Universidad de Granada

Facultad de Ciencias

# The Wilsonian Renormalization Group in Gauge/Gravity Duality

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July 2017

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Departamento de Física Teórica y del Cosmos

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Editor: Universidad de Granada. Tesis Doctorales
Autor: Francisco Javier Martínez Lizana
ISBN: 978-84-9163-639-7
URI: http://hdl.handle.net/10481/48622

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# Agradecimientos

La investigación seguramente sea una de las actividades intelectuales más gratificantes para quienes la realizamos. Los momentos fructíferos, en los que se entiende por primera vez alguna cuestión, son realmente excitantes. Sin embargo, ésta también conlleva momentos de falta de inspiración, en los cuales persiste la sensación de andar en círculos, y los cuales son realmente frustrantes. Creo que ninguna tesis está exenta de ambos, y que de hecho, probablemente, abunden los segundos.

Saber terminar una tesis tampoco es fácil. Por un lado, siempre quedarán líneas y caminos por explorar, que animan al científico a avanzar continuamente, siendo difícil saber dónde concluir un trabajo de investigación. Por otro, una tesis no puede eternizarse, y las presiones directas o indirectas para terminarla, aunque necesarias, no son nada despreciables.

A pesar de estas dificultades, he tenido la infinita suerte de contar con el apoyo de unos estupendos compañeros, familiares, y amigos, que siempre me han ayudado en los momentos más difíciles, y a los cuales me gustaría dedicar las siguientes líneas.

En primer lugar, a mi director de tesis y amigo, Manolo Pérez-Victoria, uno de los físicos que más sabe de Teoría Cuántica de Campos que conozco. Por escucharme siempre. Porque algunas ideas, aunque en su concepción parezcan bien definidas, a menudo necesitan ser expresadas y discutidas para que tomen forma completa. Gracias por toda la física que he aprendido contigo. Por hacerme ver la importancia de los detalles, y del ser cuidadoso. Gracias por ayudarme con mis puntos débiles y en mis momentos difíciles. Esta tesis no sería lo mismo (si es que sería) sin ti.

Porque la ciencia no es individualista y no se puede llevar a cabo en solitario, gracias a los geniales físicos con los que he tenido la oportunidad de trabajar a lo largo de este tiempo. A Jorge de Blas en mis inicios en física de partículas. A Mario

Araújo, Daniel Areán y Johanna Erdmenger en mi estancia en Munich, y posterior continuación de nuestra colaboración. A Tim Morris en Southampton, el cual ha sido un referente para uno de los temas principales de esta tesis. Debo mencionar también a Vijay Balasubramanian. Si bien nuestro proyecto no llegó a dar frutos, tuve la inmensa oportunidad de poder discutir mucha física con él durante mi estancia en Filadelfia. Le agradezco además a Daniel y a Emilian Dudas que se hayan tomado la molestia de leer esta tesis, como expertos para la mención internacional; y a Tim, Vijay, Johanna, Manolo y José Santiago, por sus respectivas cartas de recomendación.

Porque una tesis requiere muchas horas, quisiera agradecer a mis compañeros de despacho, aquellos que han estado sentados a escasos metros mía, por hacer estas horas mucho más llevaderas. A Olaf Kittel, con el cual solo compartí algunos meses antes de que volviera a Alemania. A Mikael Chala, cuyos pasos son un claro ejemplo a seguir para mí (o al menos a intentarlo). A Pablo Guaza, por su ayuda y consejos, no solo en cuestiones de informática, sino también de la vida misma. Y a José Alberto Orejuela, porque echaré mucho de menos nuestras discusiones si alguna vez faltan.

A todos mis compañeros de doctorado y amigos ligados a la universidad de estos años, por ser como son, y por todos esos buenos momentos juntos que hemos pasado. A Alice, por su alegría contagiosa, a Ben, por su idiosincrasia británica, a Adriano, por su bondad brasileña, a Leo, por preocuparse siempre por todos, a Migue, por su integridad, a Irene, a la que considero mi *madrina* de tesis, a Nico, por el arte que tiene, a Adrián, por su gran ingenio, a Álvaro, por su amistad, a Pablo Martín, por esas noches en vela que no fueron solitarias, a Juanpe, por tener siempre la mejor historia, a Laura, por ser tan tan pequeña, a María del Mar, por sus abrazos, a Elisa, por su continua disposición a ayudar, a Pedro, por su simpatía sobrenatural, a Paula, por su constante sonrisa, a Fernando, porque he aprendido mucho de su forma de afrontar la vida, a Pablo Ruiz, al que quiero y también odio un poquito al mismo tiempo, a Paloma, por ser tan luchadora, a Jordi, por querer continuamente subirme la moral, a Rafa, por aportarme nuevos puntos de vista, a José Antonio, por su honestidad, a Esperanza, por su gran vitalidad, a Germán, por su increíble sabiduría emocional, y a Javi Blanco, por sus agradables visitas en momentos de agobio. Del mismo modo, no me puedo olvidar tampoco de Bruno Zamorano, Alberto Gascón, Laura Molina, Patricia Sánchez, Pablo Guerrero, Alba Soto, Juan Carlos Criado, Alejandro Jiménez, Mariano Caruso, Roberto Barceló, Adrián Carmona, Adrián Ayala, José Luis Navarro, Tomás Ruiz, Keshwad Shahrivar, Ana Belén Bonhome, Diego Noguera y José Rafael.

A mis profesores durante la carrera y el máster, y ahora compañeros de departamento: Mar Bastero, Manel Masip, José Ignacio Illana, Bert Janssen y Antonio Bueno. Así como a aquellos que no me llegaron a dar clase: Juan Antonio Aguilar, Inés Grau, Elvira Gámiz, Javier López Albacete, Sergio Navas, Roberto Pittau y Roberto Vega-Morales. A Fernando Cornet, por ejercer de forma brillante como tutor de esta tesis. A Paco del Águila y José Santiago, porque si bien no trabajé con ellos, siento que se preocuparon por mí desde el principio como si fuera su propio estudiante.

Como doctorando además, he tenido la magnífica oportunidad de asistir a numerosas escuelas y congresos, y realizar tres estancias de investigación en centros extranjeros, donde he conocido a gente realmente fantástica. Me gustaría mandar un cariñoso saludo al grupo de AdS/CFT del Max Planck para física de Munich, a Ana Solaguren-Beascoa, Shangyu Sun, Stefano Di Vita, y a mis compañeros de piso de Munich, Alessandro Manfredini y Sebastian Paßehr. A Elena Ureña, Juri Fiaschi, Luca Panizzi, Felipe Rojas Abatte y Nathan Shammah, de Southmapton. A Tasha Billings, Lucas Secco, Anu Sharma y Mariana Carrillo, de Filadelfia. Y a Miquel Triana, Daniel Cámpora, Ignacio Salazar y Amadeo Jiménez Alba, con quienes me he ido encontrando en repetidas ocasiones a lo largo y ancho del mundo.

Gracias a mi familia, a la que quiero con locura. A mi padre y mi madre, por darme ánimos constantemente y creer siempre en mí. Por preocuparse por mi persona, y no dejar que me mal alimentara nunca (especialmente, en la recta final de la tesis). A mi hermana, mi amiga y confidente de toda la vida. Por estar siempre ahí. Gracias a los tres, por cuidarme tanto y tan bien durante tantos años. Por apoyarme en todo momento, independientemente de como fueran las cosas. Gracias a mis titos, primos y abuelos (tanto a los que he tenido la suerte de conocer, como a los que no). Porque entre todos, formamos una familia única que no cambiaría por nada.

A mis compañeros de carrera, que siempre serán mis amigos por años que pasen. A Pedro, Migue, Oché, Alex, Pablo Sánchez, Martín, Maribel, Rebe, Lolo, Berni, Nuri, Cuchi, Alfonso, Alba, Victor, Trino y Juan. A Jesús y Marta, o mejor dicho, *Farruquito* y *la Cachonda*. Tendríamos que haber lanzado nuestro *hit* del verano. Habéis sido un gran apoyo estos años. A mis compañeros de piso que aún no he nombrado, Miguel y Nico. A mis amigos del instituto, mis *amigos de verdad*: Pablo, Mingo,[1] Juanmi, Juanfri, Fran, Eva, José Carlos, Lety, Rafa y Migue. A Angelilla, por este último año.

Para terminar, dejo este párrafo para dedicárselo a (y disculparme de) muchos que no he podido mencionar explícitamente. Y es que, tres páginas de agradecimientos se me antojan cortas para tantísima gente que me viene a la cabeza y a la que me gustaría dedicar unas palabras.

A todos vosotros, de nuevo, gracias.

[1]Perdona, pero si escribo tu nombre, Juan, dentro de unos años no sabré a quien me refería :P

A mi padre, mi madre y mi hermana,

*Porque siempre han estado ahí para apoyarme, y sé que siempre estarán.*

# Contents

# Acronyms

| | |
|---|---|
| AdS | Anti-de Sitter space |
| CFT | Conformal Field Theory |
| IR | Infrared |
| LHS | Left hand side |
| OPE | Operator Product Expansion |
| QCD | Quantum Chromodynamics |
| QFT | Quantum Field Theory |
| RG | Renormalisation Group |
| RHS | Right hand side |
| ST | String Theory |
| SUGRA | Supergravity |
| SUSY | Supersymmetry |
| UV | Ultraviolet |

# Notations and Conventions

*We could, of course, use any notation we want; do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations.*

Richard P. Feynman [1]

## Coordinates and Indices

Lorentz indices are as usual written with the greek letters $\mu, \nu, \ldots$. We use $x, y, \ldots$ to represent $d$-dimensional space-time coordinates, which will be written sometimes as a continuous index: $g^x \equiv g(x)$. Likewise, $p, q, \ldots$ represent $d$-dimensional covariant momentum coordinates. We mostly work with an Euclidean metric $\delta_{\mu\nu}$, and rescalings of it. We define

$$(\gamma_z)_{\mu\nu} = (\gamma^{1/z})_{\mu\nu} = \frac{1}{z^2}\delta_{\mu\nu}. \tag{1}$$

The modulus of the vectors $x$ or $p$ will be indicated writing as a subscript the metric used to compute it:

$$\begin{aligned} x_\gamma =& \sqrt{x^\mu x^\nu \gamma_{\mu\nu}}, \\ p_\gamma =& \sqrt{p_\mu p_\nu \gamma^{\mu\nu}}. \end{aligned} \tag{2}$$

If the metric is the canonical one $\gamma = \delta$, we will also use $|x| = x_\delta$. Note that $x_{\gamma_z} = |x|/z$, while $p_{\gamma_z} = z|p|$ (due to its covariant character).

Latin letters $a, b, \ldots, i, j, \ldots$ are used to represent discrete indices in general (flavour indices, including Lorentz indices if necessary).[2] We also use the DeWitt condensed notation, with the index $\alpha, \beta, \ldots, \sigma, \ldots$ indicating a set of flavour and space-time indices; for instance $g^\alpha = g^{ax} = g^a(x)$. The Einstein summation convention is used for both discrete and continuous indices, with repeated space-time indices indicating an integration in that variable. As an example,

$$\begin{aligned} k_{\alpha_1\alpha_2} g^{\alpha_1} g^{\alpha_2} &= k_{a_1x_1a_2x_2}\, g^{a_1x_1} g^{a_2x_2} \\ &= \sum_{a_1,a_2} \int d^dx_1 d^dx_2\, k_{a_1x_1a_2x_2}\, g^{a_1}(x_1) g^{a_2}(x_2). \end{aligned} \tag{3}$$

The usual parenthesis notation for the argument of functions will be only used for continuous superindices, for the reasons we explain later. The Einstein convention only applies when the involved arguments are written like indices. Thus, in the first line of (3) the integral is implicit, but in the second one, it is not. Sometimes we will find expressions in the second line more convenient, and therefore we will write integrals explicitly.

In order to keep the formulas invariant under diffeomorphisms of the $d$-dimensional space-time, the generalized tensors have to transform conveniently. If we choose $g^x = g(x)$ to transform as a scalar field, the generalized tensor with $n$ lower continuous indices will transform as a density of weight $n$ (or as a tensor density if it also has Lorentz indices).

Indices inside a parenthesis label the entries of diagonal (generalized) matrices. Therefore, there is no sum or integral in the equation

$$q^\alpha = \lambda_{(\alpha)} g^\alpha, \tag{4}$$

while

$$\lambda_{(\alpha)} k_\alpha g^\alpha = \sum_a \int d^dx \lambda_{(a)}(x)\, k_{ax}\, g^a(x). \tag{5}$$

The operator Sym acting over a tensor or a function symmetrizes over the indicated

[2] In particular, $a$ will be used to label general operators, $b$ will label double-trace operators, and $i$ and $j$ single-trace operators or bulk fields. This will also be explained in the text.

indices (including possible continuous indices):

$$\underset{\{(i_k,p_k)\}_{k=1}^n}{\mathrm{Sym}} A_{i_1\ldots i_n}(p_1,\ldots,p_n) = \frac{1}{n!}\sum_{\sigma\in S_n} A_{i_{\sigma(1)}\ldots i_{\sigma(n)}}(p_{\sigma(1)},\ldots,p_{\sigma(n)}). \tag{6}$$

Sometimes, we also use the parenthesis notation to symmetrize,

$$A_{(i_1\ldots i_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} A_{i_{\sigma(1)}\ldots i_{\sigma(n)}}. \tag{7}$$

## Generalized metric

If the $d$-dimensional space-time parametrized by the continuous coordinates $x$ has a defined metric $\gamma$, we can construct a diffeomorphism-invariant generalized metric to raise and lower continuous indices:

$$\delta_{x_1x_2} = \sqrt{|\gamma|}\,\delta(x_1-x_2), \tag{8}$$

$$\delta^{x_1x_2} = \frac{1}{\sqrt{|\gamma|}}\delta(x_1-x_2). \tag{9}$$

Therefore, $g^a_x = \sqrt{|\gamma|}\,g^{ax} = \sqrt{|\gamma|}\,g^a(x)$, and (3) and (5) can be rewritten like

$$k_{\alpha_1\alpha_2}g^{\alpha_1}g^{\alpha_2} = \sum_{a_1,a_2}\int d^dx_1d^dx_2|\gamma|\,k_{a_1a_2}(x_1,x_2)\,g^{a_1}(x_1)g^{a_2}(x_2), \tag{10}$$

$$\lambda_{(\alpha)}k_\alpha g^\alpha = \sum_a \int d^dx\sqrt{|\gamma|}\lambda_{(a)}(x)\,k_a(x)\,g^a(x). \tag{11}$$

Here, $\gamma$ is the metric in the $d$-dimensional spaces parametrized by $x_1$ and $x_2$, and we have written the double $d$-form $k_{a_1x_1a_2x_2}$ in terms of a tensor $k_{a_1a_2}(x_1,x_2) = k^{x_1,x_2}_{a_1a_2}$. The square root of $|\gamma|$, the determinant of the metric, is used to raise space-time indices.

## Fourier transform

If our $d$-dimensional spacetime is flat, and thus, the metric $\gamma$ is constant, we can define the Fourier transform of a generalized tensor $A^{x_1\cdots x_n}_{y_1\cdots y_m}$. It will be denoted by $\hat{A}$

and defined as

$$\hat{A}^{q_1\cdots q_m}_{p_1\cdots p_n} = \int d^d x\, d^d y\, \exp\left[ i \left( \sum_{j=1}^{n} p_{j\mu} x_j^\mu - \sum_{k=1}^{m} q_{k\mu} y_k^\mu \right) \right] A^{x_1\dots x_n}_{y_1\dots y_m}. \tag{12}$$

In momentum space, $\hat{B}^p = \sqrt{|\gamma|}\hat{B}_{-p}$, with $\hat{B}(p) = \hat{B}^p$. In many cases, we will work with translation-invariant functions:

$$A^{x_1\dots x_n}_{y_1\dots y_m} = A^{x'_1\dots x'_n}_{y'_1\dots y'_m}, \tag{13}$$

with ${x'}_i^\mu = x_i^\mu + a^\mu$ and ${y'}_i^\mu = y_i^\mu + a^\mu$, and $a^\mu \in \mathbb{R}^d$ any vector. In this case, the Fourier transform has an overall momentum-conserving delta function. To simplify some formulas, when there is an overall delta function of momenta conservation, we define

$$(2\pi)^d \delta\Big( (p_1 + \dots p_n) - (q_1 + \dots + q_m) \Big) \breve{A}^{p_1\dots p_n}_{q_1\dots q_m} = \hat{A}^{p_1\dots p_n}_{q_1\dots q_m}. \tag{14}$$

# Chapter 1

# Introduction

*I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small – not neglecting it just because it is infinitely great and you do not want it!*

Paul Dirac [2]

Just as Quantum Mechanics has a deep impact on any serious student of physics, its application to the relativistic domain, that is, Quantum Field Theory (QFT), strikes those who face its study for the first time. Most notably, the appearance of infinities here and there has no parallel in classical physics or in Quantum Mechanics of discrete systems (except in particular cases with singular sources and boundaries). The "shell game" of *renormalization*, necessary to make sense of this situation, may look like the last resort of desperate scientists. This feeling was actually shared not so long ago by the early formulators of quantum electrodynamics.

However, a better understanding of renormalization in QFT emerged later. The crucial insights were provided by the work of Kenneth G. Wilson and others on the *Renormalization Group* (RG), which studies the different appearance of a QFT at different energy scales. In the Wilsonian formulation, this is done by regularizing the theory with some cutoff and studying how the changes in this cutoff are exactly compensated by changes in the action. The RG allows to give a clear and deep meaning to the short-distance divergences of QFT, which essentially reflect sensitivity to higher scales, and to the process of renormalization, in which the unknown ultraviolet details are parametrized by a set of local operators with arbitrary parameters. The Wilsonian RG provides a rigorous non-perturbative definition of continuous renormalizable QFT as relevant deformations of conformal field theories. Ideas such as asymptotic safety are based on this picture. Also effective field theories, a modern paradigm of physics, are best understood from the Wilsonian point of view. And not only are the renormalization process and the RG well-defined; they introduce new concepts and tools with deep physical implications.

In this thesis, we study in detail some fundamental aspects of the Wilsonian RG, including its precise relation with renormalization. These ideas are general and can in principle be used in strongly-coupled theories. However, the limitations of calculability are strong. In actual calculations one needs drastic approximations, such as truncations, or to make use of some kind of perturbation theory. One possible perturbative expansion that can be used in non-abelian gauge theories is the expansion in inverse powers of $N$, the rank of the group [3]. The large-N theories take the form of a classical string theory [4]. In some particular cases, explicit dual formulations in terms of the classical degrees of freedom have been found. They are examples of the famous Gauge/Gravity duality, also known as holography or AdS/CFT correspondence [5].

The discovery of Gauge/Gravity duality has possibly been to the most important landmark in theoretical physics in the last twenty years. It has its origin in String Theory (ST) and establishes an equivalence between certain QFT and Gravity theories. Therefore, it provides consistent descriptions of certain models of Quantum Gravity.[1] One basic property of this correspondence is that it relates strongly-coupled theories to weakly-coupled ones, and vice versa. In particular, strongly-coupled large-N theories can be described by their classical gravity duals. This makes Gauge/Gravity duality one of the few tools (if not the only one) available to perform analytical quantitative calculations in strongly-coupled regimes. For this reason, it has been applied to many different problems in Particle and Condensed Matter Physics.

In the known examples of Gauge/Gravity duality, the gravity theory is formulated

[1]It is remarkable that, while defining a Quantum Gravity Theory as a QFT using traditional methods seems hopeless, quantum gravity and space-time itself holographically emerge as effective properties of certain quantum theories.

in a higher-dimensional space with a geometry that is asymptotically Anti de-Sitter. These spaces have a boundary, which plays a prime role in the correspondence. In the explicit calculations, large-volume divergences appear, already at the classical level, in the integrations close to the boundary. They are dual to the UV divergences in the field theory. The most popular way to regularize them is to introduce an artificial boundary that excludes the region close to the AdS boundary. Changing the position of this cutoff surface can be compensated by changes in an action localized at the new boundary. This is the holographic realization of Wilsonian RG invariance. Indeed, the position along the extra-dimensional radial direction of the gravity theory is related to the energy scale in the gauge theory. Moreover, it is possible to make sense of the near-boundary divergences by a procedure of holographic renormalization. As we show here, this procedure is intimately related to the holographic RG, just as in QFT.

In fact, the main purpose of this thesis is to offer a unified description, in both sides of the duality, of renormalization, the Wilsonian RG and their precise connection. We will refine some standard tools of QFT and show that they can be directly applied to the gravity side of the correspondence. This will allow us to solve some existing puzzles, to generalize some previous holographic methods or look at them from different perspectives, and to provide clear insights on the meaning of holographic renormalization.

The thesis is naturally organized in two distinct parts: the first one is dedicated to purely field-theoretical developments, while the second one is devoted to their implementation in holography.

Part I comprises Chapters 2 and 3. The main results presented in this part have been published in [6, 7]. In Chapter 2, a comprehensive presentation of the Wilson RG is given. We introduce a novel geometric formulation of it.[2] Even if motivated by its later application to holography, we believe this formulation is very natural in QFT and has a more general interest. We analyse the fixed points of the flow, associated to scale-invariant theories, and their neighbourhood in the space of theories. We show how the RG flows can be simplified through an appropriate choice of the coordinates that parametrize the space of theories, which are called normal coordinates. These coordinates can be used, in particular, to extract conformal anomalies or beta functions in mass-independent schemes, even though the Wilson flows are intrinsically defined

[2]Other geometric formulations of the RG that share some elements with our formulation have been previously given.

with a mass cutoff. All these developments are illustrated with perturbative examples. In one of them, we study the particular features of these techniques in large N theories.

Chapter 3 is devoted to the study of renormalization and its connection with the formulation of RG presented in Chapter 2. We focus mainly on the renormalization of correlation functions of composite operators at fixed points of the flow. We provide explicit formulas that relate renormalized operators and counterterms with perturbative expansions of the RG flows. We also prove that the normal coordinates introduced in Chapter 2 are associated with a class of minimal subtraction renormalization schemes. This formulation also allows to discern when logarithmic behaviours appear in the renormalized operators or counterterms. Finally, the relevant deformations of fixed points, which describe non-scale invariant theories, will be studied under the Wilsonian optics. We will see how the continuum limit of renormalizable theories is formulated in this general picture.

Part II comprises Chapters 4, 5, 6. The main results presented in this part have been published in [6, 8]. We apply the very same techniques of Part I to the gravity side of AdS/CFT. Actually, Chapters 5 and 6 have a structure parallel to the ones of Part I. For completeness, we also include Chapter 4 as an introduction to the AdS/CFT correspondence. There, we describe the main ideas behind the correspondence and its basic features. We also review the standard holographic renormalization method, which is slightly different from the one we follow in our work.

In Chapter 5 we formulate carefully the Wilsonian RG in holography. Most of the elements introduced in Chapter 4 will appear again, in a new guise. We use exactly the same geometric formulation as the one presented in Part I, and thereby show their equivalence. We show that the holographic RG flows exhibit the especial large N features emphasized in Chapter 2. In particular, both the large N and the classical gravity flows “factorize”, in a sense to be explained in that chapter. Our holographic developments are explicitly illustrated in a theory of scalar fields fluctuating in AdS space. The backreaction of the metric is neglected. For this theory, we calculate fixed points and RG flows in the neighbourhood of the interacting fixed point. This analysis includes not only eigendeformations, but also non-linear contributions.

Just as Chapter 3 connects the Wilsonian RG with renormalization, in Chapter 6 we apply the holographic RG to the holographic renormalization of correlation functions.

We focus, as an example, on three-point functions between scalar operators of arbitrary dimensions. We discuss different techniques that can be used to tackle the renormalization of these correlation functions. The most obvious one, which has been widely used in the past, imposes Dirichlet conditions on the cutoff boundary. However, we will note that the correlators containing irrelevant operators cannot always be renormalized with this method. Indeed, it turns out that multi-trace counterterms are required in the renormalization of single-trace operators, already at the linear level. As we explain, the reason behind this problem is the fact that the space of *bare* single-trace operators is not stable under Wilsonian RG evolution. Our formulation based on general boundary conditions provides a simple extension of holographic renormalization that solves the problem in a natural way, consistent with the general field-theoretical methods. We also pay special attention to the logarithmic behaviour that these systems can present, depending on the relation of the mass dimension of the operators involved.

The main results of this thesis, our conclusions and a few possible future research lines are presented in Chapter 7. Finally, we include Appendix A that contains a technical discussion of the Poincaré-Dulac theorem, which plays a crucial role in the developments made in Chapter 2.

During my Ph.D. time, I have also worked on the phenomenology of particle physics [9, 10], and on applications of the Gauge/Gravity duality to the study of Condensed Matter systems [11, 12]. These works have contributed enormously to the development of my vision of QFT and the Gauge/Gravity duality. However, because they are not directly related to the topic of the thesis, they are not presented here.

# Introducción

Así como la Mecánica Cuántica tiene un profundo impacto en cualquier estudiante serio de física, su aplicación al dominio relativista, esto es, la Teoría Cuántica de Campos, impresiona a aquellos que afrontan su estudio por primera vez. La aparición de infinitos aquí y allí no tiene paralelismo en física clásica o Mecánica Cuántica de sistemas discretos (excepto en casos particulares con fuentes singulares y fronteras). El "juego" de la renormalización, necesario para dar sentido a esta situación, puede parecer el último recurso desesperado de los científicos. Este sentimiento fue de hecho compartido no mucho tiempo atrás por los fundadores de la electrodinámica cuántica.

Sin embargo, una mejor comprensión de la renormalización en Teoría Cuántica de Campos apareció más tarde. Algunos avances cruciales fueron proporcionados por el trabajo de Kenneth G. Wilson y otros en el *grupo de renormalización*, que estudia la distinta apariencia de una teoría cuántica de campos a diferentes escalas. En la formulación Wilsoniana, esto se hace regularizando la teoría con algún corte o "cut-off" y estudiando como los cambios de este "cutoff" son exactamente compensados por cambios en la acción. El grupo de renormalización permite dar un claro y profundo significado a las divergencias ultravioletas de la Teoría Cuántica de Campos, que esencialmente reflejan la sensibilidad a escalas más altas, y al proceso de renormalización, en el cual, los detalles desconocidos ultravioletas son parametrizados por un conjunto de operadores locales con parámetros arbitrarios. El grupo de renormalización wilsoniano proporciona una rigurosa definición no perturbativa de las teorías cuánticas de campos renormalizables como deformaciones relevantes de teorías de campos conformes. Ideas como la *seguridad asintótica* están basadas en esta imagen. Además, las teorías de campos efectivas, un paradigma moderno en física, son mejor entendidas desde el punto de vista wilsoniano. Y no solo el proceso de renormalización y el grupo de renormalización quedan bien definidos, además permiten introducir nuevos conceptos y herramientas con profundas implicaciones físicas.

En esta tesis, estudiamos en detalle algunos aspectos fundamentales del grupo de renormalización wilsoniano, incluyendo su precisa relación con la renormalización. Estas ideas son generales y pueden usarse en principio en teorías fuertemente acopladas. Sin embargo, las limitaciones en los cálculos son importantes. En cálculos reales, se necesitan aproximaciones drásticas, tales como truncamientos, o el uso de algún tipo de teoría perturbativa. Un posible desarrollo perturbativo que puede usarse en teorías "gauge" no abelianas es las expansión en potencias inversas de N, el rango del grupo [3]. Las teorías con N grande toman la forma de una teoría de cuerdas clásica [4]. En algunos casos particulares, se han encontrado formulaciones duales explícitas en términos de grados de libertad clásicos. Ellas son ejemplos de la famosa dualidad Gauge/Gravedad, también conocida como holografía, o correspondencia AdS/CFT [5].

El descubrimiento de la dualidad Gauge/Gravedad posiblemente ha generado la revolución más importante en física teórica de los últimos veinte años. Tiene su origen en Teoría de Cuerdas y establece una equivalencia entre ciertas teorías de campos y teorías de gravedad. Por lo tanto, proporciona descripciones consistentes de ciertos modelos de Gravedad Cuántica.[3] Una propiedad básica de esta correspondencia es que relaciona teorías fuertemente acopladas con débilmente acopladas, y viceversa. En particular, teorías fuertemente acopladas con N grande pueden ser descritas por sus duales gravitatorios clásicos. Esto convierte a la dualidad Gauge/Gravedad en una de las pocas herramientas disponibles (si no la única) para hacer cálculos analíticos en regímenes fuertemente acoplados. Por esta razón, se ha aplicado a muchos problemas diferentes en Física de Partículas y Física de la Materia Condensada.

En los ejemplos conocidos de dualidad Gauge/Gravedad, la teoría de gravedad es formulada en espacios de dimensión más alta con una geometría que asintóticamente es Anti de-Sitter (AdS). Estos espacios tienen una frontera que juega un papel fundamental en la correspondencia. En los cálculos explícitos, divergencias de integrales asociadas al volumen infinito cercano a la frontera aparecen ya al nivel clásico. Éstas son duales a las divergencias ultravioletas de la teoría de campos. La forma más popular de

[3]Es remarcable que, mientras definir una teoría cuántica de gravedad como teoría cuántica de campos usando métodos tradicionales parece poco esperanzador, gravedad cuántica y el espaciotiempo en sí mismo emerjan holográficamente como propiedades efectivas de ciertas teorías cuánticas.

regularizarlas es introduciendo una frontera artificial que excluye la región cercana a la frontera del espacio AdS. Cambios de la posición de esta superficie de corte pueden ser compensados por cambios en una acción localizada en la nueva frontera. Esta es la realización holográfica de la invarianza del grupo de renormalización wilsoniano. De hecho, la posición a lo largo de la dirección radial extra de la teoría de gravedad está relacionada con la escala de energía de la teoría "gauge". Más aún, es posible dar sentido a las divergencias asociadas al volumen infinito por un procedimiento de renormalización holográfica. Como veremos aquí, este procedimiento está íntimamente relacionado con el grupo de renormalización holográfico, del mismo modo que en Teoría Cuántica de Campos.

El principal propósito de esta tesis es ofrecer una descripción unificada, en ambos lados de la dualidad, de la renormalización, del grupo de renormalización wilsoniano y la conexión precisa de ambos. Refinaremos algunas herramientas estándares y mostraremos que pueden aplicarse directamente al lado gravitatorio de la correspondencia. Esto nos permitirá resolver algunos problemas existentes, generalizar algunos métodos holográficos previos o estudiarlos desde diferentes perspectivas, y proporcionar ideas claras sobre el significado de la renormalización holográfica.

La tesis está naturalmente organizada en dos partes distintas: la primera está dedicada puramente a desarrollos en la Teoría Cuántica de Campos, mientras que la segunda está dedicada a su implementación en holografía.

La parte I comprende los capítulos 2 y 3. Los principales resultados presentados en esta parte han sido publicados en [6, 7]. En el capítulo 2 se presenta de manera extensa el grupo de renormalización wilsoniano. Introducimos una nueva formulación geométrica.[4] Aunque ésta ha sido motivada por su aplicación a holografía, creemos que aplica de manera muy natural en Teoría Cuántica de Campos, y tiene un interés más general. Analizamos puntos fijos del flujo, asociados a teorías con invarianza de escala, y entornos de éstos en el espacio de teorías. Mostramos como los flujos del grupo de renormalización pueden ser simplificados a través de una elección apropiada de las coordenadas que parametrizan el espacio de teorías: las coordenadas normales. Estas coordenadas pueden usarse, en particular, para extraer anomalías conformes o funciones beta en esquemas independientes de la masa, incluso aunque los flujos wilsonianos

[4] Otras formulaciones geométricas del grupo de renormalización que comparten algunos elementos con la nuestra han sido dadas previamente.

estén intrínsecamente definidos con un “cutoff” de masa. Todos estos desarrollos son ilustrados con ejemplos perturbativos. En uno de ellos, estudiamos las características particulares de estas técnicas en teorías con N grande.

El capítulo 3 está dedicado al estudio de la renormalización y su conexión con la formulación del grupo de renormalización presentada en el capítulo 2. Nos centramos principalmente en la renormalización de funciones de correlación de operadores compuestos en puntos fijos del flujo. Proporcionamos fórmulas explícitas que relacionan operadores renormalizados y contratérminos con las expansiones perturbativas de los flujos del grupo de renormalización. Además probamos que las coordenadas normales introducidas en el capítulo 2 están asociadas con una clase de esquemas de renormalización de sustracción mínima. Esta formulación también permite discernir cuando aparecen comportamientos logarítmicos en los operadores renormalizados o contratérminos. Finalmente, se estudian deformaciones relevantes de los puntos fijos bajo el enfoque wilsoniano. Éstas describen teorías sin invarianza de escala. Veremos cómo el límite continuo de teorías renormalizadas es formulado bajo esta imagen general.

La parte II comprende los capítulos 4, 5 y 6. Los principales resultados presentados en esta parte se encuentran en [6, 8]. Aplicamos las mismas técnicas de la parte I a la parte de gravedad de AdS/CFT. De hecho, los capítulos 5 y 6 tienen una estructura paralela a la de los capítulos de la parte I. Por completitud, también incluimos el capítulo 4 como introducción a la correspondencia AdS/CFT. Allí describimos las ideas principales detrás de la correspondencia y sus características básicas. También revisamos el método estándar de renormalización holográfica, el cual es ligeramente distinto del que seguiremos en nuestro trabajo.

En el capítulo 5 formulamos cuidadosamente el grupo de renormalización wilsoniano en holografía. Muchos de los elementos introducidos en el capítulo 4 aparecerán de nuevo, con un enfoque diferente. Usamos exactamente la misma formulación geométrica a la presentada en la parte I, y de aquí, mostramos su equivalencia. Mostramos que los flujos del grupo de renormalización holográfico exhiben las características especiales del límite de N grande enfatizados en el capítulo 2. En particular, tanto los flujos con N grande, como los de gravedad clásica “factorizan” en un sentido que será explicado en este capítulo. Nuestros desarrollos holográficos son ilustrados explícitamente con una teoría de campos escalares fluctuando en un espacio AdS. La reacción de la métrica es ignorada. Para esta teoría, calculamos puntos fijos y flujos del grupo de renormalización en el entorno de puntos fijos con interacciones. Este análisis incluye, no solo autodeformaciones, sino también contribuciones no lineales.

Del mismo modos que el capítulo 3 conecta el grupo de renormalización wilsoniano con la renormalización, en el capítulo 6 aplicamos el grupo de renormalización holográfico a la renormalización holográfica de funciones de correlación. Nos centramos, como ejemplo, en las funciones de tres puntos entre operadores escalares de dimensiones arbitrarias. Discutimos técnicas diferentes que pueden usarse para abordar la renormalización de estas funciones de correlación. La más obvia, que ha sido ampliamente usada en el pasado, impone condiciones tipo Dirichlet en la frontera de corte.

Sin embargo, notaremos que correladores con operadores irrelevantes no pueden ser siempre renormalizados con este método. En efecto, resulta que, ya al nivel lineal, contratérminos multi-traza son necesarios para la renormalización de operadores de traza única. Como explicaremos, la razón detrás de este problema se debe a que el espacio de operadores *desnudos* de traza única no es estable bajo la evolución del grupo de renormalización wilsoniano. Nuestra formulación basada en condiciones de frontera generales proporciona una extensión simple de la renormalización holográfica que resuelve el problema de una forma natural, consistente con los métodos propios de Teoría Cuántica de Campos. También prestamos atención al comportamiento logarítmico que pueden presentar estos sistemas dependiendo de la relación entre las dimensiones de los operadores involucrados.

Los principales resultados de esta tesis, nuestras conclusiones y algunas líneas futuras de investigación son presentadas en el capítulo 7. Finalmente, incluimos el apéndice A que contiene una discusión técnica del teorema de Poincaré-Dulac, que juega un papel crucial en los desarrollos hechos en el capítulo 2.

Durante mi periodo como estudiante de doctorado también he trabajado en fenomenología de física de partículas [9, 10], y en aplicaciones de la dualidad Gauge/ Gravedad al estudio de sistemas de Materia Condensada [11, 12]. Estos trabajos han contribuido enormemente al desarrollo de mi visión de la Teoría Cuántica de Campos y la dualidad Gauge/Gravedad. Sin embargo, debido a que no están directamente relacionados con el tema de la tesis, estos trabajos no son presentados aquí.

# Part I

# Field Theory

# Chapter 2

# Wilsonian Renormalization Group

*We are part of this universe; we are in this universe, but perhaps more important than both of those facts, is that the universe is in us.*

Neil deGrasse Tyson

The Wilsonian RG probably provides one of the clearest pictures of QFT [13–15]. The underlying idea consists in studying how the relevant degrees of freedom that describe a theory change with the scale we use to test it. Different implementations of this idea have been largely explored and applied so far. For instance, in real-space RG methods [16], the wave function of some state of the theory is projected to discard its non-interesting short-distance behaviour. Improvements of this method are the density matrix RG [17] or the Multiscale Entanglement Renormalization Ansatz (MERA) [18].[1] However, in this thesis, we focus on the *exact RG*. This RG implementation takes as fundamental object the regulated Euclidean partition function of a QFT, and studies how the action changes when UV degrees of freedom are integrated out. It provides a consistent definition of non-perturbative QFT and solves the problem of constructing a QFT. This implementation was initiated by Wilson [14] and further developed by others (see for instance [15, 22–25] and [26–32] for reviews).

In this chapter, we present the exact RG as developed in [7]. We borrow the language

[1] There is also a strong connection of Gauge/Gravity duality with this, and similar RG implementations. These developments fall outside the purpose of this thesis, but we recommend the following references to delve into them [19–21].

and tools of differential geometry to describe the space of theories where RG flows live. In order to use this formalism to compute correlation functions, the theory space we consider will have spacetime dependent couplings.[2]

The basic idea is to, somewhat loosely, treat the space of regulated theories as a manifold. In this formulation, the spacetime dependent couplings are understood as coordinates of theory space, the beta functions are vector fields, the operators are vectors and the correlation functions are tensors.[3] Special attention will be paid to the active role of the cutoff in the parametrization of theory space. Writing the equations in a coordinate-invariant fashion will allow us to easily change coordinates to find parametrizations that suit different purposes and put the exact RG flows in a manageable form.

For instance, given a fixed point of the RG flows, we identify *normal coordinates* around it, in which the beta functions and RG flows are particularly simple. At the linear level, this reduces to identifying the deformations of the fixed point that are eigenfunctions of the linearised RG evolution. These deformations are regularized versions of the primary operators at the fixed point, with eigenvalues simply related to their scaling dimensions.[4] The normal coordinates are an extension of this linear behaviour. When the dimensions take generic values, they are such that all the non-linear terms in the flows vanish. For exceptional values of the dimensions, on the other hand, non-linear terms are unavoidable but can be reduced to a minimal set. These terms give rise to the usual Gell-Mann-Low beta functions of mass-independent schemes and to conformal anomalies.

Our formulation is interesting since it allows us to address some fundamental issues that had not yet been worked out in full detail. In particular, using what we introduce in this chapter, we will establish in Chapter 3 the exact connection between Wilsonian

[2] This should be distinguished from the local RG [33], which goes one step further and studies evolution under Weyl transformations. We will restrict our attention to the usual RG evolution under global dilatations.

[3] This formulation is somehow similar to the one developed in [34] for renormalizable theories (see also [35–40]). The main difference is that we incorporate the spacetime dependence of the couplings into the geometry of theory space, which allows for general quasilocal changes of coordinates. Furthermore, exact RG needs a dimensionful cutoff regularization, which is also included in the description of theory space.

[4] We also allow for the possibility of non-diagonalizable linear terms, which give rise to logarithmic CFT.

RG and renormalization of correlation functions of composite operators.

The value of this formalism is also manifest in Chapters 5 and 6, in the context of holographic renormalization. There, we will see how this formulation applies in a natural way, and facilitates the solution of some puzzles.[5]

This chapter is organized as follows. Sections 2.1, 2.2 and 2.3 are devoted to the introduction of the main formalism we will use along the thesis. We will define the theory spaces we work with in Section 2.1, the Wilson flows in Section 2.2 and introduce the normal coordinates in Section 2.3. All these developments are done in an abstract way and under very general assumptions. In Section 2.4, we materialize in specific examples the previous tools. In particular, we review the Polchinski equation, and apply it to deformations of the Gaussian fixed point as example. Also, we study the features of the Wilson flows in large N theories.

## 2.1 Theory Spaces

We start this chapter presenting different spaces we use to describe the exact RG. Consider a generic local quantum field theory in $d$ flat Euclidean dimensions, defined by a classical Wilson action $s$, and the corresponding regulated partition function evaluated with a UV cutoff. The Wilson action is a quasi-local functional of a set of quantum fields $\omega$,

$$s[\omega] = \int d^d x \, \mathcal{L}\left(x; \omega(x), \partial\omega(x), \ldots\right). \tag{2.1}$$

We have allowed for an explicit spacetime dependence, which will be useful for the definition and calculation of correlation functions. The cutoff partition function is obtained by functional integration over the fields $\omega$,

$$Z_\Lambda(s) = \int \left[\mathcal{D}\omega\right]^\Lambda \, e^{-s[\omega]}. \tag{2.2}$$

For the moment we do not need to know the nature of the regularization; we just assume that it is characterized by the indicated cutoff scale $\Lambda$. Let $\mathcal{I}$ be the set of all

[5] In fact, we developed this formulation guided by holography.

Wilson actions with field content $\omega$ and given symmetry restrictions. The theory space we will work on is given by $\mathcal{W} = \mathcal{I} \times \mathbb{R}^+$. As we said, we will treat these spaces as infinite-dimensional smooth manifolds. A point in $\mathcal{W}$, i.e. an action $s$ and a scale $\Lambda$,[6] specifies a particular theory described by $Z(s,\Lambda) \equiv Z_\Lambda(s)$. This definition is the first example of the following general notation: given any map $U : \mathcal{W} \to X$, with $X$ any set, we define $U_\Lambda : \mathcal{I} \to X$ by $U_\Lambda(s) = U(s,\Lambda)$.

There are however some redundancies in this description of the space of theories. In particular, a rescaling $x = tx'$ defines the new action

$$s_t[\omega] = s[D_{t^{-1}}\omega], \tag{2.3}$$

where $D_t$ is a dilatation.[7] Changing variables $\omega \to D_t\omega$ in the path integral and neglecting the trivial Jacobian we get

$$Z_\Lambda(s) = Z_{t\Lambda}(s_t). \tag{2.4}$$

This defines the equivalence relation $(s_t, t\Lambda) \sim (s_{t'}, t'\Lambda)$. It is very convenient to introduce the rescaled flat metric $(\gamma^t)_{\mu\nu} = t^2\delta_{\mu\nu}$. The equivalence relation can then be understood as $(s_t, \gamma^t) \sim (s_{t'}, \gamma^{t'})$, where the cutoff in the partition function is to be measured in energy units associated to the metric $\gamma^t$ in the second entry: $\partial^2/(t^2\Lambda^2) = (\gamma^t)^{\mu\nu}\partial_\mu\partial_\nu/\Lambda^2$.[8] For some purposes it is useful to work with the quotient space $\mathcal{M} = \mathcal{W}/\sim$. As in any quotient space, there is a projection $[\,]$ into equivalence classes: given $(s,\Lambda) \in \mathcal{W}$, $[(s,\Lambda)] \in \mathcal{M}$ is the equivalence class it belongs to. Conversely, given a positive number $\Lambda$, we define $\rho_\Lambda : \mathcal{M} \to \mathcal{W}$ by $\rho_\Lambda(\mathbf{s}) = (s,\Lambda)$ with $[(s,\Lambda)] = \mathbf{s}$. In particular, using $\rho_1$ amounts to working with dimensionless spacetime coordinates, as

[6]The idea of including the value of the cutoff in the definition of the theory is analogous to working on a theory space of extended actions that implement the cutoff regularization.

[7]Remember that $\omega$ represents a set of fields, which may be scalars, tensors or spinors. Under the dilatation, which is a particular change of coordinates, each of these fields transforms in a definite way. For a tensor with $n^u$ ($n^d$) contravariant (covariant) indices, $(D_t\omega)(x) = t^{n^d - n^u}\omega(tx)$.

[8]Expressed in this form, we see that this is a particular case of a larger redundancy in general curved spacetime. Given a change of spacetime coordinates $x = \xi(x')$ and defining $s_\xi[\omega] = s[\omega \circ \xi^{-1}]$, we have $(s,\gamma) \sim (s_\xi, \gamma^\xi)$, with the cutoff evaluated with the indicated metrics and $\gamma^\xi_{\mu\nu} = \partial_\mu\xi^\tau\partial_\nu\xi^\sigma\gamma_{\tau\sigma}$.

done in [6]. The partition function acting on equivalence classes is

$$\mathbf{Z}(\mathbf{s}) = Z \circ \rho_\Lambda(\mathbf{s}). \tag{2.5}$$

We will also be interested in the tangent bundle $\mathrm{T}\mathcal{W}$. Any vector $v$ in the tangent space of a given point $(s, \Lambda) \in \mathcal{W}$ can be associated to an operator $\mathcal{O}|_{(s,\Lambda)}$ built with the quantum fields $\omega$. Let $S_\omega$ be the function on $\mathcal{W}$ given by

$$S_\omega(s, \Lambda) = s[\omega]. \tag{2.6}$$

Then,

$$\mathcal{O}|_{(s,\Lambda)}[\omega] = v|_{(s,\Lambda)} S_\omega. \tag{2.7}$$

Note that only the vector components along the $\mathcal{I}$ directions enter in this equation. The operator $\mathcal{O}|_{(s,\Lambda)}$, which could be non-local, represents an infinitesimal deformation of the action $s$. Conversely, given an operator $\mathcal{O}[\omega]$, we can define a curve $(s + t\mathcal{O}, \Lambda)$ (with the natural definition of the sum of functionals) and associate the vector tangent to it at $t = 0$: given any function $F$ in $\mathcal{W}$,

$$v|_{(s,\Lambda)} F = \frac{\partial}{\partial t} F(s + t\mathcal{O}, \Lambda)\bigg|_0 . \tag{2.8}$$

The relations (2.7) and (2.8) are inverse to each other if the vector $v$ is restricted to be orthogonal to the $\Lambda$ direction. So, we can use the same name for an operator and the vector along $\mathcal{I}$ identified with it, and will sometimes follow this convention. We define in a similar way the expectation value of a functional or operator $G$ at the point $(s, \Lambda)$:

$$\begin{aligned} \langle G \rangle_{(s,\Lambda)} &= \frac{1}{Z(s,\Lambda)} \frac{\partial}{\partial t} Z(s - tG, \Lambda)\bigg|_0 \\ &= -\frac{1}{Z(s,\Lambda)} v_G|_{(s,\Lambda)} Z. \end{aligned} \tag{2.9}$$

In the second line we have used the vector $v_G$, associated to $G$ by (2.8).

To parametrize the spaces $\mathcal{W}$ and $\mathcal{M}$, we use an infinite set $\mathcal{C}$ of smooth functions $g^a : \mathbb{R}^d \to \mathbb{R}$, which can be regarded as background fields or spacetime dependent couplings. Most importantly for our purposes, they can act as sources to define and calculate correlation functions. We define a class of parametrizations or coordinate systems in the following way. We choose a quasilocal functional $S$ of fields and couplings such that, for each point $(s,\Lambda)\in\mathcal{W}$,

$$\begin{aligned} s[\omega] &= S[\gamma^\Lambda; g, \omega] \\ &= \int d^dx \sqrt{|\gamma^\Lambda|}\mathcal{L}\left(\gamma^\Lambda; g(x), \omega(x), \partial\omega(x), \ldots\right), \end{aligned} \tag{2.10}$$

for some unique $g\in\mathcal{C}$. The dimensionful metric $\gamma^\Lambda$ allows to work with couplings and fields of mass dimension $n^d-n^u$, with $n^d$ ($n^u$) the number of covariant (contravariant) indices they have. This metric and its inverse are used to contract the Lorentz indices, including those in derivatives. For instance, the standard linear parametrization is given by

$$S[\gamma; g, \omega] = \int d^dx \sqrt{|\gamma|} g^a(x) \mathcal{O}_a[\gamma;\omega](x). \tag{2.11}$$

Here, $\{\mathcal{O}_a\}$ is a complete set of linearly-independent Lorentz-covariant local operators made out of the relevant quantum fields $\omega$ and their derivatives, modulo total derivatives (we do not include total derivatives of operators in this set because they can be absorbed after integration by parts into the spacetime dependent couplings). Further symmetry and consistency restrictions may apply. In this thesis we mostly concentrate on Lorentz scalar operators, but we should keep in mind that this set is not stable under RG evolution. Among these operators, we include the identity operator, which contributes to the vacuum energy. We label this operator and its constant coupling with the index $a=0$.

(2.10) defines a (generalized) coordinate chart[9] $c:\mathcal{W}\to\mathcal{C}\times\mathbb{R}^+$, $c(s,\Lambda)=(g,\Lambda)$. We will use indices $\tilde{\alpha}$ to refer to either the label $\alpha$ in $\mathcal{C}$ or to the $\mathbb{R}^+$ component, which we indicate with the symbol $\wedge$. So, $c^\alpha_\Lambda(s)=c^\alpha(s,\Lambda)=g^\alpha$ and $c^\wedge(s,\Lambda)=\Lambda$. For this component, we will also write $\bar{\Lambda}=c^\wedge$.

In order to simplify some formulas in the thesis, let us make a parenthesis to introduce the following definitions. First, we introduce the canonical projection $\pi:\mathcal{C}\times\mathbb{R}^+\to$

[9] For simplicity, we assume the regions of the spaces we work with can be covered by a single chart and neglect global issues throughout the thesis.

$\mathcal{C}$, $\pi(g, \Lambda) = g$ and call $c^{\pi} \equiv \pi \circ c$ and $c^{\pi}_{\Lambda} \equiv \pi \circ c_{\Lambda}$. Second, we define the function $\bar{\gamma} : \mathcal{W} \to T^0_2\left(\mathbb{R}^d\right)$, $(s, \Lambda) \mapsto \Lambda^2 \delta_{\mu\nu} dx^{\mu} \otimes dx^{\nu}$. In terms of it, $2\bar{\gamma}\frac{\partial}{\partial\bar{\gamma}} F(s, \Lambda) = \Lambda\partial_{\Lambda} F(s, \Lambda)$ for any $F : \mathcal{W} \to \mathbb{R}$. Sometimes we will keep the coordinates $c$ implicit. In particular, we define

$$H^{\tilde{\alpha}} = c^{\tilde{\alpha}} \circ H \tag{2.12}$$

for maps $H : X \to \mathcal{W}$ with arbitrary $X$, and introduce the coordinate-dependent square bracket notation

$$U[\bar{\gamma}; c^{\pi}] = U, \tag{2.13}$$

$$U_{\Lambda}[c^{\pi}] = U_{\Lambda} \tag{2.14}$$

for any map $U : \mathcal{W} \to X$. Using these definitions, we can for instance write (2.10) as

$$s[\omega] = S_{\omega}[\bar{\gamma}; c^{\pi}]. \tag{2.15}$$

Continuing with physics, a change of variables in the integral in (2.10) gives

$$S[\gamma^{\Lambda}; g, \omega] = S[\gamma^{t\Lambda}; D_t g, D_t \omega]. \tag{2.16}$$

Therefore, given an action functional $S$, the non-trivial component of its associated chart $c$ satisfies the relation

$$c^{\pi}_{t\Lambda}(s_t) = D_t c^{\pi}_{\Lambda}(s). \tag{2.17}$$

A given chart $c$ on $\mathcal{W}$ induces a set of scale-dependent charts on the quotient space, $\mathbf{c}_{\Lambda} : \mathcal{M} \to \mathcal{C}$, defined by

$$(\mathbf{c}_{\Lambda}(\mathbf{s}), \Lambda) = c \circ \rho_{\Lambda}(\mathbf{s}). \tag{2.18}$$

and fulfilling the relation

$$\mathbf{c}_{t\Lambda} = D_t \mathbf{c}_{\Lambda}. \tag{2.19}$$

We will often work in the coordinate basis $\{\partial^c_{\tilde{\alpha}}\}$ in the tangent space of $\mathcal{W}$. Given any real function $F$ in $\mathcal{W}$,

$$\partial^c_{\tilde{\alpha}} F = \frac{\delta F \circ c^{-1}}{\delta g^{\tilde{\alpha}}}. \tag{2.20}$$

We can then write vector fields $v$ in $\mathrm{T}\,\mathcal{W}$ as

$$\begin{aligned} v|_{(s,\Lambda)} &= v^{\tilde{\alpha}}(s,\Lambda)\; \partial^c_{\tilde{\alpha}}|_{(s,\Lambda)} \\ &= v^{\alpha}_{\Lambda}(s)\; \partial^c_{\alpha}|_{(s,\Lambda)} + v^{\wedge}_{\Lambda}(s)\; \partial^c_{\wedge}|_{(s,\Lambda)}\,. \end{aligned} \tag{2.21}$$

The components in this basis are given by, $v^{\tilde{\alpha}} = v c^{\tilde{\alpha}}$. As explained above, a vector $\mathcal{O}|_{(s_0,\Lambda)} = \mathcal{O}^{\alpha}(s_0,\Lambda)\; \partial^c_{\alpha}|_{(s_0,\Lambda)}$ is associated to an operator (a functional of the quantum fields). In coordinates,

$$\mathcal{O}|_{(s_0,\Lambda)}\,[\omega] = \mathcal{O}^{\alpha}(s_0,\Lambda)\; \left.\frac{\delta S[\gamma^{\Lambda}; g, \omega]}{\delta g^{\alpha}}\right|_{g_0}, \tag{2.22}$$

with $g_0 = c_\Lambda(s_0)$ and $S$ the action functional associated to $c$. If the components $\mathcal{O}^{\alpha}$ (with upper indices and not to be confused with the operators themselves) are of the form $\mathcal{O}^{ax} = \sum_{n=0}^{m} \mathcal{O}^{a(n)} \partial_x^{2n} \delta(x-y)$, for some spacetime point $y$, the operator will be local. This is the case of the local operators associated to the basis vectors $\partial^c_{\alpha}|_{(s,\Lambda)}$, which, as can be seen in (2.22) with $(\mathcal{O}_{\alpha})^{\alpha_1} = \delta^{\alpha_1}_{\alpha}$, depend on $\Lambda$ only through the metric,

$$\begin{aligned} \partial^c_{\alpha}|_{(s,\Lambda)}\, S_{\omega} &= \mathcal{O}_{\alpha}|_{(s,\Lambda)}\,[\omega] \\ &= \mathcal{O}^{(s)}_{\alpha}[\gamma^{\Lambda};\omega]. \end{aligned} \tag{2.23}$$

We will make extensive use of quasilocal changes of coordinates $c \to c'$, given by $\zeta^{\alpha}[\bar{\gamma}, c^{\pi}] = c'^{\alpha}$. The induced changes of coordinates in the quotient space are $\boldsymbol{\zeta}_{\Lambda} = \mathbf{c}'_{\Lambda} \circ \mathbf{c}^{-1}_{\Lambda}$. The vector components in (2.21) transform in the usual way under a change of coordinates:

$$v'^{\tilde{\alpha}} = v^{\tilde{\alpha}_1} \partial^c_{\tilde{\alpha}_1} c'^{\tilde{\alpha}}. \tag{2.24}$$

The fact that this transformation mixes in general the $\mathcal{C}$ and $\mathbb{R}^+$ components of the vectors, with

$$v'^{\alpha} = v^{\alpha_1}\partial^c_{\alpha_1} c'^{\alpha} + v^{\wedge}\partial^c_{\wedge} c'^{\alpha}, \tag{2.25}$$

will be relevant below.

## 2.2 Exact RG Flows

There exists at least a further and more interesting redundancy in the description of regularized quantum field theories. Given a Wilson action $s_0$ and a cutoff $\Lambda_0$, consider a new cutoff $\Lambda < \Lambda_0$ and let the new action $s$ be defined by integrating out the quantum degrees of freedom between $\Lambda$ and $\Lambda_0$:

$$e^{-s[\omega]} = \int [\mathcal{D}\omega]^{\Lambda_0}_{\Lambda}\ e^{-s_0[\omega]}. \tag{2.26}$$

The notation in the measure indicates that the path integral is performed with a UV cutoff $\Lambda_0$ and an IR cutoff $\Lambda$, satisfying $[\mathcal{D}\omega]^{\Lambda}\,[\mathcal{D}\omega]^{\Lambda_0}_{\Lambda} = [\mathcal{D}\omega]^{\Lambda_0}$. Although we use the same symbol $\omega$ on the LHS and RHS of (2.26), the action $s$ depends only on the degrees of freedom in $\omega$ that have not been integrated out. By construction, the actions $s$ and $s_0$ satisfy

$$Z_{\Lambda}(s) = Z_{\Lambda_0}(s_0). \tag{2.27}$$

We define the flow in theory space $f_t : \mathcal{W} \to \mathcal{W}$ such that

$$(s, \Lambda) = f_{\Lambda/\Lambda_0}(s_0, \Lambda_0), \tag{2.28}$$

with $f_1 = \mathbb{1}$. In general, $s \neq (s_0)_{\Lambda/\Lambda_0}$, so (2.27) relates different points in $\mathcal{M}$, as defined in the previous section: if $\mathbf{s} = [(s, \Lambda)]$ and $\mathbf{s}_0 = [(s_0, \Lambda_0)]$,

$$\mathbf{Z}(\mathbf{s}) = \mathbf{Z}(\mathbf{s}_0), \tag{2.29}$$

The property of exact RG invariance is given by (2.27) and (2.29). The latter defines the RG flow $\mathbf{f}_t$ in $\mathcal{M}$:

$$\begin{aligned} \mathbf{s} &= \mathbf{f}_{\Lambda/\Lambda_0}(\mathbf{s}_0) \\ &= [f_{\Lambda/\Lambda_0}(s_0, \Lambda_0)], \end{aligned} \tag{2.30}$$

with $\mathbf{f}_1 = \mathbb{1}$. This is a good definition, independent of the representative, since $f_t(s_{t'}, \Lambda t') = (f_t(s, \Lambda))_{t'}$, where $(s, \Lambda)_t \equiv (s_t, \Lambda t)$. There is also an inverse relation,

$$f_t(s, \Lambda) = \rho_{t\Lambda} \circ \mathbf{f}_t([(s, \Lambda)]). \tag{2.31}$$

These flows are generated by beta vector fields, which are tangent to the corresponding curves. They act on any real function $F$ on $\mathcal{W}$ and $\mathcal{M}$ as

$$\beta F = t\partial_t F \circ f_t|_1 \,, \tag{2.32}$$

$$\boldsymbol{\beta} F = t\partial_t F \circ \mathbf{f}_t|_1 \tag{2.33}$$

respectively. They can be used to write the Callan-Symanzik equations

$$\beta Z = 0, \tag{2.34}$$

$$\boldsymbol{\beta} \mathbf{Z} = 0, \tag{2.35}$$

which are the infinitesimal versions of (2.27) and (2.29), respectively. The usual description of RG flows follows once a coordinate system $c$ is chosen in $\mathcal{W}$,

$$\begin{aligned} f^\alpha_{t,\Lambda}[g] &= f^\alpha_t[\gamma^\Lambda; g] \\ &= c^\alpha \circ f_t(s, \Lambda), \end{aligned} \tag{2.36}$$

with $c^\pi(s) = g$, which agrees with our bracket notation. In local quantum field theory these flows are position-dependent quasilocal functionals of the couplings $g$, thanks to the IR cutoff in (2.26). Similarly, the flows in $\mathcal{M}$ can be parametrized as

$$\mathbf{f}^\alpha_t = \mathbf{c}^\alpha_1 \circ \mathbf{f}_t, \tag{2.37}$$

$$\mathbf{f}^\alpha_t[\mathbf{c}^\pi_1] = \mathbf{f}^\alpha_t. \tag{2.38}$$

Their relation with the flows of couplings in (2.36) follows from (2.19):

$$f^a_{t,\Lambda} = D_{t\Lambda}\mathbf{f}^a_t\left[D_{\Lambda^{-1}}c^\pi\right]. \tag{2.39}$$

The beta vector fields can be written in the coordinate basis associated to $c$:

$$\begin{aligned}\beta|_{(s,\Lambda)} &= \beta^{\tilde{\alpha}}(s,\Lambda)\,\partial^c_{\tilde{\alpha}}|_{(s,\Lambda)}\\ &= \beta^{\alpha}(s,\Lambda)\,\partial^c_{\alpha}|_{(s,\Lambda)} + \Lambda\,\partial^c_{\wedge}|_{(s,\Lambda)}\,.\end{aligned} \tag{2.40}$$

Note that the components are given by

$$\beta^{\tilde{\alpha}} = t\partial_t f^{\tilde{\alpha}}_t\big|_1\,. \tag{2.41}$$

These beta functions are also quasilocal functionals of the couplings,

$$\beta^\alpha_\Lambda[g] = \beta^\alpha[\gamma^\Lambda; g]. \tag{2.42}$$

In coordinates, the Callan-Symanzik equation has the more familiar form

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta^{\alpha'}_\Lambda[g]\frac{\delta}{\delta g^{\alpha'}} - \int d^dx\sqrt{|\gamma^\Lambda|}\mathcal{A}(x)\right]Z_\Lambda[g] = 0, \tag{2.43}$$

where we have separated the vacuum energy coupling from the rest: $\alpha' = ax$ runs over all couplings except the vacuum energy, $a \neq 0$, and $\mathcal{A}(x) = \beta^{0x}$. $\mathcal{A}(x)$ is the conformal anomaly of the theory. Finally, the beta functions in $\mathcal{M}$ are given by the components of $\boldsymbol{\beta}$ under the chart $\mathbf{c}_1$:

$$\boldsymbol{\beta} = \boldsymbol{\beta}^\alpha\partial^{\mathbf{c}_1}_\alpha, \tag{2.44}$$

and satisfy

$$\boldsymbol{\beta}^\alpha\circ\mathbf{c}_1 = \boldsymbol{\beta}\mathbf{c}^\alpha_1, \tag{2.45}$$

$$\boldsymbol{\beta}^\alpha = t\partial_t\mathbf{f}^\alpha_t|_1\,. \tag{2.46}$$

Using (2.39), we find the relation

$$\beta^a_\Lambda = D_\Lambda\boldsymbol{\beta}^a[D^{-1}_\Lambda c^\pi_\Lambda] + Dc^a_\Lambda, \tag{2.47}$$

with $D = \left(n^d_{(a)} - n^u_{(a)}\right) + x^\mu \partial_\mu$. Remember that $n^u_{(a)}$ and $n^d_{(a)}$ are the number of contravariant and covariant indices of $c^a$ respectively. Under a change of coordinates $c \to c' = \zeta[\bar{\gamma}; c^\pi]$, the beta functions transform into

$$\beta'^{\,\alpha} = \beta^{\alpha_1}\, \partial^c_{\alpha_1} c'^{\,\alpha} + 2\bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} \zeta^\alpha \left[\bar{\gamma}; c^\pi\right]. \tag{2.48}$$

Notice the appearance of an inhomogeneous term, in agreement with (2.25).

The fixed points $\mathbf{s}_*$ of the quotient-space RG flows, with $\boldsymbol{\beta}_{\mathbf{s}} = 0$, describe scale-invariant physics. In the space $\mathcal{W}$, they correspond to points $(s^\Lambda_*, \Lambda) = \rho_\Lambda(\mathbf{s}_*)$ with trivial RG evolution $f_t(s^\Lambda_*, \Lambda) = ((s^\Lambda_*)_t, \Lambda t)$. In our parametrizations, $g^{\Lambda\,\alpha}_* = c^\alpha_\pi(s^\Lambda_*, \Lambda)$, this translates into the trivial running $g^{\Lambda\,\alpha}_* \to g^{t\Lambda\,\alpha}_* = D_t g^{\Lambda\,\alpha}_*$. We will only consider the usual translationally invariant fixed points, with constant scalar couplings, which are thus invariant under this rescaling, $g^{\Lambda\,\alpha}_* = g^\alpha_*$ and have trivial beta functions, $\beta^\alpha_\Lambda[g_*] = 0$.[10]

## 2.3 Normal Coordinates

In this section we single out a special set of coordinates, valid in some region around a given fixed point, in which the beta functions and RG flows take a remarkably simple form. Later on we will see that these coordinates are closely related to the process of renormalization.

We start with an arbitrary chart $c$ with the fixed point of interest located at $c^\pi(s^\Lambda_*, \Lambda) = g_* = 0$. Close to this fixed point, the beta functions can be expanded as

$$\beta^\alpha = \beta^\alpha_{\alpha_1}(\bar{\gamma}) c^{\alpha_1} + \sum_{n \geq 2} \beta^\alpha_{\alpha_1 \ldots \alpha_n}(\bar{\gamma}) c^{\alpha_1} \ldots c^{\alpha_n}. \tag{2.50}$$

Both sides of this equation are maps on $\mathcal{W}$. In a local QFT the beta functions are

[10] One can also write non-vanishing tensor couplings in these fixed points using the metric and additional scalar couplings. In this case, we would have

$$\beta^\alpha_\Lambda \left[g^\Lambda_*\right] = \left(n^d_{(\alpha)} - n^u_{(\alpha)}\right) g^{\Lambda\,\alpha}_*. \tag{2.49}$$

local or quasilocal, in the sense that they can be written as a finite or infinite sum of products of Dirac delta functions, their derivatives, and (inverse) metrics, with possible contractions. Consider first the linearised approximation and let $\lambda_{(a)}$ be eigenvalues of $\beta^{\alpha_1}_{\alpha_2}$, which we assume to be real numbers. The linear part of the beta function can be maximally aligned with the couplings by a linear reparametrization

$$\bar{c}^{\alpha} = \zeta^{\alpha}_{\alpha_1}(\bar{\gamma})c^{\alpha_1}, \tag{2.51}$$

with quasilocal $\zeta^{\alpha}_{\alpha_1}(\gamma)$, which puts the linear part of the beta function in a generalized Jordan form,

$$\bar{\beta}^{\alpha} = -\lambda^{\alpha}_{\alpha_1}(\bar{\gamma})\bar{c}^{\alpha_1} + O(\bar{c}^2), \tag{2.52}$$

where the quasilocal matrix $\lambda^{\alpha}_{\alpha_1}$ has, neglecting metrics, a diagonal block structure, with each block having a unique real eigenvalue $\lambda_{(a)}$. Non-vanishing terms with $n_\partial$ derivatives in off-block positions $(a, a_1)$ are only allowed when

$$\left[\lambda_{(a)} - n^u_{(a)} + n^d_{(a)}\right] - \left[\lambda_{(a_1)} - n^u_{(a_1)} + n^d_{(a_1)}\right] = n_\partial, \tag{2.53}$$

where we have allowed the directions $a$, $a_1$ may have tensor character. The number of derivatives and tensor indices enters this condition through the non-homogenous term in (2.48). Notice that, by covariance, there is a relation between $n^u_{(a_i)}$, $n^d_{(a_i)}$, $n_\partial$ and the number of metrics $n_{(\gamma)}$ and inverse metrics $n_{(\gamma^{-1})}$ appearing in $\lambda^{\alpha}_{\alpha_1}$:

$$2n_{(\gamma^{-1})} - 2n_{(\gamma)} = n^u_{(a)} - n^d_{(a)} - n^u_{(a_1)} + n^d_{(a_1)} + n_\partial. \tag{2.54}$$

The eigenvalues $\lambda_{(a)}$ give the (quantum) scaling dimensions $\Delta_{(a)}$ of the eigendeformations of the fixed-point theory: $\Delta_{(a)} = d - \lambda_{(a)} + n^u_{(a)} - n^d_{(a)}$. By definition these dimensions are less than, equal to and greater than $d$ for relevant, marginal and irrelevant deformations, respectively. Usually, the number of relevant eigenvalues is finite [26]. In a unitary CFT, the eigendeformations span the complete space $\mathcal{I}$ and $\lambda$ can be written in completely diagonal form. Nevertheless, thinking of the possible application to logarithmic CFT, we shall proceed in the general case without the assumption of diagonalizability.

