

with $CL \in [0,1]$, where $\#G_A$ means the number of decision makers in G_A. If $CL=1$, then full consensus is achieved. Otherwise, the lower the value of CL, the lower the level of consensus.

If the consensus level obtained from Eq. (12) is acceptable, then based on Eq. (10) we compute the ranking of alternatives. Let $Z^c = (z_1^c, z_2^c, ..., z_n^c)^T$ be the collective preference vector obtained from F^c to rank alternatives, where

$$z_i^c = \sum_{j=1}^{n} \lambda_j \cdot f_{ij}^c \tag{13}$$

and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is the associated weighting vector that satisfies $\lambda_j \in [0,1]$ and $\sum_{j=1}^{n} \lambda_j = 1$. The larger the value of z_i^c, the better the ranking of alternative is.

If the consensus is not reached, then the two opposing consensus groups are used in the feedback recommendation to provide direction for decision makers to make adjustments in order to improve consensus.

C. *Feedback recommendation phase*

The aim of the proposed feedback recommendation phase is to guide the decision makers that have unacceptable consensus in G_U to be more willing to modify their preferences in achieving a consensus.

Because decision makers have similar interests it is easier for them to communicate with each other based on psychology, in the proposed feedback recommendation phase we consider two factors associated with the decision makers:

- The semantics of the linguistic terms for decision makers;
- The preferences provided by decision makers.

Therefore, to improve the willingness of decision makers in changing their preferences, we highlight two actions:

(1) We will find the decision makers with similar semantics;

(2) Among these decision makers with similar semantics, the decision makers with acceptable consensus will guide the decision makers with unacceptable consensus to modify their preferences under the recommendation rule.

In the following, we illustrate the PIS based clustering method to get the clusters of decision makers with similar semantics and then we provide the recommendation rule with PIS to guide the decision makers with unacceptable consensus to change their preferences.

1) PIS based clustering method

Fuzzy clustering methods are objective function-based methods which seek to find cluster centers for a predefined number N of fuzzy clusters and assign data objects a fuzzy membership degree to each cluster. One of the most widely used fuzzy clustering algorithms is the fuzzy c-means (FCM) algorithm [2], it attempts to partition a finite collection of data objects into a collection of N fuzzy clusters with respect to some given criterion. The FCM algorithm is one of the most useful general purpose fuzzy clustering routines [3], and has an extensive range of applications. Other fuzzy clustering methods, such as k-means clustering method, can also be used in the paper to obtain the semantics-based clusters.

To find the decision makers with similar semantics, we propose a PIS based clustering method based on the FCM algorithm. The proposed clustering method is applied to find cluster centers based on the set of the personalized numerical scales of linguistic terms associated with each decision maker, and assign them a fuzzy membership degree to each cluster, so that the decision makers having similar semantics are classified in a same cluster. The PIS based clustering algorithm is provided as follows:

Algorithm 1. PIS based clustering algorithm

Input: The personalized numerical scales of linguistic terms associated with each decision maker, $NS^k(s_i)$ ($k=1,2,...,m$; $i=0,1,...,g$). The number of cluster centers N, and degree of fuzziness b.

Output: Semantic-based clusters $G^1, G^2, ..., G^N$.

Step 1: Let $NS^k=\{NS^k(s_0), NS^k(s_1), ..., NS^k(s_g)\}$ and $t=1$.

Step 2: Initialize N cluster centers $\{C^1, C^2, ..., C^N\}$ by means of a cluster initialization technique, where $C^h=\{c_0^h, c_1^h, ..., c_g^h\}$, $h\in\{1,...,N\}$.

Step 3: For each set of semantics of linguistic terms NS^k, compute its membership degree to each cluster center $C^{h,t}$, $u_{C^{h,t}}(NS^k)\in[0,1]$, as follows,

$$u_{C^{h,t}}(NS^k)=\frac{(1/d(NS^k,C^{h,t}))^{1/(b-1)}}{\sum_{h=1}^{N}(1/d(NS^k,C^{h,t}))^{1/(b-1)}},$$

where the distance between the semantics of linguistic terms is $d(NS^k,C^{h,t})=\sum_{j=1}^{g}\left|NS^k(s_j)-C_j^{h,t}\right|$.

Step 4: Update cluster centers $C^{h,t}$ as follows,

$$C^{h,t}=\frac{\sum_{k=1}^{m}u_{C^{h,t}}(NS^k)NS^k}{\sum_{k=1}^{m}u_{C^{h,t}}(NS^k)}$$

Step 5: Setting the threshold value $\gamma>0$, compute the variation in membership degree between the iterations t and $t-1$, i.e., $z^t=\frac{\sum_{h=1}^{N}\sum_{k=1}^{m}\left|u_{C^{h,t}}(NS^k)-u_{C^{h,t-1}}(NS^k)\right|}{m\cdot N}$. If $z^t\le\gamma$, go to Step 6, otherwise, let $t=t+1$, go back to Step 3.

Step 6: Let $u_{C^h}(NS^k)=u_{C^{h,t}}(NS^k)$ for $h=1,...,N$, then decision maker e_k should belong to the semantic-based cluster G^j if $u_{C^j}(NS^k)=\max_{h=1,...,N}u_{C^h}(NS^k)$. Output the semantic-based clusters $G^1, G^2, ..., G^N$.

The semantics-based clusters show the similarity of semantics among decision makers and they will be used in the recommendation rule to provide direction in achieving consensus for the feedback for two opposing consensus groups.

In the LSGDM problems, for the setting of the number of semantic-based clusters N, the decision makers should determine this value based on the decision context. In the numerical examples in Section IV, we set N=3 to partition 20 decision makers.

2) *Recommendation rule with PIS*

In the following, based on the semantics similarity and distance among opposing consensus decision makers, we propose a recommendation rule with PIS to design a feedback for decision makers with unacceptable consensus. Generally, the recommendation rule includes the identification rule and the direction rule to guide the feedback process. In this paper, we apply these two rules based on PIS as follows,

(1) Identification rule

The identification rule is used to find out the decision maker in G_U which is needed to change their preferences. It is easier to reach an established consensus with decision makers which have a higher consensus compared with the other decision makers in G_U. Hence, the decision maker e_κ, whose consensus level satisfies $CL_\kappa=\max_{e_k\in G_U}CL_k$, should change their preferences. In other words, the decision maker e_κ in G_U, which has the closest preference to the collective preference, should adjust their preferences, i.e., $\sum_{i,j=1}^{n}\left|f_{ij}^{\kappa}-f_{ij}^{c}\right|=\min_{e_k\in G_U}\sum_{i,j=1}^{n}\left|f_{ij}^{k}-f_{ij}^{c}\right|$.

