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On the (non-)uniqueness of the Levi-Civita solution in the Einstein–Hilbert–Palatini formalism

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ABSTRACT

We study the most general solution for affine connections that are compatible with the variational principle in the Palatini formalism for the Einstein–Hilbert action (with possible minimally coupled matter terms). We find that there is a family of solutions generalising the Levi-Civita connection, characterised by an arbitrary, non-dynamical vector field $\mathcal{A}_\mu$. We discuss the mathematical properties and the physical implications of this family and argue that, although there is a clear mathematical difference between these new Palatini connections and the Levi-Civita one, both unparametrised geodesics and the Einstein equation are shared by all of them. Moreover, the Palatini connections are characterised precisely by these two properties, as well as by other properties of its parallel transport. Based on this, we conclude that physical effects associated to the choice of one or the other will not be distinguishable, at least not at the level of solutions or test particle dynamics. We propose a geometrical interpretation for the existence and unobservability of the new solutions.

1. Introduction

In the standard picture of General Relativity, gravitational physics is interpreted as physics occurring in a pseudo-Riemannian spacetime. From a mathematical point of view, spacetime is described as a D-dimensional,[1] time-orientable Lorentzian manifold, equipped with a metric $g_{\mu\nu}$ and its corresponding Levi-Civita connection,

$$\Gamma_{\mu\nu}^{\rho} = \left\{ {}^{\rho}_{\mu\nu} \right\} \equiv \tfrac{1}{2}\, g^{\rho\lambda}\Big(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big). \tag{1}$$

This connection is defined as the unique connection that is both torsionless and metric compatible,

$$T_{\mu\nu}^{\rho} \equiv \Gamma_{\mu\nu}^{\rho} - \Gamma_{\nu\mu}^{\rho} = 0, \qquad\qquad \nabla_\mu g_{\nu\rho} = 0, \tag{2}$$

where ∇ denotes the covariant derivative with respect to $\Gamma_{\mu\nu}^{\rho}$. The metric, in its turn, is a dynamical quantity, as it obeys the Einstein equations,

$$R_{\mu\nu}(g) - \tfrac{1}{2}\, g_{\mu\nu} R(g) = -\kappa\, \mathcal{T}_{\mu\nu}, \tag{3}$$

a set of second order differential equations for $g_{\mu\nu}$, which can be derived through a variational principle from the so-called Einstein–Hilbert action, minimally coupled to matter,

$$S = \int d^D x \sqrt{|g|}\Big[\frac{1}{2\kappa}\, g^{\mu\nu} R_{\mu\nu}(g) + \mathcal{L}_M(\phi, g)\Big]. \tag{4}$$

In these equations, $R_{\mu\nu}(g)$ is the Ricci tensor of the metric $g_{\mu\nu}$, $R(g)$ the Ricci scalar, $\mathcal{L}_M(\phi, g)$ the minimally coupled matter Lagrangian and $\mathcal{T}_{\mu\nu}$ its energy–momentum tensor. In a given spacetime, characterised by a metric $g_{\mu\nu}$ which is a solution of (3) for a given $\mathcal{T}_{\mu\nu}$, free test particles will follow geodesic curves, described by the geodesic equation

$$\ddot{x}^\mu + \left\{ {}^{\mu}_{\nu\rho} \right\} \dot{x}^\nu \dot{x}^\rho = 0, \tag{5}$$

where $\dot{x}^\mu \equiv dx^\mu(\tau)/d\tau$ denotes derivation with respect to the proper time τ of the test particle. In this set-up, the metric com-

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[1] Though for most physically relevant applications, D is taken to be 4, we will work in arbitrary number of dimensions $D \geq 3$.

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ponents $g_{\mu\nu}$ are the only gravitational degrees of freedom of the theory, as parallel transport, and hence also the curvature tensors, are completely determined by the metric through the Levi-Civita connection (1). Traditionally, differential geometry in manifolds equipped with the Levi-Civita connection is referred to as (pseudo-)Riemannian geometry.

However, in general in differential geometry, the metric and the affine connection are two independent quantities, that in principle play two different roles. The metric defines distances between points in the manifold and angles between vectors in the tangent space, while the affine connection provides a way of performing parallel transport of vectors and tensors along curves and hence defines the intrinsic curvature of the manifold. Only when the connection is chosen to be Levi-Civita (1), both properties are fully determined by the metric, which becomes the only dynamical quantity in the theory.

One could therefore ask whether there is a reason for the privileged status of the Levi-Civita connection in standard General Relativity and whether other choices for the connection are consistent and/or physically relevant.

There are clear mathematical reasons to choose the Levi-Civita connection. Absence of torsion and metric compatibility (2) are attractive mathematical features, which tend to simplify tensor identities considerably. Furthermore, the fact that the Levi-Civita connection is the only connection that combines these two properties yields it some kind of preferred status.

At first sight, there are also physical reasons that seem to justify this choice of connection. The Equivalence Principle, the cornerstone of General Relativity, which states that the gravitational force can be locally gauged away by a convenient choice of coordinates, is sometimes summarised mathematically as the property that, at any point p of the manifold, coordinates can be found such that the affine connection in that point vanishes,[2] $\Gamma_{\mu\nu}^{\rho}(p)=0$. However, it is clear that due to the tensorial character of the non-metric part of the connection $K_{\mu\nu}^{\rho}=\Gamma_{\mu\nu}^{\rho}-\{{}^{\rho}_{\mu\nu}\}$, this property can only be accomplished if $K_{\mu\nu}^{\rho}$ vanishes identically.

Another feature of non-Levi-Civita connections is that affine geodesics and metric geodesics do not (necessarily) coincide and, since both types of geodesics have different mathematical meanings, general connections might give rise to potential difficulties as it comes to their physical interpretation. Affine geodesics describe the straightest possible lines in a given geometry and represent the trajectory of unaccelerated particles (particles with covariantly constant four-velocities). On the other hand, metric geodesics describe the critical curve between two points (in the timelike case, locally longest for the proper time) and can easily be related to the trajectories of minimal action in absence of external forces. If both curves do not coincide, it is not clear which trajectory to adscribe to a free particle, but choosing the Levi-Civita connection the problem disappears naturally.

As convincing as some of these arguments might sound, the Levi-Civita connection (1) still seems to appear as a convenient choice, not as a necessary tool. It would therefore be nice if there was a more rigorous, mathematical procedure that selects the Levi-Civita connection amongst other potential candidates.

