MEASURING PRIMALITY IN NUMERICAL SEMIGROUPS WITH EMBEDDING DIMENSION THREE

S. T. CHAPMAN, P. A. GARCÍA-SÁNCHEZ, Z. TRIPP, AND C. VIOLA

Abstract. In this paper, we find the ω-value of the generators of any numerical semigroup with embedding dimension three. This allows us to determine all possible orderings of the ω-values of the generators. In addition, we relate the ω-value of the numerical semigroup to its catenary degree.

1. Introduction

The arithmetic of non-unique factorizations in rings and monoids has been a popular topic in the recent mathematical literature. We focus on extending the results in [3], [4] and [8], where the ω-function, an arithmetic measure of how far an element is from being prime (cf. Section 2.3), is studied in numerical semigroups. In [3], the authors present an algorithm for computing values of the omega function on any numerical semigroup, and focus on computing ω-values for the generators of an embedding dimension three semigroup. When S is a numerical semigroup with minimal set of generators $\{n_1, n_2, n_3\}$, with $n_1 < n_2 < n_3$, there are 13 possible inequalities involving the $\omega(n_i)$ (for instance $\omega(n_1) \leq \omega(n_2) < \omega(n_3)$). Examples of eight of these orderings were given in [3], and two more were given in [8]. The authors in [8] conjectured that the last three orderings are not possible. In this paper, we compute the ω-values of the generators in the embedding dimension three case (Theorems 4.6 and 4.9). As a by-product, in Theorem 6.2 we confirm the conjecture cited above in [8]. Additionally, we relate, for embedding dimension three, the ω-value of a numerical semigroup to its catenary degree (cf. Section 2.2). We open in Section 2 with a brief review of definitions needed in the paper's remaining four sections. In Section 3 we consider the structure of the set of bullets (cf. Section 2.3) for the generators. In Section 4, we produce the ω-values for the generators in the embedding dimension three case. Section 5 considers the relation between the ω-function and the catenary degree and Section 6 discusses the resolution of the conjecture mentioned above from [8].

In addition to [3], [4], and [8] previously mentioned, we note that there has been much recent work on the behavior of the ω-function in several different settings. In [1] and [2] the behavior of the ω-function is studied on commutative rings and integral domains. The software package [10] will compute values of the ω-function in any affine semigroup (finitely generated submonoids of $\mathbb{N}^k$), and was used in [11] to obtain some general and asymptotic results. We note that the package [9] now deals also with affine semigroups, and has optimizations for full affine semigroups; see Chapter 10 of the manual. We used [9] for making batteries of examples that led us to many of the results presented in the current paper. It is shown recently in [20] that the omega function on a numerical semigroup is eventually quasi-linear, and the same

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authors in [21] give a comprehensive survey of the results in [4, 11, 20]. The current paper is mostly self-contained. Undefined terminology or notation regarding numerical semigroups or factorization theory can be found in [22] and [15], respectively.

2. Preliminaries

A *numerical semigroup* S is a nonempty subset of the set of nonnegative integers $\mathbb{N}$ that is closed under addition, contains 0 and has finite complement in $\mathbb{N}$. The condition $\#(\mathbb{N}\setminus S) < \infty$ is equivalent to $\gcd(S) = 1$ (see for instance [22, Chapter 1]). The largest integer not belonging to S is its *Frobenius number* $\mathrm{F}(S)$. Recall that S is *symmetric* if for every $x \in \mathbb{Z} \setminus S$, $\mathrm{F}(S) - x \in S$. In [22, Chapter 3] the reader can find more characterizations of the symmetric property.

We say that the positive integers $n_1, \ldots, n_p$ *generate* S if $S = \{\sum_{i=1}^{p} x_i n_i \mid x_i \in \mathbb{N} \text{ for each } i\}$. For such a generating set, we use the notation $S = \langle n_1, \ldots, n_p\rangle$. Using elementary number theory, it is simple to show that every numerical semigroup has a finite generating set, and in fact a unique one with minimal cardinality. The elements of this unique minimal generating system are called *minimal generators*, and its cardinality is known as the *embedding dimension* of S, denoted $\mathrm{e}(S)$.

Assume that $A = \{n_1, \ldots, n_p\}$ is the minimal generating set of the numerical semigroup S. Consider the monoid epimorphism

$$\varphi : \mathbb{N}^p \to S, \quad \varphi(a_1, \ldots, a_p) = a_1 n_1 + \cdots + a_p n_p,$$

known as the *factorization morphism* of S. The monoid S is isomorphic to $\mathbb{N}^p/\sigma$, where $\sigma = \{(a, b) \in \mathbb{N}^p \times \mathbb{N}^p \mid \varphi(a) = \varphi(b)\}$ is the kernel congruence of φ.

A *presentation* of S is a system of generators of σ, and accordingly, a *minimal presentation* is a presentation such that none of its proper subsets generates σ.

2.1. **Numerical Semigroups of Embedding Dimension Three.** Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical semigroup with embedding dimension three. Define $c_i = \min\{t \in \mathbb{N} \setminus \{0\} \mid tn_i \in \langle n_j, n_k\rangle, \{i, j, k\} = \{1, 2, 3\}\}$. Then, whenever $\{i, j, k\} = \{1, 2, 3\}$, there exists some $r_{ij}, r_{ik} \in \mathbb{N}$ such that

$$c_i n_i = r_{ij} n_j + r_{ik} n_k. \tag{1}$$

Herzog in [18] showed that the following are equivalent: (a) S is symmetric, (b) $r_{ij} = 0$ for some $i, j \in \{1, 2, 3\}$, and (c) S is a complete intersection (has a minimal presentation with two elements; see [22, Chapter 8] for details). In embedding dimension three this is also equivalent to S being free, which means that it has a minimal presentation with staircase shape for some arrangement of the generators (see [22, Section 8.3] for the definition and characterizations of free numerical semigroup).

Theorem 2.1. [22, Theorem 10.6] *Let $p_1, p_2 \in \mathbb{N} \setminus \{0, 1\}$ be relatively prime. Let a, b and c be nonnegative integers with $a \geq 2, b + c \geq 2$ and $\gcd(a, bp_1 + cp_2) = 1$. Then $S = \langle ap_1, ap_2, bp_1 + cp_2\rangle$ is a symmetric numerical semigroup with embedding dimension three. Moreover, every embedding dimension three symmetric numerical semigroup is of this form.*

2.2. **Catenary degree.** Let S be minimally generated by $\{n_1, \ldots, n_p\}$. The *set of factorizations* of an element $n \in S$ is

$$\mathsf{Z}(n) = \varphi^{-1}(n) = \{(a_1, \ldots, a_p) \in \mathbb{N}^p \mid a_1 n_1 + \cdots + a_p n_p = n\}.$$

The *length* of a factorization $a = (a_1, \dots, a_p) \in \mathsf{Z}(n)$ is $|a| = a_1 + \cdots + a_p$. For $z = (z_1, \dots, z_p), z' = (z'_1, \dots, z'_p) \in \mathbb{N}^p$ write $\gcd(z, z') = (\min\{z_1, z'_1\}, \dots, \min\{z_p, z'_p\})$ and set

$$\mathrm{d}(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\}$$

to be the *distance* between z and z'. Given $x \in \mathbb{N}^p$ and $Y \subset \mathbb{N}^p$, we define $\mathrm{d}(x, Y) = \min\{d(x, y) \mid y \in Y\}$ (which exists by Dickson's Lemma; see for instance [22, Lemma 8.6]). The *support* of $z \in \mathbb{N}^p$ is defined as $\mathrm{Supp}(z) = \{i \in \{1, \dots, p\} \mid z_i \neq 0\}$.