In going beyond the linear approximation, it is important to distinguish certain exceptional cases. The eigenvalue $\lambda_{(a)}$ is said to be *resonant* if

$$\sum_{i=1}^{m}\left[\lambda_{(a_i)} - n^u_{(a_i)} + n^d_{(a_i)}\right] + n_\partial = \lambda_{(a)} - n^u_{(a)} + n^d_{(a)} \tag{2.55}$$

for some (possibly repeated) eigendirections $a_1, \ldots, a_m$, with $a_r \neq 0$, $n_\partial$ a non-negative integer and $m \geq 2$. The eigenvalues $\lambda_{(a_1)}, \ldots, \lambda_{(a_m)}$ are said to form a *resonance*. Scalar marginal directions, i.e. $\lambda_{(a_i)} = 0$ for some $a_i$, imply an infinite number of resonances and that all eigenvalues are resonant. Note that the condition for non-negative off-diagonal terms in the linear part has the same form as (2.55), with $m = 1$. We will say that the eigenvalues, or the associated dimensions, are *exceptional* when the relations (2.55) occur for some $m \geq 1$. Non-exceptional eigenvalues or dimensions will be called *generic*. For non-resonant eigenvalues, by the Poincaré linearisation theorem (see the Appendix A) we know that, at least as a formal series, we can find a coordinate transformation such that in the new coordinates the beta function is linear:

$$\bar{\beta}^\alpha = -\lambda^\alpha_{\alpha_1}(\bar{\gamma})\bar{c}^{\alpha_1} \quad \text{(non-resonant)}. \tag{2.56}$$

In this case, the integration of this vector field is trivial and the RG flows are given by

$$\bar{f}^\alpha_t = \mathcal{P}\exp\left\{-\int_1^t \frac{dt'}{t'}\lambda\left(t'^2\bar{\gamma}\right)\right\}^\alpha_{\alpha_1}\bar{c}^{\alpha_1}, \tag{2.57}$$

where $\mathcal{P}\exp$ is the path-ordered exponential. The flows can also be written in a more useful manner:

$$\bar{f}^\alpha_t = t^{-\lambda_{(\alpha)}}\left[\mathcal{M}_t(\bar{\gamma})\right]^\alpha_{\alpha_1}\bar{c}^{\alpha_1}, \tag{2.58}$$

where $\mathcal{M}_t(\gamma)$ is the identity matrix in a fully diagonalizable case and depends logarithmically on $t$ otherwise. In all cases, $\mathcal{M}_1 = \mathbb{1}$. It satisfies the same requirements as $\lambda(\gamma)$: it is diagonalized in subspaces with the same eigenvalue $\lambda_{(\alpha)}$ and can have non-vanishing terms with $n_\partial$ derivatives in off-block positions $(a, a_1)$ only when (2.53) is satisfied. To prove (2.58), let us introduce it in the linear differential flow equation:

$$t\frac{\partial}{\partial t}\bar{f}^\alpha_t = t^{-\lambda_{(\alpha)}}\left\{-\lambda_{(\alpha)}\left[\mathcal{M}_t(\bar{\gamma})\right]^\alpha_{\alpha_2} + t\frac{\partial}{\partial t}\left[\mathcal{M}_t(\bar{\gamma})\right]^\alpha_{\alpha_2}\right\}\bar{c}^{\alpha_2}$$

$$
\begin{aligned}
&= -\,\lambda^{\alpha}_{\alpha_1}\left(t^2\bar{\gamma}\right) t^{-\lambda_{(\alpha_1)}}\left[\mathcal{M}_t(\bar{\gamma})\right]^{\alpha_1}_{\alpha_2}\bar{c}^{\alpha_2} \\
&= -\,t^{-\lambda_{(\alpha)}}\lambda^{\alpha}_{\alpha_1}(\bar{\gamma})\left[\mathcal{M}_t(\bar{\gamma})\right]^{\alpha_1}_{\alpha_2}\bar{c}^{\alpha_2}.
\end{aligned}
\tag{2.59}
$$

In the second line we have used the linear form of the beta function (2.52) and in the third one we have commuted the first two matrices, which is allowed by the specific form of the matrix $\lambda(\gamma)$. Then, combining the first and third lines, we obtain

$$
t\frac{\partial}{\partial t}\left[\mathcal{M}_t(\gamma)\right]^{\alpha}_{\alpha_1} = -\left[\lambda^{\alpha}_{\alpha_2}(\gamma) - \lambda_{(\alpha)}\delta^{\alpha}_{\alpha_2}\right]\left[\mathcal{M}_t(\gamma)\right]^{\alpha_2}_{\alpha_1}. \tag{2.60}
$$

This is a linear and autonomous differential equation. The solution is found to be

$$
\left[\mathcal{M}_t(\gamma)\right]^{\alpha}_{\alpha_1} = \exp\left\{-\left[\lambda^{\alpha}_{\alpha_1}(\gamma) - \lambda_{(\alpha)}\delta^{\alpha}_{\alpha_1}\right]\log t\right\}. \tag{2.61}
$$

The matrix $\left[\lambda^{\alpha}_{\alpha_2}(\gamma) - \lambda_{(\alpha)}\delta^{\alpha}_{\alpha_2}\right]$ has only vanishing eigenvalues, i.e. it is idempotent. Therefore, the Taylor expansion of the exponential above has only a finite number of logarithmic terms in $t$.

When the set of eigenvalues is resonant, complete linearisation is not possible in general. However, the Poincaré-Dulac theorem (see the appendix A) implies, at least in the sense of formal power series, that we can choose coordinates in which the beta functions take the *normal form*[11]

$$
\bar{\beta}^{\alpha} = -\lambda_{(\alpha)}\bar{c}^{\alpha} + \sum_{n\geq 1}\bar{\beta}^{\alpha}_{\alpha_1\ldots\alpha_n}(\bar{\gamma})\bar{c}^{\alpha_1}\ldots\bar{c}^{\alpha_n}, \tag{2.62}
$$

where we have defined $\bar{\beta}^{\alpha}_{\alpha_1} = \lambda_{(\alpha)}\delta^{\alpha}_{\alpha_1} - \lambda^{\alpha}_{\alpha_1}$. The coefficients $\bar{\beta}^{ax}_{a_1x_1\ldots a_mx_m}$ have support at $x_1 = \ldots = x_m$ and are non-vanishing only when condition (2.55) is met for $n_\partial$ equal to the number of derivatives in them. Again, further simplifications are admitted, but the form (2.62) will be sufficient for our purposes. Obviously, (2.62) reduces to (2.56) for non-resonant eigenvalues. Analogously to (2.54), by covariance, the number of metrics

[11] Actually, we are generalising the Poincaré-Dulac theorem to the case of quasilocal vector fields in a space of functions. To prove this generalization, at least at a finite order in the coupling and derivative expansions, we can simply choose a spacetime point and treat the $n$-derivatives of couplings at that point as independent couplings.

$n_{(\gamma)}$ and inverse metrics $n_{(\gamma^{-1})}$ of $\bar{\beta}^{\alpha}_{\alpha_1\ldots\alpha_n}(\gamma)$ is constrained by

$$2n_{(\gamma^{-1})} - 2n_{(\gamma)} = n^u_{(a)} - n^d_{(a)} - \sum_{i=1}^{n}\left(n^u_{(a_i)} - n^d_{(a_i)}\right) + n_\partial. \tag{2.63}$$

The coordinates in which the beta functions take the form (2.62) will be called *normal coordinates*. They are not unique. In normal coordinates, the RG flows have the perturbative form

$$\begin{aligned}\bar{f}^{\alpha}_t =& t^{-\lambda_{(\alpha)}}\left\{\bar{c}^{\alpha} + \sum_{m=1}^{\infty}\left[\sum_{p=1}^{p^{\max}_{\alpha_1\cdots\alpha_m}} \log^p t\,[B_p]^{\alpha}_{\alpha_1\ldots\alpha_m}(\bar{\gamma})\right]\bar{c}^{\alpha_1}\cdots\bar{c}^{\alpha_m}\right\}\\ =& t^{-\lambda_{(\alpha)}}\bar{c}^{\alpha} + \sum_{m=1}^{\infty}\left[\sum_{p=1}^{p^{\max}_{\alpha_1\cdots\alpha_m}} \log^p t\,[B_p]^{\alpha}_{\alpha_1\ldots\alpha_m}\left(t^2\bar{\gamma}\right)\right]\left[t^{-\lambda_{(\alpha_1)}}\bar{c}^{\alpha_1}\right]\cdots\left[t^{-\lambda_{(\alpha_m)}}\bar{c}^{\alpha_m}\right],\end{aligned} \tag{2.64}$$

where $[B_1]^{\alpha}_{\alpha_1\ldots\alpha_m} = \bar{\beta}^{\alpha}_{\alpha_1\ldots\alpha_m}$. The functions $[B_p]^{ax}_{a_1x_1\ldots a_mx_m}$ with $p>1$ have also support at $x_1 = \ldots = x_m = x$. They can be computed (up to combinatorial coefficients) by summing all the products of $p$ functions $\bar{\beta}^{\alpha}_{\alpha_1\ldots\alpha_r}$, $r \geq 1$, with upper indices contracted with lower indices in such a way that the only free upper and lower indices are $ax$ and $a_1x_1\ldots a_mx_m$ respectively. For instance,

$$[B_3]^{\alpha}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}(\gamma) = \cdots + \frac{1}{S_T}\beta^{\alpha_7}_{\alpha_1\alpha_2\alpha_3}(\gamma)\beta^{\alpha_8}_{\alpha_4\alpha_5}(\gamma)\beta^{\alpha}_{\alpha_7\alpha_8\alpha_6}(\gamma) + \ldots \tag{2.65}$$

where $S_T$ is a combinatorial coefficient (see Appendix A). Each contribution of this type can be seen as a tree $T$, whose elements are the coefficients $\beta^{\alpha}_{\alpha'\ldots\alpha''}(\gamma)$ of the product, and the tree structure is given by the contraction of the indices. For example, the contribution written explicitly in (2.65) is the tree $T$ given by the set $\left\{\beta^{\alpha_7}_{\alpha_1\alpha_2\alpha_3}, \beta^{\alpha_8}_{\alpha_4\alpha_5}, \beta^{\alpha}_{\alpha_7\alpha_8\alpha_6}\right\}$ and represented in Fig. 2.1. Each $[B_p]^{\alpha}_{\alpha_1\ldots\alpha_n}$ is a sum of the contributions of all the possible trees with $p$ dots that connect $\alpha$ (on top) with $\alpha_1\ldots\alpha_n$ (at bottom).

Terms in $[B_p]^{\alpha}_{\alpha_1\ldots\alpha_m}$ with $n_\partial$ derivatives are non-vanishing only when the resonant condition (2.55) is satisfied for the set of the eigenvalues $\{\alpha_1\ldots\alpha_m\}$ and $\alpha$. The constraint of (2.63) also holds. Moreover, for a given order in the number of couplings, $\alpha_1\ldots\alpha_m$, the sum in $p$ is finite and stops at some finite $p^{\max}_{\alpha_1\cdots\alpha_m}$, depending on the or-

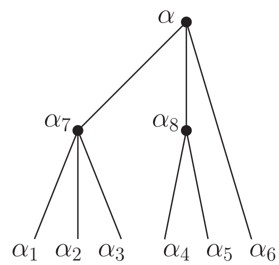


Figure 2.1: Diagrammatic representation of the contribution to $[B_p]^{\alpha}_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}$ written in (2.65): $\beta^{\alpha_7}_{\alpha_1\alpha_2\alpha_3}\beta^{\alpha_8}_{\alpha_4\alpha_5}\beta^{\alpha}_{\alpha_7\alpha_8\alpha_6}$. Each dot represents a coefficient $\beta$ of the product. The index next to a dot is the upper index of the associated coefficient $\beta$, while the indices to which the dot is linked to, downwards, are the lower indices of the associated coefficient $\beta$.

der. This is because $\bar{\beta}^{\alpha}_{\alpha_1}$ is nilpotent and thus the number of possible trees to construct $[B_p]^{\alpha}_{\alpha_1\ldots\alpha_m}$ is finite. Logarithmic differentiation of (2.64) with respect to $t$ gives the beta function (2.62) order by order in $\bar{c}$. For generic dimensions, (2.64) reduces to (2.58). Note that, non-trivially, $\bar{f}_t$ is the inverse of $\bar{f}_{t^{-1}}$, as it should. Observe also that, up to possible log terms and factors of the metric, the components $\bar{f}^{\alpha}_t$ scale homogeneously as a power of $t$. This simple feature is no longer apparent when we write the flows in other coordinates.

## 2.4 Explicit Examples

In this section, we illustrate how the previous formalism is used in practice. We use the Polchinski implementation of the cutoff [15], which allows to derive an evolution equation for the Wilson action. In Section 2.4.1 we write it using our geometric formalism, and use it to work out a simple illustrative example. In Section 2.4.2 we consider the theory space of a single real scalar field $\omega$ and define at the perturbative level the normal coordinates around the free-field fixed point. Finally, in Section 2.4.3 we study the Wilson flows and its properties in other class of theories particularly relevant for the development of the following chapters: large N theories.

### 2.4.1 Polchinski's Equation

Following [15], we implement the cutoff procedure through a modified free propagator (of the dimensionless field),

$$\begin{aligned} P^{xy}_\Lambda &= \langle \omega(x)\omega(y)\rangle_{(0,\Lambda)} \\ &= P(\gamma^\Lambda; x-y), \end{aligned} \tag{2.66}$$

where

$$P(\gamma;x) = \frac{1}{2^{d-2}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}-1\right)} D\left(-\partial^2_\gamma\right)\frac{1}{x_\gamma^{d-2}}, \tag{2.67}$$

with $\partial^2_\gamma = \gamma^{\mu\nu}\partial_\mu\partial_\nu$, $x_\gamma = (\gamma_{\mu\nu}x^\mu x^\nu)^{\frac{1}{2}}$, and $D(u)$ a function with $D(0)$=1, decreasing sufficiently fast as $u \to \infty$. We also assume that $D(u)$ is analytic at $u = 0$, which is required to keep the regularized actions quasilocal in fields and couplings.

In a canonical linear parametrization, the general Wilson action reads

$$\begin{aligned} S[\gamma^\Lambda; g, \omega] = \int d^dx \sqrt{|\gamma^\Lambda|}\big[ & g_0(x) + g_1(x)\omega(x) + g_2(x)\omega(x)^2 + g_{2,2}\gamma^{\Lambda\,\mu\nu}\partial_\mu\omega(x)\partial_\nu\omega(x) \\ & + g_3(x)\omega(x)^3 + \dots\big], \end{aligned} \tag{2.68}$$

with the dots referring to the sum of couplings times other possible monomials in $\omega$ and its derivatives of arbitrary order, up to total derivatives. The partition function is given by

$$Z_\Lambda(s) = \frac{\int \mathcal{D}\omega \exp\left\{-\int d^dx\sqrt{|\gamma^\Lambda|}\frac{1}{2}\gamma^{\Lambda\,\mu\nu}\partial_\mu\omega(x)D^{-1}(-\partial^2/\Lambda^2)\partial_\nu\omega(x) - s[\omega]\right\}}{\int \mathcal{D}\omega \exp\left\{-\int d^dx\sqrt{|\gamma^\Lambda|}\frac{1}{2}\gamma^{\Lambda\,\mu\nu}\partial_\mu\omega(x)D^{-1}(-\partial^2/\Lambda^2)\partial_\nu\omega(x)\right\}}. \tag{2.69}$$

This equation can be understood as a specific implementation of (2.2). The normalization allows us to keep track of vacuum energy terms. Since we know the form of the cutoff, the Callan-Symanzik equation (2.43) can be made more explicit. Differentiation with respect to $\Lambda$ leads to Polchinski's equation [15], which in position space and using our geometric language reads

$$\beta S_\omega - \omega^x \frac{\delta S_\omega}{\delta \omega^x} = \frac{1}{2}\dot{P}^{xy}(\bar{\gamma})\frac{\delta^2 S_\omega}{\delta\omega^x \delta\omega^y} - \frac{1}{2}\dot{P}^{xy}(\bar{\gamma})\frac{\delta S_\omega}{\delta\omega^x}\frac{\delta S_\omega}{\delta\omega^y}, \tag{2.70}$$

where

$$\begin{aligned}\dot{P}(\gamma)^{xy} &= \dot{P}(\gamma; x-y) \\ &= -\left(d - 2 + 2\gamma\frac{\partial}{\partial\gamma}\right) P(\gamma; x-y) \\ &= \frac{2}{\sqrt{|\gamma|}} D'\left(-\partial_\gamma^2\right)\delta(x-y).\end{aligned} \tag{2.71}$$

Both sides of (2.70) are functions on $\mathcal{W}$. This equation is satisfied also by the field-independent terms in the action, neglected in [15], when the cutoff dependence of the denominator in (2.69) is taken into account. Given a chart $c$, the first term of (2.70) is written in components as

$$\beta S_\omega = \beta^\alpha \partial_\alpha^c S_\omega + 2\bar{\gamma}\frac{\partial}{\partial\bar{\gamma}} S_\omega. \tag{2.72}$$

### 2.4.2 Gaussian Fixed Point

The point $s = 0$ is trivially a fixed point. It corresponds to the free massless theory, since the kinetic term is included in the cutoff procedure.[12] In the following we study the theory space of one real scalar field in a neighbourhood of this Gaussian fixed point.

To find the normal coordinates around the Gaussian fixed point and the corresponding beta functions, we could write (2.70) explicitly in an arbitrary parametrization $c$, solve for the components $\beta^\alpha$ and finally find the change of coordinates that puts the betas in normal form. We will follow instead a more direct procedure in which we impose the normal form to (2.70) (written in coordinates) from the very beginning and extract the normal beta functions and normal coordinates simultaneously. To do this, we first expand a general action close to the fixed point $s = 0$ in normal coordinates:

$$S_\omega = S_\alpha[\bar{\gamma};\omega]\,\bar{c}^\alpha + S_{\alpha_1\alpha_2}[\bar{\gamma};\omega]\,\bar{c}^{\alpha_1}\bar{c}^{\alpha_2} + O(\bar{c}^3). \tag{2.73}$$

[12] There are strong indications that this is the only fixed point for the scalar field theory in four dimensions [41, 42].

Unlike $S_\omega$ and $c^\alpha$, which are non-trivial functions on $\mathcal{W}$, the coefficients $S_{\alpha_1\dots\alpha_n}[\bar\gamma;\omega]$ are functionals of $\omega$ that do not depend on the first component of the point in $\mathcal{W}$, but only on the scale (as explicitly indicated by the $\bar\gamma$ argument). The chart $\bar c$ is perturbatively defined by these coefficients. Next, we plug (2.73) and (2.40) (with $c = \bar c$) in (2.70)), impose the normal condition (2.62) and solve order by order in $\bar c$ for the coefficients $\lambda^\alpha_{\alpha_1}$, $\bar\beta^\alpha_{\alpha_1\dots\alpha_n}$ and $S_{\alpha_1\dots\alpha_n}[\bar\gamma;\omega]$.

For definiteness, all the following calculations of this subsection will be done in dimension $d = 4$.

#### Eigendirections

We start with the linear order, assuming a diagonal[13] matrix $\lambda^\alpha_{\alpha_1} = \lambda_{(\alpha)}\delta^\alpha_{\alpha_1}$ and $\lambda_{(ax)} = \lambda_{(a)} = d - \Delta_{(a)} + n^u_{(a)} - n^d_{(a)}$, with $\Delta_{(a)}$ the conformal dimension of the operator and $n^u_{(a)}$ ($n^d_{(a)}$) the number of contravariant (covariant) indices of the coupling. At this order, (2.70) reduces to the following eigenvalue problem

$$\left[\omega^x \frac{\delta}{\delta\omega^x} + \frac{1}{2}\dot P^{xy}(\gamma)\frac{\delta^2}{\delta\omega^x\delta\omega^y} - 2\gamma\frac{\partial}{\partial\gamma}\right] S^z_a[\gamma;\omega] = \left(\Delta_{(a)} - n^u_{(a)} + n^d_{(a)}\right) S^z_a[\gamma;\omega]. \quad (2.74)$$

Recall that, in our covariant notation, $S_{ax}[\gamma;\omega] = \sqrt{|\gamma|}S^x_a[\gamma;\omega]$, with $S^x_a[\gamma;\omega]$ a scalar. This calculation of eigenoperators of the Polchinski equation in a scalar theory has been performed before in [32]. A trivial solution is given by the identity operator,

$$S^x_0[\gamma;\omega] = 1, \quad (2.75)$$

with dimension $\Delta_{(0)} = 0$.[14] To find the non-trivial solutions, we make an Ansatz consistent with our requirement of a quasilocal Wilson action:

$$S^x_a[\gamma;\omega] = Q^x_{a,x_1\dots x_{m_a}}(\gamma) S^{\langle x_1\dots x_{m_a}\rangle}[\gamma;\omega], \quad (2.76)$$

where $m_a$ is a positive integer associated to $a$ and $Q^x_{a,x_1\dots x_n}(\gamma)$ is a product of Dirac deltas and their derivatives with support at $x = x_1 = \dots = x_n$, while $S^{\langle x_1\dots x_n\rangle}[\gamma;\omega]$ are

[13] The fact that $\lambda$ is diagonalizable is an assumption that will be justified *a posteriori*.

[14] Note that the unitarity bound $\Delta \geq 1$ does not apply to this field-independent operator.

functions of $x_1, ..., x_n$ and of $\omega(x_1), ..., \omega(x_n)$, with the requirement that they and their derivatives of any order are well defined at coincident points. Inserting this form of the eigenfunctions into (2.74), we find that the $S^{\langle x_1 \dots x_{m_a}\rangle}[\gamma;\omega]$ must be solutions of the equation

$$\left[\Delta_{(a)} - n_\partial^{(a)} + 2\gamma\frac{\partial}{\partial\gamma} - \omega^x\frac{\delta}{\delta\omega^x} - \frac{1}{2}\dot{P}^{xy}(\gamma)\frac{\delta^2}{\delta\omega^x\delta\omega^y}\right] S^{\langle x_1 \dots x_{m_a}\rangle}[\gamma;\omega] = 0, \tag{2.77}$$

with $n_\partial^{(a)}$ the number of derivatives in the corresponding $Q$. The existence of a solution $S^{\langle x_1 \dots x_{m_a}\rangle}[\gamma;\omega]$ to this equation requires $\Delta_{(a)} = n_\partial^{(a)} + m_a$, which gives rise to the discrete spectrum $\Delta_{(a)} \in \mathbb{N}$ (besides $\Delta_{(0)} = 0$ for the identity). The first four explicit solutions are

$$\begin{aligned}
S^{\langle x\rangle}[\gamma;\omega] &= \omega^{x_1}, \\
S^{\langle x_1x_2\rangle}[\gamma;\omega] &= \omega^{x_1}\omega^{x_2} - P^{x_1x_2}(\gamma), \\
S^{\langle x_1x_2x_3\rangle}[\gamma;\omega] &= \omega^{x_1}\omega^{x_2}\omega^{x_3} - \left[\omega^{x_3}P^{x_1x_2}(\gamma) + \omega^{x_2}P^{x_1x_3}(\gamma) + \omega^{x_1}P^{x_2x_3}(\gamma)\right], \\
S^{\langle x_1x_2x_3x_4\rangle}[\gamma;\omega] &= \omega^{x_1}\omega^{x_2}\omega^{x_3}\omega^{x_4} - \left[\omega^{x_1}\omega^{x_2}P^{x_3x_4}(\gamma) + 5 \text{ inequivalent permutations}\right] \\
&\quad + \left[P^{x_1x_2}(\gamma)P^{x_3x_4}(\gamma) + P^{x_1x_3}(\gamma)P^{x_2x_4}(\gamma) + P^{x_1x_4}(\gamma)P^{x_2x_3}(\gamma)\right].
\end{aligned} \tag{2.78}$$

These functionals have a remarkable property: their expectation value vanishes in the free theory,

$$\left\langle S^{\langle x_1 \dots x_n\rangle}[\gamma^\Lambda, \omega]\right\rangle_{(0,\Lambda)} = 0, \tag{2.79}$$

for $n \geq 1$. For odd $n$ this statement is trivial, while for even $n$ it can be checked contracting the fields $w^x$ in (2.78) with Wick's theorem.

The local eigenoperators $S_a^x[\gamma;\omega]$ can be constructed from these solutions using (2.76). For instance,

$$\begin{aligned}
S_1^x[\gamma;\omega] &= \omega(x), \\
S_2^x[\gamma;\omega] &= (\omega(x))^2 - P(\gamma;0), \\
S_{2_2}^x[\gamma;\omega] &= \omega(x)\partial_\gamma^2\omega(x) - \partial_\gamma^2 P(\gamma;0), \\
S_3^x[\gamma;\omega] &= (\omega(x))^3 - 3P(\gamma;0)\omega(x),
\end{aligned} \tag{2.80}$$

where $1, 2, 2_2, 3, \dots$ label the eigendirections. Their eigendimensions are, respectively, $\Delta = 1, 2, 4, 3$, in agreement with the conformal dimensions of a free theory. Note that $P(\gamma; 0)$ and $\partial_\gamma^n P(\gamma; 0)$ for any $n$ are dimensionless constants. It is thus clear that any linear product of fields and their derivatives at $x$ can be written as a linear combination of the operators $S_a^x[\gamma; \omega]$ constructed in this way, together with the identity $S_0^x[\gamma; \omega]$. Therefore, these operators form a complete set and the linear part of the beta function is indeed diagonalizable. In the following we call $P_0 = P(\gamma; 0)$.

### Higher Orders

The higher orders can be obtained using (2.70) iteratively. The quadratic term $S_{a_1x_1\,a_2x_2}[\gamma; \omega] = |\gamma| S_{a_1a_2}^{x_1x_2}[\gamma; \omega]$ is given by

$$\left[2d - \lambda_{(a_1)} - \lambda_{(a_2)} + 2\gamma \frac{\partial}{\partial \gamma} - \omega^x \frac{\delta}{\delta \omega^x} - \frac{1}{2} \dot{P}^{xy}(\gamma) \frac{\delta^2}{\delta \omega^x \delta \omega^y}\right] S_{a_1a_2}^{z_1z_2}[\gamma; \omega] + \frac{1}{|\gamma|} \bar{\beta}^{\alpha}_{a_1z_1\,a_2z_2} S_\alpha[\gamma; \omega] = -\frac{1}{2} \dot{P}^{xy} \frac{\delta S_{a_1}^{z_1}[\gamma; \omega]}{\delta \omega^x} \frac{\delta S_{a_2}^{z_2}[\gamma; \omega]}{\delta \omega^y}. \tag{2.81}$$

The quadratic beta coefficients on the LHS of the equation (second line) is required to be resonant. They are necessary to cancel possible non-localities. As an example, let us solve the equation for the action coefficient $S_{33}^{x_1x_2}[\gamma; \omega]$, where the eigendirection $a = 3$ is defined in (2.80) and has $\lambda_{(3)} = 1$. Choosing $a_1 = a_2 = 3$, (2.81) reads

$$\left[6 + 2\gamma \frac{\partial}{\partial \gamma} - \omega^x \frac{\delta}{\delta \omega^x} - \frac{1}{2} \dot{P}^{xy}(\gamma) \frac{\delta^2}{\delta \omega^x \delta \omega^y}\right] S_{33}^{z_1z_2}[\gamma; \omega] + \frac{1}{|\gamma|} \left[\bar{\beta}^{2z}_{3z_13z_2} S_{2z}[\gamma; \omega] + \bar{\beta}^{0z}_{3z_13z_2} S_{0z}[\gamma; \omega]\right] = -\frac{9}{2} \dot{P}(\gamma, z_1 - z_2) \left[S^{\langle z_1z_1z_2z_2\rangle}[\gamma; \omega] + 4P^{z_1z_2}(\gamma) S^{\langle z_1z_2\rangle}[\gamma; \omega] + 2P^{z_1z_2}(\gamma)^2\right]. \tag{2.82}$$

We have included the only possible resonant beta terms:

$$\bar{\beta}^{2z}_{3z_13z_2}(\gamma) = b_{33}^2 \delta(z - z_1)\delta(z - z_2), \tag{2.83}$$

$$\bar{\beta}^{0z}_{3z_13z_2}(\gamma) = \frac{b_{33}^0}{2} \left[\delta(z - z_1)\partial_\gamma^2 \delta(z - z_2) + \partial_\gamma^2 \delta(z - z_1)\delta(z - z_2)\right]. \tag{2.84}$$

The values of the coefficients $b_{33}^{0,2}$ will be determined below. To solve (2.82), let us make the ansatz

$$S_{33}^{xy}[\gamma;\omega] = A(\gamma;x-y)\, S^{\langle xxyy\rangle}[\gamma;\omega] + B(\gamma;x-y)S^{\langle xy\rangle}[\gamma;\omega] + E(\gamma;x-y). \qquad (2.85)$$

Using (2.77), we find that the functions $A$, $B$ and $E$ must satisfy

$$\left[2 + 2\gamma_{\mu\nu}\frac{\partial}{\partial\gamma_{\mu\nu}}\right] A(\gamma;x) = -\,\frac{9}{2}\dot{P}(\gamma;x), \qquad (2.86)$$

$$\left[4 + 2\gamma_{\mu\nu}\frac{\partial}{\partial\gamma_{\mu\nu}}\right] B(\gamma;x) = -\,18\dot{P}(\gamma;x)P(\gamma;x) - b_{33}^{2}\frac{\delta(x)}{\sqrt{|\gamma|}}, \qquad (2.87)$$

$$\left[6 + 2\gamma_{\mu\nu}\frac{\partial}{\partial\gamma_{\mu\nu}}\right] E(\gamma;x) = -\,9\dot{P}(\gamma;x)P(\gamma;x)^2 - b_{33}^{0}\frac{\partial_\gamma^2\delta(x)}{\sqrt{|\gamma|}}. \qquad (2.88)$$

The most general solutions of these three equations are

$$A(\gamma;x) = \frac{9}{2}P(\gamma;x) - \frac{\xi_1}{4\pi^2 x_\gamma^2}, \qquad (2.89)$$

$$B(\gamma;x) = 9P(\gamma;x)^2 + \frac{b_{33}^2}{8\pi^2}\dot{\partial}_\gamma^2\left[\frac{\log x_\gamma^2}{x_\gamma^2}\right] + \xi_2\frac{\delta(x)}{\sqrt{|\gamma|}}, \qquad (2.90)$$

$$E(\gamma;x) = 3P(\gamma;x)^3 + \frac{b_{33}^0}{8\pi^2}\,\dot{\partial}_\gamma^4\left[\frac{\log x_\gamma^2}{x_\gamma^2}\right] + \xi_3\frac{\partial_\gamma^2\delta(x)}{\sqrt{|\gamma|}}, \qquad (2.91)$$

where the arbitrary parameters $\xi_1$, $\xi_2$ and $\xi_3$ are associated to solutions to the homogeneous part of (2.86), (2.87) and (2.88). The dot on the derivatives $\dot{\partial}^2$ and $\dot{\partial}^4$ indicates that they are defined in the sense of distributions, acting by parts on test functions and discarding (singular) surface terms. Then, $A(\gamma;x)$, $B(\gamma;x)$ and $E(\gamma;x)$ are well-defined distributions.[15] Their asymptotic behaviour when $x\to\infty$ is given by

[15] Acting instead with the derivatives on the functions inside the brackets results in

$$\partial_\gamma^2\left[\frac{\log x_\gamma^2}{x_\gamma^2}\right] = -\frac{4}{x_\gamma^4}, \qquad (2.92)$$

$$\partial_\gamma^4\left[\frac{\log x_\gamma^2}{x_\gamma^2}\right] = -\frac{32}{x_\gamma^6}, \qquad (2.93)$$

which are too singular at $x=0$ to admit a Fourier transform. The expressions with the dotted derivatives correspond to the renormalized values of these functions in differential renormalization [43].

$$A(\gamma;x) \sim \left(\frac{9}{2} - \xi_1\right) \frac{1}{4\pi^2 x_\gamma^2}, \tag{2.94}$$

$$B(\gamma;x) \sim \left(9 - 8\pi^2 b_{33}^2\right) \frac{1}{\left(4\pi^2\right)^2 x_\gamma^4}, \tag{2.95}$$

$$E(\gamma;x) \sim \left(3 - 256\pi^4 b_{33}^0\right) \frac{1}{\left(4\pi^2\right)^3 x_\gamma^6}. \tag{2.96}$$

This shows that these functions are in general non-local, with Fourier transforms that behave like $p^{-2}$, $\log p^2$ and $p^2 \log p^2$ as $p \to 0$ and are thus non-analytic at $p = 0$. To ensure that $S_{33}^{z_1 z_2}[\gamma;\omega]$ is quasilocal we need to fix $\xi_1 = 9/2$, $b_{33}^2 = 9/(8\pi^2)$ and $b_{33}^0 = 3/(256\pi^4)$. Therefore, the beta functions take the values

$$\bar{\beta}_{3x,3y}^{2z}(\gamma) = \frac{9}{8\pi^2}\delta(z-x)\delta(z-y), \tag{2.97}$$

$$\bar{\beta}_{3x,3y}^{0z}(\gamma) = \frac{3}{512\pi^4}\left[\delta(z-x)\partial_\gamma^2\delta(z-y) + \partial_\gamma^2\delta(z-x)\delta(z-y)\right]. \tag{2.98}$$

We stress that they do not depend on the particular regulator $P$. The beta function $\bar{\beta}_{3x,3y}^{0z}$ is associated to the identity operator and represents a contribution to the conformal anomaly in the presence of local couplings, see (2.43):

$$\mathcal{A}(x) = \frac{3}{256\pi^4}\bar{c}^3(x)\partial_\gamma^2\bar{c}^3(x) + \dots \tag{2.99}$$

As long as the regulating function $D(u)$ is analytic at $u = 0$, as assumed, the functions $A$, $B$ and $E$ with the selected values of $\xi_1$, $b_{33}^2$ and $b_{33}^0$ can be expanded as a series of local distributions,

$$A(\gamma;x-y) = \frac{1}{\sqrt{|\gamma|}}\left[A_0\,\delta(x-y) + A_2\,\partial_\gamma^2\delta(x-y) + A_4\,\partial_\gamma^4\delta(x-y) + ...\right], \tag{2.100}$$

$$B(\gamma;x-y) = \frac{1}{\sqrt{|\gamma|}}\left[B_0\,\delta(x-y) + B_2\,\partial_\gamma^2\delta(x-y) + B_4\,\partial_\gamma^4\delta(x-y) + ...\right], \tag{2.101}$$

$$E(\gamma;x-y) = \frac{1}{\sqrt{|\gamma|}}\left[E_0\,\delta(x-y) + E_2\,\partial_\gamma^2\delta(x-y) + E_4\,\partial_\gamma^4\delta(x-y) + ...\right], \tag{2.102}$$

with coefficients $A_n$, $B_n$ and $E_n$ depending on the chosen function $D$. Only $B_0$ and $E_2$ remain arbitrary, since they depend on the parameters $\xi_2$ and $\xi_3$. Using these functions in (2.76) we find $S^{xy}_{33}[\gamma;\omega]$, which gives a quasilocal contribution to $s$ in (2.73).

### 2.4.3 Large N Theories

The large N limit (or planar limit) of theories with fields in a representation of some internal group $G$ has been largely studied since it was introduced by 't Hooft [3] (see [44] for a review). It consists in taking the rank of $G$ and the dimensionality of the representation to infinity in the proper way. In this limit, the theory simplifies in several ways, but still it exhibits an interacting and quantum behaviour. One remarkable property is that only Feynman diagrams with a planar topology survive in this limit. Therefore, it is very useful to understand better the dynamics of strongly coupled theories (in fact, 't Hooft introduced this limit to study QCD).

In this section we analyse some simplifications of the exact RG in this limit (the ones relevant for the purposes of this thesis). Some related work on the large N exact RG can be found in [45] and the references therein. For definiteness, we will restrict ourselves to matrix theories.[16] This is, theories whose elementary degrees of freedom $\omega = \omega_{u_1 u_2}$ are $N \times N$ matrices, with some possible matrix condition (real matrices, hermitian matrices, traceless matrices, etc...). The labels $u_r$ indicate the two indices of the matrix. Additional possible flavour indices will be kept implicit. The fields $\omega$ can be rotated $\omega \to \mathfrak{g}\,\omega\,\mathfrak{g}^{-1}$, with some group of matrices $G \ni \mathfrak{g}$, which depends on the matrix condition: $G = O(N)$, $U(N)$, $SU(N)$, $\ldots$.

*Primary single-trace* operators are defined as the trace of any matrix product of $\omega$ and derivatives of $\omega$:

$$\mathcal{O}^s[\omega] = \frac{1}{N}\mathrm{Tr}[\omega(x)\ldots\partial\omega(x)\ldots\omega(x)], \tag{2.103}$$

that cannot be written as a total derivative. They are invariant under the action of $G$. We will use the letters $i$ and $j$ to label all the possible primary single-trace operators, this is, all the inequivalent products that can be built. To label a pair of a primary

[16] Other realizations of the large N limit are possible. For instance, theories with fields in a vector representation of $O(N)$ have been deeply studied in this context [45].

single-trace index and a continuous index we will use the Greek letter $\sigma = ix$.

We restrict $\mathcal{W}$ to theories symmetric under $G$, i.e., the set of actions that can be written as function of primary single-trace operators. Therefore, for some $(s, \Lambda) \in \mathcal{W}$, we will have that

$$\begin{aligned} s[\omega] &= S[\gamma^\Lambda; g, \omega] \\ &= \int d^d x \sqrt{|\gamma^\Lambda|}\, \mathcal{L}(\gamma^\Lambda; g(x), \mathcal{O}^s[\omega], \partial\mathcal{O}^s[\omega], \dots). \end{aligned} \tag{2.104}$$

Given a configuration of the primary single-trace operators $\mathcal{O}^s$, this space restriction allows to define the function $S_{\mathcal{O}^s} : \mathcal{W} \to \mathbb{R}$ that satisfies $S_{\mathcal{O}^s}(s, \Lambda) = s[\omega]$, with $\omega$ such that $\mathcal{O}^s[\omega] = \mathcal{O}^s$.

Using this definition, the partition function of (2.69) can be written as follows

$$Z = \frac{\int \mathcal{D}\omega \exp\left\{-N \int d^d x \sqrt{|\bar\gamma|}\frac{1}{2}\mathrm{Tr}\left[\bar\gamma^{\mu\nu}\partial_\mu \omega^x D^{-1}(-\partial^2_{\bar\gamma})\partial_\nu \omega^x\right] + N^2 S_{\mathcal{O}^s[\omega]}\right\}}{\int \mathcal{D}\omega \exp\left\{-N \int d^d x \sqrt{|\bar\gamma|}\frac{1}{2}\mathrm{Tr}\left[\bar\gamma^{\mu\nu}\partial_\mu \omega^x D^{-1}(-\partial^2_{\bar\gamma})\partial_\nu \omega^x\right]\right\}}. \tag{2.105}$$

The factors associated with $N$ in (2.103) and (2.105) are necessary to obtain finite correlation functions when one takes the limit $N \to \infty$ [46]. As we will see below, this normalization makes the Polchinski equation balanced in the $1/N$ orders. This can also be seen as a justification of this normalization choice.

**Exact RG Flow Equation**

For matrix theories, the Polchinski equation (2.70) becomes

$$\beta\, S_\omega - \omega^x_{u_1 u_2}\frac{\delta S_\omega}{\delta \omega^x_{u_1 u_2}} = \frac{1}{2}\dot{P}^{xy}(\bar\gamma)\mathcal{T}_{u_1 u_2 u_3 u_4}\left[\frac{1}{N}\frac{\delta^2 S_\omega}{\delta\omega^x_{u_1 u_2}\delta\omega^y_{u_3 u_4}} - N\frac{\delta S_\omega}{\delta\omega^x_{u_1 u_2}}\frac{\delta S_\omega}{\delta\omega^y_{u_3 u_4}}\right], \tag{2.106}$$

where the tensor $\mathcal{T}_{u_1 u_2 u_3 u_4}$ depends on the matrix condition. It is given by the matrix-index structure of the two point function in $s = 0$ (see (2.67)),

$$\langle \omega^x_{u_1 u_2}\omega^y_{u_3 u_4}\rangle_{(0,\Lambda)} = \frac{1}{N}\mathcal{T}_{u_1 u_2 u_3 u_4} P(\gamma^\Lambda, x - y). \tag{2.107}$$

and has necessarily the form

$$\mathcal{T}_{u_1u_2u_3u_4} = a\,\delta_{u_1u_3}\delta_{u_2u_4} + b\,\delta_{u_1u_4}\delta_{u_2u_3} + O\left(\frac{1}{N}\right). \tag{2.108}$$

for two coefficients $a$ and $b$ not depending on $N$. We are interested in writing an RG flow equation (2.106) for $S_{\mathcal{O}^s}$. For this, we first define the quasilocal coefficients $R^{(\sigma)}$, $R_\sigma^{\sigma_1\sigma_2}$, $\tilde{R}^\sigma_{\sigma_1\sigma_2}$ and $R^\sigma_{\sigma_1}$, fixed by the following equations:

$$\omega^x_{u_1u_2}\frac{\delta\mathcal{O}^\sigma}{\delta\omega^x_{u_1u_2}} = R^{(\sigma)}\mathcal{O}^\sigma, \tag{2.109}$$

$$\frac{1}{2}\dot{P}^{xy}(\gamma)\mathcal{T}_{u_1u_2u_3u_4}\frac{\delta^2\mathcal{O}^\sigma}{\delta\omega^x_{u_1u_2}\delta\omega^y_{u_3u_4}} = N\tilde{R}^\sigma_{\sigma_1\sigma_2}(\gamma)\mathcal{O}^{\sigma_1}\mathcal{O}^{\sigma_2} + O(N^0), \tag{2.110}$$

$$\frac{1}{2}\dot{P}^{xy}(\gamma)\mathcal{T}_{u_1u_2u_3u_4}\frac{\delta\mathcal{O}^{\sigma_1}}{\delta\omega^x_{u_1u_2}}\frac{\delta\mathcal{O}^{\sigma_2}}{\delta\omega^y_{u_3u_4}} = \frac{1}{N}\left[R_\sigma^{\sigma_1\sigma_2}(\gamma)\mathcal{O}^\sigma + R^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\gamma)\mathcal{O}^{\sigma_3}\mathcal{O}^{\sigma_4}\right] + O\left(\frac{1}{N^2}\right). \tag{2.111}$$

All of them are order $O(N^0)$. $R_\sigma^{\sigma_1\sigma_2}$, $\tilde{R}^\sigma_{\sigma_1\sigma_2}$ and $R^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}$ are the leading contributions in the $1/N$ expansion of the LHS of (2.111) and (2.110), keeping $\mathcal{O}^s$ finite. The structure of the RHS of these equations can be checked using the definition of primary single-trace operator (2.103) and (2.108).

With these definitions, the leading contribution in the $1/N$ expansion of (2.106) is

$$\begin{aligned}\beta\, S_{\mathcal{O}^s} =& \frac{\delta S_{\mathcal{O}^s}}{\delta\mathcal{O}^\sigma}\left[R^{(\sigma)}\mathcal{O}^\sigma + \tilde{R}^\sigma_{\sigma_1\sigma_2}(\bar{\gamma})\mathcal{O}^{\sigma_1}\mathcal{O}^{\sigma_2}\right] \\ &- \frac{\delta S_{\mathcal{O}^s}}{\delta\mathcal{O}^{\sigma_1}}\frac{\delta S_{\mathcal{O}^s}}{\delta\mathcal{O}^{\sigma_2}}\left[R_\sigma^{\sigma_1\sigma_2}(\bar{\gamma})\mathcal{O}^\sigma + R^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\gamma)\mathcal{O}^{\sigma_3}\mathcal{O}^{\sigma_4}\right] + O\left(\frac{1}{N}\right).\end{aligned} \tag{2.112}$$

This equation has a number of properties that simplifies the calculation of normal coordinates. For instance, unlike (2.106), (2.112) does not have second derivative terms, but only first derivative ones. In fact, we point out the following property:

**Large N property 1**. The RG flow equation has the form

$$\beta\, S_{\mathcal{O}^s} = F\left[\bar{\gamma};\mathcal{O}^s,\frac{\delta S_{\mathcal{O}^s}}{\delta\mathcal{O}^s}\right]. \tag{2.113}$$

The functional $F[\bar{\gamma};\mathcal{O}^s,\pi_s]$ can be read from (2.112).

### Fixed Points and Eigendirections

Let us consider a fixed point of the flow, $s_*$. We define $S_*[\gamma^\Lambda,\mathcal{O}^s]=S_{\mathcal{O}^s}(s_*,\Lambda)$. It satisfies

$$2\gamma\frac{\partial}{\partial\gamma}S_*[\gamma;\mathcal{O}^s]=F\left[\bar{\gamma};\mathcal{O}^s,\frac{\delta S_*[\gamma;\mathcal{O}^s]}{\delta\mathcal{O}^s}\right], \tag{2.114}$$

and we will assume it can be expanded like

$$S_*[\gamma;\mathcal{O}^s]=\sum_{n\geq 0}S_{*;\sigma_1\ldots\sigma_n}(\gamma)\mathcal{O}^{\sigma_1}\ldots\mathcal{O}^{\sigma_n}. \tag{2.115}$$

As explained in Section 2.4.2, $s_*=0$ gives the Gaussian fixed point. In this section however, we will not restrict ourselves to this point, and consider other possible fixed points.[17]

The function $S_{\mathcal{O}^s}$ can be expanded around the fixed point in normal coordinates:

$$S_{\mathcal{O}^s}=S_*[\bar{\gamma};\mathcal{O}^s]+S_\alpha[\bar{\gamma};\mathcal{O}^s]\bar{c}^\alpha+S_{\alpha_1\alpha_2}[\bar{\gamma};\mathcal{O}^s]\bar{c}^{\alpha_1}\bar{c}^{\alpha_2}+O(\bar{c}^3). \tag{2.116}$$

To find the coefficients of this expansion we proceed as in Section 2.4.2. We assume a normal form for the beta function and solve it order by order.

The linear order is

$$\begin{aligned}\left\{\frac{\delta F}{\delta\pi_\sigma}\left[\bar{\gamma};\mathcal{O}^s,\frac{\delta S_*[\bar{\gamma};\mathcal{O}^s]}{\delta\mathcal{O}^s}\right]\frac{\delta}{\delta\mathcal{O}^\sigma}-2\gamma\frac{\partial}{\partial\gamma}\right\}S_a^x[\gamma;\mathcal{O}^s]&=\left(\Delta_{(a)}-n^u_{(a)}+n^d_{(a)}\right)S_a^x[\gamma;\mathcal{O}^s]\\&\quad+\bar{\beta}_a^{x\,\alpha}S_\alpha[\gamma;\mathcal{O}^s].\end{aligned} \tag{2.117}$$

The second line is only necessary for non-diagonalizable fixed points. A direct consequence of the form of this equation (and therefore a consequence of the Large N property 1) is the following factorization property. Given two eigendirections $a_1$, $a_2$,

[17] Other fixed points could require the inclusion of an anomalous dimension of the quantum field $\omega$. It can be trivially done redefining $R^{(\sigma)}$ without changing the conclusions of this section. See for instance [32].

with coefficients $S^x_{a_1}[\gamma;\mathcal{O}^s]$ and $S^x_{a_2}[\gamma;\mathcal{O}^s]$, and dimensions $\Delta_{(a_1)}$ and $\Delta_{(a_2)}$, the product

$$S^x_a[\gamma;\mathcal{O}^s] = S^x_{a_1}[\gamma;\mathcal{O}^s] S^x_{a_2}[\gamma;\mathcal{O}^s], \tag{2.118}$$

gives a new eigendirection $a$ with dimension $\Delta_{(a)} = \Delta_{(a_1)} + \Delta_{(a_2)}$.

A trivial solution is again given by the identity operator $S^x_0[\gamma;\mathcal{O}^s] = 1$, with $\Delta_{(0)} = 0$. Non-trivial solutions can be found inserting the following expansion,

$$S_\alpha[\gamma;\mathcal{O}^s] = \sum_{n\geq 1} S_{\alpha;\sigma_1\ldots\sigma_n}(\gamma)\mathcal{O}^{\sigma_1}\ldots\mathcal{O}^{\sigma_n}. \tag{2.119}$$

In fact, the specific form of $F[\gamma;\mathcal{O}^s,\pi_s]$ endows (2.117) with the following property:

**Large N property 2**. The equation for non-trivial eigendirections $a$ is *triangular* in the following sense. The equation for the $n$-th order coefficient, $S_{ax;\sigma_1\ldots\sigma_n}$, only involves $m$-th order coefficients, with $m \leq n$.

Based on this property, we will classify the eigenperturbations as follows:

(1) *Single-trace eigendirections* are those solutions of (2.117) such that $S^x_{a^1;\sigma}(\gamma) \neq 0$ (for some $\sigma$). They are determined by non-trivial solutions of the equation for $S^x_{a^1;\sigma}$,

$$\begin{aligned}\left\{\delta^\sigma_{\sigma'} R^{(\sigma)} - 2S_{*;\sigma_1}(\gamma) R^{\sigma_1\sigma}_{\sigma'}(\gamma) - 2\gamma\frac{\partial}{\partial\gamma}\right\} S^x_{a^1;\sigma}(\gamma) \\ = \left(\Delta_{(a^1)} - n^u_{(a^1)} + n^d_{(a^1)}\right) S^x_{a^1;\sigma'}(\gamma) + \bar{\beta}^{x\,\alpha}_{a} S_{\alpha,\sigma'}(\gamma).\end{aligned} \tag{2.120}$$

Once the equation is solved and the eigenvalues found, (2.117) can be used to iteratively determinate higher orders.

(2) *Higher-trace eigendirections* are given by those solutions of (2.117) that, given some $n > 1$, $S_{a^n;\sigma_1\ldots\sigma_m}(\gamma) = 0$ for $m < n$, but $S^x_{a^n;\sigma_1\ldots\sigma_m}(\gamma) \neq 0$ for $m \geq n$. In particular, this would be an $n$-trace eigendirection. We will label them with $a^n$, or $\alpha^n = a^n x$.

A subgroup of the single-trace eigendirections is given by the *primary* single-trace eigendirections. They are those ones whose coefficient can be expanded like

$$S^x_{a^1}[\gamma;\mathcal{O}^s] = M_{a^1\,i}\,\mathcal{O}^{ix} + O(\partial\mathcal{O}^s) + O\left((\mathcal{O}^s)^2\right), \tag{2.121}$$

for some matrix $M_{a^1\,i}$. A convenient rotation in the space of primary single-trace operators $\mathcal{O}^{ix} \to \mathcal{O}'^{\,ix} = N^i_j\mathcal{O}^{jx}$ would allow to, given some primary single-trace direction $a^1$, make $M_{a^1\,i} = 0$ for all $i$ except one.[18] Therefore, after this rotation, we can use the same letters ($i$, $j$ and $\sigma$) to label primary single-trace directions and operators, and write

$$S^x_i[\gamma;\mathcal{O}^s] = \delta_{ij}\,\mathcal{O}^{jx} + O(\partial\mathcal{O}^s) + O\left((\mathcal{O}^s)^2\right). \tag{2.122}$$

In the Gaussian fixed point, we directly find (2.122) without the necessity of any rotation. *Descendant* single-trace eigendirections are the spacetime derivatives of primary ones.[19]

Primary single-trace eigendirections are specially important, since, taking spacetime derivatives and using the factorization property (see (2.118)), we can construct the coefficient $S_a[\gamma;\mathcal{O}]$ of any other eigendirection $a$.

For instance, given a set of single-trace eigendirections $i_1,\dots i_n$, the $n$-trace eigendirection $a^n = \langle i_1^\mu i_{2\mu}\dots i_n\rangle$ will be the one associated to

$$S^x_{a^n}[\gamma;\mathcal{O}^s] = \gamma^{\mu\nu}\partial_\mu S^x_{i_1}[\gamma;\mathcal{O}^s]\partial_\nu S^x_{i_2}[\gamma;\mathcal{O}^s]\dots S^x_{i_n}[\gamma;\mathcal{O}^s]. \tag{2.123}$$

It has dimension

$$\Delta_{(a^n)} = \sum_{r=1}^{n}\Delta_{(i_r)} + 2n_{(\gamma^{-1})} - 2n_{(\gamma)}, \tag{2.124}$$

with $n_{(\gamma)}$ and $n_{(\gamma^{-1})}$ the number of metrics and inverse metrics respectively (in this case, $n_{(\gamma)} = 0$, $n_{(\gamma^{-1})} = 1$). We will say that the directions $i_1,\dots i_n$ compose the multi-trace direction $a^n$.

An alternative way to write multi-trace eigenperturbations is

$$S^y_{a^n}[\gamma;\mathcal{O}^s] = Q^y_{a^n\sigma_1\dots\sigma_n}(\gamma)\,S^{\langle\sigma_1\dots\sigma_n\rangle}[\gamma;\mathcal{O}^s], \tag{2.125}$$

[18] We will assume that this diagonalization is possible.

[19] The spacetime derivative of any eigendirection $a$ is also an eigendirection with the same eigenvalue $\lambda_{(a)}$. This property is always true independently of the large N limit.

where

$$S^{\langle\sigma_1\ldots\sigma_n\rangle}[\gamma;\mathcal{O}^s] = S^{\sigma_1}[\gamma;\mathcal{O}^s]\ldots S^{\sigma_n}[\gamma;\mathcal{O}^s] \tag{2.126}$$

is a multilocal function constructed with $S^{\sigma}[\gamma;\mathcal{O}^s] \equiv S^{ix}[\gamma;\mathcal{O}^s] = \delta^{ij} S^x_j[\gamma;\mathcal{O}^s]$. $Q^y_{a^n\sigma_1\ldots\sigma_n}(\gamma)$ is a distribution given by a combination of delta functions and their derivatives with support only in $x = y_1 = \cdots = y_n$. In momentum space, it has the form

$$\hat{Q}_{a^n;q\,i_1\ldots i_n}^{\;\;\;\;\;p_1\ldots p_n}(\gamma) = (2\pi)^d \delta\left(q - \sum_{i=1}^n p_i\right) \mathbf{Q}_{a^n;i_1\ldots i_n}(\gamma; p_1,\ldots,p_n), \tag{2.127}$$

with $\mathbf{Q}_{a^n;i_1\ldots i_n}(\gamma;p_1,\ldots,p_n)$ an analytic and homogeneous function of $p_1,\ldots,p_n$. For example, for the eigendirection (2.123),

$$\mathbf{Q}_{a^n;j_1\ldots j_n}(\gamma;p_1,\ldots,p_n) = -\delta_{j_1 i_1}\ldots\delta_{j_n i_n}\,\gamma^{\mu\nu}\,p_{1\mu}\,p_{2\nu}\ldots \tag{2.128}$$

The locality of $S^y_{a^n}[\gamma;\mathcal{O}^s]$ is recovered in (2.125) by contraction with the $Q$ distributions. Notice that, from (2.122),

$$S^{\langle i_1 x_1\ldots i_n x_n\rangle}[\gamma;\mathcal{O}^s] = \prod_{r=1}^n \mathcal{O}^{i_r}(x_r) + \text{ higher orders.} \tag{2.129}$$

### Higher Orders

Once the set of eigendirections has been found, higher orders of the expansion can be found taking derivatives in (2.113) with respect to normal charts and evaluating them at the fixed point:[20]

$$\begin{aligned}&\left\{\sum_{r=1}^n \lambda_{(\alpha_r)} - 2\gamma\frac{\partial}{\partial\gamma} + \frac{\delta F}{\delta\pi_\sigma}\left[\gamma;\mathcal{O}^s,\frac{\delta S_*^{[\gamma;\mathcal{O}^s]}}{\delta\mathcal{O}^s}\right]\frac{\delta}{\delta\mathcal{O}^\sigma}\right\}S^{[\gamma;\mathcal{O}^s]}_{\alpha_1\ldots\alpha_n}\\ &= -\sum_{p\in\Pi_n}\frac{|p_1|!\ldots|p_{|p|}|!}{n!}\frac{\delta^{|p|}F}{\delta\pi_{\sigma_1}\ldots\delta\pi_{\sigma_{|p|}}}\left[\gamma;\mathcal{O}^s,\frac{\delta S_*^{[\gamma;\mathcal{O}^s]}}{\delta\mathcal{O}^s}\right]\frac{\delta S^{[\gamma;\mathcal{O}^s]}_{\alpha_{p_1}}}{\delta\mathcal{O}^{\sigma_1}}\ldots\frac{\delta S^{[\gamma;\mathcal{O}^s]}_{\alpha_{p_{|p|}}}}{\delta\mathcal{O}^{\sigma_{|p|}}}\end{aligned}$$

[20] To make this equation more compact, we have written the argument $[\gamma;\mathcal{O}^s]$ of some functions as a superindex.

$$+\sum_{r=1}^{n}(n-r+1)\bar{\beta}^{\alpha}_{(\alpha_1\ldots\alpha_r}(\gamma)S^{[\gamma;\mathcal{O}^s]}_{\alpha\,\alpha_{r+1}\ldots\alpha_n)}. \tag{2.130}$$

Here $\Pi_n$ is the set of partitions of $\{1,2,\ldots,n\}$, $p_r$ is the $r$-th element of the partition $p$, $\alpha_{p_r}$ is a collective index given by

$$\alpha_{p_r} = \alpha_{p_{r1}}\ldots\alpha_{p_{r|p_r|}}, \tag{2.131}$$

with $p_{ri}$ the $i$-th element of the set $p_r$, and for any set $A$, $|A|$ is its cardinality.

In (2.130), only beta coefficients $\bar{\beta}^{\alpha}_{\alpha_1\ldots\alpha_r}$ satisfying the resonance condition (2.55) can be non-vanishing. In fact, as we saw for (2.81), they are required to find quasilocal coefficients $S_{\alpha_1\ldots\alpha_n}[\gamma;\mathcal{O}^s]$. This provides a direct method to compute simultaneously from the flow equation both the beta functions in normal coordinates and the coefficients $S_{\alpha_1\ldots\alpha_n}[\gamma,\mathcal{O}^s]$, which perturbatively define the coordinates themselves: working order by order, the beta function coefficients are determined by requiring the existence of a quasilocal solution, which is then readily obtained. The solution will be unique up to different choices of the normal coordinates.

### Factorization Normal Coordinates

We have already seen how in the large N limit, the eigendirections and eigendeformations of the Wilson action factorize (multi-trace deformations are written like products of single-trace ones). In this section we show how, beyond the linear level, it is possible to select normal charts such that $S_{\mathcal{O}^s}$ also exhibits similar properties.

Let $\alpha^n = a^n x$ be a multi-trace eigendirection defined by

$$S_{\alpha^n}[\gamma;\mathcal{O}^s] = Q_{\alpha^n}{}^{\sigma_1\ldots\sigma_n}(\gamma)S_{\sigma_1}[\gamma;\mathcal{O}^s]\ldots S_{\sigma_n}[\gamma;\mathcal{O}^s]. \tag{2.132}$$

We define *factorization normal coordinates* as those ones that satisfy

$$\partial^{\bar{c}}_{\alpha^n}S_{\mathcal{O}^s} = Q_{\alpha^n}{}^{\sigma_1\ldots\sigma_n}(\bar{\gamma})\partial^{\bar{c}}_{\sigma_1}S_{\mathcal{O}^s}\ldots\partial^{\bar{c}}_{\sigma_n}S_{\mathcal{O}^s}. \tag{2.133}$$

This equation has to be understood as an equality between functions defined on $\mathcal{W}$. The evaluation of this equation at the fixed point gives exactly (2.132) and therefore is obviously true. What we stress here is that it is possible to find a subclass of normal coordinates that makes this equality also true for all points of $\mathcal{W}$ (or at least, perturbatively true for points in a neighbourhood of the fixed point).

The defining property of factorization normal coordinates (2.133) can be expressed as a relation between the coefficients that expand $S_{\mathcal{O}^s}$. Taking $n-1$ derivatives of (2.133) with respect to normal coordinates, and evaluating them at the fixed point, we find that

$$S^{[\gamma,\mathcal{O}^s]}_{\alpha^n\alpha_1\ldots\alpha_m} = Q_{\alpha^n}{}^{\sigma_1\ldots\sigma_n}(\gamma) \sum_{p\in\mathcal{P}^n_m} \frac{(|p_1|+1)!\ldots(|p_n|+1)!}{(m+1)!} S^{[\gamma,\mathcal{O}^s]}_{\sigma_1\alpha_{p_1}}\ldots S^{[\gamma,\mathcal{O}^s]}_{\sigma_n\alpha_{p_n}}. \tag{2.134}$$

Here, $\mathcal{P}^n_m$ is the set of all possible $n$-tuples of subsets of $\{1,2,\ldots,m\}$, $p=(p_1,\ldots,p_n)$, such that $p_{r_1}\bigcap p_{r_2}=\emptyset$ for $r_1\neq r_2$, and $\bigcup_{r=1}^n p_r=\{1,2,\ldots,m\}$.[21] $|A|$ is the cardinality of the set $A$ and $\alpha_{p_r}$ is the collective index as in (2.131). Applying repeatedly (2.134) to all multi-trace indices, we can express all coefficients $S_{\alpha_1\ldots\alpha_n}(\gamma)$ as function of coefficients with only single-trace indices.

To prove the existence of factorization normal coordinates, let us first take the derivative of (2.113) with respect to the normal coordinate $\alpha$,

$$\left\{\lambda_{(\alpha)} - \bar{\beta}^{\alpha'}\partial^{\bar{c}}_{\alpha'} - 2\bar{\gamma}\frac{\partial}{\partial\bar{\gamma}} + \frac{\delta F}{\delta\pi_\sigma}\left[\bar{\gamma};\mathcal{O}^s,\frac{\delta S_{\mathcal{O}^s}}{\delta\mathcal{O}^s}\right]\frac{\delta}{\delta\mathcal{O}^\sigma}\right\}\partial^{\bar{c}}_\alpha S_{\mathcal{O}^s} = \partial^{\bar{c}}_\alpha\bar{\beta}^{\alpha'}_E\ \partial^{\bar{c}}_{\alpha'}S_{\mathcal{O}^s}. \tag{2.136}$$

$\bar{\beta}^\alpha_E$ is the exceptional part of the beta function:

$$\begin{aligned}\bar{\beta}^\alpha &= -\lambda_{(\alpha)}\bar{c}^\alpha + \bar{\beta}^\alpha_E\\ &= -\lambda_{(\alpha)}\bar{c}^\alpha + \sum_{r=1}^n \bar{\beta}^\alpha_{\alpha_1\ldots\alpha_r}(\bar{\gamma})\bar{c}^{\alpha_1}\ldots\bar{c}^{\alpha_r},\end{aligned} \tag{2.137}$$

and $r=1$ is included in the sum of the second line to also consider non-diagonalizable fixed points.

(2.136) is a differential equation on $\mathcal{W}$ that $\partial^{\bar{c}}_\alpha S_{\mathcal{O}^s}$ has to satisfy. Its expansion

[21] For instance,

$$\mathcal{P}^2_2 = \left\{\Big(\{1,2\},\emptyset\Big),\Big(\emptyset,\{1,2\}\Big),\Big(\{1\},\{2\}\Big),\Big(\{2\},\{1\}\Big)\right\}. \tag{2.135}$$

in normal coordinates around the fixed point is equivalent to (2.130), and therefore, (2.136) can also be seen as a defining equation for $\partial^{\bar{c}}_{\alpha} S_{\mathcal{O}^s}$. Possible ambiguities of the solutions are associated to different choices of the normal coordinates. Thus, what we want to prove is that there are non-trivial solutions of (2.136) satisfying (2.133).

Using that $\partial_{\sigma_r} S^{\bar{c}}_{\mathcal{O}^s}$ satisfies (2.136), the RHS of (2.133) fulfils the following equation,

$$\left\{\bar{\beta}^{\alpha'}\partial^{\bar{c}}_{\alpha'} + 2\bar{\gamma}\frac{\partial}{\partial\bar{\gamma}} - \frac{\delta F}{\delta\pi_\sigma}\left[\bar{\gamma};\mathcal{O}^s,\frac{\delta S_{\mathcal{O}^s}}{\delta\mathcal{O}^s}\right]\frac{\delta}{\delta\mathcal{O}^\sigma}\right\}\left(Q_{\alpha^n}{}^{\sigma_1\ldots\sigma_n}(\bar{\gamma})\partial^{\bar{c}}_{\sigma_1}S_{\mathcal{O}^s}\ldots\partial^{\bar{c}}_{\sigma_n}S_{\mathcal{O}^s}\right)$$
$$= \left[d - \sum_{r=1}^{n}\Delta_{(\sigma_r)} - 2n_{(\bar{\gamma}^{-1})} + 2n_{(\bar{\gamma})}\right] Q_{\alpha^n}{}^{\sigma_1\ldots\sigma_n}(\bar{\gamma})\partial^{\bar{c}}_{\sigma_1}S_{\mathcal{O}^s}\ldots\partial^{\bar{c}}_{\sigma_n}S_{\mathcal{O}^s}$$
$$-Q_{\alpha^n}{}^{\sigma_1\ldots\sigma_n}(\bar{\gamma})\sum_{r=1}^{n}\partial^{\bar{c}}_{\sigma_r}\bar{\beta}^{\alpha'}_{E}\ \partial^{\bar{c}}_{\sigma_1}S_{\mathcal{O}^s}\ldots\underbrace{\partial^{\bar{c}}_{\alpha'}S_{\mathcal{O}^s}}_{\text{position r}}\ldots\partial^{\bar{c}}_{\sigma_n}S_{\mathcal{O}^s}. \quad (2.138)$$

Due to the factorization of the dimensions (2.124), for the RHS of (2.133) to satisfy (2.136) with $\alpha = \alpha^n$ (providing a non-trivial solution), it is necessary that

$$\partial^{\bar{c}}_{\alpha^n}\bar{\beta}^{\alpha'}_{E}\ \partial^{\bar{c}}_{\alpha'}S_{\mathcal{O}^s} = Q_{\alpha^n}{}^{\sigma_1\ldots\sigma_n}(\bar{\gamma})\sum_{r=1}^{n}\partial^{\bar{c}}_{\sigma_r}\bar{\beta}^{\alpha'}_{E}\ \partial^{\bar{c}}_{\sigma_1}S_{\mathcal{O}^s}\ldots\underbrace{\partial^{\bar{c}}_{\alpha'}S_{\mathcal{O}^s}}_{\text{position r}}\ldots\partial^{\bar{c}}_{\sigma_n}S_{\mathcal{O}^s}. \quad (2.139)$$

For generic dimensions, both sides of this equation vanish and therefore, the existence of factorization normal coordinates is automatically proved. For exceptional dimensions, (2.139) imposes a constraint on the beta function which has to be consistent with the normal form of the beta coefficients (2.62). Indeed, using (2.133), (2.139) is equivalent to

$$\bar{\beta}^{ax'}_{\alpha_1\ldots ix\ldots\alpha_m} = \bar{\beta}^{\langle a\ldots\rangle x'}_{\alpha_1\ldots\langle i\ldots\rangle x\ldots\alpha_m}, \quad (2.140)$$

for all beta coefficients and eigendirections. The eigendirections $\langle i\ldots\rangle$ and $\langle a\ldots\rangle$ depict the multi-trace directions that result from the composition of $i$ or $a$ with some set of eigendirections (both with the same set). The factorization of dimensions (2.124) implies that the resonance condition (2.55) is satisfied by the LHS of (2.140) if and only if it is satisfied by the RHS. Therefore, (2.140) is consistent with the resonance condition and it can be imposed for the calculation of factorization normal coordinates.

In fact, (2.140) provides the form of the beta coefficients with multi-trace lower indices once all the beta coefficients with single-trace lower indices are known.

**Reconstruction of $S_{\mathcal{O}^s}$**

To finish this chapter, we would like to show how, due to the large N properties, if $S_{\mathcal{O}^s}$ is only known on a specific submanifold of $\mathcal{W}$, it is possible to reconstruct it on the full $\mathcal{W}$ (at least perturbatively).

Let $\mathcal{T}_1$ be the submanifold of single-trace deformations of a fixed point. This is, given some factorization normal chart $\bar{c}$, $\mathcal{T}_1 \subset \mathcal{W}$ is the submanifold defined by

$$(s, \Lambda) \in \mathcal{T}_1 \;\Leftrightarrow\; \bar{c}^{\alpha^n}(s, \Lambda) = 0,\; \forall n \neq 1, \tag{2.141}$$

where $\alpha^n$ depicts $n$-trace eigendirections and $\alpha^0 \equiv 0$ is the vacuum energy eigendirection.