(2) Direction rule

The direction rule finds out the direction to change the preferences of decision makers. In order to make the decision maker e_κ more willing to modify their preferences, the main idea of the direction rule is to choose a suitable decision maker in G_A to act as a moderator, which has similar semantics and preferences with e_κ, to guide e_κ to change their preferences.

Let decision maker e_κ belong to the semantic-based cluster G^h, i.e., $e_\kappa\in G^h$. Let e_y be the corresponding moderator to help e_κ to improve the consensus. Then, we consider two cases about the conditions that decision maker e_y should satisfy.

Case A: If $G^h\cap G_A\neq\varnothing$, then e_y should meet the following three conditions:

(a) e_y belongs to the semantic-based cluster G^h, that is, $e_y\in G^h$;

(b) e_y belongs to the consensus group G_A, in which the decision makers all have acceptable consensus, i.e., $e_y\in G_A$;

(c) e_y has the closest distance to e_κ, i.e., $\sum_{i,j=1}^{n}\left|f_{ij}^{y}-f_{ij}^{\kappa}\right|=\min_{e_k\in G^h\cap G_A}\sum_{i,j=1}^{n}\left|f_{ij}^{k}-f_{ij}^{\kappa}\right|$.

Case B: If $G^h \cap G_A = \varnothing$, then e_y should satisfy the above conditions (b) and (c), that is, $e_y \in G_A$ and $\sum_{i,j=1}^{n}\left|f_{ij}^{y}-f_{ij}^{\kappa}\right| = \min_{e_k \in G_A}\sum_{i,j=1}^{n}\left|f_{ij}^{k}-f_{ij}^{\kappa}\right|$.

The main work of the decision maker e_y is to provide recommendation directions for e_κ in changing their preference, so that the consensus levels associated with e_κ and all the decision makers can be improved. Let $\overline{F^\kappa} = (\overline{f_{ij}^\kappa})_{n\times n}$ be the adjusted fuzzy preference relation associated with e_κ and satisfying the reciprocity property, that is, $\overline{f_{ij}^\kappa} + \overline{f_{ji}^\kappa} = 1$ for $i, j = 1, 2, ..., n$, the direction rule is provided as follows,

(a) If $f_{ij}^\kappa \le f_{ij}^y \le f_{ij}^c$, then e_κ should increase the preference for pairwise (x_i, x_j) to be close to f_{ij}^y, i.e., $\overline{f_{ij}^\kappa} \subseteq (f_{ij}^\kappa, f_{ij}^y]$;

(b) If $f_{ij}^\kappa \le f_{ij}^c \le f_{ij}^y$, then e_κ should increase the preference for pairwise (x_i, x_j) to be close to f_{ij}^c, i.e., $\overline{f_{ij}^\kappa} \subseteq (f_{ij}^\kappa, f_{ij}^c]$;

(c) If $f_{ij}^y \le f_{ij}^c \le f_{ij}^\kappa$, then e_κ should decrease the preference for pairwise (x_i, x_j) to be close to f_{ij}^c, i.e., $\overline{f_{ij}^\kappa} \in [f_{ij}^c, f_{ij}^\kappa)$;

(d) If $f_{ij}^c \le f_{ij}^y \le f_{ij}^\kappa$, then e_κ should decrease the preference for pairwise (x_i, x_j) to be close to f_{ij}^y, i.e., $\overline{f_{ij}^\kappa} \in [f_{ij}^y, f_{ij}^\kappa)$;

(e) Otherwise, e_κ should not change the preference for pairwise (x_i, x_j), i.e., $\overline{f_{ij}^\kappa} = f_{ij}^\kappa$.

This direction rule is to help the decision maker e_κ which has unacceptable consensus to modify their preference according to the preference of the moderator e_y and the collective preference, so that e_κ can obtain a higher consensus level.

3) *Consensus reaching process in LSGDM with PIS*

Based on the consensus framework and the above analysis, the procedure to reach a consensus in the LSGDM problems with PIS is provided below.

Algorithm 2. Algorithm of the consensus reaching process in LSGDM with PIS

Input: The individual linguistic preference relations $L^k = (l_{ij}^k)_{n\times n}$ $(k = 1, 2, ..., m)$, the weighting vectors $W = (w_1, w_2, ..., w_m)$ and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$. The individual consensus threshold ε, the collective consensus threshold $\overline{CL}$ and the established maximum number of iterations $t_{\max}$.

Output: The adjusted fuzzy preference relations $\overline{F^k} = (\overline{f_{ij}^k})_{n\times n}$ $(k = 1, 2, ..., m)$, the consensus level CL and the collective ranking of alternatives.

Step 1: Let $t = 0$. Using the PIS model (Eq. (8)) obtains the personalized numerical scales of linguistic terms NS^k. Then, based on NS^k and L^k, we get the individual fuzzy preference relations $F_0^k = (f_{ij,0}^k)_{n\times n}$.

Step 2: Based on Eq. (10), the collective fuzzy preference relation F_t^c is obtained by aggregating $\{F_t^1, F_t^2, ..., F_t^m\}$.

Step 3: Compute the consensus level $CL_{k,t}$ associated with e_k based on Eq. (11) and then classify the decision makers into opposing consensus groups $G_{A,t}$ and $G_{U,t}$. If $CL_{k,t} \ge \varepsilon$, then decision maker e_k is classified into G_A. Otherwise, decision maker e_k is classified into G_U.

Step 4: Calculate the consensus level of all the decision makers CL_t based on Eq. (12). If $CL_t \ge \overline{CL}$ or $t > t_{\max}$, compute the collective preference vectors $Z^c = (z_1^c, z_2^c, ..., z_n^c)^T$ based on Eq. (13) and obtain the collective ranking of alternatives from F_t^c, then go to Step 7, otherwise, continue with the next step.

Step 5: Based on the identification rule, the decision maker $e_{\kappa,t} \in G_{U,t}$, which has the highest consensus among the decision makers in $G_{U,t}$, i.e., $CL_{\kappa,t} = \max_{e_k \in G_{U,t}} CL_{k,t}$, should change their preference.

