Elliptic problems with singular nonlinearities and quadratic lower order terms

Programa de doctorado en Física y Matemáticas (FisyMat)

Universidad de Granada

TESIS DOCTORAL

Por

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Para mi hermana;
para nuestros maravillosos padres.
Por las sonrisas :) que ponemos en la vida del otro.

Contents

Prólogo

Esta memoria de tesis doctoral es presentada por Dña. Lourdes Moreno Mérida para optar al título de Doctora en Matemáticas por la Universidad de Granada, dentro del programa oficial de Doctorado en Física y Matemáticas (FisyMat). Se realiza por tanto de acuerdo con las normas que regulan las enseñanzas oficiales de Doctorado y del Título de Doctor en la Universidad de Granada, aprobadas por Consejo de Gobierno de la Universidad en su sesión de 2 de Mayo de 2012, donde se especifica que *"la tesis doctoral consistirá en un trabajo original de investigación elaborado por el candidato en cualquier campo del conocimiento que se enmarcará en alguna de las líneas del programa de doctorado en el que está matriculado. Para garantizar, con anterioridad a su presentación formal, la calidad del trabajo desarrollado se aportará, al menos, una publicación aceptada o publicada en un medio de impacto en el ámbito de conocimiento de la tesis doctoral firmada por el doctorando, que incluya parte de los resultados de la tesis. La tesis podrá ser desarrollada y, en su caso, defendida, en los idiomas habituales para la comunicación científica en su campo de conocimiento. Si la redacción de la tesis se realiza en otro idioma, deberá incluir un resumen en español."*.

La presente memoria ha sido redactada en base a cinco artículos de investigación, dos de los cuales fueron publicados en el año 2014 [14, 81], otros dos en el año 2015 [43, 44] y el último aún está pendiente de publicación [15]. Todos estos trabajos están sometidos o han aparecido en revistas de relevancia internacional, incluidas todas ellas en el Journal of Citations Reports e incluidas en las bases de datos MathSciNet (American Mathematical Society) y Zentralblatt für Mathematik (European Mathematical Society).

Para optar a la mención internacional en el título de doctor, la mayor parte de la memoria está escrita en inglés, idioma que actualmente es de mayoritario uso en la comunicación científica en el ámbito de las matemáticas, respetando así el idioma en que los artículos de investigación recopilados han sido o serán publicados. Al redactarse en una lengua no oficial, sin embargo, incluimos un resumen también en español.

Los resultados novedosos presentados en la memoria han sido obtenidos a lo largo de los últimos años bajo la supervisión del Profesor David Arcoya Álvarez en el Departamento de Análisis Matemático de la Universidad de Granada y del Profesor Lucio Boccardo Tangredi en el Dipartimento di Matematica dell' Università Di Roma

"La Sapienza" (Italia). En este tiempo, la doctoranda ha sido alumna del Programa Oficial de Doctorado en Física y Matemáticas (FisyMat); desde Marzo de 2013 ha disfrutado de una Beca de Formación de Profesorado Universitario (FPU12/02395), financiada por el Ministerio de Educación, Cultura y Deportes español, y ha realizado sus investigaciones en el marco del Grupo de Investigación FQM-116, financiado por la Junta de Andalucía, y de los Proyectos de Investigación MTM2012-31799, MTM2015-68210-P, financiados por la Dirección General de Investigación del Gobierno de España y cofinanciados con fondos FEDER de la Unión Europea. La doctoranda ha realizado varias estancias de investigación en la Università di Roma "La Sapienza" (Italia):

- Estancia realizada durante el mes de Marzo de 2013
- Estancia realizada durante el mes de Mayo de 2013
- Estancia realizada durante los meses Octubre, Noviembre y Diciembre de 2014

Declaración de la doctoranda

Lourdes Moreno Mérida,

DECLARA:

Que la tesis titulada *Elliptic problems with singular nonlinearities and quadratic lower order terms*, presentada para optar al Grado de Doctor en Matemáticas, ha sido realizada por ella misma, bajo la supervisión de los Profesores David Arcoya Álvarez, en el Departamento de Análisis Matemático de la Universidad de Granada y Lucio Boccardo Tangredi, en el Dipartimento di Matematica dell'Università di Roma "La Sapienza".

Granada, Junio de 2016

Lourdes Moreno Mérida

Declaración de los directores

David Arcoya Álvarez, doctor en Matemáticas y catedrático de Análisis Matemático de la Universidad de Granada

Lucio Boccardo Tangredi, catedrático de Análisis Matemático de la Università di Roma "La Sapienza"

DECLARAN:

Que la tesis titulada *Elliptic problems with singular nonlinearities and quadratic lower order terms*, presentada por Lourdes Moreno Mérida para optar al Grado de Doctor en Matemáticas, ha sido realizada bajo su supervisión, en el Departamento de Análisis Matemático de la Universidad de Granada.

