Research Article

Aureliano M. Robles-Pérez* and José Carlos Rosales

Numerical semigroups in a problem about cost-effective transport

DOI: 10.1515/forum-2015-0123
Received June 26, 2015; revised March 31, 2016

Abstract: Let $\mathbb{N}$ be the set of nonnegative integers. A problem about how to transport profitably an organized group of persons leads us to study the set T formed by the integers n such that the system of inequalities, with nonnegative integer coefficients,

$$a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p$$

has at least one solution in $\mathbb{N}^p$. We will see that $T \cup \{0\}$ is a numerical semigroup. Moreover, we will show that a numerical semigroup S can be obtained in this way if and only if $\{a + b - 1, a + b + 1\} \subseteq S$, for all $a, b \in S \setminus \{0\}$. In addition, we will demonstrate that such numerical semigroups form a Frobenius variety and we will study this variety. Finally, we show an algorithmic process in order to compute T.

Keywords: Diophantine inequalities, submonoids, numerical semigroups, non-homogeneous patterns, Frobenius varieties, trees, profitable transport

MSC 2010: 20M14, 11D07, 11D75

Communicated by: Manfred Droste

1 Introduction

Certain travel agency, which specializes in city tours for organized groups, uses small and large buses in its services. The small bus seating capacity is 30 passengers and a large bus is for up to 50 passengers. Moreover, the hire of each bus costs 310 euros, for the small ones, and 480 euros, for the large ones. Finally, the price of the city tour is 10 euros per passenger, but free for the responsible leader of the group.

At above situation, we propose the following problem: if a group of $n + 1$ persons (including the leader) wants to use its services, is the travel agency going to get profits?

It is clear that a tour is profitable if and only if there exist $x, y \in \mathbb{N}$ (where $\mathbb{N}$ is the set of nonnegative integers) such that

$$\left.\begin{aligned} n + 1 &\le 30x + 50y, \\ 10n &> 310x + 480y. \end{aligned}\right\} \tag{1.1}$$

Simplifying the second inequality of (1.1), we have the equivalent system (of strict inequalities)

$$\left.\begin{aligned} n &< 30x + 50y, \\ n &> 31x + 48y. \end{aligned}\right\} \tag{1.2}$$

And, thereby, $\{n \in \mathbb{N} \mid (1.2) \text{ has a solution in } \mathbb{N}^2\}$ is the set of nonnegative integers that give us an affirmative answer to the proposed problem.

***Corresponding author: Aureliano M. Robles-Pérez:** Departamento de Matemática Aplicada, Universidad de Granada, 18071-Granada, Spain, e-mail: arobles@ugr.es. http://orcid.org/0000-0003-2596-1249
José Carlos Rosales: Departamento de Álgebra, Universidad de Granada, 18071-Granada, Spain, e-mail: jrosales@ugr.es. http://orcid.org/0000-0003-3353-4335

The above problem can be generalized in the following way. Let us take the system of strict inequalities

$$\left.\begin{aligned} n &< b_1x_1 + \cdots + b_px_p, \\ n &> a_1x_1 + \cdots + a_px_p, \end{aligned}\right\} \quad \text{with } a_1, \ldots, a_p, b_1, \ldots, b_p \in \mathbb{N}. \tag{1.3}$$

Our first aim is to study the structure of the set $\{n \in \mathbb{N} \mid (1.3) \text{ has solution in } \mathbb{N}^p\}$. In order to do it, we need to introduce some concepts and notation.

Let M be a submonoid of $(\mathbb{N}^2, +)$ (that is, a subset of $\mathbb{N}^2$ that is closed under addition and that contains the zero element $(0, 0)$). We will say that a positive integer n is *bounded by M* if there exists $(a, b) \in M$ such that $a < n < b$. We will denote by $\mathrm{A}(M) = \{n \in \mathbb{N} \mid n \text{ is bounded by } M\}$.

A *numerical semigroup* is a submonoid S of $(\mathbb{N}, +)$ such that $\gcd(S) = 1$ (or, equivalently, $\mathbb{N} \setminus S$ is finite).

It is clear that, if M is a submonoid of $(\mathbb{N}^2, +)$ such that $\mathrm{A}(M)$ is not empty, then $\mathrm{A}(M) \cup \{0\}$ is a numerical semigroup. This fact allows us to give the following definition.

Definition 1.1. A numerical semigroup S is a *numerical $\mathcal{A}$-semigroup* if there exists M, submonoid of $(\mathbb{N}^2, +)$, such that $S = \mathrm{A}(M) \cup \{0\}$.

In Section 2 we will show that a numerical semigroup S is a numerical $\mathcal{A}$-semigroup if and only if

$$\{x + y - 1, x + y + 1\} \subseteq S \quad \text{for all } x, y \in S \setminus \{0\}.$$

From the proof of this result, we will state that, if $a_1, b_1, \ldots, a_p, b_p \in \mathbb{N}$, then the set

$$\{n \in \mathbb{N} \mid a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p \text{ has solution in } \mathbb{N}^p\} \cup \{0\}$$

is a numerical $\mathcal{A}$-semigroup or the trivial set $\{0\}$. Moreover, we will see that every numerical $\mathcal{A}$-semigroup can be obtained in this way.

As we said above, if S is a numerical semigroup, then $\mathbb{N} \setminus S$ is finite. Thus we can define a notable invariant of S. Namely, the *Frobenius number* of S, denoted by $\mathrm{F}(S)$, is the greatest integer that does not belong to S (see [8]).

A *Frobenius variety* (see [11]) is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfils the following conditions:

(i) if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$,
(ii) if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.

We will denote by $\mathcal{A} = \{S \mid S \text{ is a numerical } \mathcal{A}\text{-semigroup}\}$. In Section 3 we will show that $\mathcal{A}$ is a Frobenius variety. This fact, together with several results of [11], enables us to arrange the elements of $\mathcal{A}$ in a tree $\mathrm{G}(\mathcal{A})$ with root $\mathbb{N}$. It is clear that we can recursively build a tree from its root if we know the children of each vertex. This observation will lead us to characterize the children of a numerical $\mathcal{A}$-semigroup in $\mathrm{G}(\mathcal{A})$ and give an algorithm for recursively building $\mathcal{A}$.

In Section 4 we will see that, if X is a non-empty set of positive integers, then there exists the smallest numerical $\mathcal{A}$-semigroup, $\mathcal{A}(X)$, that contains X. We will show that

$$\mathcal{A} = \{\mathcal{A}(X) \mid X \text{ is a non-empty finite set of positive integers}\}.$$

Moreover, we will give, in explicit form, the elements of $\mathcal{A}(X)$ and we will design an algorithm to compute $\mathcal{A}(X)$ starting from X.

Let $a = (a_1, \ldots, a_p)$, $b = (b_1, \ldots, b_p) \in \mathbb{N}^p$ and let $\mathrm{S}(a, b)$ be the set

$$\mathrm{S}(a, b) = \{n \in \mathbb{N} \mid a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p \text{ for some } x_1, \ldots, x_p \in \mathbb{N}^p\}.$$

Our main aim in Section 5 will be to give an algorithmic procedure that allows us to compute $\mathrm{S}(a, b)$ starting from a and b. To achieve this goal, we will use the algorithm seen in Section 4.

The *multiplicity* of a numerical semigroup S, denoted by $\mathrm{m}(S)$, is the least positive integer that belongs to S. In Section 6 we will study the set $\mathcal{A}_m$ of all numerical $\mathcal{A}$-semigroups with multiplicity m. We will see that such a set is finite and has maximum and minimum with respect to the inclusion order. We will denote by $\Delta(m)$ and $\Theta(m)$ the maximum and the minimum, respectively, of $\mathcal{A}_m$. Moreover, we will see that $\mathcal{A}_m$ can be arranged in a tree $\mathrm{G}(\mathcal{A}_m)$ with root $\Delta(m)$ and, consequently, we will be able to show an algorithm in order to build $\mathcal{A}_m$.

If S is a numerical semigroup, then the *genus* of S, denoted by $\mathrm{g}(S)$, is the cardinality of $\mathbb{N} \setminus S$. We will give formulas for the Frobenius number and the genus of $\Delta(m)$ and $\Theta(m)$ and, in addition, we will get a formula for the height of the tree $\mathrm{G}(\mathcal{A}_m)$.

In order to justify further the study of numerical $\mathcal{A}$-semigroups, we finish this introduction making reference to two papers that also lead us to this class of numerical semigroups.

On the one hand, using the nomenclature of [3], a numerical $\mathcal{A}$-semigroup is a numerical semigroup that admits simultaneously the non-homogeneous patterns $x_1 + x_2 + 1$ and $x_1 + x_2 - 1$. Indeed, as a consequence of [3, Example 6.5], numerical $\mathcal{A}$-semigroups can be characterized as those numerical semigroups such that the maximum and minimum elements in each interval of non-gaps are minimal generators.

On the other hand, a (v, b, r, k)-configuration (see [4]) is a connected bipartite graph with v vertices on one side, each of them of degree r, and b vertices on the other side, each of them of degree k, and with no cycle of length 4. A (v, b, r, k)-configuration can also be seen as a combinatorial configuration (see [15]) with v points, b lines, r lines through every point and k points on every line. It is said that the tuple (v, b, r, k) is configurable if a (v, b, r, k)-configuration exists. In [4] was shown that, if (v, b, r, k) is configurable, then $vr = bk$ and, consequently, there exists d such that $v = d\frac{k}{\gcd(\{r,k\})}$ and $b = d\frac{r}{\gcd(\{r,k\})}$. The main result of [4] states that, if k, r are integers greater than or equal to 2, then

$$S_{(r,k)} = \left\{ d \in \mathbb{N} \;\middle|\; \left(d\frac{k}{\gcd(\{r,k\})}, d\frac{r}{\gcd(\{r,k\})}, r, k \right) \text{ is configurable} \right\}$$

is a numerical semigroup. Moreover, in [15] it was proved that, if a configuration is balanced (that is, $r = k$), then $\{x + y - 1, x + y + 1\} \subseteq S_{(r,r)}$, for all $x, y \in S_{(r,r)} \setminus \{0\}$. Therefore, $S_{(r,r)}$ is a numerical $\mathcal{A}$-semigroup.