Step 6: According to the direction rule, apply Algorithm 1 to obtain the semantic-based clusters for decision makers, $\{G^1, G^2, ..., G^N\}$. If $e_{\kappa,t} \in G^h$ $(h \in \{1, 2, ..., N\})$, then the decision maker $e_{y,t}$, which satisfies the condition $\sum_{i,j=1}^{n}\left|f_{ij,t}^{y}-f_{ij,t}^{\kappa}\right| = \min_{\substack{k\in\{1,2,...,m\}\\ e_k \in G^h \cap G_{A,t}}}\sum_{i,j=1}^{n}\left|f_{ij,t}^{k}-f_{ij,t}^{\kappa}\right|$ if $G^h \cap G_{A,t} \ne \varnothing$ or $\sum_{i,j=1}^{n}\left|f_{ij,t}^{y}-f_{ij,t}^{\kappa}\right| = \min_{\substack{k\in\{1,2,...,m\}\\ e_k \in G_{A,t}}}\sum_{i,j=1}^{n}\left|f_{ij,t}^{k}-f_{ij,t}^{\kappa}\right|$ if $G^h \cap G_{A,t} = \varnothing$, should provide the adjusted suggestions to $e_{\kappa,t}$ as follows,

$$f_{ij,t+1}^{\kappa} \in \begin{cases} (f_{ij,t}^{\kappa}, f_{ij,t}^{y}] & \text{If } f_{ij,t}^{\kappa} \le f_{ij,t}^{y} \le f_{ij,t}^{c} \\ (f_{ij,t}^{\kappa}, f_{ij,t}^{c}] & \text{If } f_{ij,t}^{\kappa} \le f_{ij,t}^{c} \le f_{ij,t}^{y} \\ [f_{ij,t}^{c}, f_{ij,t}^{\kappa}) & \text{If } f_{ij,t}^{y} \le f_{ij,t}^{c} \le f_{ij,t}^{\kappa} \\ [f_{ij,t}^{y}, f_{ij,t}^{\kappa}) & \text{If } f_{ij,t}^{c} \le f_{ij,t}^{y} \le f_{ij,t}^{\kappa} \\ f_{ij}^{\kappa} & \text{otherwise} \end{cases}. \tag{14}$$

Let $t = t + 1$, go back to Step 2.

Step 7: Let $\overline{F^k} = F_t^k$. Output the adjusted fuzzy preference relations $\overline{F^k} = (\overline{f_{ij}^k})_{n\times n}$, the consensus level CL and the collective ranking of alternatives.

IV. Numerical Example and Simulation Analysis

In this section, we propose the numerical example and simulation experiment to illustrate the use of the consensus model with PIS in LSGDM.

Here we provide a LSGDM problem, which includes a set of twenty decision makers, $E=\{e_1,e_2,...,e_{20}\}$ and a set of four alternatives, $X=\{x_1,x_2,x_3,x_4\}$. Although the number of decision makers in the example is small for LSGDM problems, it is enough to illustrate the proposed consensus model. Let S be an established linguistic term set as follows,

$$S=\{s_0=extremely\ poor, s_1=very\ poor, s_2=poor, s_3=fair,\ s_4=good, s_5=very\ good, s_6=extremely\ good\}$$

The decision makers provide the linguistic preference relations based on S, $L^k=(l_{ij}^k)_{n\times n}$, to express their preference over X. The linguistic preference relations L^k $(k=1,2,...,20)$ provided by decision makers are listed as follows.

$$L^1=\begin{pmatrix} s_3 & s_1 & s_4 & s_2 \\ - & s_3 & s_5 & s_1 \\ - & - & s_3 & s_4 \\ - & - & - & s_3 \end{pmatrix}\quad L^2=\begin{pmatrix} s_3 & s_2 & s_1 & s_3 \\ - & s_3 & s_4 & s_5 \\ - & - & s_3 & s_6 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^3=\begin{pmatrix} s_3 & s_4 & s_5 & s_6 \\ - & s_3 & s_2 & s_1 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix}\quad L^4=\begin{pmatrix} s_3 & s_5 & s_4 & s_1 \\ - & s_3 & s_4 & s_2 \\ - & - & s_3 & s_3 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^5=\begin{pmatrix} s_3 & s_1 & s_2 & s_3 \\ - & s_3 & s_4 & s_2 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix}\quad L^6=\begin{pmatrix} s_3 & s_1 & s_2 & s_4 \\ - & s_3 & s_5 & s_4 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^7=\begin{pmatrix} s_3 & s_1 & s_3 & s_4 \\ - & s_3 & s_1 & s_4 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix}\quad L^8=\begin{pmatrix} s_3 & s_4 & s_2 & s_1 \\ - & s_3 & s_5 & s_5 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^9=\begin{pmatrix} s_3 & s_3 & s_2 & s_3 \\ - & s_3 & s_1 & s_5 \\ - & - & s_3 & s_4 \\ - & - & - & s_3 \end{pmatrix}\quad L^{10}=\begin{pmatrix} s_3 & s_3 & s_4 & s_6 \\ - & s_3 & s_2 & s_5 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^{11}=\begin{pmatrix} s_3 & s_4 & s_1 & s_6 \\ - & s_3 & s_3 & s_2 \\ - & - & s_3 & s_4 \\ - & - & - & s_3 \end{pmatrix}\quad L^{12}=\begin{pmatrix} s_3 & s_3 & s_0 & s_6 \\ - & s_3 & s_4 & s_5 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^{13}=\begin{pmatrix} s_3 & s_5 & s_1 & s_3 \\ - & s_3 & s_2 & s_4 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}\quad L^{14}=\begin{pmatrix} s_3 & s_5 & s_4 & s_1 \\ - & s_3 & s_1 & s_2 \\ - & - & s_3 & s_6 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^{15}=\begin{pmatrix} s_3 & s_1 & s_2 & s_3 \\ - & s_3 & s_5 & s_4 \\ - & - & s_3 & s_6 \\ - & - & - & s_3 \end{pmatrix}\quad L^{16}=\begin{pmatrix} s_3 & s_1 & s_5 & s_4 \\ - & s_3 & s_3 & s_2 \\ - & - & s_3 & s_6 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^{17}=\begin{pmatrix} s_3 & s_2 & s_3 & s_6 \\ - & s_3 & s_4 & s_2 \\ - & - & s_3 & s_5 \\ - & - & - & s_3 \end{pmatrix}\quad L^{18}=\begin{pmatrix} s_3 & s_2 & s_4 & s_5 \\ - & s_3 & s_6 & s_1 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}$$

$$L^{19}=\begin{pmatrix} s_3 & s_2 & s_3 & s_5 \\ - & s_3 & s_4 & s_6 \\ - & - & s_3 & s_2 \\ - & - & - & s_3 \end{pmatrix}\quad L^{20}=\begin{pmatrix} s_3 & s_2 & s_4 & s_6 \\ - & s_3 & s_5 & s_5 \\ - & - & s_3 & s_1 \\ - & - & - & s_3 \end{pmatrix}$$

The following subsections illustrate the PIS process to obtain the PISs of linguistic terms for decision makers and the consensus process to get an acceptable consensus among decision makers. Finally, a simulation analysis is proposed to show the desired property of the proposed consensus model.