Note that, applying repeatedly (2.133), we can express any higher order derivative with respect to multi-trace normal coordinates of the Wilson action as a function of only single-trace derivatives. This allows to reconstruct perturbatively $S_{\mathcal{O}^s}$ in the complete $\mathcal{W}$ once it is known in $\mathcal{T}_1$. In fact, it is possible to give a closed expression to extend $S_{\mathcal{O}^s}$ if this is only defined on $\mathcal{T}_1$. Let

$$S^L_{\mathcal{O}^s}[\bar{\gamma}_\epsilon; \zeta] = \bar{c}^\sigma \zeta_\sigma - S_{\mathcal{O}^s}, \qquad \zeta_\sigma = \partial^{\bar{c}}_\sigma S_{\mathcal{O}^s}|_{\mathcal{T}_1}, \tag{2.142}$$

be the Legendre transform of the Wilson action restricted to $\mathcal{T}_1$ with respect to the single-trace coordinates $\bar{c}^\sigma$. Also, given some $\zeta$, let $G_\zeta : \mathcal{W} \to \mathbb{R}$ be the function

$$G_\zeta = \bar{c}^0 + \bar{c}^\sigma \zeta_\sigma + \sum_{n\geq 2} \bar{c}^{\alpha^n} Q^{\sigma_1 \ldots \sigma_n}_{\alpha^n}(\bar{\gamma}) \zeta_{\sigma_1} \ldots \zeta_{\sigma_n}. \tag{2.143}$$

Then, given $\mathcal{O}^s$, we can define the function on $\mathcal{W}$,

$$\tilde{S}_{\mathcal{O}^s} = G_\zeta - S^L_{\mathcal{O}^s}[\bar{\gamma}; \zeta], \tag{2.144}$$

$$\frac{\delta G_\zeta}{\delta \zeta} = \frac{\delta S^L_{\mathcal{O}^s}}{\delta \zeta}[\bar{\gamma}; \zeta], \tag{2.145}$$

where the second equality fixes the value of $\zeta$. With this definition, $\tilde{S}_{\mathcal{O}^s}$ satisfies:

(1) Its restriction to $\mathcal{T}_1$ is equal to the Wilson action, $\tilde{S}_{\mathcal{O}^s}\Big|_{\mathcal{T}_1} = S_{\mathcal{O}^s}$. For points in $\mathcal{T}_1$, (2.144) reduces to the usual Legendre transformation of $S^L_{\mathcal{O}^s}$, and by (2.142), we recover the Wilson action $S_{\mathcal{O}^s}$.

(2) It satisfies (2.133) for all points in $\mathcal{W}$. This can be directly checked inserting (2.143) in (2.144) and using (2.145).

Both properties guarantee that, at least for points perturbatively close to $\mathcal{T}_1$, $\tilde{S}_{\mathcal{O}^s} = S_{\mathcal{O}^s}$.

# Chapter 3

# Connection with Renormalization

*The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.*

David Hilbert [47]

In the previous chapter, we have described geometrically the space of theories and the Wilson flows. Also, we have singled out a special set of coordinates which simplify as much as possible the form of the beta functions and allow to compute contributions to the conformal anomaly. In this chapter we will make contact with usual renormalization methods used in QFT. In a first part, we will study the renormalization of correlation functions of composite operators at fixed points of the flow. In a second part, we will describe how scale non-invariant renormalizable theories are constructed in the Wilsonian approach.

A systematic functional formalism to renormalize general correlators of composite operators was developed by Shore in [48] (see also [49, 50] for previous developments). Let us summarize this approach. First, a source is introduced for each of the (infinitely-many) independent local operators in the theory [51]. Renormalization then proceeds quite standardly by writing the bare (spacetime dependent) couplings as convenient functions of renormalized (spacetime dependent) couplings. At the linear level, this is equivalent to the usual multiplicative renormalization of the operators, including mixing. Further counterterms are required to make finite the correlators of two or more composite operators. The main point of [48] is that all these counterterms can be generated by a non-linear dependence of the bare couplings on the renormalized ones. In general, the bare couplings at a point $x$ depend not only on the values of the renormalized couplings at $x$ but also on its derivatives at this point. Because we want to compare with the exact RG, we will use a dimensionful regulator. Then, mixing of operators of different dimension is expected already at the linear level.

This renormalization procedure has a nice geometric interpretation, which in fact provides new perspectives on it. As we will see in detail, the RG flows can be used to define spacetime dependent bare couplings that renormalize the theory. Both the linear and the non-linear terms in the renormalized couplings follow automatically. This relation had been found before in [40], but only at the linear level and without taking into account the effect of derivatives in the evolution, which turns out to be crucial. Our analysis does not have any restrictions in this sense.

We study the correlation functions at fixed points of the RG flows. The fixed points correspond to scale invariant theories, and thus, trivially describe the *continuum limit*. However, the correlators involve not only the fixed point but also infinitesimal deformations for which the continuum limit is non-trivial. Finite deformations are considered later, in Section 3.2.

Scale invariant theories, if unitary, are also expected to be conformally invariant [52–54]. A lot is known about correlation functions in conformal field theories (CFT). In particular, there has been recent progress in the bootstrap program, which aims to determine the correlation functions from minimal input and consistency conditions [55–59] (see also [60] for references to more recent work). These methods refer only to the fixed-point itself, whereas in this thesis we are investigating the relation with finite deformations in the presence of a cutoff. Of course, after renormalization the results for infinitesimal deformations must agree, so the CFT consistency conditions will impose non-trivial constraints on the behaviour of the RG flows near the fixed point.

When the fixed point of interest is Gaussian, the renormalization procedure takes its simplest form when formulated as a limit of deformations of the fixed point. Then the connection with the RG is most transparent. However, for interacting CFT it may be more convenient to consider deformations of another separate point in theory space, if they are described by simpler actions. This point must belong to the basin of attraction of the fixed point, i.e. to the so-called critical manifold. In this chapter we consider this possibility. It will be crucial for the application of these methods to holographic renormalization in Gauge/Gravity duality in the Part II.

As a consequence of the intimate connection with the exact RG, the counterterms can be found from the RG flows. Furthermore, the renormalized correlators turn out to be equal to specific bare correlators evaluated at a finite cutoff. Of course, the determination of counterterms and renormalized correlators from RG flows cannot be unique, since there is some freedom in the renormalization process. Different choices give rise to different renormalization schemes and some scheme dependence survives the continuum limit and leaks into the renormalized correlators. We will pay special attention to these ambiguities and show that when they are fixed by a minimal subtraction condition, the resulting renormalized correlators are equal to bare correlators defined by functional differentiation with respect to couplings in normal coordinates, at a finite cutoff that is identified with the renormalization scale.

In the final part of this chapter, we approach the construction of scale non-invariant theories in the Wilsonian picture, as renormalized trajectories flowing away from an ultraviolet fixed point. We will see how they can be defined with a renormalization procedure using some bare manifold, than can be chosen with a lot of freedom (the only requirement is that it has to intersect the critical manifold of the UV fixed point). In fact, this is what allows to construct them in traditional approaches without any explicit knowledge of the Wilsonian RG flows. We will also discuss different renormalization schemes in this context.

Most of the results of this chapter about the connection between Wilsonian RG flows and renormalization are published in [6, 7].

The chapter is organized as follows. In Section 3.1 we apply the methods of the previous chapter to the calculation of correlation functions of composite operators at fixed points of the flows. We show the exact connection between their renormalization and Wilson RG flows. The ideas will be applied to two examples. The first one, the Gaussian fixed point, work as an illustration of the general method. In the second one we study the impact of the large N limit on the correlation functions. Section 3.2 is dedicated to the description of general renormalizable theories in this approach.

# 3.1 Correlation Functions

## 3.1.1 Renormalization

A central quantity of interest in QFT is the continuum limit of correlation functions, evaluated at cutoff-independent spacetime points. Given a chart $c$ and a point $(s, \Lambda) \in \mathcal{W}$, we define the bare (or cutoff) $n$-point connected correlation functions as

$$\begin{aligned} G^{(s,\Lambda)}_{\alpha_1 \ldots \alpha_n} &= \partial^c_{\alpha_1} \ldots \partial^c_{\alpha_n}\big|_{(s,\Lambda)} W \\ &= \left. \frac{\delta^n W_\Lambda[g]}{\delta g^{\alpha_1} \ldots \delta g^{\alpha_n}} \right|_{c(s,\Lambda)} , \end{aligned} \tag{3.1}$$

where the generator $W$ is given

$$Z = e^W . \tag{3.2}$$

The symbol $\partial^c_{\alpha_1} \ldots \partial^c_{\alpha_n}|_{(s,\Lambda)}$ on the RHS of the first line of (3.1) refers to the sequential action of the vector fields $\partial_{\alpha_i}$ associated to the coordinates $c$ on the function $W$, eventually evaluated at $(s, \Lambda)$. These vector fields commute among themselves, so the correlators do not depend on the order of the operators. The corresponding basis vectors at $(s, \Lambda)$ are identified with the local operators $\mathcal{O}_{\alpha_i}(s, \Lambda)$ as given in (2.23). The definition (3.1) is a convenient generalization of the standard definition with linear sources. In this thesis we will be ultimately interested in correlation functions at a fixed point $\mathbf{s}_*$. Obviously, this definition of the correlation function is coordinate dependent, even if are not indicating the chart explicitly. It is important to note that for $n \geq 2$ the correlators (3.1) transform nonlinearly under changes of coordinates in $\mathcal{W}$. For instance, the *same* 2-point correlation function written in the basis associated with a different coordinate system $c'$ would read

$$\begin{aligned} G^{(s,\Lambda)}_{\alpha_1\alpha_2} = (\partial^c_{\alpha_1} c'^{\alpha_3})(\partial^c_{\alpha_2} c'^{\alpha_4}) \partial^{c'}_{\alpha_3} \partial^{c'}_{\alpha_4}\Big|_{(s,\Lambda)} W \\ + (\partial^c_{\alpha_1} c'^{\alpha_3})(\partial^c_{\alpha_3} \partial^c_{\alpha_2} c'^{\alpha_4}) \partial^{c'}_{\alpha_4}\Big|_{(s,\Lambda)} W. \end{aligned} \tag{3.3}$$

The continuum limit $\Lambda \to \infty$ of these correlators is divergent in non-trivial theories. Therefore, renormalization is necessary. Let us introduce a family of *renormalization charts* $r_t : \mathcal{W} \to \mathcal{C} \times \mathbb{R}^+$, with $t \in \mathbb{R}^+$, of the form $r_t(s, \Lambda) = (r^\pi_{t,\Lambda}(s), \Lambda/t)$. We fix the origin of these special coordinates such that, for all $t$,

$$r^\pi_t(s^\Lambda_c, \Lambda) = 0 \tag{3.4}$$

where the "critical point" $\mathbf{s}_c = [(s^\Lambda_c, \Lambda)]$ is an arbitrary point such that in the IR is attracted by the fixed point $\mathbf{s}_*$ (i.e. $\lim_{t\to 0} \mathbf{f}_t(\mathbf{s}_c) = \mathbf{s}_*$). The set of points in $\mathcal{M}$ that satisfy this condition is known as the *critical* manifold $\tilde{\mathcal{E}}$. Let $h_t = r_t^{-1}$ be the inverse of the renormalization chart. The maps $c^\alpha \circ h_t[g_R, \mu]$ play the role of *bare couplings* that depend on the cutoff $t\mu$. In a local QFT, the bare couplings can be chosen to be local functionals of the *renormalized couplings* $g^a_R$. We write the basis vectors associated to the renormalization charts as $\partial^{r_t}_\alpha$. They can be related to local *renormalized operators* $[\mathcal{O}^t_\alpha]$ by

$$\begin{aligned} \partial^{r_t}_\alpha|_{(s,\Lambda)} S_\omega &= [\mathcal{O}^t_\alpha]\big|_{(s,\Lambda)} [\omega] \\ &= [\mathcal{O}^t_\alpha]^{(s)}[\gamma^\Lambda; \omega]. \end{aligned} \tag{3.5}$$

We are now ready to define the renormalized correlation functions as the continuum limit of the correlators of renormalized operators:

$$\begin{aligned} G^R_{\alpha_1\ldots\alpha_n} &= \lim_{t\to\infty} \partial^{r_t}_{\alpha_1} \ldots \partial^{r_t}_{\alpha_n}\big|_{(s^{t\mu}_c, t\mu)} W \\ &= \lim_{\Lambda\to\infty} \left. \frac{\delta^n W[h_{\Lambda/\mu}[g_R, \mu]]}{\delta g^{\alpha_1}_R \ldots \delta g^{\alpha_n}_R} \right|_{g_R=0}, \end{aligned} \tag{3.6}$$

with $r_t$ chosen in such a way that the limit is well defined (and non-trivial). There is a large degree of freedom in the choice of renormalization scheme, i.e. in the choice of the particular renormalization charts that do the job. The renormalized correlators are scheme dependent, even if many details fade in the continuum limit. In particular, they depend on the renormalization scale $\mu$ in a way that is determined by the RG.

In actual calculations, (3.6) needs to be written in some specified coordinate system $c$. This involves in particular writing the vector fields $\partial^{r_t}_\alpha$ in the coordinate basis $\{\partial^c_\alpha\}$.

As illustrated in (3.3), non-linear terms will appear under the change of coordinates $r_t \to c$. To keep track of those terms and to preserve covariance, let us introduce a covariant derivative $\nabla^t$ acting on tensor fields, which is characterized by being trivial (with vanishing Christoffel symbols) in the $r_t$ coordinates:

$$\nabla^t_{\partial^{r_t}_{\alpha_1}} \partial^{r_t}_{\alpha_2} = 0. \tag{3.7}$$

Here, $\nabla^t_v$ is the covariant derivative along the vector $v$. In other words, by definition the covariant derivative is just an ordinary derivative in the renormalization coordinates. Consequently, the corresponding connection is symmetric and flat. In arbitrary coordinates $c$, the Christoffel symbols are given by

$$\Gamma^{t\,\tilde{\alpha}_1}_{\tilde{\alpha}_2\tilde{\alpha}_3} = (\partial^{r_t}_{\tilde{\alpha}_4} c^{\tilde{\alpha}_1})(\partial^c_{\tilde{\alpha}_2}\partial^c_{\tilde{\alpha}_3} r_t^{\tilde{\alpha}_4}). \tag{3.8}$$

The symbol $\Gamma^{t\,\alpha_1}_{\alpha_2\alpha_3}$ is quasilocal in its three spacetime indices. In this language, the renormalized correlators read

$$\begin{aligned} G^R_{\alpha_1\ldots\alpha_n} &= \lim_{t\to\infty} \nabla^t_{\partial^{r_t}_{\alpha_1}} \ldots \nabla^t_{\partial^{r_t}_{\alpha_n}} W\Big|_{(s_c^{t\mu},t\mu)} \\ &= \lim_{t\to\infty} [\mathcal{O}^t_{\alpha_1}]^{\alpha_{n+1}} \ldots [\mathcal{O}^t_{\alpha_n}]^{\alpha_{2n}} \nabla^t_{\partial^c_{\alpha_{n+1}}} \ldots \nabla^t_{\partial^c_{\alpha_{2n}}} W\Big|_{(s_c^{t\mu},t\mu)}. \end{aligned} \tag{3.9}$$

The precise meaning of these two equations deserves a short explanation. On the RHS of the first line, $\nabla^t_{\partial^{r_t}_{\alpha_1}} \ldots \nabla^t_{\partial^{r_t}_{\alpha_n}} W$ can be understood as the components of the $(0,n)$ tensor field $\nabla^t \ldots \nabla^t W$ in the $r_t$ coordinate basis. In the second line, the covariant derivatives are taken along the coordinate basis vectors associated to an arbitrary coordinate system $c$ and $\nabla^t_{\partial^c_{\alpha_1}} \ldots \nabla^t_{\partial^c_{\alpha_n}} W$ are to be understood as the components of the same tensor field $\nabla^t \ldots \nabla^t W$ in this later basis. The coefficients in front arise from the tensor transformation law. They are given by

$$[\mathcal{O}^t_{\alpha_1}]^{\alpha_2} = \partial^{r_t}_{\alpha_1} c^{\alpha_2} \tag{3.10}$$

and are just the components of the renormalized operators in the $c$ basis. For local bare couplings, $[\mathcal{O}_{a_1x_1}]^{a_2x_2}$ is a sum of terms proportional to $\delta(x_1 - x_2)$ and derivatives of it. At this point, we can extend in a natural way the definition of renormalized correlation functions to arbitrary operators:

$$G^R_{\mathcal{O}_1\ldots\mathcal{O}_n} = \lim_{t\to\infty} \; [\mathcal{O}^t_1]^{\alpha_{n+1}} \ldots [\mathcal{O}^t_n]^{\alpha_{2n}} \nabla^t_{\partial_{\alpha_{n+1}}} \ldots \nabla^t_{\partial_{\alpha_{2n}}} W\Big|_{(s^{t\mu}_c, t\mu)} . \tag{3.11}$$

However, in this thesis we only consider the particular renormalized correlators in (3.9).

The Christoffel symbols provide the non-linear counterterms that are necessary in generic coordinates. For example, the renormalized two-point functions read

$$\begin{aligned} G^R_{\alpha_1\alpha_2} &= \lim_{t\to\infty} \; [\mathcal{O}^t_{\alpha_1}]^{\alpha_3}[\mathcal{O}^t_{\alpha_2}]^{\alpha_4} \nabla^t_{\partial^c_{\alpha_3}} \nabla^t_{\partial^c_{\alpha_4}} W\Big|_{(s^{t\mu}_c, t\mu)} \\ &= \lim_{t\to\infty} \; \Big\{ [\mathcal{O}^t_{\alpha_1}]^{\alpha_3}[\mathcal{O}^t_{\alpha_2}]^{\alpha_4} \partial^c_{\alpha_3}\partial^c_{\alpha_4} W\Big|_{(s^{t\mu}_c, t\mu)} - [\mathcal{O}^t_{\alpha_1}]^{\alpha_3}[\mathcal{O}^t_{\alpha_2}]^{\alpha_4} \Gamma^{t\,\alpha_5}_{\alpha_3\alpha_4} \partial^c_{\alpha_5} W\Big|_{(s^{t\mu}_c, t\mu)} \Big\} . \end{aligned} \tag{3.12}$$

We have taken into account the fact that $\Gamma^{t\;\wedge}_{\alpha_1\alpha_2} = 0$. The first term in the last line takes care of the non-local divergences, while the second term cancels the local divergences that appear when the spacetime points in $[\mathcal{O}^t_{\alpha_1}]^{\alpha_3}$ and $[\mathcal{O}^t_{\alpha_2}]^{\alpha_4}$ coincide. From (3.8),

$$[\mathcal{O}^t_{\alpha_1}]^{\alpha_3}[\mathcal{O}^t_{\alpha_2}]^{\alpha_4} \Gamma^{t\,\alpha_5}_{\alpha_3\alpha_4} = -C^{t\,\alpha_3}_{\alpha_1\alpha_2}[\mathcal{O}^t_{\alpha_3}]^{\alpha_5}, \tag{3.13}$$

with the counterterms

$$C^{t\,\alpha}_{\alpha_1\alpha_2} = \left(\partial^{r_t}_{\alpha_1}\,\partial^{r_t}_{\alpha_2} c^{\alpha_3}\right)\left(\partial^c_{\alpha_3} r^\alpha_t\right) \tag{3.14}$$

quasilocal in the spacetime part of their indices. Because the derivatives come with inverse metrics that decrease the degree of divergence, we can truncate the derivative expansion of the expression above and get local counterterms. (3.13) used in (3.12) has the form of an OPE: in fact, in order to give finite continuum correlators, the singular parts in the counterterms $C^{t\,ax}_{a_1x_1a_2x_2}$ must cancel the singularities for coincident points $x_1 \sim x_2$ of the $[\mathcal{O}^t_{ax}]$ term in the OPE of $[\mathcal{O}^t_{a_1x_1}]$ and $[\mathcal{O}^t_{a_2x_2}]$.

The higher-point renormalized functions involve derivatives of the Christoffel symbols. They can also be written in terms of counterterms

$$C^{t\,\alpha}_{\alpha_1\alpha_2\ldots\alpha_m} = \left(\partial^{r_t}_{\alpha_1}\,\partial^{r_t}_{\alpha_2}\ldots\partial^{r_t}_{\alpha_m} c^{\alpha'}\right)\left(\partial^c_{\alpha'} r^\alpha_t\right) \tag{3.15}$$

as

$$G^R_{\alpha_1\alpha_2\ldots\alpha_n} = \lim_{t\to\infty} \sum_{p\in\Pi_n} \left(\prod_{r=1}^{|p|} C^{t\,\alpha'_r}_{\alpha_{p_r}}\right) \left[\mathcal{O}^t_{\alpha'_1}\right]^{\alpha''_1} \cdots \left[\mathcal{O}^t_{\alpha'_{|p|}}\right]^{\alpha''_{|p|}} \partial^c_{\alpha''_1} \ldots \partial^c_{\alpha''_{|p|}} W\bigg|_{(s_c^{t\mu},t\mu)} , \quad (3.16)$$

where $\Pi_n$ is the set of partitions of $\{1, 2, \ldots, n\}$, $p_r$ is the $r$-th element of the partition $p$, $\alpha_{p_r}$ is a collective index given by

$$\alpha_{p_r} = \alpha_{p_{r1}} \ldots \alpha_{p_{r|p_r|}}, \quad (3.17)$$

with $p_{ri}$ the $i$-th element of the set $p_r$, and for any set $A$, $|A|$ is its cardinality. Furthermore, we have defined $C^{t\,\alpha_1}_{\alpha_2} = \delta^{\alpha_1}_{\alpha_2}$. The counterterms in (3.16) cancel not only local but also semilocal divergences when only a subset of points coincide [6, 61, 62].

### 3.1.2 Connection with RG

Let us next connect the renormalization process with the exact RG flows. Using RG invariance, as given by (2.27), we rewrite (3.6) as

$$\begin{aligned} G^R_{\alpha_1\ldots\alpha_n} &= \lim_{t\to\infty} \nabla^t_{\partial^{rt}_{\alpha_1}} \ldots \nabla^t_{\partial^{rt}_{\alpha_n}} W \circ f_{1/t}\Big|_{(s_c^{t\mu},t\mu)} \\ &= \lim_{\Lambda\to\infty} \frac{\delta^n W(f_{\mu/\Lambda} \circ h_{\Lambda/\mu}[g_R,\mu])}{\delta g_R^{\alpha_1} \ldots \delta g_R^{\alpha_n}}\bigg|_{g_R=0}, \end{aligned} \quad (3.18)$$

for any scale $\mu$. Since $W(s,\mu)$ is finite for finite $s$, it is clear that the limit in (3.18) will be well defined as long as $f_{\mu/\Lambda} \circ h_{\Lambda/\mu}(g_R,\mu)$ stays finite when $\Lambda$ approaches infinity, at least for $g_R$ in some neighbourhood of 0. In this manner, we have rephrased the problem of removing divergences as the Wilsonian problem of finding curves $r^{-1}_{\Lambda/\mu}(g,\mu)$ that, when composed with the RG evolution, have a well defined limit. The condition (3.4) ensures that $\lim_{\Lambda\to\infty} f_{\mu/\Lambda,\Lambda} \circ h_{\Lambda/\mu}(0,\mu) = s^\mu_*$. To get non-trivial correlators we need that the combined limit, besides being finite, reaches points different from $s^\mu_*$ when $g_R \neq 0$. More precisely, the limit will be finite and non-singular if

$$\lim_{t\to\infty} r_t \circ f_t = c \quad (3.19)$$

is a well-defined chart in the neighbourhood of the fixed point.

Let us recast these statements in infinitesimal form. Because the correlation functions are given by (covariant) derivatives, for a finite set of correlators it is sufficient that the corresponding derivatives be well defined in the continuum limit. For a fixed and finite $t > 0$, the RG flow $f_t$ defines a diffeomorphism which takes points in a region $\mathcal{A} \subset \mathcal{W}$ onto points in a region $\mathcal{A}^t \subset \mathcal{W}$. This map can be used to transport any differential structure between $\mathcal{A}$ and $\mathcal{A}^t$. Recall that a vector field $v$ is transported with the differential $f_t^* : T\mathcal{A} \to T\mathcal{A}^t$,

$$(f_t^* v)F = v\, F \circ f_t \tag{3.20}$$

for any function $F$ in $\mathcal{A}^t$, while the pullback of a one-form field $\phi$ in $T^*\mathcal{A}^t$ is given by

$$((f_t)_* \phi)(v) = \phi(f_t^* v) \tag{3.21}$$

for any vector field $v$ in $T\mathcal{A}$. A tensor field $T$ of an arbitrary type $(n, m)$ can be transported from the space of tensor fields in $\mathcal{A}$ to the one in $\mathcal{A}^t$ using the pullback of $f_t$, $(f_t)_*$ and the differential of its inverse, $f_{1/t}^*$:

$$(f_t^* T)(\phi_1, \ldots, \phi_n; v_1, \ldots v_n) = T((f_t)_* \phi_1, \ldots, (f_t)_* \phi_n; f_{1/t}^* v_1, \ldots, f_{1/t}^* v_n), \tag{3.22}$$

where $\phi_i$ and $v_i$ are, respectively, dual vector and vector fields in $\mathcal{A}^t$. Similarly, any connection $\nabla$ in $\mathcal{A}$ can be transported into another connection $\nabla' = f_t^* \nabla$ in $\mathcal{A}^t$:

$$\nabla'_v T = f_t^* \left[ \nabla_{f_{1/t}^* v} (f_{1/t}^* T) \right], \tag{3.23}$$

for $v$ and $T$ arbitrary vector and tensor fields in $\mathcal{A}^t$. Using all this, we can write (3.18) as

$$G^R_{\alpha_1 \ldots \alpha_n} = \lim_{t \to \infty} f_{1/t}^* \nabla^t_{\partial^{r_t}_{\alpha_1}} \ldots f_{1/t}^* \nabla^t_{\partial^{r_t}_{\alpha_n}} W \Big|_{f_{1/t}(s_c^{t\mu}, t\mu)} . \tag{3.24}$$

The point at which the transported covariant derivatives are evaluated approaches as $t \to \infty$ the fixed point representative $(s_*^\mu, \mu)$. Therefore, the renormalization of correlation functions can be achieved by tuning the renormalized operators and the renormalization connection in a neighbourhood of the critical point in such a way that their transportation with $f_t^*$ stays finite in the limit $t \to \infty$. That is, the renormalization chart must be chosen such that the limits

$$\partial_\alpha^{r_*} = \lim_{t\to\infty} f^*_{1/t} \partial_\alpha^{r_t}, \tag{3.25}$$

$$\nabla^* = \lim_{t\to\infty} f^*_{1/t} \nabla^t \tag{3.26}$$

are non-singular in a neighbourhood of the fixed point. Then, we get

$$G^R_{\alpha_1\ldots\alpha_n} = \nabla^*_{\partial^{r_*}_{\alpha_1}} \ldots \nabla^*_{\partial^{r_*}_{\alpha_n}} W\Big|_{(s^\mu_*,\mu)} . \tag{3.27}$$

Let us stress this simple but important result:

**The renormalized correlators are exactly equal to the bare correlators of the finite operators associated to $\partial^{r_*}_{\alpha_i}$, evaluated at the fixed point.**

In arbitrary coordinates $c$, the components of the transported renormalized operators approach

$$[\mathcal{O}^*_\alpha]^{\alpha_1}|_{(s^\mu_*,\mu)} = \lim_{t\to\infty} M^{\alpha_1}_{t\,\alpha_2} \; [\mathcal{O}^t_\alpha]^{\alpha_2}\Big|_{(s^{t\mu}_c,t\mu)} , \tag{3.28}$$

where

$$\begin{aligned} M^{\alpha_1}_{t\,\alpha_2} &= \partial^c_{\alpha_2}\Big|_{(s^{t\mu}_c,t\mu)} f^{\alpha_1}_{1/t} \\ &= \frac{\delta f^{\alpha_1}_{1/t}[\gamma^{t\mu}; g]}{\delta g^{\alpha_2}}\Bigg|_{g^{t\mu}_c} \end{aligned} \tag{3.29}$$

is the Jacobian matrix of the coordinate transformation. Here, $g^\Lambda_c = c^\pi_\Lambda(s^\Lambda_c)$.[1] The components of the renormalized operators must be defined such that the limit in (3.28) is finite and non-singular. This linear (or multiplicative, in the matrix sense) renormalization can be found in the standard way, without knowledge of the RG flows, by requiring the cancellation of the non-local divergences in the correlation functions. But (3.28)

[1]Note that $g^\Lambda_c$ and $g^{\Lambda'}_c$ are related by a dilation, so $g^\Lambda_c$ will be independent of $\Lambda$ in the usual case of a homogeneous critical point.

shows that a solution to this problem always exists:

$$[\mathcal{O}^t_\alpha]^{\alpha_1}\Big|_{(s_c^{t\mu},t\mu)} = \left(M_t^{-1}\right)^{\alpha}_{\ \alpha_1}\Big|_{\text{local}}, \tag{3.30}$$

where "local" indicates a truncation of the derivative expansion that does not modify the limit in (3.28). This truncation is always possible since the derivatives come along with negative powers of $t$, which decrease the degree of divergence. With the choice in (3.30), the trivial limit $[\mathcal{O}^*_\alpha]^{\alpha_1} = \delta^{\alpha_1}_\alpha$ is obtained. Likewise, the transported Christoffel symbols approach

$$\Gamma^{*\,\alpha_1}_{\alpha_2\alpha_3}\Big|_{(s_*^\mu,\mu)} = \lim_{t\to\infty}\left(M_t^{-1}\right)^{\alpha_5}_{\alpha_2}\left(M_t^{-1}\right)^{\alpha_6}_{\alpha_3}\left(M^{\alpha_1}_{t\,\alpha_4}\,\Gamma^{t\,\alpha_4}_{\alpha_5\alpha_6}\Big|_{(s_c^{t\mu},t\mu)} - \frac{\delta^2 f^{\alpha_1}_{1/t}[\gamma^{t\mu};g]}{\delta g^{\alpha_1}\delta g^{\alpha_2}}\bigg|_{g_c^{t\mu}}\right) \tag{3.31}$$

in the limit. Observe that the transportation is non-linear, so a non-vanishing $\Gamma^t$ is necessary in general. An exception occurs for generic dimensions in normal coordinates, for which the second derivatives of the flows vanish. The renormalization of the Christoffel symbols that keep $\Gamma^*$ finite can be obtained without knowledge of the RG flows by requiring the cancellation of local and semi-local divergences in the correlation functions. (3.31) shows that, again, at least one solution exists:

$$\Gamma^{t\,\alpha_1}_{\alpha_2\alpha_3}\Big|_{(s_c^{t\mu},t\mu)} = \left(M_t^{-1}\right)^{\alpha_1}_{\alpha_4}\partial_{\alpha_2}\partial_{\alpha_3} f^{\alpha_4}_{1/t}\Big|_{(s_c^{t\mu},t\mu)}\Bigg|_{\text{local}}. \tag{3.32}$$

With this particular choice, $\Gamma^{*\,\alpha_1}_{\alpha_2\alpha_3}\big|_{(s_c^\mu,\mu)} = 0$. More generally, we can always choose coordinates for which $\partial^{r_*}_\alpha$ is a tangent vector at the fixed point. Furthermore, because the transportation preserves the flatness of the connection, it is possible to find, simultaneously, coordinates in which the Christoffel symbols vanish. In such coordinates $\tilde{c}$, the renormalized functions simply read

$$G^R_{\alpha_1\ldots\alpha_n} = \partial^{\tilde{c}}_{\alpha_1}\ldots\partial^{\tilde{c}}_{\alpha_n} W\Big|_{(s_*^\mu,\mu)}. \tag{3.33}$$

If $c(s_*^\mu,\mu) = 0$, the coordinates $\tilde{c}$ are given, perturbatively, by

$$c^\alpha_\mu = [\mathcal{O}^*_{\alpha_1}]^\alpha\Big|_{(s_*^\mu,\mu)}\tilde{c}^{\alpha_1}_\mu - \frac{1}{2}\Gamma^{*\,\alpha}_{\alpha_3\alpha_4}[\mathcal{O}^*_{\alpha_1}]^{\alpha_3}[\mathcal{O}^*_{\alpha_2}]^{\alpha_4}\Big|_{(s_*^\mu,\mu)}\tilde{c}^{\alpha_1}_\mu\tilde{c}^{\alpha_2}_\mu + O\left(\tilde{c}^3_\mu\right). \tag{3.34}$$

In the renormalization scheme given by (3.30) and (3.32), we directly have $\tilde{c} = c$.

In fact, the particular renormalized operators and Christoffel symbols in (3.30) and (3.32) result, after tructation, from the following choice of renormalization coordinates (with the same $c$ as in those equations):

$$r_t^\alpha(s,\Lambda) = c^\alpha \circ f_{1/t}(s,\Lambda) - c^\alpha \circ f_{1/t}(s_c^\Lambda,\Lambda). \tag{3.35}$$

The second term is included to ensure $r_t^\alpha(s_c^\Lambda,\Lambda) = 0$. This choice obviously gives finite correlation functions, as it fulfils (3.19), with the very same $c$ used in the definition (3.35). The result (3.33) with $\tilde{c} = c$ can then be understood as a direct consequence of this fact. Conversely, the renormalized operators and connection given by (3.30) and (3.32) are the transportation with $f_t^*$ of the coordinate vectors $\partial_\alpha^c$ and of the connection with vanishing Cristoffel symbols in $c$ coordinates.

In particular, we can work in the renormalization scheme where $r_t$ is given by (3.35) taking as $c$ a normal chart, $c = \bar{c}$. Such a renormalization scheme will be called a *exact UV scheme* in the following. Exact UV schemes have the virtue of making the renormalization operators and connection as simple as possible, when expressed in normal coordinates. In particular, when $s_c = s_*$, (3.35) can be inverted using $f_{1/t}^{-1} = f_t$:

$$\bar{c}^\alpha = f_t^\alpha[t^{-2}\bar{\gamma}; r_t^\pi]. \tag{3.36}$$

Then, from (3.15) and (2.64) it follows that

$$[O_{\alpha'}^t]^\alpha C_{\alpha_1\dots\alpha_n}^{t\,\alpha'}\Big|_{(s_*^\Lambda,\Lambda)} = n!\,t^{-\left[\lambda_{(\alpha_1)}+\dots+\lambda_{(\alpha_n)}\right]} \sum_{p=1}^{p_{\alpha_1\cdots\alpha_n}^{\max}} \log^p t\;[B_p]_{\alpha_1\dots\alpha_n}^\alpha\left(\gamma^\Lambda\right). \tag{3.37}$$

Therefore, the counterterms can be built in terms of beta coefficients, as illustrated in Figure 2.1. This structure agrees with Zimmermann's forest formula in perturbative coordinate-space renormalization [63, 64]. Each forest for a given diagram is associated to a tree in (2.65). Note in particular that in each forest only nested or disjoint (i.e. not overlapping) subtractions appear, which is an obvious property of (2.65) (see Figure 2.1). Moreover, the number of logs in a term corresponding to a given tree/forest is equal to the number of beta coefficients in the tree.

### 3.1.3 Minimal Subtraction

Working in an exact UV scheme seems to require knowledge of the exact RG flows and their maximal diagonalization. However, in the remaining of this section we show that exact UV schemes are actually equivalent to certain minimal-subtraction schemes, defined in arbitrary coordinates without any explicit reference to RG flows or normal coordinates. Our definition of minimal subtraction is given by the following restriction on the renormalization chart:

$$r_t^\alpha = t^{\tilde{\lambda}_{(\alpha)}} R_t^\alpha \left[\bar{\gamma}; c^\pi\right], \tag{3.38}$$

where $c$ is any chart, $\tilde{\lambda}_{(ax)} = \tilde{\lambda}_{(a)}$ are real numbers and $R_t^\alpha$ is a quasilocal functional of $g = c^\pi(s, \Lambda)$ that depends at most logarithmically on $t$. If this condition is met for a chart $c$, then it will also hold (with a different $R_t$) when $c$ is replaced by any other chart, in particular by a normal chart $\bar{c}$. Therefore, (3.19) can be written as

$$\begin{aligned} c^\alpha &= \lim_{t\to\infty} \left(r_t^\alpha \circ \bar{c}^{-1}\right) \circ (\bar{c} \circ f_t) \\ &= \lim_{t\to\infty} t^{\tilde{\lambda}_{(\alpha)}} \bar{R}_t^\alpha \left[t^2\bar{\gamma}; \bar{c}^\pi \circ f_t\right]. \end{aligned} \tag{3.39}$$

In the last line, the second entry in $R_t^\alpha$ is the RG flow in normal coordinates. As explained in Subsection 2.3, (2.64), the latter has coordinates of the form

$$\bar{c}^\alpha \circ f_t = \bar{f}_t^\alpha = t^{-\lambda_{(\alpha)}} F_t^\alpha \left[\bar{\gamma}; \bar{c}^\pi\right] = F_t^\alpha \left[t^2\bar{\gamma}; t^{-\lambda}\bar{c}^\pi\right], \tag{3.40}$$

where $\left(t^{-\lambda}\bar{c}^\pi\right)^\alpha = t^{-\lambda_{(\alpha)}}\bar{c}^\alpha$ and $F_t^\alpha$ is a quasilocal functional of $g^\alpha = c^\alpha(s, \Lambda)$ which can be expanded in a series of resonant monomials, with coefficients that depend at most logarithmically in $t$. Expanding the RHS of (3.39) to linear order, we have

$$c^\alpha = \lim_{t\to\infty} \left(A^\alpha_{\alpha_1} \bar{c}^{\alpha_1} t^{\tilde{\lambda}_{(\alpha)} - \lambda_{(\alpha_1)}} (1 + \text{logs}) + O(\bar{c}^2)\right). \tag{3.41}$$

But for the LHS of (3.39) to be a non-singular invertible change of coordinates, the linear term in its $\bar{c}$ expansion must be given by a finite, non-singular matrix multiplying the vector $\bar{c}$. This forces the matrix $A$ above to be non-singular and upper-triangular (if the directions are ordered with decreasing eigenvalues), and $\tilde{\lambda}_{(\alpha)} = \lambda_{(\alpha)}$. At the non-linear level, the terms with $\Pi_{i=1}^{r}\bar{c}_{\alpha_i}$, $n_{(\gamma)}$ metrics and $n_{(\gamma^{-1})}$ inverse metrics will scale like $t^{\lambda_{(\alpha)}-\sum_{i=1}^{r}\lambda_{(\alpha_i)}+2n_{(\gamma)}-2n_{(\gamma^{-1})}}$, up to logarithms. So, a finite continuum limit requires that each monomial either vanishes in the limit or is resonant (see (2.55) and (2.63)). Therefore, we learn that in minimal subtraction $c$ is a normal chart, just as in an exact UV scheme. The set of renormalization schemes with this property (i.e. $c$ in (3.19) is a normal chart) are called *UV schemes*.[2]

The restriction of minimal subtraction does not fix the renormalization scheme completely. Indeed, if $r_t$ is a valid renormalization chart in minimal subtraction, so is

$$\tilde{r}_t^{\alpha} = r_t^{\alpha} + t^{\lambda_{(\alpha)}-\lambda_{(\alpha_1)}} a^{\alpha}_{t\,\alpha_1}(\bar{\gamma})\, r_t^{\alpha_1} + t^{\lambda_{(\alpha)}-\lambda_{(\alpha_1)}-\lambda_{(\alpha_2)}} a^{\alpha}_{t\,\alpha_1\alpha_2}(\bar{\gamma})\, r_t^{\alpha_1} r_t^{\alpha_2} + \dots \tag{3.42}$$

if the coefficients $a^{\alpha}_{t\,\alpha_1\dots\alpha_n}(\gamma)$ depend at most logarithmically on $t$ and vanish when $\sum_i \lambda_{(\alpha_i)} + 2n_{(\gamma^{-1})} - 2n_{(\gamma)} \leq \lambda_{(\alpha)}$, being $n_{(\gamma)}$ and $n_{(\gamma^{-1})}$ the number of metrics and inverse metrics respectively which $a^{\alpha}_{t\,\alpha_1\dots\alpha_n}(\gamma)$ depends on. This ambiguity can be used to simplify $r_t$ and make it local.

In terms of renormalized operators and connections, minimal subtraction is characterized by the following two conditions:

(i) The operator components $[\mathcal{O}^t_{\alpha_1}]^{\alpha_2}$ are required to be proportional to $t^{-\tilde{\lambda}_{\alpha_1}}$, up to logarithms, with the same non-negative number $\tilde{\lambda}_{\alpha_1}$ for all $\alpha_2$.

(ii) The Christoffel symbols $\Gamma^{t\,\alpha}_{\alpha_1\alpha_2}$ are required to be $t$ independent in the neighbourhood where they are defined, up to logarithms.

The second condition can also be formulated in term of the counterterms $C^{t\,\alpha}_{\alpha_1\dots\alpha_n}$: they are required to be proportional to $t^{\tilde{\lambda}_{\alpha}-\tilde{\lambda}_{\alpha_1}-\dots-\tilde{\lambda}_{\alpha_n}}$, up to logarithms, with the same $\tilde{\lambda}_{\alpha}$, $\tilde{\lambda}_{\alpha_1}$, ..., $\tilde{\lambda}_{\alpha_n}$ as in condition (i).

Again, there is some remaining freedom, which can be used to make the renormalized operators and counterterms local and to simplify them. For instance, let us consider fixed points such that, for any real $r \in \mathbb{R}$, the number of eigendirections with dimension $\Delta < r$ is finite. Then, given any parametrization, the renormalization charts can be redefined order by order using (3.42) in such a way that

[2] Minimal subtraction schemes are a subclass of UV schemes. At the same time, exact UV schemes are a subclass of minimal subtraction schemes.

(i) All renormalized operators are local and have only a finite number of non-vanishing components $[\mathcal{O}^t_{\alpha_1}]^{\alpha_2}$.

(ii) All counterterms $C^{t\,\alpha}_{\alpha_1\ldots\alpha_n}$ are local and vanish if

$$\sum_{s=1}^{n}\left(\lambda_{(\alpha_s)} - n^u_{(\alpha_s)} + n^d_{(\alpha_s)}\right) > \lambda_{(\alpha)} - n^u_{(\alpha_s)} + n^d_{(\alpha_s)}. \tag{3.43}$$

With this convention, only a finite number of local renormalized operator components and counterterms appear in the renormalization of a given correlation function. This is the usual statement of renormalizability.

### 3.1.4 Exceptional Cases

In the previous example we have seen how some bare couplings can depend logarithmically on the cutoff. This is a feature of fixed points with resonances and exceptional dimensions, and thus, it is intimate related with the appearance of non-trivial beta functions.

To see explicitly this connection, let us find the relation between renormalization charts as defined in (3.35) (with normal charts), and other family of charts that do not depend on $t$ (the normal charts themselves). We assume an homogeneous critical point $g_c = g^\Lambda_c = \bar{c}^\pi_\Lambda(s^\Lambda_c)$, and define $\delta\bar{c}^\alpha = \bar{c}^\alpha - g^\alpha_c$. Using (2.64),

$$\begin{aligned} r^\alpha_t = t^{\lambda_{(\alpha)}}\Bigg\{ \delta\bar{c}^\alpha &+ \sum_{m=1}^{\infty} m \left[\sum_{p=1}^{p^{\max}_{\alpha_1\cdots\alpha_m}} (-1)^p \log^p t \, [B_p]^\alpha_{\alpha_1\ldots\alpha_m}(\bar{\gamma})\right] \delta\bar{c}^{\alpha_1} g^{\alpha_2}_c \cdots g^{\alpha_m}_c \\ &+ \sum_{m=2}^{\infty} \frac{m(m-1)}{2} \left[\sum_{p=1}^{p^{\max}_{\alpha_1\cdots\alpha_m}} (-1)^p \log^p t \, [B_p]^\alpha_{\alpha_1\ldots\alpha_m}(\bar{\gamma})\right] \delta\bar{c}^{\alpha_1}\delta\bar{c}^{\alpha_2} g^{\alpha_3}_c \cdots g^{\alpha_m}_c \\ &+ O(c^3)\Bigg\}. \end{aligned} \tag{3.44}$$

It can be inverted using that $f_t^{-1} = f_{1/t}$,

$$\delta\bar{c}^{\alpha} = t^{-\lambda_{(\alpha)}} \Bigg\{ r_t^{\alpha} + \sum_{m=1}^{\infty} m \left[ \sum_{p=1}^{p^{\max}_{\alpha_1\cdots\alpha_m}} \log^p t \, [B_p]^{\alpha}_{\alpha_1\ldots\alpha_m} \left(t^{-2}\bar{\gamma}\right) \right] r_t^{\alpha_1} \bar{f}^{\alpha_2}_{1/t}[\bar{\gamma}; g_c] \cdots \bar{f}^{\alpha_m}_{1/t}[\bar{\gamma}; g_c]$$
$$+ \sum_{m=2}^{\infty} \frac{m(m-1)}{2} \left[ \sum_{p=1}^{p^{\max}_{\alpha_1\cdots\alpha_m}} \log^p t \, [B_p]^{\alpha}_{\alpha_1\ldots\alpha_m} \left(t^{-2}\bar{\gamma}\right) \right] r_t^{\alpha_1} r_t^{\alpha_2} \bar{f}^{\alpha_3}_{1/t}[\bar{\gamma}; g_c] \cdots \bar{f}^{\alpha_m}_{1/t}[\bar{\gamma}; g_c]$$
$$+ O(r_t^3) \Bigg\}. \tag{3.45}$$

$\bar{f}^{\alpha}_{1/t}[\bar{\gamma}; g_c]$ can only be non-vanishing for scalar irrelevant and marginally irrelevant directions. We will say that an eigendirection $a$ is excited by the critical point if $\bar{f}^{ax}_{1/t}[\bar{\gamma}; g_c] \neq 0$.

Therefore, in the normal parametrization, (3.45) shows that renormalized operators (see (3.10)) have their simplest form for generic eigenvalues,

$$[\mathcal{O}^t_{\alpha_1}]^{\alpha_2} = t^{-\lambda_{(\alpha)}} \delta^{\alpha_2}_{\alpha_1}. \tag{3.46}$$

Exceptional eigenvalues can add logarithms of $t$ to $[\mathcal{O}^t_{\alpha}]$. This is the case if: i) the linear flows around the fixed point are not fully diagonalizable and $\alpha$ is not an actual eigendirection, or, ii) the eigenvalue $\lambda_{(\alpha)}$ and some eigenvalues of directions excited by the critical point form a resonance and the associated $[B_p]$ does not vanish.

Likewise, the counterterm $C^{t\,\alpha}_{\alpha_1\ldots\alpha_n}$ (see (3.15)) will depend logarithmically on $t$ if $\lambda_{(\alpha)}$ is resonant, and formed by $\lambda_{(\alpha_1)}, \ldots, \lambda_{(\alpha_n)}$ and possibly, some other eigenvalues of directions excited by the critical point. Additionally, the associated $[B_p]$ has to be non-vanishing.

A redefinition of the renormalization charts using (3.42) order by order allows to remove some logarithms. In fact, in a fully diagonalizable fixed point and if there are no marginal directions excited by the critical point, the renormalized operators can be defined without any logarithmic behaviour. Therefore, they will have the form of (3.46). Also, those logarithmic counterterms $C^{\alpha}_{\alpha_1\ldots\alpha_n}$ that appear because $\lambda_{(\alpha_1)}, \ldots, \lambda_{(\alpha_n)}$ and irrelevant eigenvalues excited by the critical point form a resonance can be removed.

So far, we have described renormalized operator components and counterterms in a very specific class of charts (the normal ones), that requires the knowledge of the exact RG flows. However, since changes of coordinates do not depend on $t$, we can still extract some consequences if we work in a general parametrization (and in minimal subtraction).

(i) The renormalized operator $[\mathcal{O}^t_\alpha]$ must have a logarithmic behaviour in $t$ if the linear flows around the fixed point are not fully diagonalizable and $\alpha$ is not an actual eigendirection.

(ii) If the linear flows are fully diagonalizable, some renormalized operators may still require logarithms if the critical point excites marginal directions.

(iii) The counterterms $C^{t\,\alpha}_{\alpha_1 \ldots \alpha_n}$ can require logarithms if a subset of the eigenvalues $\lambda_{(\alpha_1)}, \ldots, \lambda_{(\alpha_n)}$ form a resonance, or the critical point excites marginal directions.

### 3.1.5 Example I: the Gaussian Fixed Point

In this section we work out a simple example to illustrate the method and the general results found in the previous section. We consider the theory space of a single real scalar field $\omega$ in $d = 4$ and examine the RG evolution close to the Gaussian fixed point. We compare with the renormalization of composite operators in the free-field theory.

The Wilsonian information of this example was studied in Section 2.4.2. There we found beta functions and normal coordinates. It allows us to calculate renormalized correlation functions with the methods of Section 3.1. These calculations can be performed in two related ways. The most direct one is to take advantage of (3.33) to compute the renormalized correlators directly as bare correlators at the fixed point with a finite cutoff. This calculation is very simple when the fixed point is well-characterized—as for the free theory—since no counterterms are required and the continuum limit is not explicitly taken. Alternatively, we can use the RG flows to compute the corresponding renormalized operators and counterterms and then use (3.12) to obtain the renormalized correlator. We will consider both approaches in turn.

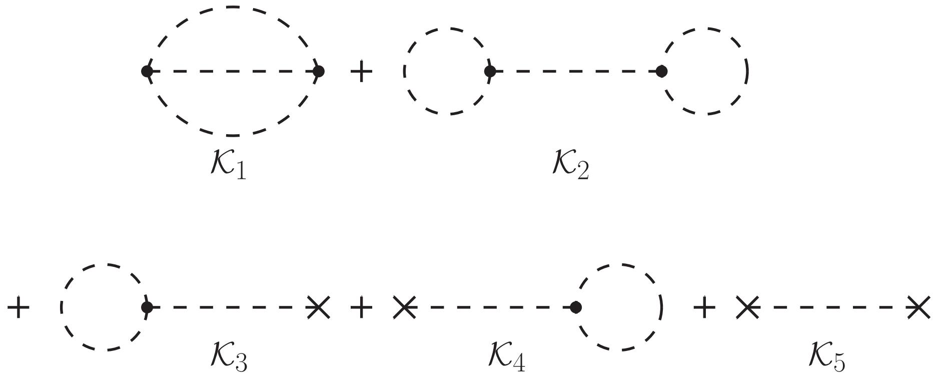


Figure 3.1: Calculation of $\left\langle \left(S_\omega^{\gamma^\mu}\right)_{3x} \left(S_\omega^{\gamma^\mu}\right)_{3y} \right\rangle_{(0,\mu)}$. Lines indicate propagators $P(\gamma^\mu; x-y)$, dots represent vertices $\sqrt{|\gamma^\mu|}\,\omega(x)^3$, and crosses, insertions of $-3\sqrt{|\gamma^\mu|}P_0\,\omega(x)$.

**Normal Correlators**

The bare correlation functions in normal coordinates are defined by functional derivatives of the generator $W$ with respect to normal couplings. We are interested in functional derivatives at the fixed point. Let us calculate (see discussion around (2.80))

$$G^R_{3x3y} = \partial^{\bar{c}}_{3x}\partial^{\bar{c}}_{3y} W\big|_{(0,\mu)}\,. \tag{3.47}$$

The notation $G^R$ already anticipates that, as shown in (3.33), this correlator computed at a finite cutoff $\mu$ is equal to the corresponding renormalized correlator, in the UV scheme given by (3.35) with $c = \bar{c}$. From (2.73), using (2.79),

$$\partial^{\bar{c}}_{3x}\partial^{\bar{c}}_{3y} W\big|_{(0,\mu)} = \left\langle \left(S_\omega^{\gamma^\mu}\right)_{3x} \left(S_\omega^{\gamma^\mu}\right)_{3y} \right\rangle_{(0,\mu)} - 2\left\langle \left(S_\omega^{\gamma^\mu}\right)_{3x3y} \right\rangle_{(0,\mu)}\,. \tag{3.48}$$

Using Wick's theorem and the explicit form in (2.80), we find that the first term on the LHS has contributions given by the diagrams $\mathcal{K}_1$–$\mathcal{K}_5$ in Figure 3.1 in the free-field theory. Taking combinatorial factors into account, it is easy to check that $\sum_{i=2}^5 \mathcal{K} = 0$. Therefore,

$$\left\langle \left(S_\omega^{\gamma^\mu}\right)_{3x} \left(S_\omega^{\gamma^\mu}\right)_{3y} \right\rangle_{(0,\mu)} = 6|\gamma^\mu| P(\gamma^\mu; x-y)^3. \tag{3.49}$$

For the second term, using the explicit form (2.85) and the property (2.79), we get

$$\left\langle \left(S_\omega^{\gamma^\mu}\right)_{3x3y} \right\rangle_{(0,\mu)} = |\gamma^\mu| E(\gamma^\mu; x-y). \tag{3.50}$$

Combining both results and inserting the solution (2.91), we finally obtain

$$\partial_{3x}^{\bar{c}} \partial_{3y}^{\bar{c}} W\big|_{(0,\mu)} = -\frac{3\mu^4}{2^{10}\pi^6}\, \dot{\partial}^4 \left[\frac{\log(x-y)^2_{\gamma^\mu}}{(x-y)^2_{\gamma^\mu}}\right] - 2\xi_3 \mu^2 \partial^2 \delta(x-y). \tag{3.51}$$

Note that the $P^3$ terms have cancelled out. The arbitrary parameter $\xi_3$ multiplies a scheme-dependent local term. It can be absorbed into a redefinition of the scale $\mu$.

### Renormalization

The standard calculation of the renormalized correlators involves a renormalization procedure: first finding universal cutoff-dependent renormalized operators and counterterms and then taking the continuum limit for the correlators of interest. Such a renormalization can be carried out in arbitrary coordinates. Usually, the renormalized operators and counterterms are determined by requiring the corresponding contributions to cancel the continuum-limit divergences of the bare correlators. Here we show how to obtain them from the exact RG flows near the fixed point, using the results in 3.1. We concentrate on the renormalized operators and counterterms that contribute to $G_{33}^R$.

We choose a renormalization chart given by (3.35) with normal coordinates $c = \bar{c}$, i.e. we choose a UV scheme. Since our fixed point is Gaussian, the simplest choice for the critical point is clearly the fixed point itself, $s_c = s_* = 0$.[3] Using (2.64) in (3.35) (with $c = \bar{c}$) we find an explicit relation between $\bar{c}$ and $r_t$. Inverting this relation perturbatively, we can write (2.73) in terms of renormalization coordinates:

$$S_\omega = r_t^\alpha\, t^{-\lambda_{(\alpha)}}\, (S_\omega^{\bar{\gamma}})_\alpha + r_t^{\alpha_1} r_t^{\alpha_2}\, t^{-\lambda_{(\alpha_1)}-\lambda_{(\alpha_2)}} \left[\log t\, \beta^\alpha_{\alpha_1\alpha_2}(\bar{\gamma})\, (S_\omega^{\bar{\gamma}})_\alpha + (S_\omega^{\bar{\gamma}})_{\alpha_1\alpha_2}\right] + O\left(r_t^3\right). \tag{3.52}$$

[3] The relation with the calculation in the previous section is quite straightforward in this case, for the renormalization chart simply counteracts the action of the RG flow. Indeed, (3.35) with $s_c = s_*$ and $c = \bar{c}$ amounts to the choice $r_t^{-1} = f_t \circ \bar{c}^{-1}$ for the bare couplings, which used in (3.18) directly gives normal correlators.

In the UV scheme, the renormalized operator associated to the eigendirection 3 is given by

$$\partial^{r_t}_{3x}|_{(0,t\mu)} = f^*_{1/t}\,\partial^{\bar{c}}_{3x}|_{(0,\mu)}\,. \tag{3.53}$$

Since the renormalization procedure is done around the critical point, all functions, vector and tensor fields on $\mathcal{W}$ in the formulas below are understood to be evaluated at the point $(0, t\mu)$, unless otherwise indicated. We can directly read the renormalized operator from (3.52):

$$\begin{aligned}\left[\mathcal{O}^t_{3x}\right] &= \partial^{r_t}_{3x}\,S_\omega \\ &= t^{-1}\left(S^{\gamma^{t\mu}}_\omega\right)_{3x} \\ &= t^{-1}\sqrt{|\gamma^{t\mu}|}\left[\omega(x)^3 - P_0\,\omega(x)\right].\end{aligned} \tag{3.54}$$

In order to compare with standard perturbative calculations, we present the calculation in terms of canonical linear coordinates which are called $c$ hereafter. Using this parametrization we can write (3.54) as

$$\partial^{r_t}_{3x} = t^{-1}\left[\partial^c_{3x} - P_0\partial^c_{1x}\right]. \tag{3.55}$$

So, in this basis the non-trivial components of $\partial^{r_t}_{3x}|_{(0,t\mu)}$ are

$$\left[\mathcal{O}^t_{3x}\right]^{3y} = t^{-1}\delta(x-y), \tag{3.56}$$

$$\left[\mathcal{O}^t_{3x}\right]^{1y} = -t^{-1}P_0\,\delta(x-y). \tag{3.57}$$

We will also need below

$$\begin{aligned}\left[\mathcal{O}^t_{0x}\right] &= \partial^{r_t}_{0x}\,S_\omega \\ &= t^{-4}\sqrt{|\gamma^{t\mu}|} \\ &= \mu^4,\end{aligned} \tag{3.58}$$

which can be read from (3.52) as well. In the canonical linear basis, its only non-vanishing component is

$$\left[\mathcal{O}^t_{0x}\right]^{0y} = t^{-4}\delta(x-y). \tag{3.59}$$

We see that the vacuum energy operator, having the lowest dimension, does not mix with other operators.

The nonlinear counterterms are defined by the transported connection $f_t^*\nabla$ in normal coordinates. They can be obtained from the non-linear part of (3.52), as we now show. We make use of the relation

$$C^{t\,\alpha}_{\alpha_1\alpha_2}\partial^{r_t}_{\alpha}S_\omega = \partial^{r_t}_{\alpha_1}\partial^{r_t}_{\alpha_2}S_\omega \qquad (\text{linear } c), \tag{3.60}$$

which is valid for any linear parametrization $c$ and follows from (3.14) and the fact that in linear coordinates $\partial^c_{\alpha_1}\partial^c_{\alpha_2}S_\omega = 0$. Choosing $\alpha_1 = 3x$, $\alpha_2 = 3y$ and taking (3.54) and (3.52) into account, we get the equation

$$C^{t\,\alpha}_{3x3y}\,t^{-\lambda_{(\alpha)}}(S^{\gamma^{t\mu}}_\omega)_\alpha = 2t^{-2}\left[\log t\,\beta^\alpha_{3x3y}(\gamma^{t\mu})\,(S^{\gamma^{t\mu}}_\omega)_\alpha + (S^{\gamma^{t\mu}}_\omega)_{3x3y}\right]. \tag{3.61}$$

To solve it for $C^{t\,\alpha}_{3x3y}$, we use (2.85) and (2.102) to expand the last term inside the brackets in the basis of eigenoperators,

$$(S^\gamma_\omega)_{3x3y} = (\tau^\gamma)^\alpha_{3x3y}\,(S^\gamma_\omega)_\alpha. \tag{3.62}$$

Note that this expansion includes not only scalar but also tensorial eigenoperators $(S^\gamma_\omega)_\alpha$ in which the collective index $\alpha$ contains Lorentz indices (contracted with the ones in the coefficients). For our purposes we can cut the series and keep only the terms that will eventually contribute in the $t\to\infty$ limit, which in this case involve only scalar operators. This is equivalent to using (3.42) to redefine the renormalization scheme, with no impact in the final renormalized correlators. We find

$$\begin{aligned}(S^\gamma_\omega)_{3x3y} =& E_0\sqrt{|\gamma|}\delta(x-y) + E_2\sqrt{|\gamma|}\partial^2_\gamma\delta(x-y) + B_0\delta(x-y)\,(S^\gamma_\omega)_{2x}\\ &+ \text{irrelevant terms.}\end{aligned} \tag{3.63}$$

Matching in (3.61) the coefficients of the identity and $(S^\gamma_\omega)^x_2$ and using the values (2.97) and (2.98) for the beta functions, we identify the counterterms

$$C^{t\,2z}_{3x3y} = \left[\frac{9}{4\pi^2}\log t + 2B_0\right]\delta(z-x)\delta(z-y), \tag{3.64}$$

$$C^{t\,0z}_{3x3y} = \left[\frac{3}{256\pi^4}\log t + E_2\right]\left[\delta(z-x)\partial^2_{\gamma^\mu}\delta(z-y) + \delta(z-y)\partial^2_{\gamma^\mu}\delta(z-x)\right] + 2t^2E_0\delta(z-x)\delta(z-y). \tag{3.65}$$

Equipped with the renormalized operators and counterterms in linear coordinates, we are ready to compute the renormalized correlator $G^R_{3x3y}$ using (3.12) and (3.13):

$$G^R_{3x3y} = \lim_{t\to\infty}\left\{[\mathcal{O}^t_{3x}]^{\alpha_1}[\mathcal{O}^t_{3y}]^{\alpha_2}\partial^c_{\alpha_1}\partial^c_{\alpha_2}W + C^{t\,\alpha_1}_{3x3y}[\mathcal{O}^t_{\alpha_1}]^{\alpha_2}\partial^c_{\alpha_2}W\right\}. \tag{3.66}$$

This limit is well-defined for any valid cutoff propagator $P$, but to make the cancellation of divergences manifest let us choose the following simple cutoff propagator:

$$P(\gamma;x) = \frac{1}{4\pi^2}\frac{1}{x^2_\gamma + 1}. \tag{3.67}$$

This corresponds to the function

$$D(u) = \sqrt{u}K_1(\sqrt{u}) = 1 + \frac{1}{4}\left(2\gamma_E - 1 + \log\frac{u}{2}\right)u + o(u). \tag{3.68}$$

Note that it does not satisfy the requirement of analiticity at $u = 0$. Even if this results in a non-quasilocal Wilson action, the non-local pieces are irrelevant for the continuum limit and we find the same renormalized propagator that could be obtained with an analytic (but more complicated) regularization.[4]

Using (3.56) and (3.57) and Wick's theorem, the first term on the RHS of (3.66) is

[4] With the propagator (3.67) the expansion of the functions $A, B, E$ needs to be modified to include non-local terms. In Fourier space,

$$\hat{A}(\gamma;p) = \frac{1}{\sqrt{|\gamma|}}\frac{9}{2}\left[\frac{2\gamma_E - 1}{4} + \frac{1}{4}\log\frac{p^2_\gamma}{4} + o\left(p^0_\gamma\right)\right],$$

$$\hat{B}(\gamma;p) = \frac{1}{\sqrt{|\gamma|}}\left[B_0 + \frac{9\,p^2_\gamma}{8\pi^2}\left(-\gamma_E + \frac{1-\gamma_E}{4} - \frac{\log p^2_\gamma}{4}\right) + o\left(p^2_\gamma\right)\right],$$

found to be given precisely by the diagrams in Figure 3.1, up to a global $t^{-2}$ factor and with $\mu$ changed by $t\mu$. Therefore,

$$\begin{aligned}
[\mathcal{O}^t_{3x}]^{\alpha_1}[\mathcal{O}^t_{3y}]^{\alpha_2}\partial^c_{\alpha_1}\partial^c_{\alpha_2}W &= 6t^{-2}|\gamma^{t\mu}|P\left(\gamma^{t\mu};x-y\right)^3 \\
&= \frac{6\mu^2}{(4\pi^2)^3}\frac{1}{(x^2+(t\mu)^{-2})^3} \\
&= \frac{6\mu^2}{(4\pi^2)^3}\Bigg[-\frac{1}{32}\dot{\partial}^4\frac{\log\left[(x-y)^2\mu^2\right]}{(x-y)^2}+\frac{\pi^2}{8}\left(1+2\log t\right)\partial^2\delta(x-y) \\
&\qquad +\frac{\pi^2}{2}t^2\mu^2\delta(x-y)\Bigg]+o\left(t^0\right). \qquad (3.70)
\end{aligned}$$

The second term on the RHS of (3.66) can be written as

$$\begin{aligned}
C^{t\,\alpha_1}_{3x3y}[\mathcal{O}^t_{\alpha_1}]^{\alpha_2}\partial^c_{\alpha_2}W &= -C^{t\,\alpha}_{3x3y}\langle[\mathcal{O}^t_{\alpha}]\rangle \\
&= -C^{t\,0z}_{3x3y}\langle[\mathcal{O}^t_{0z}]\rangle \\
&= -\left[\frac{3}{128\pi^4}\left(\log t\right)+2E_2\right]\mu^2\partial^2\delta(x-y)-2\mu^4t^2E_0\delta(x-y), \qquad (3.71)
\end{aligned}$$

where in the second equality we have used (2.79) to discard all the directions $\alpha\neq 0$ in the sum, whereas in the third one we have used (3.59) and (3.65). Finally, taking the limit $t\to\infty$ we obtain

$$G^R_{3x3y} = -\frac{3\mu^2}{2^{10}\pi^6}\dot{\partial}^4\frac{\log\left[(x-y)^2\mu^2\right]}{(x-y)^2}+\mu^2\left(\frac{3}{2^8\pi^4}-2E_2\right)\partial^2\delta(x-y). \qquad (3.72)$$

The terms with $\log t$ have cancelled out and the result precisely agrees with (3.51) for $E_2=\xi_3+3/(512\pi^4)$, as appropriate for the cutoff propagator we are using.

The two calculations of $G^R_{3x3y}$ that we have presented are based on the Wilsonian analysis of subsection 2.4.2. In the last one, the exact RG flows have been used to find renormalized operators and counterterms that render the correlator finite. These

$$\hat{E}(\gamma;p) = \frac{1}{\sqrt{|\gamma|}}\left[\frac{3}{128\pi^4}-p_\gamma^2\,E_2+\frac{3p_\gamma^4}{2^{12}\pi^4}\left(\frac{4\gamma_E-5}{2}+\log\frac{p_\gamma^2}{4}\right)+o\left(p_\gamma^4\right)\right]. \qquad (3.69)$$

Only the first terms, which are local, contribute in the $t\to\infty$ limit and we obtain the same counterterms as above, with $E_0=\frac{3}{128\pi^4}$.

objects can also be obtained without explicit Wilsonian information in the traditional way, just requiring that the UV divergences are cancelled in the correlation functions. Let us sketch the standard calculation to connect it with the one in this subsection. The starting point is the bare correlator

$$\partial^c_{3x}\partial^c_{3y}\Big|_{(0,\Lambda)} W = \left|\gamma^\Lambda\right| \langle\omega(x)^3\omega(y)^3\rangle_{(0,\Lambda)}, \tag{3.73}$$

which is given by the free-field diagrams $\mathcal{K}_1$ and $\mathcal{K}_2$ in Figure 3.1, with $\mu \to \Lambda = t\mu$. These diagrams are singular when $\Lambda \to \infty$. Both of them contain non-local divergences for separate points, $x \neq y$. The one in diagram $\mathcal{K}_1$ can be compensated by a multiplicative renormalization of the operator $\omega^3$, while the ones in diagram $\mathcal{K}_1$ can be cancelled by adding to the action a counterterm proportional to $\omega$, which gives the contributions $\mathcal{K}_{3,4,5}$ in Figure 3.1. This linear renormalization can be interpreted as the matrix renormalization of the operator $\omega^3$ given by (3.57) and (3.56). After it, only a local divergence for coincident points $x \sim y$ remains. The counterterm that cancels it is a local contribution to the vacuum energy, which can be identified with the non-linear contribution in (3.71).