Remark 1.2. An alternative problem would be to consider that all group members (including the leader) have to pay for the tour. If such is the case, the studied general system will become the next one.

$$\left. \begin{aligned} n &\le b_1x_1 + \cdots + b_px_p, \\ n &> a_1x_1 + \cdots + a_px_p. \end{aligned} \right\} \tag{1.4}$$

If $\mathcal{B}$ denotes the family of numerical semigroups associated to this new problem, then the elements of $\mathcal{B}$ satisfy the non-homogeneous pattern $x_1 + x_2 - 1$. This family has been studied in [12] at a different context. In fact, in [12] we have analogous results to the obtained ones in Sections 2, 3, and 4 of the current manuscript. Moreover, at the end of Sections 5 and 6, we include two observations (see Remarks 5.10 and 6.10) where we suggest how to get the results of these sections for the family $\mathcal{B}$. Finally, observe that the numerical semigroup $S = \langle 2, 5 \rangle \in \mathcal{B}$ is not an $\mathcal{A}$-semigroup. Therefore $\mathcal{A} \subsetneq \mathcal{B}$.

Remark 1.3. Let us consider the inequalities

$$a_1x_1 + \cdots + a_px_p \le n < b_1x_1 + \cdots + b_px_p. \tag{1.5}$$

The non-homogeneous pattern associated to them is $x_1 + x_2 + 1$. In [9] we studied the family of $\mathcal{PL}$-semigroups, which is the family of numerical semigroups fulfil such pattern. It is easy to check that $S = \langle 3, 4 \rangle$ is a $\mathcal{PL}$-semigroup but not a numerical $\mathcal{A}$-semigroup. Therefore, the family $\mathcal{A}$ is strictly contained in the family of $\mathcal{PL}$-semigroups. At last, similar ideas to those seen in Remarks 5.10 and 6.10 allow us to solve the problem corresponding to (1.5).

Remark 1.4. In order to show all the possibilities, let us consider the inequalities

$$a_1x_1 + \cdots + a_px_p \le n \le b_1x_1 + \cdots + b_px_p. \tag{1.6}$$

This case corresponds with the family of all numerical semigroups. Thus, we can consider that it is the trivial case (see [14] for more details). Observe that to achieve the correct answer for the new problem, we only need to take $b_i \ge a_i$ and intervals of the type $[a_i, b_i]$ in Theorem 5.8. Therefore, Propositions 5.4 and 5.6 will be omitted in this situation.

2 Numerical $\mathcal{A}$-semigroups

Let us recall that, if M is a submonoid of $(\mathbb{N}^2, +)$, then

$$\mathrm{A}(M) = \{n \in \mathbb{N} \mid a < n < b \text{ for some } (a, b) \in M\}.$$

Lemma 2.1. *Let M be a submonoid of $(\mathbb{N}^2, +)$. If $x, y \in \mathrm{A}(M)$, then we have $\{x + y - 1, x + y, x + y + 1\} \subseteq \mathrm{A}(M)$.*

Proof. If $x, y \in \mathrm{A}(M)$, then there exist $(a, b), (c, d) \in M$ such that $a < x < b$ and $c < y < d$. Since $(a+c, b+d) = (a, b) + (c, d) \in M$ and $a + c < x + y - 1 < x + y < x + y + 1 < b + d$, we conclude that

$$\{x + y - 1, x + y, x + y + 1\} \subseteq \mathrm{A}(M). \qquad \square$$

The next result justifies Definition 1.1.

Proposition 2.2. *Let M be a submonoid of $(\mathbb{N}^2, +)$. If $\mathrm{A}(M)$ is non-empty, then $\mathrm{A}(M) \cup \{0\}$ is a numerical semigroup.*

Proof. From Lemma 2.1, we easily deduce that $\mathrm{A}(M) \cup \{0\}$ is a submonoid of $(\mathbb{N}, +)$. In order to finish the proof, it is enough to show that $\gcd(\mathrm{A}(M)) = 1$. In fact, if $x \in \mathrm{A}(M)$, then we have that $2x + 1 \in \mathrm{A}(M)$. Since $\gcd(\{x, 2x + 1\}) = 1$, we can assert that $\gcd(\mathrm{A}(M)) = 1$. $\square$

From Definition 1.1, a numerical semigroup S is a numerical $\mathcal{A}$-semigroup if there exists M, submonoid of $(\mathbb{N}^2, +)$, such that $S = \mathrm{A}(M) \cup \{0\}$. Let us observe that there exist numerical semigroups that are not numerical $\mathcal{A}$-semigroups. In effect, by Lemma 2.1, we know that, if S is a numerical $\mathcal{A}$-semigroup and $x, y \in S \setminus \{0\}$, then $\{x + y - 1, x + y + 1\} \subseteq S$. Thus, the numerical semigroup $S = \{0, 2, 4, 6, \rightarrow\}$ (where the symbol $\rightarrow$ means that every integer greater than 6 belongs to S) is not a numerical $\mathcal{A}$-semigroup because $2 + 4 - 1 \notin S$.

Let X be a non-empty subset of a commutative monoid $(\mathfrak{M}, +)$. The monoid generated by X, denoted by $\langle X \rangle$, is the smallest (with respect to the set inclusion) submonoid of $(\mathfrak{M}, +)$ that contains X. Moreover, (see [13]) we know that

$$\langle X \rangle = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N} \setminus \{0\},\ x_1, \ldots, x_n \in X,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}.$$

If $\mathfrak{M} = \langle X \rangle$, then we say that X is a *system of generators* of $\mathfrak{M}$ or, equivalently, that $\mathfrak{M}$ is generated by X.

Let A, B be subsets of $\mathbb{N}$. As usual, we define $A + B = \{a + b \mid a \in A,\ b \in B\}$. Let x, y be nonnegative integers such that $x < y$. We will denote by $]x, y[$ the set $\{n \in \mathbb{N} \mid x < n < y\}$.

Lemma 2.3. *Let S be the numerical semigroup generated by the set of positive integers $\{n_1, \ldots, n_p\}$. Let us assume that $\{x + y - 1, x + y + 1\} \subseteq S$, for every $x, y \in S \setminus \{0\}$. If $\lambda_1, \ldots, \lambda_p, x \in \mathbb{N}$ and*

$$\lambda_1(n_1 - 1) + \cdots + \lambda_p(n_p - 1) < x < \lambda_1(n_1 + 1) + \cdots + \lambda_p(n_p + 1),$$

then $x \in S$.

Proof. We are going to use induction over $\Lambda = \lambda_1 + \cdots + \lambda_p$. If $\Lambda = 0$, then $\lambda_1 = \cdots = \lambda_p = 0$ and there does not exist an integer x such that $0 < x < 0$. If $\Lambda = 1$, then there exists $i \in \{1, \ldots, p\}$ such that $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \in \{1, \ldots, p\} \setminus \{i\}$. Thus, $n_i - 1 < x < n_i + 1$ and, consequently, $x = n_i \in S$.

Now, let us assume that $\Lambda \geq 2$ and let $i \in \{1, \ldots, p\}$ such that $\lambda_i \neq 0$. By hypothesis of induction, if x is an integer such that

$$\lambda_1(n_1 - 1) + \cdots + (\lambda_i - 1)(n_i - 1) + \cdots + \lambda_p(n_p - 1) < x < \lambda_1(n_1 + 1) + \cdots + (\lambda_i - 1)(n_i + 1) + \cdots + \lambda_p(n_p + 1),$$

then $x \in S$. Therefore,

$$\left]\sum_{k=1}^{p} \lambda_k(n_k - 1) - (n_i - 1), \sum_{k=1}^{p} \lambda_k(n_k + 1) - (n_i + 1)\right[+ \{n_i\} + \{-1, 0, 1\} \subseteq S,$$

and the conclusion is clear. $\square$

We say that a commutative monoid $(\mathfrak{M}, +)$ is *finitely generated* if there exists a finite set X such that $\mathfrak{M} = \langle X \rangle$. It is well known (see [14]) that the submonoids of $(\mathbb{N}, +)$ (in particular, the numerical semigroups) are finitely generated.

In the following result we give a characterization of numerical $\mathcal{A}$-semigroups.

Theorem 2.4. *Let S be a numerical semigroup. The following conditions are equivalent.*
(1) *S is a numerical $\mathcal{A}$-semigroup.*
(2) *If $x, y \in S \setminus \{0\}$, then $\{x + y - 1, x + y + 1\} \subseteq S$.*

Proof. (1) $\Rightarrow$ (2) It is an immediate consequence of Lemma 2.1.

(2) $\Rightarrow$ (1) Let us consider a set of positive integers, $\{n_1, \dots, n_p\}$, such that $S = \langle \{n_1, \dots, n_p\} \rangle$. Let M be the submonoid of $(\mathbb{N}^2, +)$ that is generated by $\{(n_1 - 1, n_1 + 1), \dots, (n_p - 1, n_p + 1)\}$. It is clear that $\{n_1, \dots, n_p\} \subseteq \mathrm{A}(M)$ and, by applying Proposition 2.2, that $S \subseteq \mathrm{A}(M) \cup \{0\}$.

Let us see the other inclusion. If $x \in \mathrm{A}(M)$, then there exists $(a, b) \in M$ such that $a < x < b$. Since $(a, b) \in M$, there exist $\lambda_1, \dots, \lambda_p \in \mathbb{N}$ such that $(a, b) = \lambda_1(n_1 - 1, n_1 + 1) + \cdots + \lambda_p(n_p - 1, n_p + 1)$. Therefore, x is an integer such that $\lambda_1(n_1 - 1) + \cdots + \lambda_p(n_p - 1) < x < \lambda_1(n_1 + 1) + \cdots + \lambda_p(n_p + 1)$. From Lemmas 2.1 and 2.3, we have that $x \in S$. □

In [13, Chapter 1, Exercise 2] it is shown that there exist submonoids of $(\mathbb{N}^2, +)$ that are not finitely generated. The next result guarantees us that, in order to study numerical $\mathcal{A}$-semigroups, we can focus in finitely generated submonoids of $(\mathbb{N}^2, +)$.

Corollary 2.5. *Let S be a numerical semigroup. The following conditions are equivalent.*
(1) *S is a numerical $\mathcal{A}$-semigroup.*
(2) *$S = \mathrm{A}(M) \cup \{0\}$ for some finitely generated submonoid M of $(\mathbb{N}^2, +)$.*
(3) *There exist $a_1, b_1, \dots, a_p, b_p \in \mathbb{N}$ such that S is the set*

$$\{n \in \mathbb{N} \mid a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p \textit{ for some } x_1, \dots, x_p \in \mathbb{N}\} \cup \{0\}.$$

Proof. (1) $\Rightarrow$ (2) It is an immediate consequence of the proof of Theorem 2.4.