Given $n \in S$, $z, z' \in \mathsf{Z}(n)$, and $N \in \mathbb{N}$, an *N-chain of factorizations* from z to z' is a sequence $z_1, \dots, z_k \in \mathsf{Z}(n)$ such that $z_1 = z, z_k = z'$ and $\mathrm{d}(z_i, z_{i+1}) \leq N$ for all i. The *catenary degree* of n, denoted $\mathsf{c}(n)$, is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for any two factorizations $z, z' \in \mathsf{Z}(n)$, there is an N-chain from z to z'. The catenary degree of S, denoted $\mathsf{c}(S)$, is defined by

$$\mathsf{c}(S) = \sup\{\mathsf{c}(n) \mid n \in S\}.$$

Given $z, z' \in \mathsf{Z}(n)$ with $n \in S$, we say that z and z' are $\mathcal{R}$-related if there exists a chain $z_1, \dots, z_k$ such that

- $z_1 = z$, $z_k = z'$, and
- $z_i \cdot z_{i+1} \neq 0$ (equivalently $\mathrm{Supp}(z_i) \cap \mathrm{Supp}(z_{i+1})$ is not empty) for all $i \in \{1, \dots, k-1\}$.

The $\mathcal{R}$-relation is an equivalence relation and the set of $\mathcal{R}$-classes of $\mathsf{Z}(n)$ is the set of connected components of the graph ∇_n, whose vertices are the factorizations of n and zz' is an edge if $z \cdot z' \neq 0$. This graph has the same connected components as the graph G_n whose vertices are the minimal generators n_i of S such that $n - n_i \in S$, and $n_i n_j$ is an edge provided that $n - (n_i + n_j) \in S$. Since there are only finitely many elements n such that the graph G_n is disconnected (see [22, Chapter 7]), there are only finitely many n for which the relation $\mathcal{R}$ yields more than one $\mathcal{R}$-class.

An element $s \in S$ is said to be a *Betti element* if $\mathsf{Z}(s)$ has more than one $\mathcal{R}$-class. The set of Betti elements of S is denoted by $\mathrm{Betti}(S)$. Let $n \in S$ be a Betti element, and let $\mathcal{R}^n_1, \dots, \mathcal{R}^n_{k_n}$ be the distinct $\mathcal{R}$-classes of $\mathsf{Z}(n)$. Set $\mu(n) = \max\{r^n_1, \dots, r^n_{k_n}\}$, where $r^n_i = \min\{|z| \mid z \in \mathcal{R}^n_i\}$. Define $\mu(S) = \max\{\mu(n) \mid n \in \mathrm{Betti}(S)\}$.

Theorem 2.2. [7, Theorem 3.1] *Let S be a numerical semigroup. Then*

$$\mathsf{c}(S) = \mu(S).$$

From the proof of this result, it follows that the catenary degree of a numerical semigroup is attained by one of its Betti elements. Several papers have recently appeared in the literature that address problems involving the catenary degree on numerical semigroups (see [5], [6] and [19]).

2.3. **ω-primality.** An element in a numerical semigroup is *irreducible* if it cannot be written as a sum of nonzero elements. Obviously, the only irreducible elements in a numerical semigroup are its minimal generators.

Let S be a numerical semigroup minimally generated by $\{n_1, \dots, n_p\}$. The *ω-primality function* assigns to each element $n \in S$ the value $\omega(S, n) = m$ if m is the smallest positive integer with the property that whenever $(\sum_{i=1}^p a_i n_i) - n \in S$ with $|a| > m$ ($a = (a_1, \dots, a_p)$), there exists $b = (b_1, \dots, b_p) \in \mathbb{N}^p$ with $b \leq a$ (with the usual partial ordering on $\mathbb{N}^p$) such that $(\sum_{i=1}^p b_i n_i) - n \in S$ and $|b| \leq m$. Note that $\omega(S, n) < \infty$ for each n in a numerical semigroup (see [16]). When S is clear from the context, we simply write $\omega(n)$. Furthermore, we set

$$\omega(S) = \sup\{\omega(S, n_i) \mid i \in \{1, \dots, p\}\}.$$

By definition, an element $b \in S$ is a prime element if and only if $\omega(S, b) = 1$, and S is factorial if and only if $\omega(S) = 1$. Numerical semigroups other than $\mathbb{N}$ have no prime elements.

We say that $a = (a_1, \dots, a_p) \in \mathbb{N}^p$ is a *bullet* for n if $(\sum_{i=1}^p a_i n_i) - n \in S$ and, whenever $a_j \neq 0$, $(\sum_{i=1}^p a_i n_i) - n_j - n \notin S$. Bullets correspond to minimal elements with respect to $\leq$ in the set of all factorizations of elements of $n + S$. The set of bullets for $n \in S$ is denoted $\mathcal{B}(n)$. In numerical semigroups, the set of bullets for a fixed element n lies in a bounded subset of $\mathbb{N}^p$.

Lemma 2.3. [21, Lemma 3.7] *Fix a numerical semigroup S minimally generated by $\{n_1, \dots, n_p\}$ and $n \in S$.*

(1) For every i with $1 \leq i \leq p$, there exists $b_i > 0$ such that $b_i \mathbf{e}_i$ is a bullet for n, where $\mathbf{e}_i$ is the ith row of the $p \times p$ identity matrix.
(2) $\mathcal{B}(n) \subseteq \{a \in \mathbb{N}^p \mid a \leq b\}$, where $b = (b_1, \dots, b_p)$.

From the above lemma, for every $i \in \{1, \dots, p\}$, we define $\mathsf{m}_i(n) = b_i$ when $n \in S$. The following proposition, which is a particular instance of [4, Proposition 3.3], is useful for computing the ω invariant.

Proposition 2.4. *Let S be a numerical semigroup. For every $s \in S$ we have*

$$\omega(s) = \sup\{|x| \mid x \in \mathcal{B}(n)\}.$$

3. Bullets and $\mathcal{R}$-classes

We begin with a general result on bullets.

Lemma 3.1. *Let S be a numerical semigroup minimally generated by $= \{n_1, \dots, n_p\}$. Let $n \in \operatorname{Betti}(S)$ with $a, b \in \mathsf{Z}(n)$ in different $\mathcal{R}$-classes. For every $i \in \operatorname{Supp}(b)$ it follows that $a \in \mathcal{B}(n_i)$.*

Proof. Assume to the contrary that there exists $c \in \mathsf{Z}(n_i + S)$ and $x \in \mathbb{N}^k \setminus \{0\}$ such that $c+x = a$. From $c < a$, $a \cdot b = 0$, and $i \in \operatorname{Supp}(b)$, we deduce that $i \notin \operatorname{Supp}(c)$. As $c \in \mathsf{Z}(n_i+S)$, there exists $d \in \mathsf{Z}(n_i+S)$ with $i \in \operatorname{Supp}(d)$ and $\varphi(c) = \varphi(d)$. Hence $\varphi(d+x) = \varphi(c+x) = \varphi(a)$.