Such a procedure does in fact exist and is called the Palatini formalism[3] [2] (as opposed to the metric formalism, which simply assumes the Levi-Civita connection from the beginning). In the Palatini formalism, the connection is assumed to be a general affine connection $\Gamma_{\mu\nu}^{\rho}$ and hence independent of the metric. The starting point of the Einstein–Hilbert–Palatini theory is then the Einstein–Hilbert action (4), where now the Ricci tensor is written purely in terms of the general connection. On the one hand, the Euler–Lagrange equation for the metric yields the Einstein equation, though in terms of a yet unknown connection, while on the other hand the Palatini equation, the equation of motion for the $\Gamma_{\mu\nu}^{\rho}$, imposes conditions on the connection, which are clearly compatible with the Levi-Civita connection. The Levi-Civita connection arises thus in the Palatini formalism, not as a mere choice, but as a solution to the equations of motion, obtained from a variational principle, much in the same way as the Einstein equation.

The Palatini formalism has been widely studied in different contexts, such as $f(R)$-gravity, Ricci-squared gravities and other extensions of standard General Relativity. For general Lagrangians, the Palatini formalism usually admits connections other than Levi-Civita, with different physics, which might yield alternatives to dark matter and/or dark energy or resolution of singularities [6–17]. On the other hand, it has also been proven [18–20] that within the class of gravity theories with Lagrangians of the form $\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda})$ (i.e. Lagrangians that are functionals of metric and the curvature tensors, but not of its derivatives), the Palatini formalism yields the Levi-Civita connection as a solution only for those Lagrangians that are Lovelock gravities (and their equivalent Palatini counterparts). In other words, for Lovelock gravities, metric formalism is a consistent truncation of the Palatini formalism.

It is sometimes claimed that the Levi-Civita connection is the only solution of the Palatini formalism, at least for the Einstein–Hilbert action. However, this assertion assumes implicitly either the symmetry or the metric compatibility of the connection. In fact, it has been known [21] (though it is often overlooked) that the Einstein–Hilbert action is invariant under the projective symmetry

$$\Gamma_{\mu\nu}^{\rho} \;\rightarrow\; \Gamma_{\mu\nu}^{\prime\rho}=\Gamma_{\mu\nu}^{\rho}+\mathcal{A}_{\mu}\,\delta_{\nu}^{\rho}\,, \tag{6}$$

for an arbitrary vector field $\mathcal{A}_{\mu}$, yielding the latter a gauge character. Yet it is [22] that deals with the problem we are interested in: the “traditional” Palatini problem of finding the most general connection allowed by the variational principle of the Einstein–Hilbert action and its physical and mathematical properties.

As shown in [22], the most general solution for the Palatini equation of the Einstein–Hilbert action is given by a family of connections we will refer to as the “Palatini connections”,[4]

$$\bar{\Gamma}_{\mu\nu}^{\rho}=\{{}^{\rho}_{\mu\nu}\}+\mathcal{A}_{\mu}\,\delta_{\nu}^{\rho}\,. \tag{7}$$

It is clear that the Palatini connections include the Levi-Civita connection as a special case, but are generically non-metric compatible and non-symmetric. From the physical point of view, the connections contain a non-dynamical degree of freedom $\mathcal{A}_{\mu}$, but it can easily be shown [22] that the (symmetric part of) the Ricci tensor and hence the Einstein equation are not affected by the presence of this vector field. Based in this property, the authors of [22] argue the vector field $\mathcal{A}_{\mu}$ is undetectable and hence that the Palatini and the Levi-Civita connections are indistinguishable from a physical point of view.

[2] See, for example, the discussion in [1], pp. 74–75, which uses formula (3.3.7) and, thus, (taking into account the version of the principle of equivalence in p. 74), arrives at a symmetric connection; however, other approaches allow the existence of torsion explicitly, see for example [3, Ch. 4].

[3] As stressed in [3, p. 23], such a name is unfortunate, recall [4,5].

[4] For completeness, we point out that the Palatini connection (7) has also been studied in [23,24], though in different contexts and with different conclusions to ours. Indeed, in [23] a specific quadratic curvature term is added to the Einstein–Hilbert term, which induces a particular, non-trivial dynamics for the connection, while in [24] the Palatini connection is introduced as an auxiliary field in the context of non-symmetric gravity theories. See also [25] for relevant results in the context of Massive Topological Gravity.

Even though we agree with most of interpretation of [22], we think that the invariance of the Einstein equation alone is insufficient to prove the undetectability of $\mathcal{A}_\mu$. Indeed, each member of the Palatini family not only provides an Einstein equation, but also a specific parallel transport and its corresponding geodesics. As timelike and null geodesics are the trajectories of free-falling test particles, any difference between Levi-Civita and Palatini geodesics would be physically observable. Hence, the physical indistinguishability of the Palatini connection from the Levi-Civita one can only be claimed if one succeeds in proving that the (timelike and null) geodesics of both connections coincide.

The aim of this paper is to show that the Palatini connections are indeed physically indistinguishable from Levi-Civita, not only in their Einstein equation, but also in their geodesics. As we will see, the parallel transport of the new Palatini solutions is different to the Levi-Civita one, but they differ only in a path-dependent homothety. Precisely this homothety makes that the Palatini and the Levi-Civita connections share the same pre-geodesics, i.e. they have the same geodesics, up to reparametrisations. As we will show, this property is unique for the Palatini connections and provides a strong support to the claimed undetectability. At the same time, the homothetic difference between the geodesics of both connections suggests an interpretation for the projective symmetry of the Einstein–Hilbert action and the Einstein equations observed on [21,22] as an unphysical degree of freedom, related to the freedom of reparametrisation of the geodesics.

The organisation of this paper is as follows: in Section 2 we derive the equations of motion of the Einstein–Hilbert–Palatini theory and, for the sake of completeness, we deduce the most general solution for the Palatini equation, generically non-metric compatible and non-symmetric. In Section 3 we discuss the geometrical properties of the solutions, pointing out the mathematical differences between the Palatini and the Levi-Civita connection. In Section 4, we argue that the Palatini connections have no observable effects on the physics of solutions and test particle dynamics and hence turn out to be indistinguishable from the Levi-Civita connection. Finally in Section 5 we propose a physical interpretation of the Palatini connections and elaborate on what remains of the preferred status of the Levi-Civita connection in General Relativity.