(2) $\Rightarrow$ (3) If $\{(a_1, b_1), \dots, (a_p, b_p)\}$ is a system of generators of M, then

$$S = \mathrm{A}(M) \cup \{0\} = \{n \in \mathbb{N} \mid a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p \text{ for some } x_1, \dots, x_p \in \mathbb{N}\} \cup \{0\}.$$

(3) $\Rightarrow$ (1) Let M be the submonoid of $(\mathbb{N}^2, +)$ that is generated by the set $\{(a_1, b_1), \dots, (a_p, b_p)\}$. Then it is obvious that $S = \mathrm{A}(M) \cup \{0\}$. □

Remark 2.6. Let us observe that, from (3) of Corollary 2.5, numerical $\mathcal{A}$-semigroups can be characterized as sets that contain the integers n such that the system of inequalities

$$a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p$$

has at least one solution in $\mathbb{N}^p$ (where $\{a_1, b_1, \dots, a_p, b_p\} \subseteq \mathbb{N}$ is a given set for each numerical semigroup).

Let $(\mathfrak{M}, +)$ be a commutative monoid and let X be a system of generators of $\mathfrak{M}$. If $\mathfrak{M} \neq \langle Y \rangle$ for all $Y \subsetneq X$, then we say that X is a *minimal system of generators* of $\mathfrak{M}$. We will denote by $\mathrm{msg}(\mathfrak{M})$ a minimal system of generators of $\mathfrak{M}$.

If M is a submonoid of $(\mathbb{N}, +)$, then we write $M^* = M \setminus \{0\}$. The following result is a consequence of [14, Lemma 2.3, Corollary 2.8].

Lemma 2.7. *If M is a submonoid of $(\mathbb{N}, +)$, then $\mathrm{msg}(M) = M^* \setminus (M^* + M^*)$ is the unique minimal system of generators of M. In addition, $\mathrm{msg}(M)$ is finite and contained in every system of generators of M.*

Let S be a numerical semigroup and let $\{n_1, \dots, n_p\} \subseteq \mathbb{N} \setminus \{0\}$ be a system of generators of S. If $s \in S$, we define the *order* of s (in S) by (see [5])

$$\mathrm{ord}(s; S) = \max\{a_1 + \cdots + a_p \mid a_1n_1 + \cdots + a_pn_p = s, \text{ with } a_1, \dots, a_p \in \mathbb{N}\}.$$

If no ambiguity is possible, then we write $\mathrm{ord}(s)$.

Remark 2.8. Let us observe that, from Lemma 2.7, the definition of $\operatorname{ord}(s; S)$ is independent of the considered system of generators of S, that is, $\operatorname{ord}(s; S)$ only depends on s and S. Thereby, we can take $\operatorname{msg}(S)$ in order to define $\operatorname{ord}(s; S)$.

The next result is easy to prove.

Lemma 2.9. *Let S be a numerical semigroup with minimal system of generators given by $\{n_1, \dots, n_p\}$ and let $s \in S$.*
(1) *If $s - n_i \in S$, then* $\operatorname{ord}(s - n_i) \le \operatorname{ord}(s) - 1$.
(2) *If $s = a_1 n_1 + \cdots + a_p n_p$, with* $\operatorname{ord}(s) = a_1 + \cdots + a_p$ *and $a_i \neq 0$, then* $\operatorname{ord}(s - n_i) = \operatorname{ord}(s) - 1$.

In the following proposition we show another characterization of numerical $\mathcal{A}$-semigroups.

Proposition 2.10. *Let S be a numerical semigroup with minimal system of generators given by $\{n_1, \dots, n_p\}$. The following conditions are equivalent.*
(1) *S is a numerical $\mathcal{A}$-semigroup.*
(2) *If $i, j \in \{1, \dots, p\}$, then $\{n_i + n_j - 1, n_i + n_j + 1\} \subseteq S$.*
(3) *If $s \in S \setminus \{0, n_1, \dots, n_p\}$, then $\{s - 1, s + 1\} \subseteq S$.*
(4) *If $s \in S \setminus \{0\}$, then $s + z \in S$ for all $z \in \mathbb{Z}$ such that $|z| <$* $\operatorname{ord}(s)$.

Proof. (1) ⇒ (2) It is an immediate consequence of Theorem 2.4.

(2) ⇒ (3) If $s \in S \setminus \{0, n_1, \dots, n_p\}$, then it is clear that there exist $i, j \in \{1, \dots, p\}$ and $s' \in S$ such that $s = n_i + n_j + s'$. Thus, $\{s\} + \{-1, 1\} = \{n_i + n_j\} + \{-1, 1\} + \{s'\} \subseteq S$.

(3) ⇒ (4) We are going to reason by induction over $\operatorname{ord}(s)$. The result is trivially true if $\operatorname{ord}(s) = 1$. Thereby, let us assume that $\operatorname{ord}(s) \ge 2$ and that $a_1, \dots, a_p$ are nonnegative integers such that $s = a_1 n_1 + \cdots + a_p n_p$ and $\operatorname{ord}(s) = a_1 + \cdots + a_p$, with $a_i \neq 0$ for some $i \in \{1, \dots, p\}$. By Lemma 2.9, we have $\operatorname{ord}(s - n_i) = \operatorname{ord}(s) - 1$ and, by hypothesis of induction, that $\{s - n_i\} +]-(\operatorname{ord}(s) - 1), \operatorname{ord}(s) - 1[\subseteq S$. Since $\operatorname{ord}(s) \ge 2$, it follows that $a_i \ge 2$ or $a_i = 1$ and $a_j \neq 0$, for some $j \in \{1, \dots, p\} \setminus \{i\}$. Thus, we have that

$$s = 2n_i + a_1 n_1 + \cdots + (a_i - 2) n_1 + \cdots + a_p n_p$$

or (assuming, without loss of generality, that $i < j$)

$$s = n_i + n_j + a_1 n_1 + \cdots + (a_i - 1) n_i + \cdots + (a_j - 1) n_j + \cdots + a_p n_p.$$

Anyway, $s - n_i - (\operatorname{ord}(s) - 2) > 0$ and, consequently, we have that

$$\{s - n_i\} +]-(\operatorname{ord}(s) - 1), \operatorname{ord}(s) - 1[+ \{n_i\} + \{-1, 0, 1\} \subseteq S.$$

Thereby, $\{s\} +]-\operatorname{ord}(s), \operatorname{ord}(s)[\subseteq S$.

(4) ⇒ (1) If $a, b \in S \setminus \{0\}$, then it is clear that $\operatorname{ord}(a + b) \ge 2$. Thus, we get that $\{a + b\} + \{-1, 0, 1\} \subseteq S$ and, by applying Theorem 2.4, we can conclude that S is a numerical $\mathcal{A}$-semigroup. □

Remark 2.11. Let us observe that (2) of the above proposition allows us to decide, faster than with Theorem 2.4, whether a numerical semigroup is a numerical $\mathcal{A}$-semigroup.

Remark 2.12. As a consequence of (3) of Proposition 2.10, we have that the numerical $\mathcal{A}$-semigroups can be characterized as those numerical semigroups which satisfy that the maximum and the minimum elements in each interval of non-gaps are minimal generators of the numerical semigroup or are equal to zero. (Remember that the gaps of a numerical semigroup S are the elements of the set $\mathbb{N} \setminus S$.) As we commented on in the introduction, this characterization can be also deduced from [3, Example 6.5].

Now we give two illustrative examples on the content of this section.

Example 2.13. Let us see that, if S is the numerical semigroup with minimal system of generators $\{3, 5, 7\}$, then S is a numerical $\mathcal{A}$-semigroup. Indeed, by applying (2) of Proposition 2.10 (see Remark 2.11), since $\{3 + 3\} + \{-1, 1\}$, $\{3 + 5\} + \{-1, 1\}$, $\{3 + 7\} + \{-1, 1\}$, $\{5 + 5\} + \{-1, 1\}$, $\{5 + 7\} + \{-1, 1\}$, and $\{7 + 7\} + \{-1, 1\}$ are subsets of S, then we can assert that S is a numerical $\mathcal{A}$-semigroup.

On the other hand, we have that $S = \{0, 3, 5, 6, 7, \rightarrow\}$ and, thereby, its intervals of non-gaps are $\{0\}$, $\{3\}$ and $\{5, \rightarrow\}$. Inasmuch as the maximum and the minimum of such a sets are zero or a minimal generator, from Remark 2.12, we have another way to state that S is a numerical $\mathcal{A}$-semigroup.

Example 2.14. Let T be the numerical semigroup with minimal system of generators $\{5, 7, 9\}$. Then we have $T = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ and its intervals of non-gaps are $\{0\}$, $\{5\}$, $\{7\}$, $\{9, 10\}$, $\{12\}$, and $\{14, \rightarrow\}$. Since $\max\{9, 10\} = 10$ is different from zero and it is not a minimal generator of T, then we conclude that T is not a numerical $\mathcal{A}$-semigroup.

3 The Frobenius variety of the numerical $\mathcal{A}$-semigroups

Let us recall that, following [11], a *Frobenius variety* is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfils the following conditions,

(i) if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$,

(ii) if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.

The next result is straightforward to prove and appears in [14].

Lemma 3.1. *Let S, T be numerical semigroups.*

(1) *$S \cap T$ is a numerical semigroup.*

(2) *If $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup.*

Remark 3.2. Let us denote by $\mathcal{L} = \{S \mid S$ is a numerical semigroup$\}$. By Lemma 3.1, we have that $\mathcal{L}$ is a Frobenius variety.

Having in mind that $\mathcal{A} = \{S \mid S$ is a numerical $\mathcal{A}$-semigroup$\}$, our first aim in this section will be to show that $\mathcal{A}$ is a Frobenius variety.

Proposition 3.3. *The set $\mathcal{A}$ is a Frobenius variety.*

Proof. First of all, being as $\mathbb{N} \in \mathcal{A}$, we have that $\mathcal{A}$ is a non-empty set.

Now, let $S, T \in \mathcal{A}$ and $x, y \in (S \cap T) \setminus \{0\}$. By Lemma 3.1, $S \cap T$ is a numerical semigroup. Moreover, by applying Theorem 2.4, we have that $\{x + y - 1, x + y + 1\} \subseteq S \cap T$. Consequently, $S \cap T \in \mathcal{A}$.