As $0 \neq x$ and $c+x = a$, we deduce $(d+x)\cdot(c+x) = (d+x)\cdot a \neq 0$. Also, $i \in \operatorname{Supp}(b) \cap \operatorname{Supp}(d)$, and consequently $(d + x) \cdot b \neq 0$. This leads to $a \mathcal{R} b$, a contradiction. □

With this lemma and Proposition 2.4, we recover the well known equality $\mathsf{c}(S) \leq \omega(S)$ for numerical semigroups (see [17]). Lemma 3.1 has another direct consequence in embedding dimension three. Let $S = \langle n_1, n_2, n_3 \rangle$ be a numerical semigroup with embedding dimension three. Let c_i and r_{jk} be as in (1). According to [22, Example 8.23], $\operatorname{Betti}(S) = \{c_1 n_1, c_2 n_2, c_3 n_3\}$ (though it might be the case that $c_i n_i = c_j n_j$ for some $i \neq j$).

Corollary 3.2. *If $S = \langle n_1, n_2, n_3 \rangle$ is a numerical semigroup with embedding dimension three, then for every $i \in \{1, 2, 3\}$ it follows that $\mathsf{Z}(c_i n_i) \setminus \{c_i \mathbf{e}_i\} \subseteq \mathcal{B}(n_i)$. Also, if $r_{ij} \neq 0$ for some $i, j \in \{1, 2, 3\}$ with $i \neq j$, then $c_i \mathbf{e}_i \in \mathcal{B}(n_j)$.*

Proof. If for all i, j, $r_{ij} \neq 0$, from [4, Proposition 5.5] and [12, Corollary 5], $\mathsf{Z}(c_i n_i) = \{c_i \mathbf{e}_i, r_{ij} \mathbf{e}_j + r_{ik} \mathbf{e}_k\}$. The assertion now follows from Lemma 3.1. If to the contrary $r_{ij} = 0$, for some i, j, then we use 2) in [14, Lemma 11] when $\#\operatorname{Betti}(S) = 2$, or [13, Proposition 1 b)] if $\#\operatorname{Betti}(S) = 1$. □

We now focus on the case where S is nonsymmetric with embedding dimension three. Let c_i, r_{ij} be as in Section 2.1. For such an S, it follows that $r_{ij} \neq 0$ for all $i, j \in \{1, 2, 3\}$. Also

from [4, Proposition 5.5] and [12, Corollary 5] $\#\mathsf{Z}(c_in_i) = 2$ and, more precisely,

$$\mathsf{Z}(c_in_i) = \{c_i\mathbf{e}_i, r_{ij}\mathbf{e}_j + r_{ik}\mathbf{e}_k\}.$$

Proposition 3.3. *Let $S = \langle n_1, n_2, n_3\rangle$ be a nonsymmetric numerical semigroup with embedding dimension three. Whenever $\{i, j, k\} = \{1, 2, 3\}$, we have that*

$$\mathcal{B}(n_i) = \{\mathbf{e}_i, c_j\mathbf{e}_j, c_k\mathbf{e}_k, r_{ij}\mathbf{e}_j + r_{ik}\mathbf{e}_k\}.$$

Proof. The inclusion $\{\mathbf{e}_i, c_j\mathbf{e}_j, c_k\mathbf{e}_k, r_{ij}\mathbf{e}_j + r_{ik}\mathbf{e}_k\} \subseteq \mathcal{B}(n_i)$ is a consequence of Corollary 3.2. This also implies that no other element with support of cardinality one belongs to $\mathcal{B}(n_i)$, and that i does not belong to the support of any bullet other than $\mathbf{e}_i$. Assume that there exists $x\mathbf{e}_j + y\mathbf{e}_k \in \mathcal{B}(n_i)$, with $x, y \in \mathbb{N} \setminus \{0\}$. Then $xn_j + yn_k \in n_i + S$, and consequently there exist $a, b, c \in \mathbb{N}$ with $a \neq 0$ such that $xn_j + yn_k = an_i + bn_j + cn_k$. The minimality of $x\mathbf{e}_i + y\mathbf{e}_j$ (this element is a bullet), forces $b = c = 0$. Hence $a \geq c_i$. Notice that $a = c_i$ implies $(x, y) = (r_{ij}, r_{ik})$, since $\mathsf{Z}(c_in_i) = \{c_i\mathbf{e}_i, r_{ij}\mathbf{e}_j + r_{ik}\mathbf{e}_k\}$. If $a > c_i$, then $(a - c_i)n_i + r_{ij}n_j + r_{ik}n_k = xn_j + yn_k$, contradicting the minimality of $x\mathbf{e}_i + y\mathbf{e}_j$, because $r_{ij} \neq 0 \neq r_{ik}$. Hence we have an equality. □

From Proposition 3.3, we deduce this important corollary.

Corollary 3.4. *Assume that $S = \langle n_1, n_2, n_3\rangle$ is a nonsymmetric numerical semigroup with embedding dimension three with $n_1 < n_2 < n_3$. Then $\omega(n_1) \leq \omega(n_3)$.*

Proof. According to Propositions 3.3 and 2.4, $\omega(n_1) = \max\{c_2, c_3, r_{12} + r_{13}\}$ and $\omega(n_3) = \max\{c_1, c_2, r_{31} + r_{32}\}$. Notice also that $c_1 > r_{12} + r_{13}$ and $c_3 < r_{31} + r_{32}$. The proof now follows by considering the possible values of $\omega(n_3)$.

- If $\omega(n_3) = c_1$, then $c_1 \geq c_2$ and $c_1 \geq r_{31} + r_{32} > c_3$. Also $c_1 > r_{12} + r_{13}$, and thus we obtain $c_1 \geq \omega(n_1)$.
- If $\omega(n_3) = c_2$, then $c_2 \geq c_1 > r_{12} + r_{13}$ and $c_2 \geq r_{31} + r_{32} > c_3$. Hence $c_2 \geq \omega(n_1)$.
- If $\omega(n_3) = r_{31} + r_{32}$, then $r_{31} + r_{32} \geq c_1 > r_{12} + r_{13}$, $r_{31} + r_{32} \geq c_2$ and $r_{31} + r_{32} > c_3$. Thus, $r_{31} + r_{32} \geq \omega(n_1)$. □

As an immediate consequence of the corollary we get the following nice result.

Corollary 3.5. *Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical semigroup with embedding dimension three with $n_1 < n_2 < n_3$. If $\omega(n_1) > \omega(n_3)$, then S is symmetric.*

4. The symmetric case

Now, we wish to compute the ω-values of the generators in any symmetric numerical semigroup of embedding dimension three. To do so, we will divide the claim into two cases based on the values of b and c in $\langle ap_1, ap_2, bp_1 + cp_2\rangle$ as illustrated in Theorem 2.1. In particular $n_1 = ap_1$, $n_2 = ap_2$, $n_3 = bp_1 + cp_2$, with $a, b, c \in \mathbb{N}$, $a \geq 2$, $b \geq 2$ and $\gcd(p_1, p_2) = \gcd(a, n_3) = 1$.