2. The solution

Consider the D-dimensional Einstein–Hilbert action in the Palatini formalism, minimally coupled to a generic matter field ϕ,

$$S(g,\Gamma)=\int d^Dx\sqrt{|g|}\Big[\tfrac{1}{2\kappa}\,g^{\mu\nu}R_{\mu\nu}(\Gamma)+\mathcal{L}_M(\phi,g)\Big] \tag{8}$$

where the metric $g_{\mu\nu}$ and the connection $\Gamma^\rho_{\mu\nu}$ are treated as independent variables. We assume the connection to be completely general, without imposing neither symmetry, nor metric compatibility, such that the Ricci tensor, in our conventions given by

$$R_{\mu\nu}(\Gamma)\equiv R_{\mu\lambda\nu}{}^{\lambda}(\Gamma)=\partial_\mu\Gamma^\lambda_{\lambda\nu}-\partial_\lambda\Gamma^\lambda_{\mu\nu}+\Gamma^\lambda_{\mu\rho}\,\Gamma^\rho_{\lambda\nu}-\Gamma^\lambda_{\lambda\rho}\,\Gamma^\rho_{\mu\nu}, \tag{9}$$

is completely independent of the metric. Note that the action (8) is first order in the connection and zeroth order in the metric (in contrast to the metric formalism, where (4) is second order in $g_{\mu\nu}$). In the Palatini formalism it is therefore not necessary to include a Gibbons–Hawking–York term [26,27], as there are no boundary terms coming from the variation of second order terms.

The Palatini formalism prescribes that the physics of the above action is given by the Euler–Lagrange equations of the metric, the connection and the matter fields. However, as we assume the matter Lagrangian to be minimally coupled, the matter equations of motion do not couple to the connection and hence, for the purposes we are interested in, in this letter, the matter sector will not play any relevant role. Except for its energy–momentum tensor in the Einstein equation, we will omit all references to the matter fields from now on.

The Einstein equation, the variation of the action with respect to the metric, is given by

$$0=\frac{2\kappa}{\sqrt{|g|}}\,\frac{\delta S}{\delta g^{\mu\nu}}=R_{(\mu\nu)}(\Gamma)-\tfrac12 g_{\mu\nu}\,R(\Gamma)+\kappa\,\mathcal{T}_{\mu\nu}, \tag{10}$$

where $R_{(\mu\nu)}$ indicates the symmetric part of the Ricci tensor. On the other hand, the variation of the action (8) with respect to the connection can be easily done by first computing the Palatini Identity, the variation of the Ricci tensor with respect to the connection,

$$\delta R_{\mu\nu}(\Gamma)=\nabla_\mu(\delta\Gamma^\lambda_{\lambda\nu})-\nabla_\lambda(\delta\Gamma^\lambda_{\mu\nu})+T^\rho_{\mu\lambda}(\delta\Gamma^\lambda_{\rho\nu}), \tag{11}$$

where we use ∇ and $T^\rho_{\mu\nu}$ to denote the covariant derivative and the torsion associated to the connection $\Gamma^\rho_{\mu\nu}$ respectively. The variation of (8) is obtained by substituting the Palatini Identity and integrating by parts, yielding the Palatini equation (compare with [28])

$$\begin{aligned}&\nabla_\lambda g^{\mu\nu}-\nabla_\sigma g^{\sigma\nu}\,\delta^\mu_\lambda+\tfrac12 g^{\rho\tau}\nabla_\lambda g_{\rho\tau}\,g^{\mu\nu}-\tfrac12 g^{\rho\tau}\nabla_\sigma g_{\rho\tau}\,g^{\sigma\nu}\,\delta^\mu_\lambda\\&\qquad-T^\rho_{\rho\lambda}\,g^{\mu\nu}+T^\rho_{\rho\sigma}\,g^{\sigma\nu}\,\delta^\mu_\lambda+T^\mu_{\sigma\lambda}\,g^{\sigma\nu}=0.\end{aligned} \tag{12}$$

Both the Einstein equation (10) and the Palatini equation (12) can be simplified: substracting the trace of (10) and the δ^λ_μ and the $g_{\mu\nu}$ traces of (12), these equations reduce respectively to

$$R_{(\mu\nu)}(\Gamma)=-\kappa\Big[\mathcal{T}_{\mu\nu}-\tfrac{1}{D-2}\,g_{\mu\nu}\mathcal{T}\Big], \tag{13}$$

$$\nabla_\lambda g_{\mu\nu}-T^\sigma_{\nu\lambda}\,g_{\sigma\mu}-\frac{1}{D-1}\,T^\sigma_{\sigma\lambda}\,g_{\mu\nu}-\frac{1}{D-1}\,T^\sigma_{\sigma\nu}\,g_{\mu\lambda}=0. \tag{14}$$

The idea is now to solve the Palatini equation for $\Gamma^\rho_{\mu\nu}$ and substitute this solution in the Einstein equation to determine the geometry of the spacetime. Note that the Palatini equation is not a dynamical equation for $\Gamma^\rho_{\mu\nu}$, but just an algebraic constraint. This is due to the fact that there are no kinetic terms for the connection in the Einstein–Hilbert action, which in turn is intimately related to the fact that the (metric) Einstein–Hilbert action is the first order Lovelock Lagrangian [19].

It is trivial to see that the Levi-Civita connection (1) is a solution of the Palatini equation, as each term in (14) is identically zero, due to the necessary conditions (2). It is also straightforward to see that (14) forces any symmetric connection to be metric-compatible and vice versa. Hence assuming any of the two conditions (2) is sufficient in the Einstein–Hilbert–Palatini formalism for the connection to be Levi-Civita, as the other one will be automatically imposed by the Palatini equation. However, the question remains whether there exist non-symmetric and non-metric compatible connections that are solutions of (14).