Finally, let $S \in \mathcal{A}$ such that $S \neq \mathbb{N}$ and let $x, y \in (S \cup \mathrm{F}(S)) \setminus \{0\}$. From Lemma 3.1, it is clear that $S \cup \mathrm{F}(S)$ is a numerical semigroup. Now, on the one hand, if $x, y \in S$, then $\{x + y - 1, x + y + 1\} \subseteq S \subseteq S \cup \mathrm{F}(S)$. On the other hand, if $\mathrm{F}(S) \in \{x, y\}$, then $x + y - 1 \geq \mathrm{F}(S)$ and, thereby, $\{x + y - 1, x + y + 1\} \subseteq S \cup \mathrm{F}(S)$. By Theorem 2.4 again, we conclude that $S \cup \mathrm{F}(S) \in \mathcal{A}$. □

Let us recall that a *graph* G is a pair (V, E), where V is a non-empty set (of *vertices*) and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$ (the *edges* of G). A *path (of length n)* connecting two vertices x, y is a sequence of different edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

We say that a graph G is a *tree* if there exists a vertex v^* (the *root* of G) such that, for every other vertex x, there exists a unique path connecting x and v^*. Moreover, if (x, y) is an edge, then we say that x is a *child* of y.

In our framework, we define the graph $\mathrm{G}(\mathcal{A})$ where $\mathcal{A}$ is the set of vertices and $(S, S') \in \mathcal{A} \times \mathcal{A}$ is an edge if $S' = S \cup \{\mathrm{F}(S)\}$.

It is easy to show (see [14, Exercise 2.1]) that, if S is a numerical semigroup and $x \in S$, then $S \setminus \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$. Thus, as a consequence of [11, Proposition 24, Theorem 27], we have the following result.

Theorem 3.4. *The graph $\mathrm{G}(\mathcal{A})$ is a tree with root equal to $\mathbb{N}$. Moreover, the set of the children of a vertex $S \in \mathcal{A}$ is the set*

$$\{S \setminus \{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S),\ \textit{and}\ S \setminus \{x\} \in \mathcal{A}\}.$$

In the next proposition, we will characterize the elements $x \in \mathrm{msg}(S)$ such that $S \setminus \{x\} \in \mathcal{A}$.

Proposition 3.5. *Let S be a numerical $\mathcal{A}$-semigroup such that $S \neq \mathbb{N}$, and let $x \in \mathrm{msg}(S)$. Then $S \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup if and only if*

$$\{x-1, x+1\} \subseteq \{0\} \cup (\mathbb{N} \setminus S) \cup \mathrm{msg}(S).$$

Proof. *Necessity.* By Lemma 2.7, if $x + 1 \notin \{0\} \cup (\mathbb{N} \setminus S) \cup (\mathrm{msg}(S))$, then we deduce that there exist two elements $a, b \in S \setminus \{0\}$ such that $a + b = x + 1$. Let us observe that, since $S \neq \mathbb{N}$, then $1 \notin S$. Consequently, we have that $a, b \in S \setminus \{0, x\}$ and $a + b - 1 = x \notin S \setminus \{x\}$. By applying Theorem 2.4, we have that $S \setminus \{x\}$ is not a numerical $\mathcal{A}$-semigroup.

For the case $x - 1 \notin \{0\} \cup (\mathbb{N} \setminus S) \cup (\mathrm{msg}(S))$, we can argue in a similar way.

Sufficiency. Let $a, b \in S \setminus \{x, 0\}$. Since S is a numerical $\mathcal{A}$-semigroup, by Theorem 2.4, we get

$$\{a + b - 1, a + b + 1\} \subseteq S.$$

Being that $\{x-1, x+1\} \subseteq \{0\} \cup (\mathbb{N} \setminus S) \cup (\mathrm{msg}(S))$, we have that $x \notin \{a + b - 1, a + b + 1\}$. Thereby

$$\{a + b - 1, a + b + 1\} \subseteq S \setminus \{x\}.$$

By applying Theorem 2.4 again, we conclude that $S \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup. □

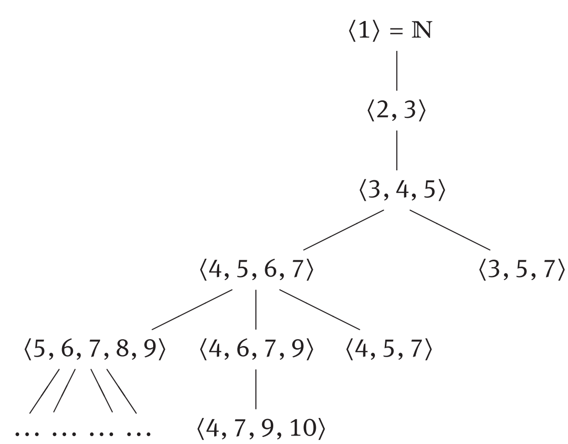
As a consequence of the previous proposition, we have the following result.

Corollary 3.6. *Let S be a numerical $\mathcal{A}$-semigroup such that $S \neq \mathbb{N}$, and let x be a minimal generator of S greater than $\mathrm{F}(S)$. Then $S \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup if and only if $\{x-1, x+1\} \subseteq \mathrm{msg}(S) \cup \{\mathrm{F}(S)\}$.*

In the next example we show that we can get the children of a vertex of $\mathrm{G}(\mathcal{A})$ by applying Theorem 3.4 together with Corollary 3.6.

Example 3.7. Let S be the numerical semigroup with minimal system of generators given by $\{4, 6, 7, 9\}$. Then $S = \{0, 4, 6, \rightarrow\}$ and, therefore, $\mathrm{F}(S) = 5$. From Proposition 2.10, we deduce that S is a numerical $\mathcal{A}$-semigroup. By applying Theorem 3.4 and Corollary 3.6, we get that S has a unique child in $\mathrm{G}(\mathcal{A})$. Namely, $S \setminus \{6\} = \langle 4, 7, 9, 10\rangle$.

Let us observe that we can recursively build a tree, from the root, if we know the children of each vertex. Therefore, we can build the tree $\mathrm{G}(\mathcal{A})$ such as it is shown in the following figure.



By Theorem 3.4, it is obvious that $\langle 2, 3\rangle = \mathbb{N} \setminus \{1\}$ is the unique child of $\mathbb{N} = \langle 1\rangle$. By applying Theorem 3.4 and Corollary 3.6, we have that

- $\langle 3, 4, 5\rangle = \langle 2, 3\rangle \setminus \{2\}$ is the unique child of $\langle 2, 3\rangle$.
- $\langle 4, 5, 6, 7\rangle = \langle 3, 4, 5\rangle \setminus \{3\}$ and $\langle 3, 5, 7\rangle = \langle 3, 4, 5\rangle \setminus \{4\}$ are the two children of $\langle 3, 4, 5\rangle$.
- $\langle 3, 5, 7\rangle$ has no children.
- $\langle 5, 6, 7, 8, 9\rangle = \langle 4, 5, 6, 7\rangle \setminus \{4\}$, $\langle 4, 6, 7, 9\rangle = \langle 4, 5, 6, 7\rangle \setminus \{5\}$, and $\langle 4, 5, 7\rangle = \langle 4, 5, 6, 7\rangle \setminus \{6\}$ are the three children of $\langle 4, 5, 6, 7\rangle$.

- $\langle 4, 5, 7\rangle$ has no children.
- $\langle 4, 7, 9, 10\rangle = \langle 4, 6, 7, 9\rangle \setminus \{6\}$ is the unique child of $\langle 4, 6, 7, 9\rangle$.
- $\langle 5, 6, 7, 8, 9\rangle$ has four children.
- $\langle 4, 7, 9, 10\rangle$ has no children.
- And so on.

Let us observe that, if S' is a child of S in $\mathrm{G}(\mathcal{A})$, then $\mathrm{F}(S') > \mathrm{F}(S)$ and $\mathrm{g}(S') = \mathrm{g}(S) + 1$. Therefore, when we go on along the branches of the tree $\mathrm{G}(\mathcal{A})$, we get numerical semigroups with greater Frobenius number and genus. Thus, we can use this construction in order to obtain all the numerical $\mathcal{A}$-semigroups with a given Frobenius number or genus.

4 The smallest numerical $\mathcal{A}$-semigroup that contains a given set of positive integers

Since $\mathcal{A}$ is a Frobenius variety, we have that the finite intersection of numerical $\mathcal{A}$-semigroups is a numerical $\mathcal{A}$-semigroup. Now observe that, if n is a nonnegative integer, then $\{0, n, \rightarrow\}$ is a numerical $\mathcal{A}$-semigroup, as a consequence of Theorem 2.4. Thus, being that $\bigcap_{n\in\mathbb{N}}\{0, n, \rightarrow\} = \{0\}$, we get that the infinite intersection of numerical $\mathcal{A}$-semigroups is not always a numerical $\mathcal{A}$-semigroup. On the other hand, it is clear that the (finite or infinite) intersection of numerical semigroups is always a submonoid of $(\mathbb{N}, +)$.

If M is a submonoid of $(\mathbb{N}, +)$, then we will say that M is an $\mathcal{A}$-monoid if it can be expressed like the intersection of numerical $\mathcal{A}$-semigroups.

The proof of the following result is straightforward.

Lemma 4.1. *The intersection of $\mathcal{A}$-monoids is an $\mathcal{A}$-monoid.*

This lemma leads us to the next definition.

Definition 4.2. Let X be a subset of $\mathbb{N}$. The *$\mathcal{A}$-monoid generated by X* (denoted by $\mathcal{A}(X)$) is the intersection of all $\mathcal{A}$-monoids containing X.

Let us observe that $\mathcal{A}(X)$ is the smallest $\mathcal{A}$-monoid containing X. The proof of the following lemma is also immediate.

Lemma 4.3. *If $X \subseteq \mathbb{N}$, then $\mathcal{A}(X)$ is the intersection of all numerical $\mathcal{A}$-semigroups that contain X.*

In the next result we show that $\mathcal{A}(X)$ is a numerical $\mathcal{A}$-semigroup (except if X is the empty set or $X = \{0\}$).

Proposition 4.4. *If X is a non-empty subset of $\mathbb{N} \setminus \{0\}$, then $\mathcal{A}(X)$ is a numerical $\mathcal{A}$-semigroup.*

Proof. Since $\mathcal{A}(X)$ is a submonoid of $(\mathbb{N}, +)$, in order to show that $\mathcal{A}(X)$ is a numerical semigroup, it will be enough to see that $\gcd(\mathcal{A}(X)) = 1$.

Let $x \in X$. If S is a numerical $\mathcal{A}$-semigroup containing X, then (by Theorem 2.4) we have $\{x, 2x + 1\} \subseteq S$. By applying Lemma 4.3, we get that $\{x, 2x + 1\} \subseteq \mathcal{A}(X)$. As $\gcd(\{x, 2x + 1\}) = 1$, it follows that $\gcd(\mathcal{A}(X)) = 1$.