A. *Illustration of the PIS process*

According to Section II.B, let λ=0.01, solving model (8) obtains the personalized numerical scales of linguistic terms for each decision maker, $\{NS^k(s_0), NS^k(s_1),..., NS^k(s_6)\}$ $(k=1,2,...,20)$. They are listed in Table 2.

Using the obtained numerical scale NS^k associated with e_k transforms L^k into the fuzzy preference relation $F^k=(f_{ij}^k)_{4\times 4}$. To save space, we only provide the fuzzy preference relation F^1 as an example to show the transformation, i.e.,

$$F^1=\begin{pmatrix} 0.5 & 0.333 & 0.51 & 0.49 \\ 0.677 & 0.5 & 0.677 & 0.333 \\ 0.49 & 0.333 & 0.5 & 0.51 \\ 0.51 & 0.677 & 0.49 & 0.5 \end{pmatrix}$$

B. *Illustration of the consensus measure based on PIS*

According to Eq. (10), in this paper the weighted average operator is used to aggregate the individual preferences into the collective preference. In this illustration study, we use the average weights of decision makers with same value $\frac{1}{20}$, and then the collective fuzzy preference relation $F^c=(f_{ij}^c)_{4\times 4}$, where $f_{ij}^c=\frac{1}{20}\sum_{k=1}^{20} f_{ij}^k$, is obtained as follows,

$$F^c=\begin{pmatrix} 0.5 & 0.466 & 0.476 & 0.671 \\ 0.534 & 0.5 & 0.565 & 0.606 \\ 0.524 & 0.435 & 0.5 & 0.63 \\ 0.329 & 0.494 & 0.37 & 0.5 \end{pmatrix}$$

Table 2. Personalized numerical scales of linguistic terms for each decision maker

	$NS^k(s_0)$	$NS^k(s_1)$	$NS^k(s_2)$	$NS^k(s_3)$	$NS^k(s_4)$	$NS^k(s_5)$	$NS^k(s_6)$
e_1	0	0.333	0.49	0.5	0.51	0.677	1
e_2	0	0.157	0.167	0.5	0.51	0.833	1
e_3	0	0.333	0.49	0.5	0.74	0.75	1
e_4	0	0.333	0.49	0.5	0.51	0.667	1
e_5	0	0.333	0.343	0.5	0.51	0.667	1
e_6	0	0.323	0.49	0.5	0.657	0.667	1
e_7	0	0.333	0.49	0.5	0.51	0.667	1
e_8	0	0.333	0.416	0.5	0.51	0.667	1
e_9	0	0.333	0.343	0.5	0.657	0.667	1
e_{10}	0	0.333	0.49	0.5	0.51	0.99	1
e_{11}	0	0.333	0.49	0.5	0.833	0.843	1
e_{12}	0	0.333	0.49	0.5	0.51	0.99	1
e_{13}	0	0.333	0.49	0.5	0.51	0.667	1
e_{14}	0	0.333	0.49	0.5	0.51	0.677	1
e_{15}	0	0.333	0.49	0.5	0.667	0.677	1
e_{16}	0	0.333	0.49	0.5	0.657	0.667	1
e_{17}	0	0.333	0.49	0.5	0.51	0.677	1
e_{18}	0	0.333	0.343	0.5	0.833	0.843	1
e_{19}	0	0.245	0.255	0.5	0.745	0.755	1
e_{20}	0	0.333	0.49	0.5	0.833	0.843	1

Based on Eq. (11), we obtain the consensus levels associated with each decision maker CL_k $(k=1,2,...,20)$ (see Table 3).

Table 3. Consensus levels associated with decision makers

	CL_k (k=1,...,5)		CL_k (k=6,…, 10)
e_1	0.857	e_6	0.942
e_2	0.761	e_7	0.865
e_3	0.782	e_8	0.862
e_4	0.853	e_9	0.891
e_5	0.872	e_{10}	0.806
	CL_k (k=11,…,15)		CL_k (k=16,...,20)
e_{11}	0.8	e_{16}	0.853
e_{12}	0.736	e_{17}	0.903
e_{13}	0.861	e_{18}	0.74
e_{14}	0.784	e_{19}	0.788
e_{15}	0.859	e_{20}	0.746

Set $\varepsilon = 0.8$ as the individual consensus threshold to check the consensus of each decision maker, we obtain the two opposing consensus groups G_A and G_U of decision makers as follows,

$G_A = \{e_1, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{13}, e_{15}, e_{16}, e_{17}\}$;

$G_U = \{e_2, e_3, e_{12}, e_{14}, e_{18}, e_{19}, e_{20}\}$.

From Table 3 and Eq. (12), the consensus level of all decision makers $\{e_1, e_2, ..., e_{20}\}$ is computed as follows,

$$CL = \frac{\# G_A}{20} = \frac{13}{20} = 0.65 .$$

C. Illustration of the feedback recommendation phase based on PIS

In this subsection, we illustrate the application of feedback recommendation phase to improve the consensus among decision makers.

Set $N = 3$ to be the number of the semantic-based clusters. Based on Algorithm 1 and Table 2, three semantic-based clusters are obtained as follows,

$G^1 = \{e_2, e_{10}, e_{12}\}$;

$G^2 = \{e_1, e_4, e_5, e_6, e_7, e_8, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}\}$;

$G^3 = \{e_3, e_9, e_{11}, e_{18}, e_{19}, e_{20}\}$.

Let $\overline{CL} = 0.75$ be the collective consensus threshold. Then, we apply the recommendation rule based on PIS to help decision makers adjust their preferences.

First round

According to the identification rule, the decision maker, which has the highest consensus in consensus group G_U, should change their preferences. From Table 3, we find that the decision maker e_{19}, whose consensus $CL_{19} = 0.788$, needs to modify their preferences.