The Palatini equation (14), being an algebraic equation, is easy to solve. In fact, the general solution can be found in the same way as the expression (1) for the Levi-Civita connection is deduced from the conditions (2). Writing (14) explicitly in terms of the connections and cyclically permuting the free indices, we find

$$\partial_\lambda g_{\mu\nu}-\Gamma^\sigma_{\lambda\mu}g_{\sigma\nu}-\Gamma^\sigma_{\nu\lambda}g_{\mu\sigma}-\tfrac{1}{D-1}\,T^\sigma_{\sigma\lambda}\,g_{\mu\nu}-\tfrac{1}{D-1}\,T^\sigma_{\sigma\nu}\,g_{\mu\lambda}=0,$$

$$\partial_\mu g_{\nu\lambda}-\Gamma^\sigma_{\mu\nu}g_{\sigma\lambda}-\Gamma^\sigma_{\lambda\mu}g_{\nu\sigma}-\tfrac{1}{D-1}\,T^\sigma_{\sigma\mu}\,g_{\nu\lambda}-\tfrac{1}{D-1}\,T^\sigma_{\sigma\lambda}\,g_{\nu\mu}=0,$$

$$\partial_\nu g_{\lambda\mu} - \Gamma^\sigma_{\nu\lambda} g_{\sigma\mu} - \Gamma^\sigma_{\mu\nu} g_{\lambda\sigma} - \tfrac{1}{D-1}\, T^\sigma_{\sigma\nu}\, g_{\lambda\mu} - \tfrac{1}{D-1}\, T^\sigma_{\sigma\mu}\, g_{\lambda\nu} = 0. \tag{15}$$

Adding up the last two equations and subtracting the first one, we find that the connection $\Gamma^\rho_{\mu\nu}$ can be expressed in terms of the trace of its torsion and the Levi-Civita connection:

$$\Gamma^\rho_{\mu\nu} = \{^{\rho}_{\mu\nu}\} - \tfrac{1}{D-1}\, T^\sigma_{\sigma\mu}\, \delta^\rho_\nu. \tag{16}$$

Using group-theoretical arguments, it is easy to see that the trace of the torsion can be fully represented by a D-dimensional vector, $T^\sigma_{\sigma\mu} = -(D-1)\,\mathcal{A}_\mu$. We conclude therefore that the most general solution of the Palatini equation (14) can be written in the form (see also [21–24])

$$\Gamma^\rho_{\mu\nu} = \bar\Gamma^\rho_{\mu\nu} \equiv \{^{\rho}_{\mu\nu}\} + \mathcal{A}_\mu\, \delta^\rho_\nu, \tag{17}$$

with $\mathcal{A}_\mu$ an arbitrary, non-dynamical vector field. Note that the Levi-Civita connection is trivially recovered, choosing $\mathcal{A}_\mu = 0$. From the construction it is clear that (17) is indeed the most general solution to the Palatini equation (14).

3. Geometrical properties

Now that we have found the most general solution (17) to the Palatini equation, we will study in this section its geometrical properties and give a physical interpretation in the next one.

As we mentioned in the construction, the (non-trivial, *i.e.* non-Levi-Civita) Palatini connections (17) are *neither symmetric, nor metric compatible*, the generalisation of (2) being

$$\bar T^\rho_{\mu\nu} = \mathcal{A}_\mu\, \delta^\rho_\nu - \mathcal{A}_\nu\, \delta^\rho_\mu, \qquad \bar\nabla_\mu g_{\nu\rho} = -2\,\mathcal{A}_\mu\, g_{\nu\rho}. \tag{18}$$

The corresponding curvature tensors are given by

$$\begin{aligned} \bar R_{\mu\nu\rho}{}^\lambda &= R_{\mu\nu\rho}{}^\lambda(g) + \mathcal{F}_{\mu\nu}(\mathcal{A})\,\delta^\lambda_\rho, \\ \bar R_{\mu\nu} &= R_{\mu\nu}(g) + \mathcal{F}_{\mu\nu}(\mathcal{A}), \qquad \bar R = R(g), \end{aligned} \tag{19}$$

where $R_{\mu\nu\rho}{}^\lambda(g)$, $R_{\mu\nu}(g)$ and $R(g)$ are respectively the Riemann tensor, the Ricci tensor and the Ricci scalar with respect to the Levi-Civita connection, $\bar R = g^{\mu\nu}\bar R_{\mu\nu}$ the Ricci scalar associated to $\bar R_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}(\mathcal{A}) = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$.[5] Note that the Riemann and Ricci tensors (19) do not satisfy the symmetry properties of their Levi-Civita counterparts, due to (18). Yet it is interesting to notice that the symmetric part of the Ricci tensor, the one determined by the Einstein equations, coincides precisely with the Ricci tensor of the Levi-Civita connection: $\bar R_{(\mu\nu)} = R_{\mu\nu}(g)$.

A remarkable property of the Palatini connections (17) is that *affine geodesics turn out to be pregeodesics of the Levi-Civita connection* (*i.e.* they describe the same trajectories in the manifolds, though with a different parametrisation). Indeed, the affine geodesic equation for the Palatini connections, $\dot x^\rho \bar\nabla_\rho \dot x^\mu = 0$, written in terms of $\{^{\rho}_{\mu\nu}\}$ and $\mathcal{A}_\mu$, take the form

$$\dot x^\rho \nabla^{(g)}_\rho \dot x^\mu = -\mathcal{A}_\rho\, \dot x^\rho\, \dot x^\mu, \tag{20}$$

where $\nabla^{(g)}$ denotes the covariant derivative with respect to the Levi-Civita connection. The equation of all (non-lightlike) pregeodesics can be derived as an extremum of the arc length functional

$$s(\lambda) = \int_0^\lambda \sqrt{|g_{\mu\nu}\, \dot x^\mu\, \dot x^\nu\,|}\, d\lambda', \tag{21}$$

where $\dot x^\mu \equiv dx^\mu(\lambda')/d\lambda'$ denotes derivation with respect to an arbitrary parameter λ'. Extrema of this functional in general take the form

$$\dot x^\rho \nabla^{(g)}_\rho \dot x^\mu = \left(\frac{\ddot s}{\dot s}\right) \dot x^\mu, \tag{22}$$

but the equation (20) can be recovered with the specific parameter choice

$$s(\lambda) = \int_0^\lambda e^{-G(\lambda')}\, d\lambda' \qquad \text{with} \qquad G(\lambda) = \int_0^\lambda \dot x^\rho \mathcal{A}_\rho\, d\lambda'. \tag{23}$$

This observation proves the two points mentioned above: first of all that (20) can be interpreted as both the equation of the geodesics of $\bar\nabla$ and the equation of a particular type of reparametrisations of the Levi-Civita geodesics. Secondly, that the right-hand side of (20) can be absorbed in a conveniently chosen (though geodesic-dependent) reparametrisation of the geodesics. In particular, (20) can be transformed into the geodesic equation of the Levi-Civita connection (5) through the change of parameter

$$\frac{dx^\mu(\lambda)}{d\lambda} = \frac{dx^\mu(\tau)}{d\tau}\,\frac{d\tau}{d\lambda} \qquad \text{with} \qquad \frac{d\tau}{d\lambda} = e^{-G(\lambda)}. \tag{24}$$

Summing up, the curves described by (5) and by (20) yield the trajectories of the geodesics, with different parametrisations controlled by (24).