It is easy to see that $c_1 = p_2$, $c_2 = p_1$, $c_3 = a$, and that minimal presentation for S is given by $\{((p_2, 0, 0), (0, p_1, 0)), ((b, c, 0), (0, 0, a))\}$ (this is because $S = a\langle p_1 + p_2\rangle + (bp_1 + cp_2)\mathbb{N}$ is a gluing of $\langle p_1, p_2\rangle$ and $\mathbb{N}$; see the proof of [22, Theorem 10.6]) . As a consequence of Corollary 3.2, we obtain $p_2\mathbf{e}_1 \in \mathcal{B}(ap_2)$ and $p_1\mathbf{e}_2 \in \mathcal{B}(ap_1)$. Thus we obtain the following result.

Corollary 4.1. *Let $S = \langle n_1, n_2, n_3\rangle$ be an embedding dimension three symmetric numerical semigroup. Then $m_j(ap_i) = p_i$ for $\{i, j\} = \{1, 2\}$.*

4.1. **The case when S is representable as $\langle ap_1, ap_2, bp_1 + cp_2\rangle$ with $bc \neq 0$.** In this setting, we can again use Corollary 3.2 to obtain $a\mathsf{e}_3 \in \mathcal{B}(n_1) \cap \mathcal{B}(n_2)$. Hence, we obtain the following consequence.

Corollary 4.2. *Suppose that $S = \langle n_1, n_2, n_3\rangle$ is a symmetric numerical semigroup that is representable as $\langle ap_1, ap_2, bp_1 + cp_2\rangle$ with $bc \neq 0$. Then $m_3(ap_1) = m_3(ap_2) = a$.*

Lemma 4.3. *Suppose that $S = \langle n_1, n_2, n_3\rangle$ is a symmetric numerical semigroup that is representable as $\langle ap_1, ap_2, bp_1 + cp_2\rangle$ with $bc \neq 0$.*
(a) $\mathcal{B}(ap_1) = \{(1,0,0), (0,p_1,0), (0,0,a)\}$.
(b) $\mathcal{B}(ap_2) = \{(p_2,0,0), (0,1,0), (0,0,a)\}$.

Proof. We already know that $\{(1,0,0), (0,p_1,0), (0,0,a)\} \subseteq \mathcal{B}(ap_1)$. Assume that $(0,x,y) \in \mathcal{B}(ap_1) \setminus \{(1,0,0), (0,p_1,0), (0,0,a)\}$. Hence $xap_1 + y(bp_1 + cp_2) = uap_1 + vap_2 + w(bp_1 + cp_2)$ for some $u, v, w \in \mathbb{N}$, with $u \neq 0$. As in the proof of Corollary 3.3, the minimality of $(0,x,y)$ implies that $v = w = 0$, and $yap_1 = xap_1 + y(bp_1 + cp_2)$. In particular, $a \mid y(bp_1 + cp_2)$, and as $\gcd(a, n_3) = 1$, it follows that $a \mid y$. This forces $a \leq y$, whence $(0,0,a) \leq (0,x,y)$ which is a contradiction.

The proof of $\mathcal{B}(ap_2) = \{(p_2,0,0), (0,1,0), (0,0,a)\}$ follows in the same way. □

Now, we wish to prove similar results for $bp_1 + cp_2$.

Lemma 4.4. *Suppose that $S = \langle n_1, n_2, n_3\rangle$ is a symmetric numerical semigroup that is representable as $\langle ap_1, ap_2, bp_1 + cp_2\rangle$ with $bc \neq 0$. If*

$$\overline{b} = \max\{x \in \mathbb{N} \setminus \{0\} \mid bp_1 + cp_2 = xp_1 + yp_2, \text{ for some } y \in \mathbb{N} \setminus \{0\}\},$$

and

$$\overline{c} = \max\{y \in \mathbb{N} \setminus \{0\} \mid bp_1 + cp_2 = xp_1 + yp_2, \text{ for some } x \in \mathbb{N} \setminus \{0\}\},$$

then
(a) $m_1(bp_1 + cp_2) = p_2 + \overline{b}$ and
(b) $m_2(bp_1 + cp_2) = p_1 + \overline{c}$.

Proof. (a) Note that the set $\{x \in \mathbb{N} \setminus \{0\} \mid bp_1 + cp_2 = xp_1 + yp_2, \text{ for some } y \in \mathbb{N} \setminus \{0\}\}$ is not empty (and finite) since b and c are both nonzero. Let $c' \in \mathbb{N}$ such that $bp_1 + cp_2 = \overline{b}p_1 + c'p_2$. Then

$$\begin{aligned}(p_2 + \overline{b})(ap_1) - (bp_1 + cp_2) &= ap_1p_2 + a(\overline{b}p_1) - (bp_1 + cp_2)\\ &= ap_1p_2 + a(bp_1 + cp_2 - c'p_2) - (bp_1 + cp_2)\\ &= (a-1)(bp_1 + cp_2) + (p_1 - c')(ap_2).\end{aligned}$$

Note that if $c' > p_1$, then

$$bp_1 + cp_2 = \overline{b}p_1 + c'p_2 = \overline{b}p_1 + (c' - p_1)p_2 + p_1p_2 = (\overline{b} + p_2)p_1 + (c' - p_1)p_2,$$

contradicting the maximality of $\overline{b}$. Therefore, $(p_2 + \overline{b})(ap_1) - (bp_1 + cp_2) = (a-1)(bp_1 + cp2) + (p_1 - c')(ap_2) \in S$ since $p_1 \geq c'$.

Now, note that if

$$\begin{aligned}(p_2 + \overline{b} - 1)(ap_1) - (bp_1 + cp_2) &= (p_2 - 1)ap_1 + a(\overline{b}p_1) - bp_1 - cp_2\\ &= (p_2 - 1)ap_1 + a(bp_1 + cp_2 - c'p_2) - bp_1 - cp_2\\ &= (a-1)(bp_1 + cp_2) + a(p_1p_2 - c'p_2 - p_1)\end{aligned}$$

is in S, then $(a-1)(bp_1+cp_2)+a(p_1p_2-p_1-p_2) = (p_2+\overline{b}-1)(ap_1)-(bp_1+cp_2)+(c'-1)p_2 \in S$. However [22, Remark 10.7] states that $\mathrm{F}(S) = (a-1)(bp_1+cp_2)+a(p_1p_2-p_1-p_2)$, a contradiction. Therefore, $(p_2+\overline{b}-1)(ap_1)-(bp_1+cp_2) \notin S$, so $m_1(bp_1+cp_2) = p_2+\overline{b}$.

The proof of (b) is similar. □

Finally, we will find $\mathcal{B}(bp_1+cp_2)$. Recall that, in our setting, a minimal presentation for S is $\{((p_2,0,0),(0,p_1,0)),((b,c,0),(0,0,a))\}$, and as $bc \neq 0$, the number of Betti elements of S is two. It is easy to deduce (see for instance [14, Lemma 11]) that the set of $\mathcal{R}$-classes of $\mathsf{Z}(a(bp_1+cp_2))$ is $\{\{(0,0,a)\}, \mathsf{Z}(a(bp_1+cp_2))\setminus\{(0,0,a)\}\}$. The second class equals $\{(x,y,0) \in \mathbb{N}^3 \mid a(bp_1+cp_2) = xap_1+yap_2\} = \{(x,y,0) \in \mathbb{N}^3 \mid bp_1+cp_2 = xp_1+yp_2\}$.