Granada, Junio de 2016

Directores de la tesis

David Arcoya Álvarez

Lucio Boccardo Tangredi

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La doctoranda Dña. Lourdes Moreno Mérida y los directores de la tesis D. David Arcoya Álvarez y D. Lucio Boccardo Tangredi, garantizan que, hasta donde su conocimiento alcanza, en la realización de la presente tesis doctoral se han respetado los derechos de otros autores a ser citados, cuando se han utilizado sus resultados o publicaciones.

Granada, Junio de 2016

Doctoranda

Lourdes Moreno Mérida

Directores de la tesis

David Arcoya Álvarez

Lucio Boccardo Tangredi

Sobre la Mención Internacional

Con el fin de obtener la Mención Internacional en el Título de Doctor, se han cumplido en lo que atañe a esta tesis y a su defensa los siguientes requisitos:

1. Esta Memoria ha sido escrita en inglés con un resumen en español.
2. Esta tesis ha sido evaluada por dos investigadores externos pertenecientes a centros no españoles.
3. Uno de los miembros del tribunal proviene de una universidad no española.
4. La defensa de la tesis se realiza en inglés.
5. La doctoranda, entre otras, ha realizado una estancia de tres meses (Octubre, Noviembre y Diciembre de 2014) en la Università di Roma "La Sapienza" (Italia).

Granada, Junio de 2016

Doctoranda

Lourdes Moreno Mérida

Directores de la tesis

David Arcoya Álvarez

Lucio Boccardo Tangredi

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Introduction

This thesis contributes to the study of relevant questions in the theory of quasilinear and semilinear elliptic equations. In particular, most of the results we present here are stated for problems with a singular nonlinearity. There are several motivations for our work coming not only from problems in applied mathematics but also from pure mathematical interest, as the one that arises out of Calculus of Variations.

This memory is divided into five chapters and a conclusion chapter in Spanish. All chapters can be read independently, although most of the terminology and some technical arguments are shared among them. Apart from a few minor notation changes that have been made to unify our presentation, and that the full bibliography has been collected at the end of the thesis, Chapter 1 is a joint work with D. Arcoya [14] published in Nonlinear Anal., Chapter 3 is [81] published in Nonlinear Anal., Chapter 4 is a joint work with L. Boccardo and L. Orsina [44] published in Milan J. Math., Chapter 5 is a joint work with L. Boccardo [43] published in Milan J. Math., while Chapter 2 corresponds to a joint work with D. Arcoya [15] which is submitted to publication.

Despite each chapter having its own introduction, we consider appropriate to include here a global treatment for all the results exposed in this manuscript.

In the first Chapter we deal with some problems whose basic model equation is the following one

$$-\Delta u = g(x, u), \quad \text{in } \Omega\,,$$

where Ω is an open, bounded, subset of $\mathbb{R}^N$ and g is a Carathéodory function which exhibit a singularity at $u = 0$. These kind of problems have been thoroughly studied during the last decades since the pioneer works by Stuart [88] and by Crandall, Rabinowitz and Tartar [65]. In the first one, the author considered a function $g(x, s)$ which "blows-up at $s = 0$" when x goes to a point belonging to the boundary of Ω. On the other hand, in the second one, the authors considered a singular function $g(x, s) = g(s)$ independent of x and they proved the existence of a solution together with some regularity properties of it. Afterwards, in 1991, Lazer and McKenna [78] studied the existence of a classical solution for the Dirichlet problem associated to the above equation in the case

$$g(x, u) = \frac{f(x)}{u^\gamma}\,,$$

where f is an Hölder continuous function which is strictly positive in $\overline{\Omega}$ and γ is a strictly positive parameter. In particular, they proved that

> "*If for some* $0 < \alpha < 1$ *one has that* $\partial\Omega \in C^{2,\alpha}$, $f \in C^{0,\alpha}(\overline{\Omega})$, $f(x) > 0$ *in* $\overline{\Omega}$ *and* $\gamma > 0$, *then there exists an unique solution* u *of the Dirichlet problem*
> $$\begin{cases} -\Delta u = \dfrac{f(x)}{u^\gamma}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$
> *such that* $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ *and* $u > 0$ *in* Ω".

Observe that the prescribed boundary condition in (1) makes the study of these singular equations hard. Actually, the assumption "$u = 0$ on $\partial\Omega$" together with the singular nonlinearity implies that, for every solution u, the term $1/u(x)^\gamma$ diverges as x goes to the boundary of Ω.