It is clear that $e_{19} \in G^3$, then we find that the decision maker e_{11}, which satisfies $\sum_{i,j=1}^{4} \left| f_{ij}^{19} - f_{ij}^{11} \right| = \min_{\substack{y \in \{1,2,\ldots,m\} \\ e_y \in G_A; e_y \in G^3}} \sum_{i,j=1}^{n} \left| f_{ij}^{19} - f_{ij}^{y} \right|$, is the suitable moderator to help e_{19} improve the consensus. According to the direction rule and Eq. (14), the new fuzzy preference relation F^{19} is obtained as follows,

$$F^{19} = \begin{pmatrix} 0.5 & 0.46 & 0.47 & 0.755 \\ 0.54 & 0.5 & 0.56 & 0.6 \\ 0.53 & 0.44 & 0.5 & 0.6 \\ 0.245 & 0.4 & 0.4 & 0.5 \end{pmatrix}$$

Based on Eqs. (10) and (11), we obtain the consensus level associated with each decision maker (see Table 4).

Let $\varepsilon = 0.8$ be the individual consensus threshold, then we obtain the two opposing consensus groups G_A and G_U of decision makers as follows,

$G_A = \{e_1, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{13}, e_{15}, e_{16}, e_{17}, e_{19}\}$;

$G_U = \{e_2, e_3, e_{12}, e_{14}, e_{18}, e_{20}\}$.

Table 4. Consensus levels associated with decision makers in the first round

	CL_k (k=1,…,5)		CL_k (k=6,…,10)
e_1	0.854	e_6	0.938
e_2	0.761	e_7	0.868
e_3	0.789	e_8	0.857
e_4	0.856	e_9	0.893
e_5	0.879	e_{10}	0.803
	CL_k (k=11,…,15)		CL_k (k=16,…,20)
e_{11}	0.807	e_{16}	0.859
e_{12}	0.733	e_{17}	0.912
e_{13}	0.865	e_{18}	0.742
e_{14}	0.793	e_{19}	0.97
e_{15}	0.855	e_{20}	0.74

Thus, the consensus level of all decision makers $\{e_1, e_2, \ldots, e_{20}\}$ is $CL = \frac{\# G_A}{20} = \frac{14}{20} = 0.7$.

Second round

Based on the identification rule and Table 4, we find that the decision maker e_{14}, which satisfies $CL_{14} = \max_{e_k \in G_U} CL_k = 0.793$, needs to change their preferences. It is clear that $e_{14} \in G^2$, then we find that the decision maker e_4, which satisfies $\sum_{i,j=1}^{4} \left| f_{ij}^{14} - f_{ij}^{4} \right| = \min_{\substack{y \in \{1,2,\ldots,m\} \\ e_y \in G_A; e_y \in G^2}} \sum_{i,j=1}^{n} \left| f_{ij}^{14} - f_{ij}^{y} \right|$, is the suitable moderator to help e_{14} to adjust its preferences. According to the direction rule and Eq. (14), the new fuzzy preference relation F^{14} is obtained as follows,

$$F^{14} = \begin{pmatrix} 0.5 & 0.604 & 0.498 & 0.333 \\ 0.396 & 0.5 & 0.368 & 0.49 \\ 0.502 & 0.632 & 0.5 & 0.78 \\ 0.667 & 0.51 & 0.22 & 0.5 \end{pmatrix}$$

Applying the consensus measure again, we obtain the consensus level associated with each decision maker (see Table 5).

Table 5. Consensus levels associated with decision makers in the second round

	CL_k (k=1,...,5)		CL_k (k=6,…,10)
e_1	0.855	e_6	0.937
e_2	0.76	e_7	0.869
e_3	0.788	e_8	0.857
e_4	0.857	e_9	0.892
e_5	0.878	e_{10}	0.804
	CL_k (k=11,…,15)		CL_k (k=16,...,20)
e_{11}	0.805	e_{16}	0.858
e_{12}	0.734	e_{17}	0.911
e_{13}	0.865	e_{18}	0.742
e_{14}	0.82	e_{19}	0.972
e_{15}	0.854	e_{20}	0.741

Based on Table 5, two opposing consensus groups are structured as follows,

$G_A = \{e_1, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{13}, e_{15}, e_{16}, e_{17}, e_{19}\}$;

$G_U = \{e_2, e_3, e_{12}, e_{18}, e_{20}\}$.

Thus, the consensus level of all decision makers is

$$CL = \frac{\#(\{e_k \mid CL_k \geq 0.8\})}{20} = 0.75.$$

Due to our interest being focused on the consensus process, we simply use the average weights to compute the collective preference vector and the other aggregation operators will be similar. Then, based on Eq. (13) we obtain the collective ranking of alternatives $A_2 \succ A_1 \succ A_3 \succ A_4$.

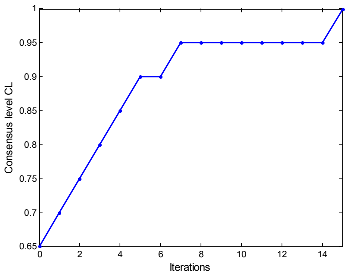
D. Simulation analysis

In this subsection, we further explore the use of the consensus process by means of simulation experiments from two aspects: the consensus level and the effect of the changing extent of the decision makers' preferences in the direction rule based on PIS.

To carry out simulation analysis of the consensus process, we replace Eq. (14) in Step 6 in Algorithm 2 with Eq. (14'), in order to automatically revise the decision makers' preferences. The replacement of Eq. (14) will not change the essence of Algorithm 2. Eq. (14') is provided as follows,

$$f_{ij,t+1}^{\kappa}=\begin{cases}(f_{ij,t}^{y}-f_{ij,t}^{\kappa})/\alpha & If\ f_{ij,t}^{\kappa}\le f_{ij,t}^{y}\le f_{ij,t}^{c}\\(f_{ij,t}^{c}-f_{ij,t}^{\kappa})/\alpha & If\ f_{ij,t}^{\kappa}\le f_{ij,t}^{c}\le f_{ij,t}^{y}\\(f_{ij,t}^{\kappa}-f_{ij,t}^{c})/\alpha & If\ f_{ij,t}^{y}\le f_{ij,t}^{c}\le f_{ij,t}^{\kappa}\\(f_{ij,t}^{\kappa}-f_{ij,t}^{y})/\alpha & If\ f_{ij,t}^{c}\le f_{ij,t}^{y}\le f_{ij,t}^{\kappa}\\f_{ij}^{\kappa} & otherwise\end{cases} \quad (14')$$

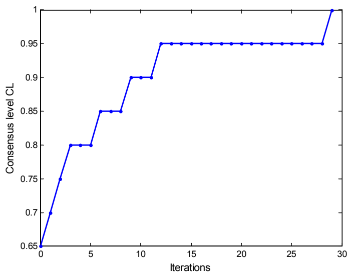
Continuing the feedback recommendation phase in Section III.C, we further investigate the process to improve the consensus. Based on Eq. (14'), we set $\alpha=5$, and Fig.2 shows the variation trends of the consensus level of all the decision makers. Table 6 presents the identified decision makers which need to change their preferences and the corresponding moderators in each iteration.