Lemma 4.5. *Suppose that $S = \langle n_1,n_2,n_3\rangle$ is a symmetric numerical semigroup that is representable as $\langle ap_1, ap_2, bp_1+cp_2\rangle$ with $bc \neq 0$. Then*

$$\begin{aligned}\mathcal{B}(bp_1+cp_2) =& \{(\mathsf{m}_1(bp_1+cp_2),0,0),(0,\mathsf{m}_2(bp_1+cp_2),0),(0,0,1)\}\\ &\cup \{(x,y,0) \in \mathbb{N}^3 \mid bp_1+cp_2 = xp_1+yp_2, xy \neq 0\}.\end{aligned}$$

Proof. ($\supseteq$) : This inclusion is a direct consequence of Lemma 3.1 and the definition of m_i.

($\subseteq$) : Now, suppose that $(x,y,z) \in \mathcal{B}(bp_1+cp_2)$. If $z \neq 0$, $x = 0$ or $y = 0$, then we are done. Therefore, we may now assume that $(x,y,0) \in \mathcal{B}(bp_1+cp_2)$ with $x \neq 0$ and $y \neq 0$. Then $x(ap_1)+y(ap_2)-(bp_1+cp_2) \in S$, but $(x-1)(ap_1)+y(ap_2)-(bp_1+cp_2) \notin S$ and $x(ap_1)+(y-1)(ap_2)-(bp_1+cp_2) \notin S$. Therefore, there exists $d \in \mathbb{N}$ such that $x(ap_1)+y(ap_2)-(bp_1+cp_2) = d(bp_1+cp_2)$. Since $\gcd(a, bp_1+cp_2) = 1$ and $a \mid (d+1)(bp_1+cp_2)$, $a \mid d+1$. Hence, there exists $d_0 \in \mathbb{N}$ such that $d+1 = ad_0$ and so $xp_1+yp_2 = d_0(bp_1+cp_2)$. Note that if $d_0 \geq 2$, then

$$\begin{aligned}(x-1)(ap_1)+y(ap_2)-(bp_1+cp_2) &= d(bp_1+cp_2)-ap_1\\ &= (ad_0-1)(bp_1+cp_2)-an_1 = (a-1)(bp_1+cp_2)+(d_0-1)a(bp_1+cp_2)-ap_1\\ &= (a-1)(bp_1+cp_2)+(bd_0-b-1)(ap_1)+(cd_0-c)(ap_2) \in S\end{aligned}$$

since $bd_0-b-1 \geq 2b-b-1 = b-1 \geq 0$. This contradicts $(x-1)(ap_1)+y(ap_2)-(bp_1+cp_2) \notin S$. Then $d_0 = 1$, and consequently $xp_1+yp_2 = bp_1+cp_2$. □

Theorem 4.6. *Suppose that $S = \langle n_1,n_2,n_3\rangle$ is a symmetric numerical semigroup that is representable as $\langle ap_1, ap_2, bp_1+cp_2\rangle$ with $bc \neq 0$. Let*

$$\overline{b} = \max\{x \in \mathbb{N}\setminus\{0\} \mid bp_1+cp_2 = xp_1+yp_2, \text{ for some } y \in \mathbb{N}\setminus\{0\}\},$$

and

$$\overline{c} = \max\{y \in \mathbb{N}\setminus\{0\} \mid bp_1+cp_2 = xp_1+yp_2, \text{ for some } x \in \mathbb{N}\setminus\{0\}\},$$

then

(a) $\omega(ap_1) = \max(a,p_1)$,
(b) $\omega(ap_2) = \max(a,p_2)$, *and*
(c) $\omega(bp_1+cp_2) = \max(\overline{b}+p_2, \overline{c}+p_1)$.

Proof. The assertions (a) and (b) follow directly from Proposition 2.4 and Lemma 4.3. For (c), we will divide the claim into two cases.

First suppose that $p_1 < p_2$. Let $c' \in \mathbb{N}$ such that $\overline{b}p_1+c'p_2 = bp_1+cp_2$ and let $(x,y,0) \in \mathbb{N}^3$ with $bp_1+cp_2 = xp_1+yp_2$. Then $(\overline{b}-x)p_1 = (y-c')p_2$, and the maximality of $\overline{b}$ implies $\overline{b}-x \geq 0$. As $\gcd(p_1,p_2) = 1$, $p_2 \mid (\overline{b}-x)$. This means that $x = \overline{b}-dp_2$ for some $d \in \mathbb{N}_0$. Therefore $y = c'+dp_1$, since $(y-c')p_2 = (\overline{b}-x)p_1 = dp_2p_1$. As a result, we see that $x+y = \overline{b}-dp_2+c'+dp_1 \leq \overline{b}+c'$, since $p_1 < p_2$. However, in the proof of Lemma 4.4, we

noted that $c' \leq p_1$. Consequently, $x+y \leq \overline{b}+c' \leq \overline{b}+p_1 < \overline{b}+p_2$. By Proposition 2.4, Lemma 4.4, and Lemma 4.5, the result follows.

The proof for $p_1 > p_2$ is analogous. □

Observe that we have not given a closed formula for *(c)* in the last theorem in terms of a, b, c, p_1 and p_2. As p_1 and p_2 are coprime, $\langle p_1, p_2\rangle$ is a numerical semigroup. Its minimal presentation is $\{((p_2,0),(0,p_1))\}$. Then whenever $k \in \mathbb{N}$ is such that $c-kp_1 \in \mathbb{N}$, the element $(b+kp_2)p_1 + (c-kp_1)p_2 \in \mathsf{Z}(bp_1+cp_2)$. The same holds if $b-kp_2 \in \mathbb{N}$, which yields $(b-kp_2)p_1+(c+kp_1)p_2 \in \mathsf{Z}(bp_1+cp_2)$. The largest k such that $c-kp_1 \in \mathbb{N}\setminus\{0\}$ is c/p_1-1 if p_1 divides c, and $\lfloor \frac{c}{p_1} \rfloor$ otherwise. A similar argument applies for the largest k fulfilling $b-kp_2 \in \mathbb{N}\setminus\{0\}$. From this we deduce that

$$\overline{b} = \begin{cases} b+\left(\frac{c}{p_1}-1\right)p_2 & \text{if } c \bmod p_1 = 0, \\ b+\lfloor \frac{c}{p_1} \rfloor p_2 & \text{otherwise,} \end{cases} \qquad \overline{c} = \begin{cases} c+\left(\frac{b}{p_2}-1\right)p_1 & \text{if } b \bmod p_2 = 0, \\ c+\lfloor \frac{b}{p_2} \rfloor p_1 & \text{otherwise.} \end{cases}$$