Now, let us see that $\mathcal{A}(X)$ is a numerical $\mathcal{A}$-semigroup. Let $a, b \in \mathcal{A}(X) \setminus \{0\}$. If S is a numerical $\mathcal{A}$-semigroup containing X, from Lemma 4.3, we have that $a, b \in S \setminus \{0\}$ and, from Theorem 2.4, we get that $\{a + b - 1, a + b + 1\} \subseteq S$. By applying again Lemma 4.3, we have that $\{a + b - 1, a + b + 1\} \subseteq \mathcal{A}(X)$. Therefore, by applying Theorem 2.4 once more, we conclude that $\mathcal{A}(X)$ is a numerical $\mathcal{A}$-semigroup. □

Let us observe that Proposition 4.4 is not true for every Frobenius variety. In fact, let $\mathcal{S}$ be the set of all numerical semigroups. It is clear that $\mathcal{S}$ is a Frobenius variety. If we take $X = \{2\}$, then the intersection of all elements of $\mathcal{S}$ containing X is equal to $\langle 2\rangle$ which is not a numerical semigroup (observe that $\bigcap_{k\in\mathbb{N}}\langle 2, 2k + 1\rangle = \langle 2\rangle$).

Let us also observe that, as a consequence of Proposition 4.4, we have that M is an $\mathcal{A}$-monoid if and only if M is a numerical $\mathcal{A}$-semigroup or $M = \{0\}$.

Theorem 4.5. *The set $\mathcal{A}$ is equal to the set $\{\mathcal{A}(X) \mid X$ is a non-empty finite subset of $\mathbb{N} \setminus \{0\}\}$.*

Proof. By Proposition 4.4, we have that

$$\{\mathcal{A}(X) \mid X \text{ is a non-empty finite subset of } \mathbb{N} \setminus \{0\}\} \subseteq \mathcal{A}.$$

Let us see the other inclusion. If $S \in \mathcal{A}$, then S is a numerical semigroup and, by Lemma 2.7, we deduce that there exists a non-empty finite subset X of $\mathbb{N} \setminus \{0\}$ such that $S = \langle X \rangle$. Thereby, S is the smallest numerical semigroup that contains X. In fact, S is the smallest numerical $\mathcal{A}$-semigroup that contains X. Consequently, $S = \mathcal{A}(X)$. ☐

If M is an $\mathcal{A}$-monoid and X is a subset of $\mathbb{N}$ such that $M = \mathcal{A}(X)$, then we will say that X is an *$\mathcal{A}$-system of generators* of M. In addition, if $M \neq \mathcal{A}(Y)$ for all $Y \subsetneq X$, then we will say that X is a *minimal $\mathcal{A}$-system of generators* of M.

Since $\mathcal{A}$ is a Frobenius variety, by applying [11, Corollary 19], we have the following result.

Proposition 4.6. *Every $\mathcal{A}$-monoid has a unique minimal $\mathcal{A}$-system of generators, which in addition is finite.*

The next result follows from [11, Proposition 24].

Proposition 4.7. *Let M be an $\mathcal{A}$-monoid and let $x \in M$. Then $M \setminus \{x\}$ is an $\mathcal{A}$-monoid if and only if x belongs to the minimal $\mathcal{A}$-system of generators of M.*

As an immediate consequence of this proposition we have the following result.

Corollary 4.8. *Let X be a non-empty subset of $\mathbb{N} \setminus \{0\}$. Then*

$$\{x \in X \mid \mathcal{A}(X) \setminus \{x\} \textit{ is a numerical } \mathcal{A}\textit{-semigroup}\}$$

is the minimal $\mathcal{A}$-system of generators of $\mathcal{A}(X)$.

Let us illustrate the previous result with an example.

Example 4.9. Beginning in Example 3.7, we know that $S = \langle 4, 6, 7, 9 \rangle$ is a numerical $\mathcal{A}$-semigroup. By applying Proposition 3.5, we easily deduce that

$$\{x \in \{4, 6, 7, 9\} \mid S \setminus \{x\} \text{ is a numerical } \mathcal{A}\text{-semigroup}\} = \{4, 6\}.$$

Therefore, $S = \mathcal{A}(\{4, 6\})$ and $\{4, 6\}$ is its minimal $\mathcal{A}$-system of generators.

Let $x_1, \dots, x_t$ be positive integers. We will denote by $S(x_1, \dots, x_t)$ the set

$$\{a_1 x_1 + \cdots + a_t x_t + z \mid a_1, \dots, a_t \in \mathbb{N},\ z \in \mathbb{Z}, \text{ and } |z| < a_1 + \cdots + a_t\} \cup \{0\}.$$

Our next purpose will be to show that $S(x_1, \dots, x_t)$ is the smallest numerical $\mathcal{A}$-semigroup containing $\{x_1, \dots, x_t\}$, that is, $S(x_1, \dots, x_t) = \mathcal{A}(\{x_1, \dots, x_t\})$.

The next result has an easy proof, so it is omitted.

Lemma 4.10. *Let S be a numerical semigroup, let $s_1, \dots, s_t \in S \setminus \{0\}$, and let $a_1, \dots, a_t \in \mathbb{N}$. Then*

$$\operatorname{ord}(a_1 s_1 + \cdots + a_t s_t) \geq a_1 + \cdots + a_t.$$

Theorem 4.11. *If $x_1, \dots, x_t$ are positive integers, then $S(x_1, \dots, x_t)$ is the smallest numerical $\mathcal{A}$-semigroup that contains the set $\{x_1, \dots, x_t\}$.*

Proof. We divide the proof into four steps.

(i) Let us see that, if $x, y \in S(x_1, \dots, x_t) \setminus \{0\}$, then

$$\{x + y - 1, x + y, x + y + 1\} \subseteq S(x_1, \dots, x_t).$$

Let $a_1, b_1, \dots, a_t, b_t \in \mathbb{N}$ and let $z, z' \in \mathbb{Z}$ such that

$$x = a_1 x_1 + \cdots + a_t x_t + z, \quad |z| < a_1 + \cdots + a_t, \qquad y = b_1 x_1 + \cdots + b_t x_t + z', \quad |z'| < b_1 + \cdots + b_t.$$

Then

$$x + y = (a_1 + b_1)x_1 + \cdots + (a_t + b_t)x_t + z + z'$$

with $|z + z'| \leq |z| + |z'| < |z| + |z'| + 1 < (a_1 + b_1) + \cdots + (a_t + b_t)$. Consequently, $x + y \in S(x_1, \ldots, x_t)$. Moreover,

$$x + y + 1 = (a_1 + b_1)x_1 + \cdots + (a_t + b_t)x_t + z + z' + 1$$

with $|z + z' + 1| \leq |z| + |z'| + 1$. Thus, $x + y + 1 \in S(x_1, \ldots, x_t)$. In the same way, since

$$x + y - 1 = (a_1 + b_1)x_1 + \cdots + (a_t + b_t)x_t + z + z' - 1$$

with $|z + z' - 1| \leq |z| + |z'| + 1$, we have that $x + y - 1 \in S(x_1, \ldots, x_t)$.

(ii) Let $i \in \{1, \ldots, t\}$. Since $x_i = 0 \cdot x_1 + \cdots + 1 \cdot x_i + \cdots + 0 \cdot x_t + 0$, we have that $x_i \in S(x_1, \ldots, x_t)$.

(iii) From the previous steps and Theorem 2.4, we deduce that $S(x_1, \ldots, x_t)$ is a numerical $\mathcal{A}$-semigroup that contains $\{x_1, \ldots, x_t\}$.

(iv) Let T be a numerical $\mathcal{A}$-semigroup containing $\{x_1, \ldots, x_t\}$ and let $x \in S(x_1, \ldots, x_t) \setminus \{0\}$. Then there exist $a_1, \ldots, a_t \in \mathbb{N}$ and $z \in \mathbb{Z}$ such that $x = a_1x_1 + \cdots + a_tx_t + z$ with $|z| < a_1 + \cdots + a_t$. Since $\{x_1, \ldots, x_t\} \subseteq T$, we get that $a_1x_1 + \cdots + a_tx_t \in T$. By applying Proposition 2.10 and Lemma 4.10, we can conclude that $x \in T$.

In this way, we have proved the statement. □

As an immediate consequence of Theorem 4.11, we have the following result.

Corollary 4.12. *If m is a positive integer, then*

$$\mathcal{A}(\{m\}) = \{km + z \mid k \in \mathbb{N} \setminus \{0\} \text{ and } z \in \{-(k-1), \ldots, k-1\}\} \cup \{0\}.$$

By applying the previous corollary, we can easily compute the smallest numerical $\mathcal{A}$-semigroup that contains a fixed positive integer, such as we show in the next example.

Example 4.13. Let us compute the smallest numerical $\mathcal{A}$-semigroup containing $\{10\}$. By Corollary 4.12, we have that such a numerical semigroup is

$$\begin{aligned}\mathcal{A}(\{10\}) &= \{k \cdot 10 + z \mid k \in \mathbb{N} \setminus \{0\} \text{ and } z \in \{-(k-1), \ldots, k-1\}\} \cup \{0\}\\ &= \{0, 10, 19, 20, 21, 28, 29, 30, 31, 32, 37, 38, 39, 40, 41, 42, 43, 46, \rightarrow\}\\ &= \langle 10, 19, 21, 28, 32, 37, 43, 46, 54, 55\rangle.\end{aligned}$$

Let us observe that, in Theorem 4.11, are described the elements of the smallest numerical $\mathcal{A}$-semigroup which contains a given set of positive integers. However, in order to compute such a numerical $\mathcal{A}$-semigroup, we propose the following algorithm that is justified by Proposition 2.10.

Algorithm 4.14.
- INPUT: A finite set X of positive integers.
- OUTPUT: The minimal system of generators of $\mathcal{A}(X)$.
  (1) $Y = \mathrm{msg}(\langle X\rangle)$.
  (2) $Z = Y \cup (\bigcup_{a,b\in Y}\{a + b - 1, a + b + 1\})$.
  (3) If $\mathrm{msg}(\langle Z\rangle) = Y$, then return Y.
  (4) Set $Y = \mathrm{msg}(\langle Z\rangle)$ and go to (2).

Example 4.15. We are going to compute $\mathcal{A}(\{5, 7\})$ applying Algorithm 4.14.
- $Y = \{5, 7\}$.
- $Z = \{5, 7, 9, 11, 13, 15\}$.
- $\mathrm{msg}(\langle Z\rangle) = \{5, 7, 9, 11, 13\}$.
- $Y = \{5, 7, 9, 11, 13\}$.
- $Z = \{5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27\}$.
- $\mathrm{msg}(\langle Z\rangle) = \{5, 7, 9, 11, 13\}$.
- $\mathcal{A}(\{5, 7\}) = \langle 5, 7, 9, 11, 13\rangle$.