$\alpha=5$

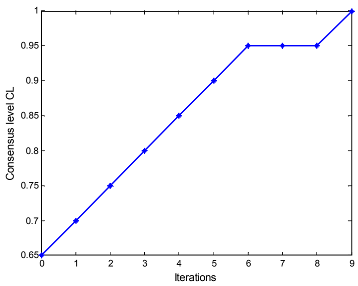
Fig.2 The process to improve the consensus based on Algorithm 2 and Eq. (14')

Next, by changing the value of α, we investigate the effect of the changing extent of decision makers' preference in computing the consensus level. Figs. 3 and 4 provide the variation trend of the consensus level by setting $\alpha=8$ and $\alpha=2$, respectively.



$\alpha=8$

Fig.3 The process to improve the consensus based on Algorithm 2 and Eq. (14')



$\alpha=2$

Fig.4 The process to improve the consensus based on Algorithm 2 and Eq. (14')

According to Figs. 2-4, the following observations can be drawn:

(1) The consensus level is improved by using the proposed consensus process with PIS.

(2) The number of the iterations depends on the value of α. When set $\alpha=8$, the consensus level can reach 1 in about 28 iterations; if set $\alpha=5$ and $\alpha=2$, the consensus level can reach 1 in about 14 iterations and 8 iterations, respectively. This means that the smaller the value of α, the easier for decision makers to reach a full consensus.

The above observations show that by applying Algorithm 2, the consensus level is improved, which demonstrates that our proposal provides an effective way of building consensus in LSGDM with PIS.

E. Comparison with other consensus models in LSGDM

LSGDM is a new topic in the GDM research area. In recent years, different consensus methods to deal with LSGDM have been proposed. They can be classified into three types:

(1) Consensus model in multi-criteria LSGDM. Different methods [26, 27, 43, 47] have been proposed to support the multi-criteria LSGDM. Regarding the consensus, Xu et al. [43] presented a consensus method for multi-attribute LSGDM by treating each cluster as a whole to form a smaller group in improving the consensus.

(2) Managing non-cooperative behaviors in LSGDM. Palomares et al. [31] proposed a consensus model to detect and manage the subgroup and individual non-cooperative behavior with weight penalizing method. Quesada et al. [33] presented a large-scale consensus reaching process based on uninorm operators to manage decision makers' behaviors according to the overall behavior. Xu et al. [42] proposed a consensus model under an emergency situation to manage non-cooperative behaviors and minority opinions by updating weights of clusters. Dong et al. [8] proposed a self-management mechanism for non-cooperative behaviors by penalizing the weights of the experts in large-scale consensus process.

(3) Consensus model with different types of preferences in LSGDM. In [39], Wu and Xu proposed a consensus model in LSGDM with hesitant fuzzy information in which the clusters are allowed to change. Palomares [32] presented an attitude-based consensus model for IT-based services management that deals with heterogeneous information. Zhang et al. [45] proposed a consensus reaching model for the LSGDM with heterogeneous preference representations considering the individual concerns and satisfactions.

Compared with these studies, our proposed consensus model is established under the linguistic context considering the individual difference in understanding the meaning of words, and our study is the first proposal in the field of LSGDM discussing the consensus with PIS. Our paper proposal involves not only dealing with the difficult issues of large groups, but also the complexity of processing the different meaning of individual linguistic expressions.

Besides, we propose a natural premise, people having similar preferences and PISs it is easier to communicate with each other, which help decision makers become more willing to change their preference, so as to achieve a consensus in LSGDM in linguistic context.

V. CONCLUSION

In this paper, we propose a consensus model with PIS in linguistic LSGDM problems. It consists of two processes: PIS process and consensus process. The PIS process is introduced to obtain the PISs of linguistic terms for decision makers. The use of the PIS process shows the difference among decision makers in understanding the meaning of words, which provides a new view for the consensus reaching in LSGDM. In the consensus process, the consensus measure and feedback recommendation phases are provided. The consensus measure computes the consensus level of decision makers and classifies the decision makers into two opposing consensus groups, which provides a basis for the feedback recommendation phase. In the feedback recommendation, we propose a PIS based clustering method to group the decision makers with similar semantics, and then a recommendation rule, based on opposing consensus groups and semantic-based clusters, is presented to help decision makers become more willing to change their preferences to reach a consensus.

The personalization of the linguistic presentation and personalized recommendation rules based on decision makers' similarities via clustering show the advantages of the proposed consensus process in dealing with the individual linguistic preferences and in improving the consensus. The limitation is the creation of the collective preference relation based on an aggregation operator without more personalized information.

In future work, we will discuss the consensus approach with PIS in social networks [40] and opinion dynamics [7]. Meanwhile, it would be necessary to also consider the clustering for this initial aggregation to get collective preferences including more personalized information. On the other hand, the duality between probability and possibility [36, 37], and its extensions to deal with probabilities and label distributions [10, 48], they will deal us to consider these kind of distributions in LSGDM.

REFERENCES

[1] S. Alonso, F. Chiclana, F. Herrera, E. Herrera-Viedma, J. Alcalá-Fdez, C. Porcel, A consistency-based procedure to estimate missing pairwise preference values. International Journal of Intelligent Systems 23(2) (2008) 155-175.

[2] J. Bezdek, Pattern recognition with fuzzy objective function algorithms, New York, NY, USA: Wiley, 2007.

[3] J.C. Bezdek, R. Ehrlich, W. Full, FCM: The fuzzy c-means clustering algorithm. Computers & Geosciences 10(2-3) (1984) 191-203.

[4] F. Chiclana, F. Mata, L. Martínez, E. Herrera-Viedma, S. Alonso, Integration of a consistency control module within a consensus model, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 16 (2008) 35–53.

[5] D. Dubois, The role of fuzzy sets in decision sciences: old techniques and new directions. Fuzzy Sets and Systems184 (2011) 3-28.