4.2. **The case when S is not representable as $\langle ap_1, ap_2, bp_1+cp_2\rangle$ with $bc \neq 0$.** Recall from Theorem 2.1 that $b+c \geq 2$. Thus the hypothesis that $bc=0$ implies that exactly one of b or c is zero. Without loss of generality, suppose that $c=0$. Then $S=\langle ap_1, ap_2, bp_1\rangle$. If $b > p_2$, then $bp_1 = (b-p_2)n_1 + (p_1)n_2$, contradicting the fact that S is not representable in the form above with $b \neq 0$ and $c \neq 0$. Therefore, $b \leq p_2$. Note that $\gcd(a,b)=1$ and $\gcd(a,p_1)=1$ since $\gcd(a,bp_1)=1$. As a result, we see that $S=\langle p_1a, p_1b, p_2a\rangle$ is of the form in Theorem 2.1 since $\gcd(p_1,p_2a)=1$ and $\gcd(a,b)=1$. Note that $p_2 \leq b$ since otherwise, $p_2a=(p_2-b)a+ab$, contradicting the fact that S is not representable in the above form with $b \neq 0$ and $c \neq 0$. Therefore, $b=p_2$, so $S=\langle ap_1, ap_2, p_1p_2\rangle$, where a, p_1, and p_2 are pairwise relatively prime.

Throughout the rest of the section, let $S=\langle ab, ac, bc\rangle$, where a, b, c are pairwise relatively prime integers greater than one. We write $n_1=ab$, $n_2=ac$ and $n_3=bc$.

If we find the ω-values of the generators of S, then by our discussion at the beginning of this section, this will complete our characterization of the ω-values of the generators for *any* symmetric numerical semigroup of embedding dimension three. Moreover, Theorem 4.9 will also cover the case when S is nonsymmetric.

According to [13, Theorem 12], $\mathrm{Betti}(S)=\{abc\}$, and by [13, Proposition 1], each $\mathcal{R}$-class in $\mathsf{Z}(abc)$ is a singleton. Thus, we deduce that the three $\mathcal{R}$-classes of $\mathsf{Z}(abc)$ are $\{(c,0,0)\}$, $\{(0,b,0)\}$ and $\{(0,0,a)\}$. As a direct consequence of Lemma 3.1, we get that $(0,0,a) \in \mathcal{B}(n_1)\cap\mathcal{B}(n_2)$, $(0,b,0) \in \mathcal{B}(n_1)\cap\mathcal{B}(n_3)$ and $(c,0,0) \in \mathcal{B}(n_2)\cap\mathcal{B}(n_3)$. In particular, we obtain the following consequence.

Corollary 4.7. *If $S=\langle ab, ac, bc\rangle$, with a, b, c pairwise relatively prime integers greater than one, then we have the following.*

(a) $m_1(ac)=m_1(bc)=c$.
(b) $m_2(ab)=m_2(bc)=b$.
(c) $m_3(ab)=m_3(ac)=a$.

Lemma 4.8. *Let $S=\langle ab, ac, bc\rangle$, with a, b, c pairwise relatively prime integers greater than one.*

(a) $\mathcal{B}(ab)=\{(1,0,0),(0,b,0),(0,0,a)\}$.
(b) $\mathcal{B}(ac)=\{(c,0,0),(0,1,0),(0,0,a)\}$.
(c) $\mathcal{B}(ab)=\{(c,0,0),(0,b,0),(0,0,1)\}$.

Proof. From Corollary 3.2, we already know that the inclusion $\{(1,0,0),(0,b,0),(0,0,a)\} \subseteq \mathcal{B}(ab)$ holds. Assume that $(0,x,y) \in \mathcal{B}(ab) \setminus \{(0,b,0),(0,0,a)\}$. Then $xac+ybc = uab+vac+wbc$ for some $u,v,w \in \mathbb{N}$ with $u \neq 0$. The minimality of $(0,x,y)$ forces $v = w = 0$. Hence $uab = xac + ybc$. Thus $b \mid xac$ and $a \mid ybc$. Since a, b and c are pairwise prime, this means that $b \mid x$ and $a \mid y$. If $x \neq 0$, then $b < x$, and $(0,b,0) < (0,x,y)$, contradicting the minimality of $(0,x,y)$. The same if $y \neq 0$. In both cases we get a contradiction.

The proofs of (b) and (c) are similar to that of (a). □

We now state a theorem that gives the ω-values of the generators of the numerical semigroup $S = \langle ab, ac, bc\rangle$. The proof follows immediately from Proposition 2.4 and Lemma 4.8.

Theorem 4.9. *Let $S = \langle ab, ac, bc\rangle$, with a, b, c pairwise relatively prime integers greater than one.*

(a) $\omega(ab) = \max(a,b)$.
(b) $\omega(ac) = \max(a,c)$.
(c) $\omega(bc) = \max(b,c)$.

Together, Theorems 4.6 and 4.9 give the ω-values of the generators for any symmetric numerical semigroup with embedding dimension three.

5. Comparing the ω-primality with the catenary degree

Let $S = \langle n_1, n_2, n_3\rangle$ with $n_1 < n_2 < n_3$ be a numerical semigroup with embedding dimension three. Recall that if we set $c_i = \min\{t \in \mathbb{N} \setminus \{0\} \mid tn_i \in \langle n_j, n_j\}$, $\{i,j,k\} = \{1,2,3\}$, then $\mathrm{Betti}(S) = \{c_1n_1, c_2n_2, c_3n_3\}$, and that if S is not symmetric some of these Betti elements coincide. We also know that $\mathsf{c}(S) \leq \omega(S)$.

We say that S is *uniquely presented* if for every two minimal presentations τ and τ' and every $(a,b) \in \tau$, either $(a,b) \in \tau'$ or $(b,a) \in \tau'$ (that is, the minimal presentation is unique up to rearranging the pairs in the presentation). We will make use of this concept in the proof of the following theorem. It is well known that if S is not symmetric, then it is uniquely presented.

Theorem 5.1. *Let $S = \langle n_1, n_2, n_3\rangle$ with $n_1 < n_2 < n_3$ be a numerical semigroup with embedding dimension three. For $\{i,j,k\} = \{1,2,3\}$, set $c_i = \min\{t \in \mathbb{N} \setminus \{0\} \mid tn_i \in \langle n_j, n_j\rangle\}$.*

(a) If $\#\,\mathrm{Betti}(S) = 3$, then $\omega(S) = \mathsf{c}(S)$.
(b) If $c_1n_1 = c_2n_2 \neq c_3n_3$, then $\mathsf{c}(S) < \omega(S)$.
(c) If $c_1n_1 = c_3n_3 \neq c_2n_2$, then $\mathsf{c}(S) < \omega(S)$.
(d) If $c_1n_1 \neq c_2n_2 = c_3n_3$ and $c_2n_2 \mid c_1n_1$, then $\mathsf{c}(S) = \omega(S)$.
(e) If $c_1n_1 = c_2n_2 = c_3n_3$, then $\mathsf{c}(S) = \omega(S)$.