Of course, there is some freedom in the choice of counterterms that do the job. However, imposing the minimal subtraction conditions written at the end of Section 3.1 we arrive at the same renormalized operators and counterterms given above. Therefore, we also obtain the same renormalized function $G^R_{3x3y}$ in (3.72) and (3.51). This illustrates the general result, proven in Section 3.1, that minimal subtraction leads to renormalized correlators that coincide with cutoff correlators in normal coordinates. In other words: in minimal subtraction schemes, the renormalized couplings can be understood as the couplings in a normal parametrization of the action. We also stress that the renormalized correlator $G^R_{3x3y}$ itself retains Wilsonian information about normal coordinates through the renormalization scale. Indeed, because the renormalized correlators are equal to bare correlators at scale $\mu$ in normal coordinates, we know that they must obey Callan-Symanzik equations. A double functional differentiation of (2.43) with $\Lambda = \mu$ leads to

$$\mu\frac{\partial}{\partial\mu} G^R_{3x3y} = 2\lambda_{(3)} G^R_{3x3y} - 2\beta^{0z}_{3x3y}(\gamma^\mu) G^R_{0z}. \tag{3.74}$$

Using (3.72) in the LHS of (3.74) we get

$$\mu\frac{\partial}{\partial\mu}G^{R}_{3x3y} = 2G^{R}_{3x3y} + \frac{3\mu^2}{128\pi^4}\dot{\partial}^2\delta(x-y). \tag{3.75}$$

Taking into account the obvious result for the renormalized vacuum energy, $G^R_{0z} = -\mu^4$, and comparing non-local and local pieces in (3.74), we find $\lambda_{(3)} = 1$ (that is, $\Delta_{(3)} = 3$) and $\beta^{0z}_{3x3y}(\gamma^\mu)$ as in (2.98). We emphasize that this also holds for any regulator and in any renormalization scheme consistent with minimal subtraction. This is particularly simple in mass-independent methods. For instance, we could simply compute the diagrams with the original unregularized propagator and use dimensional regularization to make sense of the resulting expressions. Then, diagram $\mathcal{K}_2$ of Figure 3.1 vanishes identically, while diagram $\mathcal{K}_2$ directly gives (3.51) plus a pole in $1/(d-4)$, which is cancelled by a local counterterm, see [65, 66]. The same result can be found even more directly, without explicit regularization, in differential renormalization [43]. It is remarkable that these mass-independent renormalization schemes produce renormalized correlators associated to normal coordinates, which carry all the local information near the fixed point about the exact RG flows in these coordinates. The exact beta functions are equal to their Gell-Mann-Low counterparts, up to residual scheme dependence within minimal subtraction. The fact that only resonant terms appear can be understood by dimensional analysis in the absence of dimensionful parameters in the regularization and renormalization conditions.

### 3.1.6 Example II: Large N Limit

The large $N$ limit is interesting since, as we have seen in Section (2.4.3), several simplifications occur once it is taken. One remarkable property is that the leading contribution in $1/N$ of correlation functions involving multi-trace operators factorize to single-trace operators correlation functions when we work in a factorization normal scheme. Thus, a correlation function with multi-trace and single-trace operators can be written as the sum of products of single-trace correlators. In every product, all the single-trace operators forming a multi-trace operator are split in the different correlators of the product. The sum runs over all the possible splittings avoiding loops. By loops in this context we mean products where the correlators connect in more than one way single-trace operators coming from the same multi-trace operator. Some examples are

$$G^R_{\langle ij\rangle x\, iy_1\, jy_2} = -\,2\,G^R_{ix\, iy_1} G^R_{jx\, jy_2} + O(1/N^2) \tag{3.76}$$

$$G^R_{\langle ij\rangle x_1\, \langle ij\rangle x_2\, iy_1\, iy_2} = 4\,G^R_{ix_1\, ix_2}\left[G^R_{jx_1\, jy_1} G^R_{jx_2\, jy_2} + G^R_{jx_2\, jy_1} G^R_{jx_1\, jy_2}\right] + O(1/N^2) \tag{3.77}$$

$$G^R_{\langle ij\rangle x_1\, \langle ij\rangle x_2} = O(1/N^2). \tag{3.78}$$

The last line correspond to one case where the only split that can be done is at the loop level.

This factorization can be understood as a consequence of (2.144) (where we reconstructed $S_{\mathcal{O}^s}$ from its restriction to the submanifold $\mathcal{T}_1 \subset \mathcal{W}$). The generator

$$W = \frac{1}{N^2} \log \int [\mathcal{D}\omega]^\Lambda\, e^{-N^2 S_{\mathcal{O}^s[\omega]}} \tag{3.79}$$

has a similar property (see (2.105) for a specific implementation of (3.79)). Using (2.142) – (2.145) in (3.79):

$$\begin{aligned} e^{N^2 W} &= \int [\mathcal{D}\omega]^\Lambda\, e^{-N^2 S_{\mathcal{O}^s[\omega]}} \\ &\underset{N\to\infty}{\sim} \int [\mathcal{D}\omega]^\Lambda\, \mathcal{D}\zeta\, \mathcal{D}\bar{g}^s \exp\left\{-N^2\left[S_{\mathcal{O}^s[\omega]}[\bar{\gamma};\bar{g}^s,0] + G_\zeta - \bar{g}^\sigma \zeta_\sigma\right]\right\} \\ &= \int \mathcal{D}\zeta\, \mathcal{D}\bar{g}^s \exp\left\{-N^2\left[S_{\mathcal{O}^s[\omega]}[\bar{\gamma};\bar{g}^s,0] + G_\zeta - \bar{g}^\sigma \zeta_\sigma\right]\right\} \\ &\underset{N\to\infty}{\sim} \exp\left\{N^2\left[W\left[\bar{\gamma};\bar{g}^s,0\right] - G_\zeta + \bar{g}^\sigma \zeta_\sigma\right]\right\}, \\ &\quad \text{with } \bar{g}^\sigma = \frac{\delta G_\zeta}{\delta \zeta_\sigma},\ \zeta_\sigma = -\frac{\delta W}{\delta \bar{g}^\sigma}\left[\bar{\gamma};\bar{g}^s,0\right]. \end{aligned} \tag{3.80}$$

In the second line we have used a saddle point approximation to introduce the Legendre transform of (2.142). Therefore, working in a factorization UV scheme, the generator $W$ can be written as a single-trace generator with a modified source, $\delta G_\zeta/\delta\zeta_\sigma$, and an additional term, $-G_\zeta + \delta G_\zeta/\delta\zeta_\sigma \zeta$. Taking derivatives with respect to the coordinates, one can easily find the factorization properties of correlation functions.

**Factorization Renormalization Charts**

In the same way that normal coordinates can be chosen to satisfy the factorization condition (2.133), it would be interesting to select a family of renormalization charts $r_t$ that reproduce the correlation functions in a factorization UV scheme. If we choose the renormalization charts in minimal subtraction and such that

$$\partial^{r_t}_{\alpha^n} S_{\mathcal{O}^s} = Q_{\alpha^n}{}^{\sigma_1 \ldots \sigma_n}(t^2 \bar{\gamma}) \partial^{r_t}_{\sigma_1} S_{\mathcal{O}^s} \ldots \partial^{r_t}_{\sigma_n} S_{\mathcal{O}^s}, \tag{3.81}$$

the limit (see (3.19))

$$\lim_{t \to \infty} r_t \circ f_t = c \tag{3.82}$$

in large N theories gives a factorization normal chart $c = \bar{c}$.

Conversely, given a factorization normal chart $\bar{c}$, and a critical point, its associated exact UV scheme given by (3.35) also satisfies (3.81).

Both statements are consequence of (2.136): if a chart $c$ satisfies

$$\partial^{c}_{\alpha^n} S_{\mathcal{O}^s} = Q_{\alpha^n}{}^{\sigma_1 \ldots \sigma_n}(t'^2 \bar{\gamma}) \partial^{c}_{\sigma_1} S_{\mathcal{O}^s} \ldots \partial^{c}_{\sigma_n} S_{\mathcal{O}^s}, \tag{3.83}$$

then, the chart $\tilde{c}^\alpha = c^\alpha \circ f_t$ must satisfy

$$\partial^{\tilde{c}}_{\alpha^n} S_{\mathcal{O}^s} = Q_{\alpha^n}{}^{\sigma_1 \ldots \sigma_n}(t^{-2} t'^2 \bar{\gamma}) \partial^{\tilde{c}}_{\sigma_1} S_{\mathcal{O}^s} \ldots \partial^{\tilde{c}}_{\sigma_n} S_{\mathcal{O}^s}. \tag{3.84}$$

The proof consists in taking a derivative with respect to $t$ in (3.83), and using (2.136), checking that both sides agree for all $t$.

## 3.2 Renormalizable Theories

Let us briefly summarize what we have achieved so far. We have described the space of regularized theories and the flows that they follow when UV degrees of freedom are integrated out. In particular, we have studied in detail the flows close to fixed points. We have shown how the flows select a set of special coordinates (known as normal coordinates) in a neighbourhood of the given fixed point. Additionally, we have analysed the renormalization of correlation functions at fixed points from a Wilsonian RG perspective. In this analysis, we have found that normal coordinates are in intimate connection with a class of minimal subtraction renormalization schemes.

All these developments take place in a space of theories that are regularized. The only theories with a continuum limit we have studied are the fixed points. In this section, we will extend some of our tools to other scale non-invariant renormalizable theories.

General renormalizable theories can be constructed from relevant or marginally relevant deformations of a particular fixed point. These trigger flows that approach the fixed point in the UV.[5] These deformations, modulo total derivative terms,[6] span a vector space of dimension $N_R$. In these cases Wilsonian actions exist no matter how large $\Lambda$ is taken, and thus describe the continuum limit. The set of points of $\mathcal{M}$ that can be reached from these perturbed theories under RG evolution towards the IR form the *renormalized* manifold $\tilde{\mathcal{R}}$ of the given fixed point.[7] Each integral curve of $\boldsymbol{\beta}$ with image in $\tilde{\mathcal{R}}$ defines a particular renormalizable theory, with definite physical predictions that do not depend on any cutoff. We can lift $\tilde{\mathcal{R}}$ to $\mathcal{W}$ and define

$$\mathcal{R} = \{(s,\Lambda) \in \mathcal{W} \text{ such that } [(s,\Lambda)] \in \tilde{\mathcal{R}}\}. \tag{3.85}$$

Likewise, the set of points of $\mathcal{M}$ that flow into the fixed point under direct RG evolution (towards the IR) is called the *critical* manifold, and is denoted by $\tilde{\mathcal{E}}$. A lift to define $\mathcal{E}$ in $\mathcal{W}$ can be done analogously to (3.85).

[5] In fact, all known unitary theories consistent at all scales are of this form. However, one could imagine a theory which in the UV approaches a limit cycle or an ergodic behaviour. These examples are widely believed not to be allowed. Although there is not a complete proof yet, there are strong indications for that (see [67]).

[6] In other words, considering two deformations equivalent if they differ by a total derivative, it is the quotient space that has dimension $N_R$.

[7] We mean the "renormalized trajectories" but we will again loosely regard the space $\tilde{\mathcal{R}}$ as a manifold of dimension $N_R$, keeping in mind that singular behaviours such as boundaries are quite possible far from the fixed point or at the fixed point itself.

### 3.2.1 Renormalization

In practice, however, instead of constructing the renormalized trajectories using exact RG flows, it is often easier to follow a renormalization procedure based on counterterms or, equivalently, bare couplings. For this, we choose some bare action at scale $\Lambda_0$ that depends on $N_B$ tunable parameters. There is a great deal of freedom in the form of the bare action, equivalently in the dependence of $g$ on these parameters (this is a statement of universality). Its description in our geometric language is similar to the one used in subsection 3.1.1, but restricting the domain of the renormalization charts.

We choose a submanifold $\tilde{\mathcal{B}} \subset \mathcal{M}$ of finite dimension $N_B$, in the same sense as above, that cuts the critical manifold at a point $\mathbf{s}_c$. Its lift to $\mathcal{W}$ is denoted by $\mathcal{B}$. The RG curves of points close to $\mathbf{s}_c$ will approach $\mathbf{s}_*$ and, before they reach it, leave the critical manifold along the relevant directions, approximately, and stay (at least for a while) close to $\tilde{\mathcal{R}}$. To parametrize $\mathcal{B}$, it is convenient to define $\mathcal{C}_B$ as the set formed by all $N_B$-tuples of smooth functions $g^{\breve{a}} : \mathbb{R}^d \to \mathbb{R}$. The indices running over such space will be denoted with the mark $\breve{}$, as in $\breve{\alpha}$ or $\breve{a}$.

Let $r_t : \tilde{\mathcal{B}} \to \mathcal{C}_B \times \mathbb{R}^+$, $(s, t\mu) \mapsto (g_R, \mu)$, be a family of renormalization charts that are defined only in $\tilde{\mathcal{B}}$. Then, $h_t(g_R, \mu) = r_t^{-1}(g_R, \mu)$ are curves in $\mathcal{B}$ parametrized by $(g_R, \mu)$. Let us impose that the equivalence classes $[h_t(g_R, \mu)]$ describe curves in $\tilde{\mathcal{B}}$ that, a $t \to \infty$, approach $\mathbf{s}_c$ at a rate characterized by $g_R$, with the condition that (see figure 3.2)

$$\lim_{t\to\infty} f_{1/t} \circ h_t(g_R, \mu) \in \mathcal{R}. \tag{3.86}$$

This defines a renormalized theory:

$$\begin{aligned} Z^R_\mu[g_R] &= \lim_{\Lambda_0\to\infty} Z\left(h_{\Lambda_0/\mu}(g_R, \mu)\right) \\ &= Z\left(\lim_{\Lambda_0\to\infty} f_{\Lambda/\Lambda_0}(h_{\Lambda_0/\mu}(g_R, \mu))\right), \end{aligned} \tag{3.87}$$

and induces a family of charts over the renormalized manifold (or a submanifold $\mathcal{A}$ of

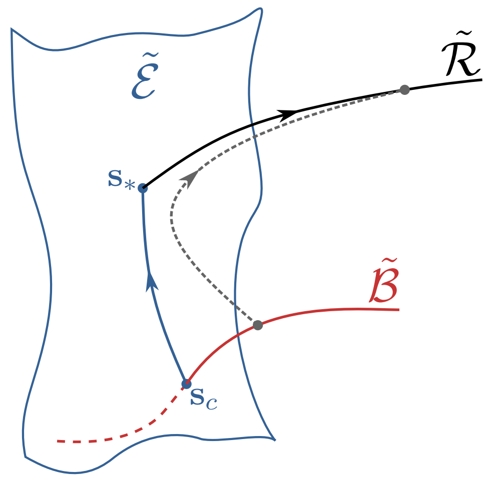


Figure 3.2: Points lying in the critical manifold are shown in blue. The bare manifold $\tilde{\mathcal{B}}$ cuts the critical manifold at a single point $\mathbf{s}_c$. The dashed grey curve illustrates the action of combined RG evolution and renormalization as in (3.86), finishing at a finite point in $\tilde{\mathcal{R}}$.

it), $c_R : \mathcal{A} \subset \mathcal{R} \to \mathcal{C}_B \times \mathbb{R}^+$:

$$c_R^{-1} = \lim_{t\to\infty} f_t \circ r_t^{-1}. \tag{3.88}$$

The use of $\mathcal{B}$ allows to work with a finite set of couplings. This makes the renormalized trajectories especial: they can be described by a finite set of parameters and it is not necessary to know the infinite number of Wilson couplings that define a generic theory.

The introduction of the renormalization scale $\mu$ in $h_t$ (or equivalently in $r_t$) is required for dimensional reasons. A change $\mu \to \mu'$ can be compensated by a change $g_R \to g'_R$ such that the same renormalized theory is obtained:

$$\begin{aligned}\lim_{t\to\infty} f_{1/t} \circ h_t\left(g_R, \mu\right) &= \lim_{t\to\infty} f_{1/t} \circ h_{t\mu/\mu'}\left(g'_R, \mu'\right) \\ &= \lim_{t\to\infty} f_{\mu/t\mu'} \circ h_t\left(g'_R, \mu'\right) \\ &= f_{\mu/\mu'}\left(\lim_{t\to\infty} f_{1/t} \circ h_t\left(g'_R, \mu'\right)\right). \end{aligned} \tag{3.89}$$

Therefore, the flows associated to changes of the renormalization scale are described by the Wilson flows of the renormalized theories:

$$c_R^{-1}(g_R, \mu) = f_{\mu/\mu'} \circ c_R^{-1}(g'_R, \mu'), \tag{3.90}$$

and the functions $c_R^{\pi} \circ f_t$ play the role of running constants of the renormalized theory. The corresponding vector fields

$$\beta_\mu^{\breve{\alpha}}[g_R] = \frac{\partial}{\partial t} f_t^{\breve{\alpha}} \circ c_R^{-1}(g_R, \mu)\bigg|_{t=1}, \tag{3.91}$$

are the local versions of the Gell-Mann-Low beta functions of the renormalized theory. They coincide with the Wilson vector fields $\beta$ restricted to $\mathcal{R}$, that by definition, are contained in $T\mathcal{R}$:

$$\beta^{\breve{\alpha}} = \beta^{\alpha} \partial_\alpha^c c_R^{\breve{\alpha}} + 2\bar{\gamma} \frac{\partial}{\partial \bar{\gamma}} c_R^{\breve{\alpha}}. \tag{3.92}$$

The Wilsonian Callan-Symanzik equation (2.43) continues applying when it is restricted to the renormalized manifold, giving the Callan-Symanzik equation for the renormalized theory,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_\mu^{\breve{\alpha}}[g_R] \frac{\delta}{\delta g_R^{\breve{\alpha}}}\right] Z_\mu^R[g_R] = 0. \tag{3.93}$$

Of course, there is a large degree of freedom in the choice of the renormalization charts, i.e. in the renormalization scheme. Different renormalization schemes lead to different parametrizations of the renormalized manifold using (3.88). We will use the term "renormalization scheme" to refer both the choice of renormalization charts and the parametrization of $\tilde{\mathcal{R}}$.

### 3.2.2 Minimal Subtraction Schemes

A very natural set of schemes are the minimal subtraction schemes, as defined in (3.38), where $c$ has to be understood as a chart of $\mathcal{B}$.

Given any chart in $\mathcal{B}$, $c_B : \mathcal{B} \to \mathcal{C}_B$, it can be written in terms of the minimal subtraction charts $r_t$ inverting (3.38). Assuming a homogeneous critical point $g_c =$

$g_c^\Lambda = \bar{c}_\Lambda^\pi(s_c^\Lambda)$, we obtain

$$c_B^{\breve{\alpha}} - g_c^{\breve{\alpha}} = t^{-\lambda_{(\breve{\alpha}_1)}} A^{\breve{\alpha}}_{t\,\breve{\alpha}_1}(\bar{\gamma}) r_t^{\breve{\alpha}_1} + t^{-\lambda_{(\breve{\alpha}_1)}-\lambda_{(\breve{\alpha}_2)}} A^{\breve{\alpha}}_{t\,\breve{\alpha}_1\breve{\alpha}_2}(\bar{\gamma}) r_t^{\breve{\alpha}_1} r_t^{\breve{\alpha}_2} + O(r_t^3), \tag{3.94}$$

with $A^{\breve{\alpha}}_{t\,\breve{\alpha}_1\ldots\breve{\alpha}_n}$ depending at most logarithmically on $t$. This equation can be used to examine the curves that $h_t$ describes in $\mathcal{B}$. Consider first the generic case in which all the eigenvalues of relevant directions fulfil the condition $\lambda_{(a_1)} + \lambda_{(a_2)} > \lambda_{(a)}$. We can then perform a linear transformation of the chart $c_B \to \tilde{c}_B$, and write

$$\tilde{c}^{\breve{\alpha}} \circ h_{\Lambda/\mu}(g_R, \mu) = \left(\frac{\mu}{\Lambda}\right)^{\lambda_{(\breve{\alpha})}} g_R^{\breve{\alpha}} \quad \text{as} \quad \Lambda \to \infty. \tag{3.95}$$

Therefore, we could conclude that the leading order of the bare couplings when $\Lambda \to \infty$ is dictated by the eigenvalues $\lambda_{(a)}$. However, when we withdraw the condition on the relevant eigenvalues, the second and possibly higher-order terms in $g_R$ can give contributions that are more important than the RHS of (3.95). Also, logarithmic terms will appear for exceptional dimensions.

### UV Schemes

The parametrization of the renormalized manifold, $\bar{c}_R$, that minimal subtraction schemes generate using (3.88), is necessarily a normal parametrization (since minimal subtraction schemes are a subclass of UV schemes).

These type of schemes are purely Wilsonian, as they can be defined in a neighbourhood of the fixed point without integrating out the IR degrees of freedom. Let us notice that, in general, the variation of all relevant *normal* couplings, fixing to zero all irrelevant and marginal ones, covers just a submanifold of the full renormalized manifold. This *relevant* submanifold, $W_R \subset \mathcal{R}$, is also invariant under the Wilson flow, just as (2.64) proves: if initially, only relevant couplings are non-vanishing, the resonant condition of $[B_p]^{\alpha}_{\alpha_1\ldots\alpha_m}$ guarantees that this condition will be maintained along the flow.[8] In any case, the full parametrization of $\mathcal{R}$ can require the inclusion of suitable marginal coordinates: the marginally relevant coordinates.

Normal coordinates characterize the flows along the renormalized manifold by their

[8] In fact, $W_R$ has better properties than $\mathcal{R}$. For instance, it cannot develop boundaries at the fixed point itself.

linearised rates as they leave the fixed point. This follows immediately from (2.64), but it is also true for any choice of the renormalized scheme (i.e. the charts that parametrize the renormalized manifold, $c_R$). Assuming a fully diagonalizable fixed point,

$$f^{\breve{\alpha}}_{\Lambda/\mu,\Lambda} = c^{\breve{\alpha}}_R(s^\Lambda_*, \Lambda) + (\Lambda/\mu)^{-\lambda_{(\breve{\alpha}')}} \left(\partial^{\bar{c}}_{\breve{\alpha}'} c^{\breve{\alpha}}_R\right)\Big|_{(s^\Lambda_*,\Lambda)} \bar{c}^{\breve{\alpha}'} + O\left(\bar{c}^2\right), \quad \text{as} \quad \Lambda/\mu \to \infty. \quad (3.96)$$

If the eigenvalues satisfy $\lambda_{(a_1)} + \lambda_{(a_2)} \leq \lambda_{(a_3)}$, for some $a_1, a_2, a_3$, then generically as $\Lambda/\mu \to \infty$, there are higher order terms that are as important or more important than the linearised terms shown in (3.96). In particular this is always true if $a_1$ or $a_2$ corresponds to a non-vanishing marginally relevant coupling. Also, in a logarithmic conformal field theory, (3.96) has to be corrected with logarithmic contributions of $\Lambda/\mu$.[9]

### 3.2.3 Other Renormalization Schemes

#### Physical Schemes

The usual mass-dependent schemes used in QFT are defined in terms of correlation functions of the elementary fields. They require the integration of all the quantum fluctuations. In this thesis we are interested in the gravity duals of gauge theories, which are manifestly gauge-invariant, so the correlation functions of elementary fields do not have a gravity counterpart. However, we can define a similar renormalization scheme in terms of other observables, like Wilson loops or correlation functions of gauge-invariant operators. This requires the intermediate usage of another renormalization scheme, such as the UV scheme above, in order to calculate them. For example we can *choose to define* the Yang-Mills coupling $g_{\rm YM}$ through the expectation of a Wilson loop $\langle W(\mathcal{C})\rangle$ in general, by setting it equal to the exact formula for $\mathcal{N} = 4$ Yang-Mills at large 't Hooft coupling $Ng^2_{\rm YM}$ [5] even when the theory no longer corresponds exactly to $\mathcal{N} = 4$ Yang-Mills in this limit. At least for small perturbations away from such a theory, we can expect this definition of $g_{\rm YM}$ to remain sensible. An interesting property

[9] In [6], the UV scheme is in fact defined in base of these rates. However, this definition presents some difficulties that are particularly important in resonant cases. Normal coordinates avoid such difficulties and allow to define the UV scheme with complete generality.

of such *physical* schemes is that the beta functions are sensitive to IR details, such as mass thresholds or the choice of vacuum state, if degenerate.

### Projection Scheme

A natural scheme for defining renormalized couplings in Wilsonian flows is by *projection*, by which we mean that they are defined through the coefficient of the natural operator in the Wilsonian effective action. Thus we pick a subset of the coordinates in a linear parametrization of $S[\gamma;\omega]$ as defined in (2.11) to play the role of the renormalized couplings. An example should make this clearer. In Yang-Mills theory a natural way to define $g_{\rm YM}$ directly from the Wilsonian action is to define the coefficient of the field-strength squared term in the Wilsonian action to be $F^2/4g^2_{\rm YM}(\Lambda)$. This defines a coupling that runs with $\Lambda$ under (2.36). It can be considered to be renormalized if it is chosen to be finite when the integrating out is continued down to values of $\Lambda$ corresponding to finite energies. Once we are on $\mathcal{R}$, all the couplings $g^\alpha$ then become functions of these renormalized couplings. In this example we would have $g^\alpha \equiv g^\alpha(g_{\rm YM})$. Clearly, this scheme breaks down when the projection is not injective. The evolution of renormalized couplings in projection schemes is sensitive to the dynamics of the theory at the probed scales. However, unlike the physical schemes, they are of Wilsonian nature and the value of the renormalized couplings at a given finite renormalization scale does not depend on lower scales.

# Part II

# Holography

# Chapter 4

# AdS/CFT Correspondence

*Don't you hear my call though you're many years away*
*Don't you hear me calling you*
*Write your letters in the sand*
*For the day I take your hand*
*In the land that our grandchildren knew.*
From '39, Queen

In chapter 2 we have introduced the Wilsonian renormalization group, which essentially consists in rewriting a theory using more suitable degrees of freedom in the path integral formulation according to the tested energy scale. More generally, by virtue of the equivalence theorem [68, 69], any redefinition of the Lagrangian (or microscopic) degrees of freedom leaves unchanged the quantum theory.[1] The existence of different microscopic descriptions of the same theory[2] is highly interesting. It allows to use one description or the other depending on the regime of the theory we are interested in. This is why the existence of dualities relating different quantum theories is one of the most remarkable facts in theoretical physics. We say that there is a duality between two microscopic descriptions apparently different if they describe the same theory. For instance, the 1+1 dimensional theories known as the sine-Gordon model and the Thirring

[1] In fact, the renormalization group can be understood precisely as a continuum redefinition of the quantum fields [70].

[2] Here, by theory we mean a Hilbert space with a corresponding algebra of observables.

model:

$$\begin{aligned} S_{SG} &= \int dx^2 \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha}{\beta^2}\left(\cos\beta\phi - 1\right)\right], \\ S_T &= \int dx^2 \left(\psi i\gamma_\mu\partial^\mu\psi + m\bar{\psi}\psi - \frac{g}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi\right), \end{aligned} \tag{4.1}$$

with $\phi$ and $\psi$ scalar and fermionic fields respectively seem to describe completely different physics. However, thanks to the phenomenon of bosonization [71, 72], we know they are two descriptions of the same theory with the identification

$$\frac{\beta}{4\pi} = \frac{1}{1+g/\pi}. \tag{4.2}$$

Notice also how this duality relates the weak coupling regime of one with the strong coupling regime of the other.[3] This is a weak-strong coupling duality. This kind of dualities allows to make calculations in a strong-coupling regime of a theory, otherwise hardly treatable.

Other examples of weak-strong coupling dualities in quantum field theory are electromagnetic dualities in supersymmetric theories like S-duality and Seiberg duality [73–76].

There are also a lot of dualities in ST (like T and S-dualities). Some of them, relate weak and strong coupling regimes of different perturbative string theories [77]. It was the discovery of these dualities in the nineties what gave room to the second superstring revolution [78]. But maybe, the duality which has shaken more deeply theoretical physics has been the Gauge/Gravity duality (also referred to AdS/CFT correspondence). It is a duality between string theories (and thus, gravity) in $d+1$ dimensions and quantum field theories in $d$ dimensions. This particular relation between the dimensions motivates to use the name of holography for this kind of dualities. In fact, the holographic character of gravity had been proposed time before the discover of the duality by 'tHooft [79] and further developed by Susskind [80]. Their ideas were based on the physics of black holes and their thermodynamic properties [81]. They proposed the *holographic principle*, which says that, in a quantum theory of gravity,

[3]The elementary fermionic degrees of freedom of the Thirring model are mapped into solitons of the sine-Gordon model and the elementary bosonic degrees of freedom of the sine-Gordon model into fermion anti-fermion bound states of the Thirring model.

the degrees of freedom of a given volume are encoded on its boundary.

It was in 1997 when Mandalcena conjectured the first example of Gauge/Gravity duality [5]. He found that the $SU(N)$ gauge theory in 3+1 dimensions with 4 supersymmetric charges[4] (commonly known as $\mathcal{N} = 4$ SYM theory), which is conformal invariant, was dual to type $IIB$ superST on $AdS_5 \times S^5$ space with a specific background RR field. This explains the name of the correspondence, AdS from the gravity side, and CFT from the field theory side. In fact, this is not the only realization of the correspondence, many of less supersymmetric Gauge/Gravity dualities have been found (see for instance [5, 82–87]). Since it is a weak-strong coupling duality, it opens a new door to make calculations in strongly coupled QFT through their weakly coupled gravity duals. It has applications in particle physics (quantum chromodynamics [88, 89], physics beyond standard model [90, 91]) and condensed matter physics (superconductivity [92], topological states [11, 93, 94], disordered systems [12, 95]). But not only is it used to describe strongly coupled systems, it has allowed to make important progress in the understanding of quantum gravity [96–102].

There is a large amount of reviews and books about the Gauge/Gravity duality. We recommend, for instance, [103–106].

This chapter does not contain original work. It is a basic review of Gauge/Gravity duality with emphasis in the main points relevant for this thesis. It is structured as follows. Section 4.1 is devoted to review the Maldacena conjecture [5]. In section 4.2 we motivate and analyse different crucial aspects of the Correspondence. Along this section, several entries of the holographic dictionary are discussed. In section 4.3 we review the holographic renormalization program in its standard form. Many aspects of AdS/CFT are carefully developed in this section.

## 4.1 The Maldacena Conjecture

In order to understand the Maldacena's argument of [5], notions of supersymmetry (SUSY), supergravity (SUGRA) and string theory (ST) are necessary. However, since we do not need of these topics in the rest of this thesis, we will not give a detailed

[4]Given the gauge group, the big amount of symmetries define uniquely the theory. In 3+1 dimensions it is the most supersymmetric quantum field theory one can build.

introduction to them.[5] In any case, in this section we sketch the argument and explain the main points of it.

We need to consider type IIB ST. It consists of a 10 dimensional supersymmetric theory of closed strings projected in a specific way.[6] The dimensionality of the spacetime is necessary to cancel the Weyl anomaly of the worldsheet. The low energy limit is given by type IIB SUGRA, which has the following spectum. The bosonic part is divided in the NSNS-sector, formed by the metric (graviton), the dilaton and a two-form; and the RR-sector, with the axion, another two-form and a four-form (all these fields are real).[7] The fermionic part completes the spectrum with two gravitinos and two dilatinos. These are precisely the massless states of the type IIB string. The mass of the next massive states is controlled by the Regge slope $\alpha'$, related with the tension of the string $\tau = (2\pi\alpha')^{-1}$.

Strings propagating in some background of these massless fields are perturbative excitations of the theory around this background, which can be thought of some coherent state of strings. In fact, the expectation value of the dilaton $\phi$ determines the strength of the coupling between strings $g_s = e^{\phi}$.

String theory requires also the introduction of a new kind of non-perturbative objects, the D$p$-branes [111]. They are objects extended in $(p+1)$ dimensions where strings can end. Thus, they allow the existence of open strings in type IIB ST (as long as their extremes are attached to the D$p$-brane). They are also sources of the RR forms introduced before. In particular, a D$p$-brane can be source of a $(p+1)$-form (in which case we say the brane is electrically charged under the form) or of a $(d-p-3)$-form (the brane is magnetically charged under the form). Thus, type IIB ST can only have D$p$-branes with $p$ odd.

The Madacena's argument starts considering a stack of $N$ coincident D3-branes. They are charged under the four-form RR-field. Depending on the strength of the string coupling $g_s$, there are two pictures for this system:

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[5] We recommend [107, 108] as references for SUSY and SUGRA, and [77, 109] for String Theory.

[6] When one consider the spectrum of a quantized closed string, tachyonic states appear in it. To get rid of these states, one needs to project out part of the theory. This projection is also chosen to leave the same number of bosonic and fermionic states, and thus obtain a supersymmetric theory. This is the known *GSO* projection [110].

[7] These names come from the boundary condition imposed for the two modes of the closed string, namely, the Neveu-Schwarz condition (NS) and the Ramond condition (R).

- In the limit of weak string coupling, $g_s N << 1$, a stack of coincident branes will not perturb the spacetime, which will stay as 10-d Minkowski. Excitations of the branes are given by open strings attached to them, that can interact with closed strings in the flat spacetime (excitations of the background). Open strings can start and end in different branes, and so, they are labeled by two indexes, $i, j = 1, \ldots, N$. In fact, the massless excitations of the open string sector can be seen as fields living in the world-volume in the adjoint representation of $SU(N)$. This configuration of branes breaks one half of the original 32 supersymmetries of type IIB, and thus, massless excitations have to be described by the $\mathcal{N} = 4$ $SU(N)$ supermultiplet, which has the same number of supersymmetries. This is the open string perspective.

- If $g_s N >> 1$, the previous picture is no-longer valid. However, there are solitonic solutions in SUGRA known as black branes, that source the same RR-fields (the four form $C_{(4)}$) and preserve the same symmetries. Thus, they are assumed to describe the background produced by a stack of branes in this limit. The geometry and RR-field in the background are given by

$$\begin{aligned} ds^2 &= H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}(r)\delta_{mn}dx^m dx^n, \\ &\qquad \mu, \nu = 0, \ldots, 3, \qquad m, n = 4, \ldots, 9, \\ H(r) &= 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N \alpha'^2, \quad r^2 = \delta_{mn}dx^m dx^n, \\ C_{(4)} &= \left(1 - H(r)^{-1}\right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \ldots \end{aligned} \tag{4.3}$$

  were the ellipsis represent additional terms necessary to make the field strength $F_{(5)} = dC_{(4)}$ self-dual. We obtain a picture of closed strings propagating in a background given by (4.3). This is the closed string perspective.

The argument follows considering the leading contribution of both perspectives in the $\alpha' \to 0$ limit. In the open string perspective, closed strings and open strings decouple. Not only that, the dynamics of the open string sector is given by $\mathcal{N} = 4$ SYM theory with the $SU(N)$ gauge group and gauge coupling $g_{YM}^2 = 2\pi g_s$. Furthermore, the $\alpha' \to 0$ limit for the closed string sector is type IIB SUGRA on Minkowski space.

The closed string perspective is also described by two decoupling sectors. On the one hand, we have closed strings propagating in the $r >> L$ region, which is a 10-d Minkowski space with no RR-field. When $\alpha' \to 0$, $L \to 0$, and the absorption cross section goes to zero. Thus, its limit is given also by type IIB SUGRA on Minkowski space. On the other hand, closed strings propagating with $r \approx 0$ keep trapped in the throat. To describe them, one introduces a new coordinate $z = L^2/r$. In the $\alpha' \to 0$ limit, the spacetime and RR-field become

$$\begin{aligned} ds^2 &= \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) + L^2 ds^2_{S^5}, \\ C_{(4)} &= \frac{L^4}{z^4}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + \ldots \end{aligned} \tag{4.4}$$

The geometry is the expected $AdS_5 \times S^5$ space.

Summarizing, one obtains that in both cases, the leading contribution when $\alpha' \to 0$ is given by two decoupled theories: type IIB SUGRA in Minkowski space and another theory depending on the perspective:

- $\mathcal{N} = 4$ SYM in four dimensions with a $SU(N)$ gauge group and a gauge coupling

  $$g^2_{YM} = 2\pi g_s \tag{4.5}$$

  when $g_s N << 1$.

- Type IIB ST on $AdS_5 \times S^5$, with radius of curvature $L$ given by

  $$L^4/\alpha'^2 = 2g^2_{YM}N \tag{4.6}$$

  and a background RR four-form given by (4.4) when $g_s N >> 1$.

Thus, it is natural to conjecture that both theories are two descriptions of the same theory (each one accurate in its respective limit).

## 4.2 Basics of AdS/CFT

Let us review (4.5) and (4.6) the relation between the couplings in both theories. When we take the number of colors $N$ large in the quantum field theory, and keep

$\lambda = g_{YM}^2 N$ fixed (this is the 't Hooft or planar limit [3]), the string coupling $g_s$ goes to zero, making the ST classical. Also, the strong coupling limit $\lambda \to \infty$ in the field theory, is dual to the $\alpha' \to 0$ limit in the ST, or what is the same, the SUGRA limit.[8]

Other Gauge/Gravity realizations have been found that share this relation: the large $N$ limit of a field theory in $d$ dimensions is mapped to the classical limit of the dual gravity description with $d+1$ extended dimensions. These other realizations of the correspondence are found using different configuration of branes in ST [5, 82, 83, 85], but also in other non-supersymmetric contexts, like the higher spin/vector model duality [86, 87].

There are also diagrammatic arguments in favor of the correspondence: the Feynman diagrams in the large $N$ limit are arranged in a similar way as the stringy diagrams in the genus expansion [103]. It is also possible to justify the holographic extra dimension as the Liouville field necessary to avoid the Weyl anomaly problem of the ST in a non-critical dimension [112, 113].

In light of these facts, it seems quite sensible to assume there is a general duality between gravity theories with $AdS_{d+1}$ vacua and $d$-dimensional conformal field theories with large N expansion, relating the classical limit of one theory with the large $N$ limit of the other. If so, a general dictionary translating objects of one theory to the other should exist. This is the known holographic dictionary. Along this subsection, we will try to motivate those entries relevant for this thesis.

### 4.2.1 AdS Space

Global symmetries of both descriptions have to agree, since they are properties of the common theory.[9] To start with, we analyse the AdS space and the symmetries that it possesses.

The $(d+1)$ Anti-de Sitter space ($AdS_{d+1}$) is the maximally symmetric Lorentzian manifold with constant negative curvature. It can be defined as the submanifold of the

[8] In fact, the complete equivalence of the theories at any value of the couplings is known as the strongest form of the correspondence. If one only trusts in one of these limits, we speak about the strong and weak form of the correspondence respectively.

[9] This is not what one expect of gauge symmetries, since they are a redundancy of the description more than an actual symmetry of the theory. However, we will see later that there is an entry of the dictionary involving gauge symmetries.

flat space $\mathbb{R}^{d+2} = \left\{(X^0, \ldots, X^{d+1})\right\}$ with metric $\tilde{\eta} = \mathrm{diag}(+, -, -, \ldots, -, +)$ satisfying $\tilde{\eta}_{MN} X^M X^N = L^2$, where $L$ is the radius. This construction makes explicit that the connected isometry group is $SO(d, 2)$. The coordinates ($z \in \mathbb{R}^+, x^\mu \in \mathbb{R}$), with $\mu = 1, \ldots, d$, can parametrize this submanifold in the following way:

$$\begin{aligned} X^0 &= \frac{z}{2}\left(1 - \frac{\eta_{\mu\nu}x^\mu x^\nu - L^2}{z^2}\right), \quad X^i = \frac{x^i}{z}L, \\ X^d &= \frac{z}{2}\left(1 - \frac{\eta_{\mu\nu}x^\mu x^\nu + L^2}{z^2}\right), \quad X^{d+1} = \frac{x^d}{z}L. \end{aligned} \tag{4.7}$$

with a inherited metric

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right). \tag{4.8}$$

Notice that these are the coordinates used in (4.4), and are the Poincaré patch coordinates. They only parametrize one half of the total $AdS_{d+1}$, known as Poincaré patch: if $X^N$ is in the covered piece, $-X^N$, which trivially also belong to the $AdS_{d+1}$, is not. Other set of coordinates that do cover the complete $AdS_{d+1}$ are the global coordinates:

$$\begin{aligned} X^0 &= L\cosh\rho\cos\tau, \\ X^{d+1} &= L\cosh\rho\sin\tau, \\ X^i &= L\,\Omega^i\sinh\rho, \qquad i = 1, \ldots, d, \end{aligned} \tag{4.9}$$

where $\sum_{i=1}^d \Omega_i^2 = 1$, $-\pi < \tau \leq \pi$ and $\rho > 0$. The metric is written as

$$ds^2 = L^2\left(\cosh^2\rho\, d\tau^2 - d\rho^2 - \sinh^2\rho\, d\Omega_{d-1}^2\right), \tag{4.10}$$

where $d\Omega_{d-1}^2$ is the volume element of the $S^{d-1}$ sphere. Under a Weyl rescaling of the metric

$$d\tilde{s}^2 = ds^2/\left(L^2\cosh^2\rho\right), \tag{4.11}$$

and a redefinition of the coordinate $d\tilde{\rho} = d\rho/\cosh\rho$, we see how the $AdS_{d+1}$ space is conformally equivalent to a warped solid $(d+1)$-cylinder of height $2\pi$ and radius $\pi/2$:

$$d\tilde{s}^2 = d\tau^2 - d\tilde{\rho}^2 - \sin^2\tilde{\rho}\; d\Omega_{d-1}^2. \tag{4.12}$$

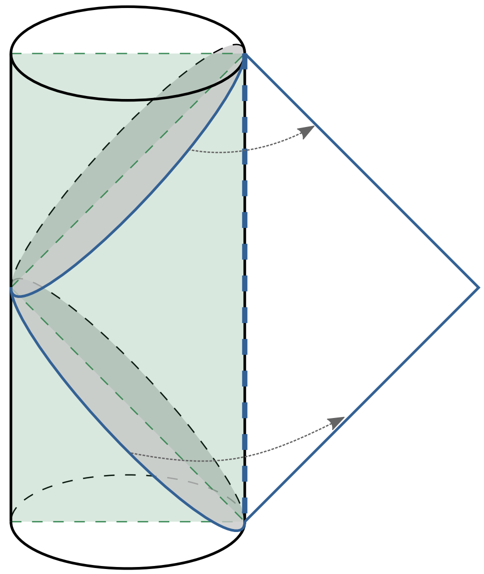

Figure 4.1: $(d+1)$-Lorentzian anti-de Sitter space. At every point there is a $(d-2)$-sphere. On the green vertical plane, the spheres collapse to a point. Also, we have to identify every point of the spheres with its antipode in the sphere which is located symmetrically under the green plane. The two grey oblique planes delimit the Poincaré patch. Its conformal boundary can be understood as the Penrose diagram of the Minkowski space.

It is represented in Figure 4.1. We have only drawn a $S^1$ component of the full $S^{d-1}$ to make the cylinder representable.[10] Let $\psi \in ]-\pi, \pi]$ be the polar angle of the cylinder with origin in the blue dashed line. Every point in the picture represents a $(d-2)$-sphere of radius $\sin^2 \tilde{\rho} \sin \psi$ (over the green vertical plane, they collapse to a point). Additionally, we have to identify every point of the spheres with its antipode in the sphere located symmetrically under the green plane.

Notice also that due to the periodicity of $\tau$, the upper and lower faces are connected with periodic boundary conditions.[11]

[10] A description of the $S^{d-1}$ of radius $R$ can be done in the following way. We start with a $S^1$ of radius $R$, parametrized with the angle $\psi \in ]-\pi, \pi]$, times a $(d-2)$-sphere. Then, we make the radius of the $(d-2)$-sphere to be $R \sin \psi$. Notice that at $\psi = 0$ and $\psi = \pi$, the $S^{d-2}$ shrinks to a point. If we identify the points $(\psi, v) \sim (-\psi, -v)$, being $-v$ the antipode of $v \in S^{d-2}$, we obtain the $S^{d-1}$.

[11] The $AdS_{d+1}$ is not simply connected and has closed timelike curves. One can also work with the universal covering space, which removes closed timelike curves and is obtained decompactifying $\tau$. It is thus given by a cylinder with infinite height.

The $AdS_{d+1}$ space does not have any geometric boundary. However, the Weyl rescaling of (4.11) shows it does have a conformal boundary $S^{d-1} \times S^1$. In this boundary, the AdS metric induces a specific conformal class we analyse below.[12] Furthermore, an isometry in $AdS_{d+1}$ induces a conformal transformation in the boundary.

The Poincaré patch described by (4.7) is the open set comprised between the two grey oblique planes of Figure 4.1. The conformal boundary of such region is a $(d-2)$-sphere times the half square given by $\rho = \pi/2$, $0 < \psi < \pi$, and $|\tau| < \pi - \psi$ (see Figure 4.1). The coordinates $(\tau, \psi)$ can be replaced by

$$r = \frac{\sin(\psi)}{\cos(\tau) + \cos(\psi)}, \quad t = \frac{\sin(\tau)}{\cos(\tau) + \cos(\psi)}, \tag{4.13}$$

with $r \geq 0$ and $t \in \mathbb{R}$. The inherited metric from (4.12) is then

$$d\tilde{s}^2\big|_{\text{boundary}} = \frac{4}{[1 + (t-r)^2]\,[1 + (t+r)^2]} \left(dt^2 - dr^2 - r^2 d\Omega_{d-2}^2\right). \tag{4.14}$$

Therefore, we obtain the conformal class of the $d$-dimensional Minkowski space.

In the Poincaré patch coordinates, the conformal boundary is reached when $z \to 0$. From (4.8), it is manifest that it inherits the conformal structure of the Minkowski space.

The set of conformal transformations of the $d$-dimensional Minkowski has the structure of $SO(d,2)$, exactly as the connected component of the isometry group of $AdS_{d+1}$. This shows the fundamental relation between conformallity of $d$-dimensional Minkowski space and geometry of $AdS_{d+1}$. The first entry of the holographic dictionary is thus that isometries in the $AdS_{d+1}$ are in one-to-one correspondence with conformal transformations in the field theory. It suggests that the field theory can be thought as living in the $AdS_{d+1}$ boundary. Also, we will refer to the gravity theory as living in the "bulk". We thus find the holographic principle in all its glory.

In this thesis we will mainly work in Euclidean time, after a Wick rotation. The Euclidean version of $AdS_{d+1}$ is given by the hyperbolic space $H^{d+1}$ (in the rest of the thesis, we will call it $AdS_{d+1}$ too). To define it, we can perform a Wick rotation in the

[12]In general, any submanifold of a given pseudo-Riemannian manifold inherits a metric. In this case, since we first have to "bring the boundary from infinity" with a Weyl transformation, the inherited metric will be only defined up to Weyl transformations.

$d+2$ flat space where $AdS_{d+1}$ can be embedded, $X^{d+1} \to iX^{d+1}$. The metric becomes $\tilde{\eta} = \text{diag}(+,-,-,\ldots,-,-)$, and thus, the space $\tilde{\eta}_{MN}X^M X^N = L^2$ is Riemannian. The connected component of the group of isometries in this case is thus $SO(d+1,1)$. A possible parametrization is given by $(y^A \in \mathbb{R})$, with $A = 1,\ldots,d+1$ with the restriction $\sum_A \left(y^A\right)^2 < 1$ :

$$X^0 = L\frac{1+\sum_A \left(y^A\right)^2}{1-\sum_A \left(y^A\right)^2}, \quad X^A = \frac{2L}{1-\sum_A \left(y^A\right)^2}. \tag{4.15}$$

The Euclidean metric in these coordinates results:

$$ds^2 = \frac{4L^2}{\left[1-\sum_A \left(y^A\right)^2\right]^2}\delta_{AB}dy^A dy^B. \tag{4.16}$$

The hyperbolic space $H^{d+1}$ is thus not only topologically equivalent to a $d+1$ solid ball (without boundaries) but also conformally equivalent. The conformal boundary of $H^{d+1}$ is then $S^d$, which is also conformally equivalent to the compactification of the flat space $\mathbb{R}^d$.

One can also use the Poincaré patch coordinates for $H^{d+1}$. Using the parametrization of (4.7) changing $\eta_{\mu\nu} \to -\delta_{\mu\nu}$, one obtains the Euclidean metric

$$dz^2 = \frac{L^2}{z^2}\left(dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu\right). \tag{4.17}$$

In contrast with the Lorentzian case, in this case, the Poncaré patch coordinates do cover the complete space.[13] The conformal boundary is again located at $z \to 0$. It is clear it inherits the conformal structure of the flat space $\mathbb{R}^d$. The extra point necessary to compactify it to $S^d$ is reached in the $z \to \infty$ limit. As for the Lorentzian case, the conformal group of the flat $\mathbb{R}^d$, $SO(d+1,1)$, agrees with the isometry group of $H^d$.

[13]One should not be surprised of this fact. For instance, in $d$-Minkowski space, the Rindler coordinates cover only the Rindler wedge (a non-dense subset of the full Minkowski space). However their euclidean version (polar coordinates) do cover a dense part of the full flat space $\mathbb{R}^d$.

### 4.2.2 Field/Operator Correspondence

Let us now consider scalar field theories living in AdS spaces. They can be seen as truncations of UV-finite theories with conformal duals. As in the rest of the thesis, we work in the Poincaré patch coordinates. The action is given by

$$S = \int d^{d+1}x^A \sqrt{g} \left[\nabla^A \Phi^i \nabla_A \Phi_i + U\left(\Phi^i\right)\right] + S_{\text{other fields}}, \tag{4.18}$$

with $A = 0, 1, \ldots, d$, $x^A = (z, x^\mu)$, $g_{AB}$ the metric of $AdS_{d+1}$ and the potential,

$$U = u_0 + \frac{u_{(i)}}{2}\Phi^i\Phi^i + \frac{u_{ijk}}{3!}\Phi^i\Phi^j\Phi^k + O(\Phi^4), \tag{4.19}$$

with $u_{(i)} = M^2_{(i)}$ the diagonalized mass of the fields. In [114, 115], the quantization of a scalar field in AdS spaces was studied. The authors found that although the mass of the fields can be negative, only fields with $M^2 > -d^2/4L^2$ are stable and can be quantized preserving the AdS symmetries. This is the known Breitenlohner-Freedman bound. Even more, they proved that when $-d^2/4 < L^2M^2 < -d^2/4+1$ two different boundary conditions in the conformal boundary can be used, giving different quantizations for the same field theory living in the bulk. For $M^2L^2 \geq -d^2/4 + 1$, only one boundary condition can be used. What they did was to search for the boundary conditions which allow to construct Hilbert spaces for the free case with the scalar product

$$\langle \Phi_1, \Phi_2 \rangle = i \int_\Sigma d^d x \sqrt{|g|} g^{dd} \left(\Phi_1^* \partial_d \Phi_2 - \partial_d \Phi_1^* \Phi_2\right), \tag{4.20}$$

and that keep the energy finite. In (4.20), $\Sigma$ is a spacelike slice.

The equation of motion is

$$z^{d+1}\frac{\partial}{\partial z}\left[z^{-d+1}\frac{\partial}{\partial z}\Phi^i\right] + \left(g^{\mu\nu}k_\mu k_\nu - m^2_{(i)}\right)\Phi^i = \frac{\partial}{\partial \Phi^i}U_{\text{int}}\left(\Phi^i\right), \tag{4.21}$$

with $k_\mu = i\partial_\mu$, $m^2_{(i)} = L^2 M^2_{(i)}$ and $U_{\text{int}}$ the cubic and higher orders of the potential. Let us restrict to the non-interacting case. The asymptotic form of the solution close to the boundary is

$$\Phi^i = z^{\Delta^-_{(i)}}\Big\{\Phi^i_{(0)}(x) + z^2\Phi^i_{(2)}(x) + \dots + z^{2\nu_{(i)}}\Big[\tilde{\Phi}^i_{(2\nu)}(x) + z^2\tilde{\Phi}^i_{(2\nu+2)}(x) + \dots$$
$$+2\log z\left(\Phi^i_{(2\nu)}(x) + z^2\Phi^i_{(2\nu+2)}(x) + \dots\right)\Big]\Big\}, \qquad (4.22)$$

where

$$\nu_{(i)} = \sqrt{\frac{d^2}{4} + m^2_{(i)}}, \qquad \Delta^\pm_{(i)} = \frac{d}{2} \pm \nu_{(i)}, \qquad (4.23)$$

and the ellipsis are generically an infinite tower of higher order terms following the same pattern. The logarithmic terms appear only if $\nu_{(i)} \in \mathbb{N}_0$. In that case, every $\Phi^i_{(2\nu+2n)}(x)$, with $n \in \mathbb{N}_0$, will be accompanied by $\log z$. If $\nu = 0$, there are no terms $\Phi^i_{(n)}$ without $\log z$.

The function $\Phi^i_{(0)}(x)$, with the equations of motion determine all the remaining $\Phi^i_{(n)}$ functions. The boundary condition valid for all the range $m^2_{(i)} > -d^2/4$ and preserving the AdS symmetries consists in fixing $\Phi^i_{(0)}(x) = 0$. Then, the whole tower of functions that depend on it vanish. The remaining modes are normalizable under (4.20). This is the *standard* boundary condition. The boundary condition that only applies if $-d^2/4 < m^2_{(i)} < -d^2/4 + 1$ fixes $\tilde{\Phi}^i_{(2\nu)}(x) = 0$ (it fixes also the tower of $\tilde{\Phi}^i_{(n)}(x)$ to zero). In this case, the remaining modes are also normalizable under (4.20). This is the *alternate* boundary condition.[14] As it is emphasized in [116], when a scalar has a mass in the $-d^2/4 < m^2_{(i)} < -d^2/4 + 1$ range, two different quantizations are possible, and thus, two different dual conformal field theories exist, but typically, only one is supersymmetric.

In the following, we restrict ourselves to the standard quantization case, and postpone for later this discussion for the alternate quantization. One can wonder what is the dual (if it exists) of a bulk theory with different boundary conditions. This is, if one can fix the non-normalizable modes to some non-vanishing value in the boundary, and obtain something physical in the dual quantum field theory. Taking into account that such a boundary condition would break the AdS symmetries (and thus, the conformal symmetry of the quantum field theory), and that the field theory can be thought as

[14] See [116] for a different argument with the same conclusion.

living in the conformal boundary, [117, 118] proposed to interpret the non-normalizable modes as sources of operators of the conformal theory. This is, a theory in the AdS-space with a boundary condition fixing the non-normalizable mode to be some function $J(x)$ would be dual to the conformal theory deformed with a source $J(x)$ coupled to some operator $\mathcal{O}_i(x)$ whose form we will specify below.[15]

For the standard case, the central formula capturing this relation is

$$\langle e^{-N^2 \int d^d x J^i(x) \mathcal{O}_i(x)} \rangle_{CFT} = \int_{\Phi^i_{(0)}=J^i} \mathcal{D}\Phi \, e^{-\kappa^{-2} S^G[\Phi]}, \tag{4.24}$$

where $\Phi$ stands for all the fields living in the gravity side and $\kappa$ is related to the Newton constant and $g_s$, $\kappa^2 \sim G_N \sim g_s^{-2}$. In the $\kappa \to \infty$ limit, the path integral of the right hand side of (4.24) is dominated by the classical solution, and can be approximated by the exponential of the on shell action. From (4.5) and (4.6), this limit corresponds to the large $N$ limit of the conformal theory and thus it can be used to compute the generating functional of the dual field theory.

This gives us other entry of the dictionary: there is a correspondence between fields in the gravity side and operators of the conformal dual. Since gauge symmetries are redundancies of the description, physical operators of the conformal theory have to be neutral under gauge charges. We will assume that the conformal theory is described by a matrix theory, i.e. a theory with all the fields in adjoint representations of the gauge group.[16] In Sections 2.4.3 and 3.1.6 we have reviewed many properties of these theories from the Wilsonian point of view. In section 4.3 we will see how to introduce insertions of multi-trace operators generalizing the boundary conditions. Insertions of descendant operators can be achieved by means of derivatives of the sources. Therefore, only primary single-trace operators are in one-to-one correspondence with bulk fields.

The conformal dimension of the operator is related to the mass of the dual field. Dilatations are part of the conformal group: $x^\mu \to \lambda x^\mu$. In Poincaré patch coordinates,

[15] Actually, in [117, 118], they only work in the Euclidean version of the correspondence and with the standard quantization. The correct understanding of the division between modes and the different forms of quantization was developed in [116, 119].

[16] This is the case of $\mathcal{N} = 4$ $SYM$. There are however other examples of the duality with vector representations [86, 87]. The large $N$ normalization done here can be easily generalized to other cases.

the associated isometry is particularly simple:

$$z \to \lambda z, \qquad x^\mu \to \lambda x^\mu. \tag{4.25}$$

Given the behaviour of the source of an operator $\mathcal{O}_i(x)$ with the $z$ coordinate in (4.22), the actuation of a infinitesimal dilatation $D$ will be then

$$[D, \mathcal{O}_i(0)] = -i\Delta^+_{(i)}\mathcal{O}_i(0), \tag{4.26}$$

Thus, the conformal dimension of a scalar single-trace operator dual to the field $\Phi^i$ is $\Delta^+_{(i)}$ (see (4.23)).

### 4.2.3 Vector Fields and Backreaction

Of course, not only scalar fields are dual to operators. Every field of any spin has a corresponding dual operator. For example, a vector gauge field in the bulk will be dual to a conserved current in the boundary. As every conserved current, it will generate a global symmetry of the boundary theory. This global symmetry corresponds to the large gauge transformations associated with the original gauge symmetry, that are actual symmetries of the bulk theory. In any case, we have found another entry of the dictionary: global symmetries in the boundary are associated with gauge symmetries in the bulk. See [120] for a precise discussion of the normalizable and non-normalizable modes of vector fields in AdS spaces.

Also, the metric field in the bulk, which is a dynamical field, has a dual operator: the energy-momentum tensor of the dual theory $T_{\mu\nu}$. Any complete description has to take into account back-reaction of the metric described by the Einstein-Hilbert action. This of course deforms the AdS-space. However, if the only activated sources are those associated with fields going fast enough to zero at the boundary (for example, negative mass scalar fields), the space will stay asymptotically AdS (AAdS). An AAdS-space can be defined as any space with the same conformal boundary as the AdS space, both in the Lorentzian and Euclidean versions [103]. Its metric can be expanded as [121, 122]:

$$\frac{ds^2}{L^2} = \frac{dz^2}{z^2} + \gamma_{\mu\nu}(x, z)dx^\mu dx^\nu, \tag{4.27}$$

with

$$\gamma_{\mu\nu}(x,z)=\frac{1}{z^2}\left[\gamma^{(0)}_{\mu\nu}(x)+z^2\gamma^{(2)}_{\mu\nu}(x)+\cdots+z^d\log z\gamma^{(d)}_{\mu\nu}(x)+z^d\tilde{\gamma}^{(d)}_{\mu\nu}(x)+O(z^{d+1})\right]. \tag{4.28}$$

The logarithmic term only appear when the $d$ dimension is even. Notice how the field $\gamma^{(0)}_{\mu\nu}(x)$ is the metric of the conformal dual theory: a conformal transformation converts the conformal boundary into a geometric boundary with $\gamma^{(0)}_{\mu\nu}(x)$ as inherited metric. This is the source of the energy-momentum tensor of the dual conformal theory. Analogously to the scalar case, fixing $\gamma^{(0)}_{\mu\nu}(x)$ and using the equation of motion (Einstein equations), one can obtain all the coefficient $\gamma^{(2)}_{\mu\nu}(x)$, $\gamma^{(4)}_{\mu\nu}(x)$, ... up to $\gamma^{(d)}_{\mu\nu}(x)$. They depend locally on $\gamma^{(0)}_{\mu\nu}(x)$. In this case, part of $\tilde{\gamma}^{(d)}_{\mu\nu}(x)$ is also fixed: the trace $\gamma^{(0)\,\mu\nu}(x)\tilde{\gamma}^{(d)}_{\mu\nu}(x)$ and the divergence $\nabla^{\mu}_{(0)}\tilde{\gamma}^{(d)}_{\mu\nu}$ (where $\nabla_{(0)}$ is the Levi-Civita connection of $\gamma^{(0)}$) are determined by $\gamma^{(0)}$. The other components are not fixed by the near boundary analysis, and they can only be fixed after solving the full equation of motions in the whole space. If the dimension $d$ is odd,

$$\begin{aligned}\gamma^{(0)\,\mu\nu}(x)\tilde{\gamma}^{(d)}_{\mu\nu}(x)&=0,\\ \nabla^{\mu}_{(0)}\tilde{\gamma}^{(d)}_{\mu\nu}&=0.\end{aligned} \tag{4.29}$$

See [123] for a detailed near boundary analysis.

## 4.3 Holographic Renormalization

There is something important that remains to be said. If one tries to use (4.24) just like it is to make actual calculations, one will run into problems soon. In fact, both the left hand side and the right hand side of (4.24) are ill-defined. The left hand side contains UV divergences characteristic of any continuum quantum field theory (as we discussed in section 2). The right hand side has divergences coming from the infinite volume integration of the AdS-space close to the boundary. Both divergences are in fact related [124]. Thus, we will call UV zone the near boundary region of the AdS (i.e. the zone where $z$ approaches zero).

It is known what to do in the quantum field theory side: first, the theory has to be regularized with a cutoff in a consistent way, and then, the cutoff has to be taken to the infinity with a well-defined renormalization procedure. The gravity side requires the same treatment.

There are different ways to treat these divergences carefully.[17] For instance, in [125], this was done regulating the AdS and removing the region $z < \epsilon$ for some $\epsilon > 0$ in the Poincaré patch coordinates.[18] Then, one can perform computations in a manifold with a geometric boundary (and not a conformal boundary) at $z = \epsilon$. Boundary conditions are therefore imposed at this geometric boundary. After all computations are done, one can take the $\epsilon \to 0$ limit removing local divergences if necessary, to isolate the finite and physical answers.

Another method is the one used in the largely-studied holographic renormalization program [126]. In this section, we will introduce and summarize the main ideas and concepts related to it that are necessary for the understanding of the present thesis.

The essential idea is the following one. One keeps the boundary conditions of (4.31) in the far UV, but also performs bulk integrations only in the region $z > \epsilon$. The limit $\epsilon \to 0$ is of course divergent, but one can add suitable local counterterms at $z = \epsilon$ to achieve a finite limit. The counterterm action

$$S_{ct}(\epsilon; \varphi, \dots) = \int d^d x \, \mathcal{S}_{ct}(\epsilon; \varphi^i(x), \partial\varphi^i(x), \dots), \tag{4.30}$$

is chosen to cancel all the infinities. It can depend on $\epsilon$ and is covariant and local in its arguments, which are the bulk fields localized at $z = \epsilon$. We represent them generically by $\varphi$. Locality in this context means $\mathcal{S}_{ct}$ depends only on the fields and a finite number of their derivatives evaluated on the point $x$.

Thus, the renormalized version of (4.24) is

$$\langle e^{N^2 \int d^d x J^i \mathcal{O}_i} \rangle_{CFT}^{ren} = \lim_{\epsilon \to 0} \int_{\Phi^i_{(0)} = J^i} \mathcal{D}\Phi \, \exp\left\{ -\kappa^{-2} \left[ \int_{z > \epsilon} dz \, d^d x \, \mathcal{L}^G + \int d^d x \, \mathcal{S}_{ct}(\epsilon; \Phi(\epsilon, x)) \right] \right\}, \tag{4.31}$$

[17] In fact, one of the purposes of this thesis is to study different treatments in light of the Wilsonian RG.

[18] Notice this breaks the isometries of the bulk associated with the dilatation in the boundary (and also special conformal transformations), but keep conserved those associated with the Poincaré transformation in the boundary.

where $\Phi^i$ not only represent the scalar fields, but all kind of fields (vector fields, metric, etc. . . ). The main aim of the holographic renormalization program is thus to find the required counterterm action.

### 4.3.1 Standard Quantization

The holographic renormalization method for the standard quantization was systematically defined in [123]. Before that, it had been already used in [126] to compute the Weyl anomaly in gravity duals. It is also used in [127, 128] to systematically compute the correlation functions for domain wall holographic solutions. See also [129] for a pedagogical introduction.

The spirit of the holographic renormalization program is to find the counterterm action using the asymptotic behaviour of fields and equation of motion. In fact, this asymptotic behaviour is enough to calculate the required counterterms, and there is no necessity of solving the full system, which would make the problem intractable.

For now, we will restrict to relevant and marginal operators, i.e., operators with dimension $\Delta \leq d$. For scalar fields, this condition is translated to considering only fields with negative or vanishing mass.

From the equation of motion, we already noticed that the asymptotic form of a free scalar field is of the form of (4.22). In general, for any field, from the equations of motion, one can find the asymptotic behaviour depending locally on two fields which we will call $\Phi_{(0)}(x)$ and $\tilde{\Phi}_{(2\nu)}(x)$ in analogy to the scalar field case. One can then find the divergent terms of the on shell action as $\epsilon$ goes to zero without knowledge of the IR region as function of $\Phi^i_{(0)}(x)$ and $\tilde{\Phi}^i_{(2\nu)}(x)$. This will be $S_{div}[\epsilon; \Phi^i_{(0)}(x), \tilde{\Phi}^i_{(2\nu)}(x)]$. Inverting (4.22), one can write $S_{div}$ as function of $\varphi^i(\epsilon, x)$. In principle, it should depend also on $\tilde{\Phi}^i_{(2\nu)}(x)$, but it contributes only to finite terms, so its dependence can be discarded. One finally arrives to the covariant couterterms:

$$S_{ct}(\epsilon; \varphi^i(x)) = -S_{div}[\epsilon; \varphi^i(x)]. \tag{4.32}$$

With this procedure, one can easily find that for a free massive scalar without backreaction, with $\nu \notin \mathbb{N}_0$, the required counterterms are

$$\begin{aligned} S_{ct}[\gamma,\varphi] &= -\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\sqrt{\gamma}\,\varphi(-q)\left[\frac{d}{2}+q_\gamma\frac{I'_{-\nu}(q_\gamma)}{I_{-\nu}(q_\gamma)}\right]_{\text{local}}\varphi(q) \\ &= -\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\sqrt{\gamma}\,\varphi(-q)\left[\Delta^- + \frac{q_\gamma^2}{2-2\nu}+\dots\right]\varphi(q), \end{aligned} \tag{4.33}$$

where $q_\gamma = \sqrt{\gamma^{\mu\nu}q_\mu q_\nu}$, and the subscript "local" indicates a truncation of irrelevant terms that go to zero in the limit, and thus, they can be discarded to make the expression local. In this case, irrelevant terms are those order $O(q_\gamma^n)$ with $n > 2\nu_{(i)}$. Notice how in this case $S_{ct}$ does not depend explicitly on $\epsilon$. For $\nu \in \mathbb{N}^+$ the expression is

$$\begin{aligned} S_{ct}[\epsilon;\gamma,\varphi] &= -\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\sqrt{\gamma}\,\varphi(-q)\left[\frac{d}{2}+q_\gamma\frac{K'_{\nu}(q_\gamma)}{K_{\nu}(q_\gamma)}\bigg|_{\text{local}} + a_\nu\, q_\gamma^{2\nu}\log\frac{\epsilon}{\mu}\right]\varphi(q) \\ &= -\frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\sqrt{\gamma}\,\varphi(-q)\left[\Delta^- + \frac{q_\gamma^2}{2-2\nu}+\dots + a_\nu\, q_\gamma^{2\nu}\log\frac{\epsilon}{\mu}\right]\varphi(q), \end{aligned} \tag{4.34}$$

where

$$a_\nu = \frac{(-1)^\nu}{2^{2\nu-2}\Gamma(\nu)^2}. \tag{4.35}$$

In this case, "local" stands for the terms with lower powers in $q_\gamma$: $q_\gamma^n$ with $n < 2\nu$. In fact, starting in $q_\gamma^{2\nu}$, divergent terms containing $\log q_\gamma$ appear too, but are discarded.

### 4.3.2 Hamiltonian Renormalization

Although the method explained above is well-defined and can be used systematically, it becomes quickly tedious as soon as one increases a bit the complexity of the system. The inversion of the asymptotic series can be highly non-trivial. An alternative method, known as Hamiltonian renormalization, has been developed giving same results [130, 131]. Let $S_{reg}(\epsilon;\Phi)$ be the on shell action integrated until $z=\epsilon$ with boundary conditions on $\epsilon$ given by $\Phi(x)$. Making variations with respect to the boundary condition one finds

$$\delta S_{reg} = \int d^dx\,\Pi_\Phi\,\delta\Phi, \tag{4.36}$$

where $\Pi_\Phi$ is the canonical radial momentum associated with the field $\Phi$. If we want just to solve the regulated problem without changing $\epsilon$, a Dirichlet condition imposed on $\Phi$ is a well defined boundary condition $\delta\Phi = 0$. However, as remarked in [132], the integrand in this equation does not have a well defined transformation under shifts in the radial coordinate $z$. Since we want to send $\epsilon$ to zero, the naive Dirichlet condition on $\Phi$ gives an ill-defined and divergent result.