The Palatini connections (17) are not the only connections that have the same pregeodesics as $\{^{\rho}_{\mu\nu}\}$. Indeed, affine connections with the same pregeodesics are called *projectively related* and any affine connection projectively related to Levi-Civita's has the form $\tilde\Gamma^\rho_{\mu\nu} = \{^{\rho}_{\mu\nu}\} + \mathcal{A}_\mu\delta^\rho_\nu + \mathcal{B}_\nu\delta^\rho_\mu$ [29]. However it is interesting to notice that the curvature tensors coming from this connection have more complicated expressions than the ones given in (19). For example, the Riemann tensor associated with $\tilde\Gamma^\rho_{\mu\nu}$ is given by

$$\begin{aligned} \tilde R_{\mu\nu\rho}{}^\lambda = {} & R_{\mu\nu\rho}{}^\lambda(g) + \mathcal{F}_{\mu\nu}(\mathcal{A})\,\delta^\lambda_\rho + (\nabla_\mu \mathcal{B}_\rho - \mathcal{B}_\mu \mathcal{B}_\rho)\,\delta^\lambda_\nu \\ & + (\nabla_\nu \mathcal{B}_\rho - \mathcal{B}_\nu \mathcal{B}_\rho)\,\delta^\lambda_\mu. \end{aligned} \tag{25}$$

Notice in particular, that the symmetric part is of the Ricci tensor in general does not coincide with the Levi-Civita Ricci tensor.[6]

Since the Palatini geodesics are pregeodesics of the Levi-Civita ones, it should be clear that they have the same geodesic deviation (modulo the pointwise direction of the velocity of the geodesic), as their trajectories in the spacetime manifold coincide. However this can also be made explicit, starting from the geodesic deviation equation for the arbitrary connections (see for example [30]), applied to the Palatini connections (17),

$$\begin{aligned} &\tfrac{\partial x^\mu}{\partial\lambda}\bar\nabla_\mu\Big[\tfrac{\partial x^\nu}{\partial\lambda}\bar\nabla_\nu\tfrac{\partial x^\alpha}{\partial\eta}\Big] + \bar R_{\mu\nu\rho}{}^\alpha\,\tfrac{\partial x^\mu}{\partial\eta}\,\tfrac{\partial x^\nu}{\partial\lambda}\,\tfrac{\partial x^\rho}{\partial\lambda} - \tfrac{\partial x^\nu}{\partial\lambda}\bar\nabla_\mu\Big[\bar T^\alpha_{\nu\rho}\,\tfrac{\partial x^\nu}{\partial\lambda}\,\tfrac{\partial x^\rho}{\partial\eta}\Big] \\ &\quad = 0, \end{aligned} \tag{26}$$

and seeing that its maps to the geodesic deviation equation for the Levi-Civita connection,

[5] The way the vector field $\mathcal{A}_\mu$ appears in the curvature tensors reminds strongly of the electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in Maxwell theory, as the curvature tensors are invariant under the transformation $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu\Lambda$. This is the reason why $\mathcal{A}_\mu$ is interpreted as a $U(1)$ gauge field in [23], even though the torsion and the parallel transport are not. We will show that $\mathcal{A}_\mu$ is not a Maxwell-like gauge field, but that the whole of $\mathcal{F}_{\mu\nu}$ is undetectable.

[6] It does if and only if $\mathcal{B}_\mu$ satisfies the condition $\nabla_\mu\mathcal{B}_\nu = \mathcal{B}_\mu\mathcal{B}_\nu$. This a well-known condition of recurrence, see [31] or [32]. However this particular condition breaks the generic character of the projectively related connections. Indeed, it is not difficult to show that such a $\mathcal{B}_\mu$ implies the existence of a parallel (covariantly constant) 1-form $\mathcal{P}_\mu$, pointwise proportional to $\mathcal{B}_\mu$. We will briefly comment on this case later in this paper.

$$\frac{\partial x^\mu}{\partial \tau}\nabla_\mu\left[\frac{\partial x^\nu}{\partial \tau}\nabla_\nu\frac{\partial x^\alpha}{\partial \sigma}\right]+R_{\mu\nu\rho}{}^{\alpha}\,\frac{\partial x^\mu}{\partial \sigma}\,\frac{\partial x^\nu}{\partial \tau}\,\frac{\partial x^\rho}{\partial \tau}=0, \tag{27}$$

under the reparametrisation

$$\frac{\partial x^\mu}{\partial \lambda}=\frac{\partial \tau}{\partial \lambda}\,\frac{\partial x^\mu}{\partial \tau}, \qquad \frac{\partial x^\mu}{\partial \eta}=\frac{\partial \tau}{\partial \eta}\,\frac{\partial x^\mu}{\partial \tau}+\frac{\partial x^\mu}{\partial \sigma}, \tag{28}$$

where $\tau=\tau(\lambda,\eta)$ is defined as

$$\tau(\lambda,\eta)=\int_0^\lambda e^{-G(\lambda',\eta)}d\lambda' \qquad \text{with} \qquad G(\lambda,\eta)=\int_0^\lambda \dot{x}^\rho\mathcal{A}_\rho\, d\lambda'. \tag{29}$$

Hence, the Palatini and the Levi-Civita connections have the same geodesic deviation, as solutions of (26) are also solutions of (27) and vice versa.