[6] Y.C. Dong, Y.F. Xu, S. Yu, Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model, IEEE Transactions on Fuzzy Systems 17(6) (2009) 1366-1378.

[7] Y.C. Dong, M. Zhan, G. Kou, Z.G. Ding, H.M. Liang, A survey on the fusion process in opinion dynamics, Information Fusion 43 (2018) 57-65.

[8] Y.C. Dong, S.H. Zhao, H.J. Zhang, F. Chiclana, E. Herrera-Viedma, A self-management mechanism for non-cooperative behaviors in large-scale group consensus reaching processes. IEEE Transactions on Fuzzy Systems, DOI: 10.1109/TFUZZ.2018.2818078.

[9] Y.C. Dong, C.C. Li, F. Herrera, Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information, Information Sciences 367-368 (2016) 259-278.

[10] M. Durand, I. Truck, A new proposal to deal with hesitant linguistic expressions on preference assessments. Information Fusion 41 (2018) 175-181.

[11] X.J. Gou, H.C. Liao, Z.S. Xu, F. Herrera, Double hierarchy hesitant fuzzy linguistic term set and MULTIMOORA method: A case of study to evaluate the implementation status of haze controlling measures, Information Fusion 38 (2017) 22-34.

[12] J.L. García-Lapresta, D. Pérez-Román, Ordinal proximity measures in the context of unbalanced qualitative scales and some applications to consensus and clustering. Applied Soft Computing 35 (2015) 864-872.

[13] F. Herrera, L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Transaction on Fuzzy Systems 8(6) (2000) 746-752.

[14] F. Herrera, E. Herrera-Viedma, J. L. Verdegay, A model of consensus in group decision making under linguistic assessments, Fuzzy Sets and Systems 78 (1996) 73–87.

[15] F. Herrera, E.Herrera-Viedma, L. Martínez, A fuzzy linguistic methodology to deal with unbalanced linguistic term sets, IEEE Transaction on Fuzzy Systems 16(2) (2008) 354-370.

[16] E. Herrera-Viedma, F. Chiclana, F. Herrera, S. Alonso, Group decision-making model with incomplete fuzzy preference relations based on additive consistency, IEEE Transactions Systems, Man, and Cybernetics, Part B 37(1) (2007) 176-189.

[17] E. Herrera-Viedma, S. Alonso, F. Chiclana, F. Herrera, A consensus model for group decision making with incomplete fuzzy preference relations, IEEE Transaction on Fuzzy Systems 15(5) (2007) 863-877.

[18] E. Herrera-Viedma, F. J. Cabrerizo, J. Kacprzyk, W. Pedrycz, A review of soft consensus model in a fuzzy environment, Information Fusion 17 (2014) 4-13.

[19] R. Hegselmann, U. Krause, Opinion dynamics and bounded confidence models, analysis, and simulation. Journal of Artificial Societies and Social Simulation 5(3) (2002) 1-33.

[20] J. Kacprzyk, M. Fedrizzi, A "soft" measure of consensus in the setting of partial (fuzzy) preferences, European Journal Operational Research 34 (1988) 316-325.

[21] J. Kacprzyk, M. Fedrizzi, H. Nurmi, Group decision making and consensus under fuzzy preferences and fuzzy majority, Fuzzy Sets and Systems 49 (1992) 21-31.

[22] C.C. Li, Y.C. Dong, F. Herrera, E. Herrera-Viedma, L. Martínez, Personalized individual semantics in computing with words for supporting linguistic group decision making. An application on consensus reaching, Information Fusion 33 (2017) 29-40.

[23] C.C. Li, R.M. Rodríguez, L. Martínez, Y.C. Dong, F. Herrera, Personalized individual semantics based on consistency in hesitant linguistic group decision making with comparative linguistic expressions, Knowledge-Based Systems 145 (2018) 156-165.

[24] Y. Liu, Z.P. Fan, X. Zhang, A method for large group decision-making based on evaluation information provided by participators from multiple groups, Information Fusion 29 (2016) 132-141.

[25] Y. Liu, J.W. Bi, Z.P. Fan, Ranking products through online reviews: A method based on sentiment analysis technique and intuitionistic fuzzy set theory. Information Fusion 36 (2017) 149-161.

[26] B. Liu, Y. Shen, Y. Chen, X. Chen, Y. Wang, A two-layer weight determination method for complex multi-attribute large-group decision-making experts in a linguistic environment, Information Fusion 23 (2015) 156-165.

[27] B. Liu, Y. Shen, X. Chen, Y. Chen, X. Wang, A partial binary tree DEA-DA cyclic classification model for decision makers in complex multi-attribute large-group interval-valued intuitionistic fuzzy decision-making problems, Information Fusion 18 (2014) 119-130.

[28] J.M. Mendel, D. Wu, Perceptual computing: aiding people in making subjective judgments. IEEE-Wiley, 2010.

[29] J.M. Mendel, L.A. Zadeh, E. Trillas, R.R. Yager, J. Lawry, H. Hagras, S. Guadarrama, What computing with words means to me: Discussion forum, IEEE Computational Intelligence Magazine 5(1) (2010) 20-26.

[30] I. Palomares, F.J. Estrella, L. Martínez, F. Herrera, Consensus under a fuzzy context: Taxonomy, analysis framework AFRYCA and experimental case of study, Information Fusion 20 (2014) 252-271.

[31] I. Palomares, L. Martínez, F. Herrera, A consensus model to detect and manage noncooperative behaviors in large-scale group decision making, IEEE Transactions on Fuzzy Systems 22(3) (2014) 516-530.

[32] I. Palomares, Consensus model for large-scale group decision support in IT services management, Intelligent Decision Technologies 8(2) (2014) 81-94.