Observe that the most complex process in Algorithm 4.14 is the computation of $\mathrm{msg}(\langle Z\rangle)$, that is, compute the minimal system of generators of a numerical semigroup S starting from any system of generators of it. For this purpose, we can use the `GAP` package `numericalsgps` (see [6]).

5 The computation of S(a, b)

Let $a = (a_1, \dots, a_p)$, $b = (b_1, \dots, b_p) \in \mathbb{N}^p$ and let $\mathrm{S}(a, b)$ be the set

$$\mathrm{S}(a, b) = \{n \in \mathbb{N} \mid a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p \text{ for some } x_1, \dots, x_p \in \mathbb{N}^p\}.$$

Our main aim in this section will be to describe an algorithmic procedure that allows us to compute $\mathrm{S}(a, b)$ starting from a and b. In order to do that we need to introduce several concepts and results.

Let $z = (z_1, \dots, z_p) \in \mathbb{Z}^p$ and

$$\mathrm{A}(z) = \{(x_1, \dots, x_p) \in \mathbb{N}^p \mid z_1x_1 + \cdots + z_px_p \geq 0\}.$$

It is well known that $\mathrm{A}(z)$ is a finitely generated submonoid of $(\mathbb{N}^p, +)$ and, moreover, in [1] it is shown an algorithm to compute a finite system of generators for $\mathrm{A}(z)$. Since we want a self-contained paper and to describe examples without make references to [1], we are going to describe an algorithmic process to compute a finite system of generators for $\mathrm{A}(z)$.

Let

$$\mathrm{B}(z) = \{(x_1, \dots, x_p, x_{p+1}) \in \mathbb{N}^{p+1} \mid z_1x_1 + \cdots + z_px_p - x_{p+1} = 0\}.$$

It is well known (see [13]) that $\mathrm{B}(z)$ is a finitely generated submonoid of $(\mathbb{N}^{p+1}, +)$ and its minimal generators are precisely the minimal elements (with the usual order in $\mathbb{N}^{p+1}$) of the set $\mathrm{B}(z) \setminus \{(0, \dots, 0)\}$. In addition, we have (see [7]) that, if $(x_1, \dots, x_p, x_{p+1})$ is a minimal element of $\mathrm{B}(z) \setminus \{(0, \dots, 0)\}$, then $x_1 + \cdots + x_{p+1} \leq |z_1| + \cdots + |z_p| + 2$. Finally, it is easy to see that, if $\{b_1, \dots, b_q\}$ is a minimal system of generators of $\mathrm{B}(z)$, then $\{\pi(b_1), \dots, \pi(b_q)\}$ is a system of generators for $\mathrm{A}(z)$ (where $\pi(x_1, \dots, x_p, x_{p+1}) = (x_1, \dots, x_p)$). Therefore, from the comments in this paragraph, it is clear that we have an algorithmic procedure in order to compute the finite system of generators for $\mathrm{A}(z)$. Let us see an example which illustrates such a process.

Example 5.1. We are going to compute a system of generators for $A = \{(x, y) \in \mathbb{N}^2 \mid x - y \geq 0\}$. For that, we begin computing the minimal elements of $B \setminus \{(0, 0, 0)\}$, where $B = \{(x, y, z) \in \mathbb{N}^3 \mid x - y - z = 0\}$. By applying that, if (x, y, z) is a minimal element of $B \setminus \{(0, 0, 0)\}$, then $x + y + z \leq 4$, we easily deduce that the set of minimal elements of $B \setminus \{(0, 0, 0)\}$ is $\{(1, 1, 0), (1, 0, 1)\}$. Thereby, $\{\pi(1, 1, 0), \pi(1, 0, 1)\} = \{(1, 1), (1, 0)\}$ is a system of generators for A.

The idea of the algorithmic procedure, which we want to show in this section, is to make a series of transformation on (a, b) in order to simplify the computation of $\mathrm{S}(a, b)$ in each step.

Let $\{m_1, \dots, m_q\}$ be the minimal system of generators of

$$\mathrm{A}(b - a) = \{(x_1, \dots, x_p) \in \mathbb{N}^p \mid (b_1 - a_1)x_1 + \cdots + (b_p - a_p)x_p \geq 0\},$$

where $m_i = (m_{i1}, \dots, m_{ip})$ for all $i \in \{1, \dots, q\}$. Moreover, let $\alpha = (\alpha_1, \dots, \alpha_q)$ and $\beta = (\beta_1, \dots, \beta_q)$, where $\alpha_i = a_1m_{i1} + \cdots + a_pm_{ip}$ and $\beta_i = b_1m_{i1} + \cdots + b_pm_{ip}$ for all $i \in \{1, \dots, q\}$. (Let us observe that $\alpha_i \leq \beta_i$ for all $i \in \{1, \dots, q\}$.)

Proposition 5.2. *Under the stated notation,* $\mathrm{S}(a, b) = \mathrm{S}(\alpha, \beta)$.

Proof. If $n \in \mathrm{S}(a, b)$, then there exists $(x_1, \dots, x_p) \in \mathbb{N}^p$ such that

$$a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p.$$

Therefore, $(x_1, \dots, x_p) \in \mathrm{A}(b - a)$ and, consequently, there exist $\lambda_1, \dots, \lambda_q \in \mathbb{N}$ such that

$$(x_1, \dots, x_p) = \lambda_1m_1 + \cdots + \lambda_qm_q.$$

Thus,

$$\begin{aligned} &a_1(\lambda_1m_{11} + \cdots + \lambda_qm_{q1}) + \cdots + a_p(\lambda_1m_{1p} + \cdots + \lambda_qm_{qp}) \\ &\qquad < n < b_1(\lambda_1m_{11} + \cdots + \lambda_qm_{q1}) + \cdots + b_p(\lambda_1m_{1p} + \cdots + \lambda_qm_{qp}). \end{aligned}$$

Now, by rearranging the terms in each one of these inequalities, we get that

$$\alpha_1\lambda_1+\cdots+\alpha_q\lambda_q < n < \beta_1\lambda_1+\cdots+\beta_q\lambda_q,$$

and, accordingly, $n \in S(\alpha,\beta)$.

Conversely, if $n \in S(\alpha,\beta)$, then there exists $(x_1,\ldots,x_q) \in \mathbb{N}^q$ such that

$$\alpha_1 x_1+\cdots+\alpha_q x_q < n < \beta_1 x_1+\cdots+\beta_q x_q.$$

Therefore,

$$\begin{aligned}&(a_1m_{11}+\cdots+a_pm_{1p})x_1+\cdots+(a_1m_{q1}+\cdots+a_pm_{qp})x_q\\&\quad< n < (b_1m_{11}+\cdots+b_pm_{1p})x_1+\cdots+(b_1m_{q1}+\cdots+b_pm_{qp})x_q,\end{aligned}$$

and

$$\begin{aligned}&a_1(m_{11}x_1+\cdots+m_{q1}x_q)+\cdots+a_p(m_{1p}x_1+\cdots+m_{qp}x_q)\\&\quad< n < b_1(m_{11}x_1+\cdots+m_{q1}x_q)+\cdots+b_p(m_{1p}x_1+\cdots+m_{qp}x_q).\end{aligned}$$

Thus, we conclude that $n \in \mathrm{S}(a,b)$. □

Let us observe that, as a consequence of the previous proposition, in the following we can assume that $a_i \le b_i$ for all $i \in \{1,\ldots,p\}$. We illustrate this fact with the next example.

Example 5.3. Let $S=\{n\in\mathbb{N} \mid 2x+3y<n<3x+2y \text{ for some } (x,y)\in\mathbb{N}^2\}$. From Example 5.1, we know that $\{(1,0),(1,1)\}$ is a system of generators of $A=\{(x,y)\in\mathbb{N}^2 \mid x-y\ge 0\}$. Then $\alpha_1=2$, $\beta_1=3$, $\alpha_2=5$, and $\beta_2=5$. Thus, by applying Proposition 5.2, we have $S=\{n\in\mathbb{N}\mid 2x+5y<n<3x+5y \text{ for some } (x,y)\in\mathbb{N}^2\}$.

Proposition 5.4. *If* $\{1,\ldots,r\}=\{i\in\{1,\ldots,p\}\mid a_i=b_i\}$, *then* $\mathrm{S}(a,b)=S'+\langle a_1\ldots,a_r\rangle$, *where*

$$S'=\{n\in\mathbb{N}\mid a_{r+1}x_{r+1}+\cdots+a_px_p<n<b_{r+1}x_{r+1}+\cdots+b_px_p \textit{ for some } (x_{r+1},\ldots,x_p)\in\mathbb{N}^{p-r}\}.$$

Proof. Let us see that $\mathrm{S}(a,b)\subseteq S'+\langle a_1\ldots,a_r\rangle$. Indeed, if $n\in\mathrm{S}(a,b)$, then there exists $(x_1,\ldots,x_p)\in\mathbb{N}^p$ such that

$$a_1x_1+\cdots+a_px_p<n<b_1x_1+\cdots+b_px_p.$$

Therefore,

$$a_{r+1}x_{r+1}+\cdots+a_px_p<n-(a_1x_1+\cdots+a_rx_r)<b_{r+1}x_{r+1}+\cdots+b_px_p.$$

Accordingly, $n-(a_1x_1+\cdots+a_rx_r)\in S'$ and, consequently, $n\in S'+\langle a_1\ldots,a_r\rangle$.

Now, let us see that $S'+\langle a_1\ldots,a_r\rangle\subseteq\mathrm{S}(a,b)$. In effect, if $n\in S'$ and $m\in\langle a_1\ldots,a_r\rangle$, then there exists $(x_1,\ldots,x_r,x_{r+1},\ldots,x_p)\in\mathbb{N}^p$ such that

$$a_{r+1}x_{r+1}+\cdots+a_px_p<n<b_{r+1}x_{r+1}+\cdots+b_px_p$$

and $m=a_1x_1+\cdots+a_rx_r=b_1x_1+\cdots+b_rx_r$. Therefore,

$$a_1x_1+\cdots+a_px_p<n+m<b_1x_1+\cdots+b_px_p$$

and, in consequence, $n+m\in\mathrm{S}(a,b)$. □

Let us observe that, as a consequence of Propositions 5.2 and 5.4, in the following we can assume that $a_i<b_i$ for all $i\in\{1,\ldots,p\}$. We illustrate this fact with the next example.