To modify (4.36) in a covariant way without changing the bulk dynamics, one needs to introduce the counter-term action:

$$S_{ren} = S_{reg} + S_{ct}. \tag{4.37}$$

To split the regulated action between the renormalized action $S_{ren}$ and the counter-term action, it is useful to introduce the dilatation operator $\delta_D$. It acts over any functional of $\Phi(x)$ as

$$\delta_D = \int d^dx 2\gamma_{\mu\nu}\frac{\delta}{\delta\gamma_{\mu\nu}} - \int d^dx \Delta^-_{(i)}\Phi^i\frac{\delta}{\delta\Phi^i}, \tag{4.38}$$

where we have assumed that the only fields are scalars and the metric. For the sake of simplicity, we restrict the analysis to the scalar sector. The treatment with the metric is similar. Using the asymptotic expansion of the fields of (4.22) and (4.28) one can expand $\Pi_\Phi(\epsilon;\Phi)$ in eigenstates of $\delta_D$. The point of doing so is that each eigenstate does have a well-defined behaviour under shifts in the radial coordinate $z$. Then,

$$\Pi_\Phi = \sqrt{\gamma}\left(\Pi_\Phi^{(\Delta^-)} + \Pi_\Phi^{(\Delta^-+2)} + \cdots + 2\log\epsilon\,\Pi_\Phi^{(\Delta^+)} + \tilde{\Pi}_\Phi^{(\Delta^+)} + \dots\right), \tag{4.39}$$

where

$$\begin{aligned}
&\delta_D\Pi_\Phi^{(\Delta^-)} = -\Delta^-\Pi_\Phi^{(\Delta^-)}, \quad \delta_D\Pi_\Phi^{(\Delta^-+2)} = -(\Delta^-+2)\Pi_\Phi^{(\Delta^-+2)}, \quad \dots \\
&\delta_D\Pi_\Phi^{(\Delta^+)} = -\Delta^+\Pi_\Phi^{(\Delta^+)}, \quad \delta_D\tilde{\Pi}_\Phi^{(\Delta^+)} = -\Delta^+\tilde{\Pi}_\Phi^{(\Delta^+)} - 2\Pi_\Phi^{(\Delta^+)}.
\end{aligned} \tag{4.40}$$

The logarithmic term only appears if $(\Delta^+ - \Delta^-)/2 = \nu \in \mathbb{N}_0$. Also, functionals $\Pi_\Phi^{(n)}$ with $n \leq \Delta^+$ are local in $\Phi$, and can be found without requiring regularity in the deep interior of the bulk. $\tilde{\Pi}_\Phi^{(\Delta^+)}$ is the first term with non-trivial dependence on the bulk interior dynamics. It is thus natural to require

$$\delta(S_{reg} + S_{ct}) = \int d^d x \sqrt{\gamma}\, \Pi^R_\Phi \,\delta\Phi = \int d^d x \sqrt{\gamma} \left[\tilde{\Pi}^{(\Delta^+)}_\Phi + \dots\right] \delta\Phi, \tag{4.41}$$

where $\Pi^R_\Phi$ is the called renormalized momentum, and the ellipsis represents vanishing terms in the $\epsilon \to 0$ limit. This is achieved imposing

$$-\,\delta S_{ct} = \int d^d x \sqrt{\gamma} \left[\sum_{n<\Delta^+} \Pi^{(n)}_\Phi + 2\log\epsilon\, \Pi^{(\Delta^+)}_\Phi\right] \delta\Phi. \tag{4.42}$$

Alternatively, we can expand $S_{reg}$ in eigenstates of $\delta_D$:

$$\begin{aligned} S_{reg} &= S^{(n)}_{reg} + S^{(n+2)}_{reg} + \dots + 2\log\epsilon\, S^{(0)}_{reg} + \tilde{S}^{(0)}_{reg} + \dots \\ \delta_D S^{(n)}_{reg} &= -n S^{(n)}_{reg}, \;\; \dots,\; \delta_D S^{(0)}_{reg} = 0, \;\; \delta_D \tilde{S}^{(0)}_{reg} = -2 S^{(0)}_{reg}. \end{aligned} \tag{4.43}$$

The value of $n$ depends on the field content. Thus, from (4.36), a scalar field $\Phi$ with dimension $\Delta^+ = \nu + d/2$ gives contributions $n = -2\nu, -2\nu + 2, \dots$ Also, if we are considering dynamical gravity in the bulk, there will be a contributions with $n = -d, -d + 2 \dots$ As for the momentum expansion, terms $S^{(n)}_{reg}$ with $n \leq 0$ are local in their dependence on $\Phi$ and can be computed from the asymptotic expansion of the fields. From (4.42), they give precisely the counterterm action:

$$S_{ct} = -\left(\sum_{n<0} S^{(n)}_{reg} + 2\log\epsilon\, S^{(0)}_{reg}\right). \tag{4.44}$$

Also, the renormalized action in the $\epsilon \to 0$ limit is simply $\tilde{S}^{(0)}_{reg}$.

### 4.3.3 Expectation Values

Until now, we have identified the leading term in the asymptotic behaviour $\Phi^i_{(0)}(x)$ as the source $J^i(x)$ of primary single trace operators of the dual theory $\mathcal{O}_i(x)$. In the asymptotic expansion of the fields of (4.22) the source determines all terms but $\tilde{\Phi}^i_{(2\nu)}(x)$ and its tower of terms. To fix this function we need to solve the problem in the whole space requiring regularity in the interior of the AdS.[19] In [133] it was shown it is intimately related with the expectation value of the associated operator $\langle\mathcal{O}_i(x)\rangle$.

The expectation value of an operator dual to the field $\Phi^i$ is given by

$$\langle\mathcal{O}_i(x)\rangle = \lim_{\epsilon\to 0}\frac{1}{\sqrt{|\gamma(\epsilon,x)|}}\frac{\delta}{\delta\Phi^i_{(0)}(x)}\left[\int_{z>\epsilon} dz\, d^dx\, \mathcal{L}^G + \int d^dx\, \mathcal{S}_{ct}(\epsilon;\Phi(\epsilon,x))\right]. \qquad (4.45)$$

Due to (4.41) and the asymptotic behaviour of the fields in (4.22), the functional

$$\hat{\Pi}_i^{(\Delta^+)} = \begin{cases} \lim\limits_{z\to 0} z^{-d/2}(\log z)^{-1}\,\tilde{\Pi}^{(d/2)}_{\Phi^i} & if \;\; \text{if } \nu_{(i)} = 0, \\ \lim\limits_{z\to 0} z^{-\Delta^+_{(i)}}\tilde{\Pi}^{(\Delta^+)}_{\Phi^i} & if \;\; \text{if } \nu_{(i)} > 0 \end{cases} \qquad (4.46)$$

is the expectation value $\langle\mathcal{O}_i(x)\rangle$ after taking the $\epsilon\to 0$ limit.

As noticed in [132],

$$\hat{\Pi}_i^{(\Delta^+)}(x) = -2\nu_{(i)}\tilde{\Phi}^i_{(2\nu)}(x) + C[\Phi^i_{(0)}](x), \qquad (4.47)$$

where $C$ is a local functional. If $\nu_{(i)} < 1$, it exactly vanishes and the expectation value is given by the normalizable mode $\tilde{\Phi}^i_{(2\nu)}(x)$. In general however, $\tilde{\Phi}^i_{(2\nu)}(x)$ only gives the $\langle\mathcal{O}_i(x)\rangle$ up to a local functional of the source, and $\hat{\Pi}_i^{(\Delta^+)}(x)$ is the actual expectation value.

### 4.3.4 Alternate Quantization

In the subsection 4.2.2 we reviewed how two possible quantizations are possible for scalar fields in a AdS when $-d^2/4 < m^2_{(i)} < -d^2/4 + 1$. However, we have only shown how to deform the theory and make computable calculations for the standard quantization. It is also possible to change the boundary conditions of the alternate theory to achieve deformations. This was first introduced in [116]. However, the complete formulation including holographic renormalization is given in [132].

If in the standard case $\Phi_{(0)}$ and $\hat{\Pi}_\Phi$ are sources and expectation values respectively, the alternate quantization exchanges the roles: $\hat{\Pi}_j$ will be sources and $\Phi^j_{(0)}$ expectation

[19] In a Lorentzian AdS, one also have to specify additional boundary conditions associated with the correct choice of the state [119].

values. To achieve this, one has to add a suitable term $S_J$ to the action, function of the asymptotic modes of the fields, $\Phi^j_{(0)}$ and $\hat{\Pi}_j$, such that

$$\lim_{\epsilon\to 0}\delta(S_{reg}+S_{ct}+S_J)=\int d^dx\sqrt{|\gamma^{(0)}(x)|}\,\Phi^j_{(0)}(x)\,\delta\hat{\Pi}_j(x). \tag{4.48}$$

Using (4.41), one finds

$$S_J=\int d^dx\,\sqrt{|\gamma^{(0)}(x)|}\,\Phi^j_{(0)}(x)\,\hat{\Pi}_j(x). \tag{4.49}$$

Then, when some fields are quantized in the alternate way, (4.31) becomes

$$\langle e^{-N^2\int d^dx\left(J^i\mathcal{O}_i+J^-_j\mathcal{O}^j_-\right)}\rangle^{ren}_{CFT}=\lim_{\epsilon\to 0}\int_{\substack{\Phi^i_0=J^i\\ \hat{\Pi}_j=J^-_j}}\mathcal{D}\Phi\,\exp\Bigg\{-\kappa^{-2}\Bigg[S_{reg}(\epsilon;\Phi(\epsilon,x))$$
$$+S_{ct}(\epsilon;\Phi(\epsilon,x))+\int d^dx\sqrt{|\gamma^{(0)}(x)|}\,\Phi^j_{(0)}(x)\,\hat{\Pi}_j(x)\Bigg]\Bigg\}, \tag{4.50}$$

where the label $i$ runs over fields quantized in the standard way and $j$ over fields quantized in the alternate way. Due to the asymptotic behaviour of $\hat{\Pi}_i$, the dimension of the dual operator in the alternate quantization is $\Delta^-_{(i)}$. Notice how the allowed values for the mass $-d^2/4<m^2_{(i)}<-d^2/4+1$ of this quantization implies $\Delta^-_{(i)}>d/2-1$. This is exactly the unitary bound for the dimension of operators in a unitary theory.

Comparing (4.50) and (4.31) one finds easily that the relation between both quantizations in the large $N$ limit is a Legendre transform:

$$\langle e^{-N^2\int d^dxJ^-_j\mathcal{O}^j_-}\rangle^{ren}_{CFT}=\int\mathcal{D}J\,e^{-\kappa^{-2}\int d^dxJ^i(x)J^-_i(x)}\langle e^{-N^2\int d^dxJ_i\mathcal{O}^i}\rangle^{ren}_{CFT}. \tag{4.51}$$

### 4.3.5 Multi-trace Deformations

In Section 3.1.6, we have found how (3.80) gives a natural generalization of (4.31) and (4.50) for general deformations including multi-trace operators given by $G[\mathcal{O}]$. The single-trace generating functional with the modified source can be achieved in the gravity dual with a mixed boundary condition:

$$\alpha^i(x) = \frac{\delta G[\beta]}{\delta \beta_i(x)}, \tag{4.52}$$

where now $\alpha^i(x)$ is the source of the operator $\mathcal{O}_i(x)$: $\Phi^i_{(0)}(x)$ ($\hat{\Pi}_i(x)$) for the standard (alternate) quantization and $\beta^i(x)$ the expectation value of $\langle \mathcal{O}_i(x) \rangle$: $\hat{\Pi}_i(x)$ ($\Phi^i_{(0)}(x)$) for the standard (alternate) quantization. The use of mixed boundary conditions to add multi-trace deformations to the theory was proposed in [46].[20]

The additional term which appears in (3.80) can be motivated from the gravity side: in order to compute correctly the generating functional, one has to add a suitable $S_J$. As we did for the alternate quantization, it is necessary to get

$$\lim_{\epsilon \to 0} \delta(S_{reg} + S_{ct} + S_J) = \int d^d x \sqrt{|\gamma^{(0)}(x)|}\, B(\Phi_{(0)}, \hat{\Pi}_\Phi)\, \delta \left[ \alpha^i(x) - \frac{\delta G[\beta]}{\delta \beta_i(x)} \right], \tag{4.53}$$

being $B$ some local functional of $\Phi^i_{(0)}$ and $\hat{\Pi}_i$. This is achieved with

$$S_J = \int d^d x \sqrt{|\gamma^{(0)}(x)|} \left\{ \mathcal{G}(\beta_i) - \beta_i(x) \frac{\delta G[\beta]}{\delta \beta_i(x)} \right\}, \tag{4.54}$$

where as before, $\beta_i(x)$ is $\hat{\Pi}_i(x)$ ($\Phi^i_{(0)}(x)$) for the standard (alternate) quantization.

### 4.3.6 The Callan-Symanzik Equation and Conformal Anomalies

The Callan-Symanzik equation for a standard renormalized theory (analogous to (3.93)) can be extracted analysing the limit

$$\lim_{\epsilon \to 0} \frac{d}{d\epsilon} (S_{reg} + S_{ct}) = 0. \tag{4.55}$$

This equality is trivial since the limit of the parenthesis must be finite. Notice that, when the counterterm action $S_{ct}$ depends explicitly on $\epsilon$ there will appear an indepen-

[20]Actually, the argument in [46] uses $\tilde{\Phi}^i_{(2\nu)}$ instead of $\hat{\Pi}_i$. However, a well defined renormalization prescription requires to use $\hat{\Pi}_i$ [132].

dent term. This happens in exceptional cases through logarithms (see (4.34)). This term has to be interpreted as a contribution to the conformal anomaly. Therefore, the coefficient of the logarithms appearing in the counterterm action gives the conformal anomaly of the theory.

### 4.3.7 Irrelevant Operators

Irrelevant operators deserve special mention. Fields with positive mass can only be quantized in the standard way (if $d \geq 2$), and are dual to operators with $\Delta > d$. This implies that they are dual to irrelevant operators. Since they blow up in the UV region, solutions where they are activated change radically the geometry near the boundary. From the field theory point of view, the activation of irrelevant operators changes dramatically the UV theory. However, they can always be treated perturbatively, assuming they are only coupled to infinitesimal sources (for example to compute correlation functions).

In [134, 135], it was noticed how when irrelevant operators are present, the prescription given above can fail. In particular, terms in (4.44) can be non-local in $\Phi(\epsilon)$, which seems to destroy fundamental requirements for a quantum field theory to be local. It was proposed then to rewrite the counterterms action as a local functional, not only of $\Phi(\epsilon)$, but also of $\Pi^R_\Phi(\epsilon)$. Now, $S_{ct}$ is written like

$$S_{ct} = S_{ct}^{st}[\Phi(\epsilon)] + S_{ct}^{mt}[\Phi(\epsilon), \Pi^R(\epsilon)], \tag{4.56}$$

where $S_{ct}^{mt}$ depends at least quadratically in $\Pi^R(\epsilon)$ and

$$\Pi^R_\Phi(\epsilon) = \frac{1}{\sqrt{|\gamma(\epsilon)|}} \frac{\delta(S_{reg} + S_{ct}^{st})}{\delta\Phi(\epsilon)}. \tag{4.57}$$

Then, the non-local dependence of $S_{ct}$ with $\Phi(\epsilon)$ would be hidden in the non-local dependence of $\Pi^R_\Phi(\epsilon)$ with $\Phi(\epsilon)$. This additional dependence was interpreted as non-local counterterms due to multitrace operators.

This observation is supported by power counting arguments as were presented at the end of subsection 3.1.3. One example considered in [134] is a classical scalar field with positive squared mass $m^2$ in a fixed AdS background. This is dual to a theory with an operator $\mathcal{O}$ with dimension $\Delta > d$ and eigenvalue $\lambda = d - \Delta < 0$, and a tower of multitrace operators with dimension $n\Delta + m$, with $n \in \mathbb{N}$, $n \geq 2$, counting the times that $\mathcal{O}$ appears and $m \in \mathbb{N}_0$ the number of derivatives. In the calculation of correlation functions with the operator $\mathcal{O}$, when $p$ points approach each other, we require non-linear (or semilocal) counterterms made of operators with an eigenvalue lower or equal than $p\lambda$. This is, we require a multitrace counterterm if

$$p(d - \Delta) \leq d - (n\Delta + m) \Rightarrow \Delta(p - n) \geq d(p - 1) + m, \tag{4.58}$$

for some $n$ and $m$. This condition can only be satisfied if $\Delta > d$. Also, it implies than the $\mathcal{O}^n$ counterterm will appear when the number $p$ of operators $\mathcal{O}$ approaching the same point satisfies

$$p \geq \frac{n\Delta - d}{\Delta - d}, \tag{4.59}$$

which is exactly what the holographic calculations in [134] shows.

All the analyzed examples in [134] give rise to contributions to $S_{ct}$ with quadratic or higher powers in $\Pi^R_\Phi(\epsilon)$ terms, but never with linear terms in $\Pi^R_\Phi(\epsilon)$. While non-linear terms can be explained as multitrace counterterms, the linear ones do not. Thus, it was suggested that this would be the case for any renormalizable theory. In chapter 6 we apply a different renormalization procedure to renormalize correlation functions. In it, we also find similar features in the renormalization of irrelevant operators. With this perspective, these features have a clear and transparent interpretation as multi-trace counterterms.

# Chapter 5

# Holographic Wilsonian Renormalization

*Things that seem incredibly different can really be manifestations of the same underlying phenomena.*

Nima Arkani-Hamed

One of the most basic and interesting features of Gauge/Gravity dualities explained in Chapter 4 is the holographic RG, which relates the radial flow of classical gravity solutions in asymptotically anti-de Sitter spaces and the RG evolution of their field-theory duals in the large-$N$ approximation [124, 136–138]. As we have seen in Section 4.3, the regions near the boundary of the space on which the gravity theory is defined correspond to the UV (ultraviolet) of the field theory, while the deep interior of that space is related to its IR (infrared). As we have emphasized in Chapter 2, on the field theory side the deepest understanding of renormalization and the RG comes from a Wilsonian perspective. Therefore one might hope to understand holography itself at a deeper level through this framework (in the line, for instance, of [20, 99, 139–142]). A number of attempts have been made to formulate the holographic RG in Wilsonian terms, but making this map precise has proved challenging.

A first proposal of a holographic Wilsonian RG was made in [143], with the Wilson action given by the gravity action with an IR boundary cutoff, evaluated on solutions to the bulk equations. The solutions are determined by specific boundary conditions at the UV and IR ends of the space. As nicely explained in [144], this is not yet a truly Wilsonian approach, as this Wilson action depends on physics below the IR cutoff. In [61], it was proposed to use as an effective action the cutoff gravity action evaluated on solutions with given UV conditions and Dirichlet conditions on the IR boundary. This object, which we call boundary action in this thesis, is a functional of the restrictions of the bulk fields to the IR boundary. It only depends on UV data and can be used to calculate observables at large N by integration of the remaining degrees of freedom. The boundary action is the gravity counterpart of the Wilson action in field theory. The RG evolution of the sliding boundary action was studied in [145].

Major progress has been made in [146] and [147]. These works put together and correctly interpreted the previous advancements from the field theory side. Additionally, they showed that the holographic boundary obeys a Hamilton-Jacobi equation that describe its dependence on the position of a sliding cutoff surface. Beyond large N, they obey a Schrödinger equation. This is a holographic formulation of the genuine Wilsonian RG. However, as emphasized in [146], the nature of the boundary cutoff on the field-theory side remains unknown. [146] also conjectures an exact connection between the boundary action and the Wilson action: in the large N limit, the Wilson action reduces to a Legendre transform of the boundary action, and therefore, follows a dual Hamilton-Jacobi equation.

This chapter follows a structure parallel to Chapter 2, and we apply what we learnt there to holographic theories. We explore in greater detail the precise relation between the Wilsonian RG in both sides of the holographic correspondence, in the strong 't Hooft coupling and large N limits. Using the geometric language we have developed in Chapter 2, we will not need to use the proposed relation between Wilsonian and boundary actions of [146]. We find fixed points of the RG/Hamilton-Jacobi evolution for a model of scalar fields living in AdS. Also, we study small deformations of them and single out normal coordinates of the flow, as we did for the Gaussian fixed point in 2.4. Many concepts introduced in Chapter 4 are reviewed in the Wilson approach. The results of this chapter are given in [6, 8].

This chapter is organized as follows. In Section 5.1, the holographic theory space, in analogy to $\mathcal{W}$ of Section 2.1 is presented. Section 5.2 is devoted to the study of the flows, in a similar way to the description of Section 2.2. In Section 5.3.1 the different fixed points of the flow are analysed and classified (standard and alternate quantizations). Normal coordinates around these fixed points are studied in Section 5.3.2. Finally, in Section 5.4, we make some comments on the connection between the formalism of this chapter and Chapter 4.

## 5.1 Holographic Theory Space

First of all, we will fix some notation of this chapter. As we mainly did in Chapter 4, we will continue using the Poincaré patch coordinates to describe $d+1$-dimensional Euclidean AdS (reescaling the metric to be measured in units of AdS radius):

$$ds^2 = \frac{dz^2}{z^2} + h_{\mu\nu}(z) dx^\mu dx^\nu. \tag{5.1}$$

where $h_{\mu\nu}(z) = (\gamma_z)_{\mu\nu} = \delta_{\mu\nu}/z^2$. We will neglect the metric backreaction in this thesis. Also, $\phi$ depicts all the fields living in the AdS. They will be labelled by the flavour indices $i$ and $j$, or $\sigma = ix$ when we include the continuous $d$-dimensional position (the radial coordinate $z$ is treated independently in this notation).

In holographic duals, energy scales in the gauge theory are related to the position in the radial direction of the higher-dimensional gravity dual [124]. In particular, the UV divergences of the field theory manifest in the dual gravity theory as IR divergences in the integration of an infinite volume in the neighbourhood of the AdS boundary.

One possible regularization method (but not the only one) consists in cutting off the region of spacetime near the boundary, i.e. restricting the domain of all fields to values $z \geq \epsilon > 0$ [125]. In the holographic renormalization method explained in Section 4.3 of the previous chapter, boundary conditions are imposed in the far UV. However, in this chapter, we put the boundary conditions at $z = \epsilon$. General consistent boundary conditions can be imposed dynamically by adding a boundary action $s$, which is a quasilocal functional of the fields restricted to the cutoff surface at $z = \epsilon$,

$$s[\varphi] = \int d^d x \, \mathcal{S}(x, \varphi(x), \partial\varphi(x), ...). \tag{5.2}$$

Therefore, the complete dynamic of the system is described by

$$S_{\text{Tot}}[\phi] = N^2 \left[ s[\phi(\epsilon)] + S^G[\phi] \right] , \tag{5.3}$$

where $S^G$ is the classical action of the gravity theory,

$$S^G = \int \frac{dz}{z} \int d^d x \, \mathcal{L}^G, \tag{5.4}$$

and $\phi$ is rescaled in order to write $\mathcal{L}^G$ canonically normalized. Extremizing the action, one obtains the equations of motion of the system. In $z = \epsilon$, they give the boundary condition generated by $s$:

$$\mathbb{BC}_\epsilon := \left\{ \Pi_i(\epsilon, x) = \frac{1}{\sqrt{|\gamma_\epsilon|}} \left. \frac{\delta s[\varphi]}{\delta \varphi^i(x)} \right|_{\varphi = \phi(\epsilon)} \right\}, \tag{5.5}$$

with $\Pi_i$ the canonical momentum in the radial direction (4.36),

$$\Pi_i(z, x) = \frac{1}{\sqrt{|\gamma_z|}} \frac{\partial \mathcal{L}^G}{\partial(z \partial_z \phi^i(z, x))}. \tag{5.6}$$

The set of possible boundary actions $s$ will be called $\mathcal{I}^G$. For a given gravity theory dual to a class of quantum field theories, we define the holographic theory space, $\mathcal{W}^G$, as the set of all pairs $(s, \epsilon)$ of boundary actions and radial cutoffs. For convenience, we work here directly with a length cutoff; the definition of theory-space points of the type defined in Chapter 2 is recovered changing $\epsilon \to 1/\epsilon$.

Similar objects to the ones defined in Chapter 2 can be used now. For instance, we define the holographic quotient space $\mathcal{M}^G = \mathcal{W}^G / \sim$, with $(s, \epsilon) \sim (s^B_t, \epsilon/t)$, being $s^B_t[\varphi] = s^B[D_{t^{-1}} \varphi]$. In analogy to (2.6), the function $S^B_\varphi : \mathcal{W}^G \to \mathbb{R}$ is

$$S^B_\varphi(s, \epsilon) = s[\varphi]. \tag{5.7}$$

Also, for any map $U : \mathcal{W}^G \to X$ (being $X$ any set), we define $U_\epsilon : \mathcal{I}^G \to X$ by $U_\epsilon(s) = U(s, \epsilon)$.

The partition function is a function in $\mathcal{W}^G$ defined by

$$Z(s,\epsilon) = \int [\mathcal{D}\phi]_\epsilon e^{-N^2\left\{s[\phi(\epsilon)]+S^G[\phi]\right\}}, \tag{5.8}$$

where $[\mathcal{D}\phi]_\epsilon$ indicates functional integration in the corresponding fields $\phi$, with support restricted to $z \geq \epsilon$. The spacetime integrals inside the functional integrals are understood to be restricted to the support of the fields. If Gauge/Gravity duality holds at the regularized level, then $Z$ in (5.8) represents the partition function of the gauge theory, with $s$ playing the role of a Wilson action and $1/\epsilon$ giving the scale of the UV cutoff. This statement calls for the following qualifications:

(i) The corresponding cutoff procedure in the gauge theory is unknown, and it is not even clear that it can be formulated in a closed form in terms of the field-theory degrees of freedom.

(ii) The action $s$ is a functional of the gravity degrees of freedom $\varphi = \phi(\epsilon)$, which are associated to single-trace gauge-invariant operators. It is not, however, the same functional that appears in the field-theoretical path-integral. It has been argued in [146], that both Wilson actions are related by an specific integral transform that reduces to a Legendre transform in the classical limit. For our purposes, we only need to assume the existence of a one-to-one map between the gravity and gauge Wilson actions and cutoffs, such that $\mathcal{W}^G$ represents the theory space of the gauge theory. Actually, even though our field-theoretical analysis and language are motivated by Gauge/Gravity duality, most of the developments in part II of the thesis would apply to the gravity theory independently of the very existence of a holographic dual.

To parametrize $\mathcal{W}^G$, we use again an infinite set $\mathcal{C}$ of smooth functions $g^a : \mathbb{R}^d \to \mathbb{R}$, which can be understood as local couplings. A specific parametrization or coordinate system is given by a quasilocal functional $S^B$ of local couplings $g(x) \in \mathcal{C}$ and boundary fields $\varphi(x)$, which defines a chart $c$ in $\mathcal{W}^G$ by

$$\begin{aligned} S^B_\varphi \circ c^{-1}(g,\epsilon) &= S^B_\varphi[\gamma_\epsilon; g] \\ &= \int d^dx \sqrt{|\gamma_\epsilon|}\mathcal{S}^B(\gamma_\epsilon; g(x), \varphi(x), \partial\varphi(x), \ldots). \end{aligned} \tag{5.9}$$

This equation defines the coordinate chart $c : \mathcal{W}^G \to \mathcal{C} \times \mathbb{R}^+$, $c(s, \epsilon) = (g, \epsilon)$. Similar expressions and definitions between (2.11) and (2.14) applies. Additionally, we define $\bar{\epsilon} : \mathcal{W}^G \to \mathbb{R}^+$, $\bar{\epsilon}(s, \epsilon) = \epsilon$, and $\bar{\gamma} : \mathcal{W} \to T^0_2\left(\mathbb{R}^d\right)$, $(s, \epsilon) \mapsto \epsilon^{-2}\delta_{\mu\nu} dx^\mu \otimes dx^\nu$. In terms of it, $2\bar{\gamma}\frac{\partial}{\partial\bar{\gamma}} = -\bar{\epsilon}\partial_{\bar{\epsilon}}$.

It can be checked that the charts (5.9) fulfil a similar scaling relation to (2.17):

$$c^\pi_{\epsilon/t}(s_t) = D_t c^\pi_\epsilon(s). \tag{5.10}$$

## 5.2 Exact Holographic RG Flows

In Chapter 2 we integrated out high energy modes to obtain exact RG flows. In this section we will do the same for holographic theories. As discussed above, the energy cutoff is associated to the radial direction of AdS, and therefore, we will integrate out fluctuations of fields "close" to the boundary.[1]

In this section, we adapt the objects introduced in Section 2.2 of Chapter 2 to the holographic framework.

### 5.2.1 Hamilton-Jacobi Equation

Given a point $(s_0, \epsilon_0) \in \mathcal{W}^G$, we can integrate out the fluctuations of the fields between $z = \epsilon_0$ and $z = \epsilon > \epsilon_0$ to find

$$e^{-N^2 s_{\langle\epsilon\rangle}[\varphi]} = \int [\mathcal{D}\phi]^{\epsilon,\varphi}_{\epsilon_0} e^{-N^2\left\{s_0[\phi(\epsilon_0)] + S^G[\phi]\right\}}, \tag{5.11}$$

Here, $[\mathcal{D}\phi]^{\epsilon,\varphi'}_{\epsilon_0,\varphi}$ indicates functional integration for fields $\phi$ with support in $\epsilon_0 < z < \epsilon$, and Dirichlet boundary conditions $\phi(\epsilon_0) = \varphi$ and $\phi(\epsilon) = \varphi'$. If $\varphi$ ($\varphi'$) is not explicitly written, we understand that the path integration include the fields $\phi(\epsilon_0)$ ($\phi(\epsilon)$), and therefore, the boundary condition is dynamically generated. (5.11) fulfils

$$Z(s_{\langle\epsilon\rangle}, \epsilon) = Z(s_0, \epsilon_0). \tag{5.12}$$

[1] AdS is an homogeneous space and so, it makes no sense the notion of closeness to the boundary. Here, we are referring to the distance in terms of the radial coordinate of some Poincaré patch coordinate system.

The RG flows $f_t : \mathcal{W}^G \to \mathcal{W}^G$, are defined by

$$(s_{\langle \epsilon/t \rangle}, \epsilon/t) = f_t(s_{\langle \epsilon \rangle}, \epsilon), \quad t > 0. \tag{5.13}$$

In terms of them, the holographic RG invariance reads

$$Z \circ f_t = Z. \tag{5.14}$$

The RG flows are generated by beta vector fields, tangent to the corresponding curves. They can be defined by their action on an arbitrary real function $F$ in $\mathcal{W}^G$:

$$\beta F = t\partial_t F \circ f_t|_1 \,. \tag{5.15}$$

We will use the same definitions to simplify the notation that the ones used in Chapter 2. Thus, given a chart $c$, we will write the coordinates of the flows as a functional of the local couplings:

$$f_t^\alpha[\gamma_\epsilon; g] = c^\alpha \circ f_t \circ c^{-1}(g, \epsilon). \tag{5.16}$$

Likewise,the beta vector fields in the corresponding coordinate basis read

$$\begin{aligned} \beta &= \beta^{\tilde{\alpha}} \partial^c_{\tilde{\alpha}} \\ &= \beta^\alpha \partial^c_\alpha + 2\bar{\gamma} \frac{\partial}{\partial \bar{\gamma}}, \end{aligned} \tag{5.17}$$

where the components $\beta^\alpha$ are functions in $\mathcal{W}^G$. We also have

$$\beta^\alpha[\gamma_\epsilon; g] = \beta^\alpha \circ c^{-1}(g, \epsilon), \tag{5.18}$$

which are quasilocal functionals of the local couplings $g$.

The infinitesimal version of (5.14) is the Callan-Symanzik equation

$$\beta Z = 0. \tag{5.19}$$

Using the path integral (5.11), the beta function can be extracted from a Schrödinger equation:

$$\beta\, e^{-N^2 S^B_\varphi} = -N^2 H\left[\bar{\gamma};\varphi, -iN^{-2}\frac{\delta}{\delta\varphi}\right] e^{-N^2 S^B_\varphi}. \tag{5.20}$$

Here, $H$ is the Hamiltonian that generates motion in the radial direction:

$$H\left[\gamma_z;\phi,\Pi\right] = \int d^d x\, \left[z\partial_z\phi^i(z,x)\, \Pi_i(z,x) - \mathcal{L}^G(\gamma_z, \phi(z,x), \partial_z\phi(z,x))\right]. \tag{5.21}$$

In the large N limit, (5.20) takes the form of a Hamilton-Jacobi equation for the boundary action. This can also be seen evaluating the path integral (5.11) at large $N$, i.e. in a saddle point approximation and using (5.15) [146, 147]:

$$\beta S^B_\varphi = H\left[\bar{\gamma};\varphi, \frac{\delta S^B_\varphi}{\delta\varphi}\right]. \tag{5.22}$$

(5.20) and (5.22) are equations between functions in $\mathcal{W}^G$. This is the holographic analogue of the field-theory equations (2.70) or (2.112).

### 5.2.2 Legendre Transformed Actions

The form of the Hamilton equations of any dynamical system is symmetric under the exchange of coordinates and momenta,

$$\begin{aligned} \phi^i &\to \Pi_i, \\ \Pi_i &\to -\phi^i. \end{aligned} \tag{5.23}$$

Indeed, this is a canonical transformation. This fact suggests an approach equivalent to the one developed above, but with $\Pi$ playing the role of $\phi$ and vice versa. This can be achieved defining a new boundary action, function of the canonical momenta. Given a configuration of the canonical momenta of the fields at $\epsilon$, $\pi = \Pi(\epsilon)$, $S_\pi : \mathcal{W}^G \to \mathbb{R}$ is given by the integral transformation,

$$e^{N^2 S_\pi} = \int \mathcal{D}\varphi\, e^{N^2\left[\varphi^\sigma\pi_\sigma - S^B_\varphi\right]}. \tag{5.24}$$

Note that $S^B_\varphi$ should be bounded from below for this definition to make sense.

In the large-$N$/classical-gravity limit, (5.24) reduces to a Legendre-Fenchel transform. For this reason, $S_\pi$ will be called Legendre action. One general property of the Legendre action defined in this manner is that it is convex as a functional of $\pi$. The Legendre-Fenchel transform is not invertible in general, but only when $S^B_\varphi$ is convex in $\varphi$. To be more explicit, when using this transform we will assume that $S^B_\varphi$ is convex.[2] In this case, the Legendre and boundary actions are related by the invertible Legendre transform

$$S_\pi = \pi_\sigma \varphi^\sigma - S^B_\varphi, \qquad \pi_\sigma = \frac{\delta S^B_\varphi}{\delta \varphi^\sigma}. \tag{5.25}$$

Therefore, $S_\pi$ defines a point $s \in \mathcal{I}^G$, and it can be used as fundamental object, instead of $S^B_\varphi$. For instance, the partition function can be defined in terms of $S_\pi$ like

$$Z = \int [\mathcal{D}\phi]_{\bar\epsilon}\, \mathcal{D}\pi\, e^{-N^2\left\{\pi_\sigma\varphi^\sigma - S_\pi + S^G[\phi]\right\}}. \tag{5.26}$$

We will also consider cases with $S[\pi]$ linear in the variables $\pi$, for which (5.25) is singular. In fact, (5.24) gives the linear Legendre action $S_\pi = c^\sigma \pi_\sigma$ when $\exp\{-S^B_\varphi\} = \delta(\varphi - c^\sigma)$, which can be considered as a singular boundary action that imposes a Dirichlet boundary condition.

All the equations above involving $S^B$ can be equivalently formulated in terms of the Legendre action. Using Legendre conjugates, the boundary condition (5.5) reads

$$\mathbb{BC}_\epsilon := \left\{ \phi^i(\epsilon, x) = \frac{1}{\sqrt{|\gamma_\epsilon|}} \left.\frac{\delta S_\pi}{\delta \pi_i(x)}\right|_{\pi(x)=\Pi(\epsilon,x)} \right\}. \tag{5.27}$$

Also, the flowing Legendre action $S_\pi \circ f_t$ obeys a dual Schrödinger equation in general,

$$\beta\, e^{N^2 S_\pi} = -N^2 H\left[\bar\gamma; -iN^{-2}\frac{\delta}{\delta\pi}, \pi\right] e^{N^2 S_\pi}. \tag{5.28}$$

[2] It is of course perfectly possible that these properties hold only in some regions of theory space and/or only when the possible values of $\varphi$ and $\pi$ are restricted. A careful study of these basic issues would be interesting, but we will not pursue this course here. We simply note in this regard that our restriction to quasilocal Legendre actions and (5.24) require $S^B(g)$ to be strictly convex at $\varphi_0$, the $\varphi$ value dual to $\pi = 0$, which is an extremum of $S^B$.

which reduces to a dual Hamilton-Jacobi equation in the large N limit,

$$\beta S_\pi = -H\left[\bar{\gamma}; \frac{\delta S_\pi}{\delta \pi}, \pi\right]. \tag{5.29}$$

In this chapter, we will mostly work with the boundary action $S^B$. However, the Legendre action will be useful in Chapter 6.

In [146], the Legendre action was proposed to be exactly the Wilson action as function of single-trace operators ($S_{\mathcal{O}^s}$ of Section 2.4.3). An inconvenient of such proposal is that typical Hamiltonians of gravity theories do not reproduce naively the flow equation (2.112). This problem seems to be overcome in $O(N)$ vector models using different cutoff procedures to the one used in Chapters 2 and 3, [148].

Other attempts to deduce the holographic form of these equations starting from their field-theoretical version have been performed so far [142, 149, 150]. This is a highly nontrivial problem which is deeply related with the emergence of the dynamical spacetime in holography. In this thesis we will not explore this interesting direction. What we do here is to develop a common framework based on the formal equivalence between both equations, and use it to shed light on some features of holographic renormalization.

Finally, notice that both (5.22) and (5.29) satisfy the Large N property 1 of Section (2.4.3). Of course, this is a feature of any Hamilton-Jacobi equation that the Schrödinger equations (5.20) and (5.29), before taking the classical limit, do not satisfy.

## 5.3 Scalar Fields in AdS

We will study a theory holographically described by a set of scalar fields living in AdS neglecting backreaction. It is described by the Lagrangian,

$$\mathcal{L}^G = \sqrt{|\gamma_z|}\left[\frac{1}{2}(z\partial_z\phi_i)^2 + \frac{1}{2}(\gamma_z)^{\mu\nu}\,\delta_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + V(\phi)\right], \tag{5.30}$$

where $\delta_{ij}$ is a metric in the space of flavour $i$, that is assumed to be diagonalized, and a bulk potential

$$V(\phi) = v_0 + \frac{m_{(i)}^2}{2}\phi_i\phi^i + \sum_{n\geq 3}\frac{1}{n!}v_{i_1\dots i_n}\phi^{i_1}\dots\phi^{i_n}. \tag{5.31}$$

The term $v_0$ is related to the radius of the AdS, $v_0 = d(d-1)/L^2$. The Hamiltonian density of the system is

$$\mathcal{H} = \sqrt{|\gamma_z|}\left[\frac{1}{2}\delta^{ij}\,\Pi_i\Pi_j - \frac{1}{2}\left(\gamma_z\right)^{\mu\nu}\delta_{ij}\partial_\mu\phi^i\partial_\nu\phi^j - V(\phi)\right]. \tag{5.32}$$

## 5.3.1 Fixed Points

The fixed points of the RG flows in the quotient space $\mathcal{W}^G/\sim$ describe scale-invariant physics. In the parent space $\mathcal{W}^G$, they correspond to the set of equivalent points $(s_*^{B\epsilon}, \epsilon)$ with trivial RG evolution $f_t(s_*^{B\epsilon}, \epsilon) = ((s_*^{B\epsilon})_t, \epsilon/t)$. In our parametrization, this translates into the trivial running $g \to D_t g$. We will concentrate on translationally and Lorentz invariant fixed points with constant scalar couplings $g_*$, which are thus invariant under this rescaling. They are characterized by $\beta^\alpha(s_*^{B\epsilon}, \epsilon) = 0$.

We will work with boundary actions $S_\varphi^B$ instead of using Legendre actions $S_\pi$. Equations are simpler this way. All results can be written using $S_\pi$ by means of the Legendre transform.

Using (5.22) and (5.17) we see that the fixed points obey the following equation

$$H\left[\gamma;\varphi,\frac{\delta}{\delta\varphi}S_*[\gamma;\varphi]\right] - 2\gamma\frac{\partial}{\partial\gamma}S_*[\gamma;\varphi] = 0, \tag{5.33}$$

with $S_*[\gamma_\epsilon;\varphi] = S_\varphi^B(s_*^\epsilon, \epsilon)$.

(5.33) was first studied in this theory in [146, 147] and a complete (recursive) solution was found in [6].

### Potential Approximation

As a warm up, we start with the potential approximation (ignoring derivatives), which was also studied in [146]. In this approximation,

$$S_*[\gamma;\varphi] = \int d^dx \sqrt{|\gamma|}\, \mathcal{S}_*(\varphi(x), \partial\varphi(x), \dots)$$
$$= \int d^dx \sqrt{|\gamma|}\, \mathcal{S}_*^{(0)}(\varphi(x)) + O(\partial\varphi). \tag{5.34}$$

(5.33) gives

$$0 = \frac{1}{2}\partial_i^\varphi \mathcal{S}_*^{(0)}(\varphi)\partial^{\varphi i}\mathcal{S}_*^{(0)}(\varphi) - d\mathcal{S}_*^{(0)}(\varphi) - V(\varphi), \tag{5.35}$$

where $\partial_i^\varphi = \partial/\partial\varphi^i$. This equation can be written as

$$\left|\partial_i^\varphi \mathcal{S}_*^{(0)}(\varphi)\right| = \sqrt{2\left[V(\varphi) + d\mathcal{S}_*^{(0)}(\varphi)\right]}. \tag{5.36}$$

Real solutions require

$$\mathcal{S}_*^{(0)}(\varphi) \geq -\frac{1}{d}V(\varphi). \tag{5.37}$$

At the points where this inequality is strict, the solutions will be analytic. On the other hand, even if the solutions are generically non-analytic at points where the inequality is saturated, we will see that analytic solutions exist about certain points. These are actually the solutions that lead to physically meaningful renormalizable theories.

Let us look for analytic solutions of (5.35) about the equilibrium point $\varphi = 0$ and work in perturbation theory. We expand $V$ and $\mathcal{S}_*^{(0)}$ in powers of $\varphi^i$,

$$V(\varphi) = v_0 + \frac{m_{(i)}^2}{2}\varphi^i\varphi^i + \sum_{n\geq 3}\frac{1}{n!}v_{i_1\dots i_n}\varphi^{i_1}\dots\varphi^{i_n}, \tag{5.38}$$

$$\mathcal{S}_*^{(0)}(\varphi) = w_{i_1\dots i_n}\varphi^{i_1}\dots\varphi^{i_n}, \tag{5.39}$$

and insert these expansions in (5.35). Then we get the algebraic equations

$$w_iw^i = 2(v_0 + dw_0) \tag{5.40}$$

and (using $v_{ij} = \delta_{ij} m^2_{(i)}$)

$$(n+1) w_{j_1 \dots j_n i} w^i$$
$$= \frac{1}{n!} v_{j_1 \dots j_n} + d w_{j_1 \dots j_n} - \frac{1}{2} \sum_{k=1}^{n-1} (k+1)(n-k+1) w^i{}_{(j_1 \dots j_k} w_{j_{k+1} \dots j_n) i}, \quad n \geq 1. \tag{5.41}$$

If the inequality (5.37) is strictly satisfied at $\varphi_0$, (5.40) has a set of solutions $w_i \neq 0$, and for each of them the tower of equations (5.41) can be iteratively solved. At each order, the new integration constants are needed. This is related to the fact that we are solving a non-linear partial differential equation, so the solution is not determined in general by a finite set of integration constants. These solutions however, can not be associated to physical fixed points, since the boundary conditions that generate (5.5) do not admit the solution $\phi(z, x) = 0$.[3]

In fact, we are interested in power expansions at critical points of the boundary action with $w_i = 0$. Both (5.36) and (5.40) show that $w_i = 0$ if and only if the inequality (5.37) is saturated at $\varphi = 0$, which implies $w_0 = -v_0/d$. The situation is pretty different in this case. (5.41) gives

$$2 w^i{}_{(j_1} w_{j_2) i} - d w_{j_1 j_2} = \frac{1}{2} \delta_{j_1 j_2} m^2_{(j_1)} \qquad (n = 2), \tag{5.42}$$

which can be easily solved. If there are $M$ fields, we have $2^M$ solutions (a sign $\pm$ is to be chosen for every field):

$$w_{ij} = \frac{\Delta^{\pm}_{(i)}}{2} \delta_{ij}, \tag{5.43}$$
$$\Delta^{\pm}_{(i)} = \frac{d}{2} \pm \nu_{(i)} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2_{(i)}}. \tag{5.44}$$

As we will see in the next section, the sign + (−) is associated to the standard (alternate)

[3]Since we are neglecting backreaction of the metric, all this analysis only makes sense for perturbative deformations of this solution.

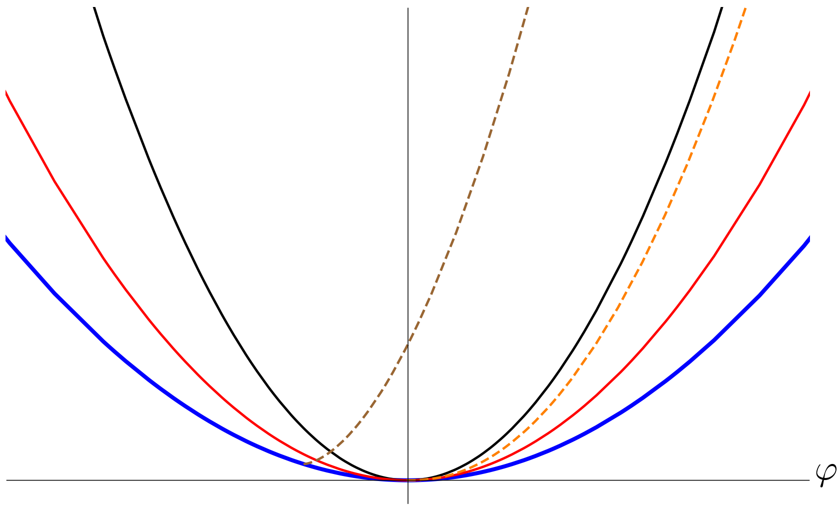


Figure 5.1: Different numerical solutions of the one-dimensional (5.36). The lowest blue curve corresponds to $-V(\varphi)/d$, which gives a lower bound to the solutions. The other two solid curves are the only analytic solutions around the point where their derivative vanishes. From top to bottom, they are associated to the standard (black) and alternate (red) quantization. The dashed curves are other generic non-analytic solutions where their derivative vanishes. From left to right, they correspond to a solution with an asymptotically behaviour $\mathcal{S}_*^{(0)} \sim (\varphi - \varphi_0)^{3/2}$ (brown curve), and $\mathcal{S}_*^{(0)} \sim \frac{\Delta^-}{2}\varphi^2 + w\varphi^{d/\Delta^-}$ (orange curve), both around the point where their derivative vanishes.

quantization of the field $\varphi^i$. The remaining equations can then be written as,

$$\left(\sum_{i=1}^{n} \Delta_{(i)} - d\right) w_{j_1 \dots j_n} = -\frac{1}{2}\sum_{k=2}^{n-2}(k+1)(n-k+1)w^i{}_{(j_1 \dots j_k} w_{j_{k+1}\dots j_n)i} + \frac{1}{n!}v_{j_1 \dots j_n} \qquad n \geq 3, \tag{5.45}$$

that can be solved iteratively. In Figure 5.1 we plot the different kinds of solutions to (5.36), obtained numerically, in the case of only one active scalar field. The standard and alternate solutions are the only ones with the property of being analytic at the point where their derivative vanishes.

### Standard and Alternate Quantizations

We have found that there are exactly $2^M$ analytic solutions about the critical point of the potential ($\varphi = 0$), characterized by the quadratic coefficient of each field:

$$\mathcal{S}_* = -\frac{v_0}{d} + \frac{1}{2}\Delta^{\pm}_{(i)}\varphi^{i\,2} + O(\varphi^3) + \text{derivatives.} \tag{5.46}$$

At the quadratic level, (5.46) imposes the boundary condition

$$\Delta^{\pm}_{(i)}\phi^i(z,x) = z\frac{\partial}{\partial z}\phi^i(z,x), \tag{5.47}$$

when $p_{\gamma_z} \ll 1$ (here $p$ is the $d$-dimensional dimensionful momentum of $\phi$). For $\nu \notin \mathbb{N}_0$, the solutions close to the boundary have the general form (4.22),

$$\phi^i(z,x) = z^{d-\Delta^{\pm}_{(i)}}\left[\phi^i_{(0)}(x) + O(z^2)\right] + z^{\Delta^{\pm}_{(i)}}\left[\tilde{\phi}^i_{(0)}(x) + O(z^2)\right]. \tag{5.48}$$

The boundary condition requires $\phi^i_{(0)}(x) = 0$ and thus selects the solutions $\phi^i(z,x)$ that go like

$$\phi^i(z,x) \sim z^{\Delta^{\pm}_{(i)}} \quad \text{when} \quad z \to 0. \tag{5.49}$$

Because the field solutions then approach zero in the limit $z \to 0$,[4] the non-linear corrections are suppressed and the same conclusion holds for the complete $S^B_*$. In fact, since $S^B_*$ is a fixed point of the Hamilton-Jacobi equation, its boundary condition (5.5) produce the same solutions independently of the cutoff $\epsilon$. Therefore, they are equivalent to the condition (5.49) in the far UV. This is, if we include all orders of the expansion (5.46), the boundary condition at finite $\epsilon$ also implies exactly $\varphi_{(0)} = 0$.

This justifies the association of the sign + (−) to the standard (alternate) quantization.

[4]For the alternate quantization, this is true if we apply the unitary bound, $\nu_{(i)} < 1$.

### Complete Expansion

Now, let us proceed and study (5.33) taking into account the (unavoidable) derivative terms. We expand the fixed point boundary action as

$$S_*[\gamma;\varphi] = S_{*;0}(\gamma) + \sum_{m\geq 2} S_{*;\sigma_1\ldots\sigma_m}(\gamma)\varphi^{\sigma_1}\ldots\varphi^{\sigma_m}. \tag{5.50}$$

As we did in the potential approximation, we have imposed $\delta S_*[\gamma;\varphi]/\delta\varphi|_{\varphi=0} = 0$. Working in momentum space, the coefficients, which are densities, are written using the scalar functions

$$\begin{aligned}\hat{S}_{*;i\,j}^{\;\,pq}(\gamma) =& (2\pi)^d\sqrt{|\gamma|}\delta(p+q)\delta_{ij}\,T_{*;(j)}(\gamma;p),\\ \check{S}_{*;\,i_1\ldots i_m}^{\;\,q_1\ldots q_m}(\gamma) =& \sqrt{|\gamma|}\;T_{*;\,i_1\ldots\,i_m}(\gamma;q_1,\ldots,q_m).\end{aligned} \tag{5.51}$$

Inserting this expansion in (5.33), we find for the quadratic order

$$2T_{*;(i)}(\gamma;q)^2 - \frac{m^2_{(i)}}{2} - \frac{q^2}{2} - \left(d + 2\gamma\frac{\partial}{\partial\gamma}\right)T_{*;(i)}(\gamma;q) = 0. \tag{5.52}$$

If $\nu_{(i)} \notin \mathbb{N}_0$, there are two possible analytic solutions around $p = 0$,

$$T^+_{*;(i)}(\gamma;q) = \frac{d}{4} + q_\gamma\frac{I'_{\nu_{(i)}}(q_\gamma)}{2I_{\nu_{(i)}}(q_\gamma)} = \frac{\Delta^+_{(i)}}{2} + \frac{q^2_\gamma}{4+4\nu_{(i)}} + \mathcal{O}(q^4_\gamma), \tag{5.53}$$

$$T^-_{*;(i)}(\gamma;q) = \frac{d}{4} + q_\gamma\frac{I'_{-\nu_{(i)}}(q_\gamma)}{2I_{-\nu_{(i)}}(q_\gamma)} = \frac{\Delta^-_{(i)}}{2} + \frac{q^2_\gamma}{4-4\nu_{(i)}} + \mathcal{O}(q^4_\gamma), \tag{5.54}$$

where the sign + (−) stands for the standard (alternate) quantization. $I_\nu(\cdot)$ is the modified Bessel function of first kind.

From the unitary bound discussed in Section 4.3.4, to obtain an unitary theory, fields in the alternate quantization have to be restricted to $0 < \nu_{(i)} < 1$. This Wilsonian analysis seems to be blind to this bound, and at least perturbatively, a solution exists. Nevertheless, it should be noticed that in these cases and when the integer part of $\nu_{(i)}$ is odd, $T^-_{(i)}(q)$ diverges at finite values of $q$. If $\nu_{(i)} \in \mathbb{N}_0$, the only analytic solution is (5.53) (the standard one).

Higher orders are given by

$$
\begin{aligned}
&\left[\sum_{k=1}^{n} 2T_{*;(i_k)}(\gamma;q_k) - 2\gamma\frac{\partial}{\partial\gamma} - d\right] T_{*;i_1\ldots i_n}(\gamma;q_1,\ldots,q_n)\\
&= \frac{1}{n!}v_{i_1\ldots i_n} - \frac{1}{2}\sum_{k=2}^{n-2}(k+1)(n-k+1)\\
&\times \underset{\{(i_k,q_k)\}_{k=1}^{n}}{\mathrm{Sym}} \; T_{*;i\,i_1\ldots i_k}(\gamma;-\textstyle\sum_{s=1}^{k} q_s, q_1,\ldots,q_k) T_{*;i_{k+1}\ldots i_n}{}^{i}(\gamma;q_1,\ldots,q_k,-\textstyle\sum_{s=1}^{k} q_s),\;\; n\geq 3.
\end{aligned}
\tag{5.55}
$$

This set of equations allows to find recursively all the orders in the expansion. Because it is a first order differential equation in $T_{*;i_1\ldots i_n}$, there are infinitely many solutions, but only one of them is analytic in momenta at $q_i = 0$ (in their components). This can be shown expanding in powers of momenta and noticing than the whole expansion is then determined.

For instance, the first correction is given by

$$
\left[\sum_{k=1}^{3} 2T_{*;(i_k)}(\gamma;q_k) - 2\gamma\frac{\partial}{\partial\gamma} - d\right] T_{*;i_1i_2i_3}(\gamma;q_1,q_2,q_3) = \frac{1}{3!}v_{i_1i_2i_3}. \tag{5.56}
$$

The only analytic solution of this equation can be written in the integral representation,

$$
\begin{aligned}
T_{*;i_1i_2i_3}(\gamma_l;q_1,q_2,q_3) =& \frac{v_{i_1i_2i_3}}{3!}\int_0^l \frac{dz}{z}\left(\frac{z}{l}\right)^{\frac{d}{2}} \frac{I_{\pm\nu_{(i_1)}}(q_{1\gamma_z})I_{\pm\nu_{(i_2)}}(q_{2\gamma_z})I_{\pm\nu_{(i_3)}}(q_{3\gamma_z})}{I_{\pm\nu_{(i_1)}}(q_{1\gamma_l})I_{\pm\nu_{(i_2)}}(q_{2\gamma_l})I_{\pm\nu_{(i_3)}}(q_{3\gamma_l})}\\
=& \frac{v_{i_1i_2i_3}}{3!}\left[\frac{1}{d/2\pm\nu_{(i_1)}\pm\nu_{(i_2)}\pm\nu_{(i_3)}} + O(q_{\gamma_l}^2)\right].
\end{aligned}
\tag{5.57}
$$

The insertion of (5.57) in (5.56) proves it is solution. We have however found this solution by a holographic renormalization method we will explain in Chapter 6.

### 5.3.2 Normal Coordinates

Let us consider one of these fixed points. As shown in general in Chapter 2, the form of the RG flows around any given fixed point can be greatly simplified by choosing certain coordinates $\bar{c}$, called *normal*, in that region. In these coordinates, the beta functions depend linearly on the couplings, up to a minimal set of nonlinear terms that are only present in exceptional cases. Normal coordinates are specially interesting for our purposes because they are related to minimal subtraction renormalization schemes. To find them perturbatively, we expand the beta and $S^B_\varphi$ functions in powers of $\bar{c}$,

$$\bar{\beta}^\alpha = -\lambda_{(\alpha)}(\bar{\gamma})\bar{c}^\alpha + \bar{\beta}^\alpha_{\alpha_1}(\bar{\gamma})\bar{c}^{\alpha_1} + \bar{\beta}^\alpha_{\alpha_1\alpha_2}(\bar{\gamma})\bar{c}^{\alpha_1}\bar{c}^{\alpha_2}, \tag{5.58}$$

$$S^B_\varphi = S_*[\bar{\gamma};\varphi] + S_\alpha[\bar{\gamma};\varphi]\,\bar{c}^\alpha + S_{\alpha_1\alpha_2}[\bar{\gamma};\varphi]\bar{c}^{\alpha_1}\bar{c}^{\alpha_2} + \dots, \tag{5.59}$$

where $\bar{\beta}^{ax}_{a_1x_1\dots a_nx_n}(\gamma)$ and $S_{a_1x_1\dots a_nx_n}[\gamma;\varphi]$ are functions of $x_1, ..., x_n$ with support in $x_1 = x_2 = ... = x_n$, which can be expanded as series of products of Dirac deltas and their derivatives, with Lorentz indices contracted with $\gamma$. The second term of the RHS of (5.58) stands to take into account possible logarithmic CFTs.

The functionals of (5.59) can be also expanded in number of fields $\varphi$ as

$$S_{\alpha_1\dots\alpha_n}[\gamma;\varphi] = S_{\alpha_1\dots\alpha_n;0}(\gamma) + \sum_{m\geq 1} S_{\alpha_1\dots\alpha_n;\sigma_1\dots\sigma_m}(\gamma)\varphi^{\sigma_1}\dots\varphi^{\sigma_m}, \tag{5.60}$$

with

$$\check{S}^{p_1\dots p_n;\,q_1\dots q_m}_{a_1\dots a_n\,;\,i_1\dots\,i_m}(\gamma) = \sqrt{|\gamma|}\; T_{a_1\dots a_n;\,i_1\dots\,i_m}(\gamma;p_1,\dots,p_n;q_1,\dots,q_m). \tag{5.61}$$

**Eigendirections**

We start with the calculation of eigendirections. Let $\Delta_{(a)} = d - \lambda_{(a)} + n^u_{(a)} - n^d_{(a)}$ be the conformal dimension of the eigendirection $a$, and $n^u_{(a)}$ ($n^d_{(a)}$) the number of contravariant (covariant) indices of the associated coupling. If we insert (5.59) in (5.22) we find at linear order in $\bar{c}$ the eigenvalue problem,

$$\left\{\frac{\delta S_*[\gamma;\varphi]}{\delta\varphi_\sigma}\frac{\delta}{\delta\varphi^\sigma} - 2\gamma\frac{\partial}{\partial\gamma}\right\} S^x_a[\gamma;\varphi] = \left(\Delta_{(a)} - n^u_{(a)} + n^d_{(a)}\right) S^x_a[\gamma;\varphi] + \bar{\beta}^{x\,\alpha}_a S_\alpha[\gamma;\varphi]. \tag{5.62}$$

The last term will be assumed to be zero unless it is necessary to find quasilocal solutions. The eigenfunctions $S^x_a[\gamma;\varphi]$ will be called hereafter eigendeformations or eigenoperators.

The fact that the physical fixed points relevant for our analysis have the form (5.46) is crucial. In particular, the lack of linear terms in $\varphi$ in (5.46) makes the eigendirection equation (5.62) triangular in the sense of the Large N property 2 of Section 2.4.3. Also, as we have stressed above, Hamilton-Jacobi equations automatically satisfy the Large N property 1. Since all the results achieved in Section 2.4.3 only rely on this two properties, we can make use of them.

Then, eigendirections can be divided into:

(1) The trivial solution $S_0^x[\gamma;\varphi] = 1$, with eigenvalue $\lambda_{(0)} = d$, associated to the identity operator, whose coupling is the vacuum energy term.

(2) Primary single-trace eigendirections:

$$S_i^x[\gamma;\varphi] = \delta_{ij}\varphi^{jx} + O(\partial\varphi) + O\left(\varphi^2\right). \tag{5.63}$$

The form of the fixed point boundary action (5.46) in (5.62) implies that the expansion in fields appears directly diagonalized. This is a direct consequence of having chosen a basis of bulk fields that diagonalize the $d+1$-masses. Other normalizations are possible, but we will work with the one in (5.63).

(3) Descendant and multi-trace eigendirections, constructed from primary single-trace ones:

$$S_a^x[\gamma;\varphi] = Q^x_{a;\sigma_1\ldots\sigma_n}(\gamma)\varphi^{\sigma_1}\ldots\varphi^{\sigma_n}. \tag{5.64}$$

Since the calculations get simplified with the use of factorization normal coordinates, we will work with them. Using the factorization property (2.134), all the coefficients of the normal expansions of $S_\varphi^B$ or $S_\pi$ can be expressed as function of coefficients with only single-trace indices. Therefore, we only need to compute these single-trace coefficients.

Inserting the expansion in the number of fields (5.60) and (5.61) in (5.62), one finds scalar primary single-trace eigendirections with a first order given by

$$\begin{aligned} T^{\pm}_{i;j}(\gamma;q,q') &= \delta_{ij}T^{\pm}_{(i)}(\gamma;q) \\ &= \frac{q_\gamma^{\pm\nu_{(i)}}}{\Gamma(1\pm\nu_{(i)})2^{\nu_{(i)}}I_{\pm\nu_{(i)}}(q_\gamma)} \end{aligned}$$

$$= 1 - \frac{q_\gamma^2}{4 \pm 4\nu_{(i)}} + O(q_\gamma^4). \tag{5.65}$$

Or in position space,

$$S^{\pm}{}_{i}^{x}[\gamma;\varphi] = \delta_{ij}\left[1 + \frac{\partial_\gamma^2}{4 \pm 4\nu_{(i)}} + O(\partial_\gamma^4)\right]\varphi^{jx} + O(\varphi^2). \tag{5.66}$$

The mass dimension is fixed to $\Delta_{(i)} = \Delta^{\pm}_{(i)} = d/2 \pm \nu_{(i)}$, and the eigenvalue to $\lambda_{(i)} = \Delta^{\mp}_{(i)}$. Here, the upper (lower) sign is taken when the fixed point involves standard (alternate) quantization of the field $\varphi_i$. Higher orders satisfy the iterative equations

$$\begin{aligned}
&\left[\left(\sum_{r=1}^{n} 2T^{\pm}_{*;(i_r)}(\gamma;q_r) - 2\gamma\frac{\partial}{\partial\gamma} - \Delta^{\pm}_{(i)}\right)\delta_i^a - B_i^a\left(\gamma;\sum_{r=1}^{n} q_r\right)\right] T_{a;i_1\ldots i_n}(\gamma;q;q_1,\ldots,q_n) \\
&\quad = -\sum_{m=2}^{n}\Bigg[(m+1)(n-m+1)\operatorname*{Sym}_{\{(i_k,q_k)\}_{k=1}^{n}} T_{*;}{}^{j}{}_{i_1\ldots i_m}\left(\gamma;-\sum_{r=1}^{m} q_r, q_1,\ldots,q_m\right) \\
&\qquad \times T_{a;j i_{m+1}\ldots i_n}\left(\gamma;q;\sum_{r=1}^{m} q_r, q_{m+1},\ldots,q_n\right)\Bigg].
\end{aligned} \tag{5.67}$$

Here $B_i^a$ appear due to possible resonant off-diagonal terms $i \neq a$,

$$\hat{\bar{\beta}}^{aq}_{ip} = (2\pi)^d\delta(p-q)\delta_i^a\lambda_{(i)} - \hat{\lambda}^{aq}_{ip} = (2\pi)^d\delta(p-q)B_i^a(\gamma;p). \tag{5.68}$$

By the resonance condition, $B_i^a(\gamma;p)$ must be analytic in momenta, with a number of metrics $n_{(\gamma)}$ and inverse metrics $n_{(\gamma^{-1})}$ such that

$$2\left(\lambda_{(a)} - \lambda_{(i)}\right) = n_{(\gamma^{-1})} - n_{(\gamma)}. \tag{5.69}$$

The quadratic correction $O(\varphi^2)$ to the single-trace eigendirection satisfies

$$\begin{aligned}
&\left[\left(\sum_{r=1}^{2} 2T^{\pm}_{*;(i_r)}(\gamma;q_r) - 2\gamma\frac{\partial}{\partial\gamma} - \Delta^{\pm}_{(i)}\right)\delta_i^a - B_i^a(\gamma;q_1+q_2)\right] T_{a;i_1 i_2}(\gamma;q,q_1,q_2) \\
&\quad = -3\,T_{*;}{}^{i}{}_{i_1 i_2}(\gamma;-q_1-q_2,q_1,q_2)\,T_{(i)}(\gamma;q_1+q_2).
\end{aligned} \tag{5.70}$$

An expansion in momenta of this equation, shows that it has a unique quasilocal solution with the $B$-term fixed to zero if

$$\Delta_{(i_1)} + \Delta_{(i_2)} - \Delta_{(i)} \notin -2\mathbb{N}_0^+. \tag{5.71}$$

It is possible to check that

$$\begin{aligned} T_{i;i_1 i_2}(\gamma_l; q, q_1, q_2) = &- \frac{v_{i\, i_1 i_2}}{2} T_{(i_1)}(\gamma_l; q_1) T_{(i_2)}(\gamma_l; q_2) T_{(i)}(\gamma_l; q) \\ &\times \int \frac{dz}{z} \left(\frac{z}{l}\right)^{\pm\nu_{(i_1)} \pm \nu_{(i_2)}} \left. \frac{\Psi_{\pm\nu_{(i)}}(l, z, q)}{T_{(i_1)}(\gamma_z; q_1) T_{(i_2)}(\gamma_z; q_2)} \right|_{z=l}, \end{aligned} \tag{5.72}$$

indeed satisfies (5.70) with vanishing $B$-terms.[5] $\Psi_\nu(l, z, p)$ is an analytic function in $p$ that is defined by

$$\begin{aligned} \Psi_\nu(\epsilon, z, p) &= \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \left[ I_\nu(p_z) K_\nu(p_\epsilon) - K_\nu(p_z) I_\nu(p_\epsilon) \right] \\ &= \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \frac{\pi}{2 \sin(\pi\nu)} \left[ I_\nu(p_z) I_{-\nu}(p_\epsilon) - I_{-\nu}(p_z) I_\nu(p_\epsilon) \right], \end{aligned} \tag{5.73}$$

where the second equality only applies if $\nu \notin \mathbb{N}_0$, and $K_\nu(\cdot)$ is the modified Bessel function of second kind. The indefinite integrals above and hereafter are defined as the primitive with vanishing constant term in the $z$ power expansion at $z = 0$. This definition could be ambiguous if the integrand has $1/z$ terms. However, condition (5.71) avoids such possibility.

Before studying the case in which (5.71) is not satisfied, let us review the multitrace eigendirections. We use the same notation as in Section 2.4.3. The operator products $S^x_{\langle ij \rangle}[\gamma; \varphi] = S^x_i[\gamma; \varphi] S^x_j[\gamma; \varphi]$, possibly including derivatives $S^x_{\langle i_\mu j_\nu \rangle}[\gamma; \varphi] = \partial_\mu S^x_i[\gamma; \varphi] \partial_\nu S^x_j[\gamma; \varphi]$ or $S^x_{\langle i^\mu j_\mu \rangle}[\gamma; \varphi] = \gamma^{\mu\nu} \partial_\mu S^x_i[\gamma; \varphi] \partial_\nu S^x_j[\gamma; \varphi]$ are also eigendeformations, with dimension $\Delta = \Delta_{(i)} + \Delta_{(j)} + n_\partial$, with $n_\partial$ the total number of derivatives. We also use the notation $\langle i^{(n_1)} j^{(n_2)} \rangle$, where $n_1$ and $n_2$ are the number of spacetime derivatives without indicating explicitly if and how they are contracted. They represent scalar double-trace operators.