[33] F. J. Quesada, I. Palomares, L. Martínez, Managing experts behavior in large-scale consensus reaching processes with uninorm aggregation operators, Applied Soft Computing 35 (2015) 873-887.
[34] L.E. Susskind, S. McKearnen, J. Thomas-Lamar, The consensus building handbook: A comprehensive guide to reaching agreement. Sage Publications,1999.
[35] R. Yager, Multicriteria Decision Making with Ordinal/Linguistic Intuitionistic Fuzzy Sets for Mobile Apps. IEEE Transactions on Fuzzy Systems 24(3) (2016) 590-599.
[36] S. Wang, G.H. Huang, B.W. Baetz, An inexact probabilistic–possibilistic optimization framework for flood management in a hybrid uncertain environment. IEEE Transactions on Fuzzy Systems 23(4) (2015) 897-908.
[37] S. Wang, B.C. Ancell, G.H. Huang, B.W. Baetz, Improving robustness of hydrologic ensemble predictions through probabilistic pre- and post-processing in sequential data assimilation. Water Resources Research 54 (2018) 2129-2151.
[38] Z.B. Wu, J.P. Xu, A concise consensus support model for group decision making with reciprocal preference relations based on deviation measures, Fuzzy Sets and Systems 206(11) (2012) 58-73.
[39] Z.B. Wu, J.P. Xu, A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters, Information Fusion 41 (2018) 217-231.
[40] J. Wu, F. Chiclana, H. Fujita, E. Herrera-Viedma, A visual interaction consensus model for social network group decision making with trust propagation, Knowledge-Based Systems 122 (2017) 39-50.
[41] J.H. Wang, J.Y. Hao, A new version of 2-tuple fuzzy linguistic representation model for computing with words, IEEE Transactions on Fuzzy Systems 143(3) (2006) 435-445.
[42] X.H. Xu, Z.J. Du, X.H. Chen, Consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions, Decision Support Systems 79 (2015) 150-160.
[43] Y.J. Xu, X.W. Wen, W.C. Zhang, A two-stage consensus method for large-scale multi-attribute group decision making with an application to earthquake shelter selection. Computers & Industrial Engineering 116 (2018) 113-129.
[44] W.Y. Yu, Z. Zhang, Q.Y. Zhong, A TODIM-based approach to large-scale group decision making with multi-granular unbalanced linguistic information. 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Naples, Italy.
[45] H.J. Zhang, Y.C. Dong, E. Herrera-Viedma, Consensus building for the heterogeneous large-scale GDM with the individual concerns and satisfactions. IEEE Transactions on Fuzzy Systems 26(2) (2018) 884-898.
[46] Z. Zhang, X. Kou, Q. Dong, Additive consistency analysis and improvement for hesitant fuzzy preference relations. Expert Systems with Applications 98 (2018) 118-128.
[47] Z. Zhang, C. Guo, L. Martínez, Managing multi-granular linguistic distribution assessments in large-scale multi-attribute group decision making. IEEE Transactions on Systems, Man, and Cybernetics: Systems 47(11) (2017) 3063-3076.
[48] G.Q. Zhang, Y.C. Dong, Y.F. Xu, Consistency and consensus measures for linguistic preference relations based on distribution assessments, Information Fusion 17 (2014) 46–55.
[49] J. Zhu, S. Zhang, Y. Chen, L. Zhang, A hierarchical clustering approach based on three-dimensional gray relational analysis for clustering a large group of decision makers with double information, Group Decision and Negotiation 25(2) (2016) 325-354.

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Bibliography

[ACC+09] Alonso S., Cabrerizo F., Chiclana F., Herrera F., and Herrera-Viedma E. (2009) Group decision making with incomplete fuzzy linguistic preference relations. *International Journal of Intelligent Systems* 24(2): 201–222.

[ACH+08] Alonso S., Chiclana F., Herrera F., Herrera-Viedma E., Alcalá-Fdez J., and Porcel C. (2008) A consistency-based procedure to estimate missing pairwise preference values. *International Journal of Intelligent Systems* 23(2): 155–175.

[CMM+08] Chiclana F., Mata F., Martínez L., Herrera-Viedma E., and Alonso S. (2008) Integration of a consistency control module within a consensus model. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 16(supp01): 35–53.

[DeG74] DeGroot M. (1974) Reaching a consensus. *Journal of the American Statistical Association* 69(345): 118–121.

[DHV15] Dong Y. and Herrera-Viedma E. (2015) Consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets and its use in the linguistic gdm with preference relation. *IEEE transactions on cybernetics* 45(4): 780–792.

[Dub11] Dubois D. (2011) The role of fuzzy sets in decision sciences: Old techniques and new directions. *Fuzzy Sets and Systems* 184(1): 3–28.

[DXY09] Dong Y., Xu Y., and Yu S. (2009) Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model. *IEEE Transactions on Fuzzy Systems* 17(6): 1366–1378.

[DZHY13] Dong Y., Zhang G., Hong W., and Yu S. (2013) Linguistic computational model based on 2-tuples and intervals. *IEEE Transactions on Fuzzy Systems* 21(6): 1006–1018.

[HHVM08] Herrera F., Herrera-Viedma E., and Martínez L. (2008) A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on fuzzy Systems* 16(2): 354–370.

[HK02] Hegselmann R. and Krause U. (2002) Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of artificial societies and social simulation* 5(3): 1–33.

[HM00] Herrera F. and Martínez L. (2000) A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on fuzzy systems* 8(6): 746–752.

[HM01] Herrera F. and Martínez L. (2001) A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making.

IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics) 31(2): 227–234.

[MH12] Martı L. and Herrera F. (2012) An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges. *Information Sciences* 207: 1–18.

[MW10] Mendel J. and Wu D. (2010) *Perceptual computing: Aiding people in making subjective judgments*, volumen 13. John Wiley & Sons.

[RMH12] Rodriguez R., Martinez L., and Herrera F. (2012) Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions on Fuzzy Systems* 20(1): 109–119.

[RMH13] RodríGuez R., MartıNez L., and Herrera F. (2013) A group decision making model dealing with comparative linguistic expressions based on hesitant fuzzy linguistic term sets. *Information Sciences* 241: 28–42.

[Tor10] Torra V. (2010) Hesitant fuzzy sets. *International Journal of Intelligent Systems* 25(6): 529–539.

[WCFHV17] Wu J., Chiclana F., Fujita H., and Herrera-Viedma E. (2017) A visual interaction consensus model for social network group decision making with trust propagation. *Knowledge-Based Systems* 122: 39–50.

[WH06] Wang J. and Hao J. (2006) A new version of 2-tuple fuzzy linguistic representation model for computing with words. *IEEE transactions on fuzzy systems* 14(3): 435–445.

[Yag04] Yager R. (2004) On the retranslation process in zadeh's paradigm of computing with words. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)* 34(2): 1184–1195.

[Zad75a] Zadeh L. (1975) The concept of a linguistic variable and its application to approximate reasoning-III. *Information sciences* 9(1): 43–80.

[Zad75b] Zadeh L. (1975) The concept of a linguistic variable and its application to approximate reasoning—I. *Information sciences* 8(3): 199–249.

[Zad75c] Zadeh L. (1975) The concept of a linguistic variable and its application to approximate reasoning—II. *Information sciences* 8(4): 301–357.