Proof. We will use $\overline{b}$ and $\overline{c}$ as defined in Theorem 4.6:

$$\overline{b} = \max\{x \in \mathbb{N} \setminus \{0\} \mid bp_1 + cp_2 = xp_1 + yp_2, \text{ for some } y \in \mathbb{N} \setminus \{0\}\},$$

$$\overline{c} = \max\{y \in \mathbb{N} \setminus \{0\} \mid bp_1 + cp_2 = xp_1 + yp_2, \text{ for some } x \in \mathbb{N} \setminus \{0\}\}.$$

(a) This is a consequence of [4, Corollary 5.8].
(b) In this setting, $n_1 = ap_1$, $n_2 = ap_2$ and $n_3 = bp_1 + cp_2$, with p_1, p_2, a, b and c as in Theorem 2.1. As we previously mentioned in the second paragraph of Section 4, it easily follows that $c_1 = p_2$, $c_2 = p_1$ and $c_3 = a$. Notice also that $c_1 > c_2$. Let R be the $\mathcal{R}$-class of $\mathsf{Z}(c_3n_3)$ not containing $(0,0,c_3)$. From Theorem 2.2 we know that the catenary degree of S is $\mathsf{c}(S) = \max\{c_1, \min\{r+s \mid (r,s,0) \in R\}\} = \max\{p_2, \min\{b+c \mid bp_1 + cp_2 = n_3\}\}$. We also know that the ω-primality of S is $\omega(S) = \max\{a, p_2 + \overline{b}, p_1 + \overline{c}\}$ (Theorem 4.6).

Observe that if S is uniquely presented we easily derive that $\overline{b} < p_2$ and $\overline{c} < p_1$. From $\overline{c} < p_1$ we obtain $\overline{c} < p_1 < p_2$ and thus $\min\{b+c \mid bp_1 + cp_2 = n_3\} = \overline{b} + \overline{c} < \overline{b} + p_2$. We have $\mathsf{c}(S) = \max\{p_2, \overline{b} + \overline{c}\} < \overline{b} + p_2 \leq \omega(S)$.

If S is not uniquely presented, let c' be such that $(\overline{b}, c', 0) \in \mathsf{Z}(an_3)$ fulfills $\max\{b+c \mid bp_1 + cp_2 = n_3\} = \overline{b} + c'$. From $\min\{b+c \mid bp_1 + cp_2 = n_3\} < \max\{b+c \mid bp_1 + cp_2 = n_3\} = \overline{b} + c'$, and $p_2 < p_2 + \overline{b}$, we obtain $\mathsf{c}(S) < \omega(S)$.

(c) As in the preceding case, we obtain $n_1 = ap_1$, $n_3 = ap_2$ and $n_2 = bp_1 + cp_2$, and $c_1 = p_2$, $c_2 = a$ and $c_3 = p_1$. Let R be the $\mathcal{R}$-class of $\mathsf{Z}(c_2 n_2)$ not containing $(0, c_2, 0)$. From Theorem 2.2 we know that the catenary degree of S is $\mathsf{c}(S) = \max\{c_1, c_2, \min\{r+s \mid (r,0,s) \in R\}\} = \max\{p_2, a, \min\{b+c \mid bp_1 + cp_2 = n_2\}\}$. As a consequence of Theorem 4.6, $\omega(S) = \max\{a, p_2 + \overline{b}, p_1 + \overline{c}\}$.

Observe that if S is uniquely presented we obtain $\overline{c} < p_1 < p_2$ and then $\overline{b} + \overline{c} < \overline{b} + p_2$. Furthermore $n_1 = ap_1 < n_2 = \overline{b}p_1 + \overline{c}p_2 < \overline{b}p_1 + p_1p_2 = (\overline{b} + p_2)p_1$, so $\mathsf{c}(S) = \max\{p_2, a, \overline{b} + \overline{c}\} < \overline{b} + p_2 \leq \omega(S)$.

If S is not uniquely presented, from $\min\{b+c \mid bp_1 + cp_2 = n_2\} < \max\{b+c \mid bp_1 + cp_2 = n_2\} = \overline{b} + c'$, with c' such that $(\overline{b}, 0, c') \in \mathsf{Z}(an_2)$. From the proof of Lemma 4.4, we know that $c' \leq p_1$, which is smaller than p_2. Also $p_2 < p_2 + \overline{b}$. Moreover, as in the preceding paragraph, $n_1 = ap_1 < (\overline{b} + p_2)p_1$ and then $a < \overline{b} + p_2$. Furthermore, if b' is such that $(b', 0, \overline{c}) \in \mathsf{Z}(n_2)$, we have $b' + \overline{c} < \overline{b} + c' < \overline{b} + p_2$. Thus we obtain $\mathsf{c}(S) = \max\{p_2, a, b' + \overline{c}\} < \overline{b} + p_2 \leq \omega(S)$.

(d) This is a consequence of the main result in [14].

(e) Follows from [13, Theorem 19]. □

Remark 5.2. In the case $c_1n_1 \neq c_2n_2 = c_3n_3$, we have $c_1 = a, c_2 = p_2, c_3 = p_1$. We now consider several examples to illustrate that the statement of Theorem 5.1 is best possible.

We know from Theorem 2.2 that $\mathsf{c}(S) = \max\{c_1, c_2\} = \max\{a, p_2\}$. Observe, we have $(b+c)p_1 < bp_1 + cp_2 < ap_1$ and then $b + c < a$. In light of Theorem 4.6, $\omega(S) = \max\{a, \overline{b} + p_2, \overline{c} + p_1\}$. So we can say that if $a \geq \max\{\overline{b} + p_2, \overline{c} + p_1\}$, then $\mathsf{c}(S) = \omega(S) = a$, while if $a < \max\{\overline{b} + p_2, \overline{c} + p_1\}$, then $\mathsf{c}(S) < \omega(S)$.

If $an_1 \in \langle n_2 \rangle$, $\omega(S) = \max\{a, p_1 + \overline{c}, \overline{b}\}$, then for $a \geq \max\{\overline{b}, \overline{c} + p_1\}$ we get $\mathsf{c}(S) = \omega(S) = a$ and otherwise $\mathsf{c}(S) < \omega(S)$. If $an_1 \in \langle n_3 \rangle$, $\omega(S) = \max\{a, p_2 + \overline{b}\}$ and then, for $a \geq p_2 + \overline{b}$, $\mathsf{c}(S) = \omega(S) = a$, while $\mathsf{c}(S) < \omega(S)$ otherwise. We have examples of the two situations, in both cases uniquely and nonuniquely presented. The examples have been obtained by using the idea of gluing and the package [9]. For instance, if we want to produce an example $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < b_2 < n_3$ and $c_1n_1 \neq c_2n_2 = c_3n_3$, then we do the following. We start with the semigroup generated by $T = \langle p_1, p_2 \rangle$ with $p_1 < p_2$ and $\gcd(p_1, p_2) = 1$. We know that a minimal presentation for T is $\{((p_2, 0), (p_1, 0))\}$ and $\mathrm{Betti}(T) = \{p_1p_2\}$. Next we take $a, b, c \in \mathbb{N} \setminus \{0\}$ such that $ap_1 > bp_1 + cp_2$ (and thus $bp_1 + cp_2 < ap_1 < ap_2$), and $\gcd(a, bp_1 + cp_2) = 1$. Define $n_1 = bp_1 + bp_2$, $n_2 = ap_1$ and $n_3 = bp_2$. If follows that S is the gluing of T and $\mathbb{N}$ (see [22, Chapter 8] for the definition of gluing), and that a minimal presentation for S is given by $\{((a,0,0),(0,b,c)),((0,p_2,0),(0,0,p_1))\}$ ([22, Theorem 9.2]).