We can construct general scalar multi-trace eigenoperators of dimension $\Delta_{(a^n)} =$

[5] The solution has been found using techniques that will be explained in Chapter 6.

$\sum_{i=1}^{n} \Delta_{(i)} + n_\partial$, as in (5.64):

$$S^y_{a^n}[\gamma;\varphi] = Q^y_{a^n\sigma_1\ldots\sigma_n}(\gamma)\, S^{\langle\sigma_1\ldots\sigma_n\rangle}[\gamma;\varphi], \tag{5.74}$$

where

$$S^{\langle\sigma_1\ldots\sigma_n\rangle}[\gamma;\varphi] = S^{\sigma_1}[\gamma;\varphi]\ldots S^{\sigma_n}[\gamma;\varphi] \tag{5.75}$$

is a multilocal operator constructed with $S^\sigma[\gamma;\varphi] \equiv S^{ix}[\gamma;\varphi] = \delta^{ij} S^x_j[\gamma;\varphi]$. We remind the reader that we are using the labels $i, j$ and $\sigma$ to refer to single-trace eigendeformations whereas $a^n$ and $\alpha^n$ indicate $n$-trace eigendeformations. For double-trace deformations, we reserve the indices $b$, and $\beta = bx$. The locality of $S^y_{a^n}[\gamma;\varphi]$ is recovered in (5.74) by contraction with the $Q$ distributions. From (5.63),

$$S^{\langle i_1x_1\ldots i_nx_n\rangle}[\gamma;\varphi] = \prod_{r=1}^{n} \varphi^{i_r}(x_r) + \text{ higher orders.} \tag{5.76}$$

### Non-diagonalizable Linear Perturbations

If (5.71) is not satisfied, i.e.

$$\Delta_{(i_1)} + \Delta_{(i_2)} + n = \Delta_{(i)}, \tag{5.77}$$

with $n \in 2\mathbb{N}_0$, the $B$-terms of (5.70) are necessary. Notice that in this case, the eigenvalue of double-trace directions $b = \langle i_1^{(n_1)} i_2^{(n_2)}\rangle$ is given by $\lambda_{(b)} = \lambda_{(i)} + n - n_1 - n_2$. Non-diagonal beta terms $\beta^b_i$ or $B^b_i$ are therefore allowed (see (2.53)). (5.70) becomes

$$\left(\sum_{r=1}^{2} 2T^{\pm}_{*;(i_r)}(\gamma;q_r) - 2\gamma\frac{\partial}{\partial\gamma} - \Delta^{\pm}_{(i)}\right) T_{i;i_1i_2}(\gamma;q,q_1,q_2) - B^b_i\,(\gamma;q_1+q_2)\,\mathbf{Q}_b{}^{i_1i_2}(\gamma;q_1,q_2)$$
$$\times T_{(i_1)}\,(\gamma;q_1)\,T_{(i_2)}\,(\gamma;q_2) = -3\,T_{*;}{}^{i}{}_{i_1i_2}\,(\gamma;-q_1-q_2,q_1,q_2)\,T_{(i)}\,(\gamma;q_1+q_2)\,. \tag{5.78}$$

As first example, let us consider $n = 0$,

$$\Delta_{(i_1)} + \Delta_{(i_2)} = \Delta_{(i)}. \tag{5.79}$$

If all fields are in the standard quantization, this case is called extremal. In the dimensional reduction of type IIB SUGRA in $AdS^5 \times S_5$, the fields that satisfy this relation have only derivative couplings, and therefore they do not fit in this analysis [61, 151, 152]. In this case, $b = \langle i_1 i_2 \rangle$, and $B_i^{\langle i_1 i_2 \rangle}$ does not depend on $p$. If we expand in momenta,

$$T_{i;i_1 i_2}(\gamma; q, q_1, q_2) = T^{(0)}_{i;i_1 i_2} + T^{(2)}_{i;i_1 i_2}(\gamma; q_1, q_2) + \dots, \tag{5.80}$$

where

$$2\gamma \frac{\partial}{\partial \gamma} T^{(n)}_{i;i_1 i_2}(\gamma; q_1, q_2) = -\, n T^{(n)}_{i;i_1 i_2}(\gamma; q_1, q_2), \tag{5.81}$$

(5.78) for the first coefficient becomes

$$\left(\Delta_{(i_1)} + \Delta_{(i_2)} - \Delta_{(i)}\right) T^{(0)}_{i;i_1 i_2} - B_i^{\langle j_1 j_2 \rangle} \delta_{j_1 i_1} \delta_{j_2 i_2} = -\frac{v_{i i_1 i_2}}{2} \frac{1}{\Delta_{(i_1)} + \Delta_{(i_2)} + \Delta_{(i)} - d}. \tag{5.82}$$

If the relation (5.79) is satisfied, the first term vanishes and fixes

$$B_i^{\langle i_1 i_2 \rangle} = \frac{v_i{}^{i_1 i_2}}{2} \frac{1}{2\Delta_{(i)} - d}, \tag{5.83}$$

leaving undetermined $T^{(0)}_{i;i_1 i_2}$ (this is a consequence of the normal charts ambiguity in exceptional cases). Consequently, the linear order of the beta function for the eigendirections $i$ and $\langle i_1 i_2 \rangle$ will have a Jordan form and will not be diagonalizable. This is a logarithmic CFT.

In the case with $n \in 2\mathbb{N}_0$, a general solution can also be given as follows. To find general expressions valid for any value of $\nu$ (including integers), we restrict ourselves to the standard fixed point. The generalization to include fields in the alternate quantization is straightforward. The integrand of (5.72) has $1/z$ terms in its power expansion if (5.77) is satisfied. Therefore, its primitive as described above has logarithms in $l$ that should not appear by covariance ($T_{i;i_1 i_2}$ only depends on $l$ through $\gamma$):

$$T_{i;i_1 i_2}(\gamma_l; -p_1 - p_2, p_1, p_2) \to \dots + T_{(i)}(\gamma_l; p_1) T_{(i)}(\gamma_l; p_2) L_{i;i_1 i_2}(\gamma_l; p_1, p_2) \log(\mu l) + \dots \tag{5.84}$$

where in general

$$L_{i;i_1i_2}(\gamma_l;p_1,p_2) = \frac{v_{ii_1i_2}}{2}\frac{\Gamma(1+\nu_{(i_1)})\Gamma(1+\nu_{(i_2)})}{2^{\nu_{(i)}-\nu_{(i_1)}-\nu_{(i_2)}}\Gamma(1+\nu_{(i)})} \times l^{\Delta^+_{(i)}-\Delta^+_{(i_1)}-\Delta^+_{(i_2)}} C_0\left[z^{\frac{d}{2}}\frac{I_{\nu_{(i_1)}}(z|p_1|)I_{\nu_{(i_2)}}(z|p_2|)K_{\nu_{(i)}}(z|p_1+p_2|)}{|p_1|^{\nu_{(i_1)}}|p_2|^{\nu_{(i_2)}}|p_1+p_2|^{-\nu_{(i)}}}\right]. \tag{5.85}$$

Here $C_n[f(z)]$ gives the $z^n$ coefficient of the power expansion of $f(z)$ around $z=0$, and as usual, the $+$ $(-)$ sign stands for the standard (alternate) quantization. Let $\bar{T}_{i;i_1i_2}$ be equal to the RHS of (5.84), but substituting the argument of the logarithm as follows,

$$\bar{T}_{i;i_1i_2}(\gamma_l;-q_1-q_2,q_1,q_2,\xi) = \cdots + T_{(i_1)}(\gamma_l;q_1)T_{(i_2)}(\gamma_l;q_2)L_{i;i_1i_2}(\gamma_l;q_1,q_2)\log(\xi) + \ldots \tag{5.86}$$

This function is a solution of (5.78) with

$$B^b_i(\gamma;q_1+q_2)\,\mathbf{Q}_b{}^{i_1i_2}(\gamma;q_1,q_2) = -L_{i;i_1i_2}(\gamma_l;q_1,q_2). \tag{5.87}$$

This proves that

$$T_{i;i_1i_2}(\gamma_l;-q_1-q_2,q_1,q_2;\xi) = \bar{T}_{i;i_1i_2}(\gamma_l;-q_1-q_2,q_1,q_2,\xi), \tag{5.88}$$

where $\xi$ parametrizes the normal coordinates ambiguity. (5.87) gives the value of the linear beta terms. An expansion in momenta proves their uniqueness.

**Higher Orders**

Let us next consider the non-linear terms in the $\bar{c}$ expansion of the Hamilton-Jacobi equation (5.22). Since we are working with factorization normal coordinates, we will only consider coefficients with single-trace lower indices. Once they are known, the coefficients with higher order indices can be constructed with (2.134). The $n$-th order equation reads

$$\left\{\sum_{i=1}^{n}\lambda_{(\sigma_i)} - 2\gamma\frac{\partial}{\partial\gamma} + \frac{\delta S_*[\gamma;\varphi]}{\delta\varphi_\sigma}\frac{\delta}{\delta\varphi^\sigma}\right\} S_{\sigma_1\ldots\sigma_n}[\gamma;\varphi]$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{r=1}^{n-1}\frac{\delta S_{(\sigma_1\ldots\sigma_r}[\gamma;\varphi]}{\delta\varphi^\sigma}\frac{\delta S_{\sigma_{r+1}\ldots\sigma_n)}[\gamma;\varphi]}{\delta\varphi_\sigma} \\
&\quad -\sum_{r=1}^{n}(n-r+1)\bar\beta^\alpha_{(\sigma_1\ldots\sigma_r}(\gamma)S_{\alpha\,\sigma_{r+1}\ldots\sigma_n)}[\gamma;\varphi]. \qquad (5.89)
\end{aligned}
$$

By definition, the normal coordinates are chosen in such a way that only a minimal number of non-linear coefficients $\bar\beta^\alpha_{\sigma_1\ldots\sigma_r}$ are non-vanishing. To see which are these coefficients, we need to single out exceptional cases. For generic (i.e. non-resonant) eigenvalues, all the functions $\bar\beta^\alpha_{\sigma_1\ldots\sigma_n}(\gamma)$ vanish. In this case the beta functions are linear functions of the normal couplings and all the terms in the third line of (5.89) vanish. A solution to (5.89) at a given order $n$ can then be obtained recursively. Therefore, each eigendeformation determines a complete perturbative solution of the Hamilton-Jacobi equation around the fixed point.

In the presence of resonances, on the other hand, the beta functions cannot be linearised in general by a change of coordinates. However, it is possible to find coordinates in which only the coefficients $\bar\beta^\alpha_{\sigma_1\ldots\sigma_m}$ that fulfil the resonance condition (2.55) are non-vanishing. This requirement defines the class of normal coordinates, although it does not determine the coordinates completely. All this is reflected in the structure of (5.89). Indeed, expanding this equation in powers of $\varphi^\sigma$, it can be seen that the leading power on the LHS vanishes when a resonance occurs. Then, for a solution to exist, resonant beta terms must be included on the RHS to cancel the non-vanishing contribution of the product of lower-order terms in the second line of (5.89) . The equation at leading order in the $\varphi$ expansion becomes trivial and does not fix its coefficient in $S_{\sigma_1\ldots\sigma_n}[\gamma;\varphi]$, which accounts for the residual ambiguity in the definition of normal coordinates. These observations provide a direct method to compute simultaneously from the Hamilton-Jacobi equation both the beta functions in normal coordinates and the coefficients $S_{\sigma_1\ldots\sigma_n}[\gamma;\varphi]$, which perturbatively define the coordinates themselves: working order by order, the beta function coefficients are determined by requiring the existence of a solution, which is then readily obtained.[6] Inserting the expansion (5.61) in (5.89), a set of recursive equations for the functions $T$ is found.

[6] As proved in Section 3.1, the beta functions in normal coordinates are associated to Gell-Mann-Low beta functions and conformal anomalies in minimal-substraction mass-independent schemes, which can thus be calculated in a purely Wilsonian fashion as described.

Finally, using the factorization properties of this scheme, beta coefficients with lower multi-trace indices are determined by (2.140).

Let us study and solve the first orders of these expansions around a fixed point of the flow. For definiteness, we will restrict to the fixed point with all fields in the standard quantization. Many of the results we find can be easily generalized to other fixed points.

### Calculation of the Quadratic Order $S_{\sigma_1\sigma_2}[\gamma;\varphi]$

Using (5.60) and (5.61), this coefficient has an expansion

$$\hat{S}^{pq}_{ij}[\gamma;\varphi] = \sqrt{|\gamma|}(2\pi)^d\delta(p+q)T_{ij;0}(\gamma,p)+T_{ij;j'}(\gamma;p,q;-p-q)\hat{\varphi}^{j'}(p+q)+O(\varphi^2). \quad (5.90)$$

The general equation (5.89) particularized for $T_{ij;0}(\gamma;p)$ is

$$\begin{aligned}\left[\nu_{(i)}+\nu_{(j)}+2\gamma\frac{\partial}{\partial\gamma}\right]T_{ij;0}(\gamma;p) =&\delta_{ij}T_{(i)}(\gamma;p)T_{(i)}(\gamma;-p)\\ =&\frac{\delta_{ij}}{2}\left[\frac{q_\gamma^{\nu_{(i)}}}{\Gamma(1+\nu_{(i)})2^{\nu_{(i)}}I_{\nu_{(i)}}(p_\gamma)}\right]^2. \end{aligned} \quad (5.91)$$

If $\nu_{(i)}\notin\mathbb{N}_0$, the only analytic solution is

$$\begin{aligned}T_{ij;0}(\gamma;p) =&\delta_{ij}T_{(i);0}(\gamma;p)\\ =&\delta_{ij}\frac{\pi\ p_\gamma^{2\nu_{(i)}}}{2^{2\nu_{(i)}+2}\sin(\pi\nu_{(i)})\Gamma(\nu_{(i)}+1)^2}\frac{I_{-\nu_{(i)}}(p_\gamma)}{I_{\nu_{(i)}}(p_\gamma)}\\ =&\delta_{ij}\left[\frac{1}{4\nu_{(i)}}+\frac{p_\gamma^2}{8-8\nu_{(i)}^2}+O(p_\gamma^4)\right]. \end{aligned} \quad (5.92)$$

If $\nu_{(i)}\in\mathbb{N}_0$, (5.91) does not have analytic solutions in $p=0$. However, in this case, the resonance condition $2\lambda_{(i)}+2\nu_{(i)}=\lambda_{(0)}=d$ is satisfied and (5.91) has to be modified to include the beta term

$$\begin{aligned}&\beta^0_{ixjy}(\gamma) = \delta_{ij}B^0_{(i)}\partial_\gamma^{2\nu_{(i)}}\delta(x-y)\\ \Rightarrow\ &\breve{\beta}^{0\,pq}_{\ ij}(\gamma) = (-1)^{\nu_{(i)}}(p_\gamma)^{2\nu_{(i)}}\sqrt{|\gamma|}\,\delta_{ij}B^0_{(i)}. \end{aligned} \quad (5.93)$$

We obtain the equation

$$\left[2\nu_{(i)} + 2\gamma\frac{\partial}{\partial\gamma}\right] T_{(i);0}(\gamma;p) = T_{(i)}(\gamma;p)^2 - (-1)^{\nu_{(i)}}(p_\gamma)^{2\nu_{(i)}} B^0_{(i)}. \tag{5.94}$$

To get an analytic solution around $p = 0$, we need to fix $B^0_{(i)} = 1/2^{2\nu_{(i)}+1}(\nu_{(i)}!)^2$. Therefore,

$$T_{(i);0}(\gamma;p) = \frac{(p_\gamma)^{2\nu_{(i)}}}{2^{2\nu_{(i)}+1}\nu_{(i)}!^2}\frac{K_{\nu_{(i)}}(p_\gamma) + (-1)^{\nu_{(i)}}\log(\xi p_\gamma)\, I_{\nu_{(i)}}(p_\gamma)}{I_{\nu_{(i)}}(p_\gamma)}, \tag{5.95}$$

where $\xi$ is an arbitrary parameter associated to a solution to the homogeneous part of (5.94). It parametrizes the freedom in the exact choice of the normal chart. We thus find a contribution to $\beta^0$ or, what is the same, to the conformal anomaly,[7]

$$\mathcal{A}(x) = \frac{1}{2^{2\nu_{(i)}+1}\nu_{(i)}!^2}\,\bar{c}^i(x)\,\partial_\gamma^{2\nu_{(i)}}\bar{c}_i(x) + \dots \tag{5.96}$$

The next coefficient in the expansion of $S_{\sigma_1\sigma_2}$ is $T_{i_1 i_2;i}(\gamma;p_1,p_2;p)$. It satisfies the equation

$$\left[\nu_{(i_1)} + \nu_{(i_2)} + 2\gamma\frac{\partial}{\partial\gamma} - T_{*;(i)}(\gamma;q)\right] T_{i_1 i_2;i}(\gamma;p_1,p_2,p) = T_{(i_1)}(\gamma;p_1)T_{i_2;i_1 i}(\gamma;p_2,p_1,p) \\ + T_{(i_2)}(\gamma;p_2)T_{i_1;i_2 i}(\gamma;p_1,p_2,p). \tag{5.97}$$

The expansion in momenta shows that this equation has analytic solution if

$$\Delta^-_{(i)} - \Delta^-_{(i_1)} - \Delta^-_{(i_2)} \notin 2\mathbb{N}_0. \tag{5.98}$$

In this case, the solution is given by

$$T_{i_1\,i_2\,;i}(\gamma_l;p_1,p_2,p) = -\frac{v_{i\,i_1 i_2}}{2}T_{(i)}(\gamma_l;p)T_{(i_1)}(\gamma_l;p_1)T_{(i_2)}(\gamma_l;p_2)$$

[7] This result differs in a factor $4\nu^2_{(i)}$ from the one found with different methods in (5.14) of [123]. This factor is due to the normalization we choose. In our formalism, we have chosen $S^\sigma[\gamma;\varphi] = \varphi^\sigma + \dots$. As we will discuss in Section 5.4 and in Chapter 6, other approaches (like the one of [123]) are equivalent to choosing $S^\sigma[\gamma;\varphi] = -2\nu_{(\sigma)}\varphi^\sigma + \dots$.

$$\times \int \frac{dz}{z}\left(\frac{l}{z}\right)^{\Delta^-_{(i)}} \frac{\Psi_{\nu_{(i_1)}}(l,z,p_1)\Psi_{\nu_{(i_2)}}(l,z,p_2)}{T_{(i)}(\gamma_z;p)}\Bigg|_{z=l}. \tag{5.99}$$

(5.98) guarantees that the integrand of (5.99) will not have $1/z$ terms. On the other hand, if (5.98) is not satisfied and

$$\Delta^-_{(i)} - \Delta^-_{(i_1)} - \Delta^-_{(i_2)} = n \in 2\mathbb{N}_0, \tag{5.100}$$

the resonant beta term, ${\beta^i}_{i_1 i_2}$, may be necessary to find quasilocal solutions. In momentum space it takes the form

$$\hat{\bar{\beta}}^{i\ p_1 p_2}_{p\ i_1 i_2} = (2\pi)^d \delta(p - p_1 - p_2) B^i_{i_1 i_2}(\bar{\gamma}; p_1, p_2), \tag{5.101}$$

and (5.97) becomes

$$\left[\nu_{(i_1)} + \nu_{(i_2)} + 2\gamma\frac{\partial}{\partial\gamma} - T_{*;(i)}(\gamma;p)\right] T_{i_1 i_2;i}(\gamma;p_1,p_2,p) = -B^i_{i_1 i_2}(\gamma;p_1,p_2)T_{(i)}(\gamma;p)$$
$$+T_{(i_1)}(\gamma;p_1)T_{i_2;i_1 i}(\gamma;p_2,p_1,p) + T_{(i_2)}(\gamma;p_2)T_{i_1;i_2 i}(\gamma;p_1,p_2,p). \tag{5.102}$$

To obtain a general solution, we follow a similar method to the one used for $T_{i\,;i_1 i_2}$ (see (5.84) – (5.88)). Logarithmic terms $\log(\mu l)$ will appear in (5.99) when (5.100) is satisfied,

$$T_{i_1 i_2;i}(\gamma_l;p_1,p_2,-p_1-p_2) \to \cdots + T_{(i)}(\gamma_l;p_2)L_{i_1 i_2;i}(\gamma_l;p_1,p_2)\log(\mu l) + \ldots \tag{5.103}$$

with

$$L_{i_1 i_2;i}(\gamma_l;p_1,p_2) = -\frac{v_{i i_1 i_2}}{2}\frac{2^{\nu_{(i)}-\nu_{(i_1)}-\nu_{(i_2)}}\Gamma(1+\nu_{(i)})}{\Gamma(1+\nu_{(i_1)})\Gamma(1+\nu_{(i_2)})}$$
$$\times\, l^{\Delta^-_{(i)}-\Delta^-_{(i_1)}-\Delta^-_{(i_2)}} C_0\left[z^{\frac{d}{2}}\frac{K_{\nu_{(i_1)}}(z|p_1|)K_{\nu_{(i_2)}}(z|p_2|)I_{\nu_{(i)}}(z|p_1+p_2|)}{|p_1|^{-\nu_{(i_1)}}|p_2|^{-\nu_{(i_2)}}|p_1+p_2|^{\nu_{(i)}}}\right]. \tag{5.104}$$

If we substitute $\log(\mu l) \to \log \xi$:

$$T_{i_1 i_2;i}(\gamma_l; q_1, q_2, -q_1 - q_2; \xi) = \dots + T_{(i)}(\gamma_l; q_2) L_{i_1 i_2;i}(\gamma_l; q_1, q_2) \log(\xi) + \dots \tag{5.105}$$

results in a good candidate for the coefficient. Indeed, inserting the new $T_{i_1 i_2;i}$ in (5.102), the beta term is obtained:

$$B^i_{i_1 i_2}(\gamma; p_1, p_2) = -\delta^{ij} L_{i_1 i_2;j}(\gamma_l; p_1, p_2), \tag{5.106}$$

and the equation is satisfied.

### Calculation of the Cubic Order $S_{\sigma_1\sigma_2\sigma_3}[\gamma; \varphi]$

Let us next calculate the first order in the expansion in number of fields of the cubic order coefficient,

$$\hat{S}^{p_1 p_2 p_3}_{i_1\, i_2\, i_3}[\gamma; \varphi] = \sqrt{|\gamma|}(2\pi)^d \delta(p_1 + p_2 + p_3) T_{i_1 i_2 i_3;0}(\gamma; p_1, p_2, p_3) + O(\varphi). \tag{5.107}$$

Inserting this expansion in (5.89), we obtain

$$\begin{aligned}\left[\nu_{(i_1)} + \nu_{(i_2)} + \nu_{(i_3)} - \frac{d}{2} + 2\gamma \frac{\partial}{\partial \gamma}\right] & T_{i_1 i_2 i_3;0}(\gamma; p_1, p_2, p_3) \\ = \frac{1}{2} \operatorname*{Sym}_{\{(i_r, p_r)\}_{r=1}^3} & T_{i_1 i_2;i_3}(\gamma; p_1, p_2, p_3) T_{(i_3)}(\gamma; -p_3).\end{aligned} \tag{5.108}$$

The expansion in momenta shows that the equation has quasilocal solution if

$$\Delta^-_{(i_1)} + \Delta^-_{(i_2)} + \Delta^-_{(i_3)} - d \notin -2\mathbb{N}_0. \tag{5.109}$$

In this case, its solution is given by the integral,

$$\begin{aligned} T_{i_1\, i_2\, i_3;0}(\gamma_l; p_1, p_2, p_3) = & -\frac{v_{i\, i_1 i_2}}{3!} T_{(i_1)}(\gamma_l, p_1) T_{(i_2)}(\gamma_l, p_2) T_{(i_3)}(\gamma_l, p_3) \\ & \times \int \frac{dz}{z} \left(\frac{l}{z}\right)^d \Psi_{\nu_{(i_1)}}(l, z, p_1) \Psi_{\nu_{(i_2)}}(l, z, p_2) \Psi_{\nu_{(i_3)}}(l, z, p_3) \Bigg|_{z=l} . \end{aligned} \tag{5.110}$$

The exceptional case,

$$d - \Delta^-_{(i_1)} - \Delta^-_{(i_2)} - \Delta^-_{(i_3)} = n \in 2\mathbb{N}_0, \tag{5.111}$$

requires the introduction of a new contribution to the conformal anomaly $\beta^0_{i_1 i_2 i_3}$. We proceed as in previous sections. The integral expression for the coefficient $T_{i_1 i_2 i_3;0}$ in (5.108) has $\log(\mu l)$ terms:

$$T_{i_1 i_2 i_3;0}(\gamma_l; p_1, p_2, p_3) \to \cdots + L_{i_1 i_2 i_3}(\gamma_l; p_1, p_2, p_3) \log(\mu l) + \ldots \tag{5.112}$$

with

$$\begin{aligned} L_{i_1 i_2 i_3}(\gamma_l; p_1, p_2, p_3) =& \frac{v_{i_1 i_2 i_3}}{3!} \frac{2^{-\nu_{(i)} - \nu_{(i_1)} - \nu_{(i_2)}}}{\Gamma(1+\nu_{(i_1)})\Gamma(1+\nu_{(i_2)})\Gamma(1+\nu_{(i_3)})} \\ &\times l^{d - \Delta^-_{(i_1)} - \Delta^-_{(i_1)} - \Delta^-_{(i_2)}} C_0 \left[ z^{\frac{d}{2}} \frac{K_{\nu_{(i_1)}}(z|p_1|) K_{\nu_{(i_2)}}(z|p_2|) K_{\nu_{(i_3)}}(z|p_3|)}{|p_1|^{-\nu_{(i_1)}} |p_2|^{-\nu_{(i_2)}} |p_3|^{-\nu_{(i_3)}}} \right]. \end{aligned} \tag{5.113}$$

Substituting $\log(\mu l) \to \log \xi$,

$$T_{i_1 i_2; i}(\gamma_l; q_1, q_2, -q_1 - q_2; \xi) = \cdots + L_{i_1 i_2 i_3}(\gamma_l; p_1, p_2, p_3) \log(\xi) + \ldots \tag{5.114}$$

gives the correct $\xi$-dependent coefficient. The necessary beta coefficient is then

$$\hat{\beta}^{0\, p_1 p_2 p_3}_{\;\; i_1\, i_2\, i_3}(\gamma) = -(2\pi)^d \delta(p_1 + p_2 + p_3) L_{i_1 i_2 i_3}(\gamma; p_1, p_2, p_3). \tag{5.115}$$

**Coefficients with Multi-trace Directions**

The calculation of other coefficients of the expansion of $S^B_\varphi$ in normal coordinates with multi-trace indices is now automatic. Based on the factorization properties of large N flows (see Section 2.4.3), a consistent factorization normal chart can be chosen. In this chart, these coefficients factorize as indicated in (2.134) and can be expressed as products of the coefficients already calculated.

The method used to calculate the previous coefficients also provides the expansion of the beta function. In particular, the equations are sensitive to the beta coefficients with the same lower indices as the action coefficients (or a subset of them). Thus, studying equations that only involve action coefficients with single-trace indices, one can only extract the beta coefficients with single-trace lower indices. Then, the use of factorization normal coordinates allows to calculate any beta coefficient using (2.140). In particular, note that in the case of a field $\phi^i$ in the standard quantization with $\nu_{(i)} \in \mathbb{N}_0$, the conformal anomaly of (5.96) implies the non-vanishing beta coefficients:

$$0 \neq \bar{\beta}^{0y}_{ix_1 ix_2} = \bar{\beta}^{jy}_{\langle ij\rangle x_1 \; ix_2} = \bar{\beta}^{\langle jj'\rangle y}_{\langle ij\rangle x_1 \; \langle ij'\rangle x_2}, \quad \forall j, j'. \tag{5.116}$$

This is, these exceptional cases not only have a conformal anomaly, but also they develop a mixing between single and multi-trace eigendirections.

## 5.4 Connection Between Formalisms

Before finishing this this chapter, we would like to say a few words about the connection of our methods with the standard one reviewed in Chapter 4. In the standard approach to holographic renormalization, the deformations of the theory correspond to boundary conditions on the modes of the asymptotic expansion of the fields close to the conformal boundary.

However, the Wilsonian description introduced in this chapter is slightly different: we have described the space of boundary actions at a geometric boundary of a regulated AdS. They evolve under a Hamilton-Jacobi equation under the change of the cutoff. Then, we have identified the physically relevant fixed points of the flows and studied their neighbourhood. The Hamilton-Jacobi evolution selects diagonal perturbations of the fixed point that can be used to deform the theory.

Let us explain how both formalism are connected.

As explained in Chapter 3, renormalizable theories can be intrinsically described in terms of the renormalized space formed by the actions that can be reached, under RG evolution, from relevant or marginally relevant deformations of a given fixed-point action. Each particular renormalized theory is given by an integral curve of the beta functions along the renormalized manifold $\mathcal{R} \subset \mathcal{W}^G$, which in the gravity picture corresponds to a solution to the Hamilton-Jacobi equation that approaches the fixed point towards the UV. The renormalization schemes define parametrizations of this manifold. For instance, relevant (and marginally relevant) normal coordinates $\bar{c}$ constitute a good parametrization: the UV renormalization scheme.

The "perfect" boundary actions $(s,\epsilon) \in \mathcal{R}$, or its integral curves $f_t \circ (s,\epsilon) \in \mathcal{R}$, impose a boundary conditions on the bulk fields given by (5.5). The integral curve for relevant single-trace deformations takes the form,

$$\begin{aligned} S^B_\varphi \circ f_{l/\epsilon} =& \frac{1}{2}\Delta_{(\sigma)}\, \varphi^\sigma \varphi^\sigma + \text{constant term in } \varphi + O(\varphi^3) \\ &+ \left(\frac{\epsilon}{l}\right)^{\lambda_{(\sigma)}} \bar{c}^\sigma \left[\varphi_\sigma + O(\varphi^2)\right] + O(\bar{c}^2) + \text{derivatives}, \end{aligned} \tag{5.117}$$

where we have assumed generic dimensions. The asymptotic form of the bulk fields is,

$$\phi^i(z,x) = z^{d-\Delta_{(i)}} \left[\phi^i_{(0)}(x) + O(z^2)\right] + z^{\Delta_{(i)}} \left[\tilde{\phi}^i_{(0)}(x) + O(z^2)\right] + \dots, \tag{5.118}$$

where ellipsis depicts other possible contributions due to bulk interactions. Neglecting higher orders, and using (5.118), the boundary condition imposed by (5.117) fixes the coefficient of the asymptotic term $z^{d-\Delta_{(i)}}$ to be proportional to the renormalized UV coupling:

$$\phi^\sigma_{(0)} + \dots = l^{-\lambda_{(\sigma)}} \frac{\bar{c}^\sigma}{d - 2\Delta_{(\sigma)}} + O(\bar{c}^2), \tag{5.119}$$

where the ellipsis represents quadratic and higher orders in the modes $\phi_{(0)}$ and $\tilde{\phi}_{(0)}$. This works both for standard and alternate quantization, with the corresponding values $\Delta_{(\sigma)} = \Delta^\pm_{(\sigma)}$.

To see the implications at finite $\epsilon$, let us assume first the following relation between dimensions. Any trio of bulk fields $\sigma$, $\sigma_1$, $\sigma_2$, must satisfy

$$\lambda_{(\sigma_1)} + \lambda_{(\sigma_2)} > \lambda_{(\sigma)}. \tag{5.120}$$

In this case, in (5.119), $O(\bar{c}^2) \sim \epsilon^{\lambda_{(\sigma_1)}+\lambda_{(\sigma_2)}-\lambda_{(\sigma)}}$, and therefore, these orders vanish in the $\epsilon \to 0$ limit. Because in this limit, the renormalized trajectories flow to the fixed point, and therefore, the field solutions approach zero, the non-linear corrections of the fields are also suppressed. We find that (5.119) is then exact in the $\epsilon \to 0$ limit.

However, since the boundary actions $S^B_\varphi \circ f_{l/\epsilon}$ satisfy the Hamilton-Jacobi equation, their boundary conditions must produce the same solutions independently of the cutoff $\epsilon$. This is, if we include all orders of the expansion (5.117), the boundary condition at finite $\epsilon$ must imply exactly

$$\phi^{\sigma}_{(0)} = l^{-\lambda_{(\sigma)}} \frac{\bar{c}^{\sigma}}{d - 2\Delta_{(\sigma)}}. \tag{5.121}$$

If (5.120) is not satisfied, some contributions $O(\bar{c}^2)$ in (5.119) can be divergent in the $\epsilon \to 0$ limit. However, in this case, higher orders represented with ellipsis in the LHS of (5.119), produced by bulk interactions, also give divergent terms. Since by definition, $S^B \circ f_t$ must generate finite and consistent boundary conditions, both divergent terms must cancel between them. Then, (5.121) is exact again.

Finally, we note that the factors $1/(d - 2\Delta_{(i)})$ in these relations explain the normalization factors we have found in (5.96).

# Chapter 6

# Application to Correlation Functions

*The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.*

Sidney R. Coleman

In the previous chapter, we have shown how the holographic space of theories has a description which is completely equivalent to the Wilsonian description of the space of theories in QFT of Chapter 2. Since Chapter 3 only relies on this structure, it is clear that all the tools and procedures explained there also apply in holography.

In this chapter we apply this formalism to the renormalization of correlation functions in AdS/CFT. In particular, we will study two and three point functions of operators with arbitrary scaling dimensions. They constitute a non-trivial example that is worth analysing under our formalism, with many subtleties that our approach helps to understand.

As we have seen in Chapters 4 and 5, the most standard regularization in (asymptotically) AdS spaces consists in introducing an artificial geometric boundary that restricts space-time to radial coordinates $z \geq \epsilon > 0$. Boundary conditions on the fields can be imposed either at the true AdS boundary, as in Chapter 4, or at the new boundary at $z = \epsilon$, as in Chapter 5. In this chapter we follow the second approach: it has a closer connection with our renormalization formalism. The divergences are then seen explicitly as singularities in the $\epsilon \to 0$ limit. We calculate the renormalized correlation function following standard renormalization techniques, and using the language of Chapter 3. As we have stressed there, although the extracted information is intimately related to the Wilson flows in a neighbourhood of the fixed point, this procedure does not require of any previous knowledge of these flows or normal coordinates. We will pay special attention to the presence of irrelevant operators, and to its consequences for renormalization.

We also wish to explicitly see in this system the connection between renormalization and Wilsonian RG developed in Part I of this thesis. Therefore, in a second part of this chapter, we will employ the results of Chapter 5 to understand the specific features we have found in the renormalization of holographic correlation functions.

The main results of this chapter are presented in [8]. It is organized as follows. In Section 6.1 we calculate the two and three point functions following the renormalization procedure described in Chapter 3. We first try to use exclusively Dirichlet conditions on the cutoff surface. However, irrelevant operators need more general boundary conditions. We discuss this extension. In Section 6.2, we connect the previous calculation with the Wilsonian RG. In order to do this, we compute perturbatively the renormalization charts in a exact UV scheme, and compare with the results found in the previous section. Section 6.3 is devoted to the computation of the correlation function as normal derivatives in the fixed point. This is connected to the equivalence between the GKPW dictionary and the BDHM dictionary in holography [153]. In Section 6.4 the resonant cases that can appear in this system are analysed. Finally, in Section 6.5, we make a brief discussion about differences and similarities between the methods explained in this section and the more standard methods reviewed in Chapter 4.

## 6.1 Minimal Subtraction Renormalization

Among all the possible boundary conditions to impose at $\epsilon$, Dirichlet boundary conditions seem to be the most direct implementation of the Witten prescription (4.24). In fact, they have been often used before (see for instance [61, 125]).

Let $\mathcal{D} \subset \mathcal{W}^G$ the submanifold of Dirichlet boundary conditions. We define the

Dirichlet chart $c : \mathcal{D} \subset \mathcal{W}^G \to \mathcal{C} \times \mathbb{R}^+$ as the chart in $\mathcal{D}$ that associates the coordinates $(g^{0x}, g^{ix}, \epsilon)$ to the boundary condition $\phi^i(\epsilon, x) = g^{ix}$ and the vacuum energy coupling $g^{0x}$. The CFT generating function is then

$$Z = \int [\mathcal{D}\phi]_{\bar{\epsilon},c} \, e^{-N^2\left\{S^G[\phi] + \int d^dx \sqrt{|\bar{\gamma}|}\left[\frac{v_0}{d} + c^0(x)\right]\right\}}. \tag{6.1}$$

where, as in the previous chapter, $[\mathcal{D}\phi]_{\epsilon,g}$ indicates functional integration in the corresponding fields $\phi$, with support restricted to $z > \epsilon$, and boundary condition $\phi^\sigma(\epsilon) = g^\sigma$. The spacetime integrals inside the functional integrals are understood to be restricted to the support of the fields. We have also added a vacuum energy term, $v_0/d + c^0(x)$. We choose to shift the origin of $g^{0x}$ this way to have $Z[\gamma; 0] = 1$.

The bare correlation functions are functional derivatives of the connected generating function $W = N^{-2} \log Z$, with respect to the Dirichlet boundary conditions of the fields:

$$\begin{aligned} G^{(s^\epsilon_c,\epsilon)}_{\sigma_1\ldots\sigma_n} &= \left.\frac{\delta^n W[\gamma_\epsilon; g]}{\delta g^{\sigma_1} \ldots \delta g^{\sigma_n}}\right|_0 \\ &= \partial^c_{\sigma_1} \ldots \partial^c_{\sigma_n}\Big|_{(s^\epsilon_c,\epsilon)} W. \end{aligned} \tag{6.2}$$

The point where we evaluate the derivatives is $g^{ix} = 0$ (i.e. the minimum of $V(\phi)$) and $g^{0x}_c = 0$: $(s^\epsilon_c, \epsilon) = c^{-1}(0, \epsilon) \in \mathcal{W}$. This point of $\mathcal{W}^G$ will be called the *critical point* (name that will be justified in the next section).

For computing bare correlation functions in a mixed position/momentum representation (doing only the Fourier transform to the Minkowski coordinates) we need the boundary-to-bulk propagator $\mathcal{P}^{(i)}_\epsilon(z, p)$ for the field $i$. It is given by the solution to the equation of motion with the boundary condition $\mathcal{P}^{(i)}_\epsilon(\epsilon, p) = 1$ and that is regular in the interior of the bulk:

$$\mathcal{P}^{(i)}_\epsilon(z, p) = \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \frac{K_{\nu_{(i)}}(p_{\gamma_z})}{K_{\nu_{(i)}}(p_{\gamma_\epsilon})}. \tag{6.3}$$

### 6.1.1 Two-Point Function

The bare two-point functions can be easily calculated, as integration by parts reduces the on-shell action to a boundary contribution:

$$\begin{aligned}\hat{G}^{(s_c^\epsilon,\epsilon)}_{ij}(p_1,p_2) &= (2\pi)^d\delta(p_1+p_2)\delta_{ij}\; z^{-d+1}\partial_z\mathcal{P}^{(i)}_\epsilon(z,p_1)\big|_{z=\epsilon}\\ &= (2\pi)^d\delta(p_1+p_2)\delta_{ij}\;\epsilon^{-d}G_{(i)}(\gamma_\epsilon;p_1),\end{aligned} \tag{6.4}$$

where

$$G_{(i)}(\gamma;q) = \frac{d}{2} + q\frac{K'_{\nu_{(i)}}(q_\gamma)}{K_{\nu_{(i)}}(q_\gamma)}. \tag{6.5}$$

This function can be split into a local and a non-local part:

$$G_{(i)}(\gamma;q) = G^{\mathrm{L}}_{(i)}(\gamma;q;t) + G^{\mathrm{NL}}_{(i)}(\gamma;q;t). \tag{6.6}$$

For non-integer $\nu_{(i)}$,

$$\begin{aligned}G^{\mathrm{L}}_{(i)}(\gamma;q;t) &= \frac{d}{2} + q\frac{I'_{-\nu_{(i)}}(q_\gamma)}{I_{-\nu_{(i)}}(q_\gamma)}\\ &= \Delta^-_{(i)} + \frac{q_\gamma^2}{2-2\nu_{(i)}} - \frac{q_\gamma^4}{8[(1-\nu_{(i)})^2(2-\nu_{(i)})]} + \mathcal{O}(q_\gamma^6)\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}G^{\mathrm{NL}}_{(i)}(\gamma;q;t) &= -\frac{1}{I_{-\nu_{(i)}}(q_\gamma)K_{\nu_{(i)}}(q_\gamma)}\\ &= -2^{1-2\nu_{(i)}}\frac{\Gamma\left(1-\nu_{(i)}\right)}{\Gamma\left(\nu_{(i)}\right)}q_\gamma^{2\nu_{(i)}} + o\left(q_\gamma^{2\nu_{(i)}}\right).\end{aligned} \tag{6.8}$$

Actually, these functions do not depend on $t$ (if we work exclusively in this case, we need not write explicitly the $t$ argument). Note that $G^{\mathrm{NL}}$ is purely non-local for generic $\nu_i$. If $\nu \in \mathbb{N}^+$, the splitting is ambiguous and it is convenient to define instead

$$G^{\mathrm{L}}_{(i)}(\gamma;q;t) = \frac{d}{2} + q_\gamma\frac{R'_{\nu_{(i)}}(\gamma;q;t)}{R_{\nu_{(i)}}(\gamma;q;t)}. \tag{6.9}$$

and

$$G^{\rm NL}_{(i)}(\gamma;q;t) = (-1)^{\nu}\frac{I_{\nu_{(i)}}(q_\gamma)K_{\nu_{(i)}}(q_\gamma)+\log(q_\gamma t)}{K_{\nu_{(i)}}(q_\gamma)R_{\nu_{(i)}}(q_\gamma;t)}, \tag{6.10}$$

with

$$R_\nu(\gamma;q;t) = K_\nu(q_\gamma) + (-1)^\nu I_\nu(q_\gamma)\log(q_\gamma t). \tag{6.11}$$

In all cases with $\nu_{(i)} \neq 0$, $G^{\rm L}$ is analytic at $q=0$. The case $\nu_{(i)}=0$ is quite special and its discussion will be postponed to Section 6.4.

If one tries to take the cutoff $\epsilon$ to 0, one finds that both contributions are divergent. A renormalization procedure is necessary to extract the renormalized two point function. As explained in Chapter 2, to find the renormalized quantity, we have to promote (6.2) to the generalized covariant equation

$$\begin{aligned} G^R_{\alpha_1\ldots\alpha_n} &= \lim_{t\to\infty}\ \nabla^t_{[\mathcal{O}^t_{\alpha_1}]}\ldots\nabla^t_{[\mathcal{O}^t_{\alpha_n}]}W\Big|_{\left(s_c^{1/t\mu},1/t\mu\right)} \\ &= \lim_{t\to\infty}\ [\mathcal{O}^t_{\alpha_1}]^{\alpha_{n+1}}\ldots[\mathcal{O}^t_{\alpha_n}]^{\alpha_{2n}}\nabla^t_{\alpha_{n+1}}\ldots\nabla^t_{\alpha_{2n}}W\Big|_{\left(s_c^{1/t\mu},1/t\mu\right)}. \end{aligned} \tag{6.12}$$

where $\nabla^t$ is a family of $t$-dependent flat connections in a neighborhood of the critical point and $[\mathcal{O}^t_\alpha]^{\alpha'}$ a set of $t$-depending local vectors (labelled by $\alpha$), known as renormalized operators, and defined at $T\mathcal{W}^G\big|_{(s_c^\epsilon,\epsilon)}$ (the tangent vector space of the critical point). Their behaviour in $t$ must be tuned in such a way that the limit (6.12) is finite. The renormalized operators will rescale and absorb non-local divergences of the bare correlators, and the Christoffel symbols will absorb local and semi-local divergences appearing when two or more operators coincide.

Both families of objects are defined by the renormalization charts, $r_t : \mathcal{W}^G \to \mathcal{C}\times\mathbb{R}^+$, $(s,\epsilon)\mapsto(g_R,\epsilon/t)$.

Coming back to the two-point function, we have that

$$\nabla^t_{[\mathcal{O}^{t\,p_1}_{i_1}]}\nabla^t_{[\mathcal{O}^{t\,p_2}_{i_2}]}W = [\mathcal{O}^{t\,p_1}_{i_1}]^{\sigma_1}[\mathcal{O}^{t\,p_2}_{i_2}]^{\sigma_2}\left(\partial_{\sigma_1}\partial_{\sigma_2}W - \Gamma^\sigma_{\sigma_1\sigma_2}\,\partial_\sigma W\right). \tag{6.13}$$

Therefore, the renormalized operators components will make finite the non-local part of the bare correlators, and the Christoffel symbols will cancel the local and semilocal divergences. The renormalized operators associated to single-trace fields should be chosen as follows

$$\partial^{rt}_{\sigma} = t^{-\Delta^{-}_{(\sigma)}} \partial^{c}_{\sigma}. \tag{6.14}$$

Then, $G^{\mathrm{L}}$ gives a divergent contribution, but it can be absorbed with a non-vanishing Christoffel symbol, ${\Gamma^{t\,0}}_{\sigma_1\sigma_2}$:

$$\hat{\Gamma}^{t\,0}{}_{j_1 j_2}(q_1, q_2) = -\sqrt{|\bar{\gamma}|}(2\pi)^d \delta(q_1 + q_2)\delta_{j_1 j_2}\, G^{\mathrm{L}}_{(j_1)}(\bar{\gamma}, q; t)\Big|_{\mathrm{local}}. \tag{6.15}$$

Then, we obtain for $\nu \notin \mathbb{N}_0$

$$\hat{G}^{R\,p_1 p_2}_{\;\;i\;\,j} = -\mu^d \delta_{ij}(2\pi)^d \delta(p_1 + p_2) 2^{1-2\nu_{(i)}} \frac{\pi \csc(\pi \nu_{(i)})}{\Gamma\left(\nu_{(i)}\right)^2} p_{1\gamma^\mu}^{2\nu_{(i)}}. \tag{6.16}$$

### 6.1.2 Three-Point Function

For the tree-point calculation, it will be useful to notice that the boundary-to-bulk propagator can be expressed as

$$\mathcal{P}^{(i)}_{\epsilon}(z, p) = \bar{\Upsilon}_{\nu_{(i)}}(\epsilon, z, p) + \Psi_{\nu_{(i)}}(\epsilon, z, p) G_{(i)}(\gamma_\epsilon; p), \tag{6.17}$$

where $\bar{\Upsilon}_{\nu_{(i)}}(\epsilon, z, p)$ and $\Psi_{\nu_{(i)}}(\epsilon, z, p)$ (already introduced in (5.73)) are analytic functions of $p$:

$$\begin{aligned} \Psi_\nu(\epsilon, z, p) &= \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \left[I_\nu(p_z)K_\nu(p_\epsilon) - K_\nu(p_z)I_\nu(p_\epsilon)\right] \\ &= \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \frac{\pi}{2\sin(\pi\nu)} \left[I_\nu(p_z)I_{-\nu}(p_\epsilon) - I_{-\nu}(p_z)I_\nu(p_\epsilon)\right], \end{aligned} \tag{6.18}$$

where the second equality only applies if $\nu \notin \mathbb{N}_0$ and

$$\bar{\Upsilon}_\nu(\epsilon, z, p) = \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \left\{ \left[\frac{d}{2} + p_\epsilon \frac{I'_{\nu_{(i)}}(p_\epsilon)}{I_{\nu_{(i)}}(p_\epsilon)}\right] \left[K_\nu(p_z)I_\nu(p_\epsilon) - I_\nu(p_z)K_\nu(p_\epsilon)\right] + \frac{I_\nu(p_z)}{I_\nu(p_\epsilon)} \right\}. \tag{6.19}$$

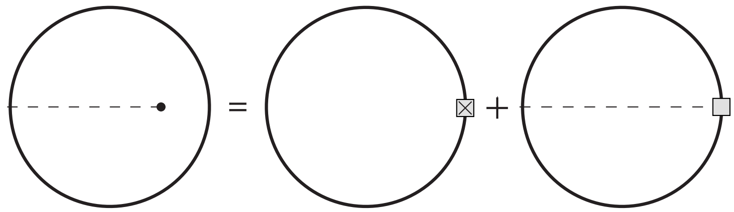


Figure 6.1: Diagrammatic representation of equality (6.17). The squared cross represent the analytic function $\bar{\Upsilon}_{\nu_{(i)}}(\epsilon, z, p)$. Likewise, the square depicts $\Psi_{\nu_{(i)}}(\epsilon, z, p)$.

This equality is diagrammatically represented in Figure 6.1. We can also separate the local and divergent parts of $G_{(i)}(\gamma_\epsilon; p)$ in (6.17) to write

$$\mathcal{P}_\epsilon^{(i)}(z,p) = \Upsilon_{\nu_{(i)}}(\epsilon, z, p) + \Psi_{\nu_{(i)}}(\epsilon, z, p) G_{(i)}^{\mathrm{NL}}(\gamma_\epsilon; p; t). \tag{6.20}$$

If $\nu_{(i)} \notin \mathbb{N}_0$,

$$\Upsilon_\nu(\epsilon, z, p) = \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \frac{I_{-\nu}(p_z)}{I_{-\nu}(p_\epsilon)}. \tag{6.21}$$

If $\nu_{(i)} \in \mathbb{N}_0$, the function $\Upsilon_{\nu_{(i)}}(\epsilon, z, p)$ will depend additionally on $t$.

The bare three-point function is given by the Witten diagram of Figure 6.2, and in momentum space it reads,

$$\begin{aligned}\check{G}_{i_1 i_2 i_3}^{(s_c^\epsilon, \epsilon)}(p_1, p_2, p_3) =& - v_{i_1 i_2 i_3} \int_\epsilon^\infty \frac{dz}{z^{d+1}} \mathcal{P}_\epsilon^{(i_1)}(z, p_1) \mathcal{P}_\epsilon^{(i_2)}(z, p_2) \mathcal{P}_\epsilon^{(i_3)}(z, p_3) \\ =& - v_{i_1 i_2 i_3} \epsilon^{-d} \int_\epsilon^\infty \frac{dz}{z} \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \frac{K_{\nu_{(i_1)}}(p_{1\gamma_z}) K_{\nu_{(i_2)}}(p_{2\gamma_z}) K_{\nu_{(i_3)}}(p_{3\gamma_z})}{K_{\nu_{(i_1)}}(p_{1\gamma_\epsilon}) K_{\nu_{(i_2)}}(p_{2\gamma_\epsilon}) K_{\nu_{(i_3)}}(p_{3\gamma_\epsilon})}.\end{aligned} \tag{6.22}$$

This integral is divergent in the $\epsilon \to 0$ limit (below we analyse its divergences). Following (6.12), the renormalized correlator is

$$\begin{aligned}G^R_{\sigma_1\sigma_2\sigma_3} =& \lim_{t\to\infty} [\mathcal{O}^t_{\sigma_1}]^{\alpha_1} [\mathcal{O}^t_{\sigma_2}]^{\alpha_2} [\mathcal{O}^t_{\sigma_3}]^{\alpha_3} \Big( \partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3} \\ & -\Gamma^{t\,\alpha}_{\alpha_1\alpha_2}\partial_\alpha\partial_{\alpha_3} - \Gamma^{t\,\alpha}_{\alpha_2\alpha_3}\partial_\alpha\partial_{\alpha_1} - \Gamma^{t\,\alpha}_{\alpha_3\alpha_1}\partial_\alpha\partial_{\alpha_2} + \Gamma^{t\,\alpha}_{\alpha_1\alpha_2\alpha_3}\partial_\alpha \Big) W \Big|_{\left(s_c^{1/t\mu}, 1/t\mu\right)}.\end{aligned} \tag{6.23}$$

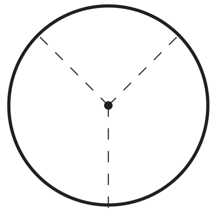

Figure 6.2: Witten diagram for the three point function.

$\Gamma^{\alpha}_{\alpha_1\alpha_2\alpha_3}$ is the contribution coming from terms quadratic in the Christoffel symbols and terms with derivatives of them.[1] In terms of $r_t$, it is given by

$$\Gamma^{t\,\alpha}_{\alpha_1\alpha_2\alpha_3} = \left(\partial^c_{\alpha_1} r_t^{\alpha'_1}\right)\left(\partial^c_{\alpha_2} r_t^{\alpha'_2}\right)\left(\partial^c_{\alpha_3} r_t^{\alpha'_3}\right)\left(\partial^{r_t}_{\alpha'_1}\partial^{r_t}_{\alpha'_2}\partial^{r_t}_{\alpha'_3} c^{\alpha}\right). \tag{6.25}$$

From the two-point analysis, we know the form of the $[\mathcal{O}_\sigma]^{\sigma'}$ components of the renormalized operators (6.14).

Using them, we analyse the $t = 1/\epsilon\mu \to \infty$ limit of the first line of (6.25), to find which further Christoffel symbols or operator components we need. Notice that, since $\partial_\alpha\partial_{0x}W = 0$, the Christoffel symbol found above, $\Gamma^0_{\sigma_1\sigma_2}$, cannot give any contribution to this correlator.

We will analyse first the generic case in which

$$\frac{d}{2} \pm \nu_{(i_1)} \pm \nu_{(i_2)} \pm \nu_{(i_3)} \notin -2\mathbb{N}_0. \tag{6.26}$$

Therefore, the integrand of (6.22) can be expressed as a power expansion around $z = 0$ (admitting real exponents) without $z^{-1}$ terms. Its primitive is therefore defined unambiguously as that one with vanishing constant term in this expansion. Using the

[1] It only make sense to define such quantity in flat space, where the covariant derivatives commute. Specifically,

$$\Gamma^{\alpha}_{\alpha_1\alpha_2\alpha_3} = \frac{2}{3}\left(\Gamma^{\alpha}_{\alpha'\alpha_1}\Gamma^{\alpha'}_{\alpha_2\alpha_3} + \Gamma^{\alpha}_{\alpha'\alpha_3}\Gamma^{\alpha'}_{\alpha_1\alpha_2} + \Gamma^{\alpha}_{\alpha'\alpha_2}\Gamma^{\alpha'}_{\alpha_3\alpha_1}\right) - \frac{1}{3}\left(\partial_{\alpha_1}\Gamma^{\alpha}_{\alpha_2\alpha_3} + \partial_{\alpha_3}\Gamma^{\alpha}_{\alpha_1\alpha_2} + \partial_{\alpha_2}\Gamma^{\alpha}_{\alpha_3\alpha_1}\right). \tag{6.24}$$

renormalized operators found in the two-point calculation, we obtain that

$$
\begin{aligned}
[\mathcal{O}^{t\,p_1}_{\;i_1}]^{\sigma_1}[\mathcal{O}^{t\,p_2}_{\;i_2}]^{\sigma_2}[\mathcal{O}^{t\,p_3}_{\;i_3}]^{\sigma_3}\ \partial_{\sigma_1}\partial_{\sigma_2}\partial_{\sigma_3}W\big|_{\left(s_c^{1/t\mu},1/t\mu\right)} &= \mu^d(2\pi)^d\delta(p_1+p_2+p_3)\\
\times\Big\{\mathcal{R}_{i_1i_2i_3}(\gamma^\mu;p_1,p_2,p_3)+t^{d-\Delta^-_{(i_1)}-\Delta^-_{(i_2)}-\Delta^-_{(i_3)}}\Big[\mathcal{Z}_{i_1i_2i_3}(\gamma^{t\mu};p_1,p_2,p_3)\\
+\mathcal{Y}_{i_1i_2i_3}(\gamma^{t\mu};p_1,p_2,p_3)+\mathcal{Y}_{i_2i_3i_1}(\gamma^{t\mu};p_2,p_3,p_1)+\mathcal{Y}_{i_3i_1i_2}(\gamma^{t\mu};p_3,p_1,p_2)\\
+\mathcal{X}_{i_1i_2i_3}(\gamma^{t\mu};p_1,p_2,p_3)+\mathcal{X}_{i_2i_3i_1}(\gamma^{t\mu};p_2,p_3,p_1)+\mathcal{X}_{i_3i_1i_2}(\gamma^{t\mu};p_3,p_1,p_2)\Big]+\ldots\Big\}, \qquad (6.27)
\end{aligned}
$$

where the ellipsis represents terms of order $O\big(t^{-\nu_{(i_1)}-\nu_{(i_2)}-\nu_{(i_3)}-\frac{d}{2}},\ t^{-\nu_{(i_1)}},\ t^{-\nu_{(i_2)}},\ t^{-\nu_{(i_3)}}\big)$ in the $t\to\infty$ limit (and therefore vanishing).

$$
\begin{aligned}
\mathcal{R}_{i_1i_2i_3}(\gamma^\mu;p_1,p_2,p_3) = &-v_{i_1i_2i_3}\,\frac{2^{3-\nu_{(i_1)}-\nu_{(i_2)}-\nu_{(i_3)}}}{\Gamma(\nu_{(i_1)})\Gamma(\nu_{(i_2)})\Gamma(\nu_{(i_3)})}\,p_{1\gamma^\mu}^{\nu_{(i_1)}}\ p_{2\gamma^\mu}^{\nu_{(i_2)}}\ p_{3\gamma^\mu}^{\nu_{(i_3)}}\\
&\times\int\frac{dz}{z}\,(\mu z)^{\frac{d}{2}}\,K_{\nu_{(i_1)}}(p_{1\gamma_z})K_{\nu_{(i_2)}}(p_{2\gamma_z})K_{\nu_{(i_3)}}(p_{3\gamma_z})\Big|_{z=\infty} \qquad (6.28)
\end{aligned}
$$

gives the leading contribution of the primitive evaluated in $z=\infty$. It is finite, and agrees with the expected 3-point function of a CFT in momentum space [62, 154].

The leading contributions coming from the evaluation in $z=\epsilon$ have been separated in the following pieces using (6.20):

$$
\mathcal{Z}_{i_1i_2i_3}(\gamma_\epsilon,p_1,p_2,p_3) = v_{i_1i_2i_3}\int\frac{dz}{z^{d+1}}\Upsilon_{\nu_{(i_1)}}(\epsilon,z,p_1)\Upsilon_{\nu_{(i_2)}}(\epsilon,z,p_2)\Upsilon_{\nu_{(i_3)}}(\epsilon,z,p_3)\Big|_{z=\epsilon} = O(\epsilon^0), \qquad (6.29)
$$

$$
\begin{aligned}
\mathcal{Y}_{i_1i_2i_3}(\gamma_\epsilon,p_1,p_2,p_3) =& v_{i_1i_2i_3}\,G^{\rm NL}_{(k)}(\gamma_\epsilon;p_3)\int\frac{dz}{z^{d+1}}\Upsilon_{\nu_{(i_1)}}(\epsilon,z,p_1)\Upsilon_{\nu_{(i_2)}}(\epsilon,z,p_2)\Psi_{\nu_{(i_3)}}(\epsilon,z,p_3)\Big|_{z=\epsilon}\\
=& O(\epsilon^{2\nu_{(i_1)}}), \qquad (6.30)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{X}_{i_1i_2i_3}(\gamma_\epsilon,p_1,p_2,p_3) =& v_{i_1i_2i_3}\int\frac{dz}{z^{d+1}}\Upsilon_{\nu_{(i_1)}}(\epsilon,z,p_1)\Psi_{\nu_{(i_2)}}(\epsilon,z,p_2)\Psi_{\nu_{(i_3)}}(\epsilon,z,p_3)\Big|_{z=\epsilon}\\
&\times G^{\rm NL}_{(i_2)}(\gamma_\epsilon;p_2)G^{\rm NL}_{(i_3)}(\gamma_\epsilon;p_3) = O(\epsilon^{2\nu_{(i_2)}+2\nu_{(i_3)}}). \qquad (6.31)
\end{aligned}
$$

All of them are potentially divergent.

If (6.26) is not satisfied, the power expansion around $z = 0$ of some of these functions will have $z^{-1}$ terms, which makes our prescription to define the primitive ill-defined. However, they can also be described by the same equations if we define a prescription for the primitive: we choose the one that, when expanding in series of $z$ around $z = 0$, has no independent term if the logarithms are written with the scale $\mu$, $\log \mu z$. After the evaluation of the primitive at $z = \epsilon$, these logarithmic terms produce a $t$ dependence of these functions.

The first function, $\mathcal{Z}_{i_1 i_2 i_3}$, is purely local (analytic in momenta) and so, appears when the tree points coincide. Terms with derivatives make the degree of divergence decrease. Therefore, only terms with a number of derivatives $n_\partial < -d/2 + \nu_{(i_1)} + \nu_{(i_2)} + \nu_{(i_3)}$ are divergent (the saturation of the inequality is discarded by (6.26)). The divergence can be removed with

$$\check{\Gamma}^0{}_{i_1 i_2 i_3}(p_1, p_2, p_3)\big|_{(s_c^\epsilon, \epsilon)} = -\epsilon^{-d}\, \mathcal{Z}_{i_1 i_2 i_3}(\gamma_\epsilon; p_1, p_2, p_3)\big|_{\text{local}}. \tag{6.32}$$

The second function, $\mathcal{Y}_{i_1 i_2 i_3}$ is semilocal (non-analytic in one of the two free momenta), and so, it appears when two operators ($i_2$ and $i_3$) coincide. It is divergent when $\Delta^-_{(i_1)} > \Delta^-_{(i_2)} + \Delta^-_{(i_3)}$, and the terms with a number of derivatives $n_\partial > \Delta^-_{(i_1)} - \Delta^-_{(i_2)} - \Delta^-_{(i_3)}$ vanish in the limit. From (6.30), we see that, up to an analytic function, it has the form of a (subtracted) two point function. If we define

$$\check{\Gamma}^{i_3}{}_{i_1 i_2}(p_3, p_1, p_2)\big|_{(s_c^\epsilon, \epsilon)} = \epsilon^{-d} v_{i_1 i_2 i_3} \int \frac{dz}{z^{d+1}} \Upsilon_{\nu_{(i_1)}}(\epsilon, z, p_1) \Upsilon_{\nu_{(i_2)}}(\epsilon, z, p_2) \Psi_{\nu_{(i_3)}}(\epsilon, z, p_3)\Big|_{z=\epsilon,\text{local}}, \tag{6.33}$$

Christoffel-symbol terms of (6.23) are

$$\begin{aligned} -[\mathcal{O}^{t\,p_1}_{\ i_1}]^{\sigma_1}[\mathcal{O}^{t\,p_2}_{\ i_2}]^{\sigma_2}[\mathcal{O}^{t\,p_3}_{\ i_3}]^{\sigma_3}\Gamma^{t\,\sigma}_{\sigma_1\sigma_2}\,\partial_\sigma\partial_{\sigma_3} W\big|_{\left(s_c^{1/t\mu}, 1/t\mu\right)} &= -(2\pi)^d \delta(p_1 + p_2 + p_3) \\ &\times t^{d-\Delta^-_{(i_1)}-\Delta^-_{(i_2)}-\Delta^-_{(i_3)}} \mu^d\, \mathcal{Y}^{p_3 p_1 p_2}_{i_3\, i_1\, i_2}(\gamma^{t\mu}), \end{aligned} \tag{6.34}$$

and therefore, cancel the semilocal divergences $\mathcal{Y}^\epsilon_{i_1 i_2 i_3}$ but produce new local ones. These

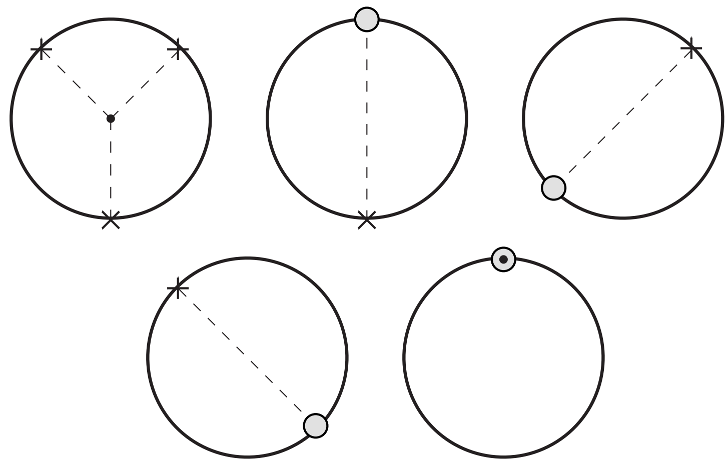

Figure 6.3: Witten diagrams contributing to the renormalization of the tree-point function in the subextremal case. Crosses indicate insertions of $[\mathcal{O}^t_\sigma]^{\sigma'}\partial^c_{\sigma'}$, the shaded circles, insertions of $-[\mathcal{O}^t_{\sigma_1}]^{\sigma'_1}[\mathcal{O}^t_{\sigma_2}]^{\sigma'_2}\Gamma^\sigma_{\sigma'_1\sigma'_2}\partial^c_\sigma$, and the dotted shaded circles, insertions of $[\mathcal{O}^t_{\sigma_1}]^{\sigma'_1}[\mathcal{O}^t_{\sigma_2}]^{\sigma'_2}[\mathcal{O}^t_{\sigma_3}]^{\sigma'_3}\Gamma^\sigma_{\sigma'_1\sigma'_2\sigma'_3}\partial^c_\sigma$.

can be cancelled redefining (6.32),

$$\check{\Gamma}^0{}_{i_1i_2i_3}(p_1,p_2,p_3)\big|_{(s^\epsilon_c,\epsilon)} = -\epsilon^{-d}\mathcal{Z}_{i_1i_2i_3}(\gamma_\epsilon;p_1,p_2,p_3) + \Big[\Gamma^{i_3}{}_{i_1i_2}(p_3,p_1,p_2)G^{\mathrm{L}}_{(i_3)}(\bar{\gamma};p_3) \\ +(1\to 2\to 3)+(1\to 3\to 2)\Big]\Big|_{(s^\epsilon_c,\epsilon),\mathrm{local}}. \tag{6.35}$$

If $\Delta^+_{(i_1)} < \Delta^+_{(i_2)} + \Delta^+_{(i_3)}$ for any choice of $i_1, i_2, i_3$, no further divergences appear (in this case, $\mathcal{X}_{i_1i_2i_3}$ vanishes in the limit), and therefore we have cancelled all the divergences. This is the subextremal case. It is represented diagrammatically in Figure 6.3. Note that, if all operators are relevant, this condition is automatically satisfied for any three operators. Nevertheless, if we consider irrelevant operators, and the relation above between dimensions is not satisfied, new non-local divergences given by $\mathcal{X}^\epsilon_{i_1i_2i_3}$ appear. They are a consequence of the integration of $z$ in a neighborhood of the point where the operator $i_1$ is inserted, and are present even when the three operators are separated in position space. Terms with a number of derivatives $n_\partial > \Delta^+_{(i_1)} - \Delta^+_{(i_2)} - \Delta^+_{(i_3)}$ vanish in the limit. Due to its non-locality, the divergence cannot be cancelled by non-linear

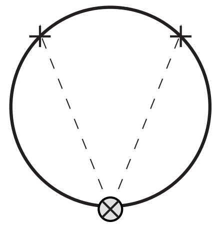

Figure 6.4: Required diagram to subtract $\mathcal{X}_{i_1 i_2 i_3}$ divergences.

counterterms. It indicates that our renormalized operators are not the correct ones, and somehow they have to be improved. From (6.31), we see that these divergences have the same structure as the diagram shown in Figure 6.4. However, no Dirichlet boundary conditions can generate such diagram. We need to extend the Dirichlet chart $c$ beyond $\mathcal{D}$.