Example 5.5. Let $S=\{n\in\mathbb{N}\mid 2x+3y<n<3x+2y \text{ for some } (x,y)\in\mathbb{N}^2\}$. From Example 5.3,

$$S=\{n\in\mathbb{N}\mid 2x+5y<n<3x+5y \text{ for some } (x,y)\in\mathbb{N}^2\}.$$

Now, by Proposition 5.4, we have that $S=S'+\langle 5\rangle$, where $S'=\{n\in\mathbb{N}\mid 2x<n<3x \text{ for some } x\in\mathbb{N}\}$. In this way, if we can compute S', then we can compute S.

We will denote by $\frac{X}{k}$ the set $\{n \in \mathbb{N} \mid kn \in X\}$, where X is a subset of $\mathbb{N}$ and k is a positive integer.

Proposition 5.6. *Under the stated notation,* $\mathrm{S}(a, b) = \frac{\mathrm{S}(2a,2b)}{2}$.

Proof. If $n \in \mathrm{S}(a, b)$, then there exists $(x_1, \dots, x_p) \in \mathbb{N}^p$ such that

$$a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p.$$

Therefore,

$$2a_1x_1 + \cdots + 2a_px_p < 2n < 2b_1x_1 + \cdots + 2b_px_p,$$

and, consequently, $2n \in \mathrm{S}(2a, 2b)$. In this way, we conclude that $n \in \frac{\mathrm{S}(2a,2b)}{2}$.

Conversely, if $n \in \frac{\mathrm{S}(2a,2b)}{2}$, then $2n \in \mathrm{S}(2a, 2b)$ and, therefore, there exists $(x_1, \dots, x_p) \in \mathbb{N}^p$ such that

$$2a_1x_1 + \cdots + 2a_px_p < 2n < 2b_1x_1 + \cdots + 2b_px_p.$$

Then

$$a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p,$$

and, thereby, $n \in \mathrm{S}(a, b)$. □

From this moment, as a consequence of Propositions 5.2, 5.4, 5.6, we can assume that $b_i \geq a_i + 2$ for all $i \in \{1, \dots, p\}$. Let us illustrate this fact with an example.

Example 5.7. Let $S = \{n \in \mathbb{N} \mid 2x + 3y < n < 3x + 2y$ for some $(x, y) \in \mathbb{N}^2\}$. From Example 5.5, we know that $S = S' + \langle 5\rangle$, where $S' = \{n \in \mathbb{N} \mid 2x < n < 3x$ for some $x \in \mathbb{N}\}$. By applying Proposition 5.6, we have that $S' = \frac{S''}{2}$, where $S'' = \{n \in \mathbb{N} \mid 4x < n < 6x$ for some $x \in \mathbb{N}\}$. Therefore, if we can compute S'', then we can compute S' and, consequently, S. By the way, observe that $S' = \{\frac{n}{2} \mid n \in S''$ and n is even$\}$.

Theorem 5.8. *Under the stated notation, if* $b_i \geq a_i + 2$ *for all* $i \in \{1, \dots, p\}$, *then* $\mathrm{S}(a, b) \cup \{0\}$ *is the smallest numerical* $\mathcal{A}$*-semigroup containing* $\bigcup_{i=1}^{p}]a_i, b_i[$, *that is,*

$$\mathrm{S}(a, b) = \mathcal{A}\left(\bigcup_{i=1}^{p}]a_i, b_i[\right) \setminus \{0\}.$$

Proof. Let us observe that

(i) $\mathrm{A}(b - a) = \{(x_1, \dots, x_p) \in \mathbb{N}^p \mid (b_1 - a_1)x_1 + \cdots + (b_p - a_p)x_p \geq 0\} = \mathbb{N}^p$ and, therefore,

$$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 0, 1)\}$$

is a system of generators of $\mathrm{A}(b - a)$,

(ii) $\mathrm{S}(a, b) = \bigcup_{(x_1,\dots,x_p)\in\mathrm{A}(b-a)\setminus\{(0,\dots,0)\}}]a_1x_1 + \cdots + a_px_p, b_1x_1 + \cdots + b_px_p[$,

(iii) if $(x_1, \dots, x_p) \in \mathrm{A}(b - a) \setminus \{(0, \dots, 0)\}$, then $b_1x_1 + \cdots + b_px_p \geq a_1x_1 + \cdots + a_px_p + 2$ and, therefore, $]a_1x_1 + \cdots + a_px_p, b_1x_1 + \cdots + b_px_p[$ is a non-empty set.

It is clear that $\mathrm{S}(a, b) \cup \{0\}$ is a numerical $\mathcal{A}$-semigroup containing $\bigcup_{i=1}^{p}]a_i, b[$. Therefore, in order to prove the theorem, it is enough to see that, if T is a numerical $\mathcal{A}$-semigroup containing $\bigcup_{i=1}^{p}]a_i, b_i[$, then $\mathrm{S}(a, b) \subseteq T$. For that, we will show by induction on $x_1 + \cdots + x_p$ that, if $(x_1, \dots, x_p) \in \mathrm{A}(b - a) \setminus \{(0, \dots, 0)\}$, then $]a_1x_1 + \cdots + a_px_p, b_1x_1 + \cdots + b_px_p[\subseteq T$.

If $x_1 + \cdots + x_p = 1$, then it is clear that there exists $i \in \{1, \dots, p\}$ such that

$$]a_1x_1 + \cdots + a_px_p, b_1x_1 + \cdots + b_px_p[=]a_i, b_i[\subseteq T.$$

If $x_1 + \cdots + x_p \geq 2$, then $(x_1, \dots, x_p)$ is not a minimal generator of $\mathrm{A}(b - a)$ and, therefore, there exist $(y_1, \dots, y_p), (z_1, \dots, z_p) \in \mathrm{A}(b - a) \setminus \{(0, \dots, 0)\}$ such that $(x_1, \dots, x_p) = (y_1, \dots, y_p) + (z_1, \dots, z_p)$. By the hypothesis of induction, we know that

$$]a_1y_1 + \cdots + a_py_p, b_1y_1 + \cdots + b_py_p[\quad \text{and} \quad]a_1z_1 + \cdots + a_pz_p, b_1z_1 + \cdots + b_pz_p[$$

are subsets of T. That is,

$$\{a_1y_1 + \cdots + a_py_p + 1, \dots, b_1y_1 + \cdots + b_py_p - 1\} \quad \text{and} \quad \{a_1z_1 + \cdots + a_pz_p + 1, \dots, b_1z_1 + \cdots + b_pz_p - 1\}$$

are subsets of T. Now, by applying that T is a numerical semigroup, we have that

$$\begin{aligned}&\{a_1x_1+\cdots+a_px_p+2,\ldots,b_1x_1+\cdots+b_px_p-2\}\\&\quad=\{a_1y_1+\cdots+a_py_p+1,\ldots,b_1y_1+\cdots+b_py_p-1\}+\{a_1z_1+\cdots+a_pz_p+1,\ldots,b_1z_1+\cdots+b_pz_p-1\}\\&\quad\subseteq T.\end{aligned}$$

Finally, by applying that T is a numerical $\mathcal{A}$-semigroup, we get that

$$a_1x_1+\cdots+a_px_p+1=(a_1y_1+\cdots+a_py_p+1)+(a_1z_1+\cdots+a_pz_p+1)-1\in T$$

and

$$b_1x_1+\cdots+b_px_p-1=(b_1y_1+\cdots+b_py_p-1)+(b_1z_1+\cdots+b_pz_p-1)+1\in T.$$

Thus, we conclude that $]a_1x_1+\cdots+a_px_p,b_1x_1+\cdots+b_px_p[\,\subseteq T$. □

In the next example we show how the previous theorem works.

Example 5.9. We are going to compute

$$S=\{n\in\mathbb{N}\mid 2x+3y<n<3x+2y\text{ for some }(x,y)\in\mathbb{N}^2\}.$$

From Example 5.7, we know that $S=\frac{T}{2}+\langle 5\rangle$, where $T=\{n\in\mathbb{N}\mid 4x<n<6x\text{ for some }x\in\mathbb{N}\}$. By applying Theorem 5.8, we have that $T\cup\{0\}$ is the smallest numerical $\mathcal{A}$-semigroup containing $]4,6[\,=\{5\}$. Now, by applying Algorithm 4.14, we easily deduce that $T=\{5,9,10,11,13,\rightarrow\}$ and, consequently, $\frac{T}{2}=\{5,7,\rightarrow\}$. Therefore, we conclude that $S=\frac{T}{2}+\langle 5\rangle=\{5,7,\rightarrow\}$.

Remark 5.10. In order to solve the problem stated in Remark 1.2, observe that we only need to make minors changes in Propositions 5.2 and 5.4. Moreover, in Theorem 5.8 we have to take $b_i\geq a_i+1$ and intervals of the type $]a_i,b_i]$. Consequently, Proposition 5.6 is not necessary to get the answer to the problem.

6 Numerical $\mathcal{A}$-semigroups with fixed multiplicity

Let m be a positive integer. On the one hand, from Corollary 4.12, we have that

$$\Theta(m)=\{km+z\mid k\in\mathbb{N}\setminus\{0\},\ z\in\mathbb{Z},\text{ and }|z|<k\}\cup\{0\}$$

is the smallest (with respect to the inclusion order) numerical $\mathcal{A}$-semigroup with multiplicity m. On the other hand, it is clear that $\Delta(m)=\{0,m,\rightarrow\}$ is the greatest (with respect to the inclusion order) numerical $\mathcal{A}$-semigroup with multiplicity m. Therefore, if we denote by $\mathcal{A}_m$ the set of all numerical $\mathcal{A}$-semigroups with multiplicity m, then we have that $\Theta(m)=\min\mathcal{A}_m$ and $\Delta(m)=\max\mathcal{A}_m$.

Remark 6.1. Let m be a positive integer. The numerical semigroup $\{0,m,\rightarrow\}$ is usually called the *ordinary semigroup with multiplicity m*.

As an application of the above comment, we have the next result.

Proposition 6.2. *Let m be a positive integer. Then the set $\mathcal{A}_m$ is finite.*

Proof. If $S\in\mathcal{A}_m$, then $\Theta(m)\subseteq S\subseteq\Delta(m)$. Since $\Delta(m)$ and $\Theta(m)$ are numerical semigroups, we have that $\Delta(m)\setminus\Theta(m)$ is finite. Consequently, $\mathcal{A}_m$ is also finite. □

Now, let us consider the graph $\mathrm{G}(\mathcal{A}_m)$, where $\mathcal{A}_m$ is the set of vertices and $(S,S')\in\mathcal{A}_m\times\mathcal{A}_m$ is an edge if $S'=S\cup\{\mathrm{F}(S)\}$. The next result is analogous to Theorem 3.4.