(Ex 1) $S = \langle 19, 350, 490 \rangle$ is uniquely presented and $c_1n_1 \neq c_2n_2 = c_3n_3$. We have $\mathsf{c}(S) = \omega(S) = 70$. Here $S = (5 + 2 \times 7)\mathbb{N} + 70\langle 5, 7 \rangle$; $p_1 = 5$, $p_2 = 7$, $a = 70$, $b = 1$ and $c = 2$.

(Ex 2) $S = \langle 17, 40, 56 \rangle$ is uniquely presented and $c_1n_1 \neq c_2n_2 = c_3n_3$. We have $\mathsf{c}(S) = 8 < \omega(S) = 9$.

(Ex 3) $S = \langle 75, 130, 234 \rangle$ is not uniquely presented and $c_1n_1 \neq c_2n_2 = c_3n_3$. We have $\mathsf{c}(S) = \omega(S) = 26$.

(Ex 4) $S = \langle 62, 63, 147\rangle$ is not uniquely presented and $c_1n_1 \neq c_2n_2 = c_3n_3$. We have $\mathsf{c}(S) = 21 < \omega(S) = 23$.

A particular case of the situation in which $a \geq \max\{\overline{b} + p_2, \overline{c} + p_1\}$ is when c_2n_2 divides c_1n_1. Notice that $a \geq \{\overline{b} + p_2, \overline{c} + p_1\}$ is a necessary condition for $c_2n_2 \mid c_1n_1$ but not sufficient, as we can see in (Ex 1). Indeed in that case $a = 70 \geq \max\{8, 7\}$ but $c_2n_2 = 7 \cdot 350$ does not divide $c_1n_1 = 70 \cdot 19 = 1330$.

6. The inequalities involving the ω-values of the generators

As in the preceding section, let S be minimally generated by $\{n_1, n_2, n_3\}$ with $n_1 < n_2 < n_3$. As we mentioned in the Introduction, there are 13 possible inequalities involving the $\omega(n_i)$.

Theorem 6.1. *Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical semigroup with embedding dimension three with $n_1 < n_2 < n_3$. When $\{i, j, k\} = \{1, 2, 3\}$, set $c_i = \min\{t \in \mathbb{N} \setminus \{0\} \mid tn_i \in \langle n_j, n_j\rangle\}$.*

(a) If $\#\operatorname{Betti}(S) = 3$, then $\omega(n_1) \leq \omega(n_3)$.
(b) If $c_1n_1 = c_2n_2 \neq c_3n_3$, then $\omega(n_1) \leq \omega(n_2) < \omega(n_3)$.
(c) If $c_1n_1 = c_3n_3 \neq c_2n_2$, then $\omega(n_1) \leq \omega(n_3) < \omega(n_2)$.
(d) If $c_1n_1 = c_2n_2 = c_3n_3$, then $\omega(n_1) < \omega(n_2) = \omega(n_3)$.

Proof. (a) This is just Corollary 3.4.

(b) From Theorem 4.6, we get that $\omega(n_1) = \max\{a, p_1\}$, $\omega(n_2) = \max\{a, p_2\}$, and $\omega(n_3) = \max\{\overline{b} + p_2, \overline{c} + p_1\}$. Obviously, $\overline{b} + p_2 > p_2$ and $\overline{c} + p_1 > p_1$, so it is sufficient to show that $\omega(n_3) > a$. Notice that $ap_2 = n_2 < n_3 = b'p_1 + \overline{c}p_2 < (b' + \overline{c})p_2$, so $a < b' + \overline{c} \leq \omega(n_3)$.

(c) From Theorem 4.6, we get that $\omega(n_1) = \max\{a, p_1\}$, $\omega(n_3) = \max\{a, p_2\}$, and $\omega(n_2) = \max\{\overline{b} + p_2, \overline{c} + p_1\}$. Obviously, $\overline{b} + p_2 > p_2$ and $\overline{c} + p_1 > p_1$, so it is sufficient to show that $\omega(n_2) > a$. Notice that $ap_1 = n_1 < n_2 = \overline{b}p_1 + c'p_2 \leq \overline{b}p_1 + p_1p_2$, so $a < \overline{b} + p_2 \leq \omega(n_2)$.

(d) By Theorem 4.9, $\omega(n_1) = \max\{a, b\} = b$, $\omega(n_2) = \max\{a, c\} = c$, and $\omega(n_3) = \max\{b, c\} = c$, so we are done. □

6.1. **Orderings when $\omega(n_1) > \omega(n_3)$.** Suppose that $\omega(n_1) > \omega(n_3)$. Then by Corollary 3.5, S is symmetric. Therefore, by Theorem 6.1, we must have $c_1n_1 \neq c_2n_2 = c_3n_3$ which implies, by Theorem 4.6, that $\omega(n_1) = \max\{\overline{b} + p_2, \overline{c} + p_1\}$, $\omega(n_2) = \max\{a, p_1\}$ and $\omega(n_3) = \max\{a, p_2\}$. Since $ap_1 = n_2 < n_3 = ap_2$, this means that $\omega(n_2) \leq \omega(n_3)$. Note that $\overline{b}+p_2 > p_2$ and $\overline{c} + p_1 > p_1$, and thus

(1) if $a < p_2$, then $\omega(n_2) < \omega(n_3) < \omega(n_1)$,
(2) if $p_2 \leq a < \omega(n_1)$, then $\omega(n_2) = \omega(n_3) < \omega(n_1)$,
(3) if $a \geq \omega(n_1)$, then $\omega(n_1) \leq \omega(n_2) = \omega(n_3)$.

Hence, we have verified the following conjecture from [8].

Theorem 6.2. *Let $S = \langle n_1, n_2, n_3\rangle$ be a numerical semigroup with embedding dimension three. The sequence $\omega(n_1)$, $\omega(n_2)$, and $\omega(n_3)$ does not satisfy any of the following three orderings:*

- $\omega(n_1) > \omega(n_2) > \omega(n_3)$;
- $\omega(n_1) = \omega(n_2) > \omega(n_3)$;
- $\omega(n_1) < \omega(n_2)$, $\omega(n_2) > \omega(n_3)$, $\omega(n_3) < \omega(n_1)$.

The interested reader is directed to the tables in both [8] and [3] for examples illustrating each of the remaining 10 possible sets of inequalities.

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Sam Houston State University, Department of Mathematics and Statistics, Box 2206, Huntsville, TX 77341-2206
E-mail address: scott.chapman@shsu.edu

Departamento de Álgebra and CITIC-UGR, Universidad de Granada, E-18071 Granada, España
E-mail address: pedro@ugr.es

Tufts University, Department of Mathematics, 503 Boston Avenue, Bromfield-Pearson, Room 103, Medford, MA 02155
E-mail address: trippzac@gmail.com

Dipartimento di Matematica e Informatica, Viale A. Doria, 6 - I 95125, Catania, Italia
E-mail address: violacaterina@gmail.com