### 6.1.3 Extension of the Dirichlet Boundary Conditions

The extension of the Dirichlet boundary conditions is not obvious. The Dirichlet condition corresponds to a singular boundary action. A way to treat them with a regular functional is using the Legendre transformed actions of Section 5.2.2,

$$S_\pi = \pi_\sigma \varphi^\sigma - S^B_\varphi, \qquad \pi_i(x) = \frac{1}{\sqrt{|\gamma_\epsilon|}} \frac{\delta S^B_\varphi}{\delta \varphi^i(x)}. \tag{6.36}$$

Since the Legendre transform is invertible, it can be used to define a chart on $\mathcal{W}^G$:

$$S_\pi \circ c^{-1}(g, \epsilon) = S_\pi[\gamma_\epsilon; g] = \int d^d x \sqrt{|\gamma_\epsilon|}\, \mathcal{S}(\gamma_\epsilon, g, \partial g, \dots, \pi, \partial \pi, \dots). \tag{6.37}$$

In particular, we will use the following canonical linear parametrization $c$:

$$S_\pi = \int d^d x \sqrt{|\bar\gamma|} \left[ -\frac{v_0}{d} - c^{0x} + c^i(x)\pi_i(x) + c^{\langle ij \rangle}(x)\pi_i(x)\pi_j(x) + c^{\langle i^\mu_\mu j \rangle} \partial^2_{\bar\gamma} \pi_i(x)\pi_j(x) + \dots \right]. \tag{6.38}$$

Quadratic and higher orders will be associated with bare multitrace deformations. An $n$-th order multitrace deformation, labelled by $a^n$ or $\alpha^n$, is given by

$$\Delta S_\pi = \int d^d x \sqrt{|\bar{\gamma}|}\, c^a(x) Q^x_{a;\sigma_1\ldots\sigma_n}(\bar{\gamma}) \pi^{\sigma_1} \ldots \pi^{\sigma_n}. \tag{6.39}$$

where $Q^x_{a;\sigma_1\ldots\sigma_n}(\bar{\gamma})$ is a local distribution with support at $y = x_1 = ... = x_n$ (see (2.127)). We reserve the letters $b$ and $\beta$ for double-trace deformations $a^2$ and $\alpha^2$.

A Dirichlet condition corresponds to the linear functional

$$S_\pi = c^\sigma\, \pi_\sigma. \tag{6.40}$$

The inverse Legendre transform of (6.40) is singular, as expected. However, perturbing $S_\pi$ with quadratic or higher orders terms in $\pi_\sigma$ gives a well defined $S^B_\varphi$ which generates mixed boundary conditions. Therefore, the chart defined by (6.38) constitutes an extension of the Dirichlet chart. We will refer to the extension of the Dirichlet charts as Legendre charts (because of its connection with the Legendre action).

The critical point we are using is given by $S_\pi(s^\epsilon_c, \epsilon) = \int d^d x \sqrt{|\gamma_\epsilon|}\, v_0/d$. Derivatives with respect to bare multitrace deformations of the partition function in the critical point are

$$\begin{aligned}
&\frac{\partial^c_{\alpha_1}}{N^2}\frac{\partial^c_{\alpha_2}}{N^2}\ldots Z\Big|_{(s^\epsilon_c,\epsilon)} \\
&\quad = Q_{\alpha_1;\sigma^1_1\ldots\sigma^1_{n_1}}(\gamma_\epsilon) Q_{\alpha_2;\sigma^2_1\ldots\sigma^2_{n_2}}(\gamma_\epsilon)\cdots \int [\mathcal{D}\phi]_\epsilon\, \mathcal{D}\pi\, \pi^{\sigma^1_1}\ldots\pi^{\sigma^1_{n_1}}\, \pi^{\sigma^2_1}\ldots\pi^{\sigma^2_{n_2}}\ldots \\
&\quad\quad \times \exp N^2 \Big\{ \int d^d x \sqrt{|\gamma_\epsilon|} \Big[\frac{v_0}{d} + \phi^i(\epsilon, x)\pi_i(x)\Big] - S^G[\phi] \Big\} \\
&\quad = Q^{\sigma^1_1\ldots\sigma^1_n}_{\alpha_1}(\gamma_\epsilon) Q^{\sigma^1_1\ldots\sigma^1_n}_{\alpha_2}(\gamma_\epsilon)\cdots \frac{\partial^c_{\sigma^1_1}}{N^2}\cdots\frac{\partial^c_{\sigma^1_{n_1}}}{N^2}\frac{\partial^c_{\sigma^2_1}}{N^2}\cdots\frac{\partial^c_{\sigma^2_{n_2}}}{N^2}\cdots Z\Bigg|_{(s^\epsilon_c,\epsilon)}.
\end{aligned} \tag{6.41}$$

The right-hand side is given by standard Witten diagrams (including disconnected ones) with subsets of boundary points contracted with $Q$. To translate this into an identity for connected correlators, one must be careful to include disconnected Witten diagrams that are connected on the boundary by the contractions. Moreover, the contraction mixes different orders in the loop (dual to 1/N) expansion, so some Witten

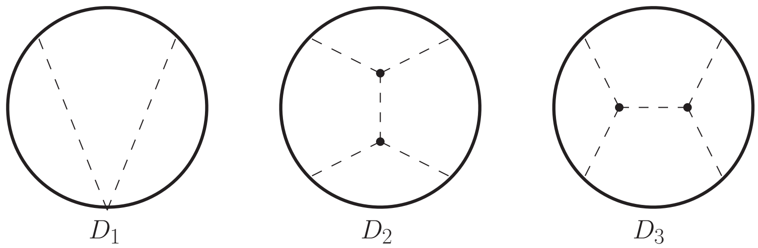


Figure 6.5: Witten diagrams for $G^{(s_c^\epsilon,\epsilon)}_{\langle\sigma_1\sigma_2\rangle\sigma_3\sigma_4} = D_1$, and $G^{(s_c^\epsilon,\epsilon)}_{\sigma_1\sigma_2\sigma_3\sigma_4} = D_2 + D_3$ at the tree-level.

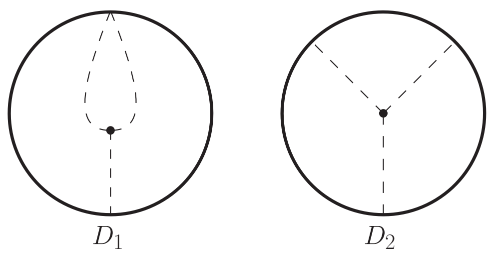


Figure 6.6: Witten diagrams for $G^{(s_c^\epsilon,\epsilon)}_{\langle\sigma_1\sigma_2\rangle\sigma_3} = D_1$, and $G^{(s_c^\epsilon,\epsilon)}_{\sigma_1\sigma_2\sigma_3} = D_2$. The first non-trivial contribution for $D_1$ is at the one-loop level, unlike $D_2$, which has non-trivial contribution at the tree level.

diagrams give subleading contributions upon contraction and need not be considered. To take these two issues into account, we define the correlators $G^{(s_c^\epsilon,\epsilon)}_{\langle\sigma_1^1\dots\sigma_{n_1}^1\rangle\dots\langle\sigma_1^m\dots\sigma_{n_m}^m\rangle}$ as those given by all (connected or disconnected) single-trace tree-level Witten diagrams of $n_1+\dots+n_m$ external legs that, upon contraction of the space-time points inside the same angle-bracket, are connected and contain no loops. In the representation of the corresponding Witten diagrams we will put together the boundary points in the same angle-bracket. We illustrate all this with the examples in Figures 6.5 and 6.6. We can write the bare connected correlators of multi-trace operators in terms of these objects:

$$G^{(s_c^\epsilon,\epsilon)}_{\alpha_1\dots\alpha_m} = Q^{\sigma_1^1\dots\sigma_n^1}_{\alpha_1}(\gamma_\epsilon)\cdots Q^{\sigma_1^m\dots\sigma_n^m}_{\alpha_m}(\gamma_\epsilon) G^{(s_c^\epsilon,\epsilon)}_{\langle\sigma_1^1\dots\sigma_{n_1}^1\rangle\dots\langle\sigma_1^m\dots\sigma_{n_m}^m\rangle}. \tag{6.42}$$

Notice that the diagram

$$G^{(s_c^\epsilon,\epsilon)}_{\langle\sigma_1\sigma_2\rangle\sigma_3\sigma_4} = G^{(s_c^\epsilon,\epsilon)}_{\sigma_1\sigma_3}G^{(s_c^\epsilon,\epsilon)}_{\sigma_2\sigma_4} + G^{(s_c^\epsilon,\epsilon)}_{\sigma_1\sigma_4}G^{(s_c^\epsilon,\epsilon)}_{\sigma_2\sigma_3} \tag{6.43}$$

has exactly the form we need to cancel the non-local divergence $\mathcal{X}_{i_1 i_2 i_3}$. This suggests that this divergence has to be removed adding a bare double-trace component to the single-trace renormalized operator:

$$\hat{\partial}_i^{r_t}(p) \to \hat{\partial}_i^{r_t}(p) = t^{-\Delta^-_{(i)}}\left[\hat{\partial}_i^c(p) + \tilde{\rho}_i^b(\bar{\gamma};p)\hat{\partial}_b^c(p)\right], \tag{6.44}$$

where $\tilde{\rho}_i^b(\bar{\gamma};p)$ will be determined shortly. This component gives an additional contribution in the triple-derivative term of (6.12):

$$\begin{aligned}[\mathcal{O}^{t\,p_1}_{i_1}]^{\sigma_1}[\mathcal{O}^{t\,p_2}_{i_2}]^{\sigma_2}[\mathcal{O}^{t\,p_3}_{i_3}]^{\beta}\,\partial_{\sigma_1}\partial_{\sigma_2}\partial_\beta W\big|_{\left(s_c^{1/t\mu},1/t\mu\right)} &= (2\pi)^d\delta(p_1+p_2+p_3)\,\mu^d\,t^{d-\Delta^-_{(i_1)}-\Delta^-_{(i_2)}-\Delta^-_{(i_3)}} \\ &\times 2\left[\mathbf{Q}_b{}^{i_1 i_2}(\gamma;p_1,p_2)\,\tilde{\rho}^b_{i_1}(\gamma;-p_1-p_2)G_{(i_1)}(\bar{\gamma};p_1)G_{(i_2)}(\bar{\gamma};p_2)\right]\Big|_{\left(s_c^{1/t\mu},1/t\mu\right)}.\end{aligned} \tag{6.45}$$

Apart from permutations in the labels $(1,2,3)$, since $G^{(s_c^\epsilon,\epsilon)}_{\sigma\beta_1\beta_2} = 0$ and $G^{(s_c^\epsilon,\epsilon)}_{\beta_1\beta_2\beta_3} = 0$, (there no diagrams at the tree level) the bare double-trace component does not give more contributions in the triple-derivative term of (6.12). Using (6.6), in order to cancel the non-local terms $\mathcal{X}_{i_1 i_2 i_3}$, the functions $\tilde{\rho}_i^b(\bar{\gamma};p)$ must satisfy

$$\begin{aligned}\mathbf{Q}_b{}^{i_1 i_2}(\gamma;p_1,p_2)\,\tilde{\rho}_i^b(\gamma;-p_1-p_2) = &-\frac{v_{i\,i_1 i_2}}{2}\int\frac{dz}{z^{d+1}}\Upsilon_{\nu_{(i)}}(\epsilon,z,-p_1-p_2) \\ &\times\Psi_{\nu_{(i_1)}}(\epsilon,z,p_1)\Psi_{\nu_{(i_2)}}(\epsilon,z,p_2)\Big|_{z=\epsilon,\text{local}}.\end{aligned} \tag{6.46}$$

Once a basis for the double-trace deformation is specified (modulo total derivatives), this equation fixes $\tilde{\rho}_i^b$ for all $b$ expanding the right hand side in momenta. A possible basis is $b = \langle ij\rangle$, $\langle i^{(2)}j\rangle$, $\langle ij^{(2)}\rangle$, $\ldots$ $\forall i,j$. For instance, $b = \langle i^{(n)}j\rangle$,

$$\mathbf{Q}_{\langle i^{(n)}j\rangle}{}^{i_1 j_1}(\gamma;p_1,p_2) = -p^n_{1\gamma}\delta_i^{i_1}\delta_j^{j_1}, \tag{6.47}$$

would give the following contribution to $S_\pi$:

$$\Delta S_\pi = \int d^dx \sqrt{|\bar{\gamma}|} c^{\langle i^{(n)} j\rangle}(x) \partial_\gamma^n \pi_i(x) \pi_j(x). \tag{6.48}$$

All quasilocal double-trace contributions to $S_\pi$ in (6.38) can be expressed as combination, with possible derivatives, of $c^b$, with $b$ running in the indices written above.

(6.46) cancels non-local divergences, but brings in new semilocal and local ones (see (6.6)). We have to introduce then new Christoffel symbols,

$$\check{\Gamma}^{t\,i_1}{}_{i_2 b}(p_1,p_2,p_3) = 2\sqrt{|\bar{\gamma}|}\ G^{\mathrm{L}}_{(i_2)}(\bar{\gamma},p_2)\Big|_{\mathrm{local}} \mathbf{Q}_b{}^{i_1 i_2}(\bar{\gamma};p_1,p_2), \tag{6.49}$$

$$\check{\Gamma}^{t\,0}{}_{i_1 i_2 b}(p_1,p_2,p_3) = 2\sqrt{|\bar{\gamma}|}\ G^{\mathrm{L}}_{(i_1)}(\bar{\gamma};p_1) G^{\mathrm{L}}_{(i_2)}(\bar{\gamma};p_2)\Big|_{\mathrm{local}} \mathbf{Q}_b{}^{i_1 i_2}(\bar{\gamma};p_1,p_2). \tag{6.50}$$

They give the following contribution in (6.46),

$$[\mathcal{O}^{t\,p_1}_{\ \ i_1}]^{\sigma_1}[\mathcal{O}^{t\,p_2}_{\ \ i_2}]^{\sigma_2}[\mathcal{O}^{t\,p_3}_{\ \ i_3}]^{\beta}\left(-\Gamma^{t\,\sigma}_{\sigma_2\beta}\partial_\sigma\partial_{\sigma_1} - \Gamma^{t\,\sigma}_{\beta\sigma_1}\partial_\sigma\partial_{\sigma_2} + \Gamma^{t\,0}_{\sigma_1\sigma_2\beta}\partial_0\right)W\Big|_{\left(s_c^{1/t\mu},1/t\mu\right)} + (1\to 2\to 3) + (1\to 3\to 2), \tag{6.51}$$

which cancels the new semilocal divergence (first two terms between parenthesis) and the local one (third term between parenthesis). Figure 6.7 shows the diagramatic representation of the cancelation of the non-local terms. The renormalized result is

$$\hat{G}^R_{i_1 i_2 i_3}(p_1,p_2,p_3) = \mu^d (2\pi)^d \delta(p_1+p_2+p_3) \mathcal{R}_{i_1 i_2 i_3}(\gamma^\mu;p_1,p_2,p_3). \tag{6.52}$$

## 6.2 Exact UV Renormalization

This section is devoted to making contact between the previous section and Chapter 5, following the line developed in Chapter 3. In the previous section we have reviewed the renormalization process applied to correlation functions. There, we essentially tuned some "counterterms" to extract the finite part of the correlation functions.

The renormalization procedure has been performed around a specific critical point

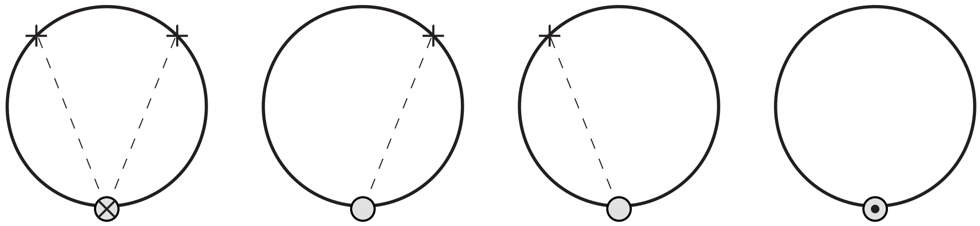

Figure 6.7: Witten diagrams involved in the cancelation of the non-local terms. Crosses indicate insertions of $[\mathcal{O}^t_\sigma]^{\sigma'}\partial^c_{\sigma'}$, the circled crosses, insertions of $[\mathcal{O}^t_\sigma]^\beta\partial^c_\beta$, shaded circles, insertions of $-[\mathcal{O}^t_{\sigma_1}]^{\sigma'}[\mathcal{O}^t_{\sigma_2}]^\beta\Gamma^\sigma_{\sigma'\beta}\partial^c_\sigma$, and the dotted shaded circles, insertions of $[\mathcal{O}^t_{\sigma_1}]^{\sigma'_1}[\mathcal{O}^t_{\sigma_2}]^{\sigma'_2}[\mathcal{O}^t_{\sigma_3}]^\beta\Gamma^0_{\sigma'_1\sigma'_2\beta}\partial^c_0$.

(the Dirichlet critical point). We will see that indeed, this point belongs to the critical manifold of the standard-quantization fixed point.

Normal coordinates are in intimate connection with the minimal subtraction class of schemes we used in Section 6.1. As proved in Chapter 2, the renormalized correlators in such schemes are equal to successive derivatives in normal coordinates of the generator $W$ at the regulated fixed point

$$G^R_{\alpha_1\ldots\alpha_n} = \partial^{\bar{c}}_{\alpha_1}\ldots\partial^{\bar{c}}_{\alpha_n}W\Big|_{\left(s_*^{1/\mu},\mu\right)}. \tag{6.53}$$

In this section we will find perturbatively the exact UV renormalization charts:

$$r_t(s,\epsilon) = \bar{c}\circ f_{1/t}(s,\epsilon) - \bar{c}\circ f_{1/t}(s^\epsilon_c,\epsilon). \tag{6.54}$$

using the Dirichlet critical point. After that, we will compare results with the charts found in Section 6.1. This will be done studying the asymptotic behaviour of Witten diagrams in a slice of the AdS space. For these computations, we will restrict ourselves to a system with generic dimensions without any resonance or exceptional case.

The UV exact renormalization charts, $r_t$, will be written perturbatively as functions of the Dirichlet chart $c$:

$$r^\alpha_t = r^\alpha_{t\ \alpha_1}(\bar{\gamma})c^{\alpha_1} + r^\alpha_{t\ \alpha_1\alpha_2}(\bar{\gamma})c^{\alpha_1}c^{\alpha_2} + r^\alpha_{t\ \alpha_1\alpha_2\alpha_3}(\bar{\gamma})c^{\alpha_1}c^{\alpha_2}c^{\alpha_3} + O\left(c^4\right). \tag{6.55}$$

The coefficients are expanded as

$$\hat{r}^{a'}_{t\,a_1\ldots a_n}(\gamma;p,q_1,\ldots,q_n) = t^{\lambda_{(a')}}(2\pi)^d\sqrt{|\gamma|}\delta(p+q_1+\cdots+q_n)\rho^{a'}_{a_1\ldots a_n}(\gamma;q_1,\ldots,q_n). \tag{6.56}$$

Using our notation, the on shell action in a slice of the AdS space can be written as

$$S^B_\varphi \circ f_{1/t} = -\frac{1}{N^2}\log\int \mathcal{D}\pi\,[\mathcal{D}\phi]^{t\bar{\epsilon},\varphi}_{\bar{\epsilon}}\; e^{N^2\left[S_\pi+\pi_\sigma\phi^\sigma(\bar{\epsilon})-S^G\right]}. \tag{6.57}$$

Derivatives of it at $(s^\epsilon_c,\epsilon)$ and $\varphi=0$ can be calculated using Witten diagrams. In particular,

$$D^{\epsilon,l}_{\alpha_1\ldots\alpha_n;\sigma'_1\ldots\sigma'_m} = -\left.\frac{\delta^m}{\delta\varphi^{\sigma'_1}\ldots\delta\varphi^{\sigma'_m}}\right|_{\varphi=0}\partial^c_{\alpha_1}\ldots\partial^c_{\alpha_n}\big|_{(s^\epsilon_c,\epsilon)}\left(S^B_\varphi\circ f_{\epsilon/l}\right) \tag{6.58}$$

will be Witten diagrams in a slice of the AdS space between $\epsilon$ and $l$ with $n$ legs on $\epsilon$ and $m$ legs on $l$. Diagrams with multitrace deformations on the left surface ($\epsilon$) are

$$D^{\epsilon,l}_{\alpha_1\alpha_2\ldots;\sigma'_1\ldots\sigma'_r} = Q^{\sigma^1_1\ldots\sigma^1_{n_1}}_{\alpha_1}(\gamma_\epsilon)Q^{\sigma^2_1\ldots\sigma^2_{n_2}}_{\alpha_2}(\gamma_\epsilon)\cdots D^{\epsilon,l}_{\langle\sigma^1_1\ldots\sigma^1_{n_1}\rangle\langle\sigma^2_1\ldots\,\sigma^2_{n_2}\rangle\ldots;\sigma'_1\ldots\sigma'_r}. \tag{6.59}$$

As in Section 6.1.3, $D^{\epsilon,l}_{\langle\sigma^1_1\ldots\sigma^1_{n_1}\rangle\langle\sigma^2_1\ldots\,\sigma^2_{n_2}\rangle\ldots;\sigma'_1\ldots\sigma'_r}$, are those diagrams given by all (connected or disconnected) single-trace tree-level Witten diagrams of $n_1+n_2+\cdots$ legs on the UV surface and $r$ legs on the IR surface that, upon contraction of the space-time points inside the same angle-bracket, are connected and contain no loops. In the representation of the corresponding Witten diagrams we will put together the boundary points in the same angle-bracket.

To compute these Witten diagrams, it is convenient to define $\mathcal{P}^{(j)}_{\epsilon,l}(z,q)$ ($\mathcal{K}^{(j)}_{\epsilon,l}(z,q)$) as the $\epsilon(l)$-boundary-to-bulk propagator. They are given by

$$\begin{aligned}\mathcal{K}^{(i)}_{\epsilon,l}(z,p) &= \left(\frac{z}{l}\right)^{\frac{d}{2}}\frac{K_{\nu_{(i)}}(p_\epsilon)I_{\nu_{(i)}}(p_z)-K_{\nu_{(i)}}(p_z)I_{\nu_{(i)}}(p_\epsilon)}{K_{\nu_{(i)}}(p_\epsilon)I_{\nu_{(i)}}(p_l)-K_{\nu_{(i)}}(p_l)I_{\nu_{(i)}}(p_\epsilon)},\\ &= \frac{\Psi_{\nu_{(i)}}(\epsilon,z,p)}{\Psi_{\nu_{(i)}}(\epsilon,l,p)},\\ \mathcal{P}^{(i)}_{\epsilon,l}(z,p) &= \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}}\frac{K_{\nu_{(i)}}(p_z)I_{\nu_{(i)}}(p_l)-K_{\nu_{(i)}}(p_l)I_{\nu_{(i)}}(p_z)}{K_{\nu_{(i)}}(p_\epsilon)I_{\nu_{(i)}}(p_l)-K_{\nu_{(i)}}(p_l)I_{\nu_{(i)}}(p_\epsilon)}\end{aligned} \tag{6.60}$$

$$= \frac{\Psi_{\nu_{(i)}}(l,z,p)}{\Psi_{\nu_{(i)}}(l,\epsilon,p)}. \tag{6.61}$$

### 6.2.1 Dirichlet Manifold and the Critical Point

The set of Dirichlet boundary conditions at $\epsilon$ is seen in our approach as a submanifold of the holographic space of theories, $\mathcal{D} \subset \mathcal{W}$. Following Section 3.2, $\mathcal{D}$ can be used as bare manifold. Renormalized trajectories emanating from the standard-quantization fixed point through relevant and marginally relevant single-trace deformations can be reached with a renormalization procedure [6]. In fact, the bare couplings as functions of the cutoff, $h_t = r_t^{-1}$, can be chosen to be a solution of the equation of motion of the gravity fields (plus a vacuum energy coupling to be determinate). This method has the advantage of providing directly the counterterms associated to semi-local divergences [6].

All this implies that the renormalization of the correlation functions of relevant operators can be done restricting the renormalization charts (or the bare couplings) to $\mathcal{D}$. However, as we have explicitly seen, renormalized irrelevant operators may require new directions that expand the Dirichlet manifold.

To show that the critical manifold cuts the Dirichlet manifold at least at one point (the critical point), let us prove that, in the quotient space $\mathcal{M}$, it flows under RG evolution towards the standard-quantization fixed point $\mathbf{s}_*$. The boundary action after a finite RG evolution is given, in path integral notation, by

$$\exp\left[-N^2 S^B_\varphi \circ f_t(s_c^{l/t}, 1/lt)\right] = \int [\mathcal{D}\phi]^{l,\varphi}_{l/t,0}\, e^{N^2\left[\int d^dx \sqrt{|\gamma_{l/t}|}\,\frac{v_0}{d} - S^G[\phi]\right]}. \tag{6.62}$$

$S^B \circ f_t(s_c^{l/t}, 1/lt)$ is thus obtained from solutions of the $S^G$ equations of motion that vanish at $l/t$. If we now take the limit $t \to \infty$, this condition forces the solutions to approach zero as fast as possible when $z \to 0$. The quadratic approximation to $S^G$ is then valid in the near-boundary region and we can use (5.118). The $t \to \infty$ boundary condition requires that the coefficient of the leading term $z^{\Delta^-_{(i)}}$ vanishes. This agrees with the boundary condition imposed by the standard-quantization fixed boundary action.

To study some properties of this critical point, let us compute (6.62) explicitly

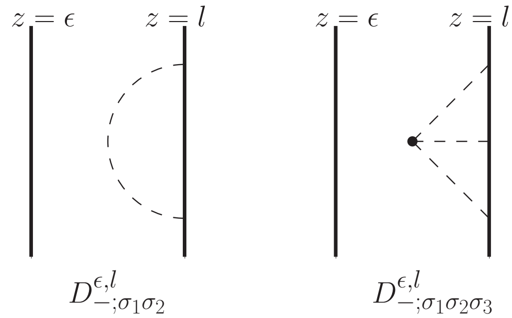


Figure 6.8: Witten diagrams $D^{\epsilon,l}_{-;\sigma_1\sigma_2}$ and $D^{\epsilon,l}_{-;\sigma_1\sigma_2\sigma_3}$.

$(D^{\epsilon,l}_{-;\sigma_1\ldots\sigma_n})$ for the first orders ($n \leq 3$), (see Figure 6.8):

$$\check{D}^{\epsilon,l}_{-;ij}(p,-p) = -\,\delta_{ij}\, l^{-d+1}\, \partial_z \mathcal{K}^{(i)}_{\epsilon,l}(z,p)\Big|_{z=l}, \tag{6.63}$$

$$\check{D}^{\epsilon,l}_{-;ijk}(p_1,p_2,p_3) = -\, v_{i_1 i_2 i_3} \int_\epsilon^l \frac{dz}{z^{d+1}} \mathcal{K}^{(i_1)}_{\epsilon,l}(z,p_1)\mathcal{K}^{(i_2)}_{\epsilon,l}(z,p_2)\mathcal{K}^{(i_3)}_{\epsilon,l}(z,p_3). \tag{6.64}$$

We will say that some normal direction, $a$, is excited by the critical point if $\bar{c}^a \circ f_t(s^\epsilon_c,\epsilon) \neq 0$ for some normal chart $\bar{c}$ and some $0 < t < 1$. Notice that, since there are no diagrams with a single leg on the right, $S_\varphi \circ f_t(s^\epsilon_c,\epsilon) = S_\alpha[\bar{\gamma};\varphi]\,\bar{c}^\alpha \circ f_t(s^\epsilon_c,\epsilon)$ cannot have linear terms in $\varphi$. Since only single-trace eigenperturbations $S_\sigma$ can give such linear terms, we conclude that only multi-trace directions are excited by the Dirichlet critical point.

From the $\epsilon \to 0$ limit of (6.63) and (6.63), the form of the boundary action for the fixed point can be calculated. In fact, (5.57) has been found following this method.

### 6.2.2 Exact Renormalized Operators

Let us study now diagrams with only one deformation on the left (which could be single or multitrace):

$$\begin{aligned} \partial^c_\alpha\big|_{(s^\epsilon_c,\epsilon)} \left(S^B_\varphi \circ f_{1/t}\right) &= \left(f^*_{1/t}\partial^c_\alpha\right) S^B_\varphi\big|_{f_{1/t}(s^\epsilon_c,\epsilon)} \\ &= \left(f^*_{1/t}\partial^c_\alpha \bar{c}^{\alpha_1}\right)\,\left(\partial^{\bar{c}}_{\alpha_1} S^B_\varphi\right)\big|_{f_{1/t}(s^\epsilon_c,\epsilon)} \\ &= \left(\partial^c_\alpha r^{\alpha_1}_t\right)\big|_{(s^\epsilon_c,\epsilon)}\; \partial^{\bar{c}}_{\alpha_1} S^B_\varphi\big|_{f_{1/t}(s^\epsilon_c,\epsilon)}\,. \end{aligned} \tag{6.65}$$

In the first line we have made use of the definition of pull-back, in the second, we have applied the chain rule and in the third one, we have used the definition of pull-back again and (6.54). Using (5.59), we have

$$\partial^{\bar{c}}_{\alpha} S^B_{\varphi}\Big|_{f_{1/t}(s^\epsilon_c,\epsilon)} = S_\alpha[\gamma_l,\varphi] + 2\,\bar{c}^{\alpha_1}\left(f_{1/t}(s^\epsilon_c,\epsilon)\right) S_{\alpha\alpha_1}[\gamma_l,\varphi] \\ + 3\,\bar{c}^{\alpha_1}\left(f_{1/t}(s^\epsilon_c,\epsilon)\right)\bar{c}^{\alpha_2}\left(f_{1/t}(s^\epsilon_c,\epsilon)\right) S_{\alpha\alpha_1\alpha_2}[\gamma_l,\varphi] + \dots \tag{6.66}$$

If we knew the expansion of $S^B_\varphi$ in normal coordinates, we would be able to extract the linear relation between the charts $r_t$ and $c$, and thus, the renormalized operators. Actually, in the non-resonant case, the knowledge of the eigenvalues is enough to split the different contributions (resonances will be studied in Section 6.4). Therefore, $r_t{}^{\alpha_1}{}_{\alpha}$ and $S_{\alpha_1}[\gamma;\varphi]$ can be extracted analysing the behaviour of (6.65), fixing $l = t\epsilon$ and taking $\epsilon \to 0$:

$$\hat{\partial}^{c\,p}_{a}\Big|_{(s^\epsilon_c,\epsilon)} \left(S^B_\varphi \circ f_{l/\epsilon}\right) = r_{l/\epsilon}{}^{\alpha}{}_{a}(\gamma_\epsilon;p)\ S_\alpha[\gamma_l;\varphi] + \text{remaining orders in } \epsilon \\ = \left(\frac{l}{\epsilon}\right)^{\lambda_{(a')}} \rho^{a'}_a(\gamma_\epsilon;p)\hat{S}^p_{a'}[\gamma_l;\varphi] + \text{remaining orders in } \epsilon. \tag{6.67}$$

The first term of the last member side carries orders $\epsilon^{-\lambda_{(\alpha_1)}+n^u_{(\alpha_1)}-n^d_{(\alpha_1)}+n_\partial}$, with $n^d_{(\alpha)}$, ($n^u_{(\alpha)}$) the number of covariant (contravariant) indices of the chart $c^\alpha$ and $n_\partial \in \mathbb{N}_0$. The remaining orders will be of the form $\epsilon^{-\sum_i\left[\lambda_{(\alpha_i)}-n^u_{(\alpha_i)}+n^d_{(\alpha_i)}\right]+n_\partial}$ and thus, if there are no resonances, the knowledge of the spectra of eigenvalues is enough to discard the remaining contributions.

Expanding (6.67) also in the number of fields, we have

$$\check{D}^{\epsilon,l}_{a;j_1\dots j_n}(p;p_1,\dots,p_n) = -\,n!\,l^{-d}\left(\frac{l}{\epsilon}\right)^{\lambda_{(a')}} \rho^{a'}_a(\gamma_\epsilon;p)\,T_{a';i_1\dots i_n}(\gamma_l;p;p_1,\dots,p_n) \\ + \text{remaining orders in } \epsilon. \tag{6.68}$$

Remember that $S_{a^n}[\gamma,\varphi]$, with $a^n$ a $n$-trace deformation, has only terms of the form $\prod_i^m \varphi_i$, with $m \geq n$. This is, $T_{a^n;i_1\dots i_m} = 0$ if $m < n$. Therefore, in (6.68), $D_{\alpha;\sigma_1\dots\sigma_n}$ receives only contributions from $a'$ running from singletrace to $n$-trace deformations. Also, to find all linear contributions to some $n$-trace renormalized chart $r_t^{\alpha^n}$, it is necessary the analysis of all diagrams of the form $D_{\alpha;\sigma_1\dots\sigma_n}$. For this reason, we organize the following analysis studying the diagrams in increasing order of number of right legs.

### Vacuum Renormalized Coupling

Let us start with the diagrams of the form $D^{\epsilon,l}_{\alpha;-}$. At tree level, the only one giving a contribution is (see (6.57) and (6.67)):

$$D^{\epsilon,l}_{0x;-} = -\, t^d \sqrt{|\gamma_l|}. \tag{6.69}$$

No further diagrams can contribute. The only eigenperturbation $S_\alpha[\gamma;\varphi]$ giving a contribution to terms $O\left((\varphi_i)^0\right)$ is the vacuum energy, which is also the only one with $\lambda_{(0)} = d$. We can conclude that

$$\begin{aligned} r^0_t(x) =& t^d\, c^0(x) + O\left(c^2\right), \\ S_{0x}[\gamma;\varphi] =& \sqrt{|\gamma|}. \end{aligned} \tag{6.70}$$

### Single-Trace Renormalized Coupling

The left diagram of Figure 6.9 is the only non-vanishing diagram with one insertion on the left and one leg on the right: $D^{\epsilon,l}_{\sigma;\sigma'}$. This gives the single-trace renormalized charts $r^\sigma_t$ at the linear level (in the Dirichlet charts). It can be expanded as

$$\begin{aligned} \check{D}^{\epsilon,l}_{i;j}(p,-p) =& \frac{\delta_{ij}}{2}\left[\, z^{-d+1}\partial_z \mathcal{K}^{(i)}_{\epsilon,l}(z,p)\Big|_{z=\epsilon} -\, z^{-d+1}\partial_z \mathcal{P}^{(i)}_{\epsilon,l}(z,p)\Big|_{z=l}\,\right] \\ =& \,\delta_{ij}\frac{2\,\epsilon^{-d/2}\, l^{-d/2}}{\Psi_{\nu_{(i)}}(\epsilon,l,p)} \\ =& \,\delta_{ij}\; l^{-d}\Bigg\{ \left(\frac{l}{\epsilon}\right)^{\Delta^-_{(i)}} \left[\frac{2^{\nu_{(i)}+1}\nu_{(i)}\, p_\epsilon^{-\nu_{(i)}}}{\Gamma(1-\nu_{(i)}) I_{-\nu_{(i)}}(p_\epsilon)}\right]\left[\frac{p_l^{\nu_{(i)}}}{\Gamma(1+\nu_{(i)})2^{\nu_{(i)}} I_{\nu_{(i)}}(p_l)}\right] \\ &+ \left(\frac{l}{\epsilon}\right)^{\Delta^-_{(i)}-2\nu_{(i)}} a'_{(i)}(p_\epsilon)b'_{(i)}(p_l) + \left(\frac{l}{\epsilon}\right)^{\Delta^-_{(i)}-4\nu_{(i)}} a''_{(i)}(p_\epsilon)b''_{(i)}(p_l) + \dots \Bigg\}. \end{aligned} \tag{6.71}$$

This diagram is only sensitive to single-trace deformations, whose eigenvalue is $\Delta^-_{(i)}$. Therefore, the first line of the last equality of (6.71) necessarily contributes to the first

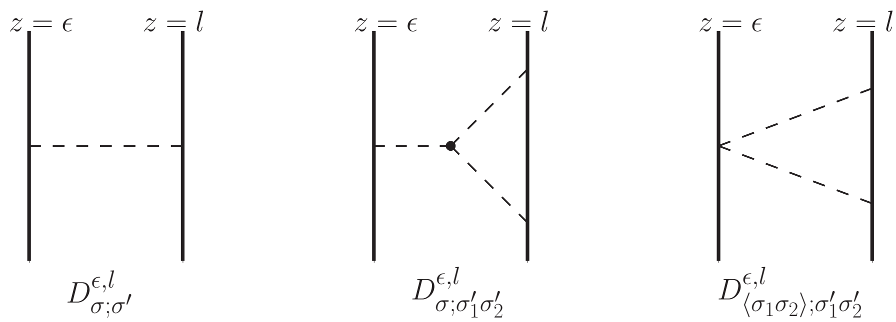


Figure 6.9: Possible Witten diagrams with one insertion on the left (including possible multitrace ones) and up to two legs on the right.

line of (6.68). Since $S_\sigma[\gamma;\varphi]$ must be a quasilocal function, the splitting between the two terms of (6.68) is unique up to a normalization. Choosing the normalization of $S_\sigma[\gamma_l,\varphi]$ to be the one of (5.63), we obtain

$$\begin{aligned}\rho_i^j(\gamma;p) &= \delta_i^j \left[-\frac{2^{\nu_{(i)}+1}\nu_{(i)}\, p_\gamma^{-\nu_{(i)}}}{\Gamma(1-\nu_{(i)})I_{-\nu_{(i)}}(p_\gamma)}\right] \\ &= \delta_i^j \left[-2\nu_{(i)} - \frac{2\nu_{(i)}}{4\nu_{(i)}-4}p_\gamma^2 + O(p_\gamma^4)\right].\end{aligned} \tag{6.72}$$

Functions $a'^{\cdots}_{(i)}$ and $b'^{\cdots}_{(i)}$ of the remaining terms of (6.71) are analytic. Then, these terms carry orders $O(\epsilon^{-\Delta^-_{(i)}-2n\nu_{(i)}-2n_\partial})$, with $n \in \mathbb{N}^+$ and $n_\partial \in \mathbb{N}_0$, that cannot be identified with any eigenvalue (in the generic case). They must correspond to the remaining orders of (6.68).

Since no further diagrams with one insertion on the left and one leg on the right can be built, $\rho^i_{a^n}(\gamma;p) = 0$ for higher trace deformations ($n > 1$), and

$$\hat{r}_t^{ip} = \rho_{(i)}(\bar{\gamma};p)\hat{c}^{ip} + O(c^2). \tag{6.73}$$

### Double-Trace Renormalized Coupling

The rest of diagrams of Figure 6.9 have one insertion on the left and two legs on the right. They give us the double-trace renormalized charts at the linear level (in the

Dirichlet charts). Starting with $D^{\epsilon,l}_{\sigma;\sigma_1\sigma_2}$:

$$\begin{aligned}\check{D}^{\epsilon,l}_{i;j_1j_2}(p,p_1,p_2) =& -v_{ij_1j_2}\int_\epsilon^l \frac{dz}{z^{d+1}}\mathcal{P}^{(i)}_{\epsilon,l}(z,p)\mathcal{K}^{(j_1)}_{\epsilon,l}(z,p_1)\mathcal{K}^{(j_2)}_{\epsilon,l}(z,p_2)\\ =& -2l^{-d}\Bigg\{\left(\frac{l}{\epsilon}\right)^{\lambda_{(b)}}\rho^b_i(\gamma_\epsilon;p)\,T_{b;j_1j_2}(\gamma_l;p;p_1,p_2)\\ &+\left(\frac{l}{\epsilon}\right)^{\Delta^-_{(j)}}\rho^j_i(\gamma_\epsilon;p)\,T_{j;j_1j_2}(\gamma_l;p;p_1,p_2)+\dots\Bigg\}. \end{aligned}\tag{6.74}$$

The integral in $z$ is hard to calculate analytically, but can be easily expanded in powers of momenta and computed order by order. We can fix the primitive to be the one with a vanishing constant term in its power expansion around $z=0$. This way, its evaluation on $\epsilon$ and $l$ is $O(\epsilon^{\nu_{(j_1)}+\nu_{(j_2)}})$, and $O(\epsilon^{-\Delta^-_{(i)}})$ respectively. They must correspond to the first and second line of the last member of (6.74). Thereby, we obtain

$$\mathbf{Q}_b^{\;j_1j_2}(\gamma;p_1,p_2)\,\rho^b_i(\gamma;-p_1-p_2) = \rho_{(j_1)}(\gamma;p_1)\rho_{(j_2)}(\gamma;p_2)\,\tilde{\rho}_i^{\langle j_1j_2\rangle}(\gamma;p_1,p_2),\tag{6.75}$$

where

$$\tilde{\rho}_i^{\langle j_1j_2\rangle}(\gamma_\epsilon;q_1,q_2) = -\frac{v_{i\,j_1j_2}}{2}\int\frac{dz}{z}\left(\frac{\epsilon}{z}\right)^d\Upsilon_{\nu_{(i)}}(\epsilon,z,-q_1-q_2)\Psi_{\nu_{(j_1)}}(\epsilon,z,q_1)\Psi_{\nu_{(j_2)}}(\epsilon,z,q_2)\Big|_{z=\epsilon}.\tag{6.76}$$

Expanding (6.75) in momenta, we obtain $\rho^b_i(p)$ for all double-trace deformation $b$. Let us remark also that (5.72) gives the quadratic contribution in $\varphi$ to the single-trace renormalized eigenperturbation. It is indeed the solution of (5.67).

Lastly, the right-hand diagram of Figure 6.9,

$$D^{\epsilon,l}_{\langle\sigma_1\sigma_2\rangle;\sigma_1'\sigma_2'} = D^{\epsilon,l}_{\sigma_1;\sigma_1'}\,D^{\epsilon,l}_{\sigma_2;\sigma_2'} + D^{\epsilon,l}_{\sigma_1;\sigma_2'}\,D^{\epsilon,l}_{\sigma_2;\sigma_1'},\tag{6.77}$$

is the only one with one double-trace insertion on the left and two legs on the right:

$$\check{D}^{\epsilon,l}_{b;j_1j_2}(p;p_1,p_2) = \frac{2}{\sqrt{|\gamma_\epsilon|}}\,\mathbf{Q}_b^{\;i_1i_2}(\gamma_\epsilon;p_1,p_2)\,\check{D}^{\epsilon,l}_{i_1;j_1}(-p_1,p_1)\check{D}^{\epsilon,l}_{i_2;j_2}(-p_2,p_2)$$

$$= 2\, l^{-d} \Bigg\{ \left(\frac{l}{\epsilon}\right)^{-\nu_{(1)}-\nu_{(2)}} \rho_{(j_1)}(\gamma_\epsilon, p_1)\rho_{(j_2)}(\gamma_\epsilon, p_2)$$
$$\times\, \mathbf{Q}_b{}^{j_1 j_2}(\gamma_\epsilon; p_1, p_2) T_{(j_1)}(\gamma_l; p_1) T_{(j_2)}(\gamma_l; p_2) + \dots \Bigg\}, \qquad (6.78)$$

where the ellipsis stand for terms whose order in $\epsilon$ is not an eigenvalue.

Comparing with (6.68), we obtain

$$\mathbf{Q}_{b'}{}^{j_1 j_2}(\gamma; p_1, p_2)\, \rho_b^{b'}(\gamma; p_1 + p_2) = -\mathbf{Q}_b{}^{j_1 j_2}(\gamma; p_1, p_2)\, \rho_{(j_1)}(\gamma; p_1)\rho_{(j_2)}(\gamma; p_2), \qquad (6.79)$$

equation that, from an expansion in momenta, can be used to extract $\rho_b^{b'}(\gamma; p_1 + p_2)$ $\forall\, b, b'$. Since no more diagrams have to be considered at this order, we can write,

$$\mathbf{Q}_b{}^{j_1 j_2}(\gamma; p_1, p_2)\, \hat{r}_t^b(p_1 + p_2) = t^{-\nu_{(j_1)}-\nu_{(j_2)}} \rho_{(j_1)}(\bar{\gamma}; p_1)\rho_{(j_2)}(\bar{\gamma}; p_2) \Big[ \tilde{\rho}_i^{\langle j_1 j_2 \rangle}(\bar{\gamma}; p_1, p_2)\, \hat{c}^i(p_1 + p_2)$$
$$- \mathbf{Q}_b{}^{j_1 j_2}(\gamma; p_1, p_2)\, \hat{c}^b(p_1 + p_2) \Big] + O(c^2). \qquad (6.80)$$

In this manner, we could continue calculating triple and higher-trace renormalized operators. However, single and double-trace renormalization charts are enough for the three point functions.

Once we know the renormalized charts at the linear level, we can invert the equations to find the single and double-trace renormalized operators at the critical point:

$$\hat{\partial}_i^{r_t}(p) = \frac{t^{-\Delta^-_{(i)}}}{\rho_{(i)}(\bar{\gamma}; p)} \left[ \hat{\partial}_i^c(p) + \tilde{\rho}_i^b(\bar{\gamma}; p)\hat{\partial}_b^c(p) + O\left(\hat{\partial}_{a^3}^c\right) \right], \qquad (6.81)$$

$$\hat{\partial}_b^{r_t}(p) = t^{-\lambda_{(b)}} \left[ -\tilde{\rho}_b^{b'}(\bar{\gamma}; p)\hat{\partial}_{b'}^c(p) + O\left(\hat{\partial}_{a^3}^c\right) \right], \qquad (6.82)$$

with

$$\mathbf{Q}_b{}^{i_1 i_2}(\gamma; p_1, p_2)\, \tilde{\rho}_i^b(\gamma; -p_1 - p_2) = \tilde{\rho}_i^{\langle i_1 i_2 \rangle}(\gamma; p_1, p_2),$$
$$\mathbf{Q}_{b'}{}^{i_1 i_2}(\gamma; p_1, p_2)\, \tilde{\rho}_b^{b'}(\gamma; -p_1 - p_2) = \frac{\mathbf{Q}_b{}^{i_1 i_2}(\gamma; p_1, p_2)}{\rho_{(i_1)}(\gamma; p_1)\rho_{(i_2)}(\gamma; p_2)}. \qquad (6.83)$$

To conclude this section, notice that, as we have seen in the cases analysed so far,

$D_{\alpha^n;\sigma_1\dots\sigma_m} = 0$ if $\forall m < n$. This implies that

$$\rho^{a^m}_{a^n}(\gamma; p) = 0 \quad \text{and} \quad [\mathcal{O}^t_{\alpha^n}]^{\alpha^m} = 0, \quad \forall m < n. \tag{6.84}$$

### 6.2.3 Higher Orders

To compute counterterms and find higher orders of the normal expansion of the boundary action, we will study diagrams with more than one deformation on the left. Performing successive derivatives of (6.65) one obtains

$$\begin{aligned}
\partial^c_{\alpha_1}\partial^c_{\alpha_2}\big|_{(s^\epsilon_c,\epsilon)}\left(S^B_\varphi \circ f_{\epsilon/l}\right) &= \left(\partial^c_{\alpha_1}\partial^c_{\alpha_2} r^{\alpha'}_{l/\epsilon}\right)\Big|_{(s^\epsilon_c,\epsilon)} \partial^{\bar c}_{\alpha'} S^B_\varphi\big|_{f_{\epsilon/l}(s^\epsilon_c,\epsilon)} \\
&\quad + \left(\partial^c_{\alpha_1} r^{\alpha'_1}_{l/\epsilon}\right)\left(\partial^c_{\alpha_2} r^{\alpha'_2}_{l/\epsilon}\right)\Big|_{(s^\epsilon_c,\epsilon)} \partial^{\bar c}_{\alpha'_1}\partial^{\bar c}_{\alpha'_2} S^B_\varphi\Big|_{f_{\epsilon/l}(s^\epsilon_c,\epsilon)}, \qquad (6.85) \\
\partial^c_{\alpha_1}\partial^c_{\alpha_2}\partial^c_{\alpha_3}\big|_{(s^\epsilon_c,\epsilon)}\left(S^B_\varphi \circ f_{\epsilon/l}\right) &= \left(\partial^c_{\alpha_1}\partial^c_{\alpha_2}\partial^c_{\alpha_3} r^{\alpha'}_{l/\epsilon}\right)\Big|_{(s^\epsilon_c,\epsilon)} \partial^{\bar c}_{\alpha'} S^B_\varphi\big|_{f_{\epsilon/l}(s^\epsilon_c,\epsilon)} \\
&\quad + 3\left(\partial^c_{(\alpha_1}\partial^c_{\alpha_2} r^{\alpha'_1}_{l/\epsilon}\right)\left(\partial^c_{\alpha_3)} r^{\alpha'_2}_{l/\epsilon}\right)\Big|_{(s^\epsilon_c,\epsilon)} \partial^{\bar c}_{\alpha'_1}\partial^{\bar c}_{\alpha'_2} S^B_\varphi\Big|_{f_{\epsilon/l}(s^\epsilon_c,\epsilon)} \\
&\quad + \left(\partial^c_{\alpha_1} r^{\alpha'_1}_{l/\epsilon}\right)\left(\partial^c_{\alpha_2} r^{\alpha'_2}_{l/\epsilon}\right)\left(\partial^c_{\alpha_3} r^{\alpha'_3}_{l/\epsilon}\right)\Big|_{(s^\epsilon_c,\epsilon)} \partial^{\bar c}_{\alpha'_1}\partial^{\bar c}_{\alpha'_2}\partial^c_{\alpha'_3} S^B_\varphi\Big|_{f_{\epsilon/l}(s^\epsilon_c,\epsilon)}, \\
&\dots \qquad (6.86)
\end{aligned}$$

Using (6.54), (6.56) and (5.61), we can extract

$$\begin{aligned}
\check{D}^{\epsilon,l}_{a_1\dots a_n;i_1\dots i_m}(p_1,\dots;q_1,\dots) = &- n!\,m!\,l^{-d}\left(\frac{l}{\epsilon}\right)^{\lambda_{(a)}} \rho^a_{a_1\dots a_n}(\gamma;p_1,\dots)T_{a;i_1\dots i_m}(\gamma;p;q_1,\dots) \\
&+ \text{remaining orders in } \epsilon.
\end{aligned} \tag{6.87}$$

where the remaining orders are of the form $\epsilon^{-\sum_i \lambda_{(a_i)}+n_\partial}$. Therefore, the identification of the orders of the form $\epsilon^{-\lambda_{(a)}+n_\partial}$ allows to calculate $\rho^a_{a_1\dots a_n}(\gamma;p_1,\dots,p_n)$.

In this section we extract the quadratic contribution to the single-trace and vacuum renormalization charts and the cubic contribution of the vacuum renormalization chart. As we will see, this is enough to compute the counterterms that the renormalization of the three point function requires. As in the previous subsection, we will organize this calculation depending on the number of right legs of the diagrams.

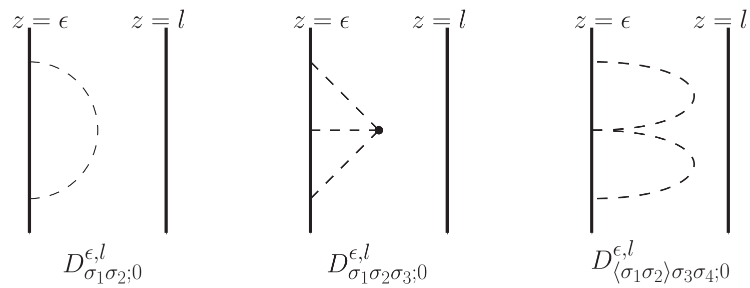


Figure 6.10: Witten diagrams with up to three insertions on the left and no legs on the right.

**Vacuum Renormalized Coupling**

Figure 6.10 shows all diagrams with no legs on the right and up to three insertions on the left (we will not compute terms with quartic and higher orders). We expand now in $\epsilon$, and write only terms that go as $\epsilon^{-d+n_\partial}$,

$$\check{D}^{\epsilon,l}_{i_1i_2;-}(p,-p) = \epsilon^{-d}\delta_{i_1i_2}G^{\mathrm{L}}_{(i_1)}(\gamma_\epsilon;p)+\dots \tag{6.88}$$

$$\check{D}^{\epsilon,l}_{i_1i_2i_3;-}(p_1,p_2,p_3) = -\epsilon^{-d}v_{i_1i_2i_3}\mathcal{Z}_{i_1i_2i_3}(\gamma_\epsilon;p_1,p_2,p_3)+\dots \tag{6.89}$$

$$\check{D}^{\epsilon,l}_{i_1i_2b;-}(p_1,p_2,p_3) = 2\epsilon^{-d}\mathbf{Q}_b{}^{i_1i_2}(\gamma_\epsilon;p_1,p_2)G^{\mathrm{L}}_{(i_1)}(\gamma_\epsilon;p_1)G^{\mathrm{L}}_{(i_2)}(\gamma_\epsilon;p_2)+\dots \tag{6.90}$$

Comparing with (6.87) we obtain $\rho^0_{a_1,\dots,a_n}(\gamma;p_1,\dots)$, with $n\leq 3$, and therefore, we can write

$$r^0_t = t^d\Bigg[c^0-\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\frac{G^{\mathrm{L}}_{(i)}(\bar{\gamma}p)}{\sqrt{|\bar{\gamma}|}}\hat{c}^{ip}\hat{c}^{i-p}$$
$$-\frac{1}{3}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}|\bar{\gamma}|}\mathbf{Q}_b{}^{i_1i_2}(\bar{\gamma};p_1,p_2)G^{\mathrm{L}}_{(i_1)}(\bar{\gamma};p_1)G^{\mathrm{L}}_{(i_2)}(\bar{\gamma};p_2)c^{i_1}(p_1)c^{i_2}(p_2)c^b(-p_1-p_2)$$
$$+\frac{1}{6}\int\frac{d^dp_1d^dp_2d^dp_3}{(2\pi)^{2d}|\bar{\gamma}|}\delta(p_1+p_2+p_3)\mathcal{Z}_{i_1i_2i_3}(\bar{\gamma};p_1,p_2,p_3)c^{i_1p_1}c^{i_2p_2}c^{i_3p_3}+O\left(c^4\right)\Bigg]. \tag{6.91}$$

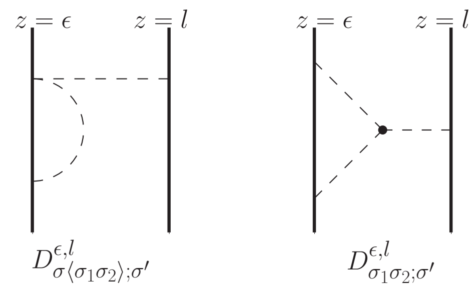


Figure 6.11: Witten diagrams with up to three insertions on the left and one leg on the right.

### Single-Trace Renormalized Coupling

Figure 6.11 shows all diagrams with one leg on the right and up to two insertions on the left (for single-trace renormalization chart, we only need to compute quadratic terms). Expanding in $\epsilon$ and writing only orders $\epsilon^{-\Delta^-_{(i)}+n_\partial}$:

$$\check{D}^{\epsilon,l}_{ib;j}(p_1,p_2;p) = 2\,l^{-d}\left(\frac{l}{\epsilon}\right)^{\Delta^-_{(i)}} \mathbf{Q}_b{}^{ij}(\gamma_\epsilon;p_1,-p)G^{\mathrm{L}}_{(i)}(\gamma_\epsilon;p_1)\rho_{(j)}(\gamma_\epsilon;p)T_{(j)}(\gamma_l;p)+\ldots \tag{6.92}$$

$$\check{D}^{\epsilon,l}_{i_1i_2;i}(p_1,p_2;p) = l^{-d}\left(\frac{l}{\epsilon}\right)^{\Delta^-_{(i)}} \rho_{(i)}(\gamma_\epsilon,p)\tilde{\rho}^i{}_{i_1i_2}(\gamma_\epsilon;p_1,p_2)T_{(i)}(\gamma_l;p)+\ldots \tag{6.93}$$

with

$$\tilde{\rho}^i{}_{i_1i_2}(\gamma_\epsilon;p_1,p_2) = v_{ii_1i_2}\int\frac{dz}{z}\left(\frac{\epsilon}{z}\right)^d \Upsilon_{\nu_{(i_1)}}(\epsilon,z,p_1)\Upsilon_{\nu_{(i_2)}}(\epsilon,z,p_2)\Psi_{\nu_{(i)}}(\epsilon,z,p_1+p_2)\Big|_{z=\epsilon}. \tag{6.94}$$

Comparing with (6.87), these diagrams give us the quadratic terms of the single-trace renormalization charts. Adding these new terms to (6.73), we obtain

$$\hat{r}^i_t(p) = t^{\Delta^-_{(i)}}\Big[\rho_{(i)}(\bar{\gamma};p)\hat{c}^i(p)$$

$$- \int \frac{d^d p_1 d^d p_2}{(2\pi)^d |\bar{\gamma}|} \delta(p + p_1 + p_2) \mathbf{Q}_b{}^{ji}(\bar{\gamma}; p_1, -p) G^{\mathrm{L}}_{(j)}(\bar{\gamma}; p_1) \rho_{(i)}(\bar{\gamma}; p) \hat{c}^j(p_1) \hat{c}^b(p_2)$$
$$\left. - \int \frac{d^d p_1 d^d p_2}{(2\pi)^d |\bar{\gamma}|} \delta(p + p_1 + p_2) \rho_{(i)}(p) \tilde{\rho}^i{}_{i_1 i_2}(\bar{\gamma}; p_1, p_2) \hat{c}^{i_1}(p_1) \hat{c}^{i_2}(p_2) + O(c^3) \right]. \tag{6.95}$$

### 6.2.4 Christoffel Symbols

We have found the first orders of the expansion of the exact relation between exact UV renormalization charts and Dirichlet charts. Thus, we can extract the following Christoffel symbols and counterterms at the critical point. All formulas of this subsection are understood to be evaluated at the critical point $(s_c^\epsilon, \epsilon)$.

At the quadratic level we obtain

$$\begin{aligned}
\hat{\Gamma}^{t\,0}{}_{j_1 j_2}(q_1, q_2) &= \left(\partial_0^{r_t} c^0\right) \left(\hat{\partial}_{j_1}^{c\, q_1} \hat{\partial}_{j_2}^{c\, q_2} r_t^0\right) \\
&= - \sqrt{|\bar{\gamma}|} (2\pi)^d \delta(q_1 + q_2) \delta_{j_1 j_2} G^{\mathrm{L}}_{(j_1)}(\bar{\gamma}, q), && (6.96) \\
\hat{\Gamma}^{t\, i}{}_{j_1 j_2}(p, q_1, q_2) &= \left(\partial_\sigma^{r_t} \hat{c}^{ip}\right) \left(\hat{\partial}_{j_1}^{c\, q_1} \hat{\partial}_{j_2}^{c\, q_2} r_t^\sigma\right) \\
&= \sqrt{|\bar{\gamma}|} (2\pi)^d \delta(p + q_1 + q_2) \tilde{\rho}^i_{j_1 j_2}(\bar{\gamma}; q_1, q_2), && (6.97) \\
\hat{\Gamma}^{t\, i}{}_{jb}(p, q_1, q_2) &= \left(\partial_\sigma^{r_t} \hat{c}^{ip}\right) \left(\hat{\partial}_{j}^{c\, q_1} \hat{\partial}_{b}^{c\, q_2} r_t^\sigma\right) \\
&= 2 \sqrt{|\bar{\gamma}|} (2\pi)^d \delta(p + q_1 + q_2) G^{\mathrm{L}}_{(j)}(\bar{\gamma}, q_1) \mathbf{Q}_b{}^{ji}(\bar{\gamma}; q_1, p). && (6.98)
\end{aligned}$$

In the first line of the equations above, we have only written the contributions that are non-vanishing. The remaining Christoffel symbols with the vacuum or a single-trace component as upper index are exactly zero. Finally, we calculate the exact counterterm $\Gamma^{t0}{}_{\sigma_1 \sigma_2 \sigma_3}$. Inverting (6.25),

$$\begin{aligned}
\hat{\Gamma}^{t\,0\,p_1 p_2 p_3}_{\;\;\; i_1\, i_2\, i_3} &= \left[- \left(\partial_{i_1}^{p_1} \partial_{i_2}^{p_2} \partial_{i_3}^{p_3} r_t^0\right) \left(\partial_0^{r_t} c^0\right) + \left(\Gamma^{t\,0\;\, p_1}_{\;\;\sigma\, i_1} \Gamma^{\sigma p_2 p_3}_{\;\; i_2\, i_3} + \Gamma^{t\,0\;\, p_2}_{\;\;\sigma\, i_2} \Gamma^{\sigma p_3 p_1}_{\;\; i_3\, i_1} + \Gamma^{t\,0\;\, p_3}_{\;\;\sigma\, i_3} \Gamma^{\sigma p_1 p_2}_{\;\; i_1\, i_2}\right)\right] \\
&= \sqrt{|\bar{\gamma}|}\, (2\pi)^d \delta\left(\sum_{r=1}^{3} p_r\right) \Bigg\{ - \mathcal{Z}_{i_1 i_2 i_3}(\bar{\gamma}; p_1, p_2, p_3) + \Big[G^{\mathrm{L}}_{(i_3)}(\bar{\gamma}; p_3)\, \tilde{\rho}^{i_3}_{i_1 i_2}(\bar{\gamma}; p_1, p_2) \\
&\qquad + (1 \to 2 \to 3) + (1 \to 3 \to 2)\Big]\Bigg\}.
\end{aligned} \tag{6.99}$$

The found renormalized operators and counterterms make finite the limit and give the same renormalized correlation function (6.52).

### 6.2.5 Comparison with Minimal Subtraction

In the previous subsection we have found the renormalized operators and counterterms of associated to an exact UV scheme. However, they can be simplified using (3.42) preserving minimal subtraction. For instance, through a linear change in $r_t$ we can redefine the renormalized operators as

$$\partial_\alpha^{r_t} \to \partial_\alpha^{\tilde{r}_t} = \partial_\alpha^{r_t} + t^{\lambda_{(\alpha')}-\lambda_{(\alpha)}} \tilde{a}_\alpha^{\alpha'}(\bar{\gamma}) \partial_{\alpha'}^{r_t}, \tag{6.100}$$

where $\tilde{a}_\alpha^{\alpha'}$ is only non-vanishing if $\lambda_{(\alpha)} + 2n_{(\gamma^{-1})} - 2n_{(\gamma)} > \lambda_{(\alpha')}$. This always allows to rewrite (6.81) as

$$\hat{\partial}_i^{\tilde{r}_t}\Big|_{(s_c,\epsilon)} = -\frac{t^{-\Delta^-_{(i)}}}{2\nu_{(i)}} \left[ \hat{\partial}^c_{ip}\Big|_{(s_c,\epsilon)} + {\rho_i}^b(\gamma_\epsilon; p)\, \hat{\partial}^c_{b\,p}\Big|_{(s_c,\epsilon)} + O\left(\hat{\partial}^c_{a^n}\right)\right], \tag{6.101}$$

which is the renormalized operator found by minimal subtraction up to a normalization factor $-2\nu$.

When $\lambda_{(b)} < \lambda_{(i)}$, the double-trace deformation $b$ can also be removed. This is always the case if all operators are relevant. That is why for relevant operators it is enough to consider

$$\hat{\partial}_i^{\tilde{r}_t}\Big|_{(s_c,\epsilon)} = -\frac{t^{-\Delta^-_{(i)}}}{2\nu_{(i)}}\, \hat{\partial}^c_{ip}\Big|_{(s_c,\epsilon)}, \tag{6.102}$$

or, using a different normalization, simply

$$\hat{\partial}_i^{\tilde{r}_t}\Big|_{(s_c,\epsilon)} = t^{-\Delta^-_{(i)}}\, \hat{\partial}^c_{ip}\Big|_{(s_c,\epsilon)}. \tag{6.103}$$

However, if $\Delta^+_{(i)} > \Delta^+_{(j_1)} + \Delta^+_{(j_2)}$ for some $j_1$, $j_2$, and $v_{ij_1j_2} \neq 0$, the renormalized operator $[\mathcal{O}_{ix}]$ will necessary include the contribution of the bare double trace operators.

The Christoffel symbols and higher order counterterms we found in this section essentially agree with the ones found in Section 6.1. This is so because linear changes of the renormalization charts $r_t$ do not affect to the Christoffel symbols. The only difference is that there, we truncated them to make them local, and here, we have found the whole tower of derivatives. (3.42) with quadratic and cubic contributions can be used to make our operator components and Christoffel symbols local.

## 6.3 Normal Correlators

As we have anticipated in (6.53), the correlation functions in minimal subtraction schemes are given by functional derivatives of the generator $W$ at the fixed point. In this section we will check this identity with the two and three-point function that we have already computed with a renormalization process:

$$\partial^{\bar{c}}_{\sigma_1}\partial^{\bar{c}}_{\sigma_2}W\big|_{(s^\epsilon_*,\epsilon)} = \Big\langle S_{\sigma_1}[\gamma_\epsilon;\varphi]S_{\sigma_2}[\gamma_\epsilon;\varphi]\Big\rangle^{\mathrm{c}}_{(s^\epsilon_*,\epsilon)} - 2\Big\langle S_{\sigma_1\sigma_2}[\gamma_\epsilon;\varphi]\Big\rangle^{\mathrm{c}}_{(s^\epsilon_*,\epsilon)}, \tag{6.104}$$

$$\begin{aligned}\partial^{\bar{c}}_{\sigma_1}\partial^{\bar{c}}_{\sigma_2}\partial^{\bar{c}}_{\sigma_3}W\big|_{(s^\epsilon_*,\epsilon)} =& -\Big\langle S_{\sigma_1}[\gamma_\epsilon;\varphi]S_{\sigma_2}[\gamma_\epsilon;\varphi]S_{\sigma_3}[\gamma_\epsilon;\varphi]\Big\rangle^{\mathrm{c}}_{(s^\epsilon_*,\epsilon)} + 2\Big\langle S_{\sigma_1\sigma_2}[\gamma_\epsilon;\varphi]S_{\sigma_3}[\gamma_\epsilon;\varphi]\Big\rangle^{\mathrm{c}}_{(s^\epsilon_*,\epsilon)}\\ &+ 2\Big\langle S_{\sigma_2\sigma_3}[\gamma_\epsilon;\varphi]S_{\sigma_1}[\gamma_\epsilon;\varphi]\Big\rangle^{\mathrm{c}}_{(s^\epsilon_*,\epsilon)} + 2\Big\langle S_{\sigma_3\sigma_1}[\gamma_\epsilon;\varphi]S_{\sigma_2}[\gamma_\epsilon;\varphi]\Big\rangle^{\mathrm{c}}_{(s^\epsilon_*,\epsilon)}\\ &- 6\Big\langle S_{\sigma_1\sigma_2\sigma_3}[\gamma_\epsilon;\varphi]\Big\rangle^{\mathrm{c}}_{(s^\epsilon_*,\epsilon)}.\end{aligned} \tag{6.105}$$

In this expressions, the brackets around any functional, $\mathcal{F}[\varphi]$, represent the connected contribution to its expectation value,

$$\Big\langle \mathcal{F}[\varphi]\Big\rangle^{c}_{(s^\epsilon_*,\epsilon)} = \frac{\int [\mathcal{D}\phi]_\epsilon\, \mathcal{F}[\phi(\epsilon)] e^{-N^2\left[S^B_*[\gamma_\epsilon,\phi(\epsilon)]+S^G[\phi]\right]}}{\int [\mathcal{D}\phi]_\epsilon\, e^{-N^2[S^B_*[\gamma_\epsilon,\phi(\epsilon)]+S^G[\phi]]}} \ - \ \text{disconnected diagrams.} \tag{6.106}$$

Using (6.62), the fixed point boundary action can be written as a UV path integral,

$$\begin{aligned} e^{-N^2 S^B_*[\gamma_\epsilon;\varphi]} &= \lim_{\epsilon_0\to 0}\int [\mathcal{D}\phi]^{\epsilon,\varphi}_{\epsilon_0,0}\ e^{-N^2\left[S^G[\phi]-\int d^dx\sqrt{|\gamma_{\epsilon_0}|}\,v_0/d\right]}\\ &= \int_{\phi^{(0)}=0} [\mathcal{D}\phi]^{\epsilon,\varphi}_0\ e^{-N^2\left[S^G[\phi]+\int d^dx\sqrt{|\gamma_0|}\,v_0/d\right]}. \end{aligned} \tag{6.107}$$

In the first line we have fixed the value of the field to 0 in $\epsilon_0$, to later take the limit $\epsilon \to 0$. This is equivalent to the second line: we have integrated starting in $z = 0$, and fixed to zero the leading mode of $\phi$ when $z \to 0$, $\phi^{(0)} = 0$. Therefore, (6.106) becomes

$$\begin{aligned}\Big\langle \mathcal{F}[\varphi] \Big\rangle^{c}_{(s^{\epsilon}_{*},\epsilon)} =& \frac{\int_{\phi^{(0)}=0} [\mathcal{D}\phi]\, \mathcal{F}[\phi(\epsilon)] e^{-N^2 S^G[\phi]}}{\int_{\phi^{(0)}=0} [\mathcal{D}\phi]\, e^{-N^2 S^G[\phi]}} \quad - \text{ disconnected diagrams} \\ \equiv& \Big\langle \mathcal{F}[\phi(\epsilon)] \Big\rangle^{c}_{+} . \end{aligned} \tag{6.108}$$

The last equation is a definition of $\langle\cdot\rangle^c_+$. We obtain that derivatives of $W$ evaluated at the fixed point can be calculated like correlation functions of fields in the bulk. This argument is similar to the one used in [153] to prove the equivalence between the GKPW dictionary [117, 118] and the BDHM dictionary [105, 155, 156].[2] In the GKPW dictionary (the most used one), the CFT sources correspond to the asymptotic boundary condition of the fields in the gravity theory. However, in the BDHM dictionary, the CFT correlators are extracted from bulk correlators. This last dictionary is understood in our approach as a renormalization procedure using the fixed point as critical point. Thus, bare correlators are identified with bulk correlators of fields located at $z = \epsilon$, and renormalized correlators are extracted when $\epsilon \to 0$ (finding renormalized operators and counterterms).