Another remarkable property of the Palatini connections is that *its parallel transport becomes homothetic with respect to the Levi-Civita connection.* From the very definition of the Palatini connections (17), it is clear that the difference between parallel transport of a vector V^μ along a curve $x^\mu=x^\mu(\lambda)$ according to $\bar{\Gamma}^\rho_{\mu\nu}$ and according to $\{^{\rho}_{\mu\nu}\}$ is proportional to the vector itself:

$$\dot{x}^\rho\bar{\nabla}_\rho V^\mu-\dot{x}^\rho\nabla^{(g)}_\rho V^\mu=\dot{x}^\rho\mathcal{A}_\rho\, V^\mu. \tag{30}$$

Concretely this means that the result of parallelly transporting vectors with the Levi-Civita or with the Palatini connections leads to different results, but the resulting vectors only differ in their norm (or, more properly for the lightlike case, in a proportionality coefficient). Indeed, if $V^\mu_g(\lambda)$ is the result of parallel transport along a curve $x^\mu=x^\mu(\lambda)$ according to $\{^{\rho}_{\mu\nu}\}$, then the result of parallel transport along the same curve according to $\bar{\Gamma}^\rho_{\mu\nu}$ is given by

$$V^\mu_{\bar{\Gamma}}(\lambda)=e^{-G(\lambda)}\,V^\mu_g(\lambda), \tag{31}$$

with $G(\lambda)$ given by (23). Note that the proportionality coefficient depends on the curve $x^\mu=x^\mu(\lambda)$, but not on V^μ. We therefore have that Palatini transport is equal to Levi-Civita transport, composed by a homothety of ratio $e^{-G(\lambda)}$.

The property of non-constant norm under parallel transport is a consequence of the fact that the Palatini connections are not metric compatible. A general feature of non-metric compatible connections is that parallel transport does not conserve the scalar product of vectors. For the Palatini transport of two vectors V^μ and W^μ, we have that

$$\begin{aligned}\dot{x}^\rho\bar{\nabla}_\rho(g_{\mu\nu}V^\mu W^\nu)&=\dot{x}^\rho\bar{\nabla}_\rho g_{\mu\nu}\,V^\mu W^\nu=2\,\dot{x}^\rho\mathcal{A}_\rho\,V_\mu W^\mu\\&=2\,G'(\lambda)\,V_\mu W^\mu.\end{aligned} \tag{32}$$

The Palatini connections (17) are the *only connections yielding this homothety property under parallel transport for all vectors along any curve.*[7] Indeed, an arbitrary connection $\Gamma^\rho_{\mu\nu}=\{^{\rho}_{\mu\nu}\}+K^\rho_{\mu\nu}$ with an arbitrary tensor $K^\rho_{\mu\nu}$ yields homothetic parallel transport with respect to Levi-Civita if and only if

$$\dot{x}^\nu K^\mu_{\nu\rho}V^\rho=f(\lambda)\,V^\mu, \tag{33}$$

for some function $f(\lambda)$, which may depend on the curve followed. The above expression is true for all vectors V^μ, if and only if $\dot{x}^\nu K^\mu_{\nu\rho}=f(\lambda)\delta^\mu_\rho$. The homothety condition (33) can therefore be written as

$$\dot{x}^\nu K^\mu_{\nu\rho}=\dot{x}^\nu\mathcal{A}_\nu\,\delta^\mu_\rho. \tag{34}$$

Then it is easy to see that, in order for this identity to be true for all curves $x^\mu=x^\mu(\lambda)$, we must have that $K^\mu_{\nu\rho}=\mathcal{A}_\nu\,\delta^\mu_\rho$ and hence that $\Gamma^\rho_{\mu\nu}=\bar{\Gamma}^\rho_{\mu\nu}$. In particular, the Levi-Civita connection can be characterised as the unique symmetric connection such that its parallel transport becomes a metric homothety (which turns out trivial, *i.e.*, a isometry). As a last remark, notice also that to take only curves x^μ that are timelike is enough to prove all these characterisations regarding homotheties.

4. Physical observability

Let us briefly summarise the main results of the previous sections. We found that the most general solution to the Palatini equation (14) is given by the family of Palatini connections (17),

$$\bar{\Gamma}^\rho_{\mu\nu}=\{^{\rho}_{\mu\nu}\}+\mathcal{A}_\mu\,\delta^\rho_\nu, \tag{35}$$

yielding curvature tensors (19) that only differ from the Levi-Civita tensor by terms involving $\mathcal{F}_{\mu\nu}(\mathcal{A})$. In particular, the symmetric part of the Ricci tensor is identical to the Levi-Civita Ricci tensor,

$$\bar{R}_{(\mu\nu)}=R_{\mu\nu}(g). \tag{36}$$

Furthermore, we have found that the Palatini connections (35) are unique in two ways:

- $\bar{\Gamma}^\rho_{\mu\nu}$ are the only connections that, for a given metric $g_{\mu\nu}$ have the same pregeodesics as $\{^{\rho}_{\mu\nu}\}$ and at the same time satisfy the relation (36) between its Ricci tensor and $R_{\mu\nu}(g)$.[8]
- $\bar{\Gamma}^\rho_{\mu\nu}$ are the only connections whose parallel transport of any vector along timelike curves (and, then, along any curve) is homothetic to the Levi-Civita transport.

Given that the Palatini connections are mathematically clearly different from the Levi-Civita connection, one wonders whether it would also lead to different physics and, in case it does, whether (any of) these connections describe correctly our universe. The question is especially important in the light of the issue about the preferred status of the Levi-Civita connection in General Relativity. If the Palatini connections have physically observable effects, then the question remains why Levi-Civita is singled out, as there seems to be no experimental or observational evidence that supports the existence of a non-trivial vector field $\mathcal{A}_\mu$. On the other hand, if the Palatini connections turn out to be physically indistinguishable from the Levi-Civita connection, then there seems to be a surprising "duality symmetry" (rather than the commented $U(1)$ gauge invariance pointed out in [23]), that relates mathematically different spaces as physically equivalent.

In our opinion, the first uniqueness property of the Palatini connections stated above, suggests the latter possibility, namely, the physical indistinguishability of all the Palatini connections at least in "rough" physics, i.e. at the level of the solutions of the Einstein equations and the dynamics of test particles.

[7] It is worth pointing out that Palatini connections present some formal analogies with the *standard volume-preserving connections*, exhibited in [33, sections 3.10, 3.12] as a natural example of connections arising from a decomposition in "Riemannian and post-Riemannian pieces". However, ours are allowed to have a non-volume preserving transport and a non-symmetric Ricci tensor.