In contrast with [78], we are interested in the study of distributional solutions for the problem (1). As usual, this means that we look for distributional solutions u of the differential equation

$$-\Delta u = \frac{f(x)}{u^\gamma}, \quad \text{in } \Omega, \tag{2}$$

which satisfy, in some sense, "$u = 0$ on $\partial\Omega$". Specifically, we search for solutions $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $u > 0$ a.e. in Ω, $\frac{f(x)}{u^\gamma} \in L^1_{\mathrm{loc}}(\Omega)$ and moreover they satisfy (2) in a distributional sense, i.e.,

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega \frac{f(x)}{u^\gamma}\phi, \quad \forall \phi \in C^1_c(\Omega).$$

With the aim of establishing what the condition "$u = 0$ on $\partial\Omega$" means, we point out the surprising result obtained by Lazer and McKenna in [78]. More precisely, the authors proved that

> "*The unique solution* u *of the Dirichlet problem* (1) *belongs to the Sobolev space* $W^{1,2}_0(\Omega)$ *if and only if the parameter* $\gamma < 3$."

As a consequence, in the distributional context, one would not expect to find solutions belonging to the Sobolev space $W^{1,2}_0(\Omega)$ for any value of $\gamma > 0$. Therefore, it is necessary to introduce a new concept for the condition "$u = 0$ on $\partial\Omega$".

Precisely, in 2010, Boccardo and Orsina [51] studied the existence of one distributional solution for the problem (1). With respect to the boundary condition "$u = 0$ on $\partial\Omega$", in contrast with [61, 75] where this condition is understood under the assumption $(u - \varepsilon)^+ \in W^{1,2}_0(\Omega)$ for all $\varepsilon > 0$, they followed the ideas of [9]. That is, an even stronger requirement is imposed based on the fact that some positive powers of the solution of the differential equation (2) belong to the Sobolev space $W^{1,2}_0(\Omega)$. In this paper, the authors needed to study the cases $\gamma < 1$, $\gamma = 1$ y $\gamma > 1$, separately, connecting each one with the regularity of f. In particular, they proved the following result

"Assume that $f \in L^m(\Omega)$ with $m \geq 1$. The following assertions hold:

1. *If $\gamma < 1$ and $m \geq \left(\frac{2^*}{1-\gamma}\right)'$, then there exists a positive solution u of (2) such that $u \in W_0^{1,2}(\Omega)$.*
2. *If $\gamma = 1$ and $m = 1$, then there exists a positive solution u of (2) such that $u \in W_0^{1,2}(\Omega)$.*
3. *If $\gamma > 1$ and $m = 1$, then there exists a positive solution u of (2) such that $u^{\frac{\gamma+1}{2}} \in W_0^{1,2}(\Omega)$".*

We point out that, in contrast with the results obtained by Lazer and McKenna, here the authors obtained a solution belonging to the Sobolev space $W_0^{1,2}(\Omega)$ only when the parameter $\gamma \leq 1$.

In a natural way, one can consider another kind of semilinear singular problems presenting new challenges such as the following model problem

$$\begin{cases} -\Delta u = \dfrac{\lambda}{u^\gamma} + u^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with λ, γ positive parameters and $p > 1$. Apart from the usual difficulties of these singular problems, in this case we also find a new one due to superlinear nature of the problem. Indeed,

$$\lim_{s \to +\infty} \frac{1}{s}\left(\frac{\lambda}{s^\gamma} + s^p\right) = +\infty\,,$$

which makes the study of *a priori estimates*, among other things, more delicate.

This model was considered in 1989 by Coclite and Palmieri [64], and afterwards it has been further studied in papers like [21, 75].

Boccardo [21] considered the last problem, following the ideas of [51], for the distributional study of the problem (1). On the one hand, we say that u is a distributional positive solution of the differential equation associated to (3) if $u \in W^{1,2}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfies $u > 0$ a.e. in Ω, $\frac{\phi}{u^\gamma} \in L^1(\Omega)$ for all $\phi \in W_0^{1,2}(\omega)$ and that

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega \frac{\phi}{u^\gamma} + \int_\Omega u^p \phi\,, \quad \forall \phi \in W_0^{1,2}(\omega)\,, \tag{4}$$

for all open subset ω of Ω, such that $\omega \subset\subset \Omega$. On the other hand, concerning the boundary condition "$u = 0$ on $\partial\Omega$", it is required that some positive powers of the function u belong to the Sobolev Space $W_0^{1,2}(\Omega)$. More precisely, using a sub-super solution method as in [28], the author proved in [21] the following result