Here, we are doing something different. We are calculating the renormalized correlators directly without taking $\epsilon \to 0$, and including the whole tower of irrelevant contributions. Therefore, the eigenperturbations $S_\alpha[\gamma, \varphi]$ can be understood as the CFT renormalized operators constructed in terms of local bulk fields.

To carry out the computation of (6.105), we need to know $\tilde{\mathcal{G}}^{(i)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$, the bulk-to-bulk correlator in the whole AdS for the field $i$, with vanishing leading mode as $z \to 0$. Bold letters represent $(d+1)$-coordinates $\boldsymbol{x}^A = (z, x^\mu)$. It is defined as the Green function:

$$\left[-g^{AB}(\boldsymbol{x}_1) \nabla^{\boldsymbol{x}_1}_A \nabla^{\boldsymbol{x}_1}_B + m^2_{(i)}\right] \tilde{\mathcal{G}}^{(i)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{\delta\left(\boldsymbol{x}_1 - \boldsymbol{x}_2\right)}{\sqrt{|g(\boldsymbol{x}_2)|}}, \tag{6.109}$$

where $g$ and $\nabla$ are the euclidean AdS metric and connection. In a mixed position/momentum representation,

$$\check{\tilde{\mathcal{G}}}^{(i)}_{pq}(z_1, z_2) = \mathcal{G}^{(i)}(z_1, z_2, p)$$

[2] We use the same names as [153] to refer to both dictionaries.

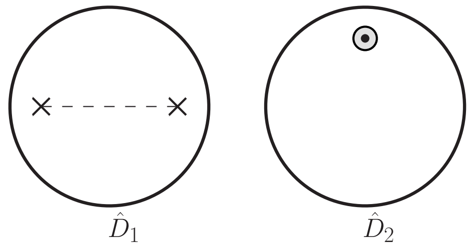


Figure 6.12: Calculation of the two point function by means of derivatives with respect of normal coordinates. The diagrams are considered in the full AdS space (with standard boundary conditions), and all insertions are at the radial coordinate $z=\epsilon$. Crosses indicate insertions of $\epsilon^{-d}T_{(i)}(\gamma_\epsilon,p)\hat{\phi}^i(\epsilon,p)$ and dotted circles, insertions of $\epsilon^{-d}T_{i_1 i_2;0}(\gamma_\epsilon;p_1,p_2)$.

$$=(z_1 z_2)^{\frac{d}{2}} I_{\nu_{(i)}}(p_{\gamma_{z_1}}) K_{\nu_{(i)}}(p_{\gamma_{z_2}}), \qquad z_1 \le z_2. \tag{6.110}$$

### 6.3.1 Two-Point Function

The two point function is then (6.104). Figure 6.12 shows diagrammatically the calculation. Diagram $\hat{D}_1$ gives the first term and diagram $\hat{D}_2$ the second one:

$$\begin{aligned}\left\langle S^p_i[\gamma_\epsilon;\varphi] S^q_j[\gamma_\epsilon;\varphi]\right\rangle^c_{(s^\epsilon_*,\epsilon)} &= \epsilon^{-2d}(2\pi)^d\delta(p+q)\delta_{ij}T_{(i)}(\gamma_\epsilon;p)T_{(i)}(\gamma_\epsilon;p)\mathcal{G}^{(i)}(\epsilon,\epsilon,p)\\ &= \epsilon^{-d}(2\pi)^d\delta(p+q)\delta_{ij}\frac{(p_\gamma)^{2\nu_{(i)}}}{2^{2\nu_{(i)}}\Gamma^2\left(1+\nu_{(i)}\right)}\frac{K_{\nu_{(i)}}(p_{\gamma_\epsilon})}{I_{\nu_{(i)}}(p_{\gamma_\epsilon})},\end{aligned} \tag{6.111}$$

$$\begin{aligned}\left\langle S^{pq}_{i\,j}[\gamma_\epsilon;\varphi]\right\rangle^c_{(s^\epsilon_*,\epsilon)} =&\epsilon^{-d}(2\pi)^d\delta(p+q)\delta_{ij}T_{(i);0}(\gamma_\epsilon;p)\\ =&\epsilon^{-d}(2\pi)^d\delta(p+q)\delta_{ij}\frac{(p_\gamma)^{2\nu_{(i)}}}{2^{2\nu_{(i)}+1}\Gamma(\nu_{(i)}+1)^2}\frac{K_{\nu_{(i)}}(p_\gamma)+a_{\nu_{(i)}}I_{\nu_{(i)}}(p_\gamma)}{I_{\nu_{(i)}}(p_\gamma)},\end{aligned} \tag{6.112}$$

with

$$a_{\nu_{(i)}} = \begin{cases}\frac{\pi}{2}\csc(\nu_{(i)}\pi) & \text{if } \nu_{(i)}\notin\mathbb{N}_0\\ (-1)^{\nu_{(i)}}\log(\xi p_\gamma) & \text{if } \nu_{(i)}\in\mathbb{N}_0.\end{cases} \tag{6.113}$$

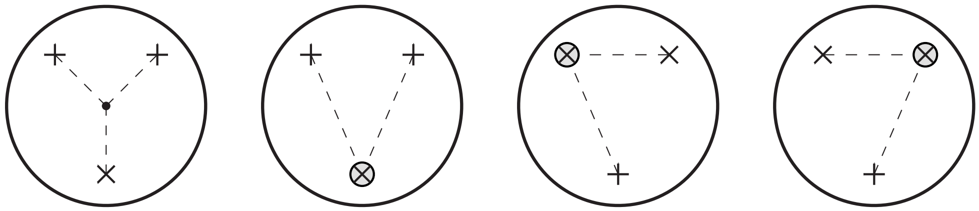


Figure 6.13: Calculation of the three-point function by means of derivatives with respect of normal coordinates. The diagrams are considered in the full AdS space (with standard boundary conditions), and all insertions are at the radial coordinate $z = \epsilon$. Crosses indicate insertions of $\epsilon^{-d}T_{(i)}(\gamma_\epsilon, p)\hat{\phi}^i(\epsilon, p)$ and circled crosses, insertions of $\epsilon^{-d}T_{i;j_1j_2}(\gamma_\epsilon; q, p_1, p_2)\hat{\phi}^{j_1}(\epsilon, p_1)\hat{\phi}^{j_2}(\epsilon, p_2)$.

Combining both results, we obtain

$$\partial^{\bar{c}}_{\sigma_1}\partial^{\bar{c}}_{\sigma_2}W\Big|_{(s^\epsilon_*,\epsilon)} = -\epsilon^{-d}(2\pi)^d\delta(p+q)\delta_{ij}\frac{(p_{\gamma_\epsilon})^{2\nu_{(i)}}}{2^{2\nu_{(i)}+1}\Gamma(\nu_{(i)}+1)^2}a_{\nu_{(i)}}, \tag{6.114}$$

which agrees with (6.16) (in the general case), changing $\epsilon \to 1/\mu$, up to the factor $4\nu^2_{(i)}$ associated to the normalization of the eigenperturbations (6.102).

### 6.3.2 Three-Point Function

The three-point function is given in (6.105). Figure 6.13 shows diagrammatically the calculation of the first term,

$$\Big\langle S^{p_1}_{i_1}[\gamma_\epsilon;\varphi]S^{p_2}_{i_2}[\gamma_\epsilon;\varphi]S^{p_3}_{i_3}[\gamma_\epsilon;\varphi]\Big\rangle^c_{(s^\epsilon_*,\epsilon)} = \hat{D}_3 + \hat{D}_4 + \hat{D}_5 + \hat{D}_6. \tag{6.115}$$

The first diagram, $\hat{D}_3$, is

$$\begin{aligned}\hat{D}_3 =& \epsilon^{-3d}T_{(i_1)}(\gamma_\epsilon;p_1)T_{(i_2)}(\gamma_\epsilon;p_2)T_{(i_3)}(\gamma_\epsilon;p_3)\Big\langle\hat{\phi}^{i_1}(\epsilon,p_1)\hat{\phi}^{i_2}(\epsilon,p_2)\hat{\phi}^{i_3}(\epsilon,p_3)\Big\rangle^c_+\\ =& -\epsilon^{-3d}(2\pi)^d\delta(p_1+p_2+p_3)v_{i_1i_2i_3}T_{(i_1)}(\gamma_\epsilon;p_1)T_{(i_2)}(\gamma_\epsilon;p_2)T_{(i_3)}(\gamma_\epsilon;p_3)\\ &\times\int_0^\infty\frac{dz}{z^{d+1}}\mathcal{G}^{(i_1)}(\epsilon,z,p_1)\mathcal{G}^{(i_2)}(\epsilon,z,p_2)\mathcal{G}^{(i_3)}(\epsilon,z,p_3).\end{aligned} \tag{6.116}$$

Using (6.110) and rearranging the equation, we find

$$\check{D}_3 = \epsilon^{-d}\frac{\mathcal{R}_{i_1i_2i_3}(\gamma_\epsilon;p_1,p_2,p_3)}{8\nu_{(i_1)}\nu_{(i_2)}\nu_{(i_3)}} - \epsilon^{-d}v_{i_1i_2i_3}T_{(i_1)}(\gamma_\epsilon;p_1)T_{(i_2)}(\gamma_\epsilon;p_2)T_{(i_3)}(\gamma_\epsilon;p_3) \times \int \left(\frac{z}{\epsilon}\right)^{\frac{d}{2}} \Big[ I_{\nu_{(i_1)}}(p_{1\gamma_z})I_{\nu_{(i_2)}}(p_{2\gamma_z})I_{\nu_{(i_3)}}(p_{3\gamma_z})K_{\nu_{(i_1)}}(p_{1\gamma_\epsilon})K_{\nu_{(i_2)}}(p_{2\gamma_\epsilon})K_{\nu_{(i_3)}}(p_{3\gamma_\epsilon}) - (z\leftrightarrow\epsilon)\Big]\Bigg|_{z=\epsilon}. \tag{6.117}$$

The remaining diagrams $\hat{D}_4$, $\hat{D}_5$, $\hat{D}_6$, appear due to the quadratic contribution in the fields of one of the eigendirections, and the linear contribution of the other two:

$$\begin{aligned}\check{D}_4 =& 2\epsilon^{-d}T_{i_1;i_2i_3}(\gamma_\epsilon;p_1,p_2,p_3)T_{(i_2)}(\gamma_\epsilon;p_2)T_{(i_3)}(\gamma_\epsilon;p_3)\mathcal{G}_{(i_2)}(\epsilon,\epsilon,p_2)\mathcal{G}_{(i_3)}(\epsilon,\epsilon,p_3)\\ =& -v_{i_1i_2i_3}T_{(i_1)}(\gamma_\epsilon;p_1)T_{(i_2)}(\gamma_\epsilon;p_2)T_{(i_3)}(\gamma_\epsilon;p_3)\\ &\times\int\frac{dz}{z}I_{\nu_{(i_2)}}(p_{2\gamma_z})I_{\nu_{(i_3)}}(p_{3\gamma_z})K_{\nu_{(i_2)}}(p_{2\gamma_\epsilon})K_{\nu_{(i_3)}}(p_{3\gamma_\epsilon})\Psi(\epsilon,z,p_1)\Big|_{z=\epsilon}.\end{aligned} \tag{6.118}$$

Figure 6.14 shows the remaining terms. The first three are associated to the two point contributions in (6.105):

$$\begin{aligned}\hat{D}_7+\hat{D}_8+\hat{D}_9 =& \Big\langle S^{p_1}_{i_1}[\gamma_\epsilon;\varphi]S^{p_2p_3}_{i_2i_3}[\gamma_\epsilon;\varphi]\Big\rangle^c_{(s^\epsilon_*,\epsilon)} + \Big\langle S^{p_2}_{i_2}[\gamma_\epsilon;\varphi]S^{p_3p_1}_{i_3i_1}[\gamma_\epsilon;\varphi]\Big\rangle^c_{(s^\epsilon_*,\epsilon)}\\ &+\Big\langle S^{p_3}_{i_3}[\gamma_\epsilon;\varphi]S^{p_1p_2}_{i_1i_2}[\gamma_\epsilon;\varphi]\Big\rangle^c_{(s^\epsilon_*,\epsilon)},\end{aligned} \tag{6.119}$$

The calculation of these diagrams give

$$\begin{aligned}\check{D}_7 =& \epsilon^{-d}T_{(i_2)}(\gamma_\epsilon;p_2)T_{(i_3)}(\gamma_\epsilon;p_3)T_{i_2i_3;i_1}(\gamma_\epsilon;p_2,p_3,p_1)\mathcal{G}_{(i_1)}(\epsilon,\epsilon,p_1)\\ =& -\frac{v_{i_1i_2i_3}}{2}T_{(i_1)}(\gamma_\epsilon;p_1)T_{(i_2)}(\gamma_\epsilon;p_2)T_{(i_3)}(\gamma_\epsilon;p_3)\\ &\times\int\frac{dz}{z}\left(\frac{l}{z}\right)^{\frac{d}{2}}I_{\nu_{(i_1)}}(p_{1\gamma_z})K_{\nu_{(i_1)}}(p_{1\gamma_\epsilon})\Psi(\epsilon,z,p_2)\Psi(\epsilon,z,p_3)\Big|_{z=\epsilon}.\end{aligned} \tag{6.120}$$

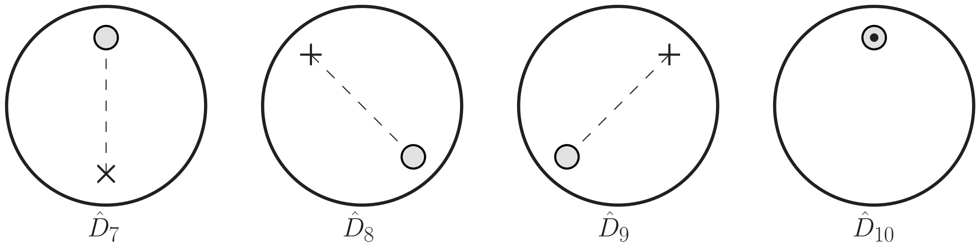


Figure 6.14: Calculation of the three-point function by means of derivatives with respect of normal coordinates. The diagrams are considered in the full AdS space (with standard boundary conditions), and all insertions are at the radial coordinate $z = \epsilon$. Crosses indicate insertions of $\epsilon^{-d} T_{(i)}(\gamma_\epsilon, p)\hat{\phi}^i(\epsilon, p)$, shaded circles, insertions of $\epsilon^{-d} T_{i_1 i_2;j}(\gamma_\epsilon; p_1, p_2; q)\hat{\phi}^j(\epsilon, q)$, and dotted circles insertions of $\epsilon^{-d} T_{i_1 i_2 i_3;0}(\gamma_\epsilon; p_1, p_2, p_3)$.

Finally, the last diagram represents the one-point function of (6.105), whose only non-vanishing contribution comes from the vacuum energy term,

$$\begin{aligned}\hat{D}_{10} &= \left\langle S^{p_1 p_2 p_3}_{i_1\, i_2\, i_3}[\gamma_\epsilon; \varphi] \right\rangle^c_{(s^\epsilon_*, \epsilon)} \\ &= \epsilon^{-d} T_{i_1 i_2 i_3;0}(\gamma_\epsilon; p_1, p_2, p_3). \end{aligned} \tag{6.121}$$

Summing all the contributions of (6.105), all contributions cancel out, except the first term of $\hat{D}_3$,

$$\begin{aligned}\partial^{\bar{c}\, p_1}_{i_1} \partial^{\bar{c}\, p_2}_{i_2} \partial^{\bar{c}\, p_3}_{i_3} W\big|_{(s^\epsilon_*, \epsilon)} &= -\,(\hat{D}_3 + \hat{D}_4 + \hat{D}_5 + \hat{D}_6) + 2(\hat{D}_7 + \hat{D}_8 + \hat{D}_9) - 6\hat{D}_{10} \\ &= -\,\epsilon^{-d}(2\pi)^d \delta(p_1 + p_2 + p_3) \frac{\mathcal{R}_{i_1 i_2 i_3}(\gamma_\epsilon; p_1, p_2, p_3)}{8\nu_{(i_1)}\nu_{(i_2)}\nu_{(i_3)}}, \end{aligned} \tag{6.122}$$

which agrees with the calculation of (6.52), changing $\epsilon \to 1/\mu$, up to the factor $-8\nu_{(i_1)}\nu_{(i_2)}\nu_{(i_3)}$ associated to the normalization of the eigenperturbations (6.102).

## 6.4 Exceptional Cases

The appearance of logarithms of the cutoff (or $t$ in our analysis) in the renormalized operators or counterterms is a signal of exceptional and resonant behaviours of the

Wilson flows close to the fixed point, and the existence of normal beta terms beyond the diagonal ones. This was discussed from the Wilson perspective in Section 3.1.4.

In this chapter, we have seen several situations with these behaviours. In particular, they appear in the following situations:

(1) At the level of the two-point function:

(a) If there is a field $\phi^i$ with $\nu_{(i)} = 0$ (i.e. saturating the Breitenlohner-Freedman bound).

(b) If there is a field $\phi^i$ with $\nu_{(i)} \in \mathbb{N}^+$.

(2) At the level of the three-point function, logarithms appear if there are three fields $\phi^{i_1}$, $\phi^{i_2}$, $\phi^{i_3}$ such that

$$\frac{d}{2} \pm \nu_{(i_1)} \pm \nu_{(i_2)} \pm \nu_{(i_3)} \in -2\mathbb{N}_0. \tag{6.123}$$

for some choice of the signs. Each particular choice is associated with a different case:

(a) If we choose all signs to be minus, the local divergence $\mathcal{Z}$ depends logarithmically on $t$.

(b) If we choose two signs to be minus and one to be plus, the semi-local divergence $\mathcal{Y}$ depends logarithmically on $t$.

(c) If we choose two signs to be plus and one to be minus, the non-local divergence $\mathcal{X}$ depends logarithmically on $t$.

(d) The all-plus condition cannot be satisfied for any choice of $\nu$ (which are always non-negative).

In this section, we analyse all of them, and find the connection with the Wilsonian analysis of the flows performed in Chapter 5.

### 6.4.1 Logarithms in Two-Point Functions

**Case (a):** $\nu_{(i)} = 0$

Due to its exceptional properties, this case was in fact avoided in the two-point function analysis of Section 6.1.1. Therefore, we will redo the calculation for this field. All the following formulas referred to functions over $\mathcal{W}^G$ are understood to be evaluated at the Dirichlet critical point $(s_c^\epsilon, \epsilon)$, and $\epsilon = 1/t\mu$.

The bare two-point function (6.4) is

$$\begin{aligned}\partial^{c\,p}_{\;i}\partial^{c\,p'}_{\;i}\,W =&\epsilon^{-d}\delta(p+p')\left[\frac{d}{2}+p_\gamma\frac{K_1(p_\gamma)}{K_0(p_\gamma)}\right]\\ =&\epsilon^{-d}\delta(p+p')\left[\frac{d}{2}+\frac{1}{\gamma_E+\log\left(\epsilon|p|/2\right)}+O(p_\gamma^2)\right]\\ =&\epsilon^{-d}\delta(p+p')\left[\frac{d}{2}-\frac{1}{\log t}\left[1+\frac{\gamma_E+\log(p_{\gamma_\mu}/2)}{\log t}+O\left((\log t)^{-2}\right)\right]+O(p_\gamma^2)\right].\end{aligned} \tag{6.124}$$

Due to the logarithmic divergence in $\epsilon \to 0$, the multiplicative renormalization coming from the renormalized operators must have a logarithmic behaviour. We must define

$$\partial^{r_t}_{ix} = t^{-\frac{d}{2}}\log t\;\partial^c_{ix}, \tag{6.125}$$

to make finite the leading non-local contribution. The renormalized two point function becomes

$$\begin{aligned}[\mathcal{O}^{t\,p}_{\;i}]^{\sigma_1}[\mathcal{O}^{t\,p'}_{\;i}]^{\sigma_1}\partial^c_{\sigma_2}\partial^c_{\sigma_2}W = (2\pi)^d\mu^d\delta(p+p')\Big[&-\log(p_{\gamma_\mu})+\gamma_E-\log 2\\ &+\frac{d}{2}\log t^2-\log t+\dots\Big],\end{aligned} \tag{6.126}$$

where the dots stand for terms that go to zero in the $t \to 0$ limit. A local divergence remains, which has to be cancelled by the Christoffel symbol

$$\hat{\Gamma}^{t\,0\,p\,p'}_{\;\;i\,i} = (2\pi)^d\epsilon^{-d}\delta(p+p')\left(\frac{d}{2}\log t^2-\log t+C\right), \tag{6.127}$$

where $C$ is arbitrary and selects the renormalization scheme.

As we discussed in Section 3.1.4, the logarithmic behaviours in renormalized operators are unavoidable if the theory is a logarithmic CFT (it has non-diagonalizable linear perturbations), or the critical point excites a marginal direction. This case corresponds to the second possibility. Indeed, if $\Delta^+_{(i)}$, the double-trace direction $\langle ii \rangle$ is marginal, $\lambda_{\langle ii \rangle} = 0$. As we have seen in Section 6.2.1, the Dirichlet critical point precisely excites double-trace deformations of the form $\langle i^{(n_1)} i^{(n_2)} \rangle$ (besides higher-trace ones). Additionally, as we have discussed below (3.45), the renormalized operators that get logarithmic corrections are those ones whose associated eigendirection $\alpha$ forms a resonance with the excited directions of the critical point; and $\beta^{\alpha'}_{\alpha_1 \ldots \alpha_n \alpha} \neq 0$, where $\alpha_1, \ldots, \alpha_n$ are directions excited by the critical point. This is precisely the case for $[\mathcal{O}_{ix}]$ if $\nu_{(i)} = 0$. As we have discussed at the end of Section 5.3.2, in (5.116), the existence of the conformal anomaly (5.96) in this case, and the factorization properties of the large N limit imply

$$\beta^{iy}_{\langle ii \rangle x_1 \, i x_2} \neq 0. \tag{6.128}$$

On the other hand, as discussed in Section 3.1.4, a counterterm or a Christoffel symbol may require a logarithmic dependence on $t$ if there are resonances and/or marginal directions excited by the critical point. Both situations appear in this case, which explains the double logarithm of the Christoffel symbol (6.127).

**Case (b):** $\nu_{(i)} \in \mathbb{N}^+$

This case also presents logarithmic corrections already in the two-point function. As it is shown in (6.9) and (6.10), the local part of the bare propagator depends logarithmically on $t$. This is also consequence of the conformal anomaly (5.96) (in fact, in usual approaches, the conformal anomaly is deduced from the logarithmic dependence of the counterterms). This case shares some similarities with the previous one. In particular, due to the conformal anomaly (5.116) and factorization properties (5.116), the eigendirections excited by the critical point also form resonances with the eigendirection $i$. (6.128) still is valid. However, $\langle ii \rangle$ (which is excited by the critical point) is not marginal in this case but completely irrelevant, and therefore it can be disregarded: the renormalized operator in the exact UV scheme will have a logarithmic dependence, but it can be removed using a redefinition of the renormalized operators with (3.42).

### 6.4.2 Logarithms in Three-Point Functions

In Section 6.1, we have isolated the divergences of the three-point function in three functions: $\mathcal{Z}$, $\mathcal{Y}$ and $\mathcal{X}$. The renormalized operators and counterterms are derived from them. If the relation between the dimensions of the operators is such that (6.123) is not satisfied, $\mathcal{Z}$, $\mathcal{Y}$ and $\mathcal{X}$ do not depend explicitly on $t$,[3] and therefore, the renormalized operators or counterterms do not have logarithmic dependence on $t$.

We analyse in the following the three possible cases. Cases (a) and (b) appear generically in the study of tree-point functions in CFT since they are consequence of quite general properties of the flows (non-vanishing anomalies and beta functions). In fact, in [62], they are studied from the field theory perspective (some holographic calculations are also done). Case (c) however does not appear in these studies.

**Case (a):** $d/2 - \nu_{(i_1)} - \nu_{(i_2)} - \nu_{(i_3)} \in -2\mathbb{N}_0$

Following the definition of the local divergence $\mathcal{Z}$ of (6.29), this particular relation between the dimensions produces a logarithmic behaviour on $t$ of $\mathcal{Z}$, and therefore, of the cubic counterterm. It is equivalent to

$$\Delta^-_{(i_1)} + \Delta^-_{(i_2)} + \Delta^-_{(i_3)} + n = d, \quad n \in 2\mathbb{N}_0. \tag{6.129}$$

This is the resonance condition to have a contribution to the conformal anomaly due to the beta term $\beta^0_{i_1x_1\, i_2x_2\, i_3x_3} \neq 0$. Indeed we have observed the existence of such anomaly in (5.115).

**Case (b):** $d/2 - \nu_{(i_1)} - \nu_{(i_2)} + \nu_{(i_3)} \in -2\mathbb{N}_0$

In this case, the semilocal divergence $\mathcal{Y}$, as defined in (6.30) has logarithms. It implies that the quadratic counterterms (or Christoffel symbols) will depend logarith-

[3]However, they always depend on $t$ through the point of $\mathcal{W}$ where they are evaluated, $(s^c_\epsilon, \epsilon)$, with $\epsilon = 1/t\mu$.

mically on $t$. Again, the condition can be rewritten as

$$\Delta^-_{(i_1)} + \Delta^-_{(i_2)} + n = \Delta^-_{(i_3)}, \quad n \in 2\mathbb{N}_0. \tag{6.130}$$

This is the resonance condition to obtain non-trivial beta functions between the single-trace operators $i_1$, $i_2$ and $i_3$, $\beta^{i_3 x_3}_{i_1 x_1\, i_2 x_2} \neq 0$. (5.106) shows this beta term is in fact non-vanishing.

**Case (c):** $d/2 + \nu_{(i_1)} + \nu_{(i_2)} - \nu_{(i_3)} \in -2\mathbb{N}_0$

This condition implies a logarithmic term in $\mathcal{X}$. Unlike the previous cases, $\mathcal{X}$ contributes to the double-trace component of the renormalized single-trace operator, and not to the counterterms or Christoffel symbols. As discussed in the case (1a), this behaviour only occurs for non-diagonalizable CFTs, or for critical points that excite marginal directions. If no field saturates the Breitenlohner-Freedman bound, the second possibility is excluded. This behaviour must be consequence of the non-diagonalizability of the perturbations at the linear level. Indeed, we have observed such behaviour: this relation between the dimensions is equivalent to

$$\Delta^-_{(i_3)} = \lambda_{\left(\langle i_1^{(n_1)} i_2^{(n_1)}\rangle\right)}. \tag{6.131}$$

Therefore, the single-trace direction $i_3$, and the double-trace direction $\langle i_1^{(n_1)} i_2^{(n_1)}\rangle$ can mix at the linear level giving a non-diagonalizable eigenvalue matrix. In (5.87), we have explicitly seen this is the case. Of course, this is not an actual CFT, but a logarithmic CFT and this behaviour will never appear in authentic CFTs.

## 6.5 Comments on Different Formalisms

In this chapter we have applied the renormalization procedure explained in Chapter 3 to holography. It essentially consists in finding a family of renormalization charts $r_t$. In holography, this is equivalent to imposing a $t$-dependent boundary condition at the cutoff surface $\epsilon = l/t$ which will be associated to the boundary action $S^B \circ r_t^{-1}(g_R, l)$.

This holographic renormalization procedure is then, by construction, completely equivalent to renormalization in QFT.

There are some differences with the standard procedure introduced in Chapter 4. Maybe the main difference is that, here, the boundary condition is imposed at $z = \epsilon$, and derived dynamically from a boundary action that depends on the renormalized couplings. On the other hand, in Chapter 4 the boundary condition is imposed on the modes of the fields as they approach $z \to 0$. Also, a counterterm action $S_{ct}$ is added on the cutoff surface at $\epsilon$. This counterterm action is a local functional of the bulk fields in $\epsilon$, but not of the renormalized couplings themselves (this distinction is particularly important for irrelevant operators [134]). Of course, bulk fields depend on the asymptotic boundary conditions, and therefore, on the renormalized couplings, but also on the AdS IR boundary conditions (typically, regularity in the interior of the bulk). Thus, the possible interpretation of the counterterm action as our counterterm to the vacuum energy could be misleading in general.

In any case, both methods work, so our formulation has to provide somehow a way to calculate the counterterm action $S_{ct}$. This connection could be very interesting, since it would allow to formulate a "holographic" version of renormalization for large N theories, or possibly, for any QFT, independently of the existence of gravity duals (this could be related with the ideas of [141, 157, 158]). We leave the study for future work.

# Chapter 7

# Conclusions

**Fry:** *What do you say? Wanna go around again?*
**Leela:** *I do.*

From the last episode of Futurama

The AdS/CFT correspondence is an outstanding and powerful tool in Theoretical Physics. It is especially useful to describe strongly coupled systems and also gives unique insights on the nature of gravity, at the classical and quantum levels. In this thesis, we have studied some fundamental aspects of renormalization and the RG in this context. We have provided a unified Wilsonian renormalization formalism that works equally well in both sides of the correspondence. Not only can this formalism be applied to actual calculations, but it also furnishes a clear picture of the structure of theory space in holography. This picture throws light on different peculiar features of gauge/gravity duals, which appear more natural than when studied in isolation.

All the developments of the thesis have been formulated in a geometric fashion. We believe that the concepts and methods associated to the RG are most naturally understood in terms of differential geometry. In fact, a geometric language is often used in qualitative discussions of RG flows. Here, we have gone one step further and have given a precise geometric description of theory space, the exact RG and renormalization, which can be used in quantitative calculations. Just as in General Relativity, the equations built with intrinsic objects highlight the invariant physics without distractions from arbitrary choices. Even more importantly, the geometric formulation has given us the flexibility of choosing, in a consistent way, different non-linearly related coordinate systems, which serve different purposes.

Part I of the thesis has been devoted to general quantum field theories, without reference to their possible duals. However, this part has actually been motivated by its application to holography. For instance, the fact that the fields in the gravity side of the duality depend on the spacetime coordinates has lead us to consider a theory space with spacetime-dependent couplings and to introduce a metric that keeps track of the scales associated to derivatives. The local couplings prove to be handy, since they serve as sources to calculate correlation functions. Each point in theory space is characterized by a Wilson action and a cutoff scale. The latter plays a non-trivial role in the presence of local couplings. In the study of the exact RG we have singled out *normal coordinates*, in which the RG flows take their simplest form. They generalize the concept of eigendeformations of fixed points beyond the linear level. A logarithmic dependence on the cutoff unavoidably appears in the RG evolution for exceptional values of the scaling dimensions of the allowed operators.

The renormalization process has been defined by means of cutoff-dependent renormalization coordinates. When studied locally, they lead to a covariant definition of renormalized correlation functions of composite operators, which incorporates renormalized operators (vectors) and counterterms (from a connection). We have studied these correlation functions at fixed points of the RG evolution, which usually (if not in all physically-admissible cases) describe conformal field theories. One of our main results is the precise connection of the renormalization process with the RG flows. This connection has some important consequences, which we have described in detail:

1. The renormalization can be performed at any point of the critical surface, not only at the fixed point. This is very useful when the fixed point is non-trivial and strongly-coupled.

2. The renormalized operators and counterterms to be used at the critical point are determined, up to scheme dependence, by the perturbative expansion of the flows close to the corresponding fixed point. The converse statement also holds.

3. Minimal substraction schemes are related to normal coordinates. The correlation

> functions in these schemes, obtained after renormalization and the continuum limit, are identical to bare correlation functions defined at a finite cutoff in normal coordinates. The exact beta functions in normal coordinates are identified with Gell-Mann-Low beta functions and conformal anomalies in mass-independent schemes.

Our findings provide a precise relation between the exact RG near a fixed point and intrinsic properties of the conformal field theory at the fixed point. Indeed, the singularities of a conformal field theory determine the scaling of renormalized operators (given by conformal dimensions) and, to a large extent, the counterterms, which as we have just emphasized contain all the local information of the RG flows. The structure of these singularities is in turn fixed by basic properties of the theory, such as the Wilson coefficients of the OPE. It would be interesting to explore along these lines how the consistency conditions of the conformal field theory used in the bootstrap program are implemented in the exact RG.

We have also moved far away from the fixed point to define general scale non-invariant theories with a well-defined continuum limit. They are given by the renormalized trajectories that flow away from an ultraviolet fixed point. We have shown how they can be described using a bare manifold that intersects with the critical manifold. The renormalization of correlators of composite operators in scale non-invariant theories, on the other hand, has not been discussed in this thesis. In principle, this can be naturally achieved in our framework by a generalization of the bare manifold to cover all the directions around the critical point. We leave for future work the analysis of this interesting problem. More generally, our formulation of the exact RG provides some new tools that might be useful to extract valuable quantitative information in specific strongly-coupled theories.

One limitation of our formulation is that it only considers theories defined in flat space. It would be highly interesting to generalize the formalism to curved spaces. In this case, the metric should be treated as a coupling, so the space-time geometry would be expected to evolve under the exact RG evolution. Such a generalization could be useful, for example, to calculate with Wilsonian methods the contributions of the energy-momentum tensor to the conformal anomaly of strongly-interacting conformal theories. A Wilsonian version of the local RG, which studies not only changes under dilatations but under general Weyl transformations [33, 54], looks feasible in this context.

In Part II, the very same formalism has been applied to holography. The fact that renormalization and the RG have the very same structure and share the same features in both sides of the AdS/CFT correspondence is possibly our most remarkable conclusion. Of course, if the duality is to hold, this conclusion is unavoidable. But the translation of field-theoretical methods to the problem of field theories in an AdS boundary could have been much more obscure. Maybe the reason behind the success of this unified treatment is the fact that the RG methods and concepts can be used to describe a very general class of dynamical systems. Indeed, all the developments in the thesis apply to gravity theories independently of the existence of a holographic dual. In particular, classical gravity theories with boundary are holographically renormalizable when there exist fixed points in the Hamilton-Jacobi flows of the boundary action.

Some specific features of holographic RG flows in the classical limit have been recognized as general properties of the exact RG flows of matrix quantum field theories in the large N limit. Most significantly, the second-derivative terms in the Polchinski equation do not contribute in the large N limit. This agrees with the fact that the Hamilton-Jacobi equation, which controls the evolution of the boundary action in the classical limit of the gravity duals, is a first-order equation. Moreover, in both cases the use of *factorization normal coordinates*, a special kind of normal coordinates, further simplifies the calculation of the flows close to the fixed point, beyond the linear order. This is the manifestation in our formalism of the usual factorization properties of large N theories.

These theoretical developments have been applied to a theory of fluctuating scalar fields of arbitrary masses in an AdS space, neglecting the metric backreaction. We have explicitly found the physically-relevant fixed points of the boundary action, which are associated to both standard and alternate quantizations / boundary conditions. We have also extracted some properties of the exact RG flows in the neighbourhood of these fixed points. These properties include an analysis of beta functions and conformal anomalies depending on the mass relation of the bulk fields. Then we have studied the renormalization of correlation functions of scalar operators with arbitrary dimensions. We have exhibited the limitations of the widely-used renormalization method based on Dirichlet boundary conditions by examining three-point functions with one or more irrelevant operators. In the so-called extremal and super-extremal cases, non-local divergences remain that cannot be substracted away. We have explained that this problem is due to a too restrictive choice of the allowed bare couplings. The problem arises because the subset of Dirichlet boundary conditions is not stable under the RG evolution and renormalized single-trace operators may mix with higher-trace ones. For relevant operators, the contribution of this mixing is irrelevant, and can be neglected, while in the presence of irrelevant operators, it must be taken into account. However, in our formalism the theory space contains general boundary actions, which implement general boundary conditions and are dual to general Legendre actions with multi-trace operators. So, we have all the necessary ingredients to renormalize any correlation function and indeed we have shown that all the non-local divergences in the three-point functions are taken care of by the renormalized operators when an off-diagonal single-double trace contribution is included in them.

The holographic renormalization method we have used in these calculations works in the neighbourhood of the point in theory space with vanishing Dirichlet boundary conditions. This is not any of the fixed points of the RG, which have a much more complicated form. However, it is attracted by the fixed point with standard quantization in all directions, i.e. it belongs to the critical manifold of that fixed point. Therefore, the renormalized correlation functions calculated in this manner are equal to bare normal correlation functions at the standard-quantization fixed point. Working away from the fixed point also brings about extra logarithmic behaviours in some exceptional cases. It is the case of a field with a mass that saturates the Breitenlohner-Freedman bound: the renormalized operator contains logarithms because the critical point flows into the fixed point along a marginally-irrelevant direction. We have also explained that the BDHM renormalization formalism corresponds to a holographic renormalization performed in the fixed point.

In our particular examples we have considered scalar operators and scalar fields. We have not needed to include effects of higher spins. However, arbitrary tensor fields are incorporated in our formalism. In fact, they are necessary in general for self-consistency, as they are generated at some order by the RG evolution. In particular, a complete treatment of RG flows should include the backreaction on the geometry, i.e. it should treat the metric as a dynamical field. Except in trivial examples, this is unavoidable to study the evolution far away from the ultraviolet fixed points. Indeed, the size of relevant deformations increases towards the infrared and their impact on the geometry cannot be neglected at arbitrarily low energies. Most of the work on non-Wilsonian holographic RG flows is actually based on complete solutions of the gravity-scalar coupled equations [84, 127, 159–161]. A holographic Wilsonian formalism that incorporates dynamical gravity has been sketched by Heemskerk and Polchinski in [146]. A key point of the proposal is that the boundary action should not satisfy the Hamiltonian constraint (or other gauge constraints in general). In this Wilsonian formulation, the treatment of the gauge-fixed metric (or any gauge field) is similar to the one of matter fields. Therefore, we expect that our formalism will qualitatively apply as well to an exact Wilsonian description with dynamical gravity without radical changes. Of course, many details, such as the form of the Hamilton-Jacobi equation and the actual fixed-points and eigenperturbations will have to be modified. Such an extension of our methods will be necessarily connected with the generalization to curved spaces of the field-theory version of our formalism we have mentioned above, and with the local RG. Studying all these issues associated to dynamical gravity forms a project that naturally continues the work in this thesis.

# Conclusiones

La correspondencia AdS/CFT es una destacable y potente herramienta en Física Teórica. Es especialmente útil para describir sistemas fuertemente acoplados. Además, proporciona avances únicos sobre la naturaleza de la gravedad, a niveles clásicos y cuánticos. En esta tesis, hemos estudiado algunos aspectos fundamentales de la renormalización y del grupo de renormalización en este contexto. Hemos proporcionado un formalismo unificado para la renormalización wilsoniana que funciona igualmente bien en ambos lados de la correspondencia. Este formalismo no solo puede ser aplicado a cálculos reales, sino que también ofrece una imagen clara de la estructura del espacio de teorías en holografía. Esta imagen arroja luz sobre diferentes peculiaridades de la dualidad Gauge/Gravedad, las cuales aparecen de forma más natural que cuando se estudian por separado a ambos lados de la correspondencia.

Todos los desarrollos de esta tesis han sido formulados de un modo geométrico. Creemos que los conceptos y métodos asociados al grupo de renormalización se entienden de forma más natural en términos de geometría diferencial. De hecho, el lenguaje geométrico es a menudo usado en discusiones cualitativas de los flujos del grupo de renormalización. Aquí, hemos ido un paso más allá y hemos dado una descripción geométrica precisa del espacio de teorías, del grupo de renormalización exacto y de la renormalización, que puede usarse en cálculos cuantitativos. Del mismo modo que en Relatividad General, las ecuaciones construidas con objetos intrínsecos destacan la física invariante sin distracciones de elecciones arbitrarias. De forma más importante, la formulación geométrica nos da la flexibilidad de elegir consistentemente diferentes sistemas de coordenadas que no están relacionados linealmente y sirven para diferentes propósitos.

La parte I de la tesis ha sido dedicada a teorías cuánticas de campos generales, sin referencia a sus posibles duales gravitatorios. Sin embargo, esta parte ha sido motivada por sus aplicaciones a holografía. Por ejemplo, el hecho de que los campos en el lado de gravedad dependan de las coordenadas espaciotemporales nos ha llevado a considerar espacios de teorías con acoplamientos dependientes del espaciotiempo y a introducir una métrica que define la escala asociada a las derivadas. Los acoplamientos locales son además útiles ya que sirven como fuentes para calcular funciones de correlación. Cada punto en el espacio de teorías es caracterizado por una acción de Wilson y una escala de corte o "cutoff". La última juega un papel no trivial en la presencia de acoplamientos locales. En el estudio del grupo de renormalización exacto hemos distinguido las *coordenadas normales*, en las cuales los flujos del grupo de renormalización toman su forma más simple. Ellas generalizan el concepto autodeformaciones de puntos fijos mas allá del orden lineal. Dependencias logarítmicas con el "cutoff" aparecen inevitablemente en la evolución del grupo de renormalización para ciertos valores excepcionales de las dimensiones de masa de los operadores involucrados.

El proceso de renormalización ha sido definido por medio de coordenadas de renormalización dependientes del "cutoff". Cuando son estudiadas localmente, dan lugar a definiciones covariantes para las funciones de correlación renormalizadas de operadores compuestos, que incorporan los operadores renormalizados (vectores) y los contratérminos (asociados a una conexión). Hemos estudiado estas funciones de correlación en puntos fijos de la evolución del grupo de renormalización, que usualmente (si no en todos los casos admisibles físicamente) describen teorías de campos conformes. Uno de nuestros resultados principales es la conexión precisa entre el proceso de renormalización y los flujos del grupo de renormalización. Esta conexión tiene algunas consecuencias importantes, que a continuación describimos en detalle:

1. La renormalización puede ser realizada desde cualquier punto de la superficie crítica, no solo desde los puntos fijos. Esto es muy útil cuando el punto fijo es no trivial y fuertemente acoplado.

2. Los operadores renormalizados y contratérminos que se usan en el punto crítico están determinados, salvo dependencia del esquema, por las expansión perturbativa de los flujos cerca del correspondiente punto fijo. La afirmación recíproca es también cierta.

3. Esquemas de sustracción mínima están relacionados con coordenadas normales. Las funciones de correlación en estos esquemas obtenidas tras renormalizar en el límite continuo son idénticas a funciones de correlación desnudas definidas con un "cutoff" finito usando coordenadas normales. Las funciones beta exactas en coordenadas normales se identifican con las funciones beta de Gell-Mann-Low y anomalías conformes en esquemas independientes de la masa.

Nuestros descubrimientos proporcionan una relación precisa entre el grupo de renormalización exacto cerca del punto fijo y propiedades intrínsecas de la teoría de campos conforme en el punto fijo. En efecto, las singularidades de la teoría de campos conforme determinan las propiedades de escalado de los operadores renormalizados (dadas por la dimensión conforme), y los contratérminos, los cuales contienen toda la información local de los flujos del grupo de renormalización. Esta estructura de las singularidades está fijada por propiedades básicas de la teoría, tales como los coeficientes de Wilson y los OPE. Sería interesante explorar en esta línea cómo las condiciones de consistencia de teorías de campos conformes usadas en el programa de "bootstrap" son implementadas en el grupo de renormalización exacto.

Además, nos hemos movido del punto fijo para definir teorías sin invarianza de escala generales con un buen límite continuo. Ellas están dadas por trayectorias renormalizadas que fluyen desde un punto fijo ultravioleta. Hemos mostrados cómo pueden ser descritas usando una variedad desnuda que intersecta con la variedad crítica. La renormalización de correladores de operadores compuestos en teorías sin invarianza de escala, por otro lado, no ha sido discutida en esta tesis. En principio, ésta se puede conseguir en nuestra imagen generalizando la variedad desnuda para que cubra todas las direcciones en torno al punto crítico. Dejamos para el futuro el análisis de este interesante problema. Más generalmente, nuestra formulación del grupo de renormalización exacto proporciona algunas nuevas herramientas que podrían ser útiles para extraer información valiosa en específicas teorías fuertemente acopladas.

Una de las limitaciones de nuestro formulación es que solo considera teorías definidas en espacio plano. Sería altamente interesante generalizar este formalismo a espacios curvos. En este caso, la métrica debería ser tratada como un acoplamiento, de modo que la geometría espaciotemporal evolucionaría bajo la evolución del grupo de renormalización exacto. Tal generalización podría ser útil, por ejemplo, para calcular con métodos wilsonianos la contribución del tensor de energía-momento a la anomalía conforme en teorías conformes fuertemente acopladas. Una versión wilsoniana del grupo de renormalización local, que estudia no solo cambios bajo dilataciones, sino también bajo transformaciones generales de Weyl [33, 54], parece factible en este contexto.

En la parte II, el mismo formalismo se ha aplicado en holografía. El hecho de que la renormalización y el grupo de renormalización tengan la misma estructura y compartan las mismas características a ambos lados de la correspondencia AdS/CFT es posiblemente nuestra conclusión más remarcable. Por supuesto, si se satisface la dualidad, esta conclusión es inevitable. Pero la traducción de los métodos de la Teoría Cuántica de Campos a teorías definidas en la frontera de espacios AdS podría haber sido mucho más oscura. Quizás, la razón detrás del éxito de este tratamiento unificado es el hecho de que los métodos y conceptos del grupo de renormalización pueden ser usados para describir una clase muy general de sistemas dinámicos. En efecto, todos los desarrollos en esta tesis aplican a teorías gravitatorias independientemente de la existencia de un dual holográfico. En particular, teorías de gravedad clásicas con frontera son holográficamente renormalizables cuando existen puntos fijos en los flujos de Hamilton-Jacobi de la acción de frontera.

Algunas características específicas de los flujos del grupo de renormalización holográfico en el límite clásico han sido reconocidos como propiedades generales de los flujos de teorías cuánticas de campos de matrices en el límite de N grande. Más concretamente, los términos con segundas derivadas en la ecuación de Polchinski no contribuyen en este límite. Esto está de acuerdo con el hecho de que la ecuación de Hamilton-Jacobi, que controla la evolución de la acción de frontera en el límite clásico de duales gravitatorios, es una ecuación de primer orden. Mas aún, en ambos casos, el uso de *coordenadas normales de factorización*, un tipo especial de coordenadas normales, simplifica el cálculo de los flujos cerca del punto fijo, más allá de el orden lineal. Esta es la manifestación en nuestro formalismo de las propiedades de factorización de las teorías con N grande.

Estos desarrollos teóricos han sido aplicados a una teoría de campos escalares fluctuantes de masa arbitraria en un espacio AdS, ignorando la reacción de la métrica. Hemos encontrado explícitamente los puntos fijos físicamente relevantes de la acción de frontera, asociados con la cuantización estándar y alternativa de los campos. Hemos extraído además algunas propiedades de los flujos del grupo de renormalización exacto en un entorno de estos puntos fijos. Estas propiedades incluyen un análisis de las funciones beta y anomalías conformes dependiendo de la relación entre las masas de los campos. Luego, hemos estudiado la renormalización de funciones de correlación de operadores escalares de dimensión arbitraria. Hemos mostrado las limitaciones del método de renormalización ampliamente usado basado en condiciones de frontera Dirichlet examinando funciones de tres puntos con uno o más operadores irrelevantes. En los llamados casos extremal y super-extremal, divergencias no locales persisten y no pueden ser sustraídas. Hemos explicado que este problema es debido a una elección demasiado restrictiva de los acoplamientos desnudos permitidos. El problema aparece porque el subespacio de condiciones de frontera Dirichlet no es estable bajo la evolución del grupo de renormalización y los operadores renormalizados de traza única pueden mezclarse con operadores de traza más alta. Para operadores relevantes, la contribución de la mezcla es irrelevante y puede despreciarse, mientras que en presencia de operadores irrelevantes, debe ser considerada. Sin embargo, en nuestro formalismo el espacio de teorías contiene acciones de frontera generales, que implementan condiciones de frontera generales y son duales a acciones de Legendre con operadores multi-traza. De este modo, tenemos todos los ingredientes necesarios para renormalizar cualquier función de correlación, y en efecto, hemos mostrado que todas las divergencias no locales en las funciones de tres puntos son correctamente tratadas cuando una contribución de doble-traza no diagonal es incluida en la definición de los operadores renormalizados de traza única.

El método de renormalización holográfico que hemos usado en estos cálculos funciona en un entorno del punto del espacio de teorías con condiciones de contorno de Dirichlet nulas. Éste no es ninguno de los puntos fijos del grupo de renormalización, que tienen una forma más complicada. Sin embargo, es atraído por el punto fijo con cuantización estándar para todos los campos, i.e. pertenece a la variedad crítica de este punto fijo. Por lo tanto, las funciones de correlación renormalizadas calculadas de esta manera son iguales a las funciones de correlación desnudas normales en el punto fijo estándar. Usar como punto crítico un punto distinto del punto fijo también trae comportamientos logarítmicos extras en algunos casos excepcionales. Es el caso de un campo con una masa que satura la cota de Breitenlohner-Freedman: el operador renormalizado asociado contiene logaritmos porque el punto crítico fluye hacia el punto fijo a lo largo de direcciones marginalmente irrelevantes. Además, hemos explicado que el formalismo BDHM corresponde a una renormalización holográfica realizada en el punto fijo.

En nuestros ejemplos particulares hemos considerado operadores y campos escalares. No hemos necesitado incluir efectos de campos con espines más altos. Sin embargo, nuestro formalismo incorpora campos tensoriales arbitrarios. De hecho, ellos son necesarios en general por autoconsistencia, ya que son generados a algún orden por la evolución del grupo de renormalización. En particular, un tratamiento completo de los flujos del grupo de renormalización debería incluir la reacción en la geometría, i.e. debería tratar la métrica como un campo dinámico. Excepto en ejemplos triviales, esto es inevitable para estudiar la evolución lejos de puntos fijos ultravioletas. En efecto, el peso de las deformaciones relevantes se incrementa hacia el infrarojo, y su impacto en la geometría no puede ser ignorado en energías arbitrariamente bajas. Mucho del trabajo en flujos del grupo de renormalización holográfico no wilsoniano está de hecho basado en soluciones completas de ecuaciones acopladas de gravedad con escalares [84, 127, 159–161]. Un formalismo wilsoniano holográfico que incorpora gravedad dinámica ha sido introducido por Heemskerk y Polchinski en [146]. La clave de la propuesta reside en que la acción de frontera no satisfaga la ligadura Hamiltoniana (y otras ligaduras "gauge" en general). En esta formulación wilsoniana, el tratamiento de la métrica en un "gauge" fijado (o cualquier otro campo "gauge") es similar al de los campos de materia. Por lo tanto, esperamos que nuestro formalismo aplique cualitativamente también en una descripción wilsoniana exacta con gravedad dinámica sin cambios radicales. Por supuesto, muchos detalles tales como la forma de la ecuación de Hamilton-Jacobi y los puntos fijos y autoperturbaciones necesariamente se verán modificadas. Tales extensiones de nuestros métodos estarán necesariamente conectadas con la generalización a espacios curvos de la versión en teoría de campos de nuestro formalismo que hemos mencionado anteriormente, y con el grupo de renormalización local. El estudio de todos estas cuestiones asociadas a gravedad dinámica forma un proyecto que de manera natural continúa el trabajo de esta tesis.

# Appendix A

# Non-Linear Flows

Let us consider an autonomous nonlinear system of real differential equations

$$t\frac{d}{dt}x^i(t) = \beta^i(x^1(t), ..., x^n(t)), \quad 1 \leq i \leq N, \tag{A.1}$$

such that $\beta(0, ..., 0) = 0$, i.e. it has a fixed point at $x^i = 0$. We will assume $\beta^i$ has a formal power expansion. Let us separate the linear part of the system and the higher orders:

$$t\frac{d}{dt}x^i(t) = -\lambda^i_j\, x^j(t) + \tilde{\beta}^i(x^1(t), ..., x^N(t)), \tag{A.2}$$

where $\tilde{\beta}^i(x^1, ..., x^N) = O(x^2)$.

## A.1 Linear Order

With a linear rotation of coordinates, $\lambda^i_j$ can be diagonalized in blocks in the following way. The eigenvalues of $\lambda^i_j$ are those numbers $\lambda \in \mathbb{C}$ such that

$$M^n_\lambda = \mathrm{Ker}\left(\lambda^i_j - \lambda\, \delta^i_j\right)^n \neq 0 \tag{A.3}$$

for $n \geq 1$. Given any matrix $\lambda^i_j \in \mathcal{M}_{N\times N}(\mathbb{R})$, its canonical Jordan form allows to write[1]

$$\mathbb{C}^N = \bigoplus_{\lambda} M_\lambda^{n_\lambda}, \tag{A.4}$$

being

$$n_\lambda = \min\left\{n \ : \ \dim\left[\mathrm{Ker}\left(\lambda^i_j - \lambda\,\delta^i_j\right)^n\right] \text{ is maximal}\right\}. \tag{A.5}$$

The subspaces $M_\lambda = M_\lambda^{n_\lambda}$ are the generalized eigenspaces. Using a basis whose elements belong to some $M_\lambda$ diagonalizes $\lambda^i_j$ in blocks, where each block is label by the eigenvalue $\lambda$. If $\lambda^i_j$ is completely diagonalizable, $n_\lambda = 1$ for all $\lambda$, and $M_\lambda$ will be the standard eigenspaces.

Neglecting higher orders and considering only the linear part (i.e. working infinitesimally close to the fixed point), eq. (A.2) has as solution

$$x^i(t) = \exp\left[-\lambda^i_j \,\log t\right]\, x^j_0. \tag{A.6}$$

Working with the matrix diagonalized in blocks as above, the exponentiation of a block with eigenvalue $\lambda$ is

$$\exp\left[\lambda^i_j|_{M_\lambda}\,\log t\right] = t^\lambda \sum_{m=0}^{n_\lambda-1} \frac{(-1)^m}{m!}\left(\lambda^i_j|_{M_\lambda} - \lambda\,\delta^i_j\right)^m (\log t)^m\,. \tag{A.7}$$

Due to eqs. (A.3) and (A.5) the sum stops at a finite $m = n_\lambda - 1$. Because of this, in the $t \to 0$ limit the power term dominates with respect to the logarithmic ones. If $\mathrm{Re}(\lambda) < 0$ ($\mathrm{Re}(\lambda) > 0$), $x(t) \in M_\lambda$ approaches the fixed point (get away from the fixed point). This motivates dividing $\mathbb{R}^N$ in three subspaces invariant under the evolution of the linear equation:

$$\mathbb{R}^N = E_R \oplus E_I \oplus E_M, \quad \text{with} \quad E_R = \left[\bigoplus_{\mathrm{Re}(\lambda)>0} M_\lambda\right] \bigcap \mathbb{R}^N,$$

[1] Although $\beta^i(x)$, and thus $\lambda^i_j$, are real, it is necessary to complexify the vector space and work in $\mathbb{C}^N$.

$$E_I = \left[ \bigoplus_{\mathrm{Re}(\lambda)<0} M_\lambda \right] \bigcap \mathbb{R}^N, \quad E_M = \left[ \bigoplus_{\mathrm{Re}(\lambda)=0} M_\lambda \right] \bigcap \mathbb{R}^N. \tag{A.8}$$

They are the relevant ($E_R$), irrelevant ($E_I$) and marginal ($E_M$) subspaces.[2]

## A.2 Some Invariant Manifolds

The previous analysis is useful to characterize the behavior of the system very close to the fixed point. However, as soon as one moves a bit away from the fixed point, quadratic and higher orders take relevance. In any case, the relevant, irrelevant and marginal subspace structure is conserved with a slightly modification: instead of subspaces, they will be manifolds. Thus, given the set of equations of (A.2), there are always three submanifolds in a neighborhood of the fixed point, the relevant, $W_R$, irrelevant $W_I$, and marginal $W_M$ manifolds, whose tangent spaces in the fixed point are $E_R$, $E_I$, and $E_M$ respectively, that are invariant under the evolution of eq. (A.2) [162].[3] Points in the irrelevant (relevant) manifold flow to the fixed point (get away from the fixed point) in the $t \to 0$ limit. The behavior of flows that belong to the marginal manifold requires however a deeper analysis.

In this thesis we make use also of two more invariant sets under the evolution, the renormalized ($\mathcal{R}$) and critical manifolds ($\mathcal{E}$):

$$\begin{aligned} \mathcal{R} &= \{x \in \mathbb{R}^N \,:\, \lim_{t\to\infty} f_t^i(x) = 0\}, \\ \mathcal{E} &= \{x \in \mathbb{R}^N \,:\, \lim_{t\to 0} f_t^i(x) = 0\}. \end{aligned} \tag{A.9}$$

Where we define $f_t : \mathbb{R}^N \to \mathbb{R}^N$ as the integral of the vector field $\beta$: $f_{t=1}(x) = x$ and $t\frac{d}{dt} f_t^i(x) = \beta^i(f_t(x))$. Notice how $W_R \subseteq \mathcal{R}$ and $W_I \subseteq \mathcal{E}$. Also, the intersection of both sets with the marginal manifold can be either vanishing or non-vanishing. In contrast with $W_R$, $W_I$ and $W_M$, we have used loosely the term “manifold” for these sets. Even

[2] In the standard terminology of the study of dynamical systems, they are called unstable, stable and center subspaces respectively [162].

[3] The relevant and irrelevant manifolds are uniquely defined. However, the marginal manifold is not unique. Nevertheless, all marginal manifolds differ in exponentially suppressed terms, and its Taylor expansion is in fact unique [162].

in a neighborhood of the fixed point, they can preset boundaries and borders.[4]

## A.3 Normal Coordinates

In subsection A.1, the linear order of the eq. (A.2) was simplified through a linear redefinition of the coordinates. One could wonder if allowing general transformations of coordinates (invertible and formally expansible in power series), higher orders in eq. (A.2) can be simplified or even removed. This is the question that the Poincaré-Dulac theorem [163–165] addresses.

Imagine the system

$$t\frac{dx^i}{dt} = -\lambda^i_j x^j + \beta^i_{j_1...j_n} x^{j_1}...x^{j_n}, \tag{A.10}$$

where $\lambda^i_j$ is already diagonalized in blocks of same eigenvalue. Let us try to remove the non-linear term by defining

$$y^i = x^i + \xi^i_{j_1...j_n} x^{j_1}...x^{j_n}, \tag{A.11}$$

with $\xi^i_{j_1...j_n}$ some tensor we have to choose. Taking the derivative of eq. (A.11) and using eq. (A.10) one finds

$$t\frac{dy^i}{dt} = -\lambda^i_j y^j + \left[\xi^l_{k_1...k_n} L^{i\,k_1...k_n}_{l\,j_1...j_n} + \beta^i_{j_1...j_n}\right] y^{j_1}...y^{j_n} + O(y^{n+1}). \tag{A.12}$$

with

$$\begin{aligned} L^{i\,k_1...k...k_n}_{l\,j_1...j...j_n} =& \lambda^i_l \delta^{k_1}_{j_1}...\delta^k_j...\delta^{k_n}_{j_n} - \left[\delta^i_l \lambda^{k_1}_{j_1}...\delta^k_j...\delta^{k_n}_{j_n} + ...\right. \\ &\left. + \delta^i_l \delta^{k_1}_{j_1}...\lambda^k_j...\delta^{k_n}_{j_n} + ... + \delta^i_l \delta^{k_1}_{j_1}...\delta^k_j...\lambda^{k_n}_{j_n}\right]. \end{aligned} \tag{A.13}$$

Thus, the $n$-order of eq. (A.10) can be removed (although higher order terms will appear) if we can find a tensor $\xi^l_{k_1...k_n}$ such that the tensorial expression between squared brackets in eq. (A.12) vanishes. This requires for $L^{i\,k_1...k...k_n}_{l\,j_1...j...j_n}$ to be invertible as an

[4] One trivial example showing this is given by the one-dimensional system $\beta(x) = x^2$. It is straightforward to check that $\mathcal{R} = \mathbb{R}^-_0$ and $\mathcal{E} = \mathbb{R}^+_0$. Both manifolds present a boundary in the fixed point itself $x = 0$.

endomorphism $L : \underbrace{\mathbb{R}^N \times ... \times \mathbb{R}^N}_{n+1 \text{ times}} \to \underbrace{\mathbb{R}^N \times ... \times \mathbb{R}^N}_{n+1 \text{ times}}$:

$$\begin{aligned}\xi^l_{k_1...k_n} &= -\left(L^{-1}\right)^{l\, j_1...j_n}_{i\, k_1...k_n} \beta^i_{j_1...j_n} \\ &= \left[\sum_{r=1}^n \lambda_{(k_r)} - \lambda_{(l)}\right]^{-1} \left(\mathbb{1} - \tilde{L} + \tilde{L}^2 - \tilde{L}^3 + ...\right)^{l\, j_1...j_n}_{i\, k_1...k_n} \beta^i_{j_1...j_n},\end{aligned} \tag{A.14}$$

where,

$$\begin{aligned}\tilde{L}^{l\, j_1...j_n}_{i\, k_1...k_n} &= \left[\lambda_{(l)} - \sum_{r=1}^n \lambda_{(k_r)}\right]^{-1} L^{l\, j_1...j_n}_{i\, k_1...k_n} - \delta^l_i \delta^{j_1}_{k_1} ... \delta^{j_n}_{n_1} \\ &= \left[\lambda_{(l)} - \sum_{r=1}^n \lambda_{(k_r)}\right]^{-1} \left\{\bar{\lambda}^i_l \delta^{k_1}_{j_1} ... \delta^k_j ... \delta^{k_n}_{j_n} - \left[\delta^i_l \bar{\lambda}^{k_1}_{j_1} ... \delta^k_j ... \delta^{k_n}_{j_n} + ... \right.\right. \\ &\qquad \left.\left. + \delta^i_l \delta^{k_1}_{j_1} ... \bar{\lambda}^k_j ... \delta^{k_n}_{j_n} + ... + \delta^i_l \delta^{k_1}_{j_1} ... \delta^k_j ... \bar{\lambda}^{k_n}_{j_n}\right]\right\},\end{aligned} \tag{A.15}$$

and we have defined $\bar{\lambda}^i_j = \lambda^i_j - \lambda_{(j)} \delta^i_j$. Notice that we are subtracting the diagonal part of $\lambda^i_j$, and thus, $\bar{\lambda}^i_j$ is nilpotent. Then, $\tilde{L}$ is also nilpotent, and the sum between the parenthesis of eq. (A.14) is finite, and thus, always well defined. We find that, if $\sum_{r=1}^n \lambda_{(k_r)} - \lambda_{(l)}$ is non-vanishing for any set of eigenvalues, $L$ will be invertible.

### A.3.1 Resonances

In the light of the previous calculations, it is worth to make the next definition. We will say there is a resonance in the system of the eq. (A.2) if some eigenvalue $\lambda_{(i)}$ of the system satisfies

$$\lambda_{(i)} = \sum_s m_s \lambda_{(j_s)}, \tag{A.16}$$

for some subset of eigenvalues $\{\lambda_{(j_1)}, ..., \lambda_{(j_s)}, ...\}$ and a set of strictly positive integer numbers $(m_1, ..., m_s)$.

Then, we will say a term in the expansion of the $\beta$-function, $\beta^i_{j_1...j_n}$, is resonant if $\sum_{r=1}^n \lambda_{(j_r)} = \lambda_{(i)}$.

### A.3.2 Poincaré-Dulac Theorem

From calculations around eqs. (A.12) and (A.14), we see how, order by order, all the non-resonant terms in the perturbative expansion of eq. (A.2) can be removed with a change of coordinates. Notice that these new coordinates are expandable in a formal power expansion of the old coordinates.

Thus, any non-lineal system can be reduced to a system with only resonant terms. A differential equation with such property will be called normal differential equation. This is the known Poincaré-Dulac theorem. An immediate corollary is the Poincaré linearisation theorem, which implies that when the system is free of resonances, it can be transformed to a lineal system.

### A.3.3 Perturbative Solution of a Normal System

Normal coordinates simplify the resolution of the differential system of equations. Let us consider the normal system:

$$t\frac{dx^i}{dt} = -\lambda^i_j x^j + \sum_{n\geq 2} \bar{\beta}^i_{j_1 \dots j_n} x^{j_1} \dots x^{j_n}, \tag{A.17}$$

where $\bar{\beta}^i_{j_1 \dots j_n}$ is only non-vanishing if $\lambda_{(i)} = \lambda_{(j_1)} + \dots + \lambda_{(j_n)}$. A perturbative solution close to the fixed point $x = 0$ is given by

$$x^i(t) = t^{-\lambda_{(i)}} \left\{ x^i_0 + \sum_{r\geq 1} \left[ \sum_{n\geq 1} (\log t)^n \, (B^n)^i_{i_1 \dots i_r} \right] x^{i_1}_0 \dots x^{i_r}_0 \right\}, \tag{A.18}$$

where $x_0 = x(t=1)$, $(B^1)^i_{i_1 \dots i_r} = \bar{\beta}^i_{i_1 \dots i_r}$, and $(B^n)^i_{i_1 \dots i_r}$ for $n > 1$ are given by

$$(B^n)^i_{i_1 \dots i_r} = \sum_P \frac{1}{S_P} \underset{\{i_1 \dots i_r\}}{\mathrm{Sym}} \, C_P \left[ \bar{\beta}_{(p_1)} \otimes \dots \otimes \bar{\beta}_{(p_n)} \right]^i_{i_1 \dots i_r}. \tag{A.19}$$

This equation requires further explanation. First of all, for sake of simplicity, we define $\bar{\beta}^i_{i_1} = -\bar{\lambda}^i_{i_1} = \lambda_{(i)} \delta^i_{i_1} - \lambda^i_{i_1}$. The product between squared brackets of eq. (A.19) is the tensor product of $n$ tensors $\bar{\beta}_{(s)} = \bar{\beta}^j_{j_1 \dots j_{m_s}}$, $m_s \geq 1$. $C_P$ indicates a contraction where the upper indexes are contracted with lower indexes in such a way that the only free upper index is $i$ and the only free lower indexes are $i_1, ..., i_r$. The sum of eq. (A.19) runs over all possible ways of doing these tensor products and contractions (represented with $P$) up to permutations of the free indexes $\{i_1, ..., i_r\}$. Lastly, $S_P$ is a combinatorial factor we define below. Notice that, for a specific product $C_P\left[\bar{\beta}_{(p_1)} \otimes ... \otimes \bar{\beta}_{(p_n)}\right]^i_{i_1...i_r}$ we can define an order relation between the elements $P = \{\bar{\beta}_{(p_1)}, ..., \bar{\beta}_{(p_n)}\}$: $\bar{\beta}_{(p_a)} \leq \bar{\beta}_{(p_b)}$ if a lower index of $\bar{\beta}_{(p_a)}$ is contracted with the upper index of $\bar{\beta}_{(p_b)}$. Using the transitivity property, one can minimally extend this order relation to make $P$ a partially ordered set. Since the coefficients $\bar{\beta}^j_{j_1...j_m}$ have only one upper index (and can have several lower indexes), every set $P$ has the structure of a finite tree with a single root in the sense of set theory [166]. The combinatorial factor $S_P$ can be defined as

$$S_P = \prod_{\bar{\beta}_{(p_i)} \in P} \text{card}\left\{\bar{\beta}_{(p_a)} \in P \text{ such that } \bar{\beta}_{(p_i)} \leq \bar{\beta}_{(p_a)}\right\}, \tag{A.20}$$

where card$(A)$ is the cardinality of the set $A$.

Since the elements $\bar{\beta}^i_j$ are nilpotent, given a fixed order in $x_0^{i_1}...x_0^{i_n}$ of the eq. (A.18), only a finite number of finite tree structures $P$ can be constructed, and thus, the logarithmic order $n$ is bounded.

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