[8] With the exception of the connection $\tilde{\Gamma}^\rho_{\mu\nu}=\{^{\rho}_{\mu\nu}\}+\mathcal{B}_\nu\delta^\lambda_\mu$ with $\nabla_\mu\mathcal{B}_\nu=\mathcal{B}_\mu\mathcal{B}_\nu$, as explained in footnote 6. These connections appear only when there exists a parallel $\mathcal{P}_\mu$ and, then, one can write locally $\mathcal{P}_\mu=dx$ and $\mathcal{B}_\mu=dx/(1-x)$. When $\mathcal{P}_\mu$ is not lightlike then the spacetime decomposes as a product $N\times\mathbb{R}$ and x can be regarded as a natural coordinate on $\mathbb{R}$. When it is lightlike, then the spacetime becomes a Brinkmann space (see [34]) and x can be regarded as a natural lightcone coordinate u. This shows the exceptionality of these connections "physically indistinguishable to Levi-Civita, but not Palatini."

First of all, the fact that the symmetric part of the Ricci tensor of the Palatini connection coincides exactly with the Ricci tensor of the Levi-Civita connection implies that the explicit form of the Einstein equation is independent of the choice of $\mathcal{A}_\mu$: any metric $g_{\mu\nu}$ that, for a given minimally coupled $\mathcal{T}_{\mu\nu}$, is a solution of the Einstein equations (13) with $\Gamma^\rho_{\mu\nu} = \bar{\Gamma}^\rho_{\mu\nu}$, is also a solution of the same Einstein equations with $\Gamma^\rho_{\mu\nu} = \{^{\rho}_{\mu\nu}\}$, and vice versa. Furthermore, these solutions coincide with the solutions of the Einstein equation (3) in the metric formalism, which proves the complete equivalence of both formalisms at the level of the solutions.

The invariance of the Einstein equation was already pointed out in [22]. However, under our viewpoint, this property alone is not enough to ensure the undetectability of Palatini connections, as it does not take into account an important issue in gravitational physics: the dynamics of test particles. Notice that if the arguments of [22] were sufficient, one should consider as physically indistinguishable from Levi-Civita all the affine connections that leave the symmetric part of the Ricci tensor invariant, whether they are solutions to the Palatini equation (14), or not. However, such connections would have their own geodesics, which may be very different to Levi-Civita ones. A trivial counterexample in Minkowski spacetime would be the affine connection whose components $\hat{\Gamma}^\rho_{\mu\nu}$ in natural coordinates all vanish except $\hat{\Gamma}^1_{00} = 1$. Obviously, this connection is flat (and hence has the same Ricci tensor as the Levi-Civita connection). Yet, the curve $x^\mu(\lambda) = \lambda\, \delta^\mu_0$, being a geodesic for the Levi-Civita connection but not for $\hat{\Gamma}^\rho_{\mu\nu}$, would represent two physically clearly distinguishable situations (free fall and accelerated motion) depending of the choice of the connection.

It is therefore necessary to consider not only the invariance of the Einstein equations, but also the equivalence of the geodesic motion of both connections. As we have shown, this is indeed the case, thanks to the fact that the (timelike) geodesics of the Palatini connection are pregeodesics of the Levi-Civita connection. In other words, the spacetime trajectories of free-falling test particles for the Palatini connection are the same as the ones described for Levi-Civita. As the latter respect the Equivalence Principle, so does the former: for any of the two connections considered, the outcome of any local experiment in a free falling system will be independent of the velocity and the location of the system in spacetime. Furthermore, also all non-local effects, which show up in tidal forces, will be the same for both connections, as the pregeodesic deviation equations for the two cases are equivalent. Hence we find that also at the level of the motion of test particles, the physics of the Palatini connections is indistinguishable from standard physics.[9]

On the other hand, the second characterisation of the Palatini connections needs a subtler analysis. One might argue that there must be physical effects that become visible in the parallel transport of vectors: as in general the results of parallel transport according to the Palatini and the Levi-Civita connection do not agree, the comparison of vectors in different points of the spacetime manifold will lead to different results, when performed with one connection or the other. In particular, one can think of configurations that would be static according to one connection, but not according to the other. A vector field $V^\mu(\lambda)$, representing some physical magnitude that evolves according to the equations of motion of the system with initial conditions $V^\mu(\lambda_0)$, is said to be unchanged by the evolution of the system if its value $V^\mu(\lambda_1)$ for that magnitude at a given time λ_1 is identical to the parallel transport of $V^\mu(\lambda_0)$ to λ_1. Now, let $V^\mu_g(\lambda)$ and $V^\mu_\Gamma(\lambda)$ be the results of parallel transport of $V^\mu(\lambda_0)$ according to the Levi-Civita and the Palatini connections respectively. As in general $V^\mu_g(\lambda)$ and $V^\mu_\Gamma(\lambda)$ will be different, $V^\mu_g(\lambda_1) - V^\mu(\lambda_1)$ and $V^\mu_\Gamma(\lambda_1) - V^\mu(\lambda_1)$ will not be simultaneously zero. The concept of staticity is therefore as much related to the choice of connection, as it is to the dynamics of the system. So, in principle, one could think that the parallel transport should be observable.[10] However, we believe that it is precisely the homothetic property of the Palatini connection that turns the Palatini and the Levi-Civita parallel transports indistinguishable.

As we have seen in Section 3, the Palatini and the Levi-Civita transports are homothetic, such that in general $V^\mu_g(\lambda_1)$ and $V^\mu_\Gamma(\lambda_1)$ only differ by a (curve-dependent) overall factor, as shown in (31). Configurations that are static according to one connection, would with respect to the other also appear static, up to a homothety. In other words, there are no additional "curvature effects" associated to the Palatini transport, except for a change of norm of the transported vector. Traditionally, the latter would be of course interpreted as non-staticity, but that is due to the fact that we are used to work with metric compatible connections, where the invariance of the norm under parallel transport is guaranteed. The real question is whether the norm of a parallel transported vector can be physically detected in an experiment, or whether it is just a (useful) mathematical construction to understand the theory.

Traditionally, in General Relativity it assumed that one can define a unit measure in any point, by defining it in one point and then transporting the measurement instrument (say a rod), using the Levi-Civita connection. The rod is assumed to maintain its length, as the different particles constituting the rod do not obey the geodesic deviation equation, as they feel the electromagnetic or nuclear forces of the neighbouring particles, which, except for the cases of extreme tidal forces, are much stronger than the curvature effects. When on the other hand, the Palatini connection is used, it seems reasonable to argue that the same physical arguments hold: the main forces acting on the individual particles of the instruments are not the geometrical ones, but the ones created by neighbouring particles.[11] Since the non-gravitational physics is unaffected as long as we are working with minimally coupled matter Lagrangian's, as we argued above, it seems reasonable to conclude that the homothetic character of the Palatini parallel transport, rather than an experimentally measurable property, becomes a mathematical issue to be taken into account when counting the descriptions of the same mensurable system.