Theorem 6.3. *The graph $\mathrm{G}(\mathcal{A}_m)$ is a tree with root equal to $\Delta(m)$. Moreover, the set of children of a vertex S (of such a tree) is*

$$\{S\setminus\{x\}\mid x\in\mathrm{msg}(S),\ x\neq m,\ x>\mathrm{F}(S)\text{ and }S\setminus\{x\}\in\mathcal{A}\}.$$

By applying Theorem 6.3 and Corollary 3.6, we can get easily $\mathrm{G}(\mathcal{A}_m)$ such as we show in the following example.

Example 6.4. We are going to depict $\mathrm{G}(\mathcal{A}_5)$, that is, the tree of the numerical $\mathcal{A}$-semigroups with multiplicity equal to 5.

$$\begin{tikzcd}[column sep=small]
 & & & \langle 5,6,7,8,9\rangle = \Delta(5) \arrow[dll, dash] \arrow[d, dash] \arrow[dr, dash] & \\
 & \langle 5,7,8,9,11\rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle 5,6,8,9\rangle \arrow[d, dash] & \langle 5,6,7,9\rangle \\
\langle 5,8,9,11,12\rangle \arrow[d, dash] & & \langle 5,7,9,11,13\rangle & \langle 5,6,9,13\rangle & \\
\langle 5,9,11,12,13\rangle \arrow[d, dash] & & & & \\
\langle 5,9,11,13,17\rangle = \Theta(5) & & & &
\end{tikzcd}$$

Remark 6.5. In [10] it was introduced the concept of Frobenius pseudo-variety, which generalizes the idea of Frobenius variety and the notion of m-variety (see [3]). Following Section 3, we can show that $\mathcal{A}_m$ is a Frobenius pseudo-variety. Thus, we have another way to get the tree associated to $\mathrm{G}(\mathcal{A}_m)$.

It is obvious that $\mathrm{F}(\Delta(m)) = \mathrm{g}(\Delta(m)) = m-1$. Now we are interested in computing the Frobenius number and the genus of $\Theta(m)$. For that, several concepts and results are introduced.

If S is a numerical semigroup and $n \in S \setminus \{0\}$, then we define the *Apéry set* of n in S (see [2]) as the set $\mathrm{Ap}(S,n) = \{s \in S \mid s-n \notin S\}$. It is clear (see [14, Lemma 2.4]) that $\mathrm{Ap}(S,n) = \{\omega(0)=0, \omega(1), \dots, \omega(n-1)\}$, where $\omega(i)$ is, for each $i \in \{0,\dots,n-1\}$, the least element of S that is congruent with i modulo n. The next result is [14, Proposition 2.12].

Lemma 6.6. *Let S be a numerical semigroup and let $m \in S \setminus \{0\}$. Then:*
(1) $\mathrm{F}(S) = \max(\mathrm{Ap}(S,m)) - m$,
(2) $\mathrm{g}(S) = \frac{1}{m}\left(\sum_{w\in \mathrm{Ap}(S,m)} w\right) - \frac{m-1}{2}$.

It is clear that $\Theta(1) = \mathbb{N}$ and, therefore, $\mathrm{Ap}(\Theta(1),1) = \{0\}$. In the following result we determine the Apéry sets $\mathrm{Ap}(\Theta(m),m)$, for all $m \geq 2$.

Proposition 6.7. *Let m be an integer greater than or equal to two.*
(1) $\mathrm{Ap}(\Theta(m),m) = \{(k+1)m \pm k \mid k \in \{1,\dots,\frac{m-1}{2}\}\} \cup \{0\}$ *if m is odd.*
(2) $\mathrm{Ap}(\Theta(m),m) = \{(k+1)m \pm k \mid k \in \{1,\dots,\frac{m-2}{2}\}\} \cup \{(\frac{m}{2}+1)m - \frac{m}{2}\} \cup \{0\}$ *if m is even.*

Proof. (1) It is obvious that $\{(k+1)m \pm k \mid k \in \{1,\dots,\frac{m-1}{2}\}\} \cup \{0\}$ is a subset of $\Theta(m)$ with cardinality equal to m. Therefore, if we show that $\{(k+1)m-k-m, (k+1)m+k-m\} \cap \Theta(m) = \{km-k, km+k\} \cap \Theta(m) = \emptyset$ for each $k \in \{1,\dots,\frac{m-1}{2}\}$, then we will finish the proof.

Indeed, from Corollary 4.12, if $km+k \in \Theta(m)$, then $im-(i-1) \leq km+k \leq im+(i-1)$ for some $i \in \mathbb{N}\setminus\{0\}$. Thus, from the right inequality, we have $1 \leq (i-k)(m+1)$ and, therefore, $1 \leq i-k$. But, from the left inequality, we get that $(i-k)\frac{m-1}{2} + \frac{1}{2} \leq k$. Then we conclude that $k > \frac{m-1}{2}$, in contradiction with the hypothesis.

For $km-k$ we can argue in the same way, simply by changing the role of inequalities.

(2) First of all, observe that, if $m=2$, then we assume that $\{1,\dots,\frac{2-2}{2}\} = \{1,\dots,0\} = \emptyset$. Thus,

$$\mathrm{Ap}(\Theta(2),2) = \left\{\left(\frac{2}{2}+1\right)2 - \frac{2}{2}\right\} \cup \{0\} = \{0,3\},$$

which is trivially true.

Now, if m is even and greater than two, then the reasoning is similar to the case in which m is odd. □

If m is even, then we have that $(\frac{m}{2}+1)m - \frac{m}{2} > (\frac{m-2}{2}+1)m + \frac{m-2}{2}$. From this inequality, Lemma 6.6 and Proposition 6.7, we have the next result.

Corollary 6.8. *Let m be an integer greater than or equal to two.*
(1) *If m is odd, then* $\mathrm{F}(\Theta(m)) = \frac{(m-1)(m+1)}{2}$ *and* $\mathrm{g}(\Theta(m)) = \frac{(m-1)(m+3)}{4}$.
(2) *If m is even, then* $\mathrm{F}(\Theta(m)) = \frac{(m-1)m}{2}$ *and* $\mathrm{g}(\Theta(m)) = \frac{(m-2)m}{4} + m - 1$.

Observe that the previous corollary is not true for $m = 1$ because $\Theta(1) = \mathbb{N}$ and $\mathrm{F}(\mathbb{N}) = -1$.

Recalling that the *height* of a tree T is the maximum of the lengths of the paths that connect each vertex with the root, we have that the height of $\mathrm{G}(\mathcal{A}_5)$ is 4 (see Example 6.4). In general, it is clear that the height of the tree $\mathrm{G}(\mathcal{A}_m)$ is equal to $\mathrm{g}(\Theta(m)) - \mathrm{g}(\Delta(m))$. Thus, having in mind that $\mathrm{g}(\Delta(m)) = m - 1$, by applying Corollary 6.8, we get the following result.

Corollary 6.9. *Let m be a positive.*
(1) *If m is odd, then* $\mathrm{G}(\mathcal{A}_m)$ *is a tree with height equal to* $\frac{(m-1)^2}{4}$.
(2) *If m is even, then* $\mathrm{G}(\mathcal{A}_m)$ *is a tree with height equal to* $\frac{(m-2)m}{4}$.

Remark 6.10. Taking $\Theta(m) = \{km - z \mid k \in \mathbb{N} \setminus \{0\},\ z \in \mathbb{N},\ \text{and } z < k\} \cup \{0\}$, we obtain analogous results for the family $\mathcal{B}$ shown in Remark 1.2.

Acknowledgment: The authors would like to thank the referee for providing constructive comments and help in improving the contents of this paper.

Funding: Both authors are supported by the project MTM2014-55367-P, which is funded by Ministerio de Economía y Competitividad and Fondo Europeo de Desarrollo Regional FEDER, and by the Junta de Andalucía Grant Number FQM-343. The second author is also partially supported by Junta de Andalucía/Feder Grant Number FQM-5849.

References

[1] F. Ajili and E. Contejean, Avoiding slack variables in the solving of linear diophantine equations and inequations, *Theoret. Comput. Sci.* **173** (1997), no. 1, 183–208.
[2] R. Apéry, Sur les branches superlinéaires des courbes algébriques, *C. R. Math. Acad. Sci. Paris* **222** (1946), 1198–1200.
[3] M. Bras-Amorós, P. A. García-Sánchez and A. Vico-Oton, Nonhomogeneous patterns on numerical semigroups, *Internat. J. Algebra Comput.* **23** (2013), 1469–1483.
[4] M. Bras-Amorós and K. Stokes, The semigroup of combinatorial configurations, *Semigroup Forum* **84** (2012), 91–96.
[5] L. Bryant, Goto numbers of a numerical semigroup ring and the Gorensteiness of associated graded rings, *Comm. Algebra* **38** (2010), 2092–2128.
[6] M. Delgado, P. A. García-Sánchez and J. Morais, "Numericalsgps": A GAP package on numerical semigroups, www.gap-system.org/Packages/numericalsgps.html.
[7] L. Pottier, Bornes et algorithme de calcul des générateurs des solutions de systèmes diophantiens linéaires, *C. R. Math. Acad. Sci. Paris* **311** (1990), 813–816.
[8] J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press, Oxford, 2005.
[9] A. M. Robles-Pérez and J. C. Rosales, The numerical semigroup of phrases' lengths in a simple alphabet, *Sci. World J.* **2013** (2013), Article ID 459024.
[10] A. M. Robles-Pérez and J. C. Rosales, Frobenius pseudo-varieties in numerical semigroups, *Ann. Mat. Pura Appl. (4)* **194** (2015), 275–287.
[11] J. C. Rosales, Families of numerical semigroups closed under finite intersections and for the Frobenius number, *Houston J. Math.* **34** (2008), 339–348.
[12] J. C. Rosales, M. B. Branco and D. Torrão, Sets of positive integers closed under product and the number of decimal digits, *J. Number Theory* **147** (2015), 1–13.
[13] J. C. Rosales and P. A. García-Sánchez, *Finitely Generated Commutative Monoids*, Nova Science, New York, 1999.
[14] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, Dev. Math. 20, Springer, New York, 2009.
[15] K. Stokes and M. Bras-Amorós, Linear, non-homogeneous, symmetric patterns and prime power generators in numerical semigroups associated to combinatorial configurations, *Semigroup Forum* **88** (2014), 11–20.