> "*There exists a positive number Λ such that for every $\lambda \in (0, \Lambda)$ the problem (3) has one positive solution $0 < u \in W^{1,2}_{loc}(\Omega) \cap L^\infty(\Omega)$ satisfying (4) with*

- $u \in W_0^{1,2}(\Omega)$, *if* $0 < \gamma \leq 1$;
- $u^{\frac{\gamma+1}{2}} \in W_0^{1,2}(\Omega)$, *if* $\gamma > 1$.*"*

Afterwards, Arcoya and Boccardo [8] studied the multiplicity of solutions for this problem. In particular, they proved

> *"For λ small enough, there are at least two positive solutions of* (3) *belonging to the space $W_0^{1,2}(\Omega)$, when the parameters satisfy $\gamma < 1$ and $2 < p+1 < 2^* := \frac{2N}{N-2}$."*

The authors deduced this result using variational methods. This is the reason why the restriction $\gamma < 1$ is needed.

Keeping all these results in mind, our first aim is to address the study of the multiplicity of solutions for all $\gamma > 0$. Specifically, the first theorem we present in the first chapter is the following one.

Theorem 1 *Let Ω be an open, bounded set of $\mathbb{R}^N$ with C^2 boundary. If $\gamma > 0$ and $2 < p+1 < 2^*$ hold, then there is $\Lambda > 0$ such that for every $\lambda \in (0, \Lambda)$ the problem* (3) *has two different strictly positive solutions u and v in $W_{loc}^{1,2}(\Omega) \cap L^\infty(\Omega)$, i.e., u and v satisfy* (4) *with*

$$u^\alpha, v^\alpha \in W_0^{1,2}(\Omega), \quad \forall \alpha > \frac{\gamma+1}{4}.$$

We remark that in the case $1 < \gamma < 3$, we have $\frac{\gamma+1}{4} < 1$ and thus we obtain that the solutions belong to $W_0^{1,2}(\Omega)$ too. In this sense, we improve the results proved in [21] about the existence of solution in this space for the case $0 < \gamma \leq 1$.

In addition, we emphasize that the tools developed in the proof of our first theorem, also allow us to handle the singular problem (1) studied by Boccardo and Orsina in [51]. Indeed, under slightly more restrictive assumptions, we improve the meaning of the boundary condition for the case $\gamma > 1$. Precisely, in the last section of the first chapter we also present the following result.

Theorem 2 *Assume that the bounded, open subset Ω in $\mathbb{R}^N$ satisfies the interior sphere condition, that $f \in L^m(\Omega)$ with $m > 1$ and that there is a positive constant f_0 such that $f(x) \geq f_0 > 0$ a.e. in Ω.*

If $1 < \gamma < \frac{3m-1}{m+1}$, then there exists a solution $u \in W_{loc}^{1,1}(\Omega)$ of (2) *such that $u^\alpha \in W_0^{1,2}(\Omega)$ for all $\alpha \in \left(\frac{(m+1)(\gamma+1)}{4m}, \frac{\gamma+1}{2}\right]$.*

Observe that when $1 < \gamma < \frac{3m-1}{m+1}$, we have $\frac{(m+1)(\gamma+1)}{4m} < 1 < \frac{\gamma+1}{2}$, so that the choice $\alpha = 1$ is allowed in the previous theorem. As a consequence, for all $m > 1$ and $1 < \gamma < \frac{3m-1}{m+1}$, we obtain a solution u of the problem (1) which belongs to $W_0^{1,2}(\Omega)$. Then, in this case, the hypotheses on the boundary of Ω and $f(x) \geq f_0 > 0$ a.e. in Ω allow us to get as close as we want to the natural threshold established by Lazer and McKenna.

Going ahead with the study of singular problems, one can also consider the existence of a solution for the following kind of problems

$$\begin{cases} -\Delta u = \mu(x)\dfrac{|\nabla u|^2}{u} + \lambda u + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $0 \lneqq f \in L^m(\Omega)$ with $m > N/2$ and $0 \lneqq \mu(x) \in L^\infty(\Omega)$. Observe that in these particular problems the singular nonlinearity has also a quadratic growth with respect to the gradient.

It is worth pointing out that in the trivial case $\mu(x) \equiv 0$ (non singular linear Dirichlet problem) it is well known that there is a positive solution iff $\lambda < \lambda_1$, where λ_1 is the eigenvalue associated to the first eigenfunction of the Laplacian operator. Even more, in this particular case we have that $\lambda = \lambda_1$ is a bifurcation point from infinity.