Finally, one could wonder whether the Palatini connection could give rise to observable quantum effects. A well-known example in quantum mechanics is the Aharonov–Bohm effect, where topologically distinct gauge fields A_μ give rise to physically different situations, even though they yield the same field strength tensor $F_{\mu\nu}(A)$. However we believe that in our case there are no observable quantum effects associated to the vector field $\mathcal{A}_\mu$: any field configuration for $\mathcal{A}_\mu$ can be reabsorbed in a geodesic reparametrisation (23), independently of its field strengths $\mathcal{F}_{\mu\nu}(\mathcal{A})$ and independently of the topological class the specific field configuration belongs to. In other words, in contrast to Maxwell theory, any choice of $\mathcal{F}_{\mu\nu}(\mathcal{A})$ is a gauge choice.

[9] In this light, it can be argued that the mathematical formulation of the Equivalence Principle should not be, as stated in the introduction, that the connection should be Levi-Civita, but rather that the connection should have the same pregeodesics as Levi-Civita.

[10] Weyl connections (introduced from different physical grounds, see [35]) also leaded to a parallel transport different to Levi-Civita's. The detectability of this transport was essential in that theory, and its consequences were criticised by Einstein. Notice, however, that Weyl tried to unify electromagnetism and gravity, while Palatini connections emerge even when only the gravitational interaction is taken into account.

[11] Note this important difference with Weyl's connections cited in footnote 10.

5. Interpretation and conclusions

The most general affine connection allowed by the Palatini formalism in the Einstein–Hilbert action (allowing also minimally coupled matter terms) is given by the non-symmetric and non-metric compatible connection

$$\bar{\Gamma}^{\rho}_{\mu\nu} = \{^{\rho}_{\mu\nu}\} + \mathcal{A}_{\mu}\,\delta^{\rho}_{\nu}, \qquad (37)$$

with $\mathcal{A}_{\mu}$ an arbitrary non-dynamical vector field. We have shown that the family of Palatini connections is furthermore unique in two ways, first of all because it is the only connection (up to the exceptional case of footnote 8) that has the same pregeodesics as Levi-Civita and at the same time conserves the form of the Einstein equations and secondly because it is the only connection that provides a parallel transport of vectors that is homothetic to the Levi-Civita transport along any curve. We have proven in the previous sections that this connection does not lead to physically observable effects at the level of the Einstein equations or the trajectories of test particles and we have argued that most likely neither it does when comparing the results of parallel transport of vectors. As the Palatini connections are the unique ones that preserve this basic physics (Einstein equations and pregeodesics), the *Palatini approach yields an exact variational characterisation of such basic physics.*

So, if our interpretation is correct and the Palatini connections indeed turn out to be unobservable in all physical situations, then this would hint to a kind of duality (beyond the gauge symmetry as stated in [23] or the invariance of the Einstein equation in [22]) between spacetimes with different geometrical properties, as these would all display the same physics. In mathematical terms, this would mean that for every (pseudo-)Riemannian geometry that is a solution of the minimally coupled Einstein equations, there is a family of non-(pseudo-)Riemannian geometries that are mathematically distinct, but physically indistinguishable.

Probably the best way to see the geometrical origin of the Palatini connection is looking at the geodesic equation (22) and its functional (21). When λ is chosen to be an affine parametrisation (proper time, in physics language), then the geodesic equation acquires its standard form (5). But when any other parametrisation is chosen, extra parametrisation-dependent terms appear in the equation for the pregeodesics. We have shown that these extra terms can be written as a scalar product $\dot{x}^{\rho}\mathcal{A}_{\rho}$ between the velocity of the curve and some specific vector field $\mathcal{A}_{\rho}$, independent of the curve, which in turn can be combined with the Levi-Civita connection and be interpreted as a new, mathematically inequivalent connections $\bar{\Gamma}^{\rho}_{\mu\nu}$. It is therefore as if the Palatini formalism allows its users to freely choose the parametrisation of their geodesics, providing as solutions of the variational principle those connections that under reparametrisation yield the standard Levi-Civita geodesics with affine parametrisation (5). However, notice that different non-symmetric connections have the same geodesics (as the latter only depend on the symmetrised part of the former) and then not all connections projectively related to Levi-Civita are allowed. Indeed, in order for the physics to be invariant, it is not enough that the new connection has the same pregeodesics, but also that the curvature tensors change in such a way that the Einstein equations are invariant. And as we have seen, the only connections that can do this, are precisely those selected by the Palatini formalism.[12]

Summing up the answer to our original problem is a bit subtler than expected: not only the Levi-Civita connection, but the entire family of Palatini connections are singled out by the variational principle and from a mathematical point of view, so there is no reason to assign a preferred status to Levi-Civita. However, *since all Palatini connections lead to the same "rough" physics, the Levi-Civita connection has the virtue of being the simplest representative of a class of physically indistinguishable connections.*

We wish to emphasise that strictly speaking we can only make a hard statement about the observability of the vector field $\mathcal{A}_{\mu}$ in the realm of "rough" physics, not excluding completely that the presence of $\mathcal{A}_{\mu}$ might acquire a physical meaning in subtler situations. However, if this were the case, we can not stop wondering why there is no experimental evidence for the existence of this vector field in our universe. We leave these possible effects for future investigations.

There are a number of ways the results of this letter can be extended. In the first place, it would be interesting to see whether the presence of $\mathcal{A}_{\mu}$ could be detected in more complicated situations, such as for example non-minimal couplings, higher curvature terms or in a Jordan frame. Secondly, an obvious question is whether the Palatini connection as the most general solution to the variational principle is limited to the Einstein–Hilbert actions, or whether it also appears in different theories. It is well known that the metric and the Palatini formalisms are equivalent for Lovelock gravities, in the sense that the Levi-Civita connections appear as a solution to the Palatini equation for these theories. However, it is not clear whether it is a unique solution and, if not, whether the Palatini connections appear also as an allowed solution by the variational principle (as far as we know, the only results in this direction appear in [21,22]). Answering this question would also give hints on whether there are physically observable effects associated with the Palatini connections. Work on these topics by some of the authors is in progress.

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