It is convenient to recall that the non singular quasilinear problems with quadratic growth with respect to the gradient have been widely studied in the literature. Precisely, Boccardo, Murat and Puel considered them in several papers, see for instance [46, 47, 48, 49] and the references therein. Specifically, in [49] the authors proved the existence of a solution in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ for the problem

$$\begin{cases} -\Delta u = \mu(x)|\nabla u|^2 + \lambda u + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6}$$

for every $\lambda < 0$, where $\mu(x)$ is a positive bounded function and $0 \lneqq f \in L^m(\Omega)$ with $m > N/2$. Meanwhile, Barles and Murat [17] proved uniqueness.

The existence of a bounded solution for the above problem with $\lambda = 0$ was obtained by Ferone and Murat [68] in 1998 when the datum f was small enough, (see also the paper by Abdellaoui, Dall'Aglio and Peral, in [1], were such a hypothesis was relaxed). Even more, Barles, Blanc, Georgelin, Kobylanski [16] proved in 1999 (see also [11]) the uniqueness of the solution in the case $\lambda = 0$ but assuming again suitable restrictions "on the size of the datum f".

In 2013, Jeanjean and Sirakov [76] observed that the uniqueness of solutions for the above problem might fail if λ is positive. Indeed, assuming that $\mu(x) \equiv \mu > 0$, the problem (6) becomes a semilinear one using a convenient change of variable. Thanks to this trick, the authors proved that for a $\lambda > 0$ small enough, the above problem admits at least two bounded solutions.

Recently, in contrast with the situation for the linear problem (i.e., $\mu \equiv 0$), Arcoya, De Coster, Jeanjean and Tanaka, in [12] (see also [11]), realized that this multiplicity phenomenon occurs for the problem (6) due to the fact that $\lambda = 0$ is a bifurcation point from infinity. In this sense, the continuum of solution which contains the pairs (λ, u_λ), with $\lambda < 0$ and u_λ the unique solution of (6) associated to this value of λ, presents two situations. Either $||u_\lambda||_{L^\infty(\Omega)}$ diverges when the negative parameter λ tends to $\lambda = 0$ (i.e., in this case (6) does not have a solution with $\lambda = 0$), or the continuum of solutions contains a point as $(0, u_0)$, being u_0 a positive bounded solution of (6) with $\lambda = 0$. In this second case, one has that $\lambda = 0$ is a bifurcation point to the right and this is the reason why the multiplicity phenomenon appears. More precisely, the authors proved the following result

> *"Assuming that $\mu(x) \geq \mu_0 > 0$ and that the problem (6) with $\lambda = 0$ has a solution, then there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, the problem (6) admits at least two different bounded solutions."*

Observe that, in this context, the assumption $\mu_0 \leq \mu(x) \in L^\infty(\Omega)$ implies that the lower order term of the problem (6) satisfies

$$0 \leq \mu_0|\nabla u|^2 \leq \mu(x)|\nabla u|^2 \leq ||\mu||_{L^\infty(\Omega)}|\nabla u|^2 .$$

Recently, as part of the results of [89] for space of dimensions up to N = 5, Souplet has proved that if $N \leq 2$, then the assumption $\mu(x) \geq \mu_0 > 0$ in Ω can be strongly weaken to $\mu(x) \geq \mu_0 > 0$ in B, for some ball $B \subset \Omega$.

One may wonder whether $\lambda = 0$ is still a bifurcation point from infinity when a singular term as in (5) is presented. We emphasize that in this situation, the lower order term $\mu(x)\frac{|\nabla u|^2}{u}$ "is not bounded below by $\mu_0|\nabla u|^2$" with $\mu_0 > 0$.

As before, we say that u is a positive solution of the differential equation associated to (5), that is the following one

$$-\Delta u = \mu(x)\frac{|\nabla u|^2}{u} + \lambda u + f(x) ,$$

if $u \in W^{1,1}_{\rm loc}(\Omega)$ is such that, $u > 0$ a.e. in Ω, $\frac{|\nabla u|^2}{u} \in L^1_{\rm loc}(\Omega)$ and moreover it satisfies

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \mu(x)\,\frac{|\nabla u|^2}{u}\,\varphi + \lambda\int_\Omega u\,\varphi + \int_\Omega f\,\varphi \tag{7}$$

for every $\varphi \in C^1_c(\Omega)$. Concerning the boundary condition "$u = 0$ on $\partial\Omega$", following the definition we used before and the ideas of [70], we require that a suitable positive power of the function u belongs to the Sobolev space $W^{1,2}_0(\Omega)$.

On the one hand, the existence of solution for (5) in the particular case $\lambda = 0$ has been proved by Arcoya, Boccardo, Leonori and Porretta in [9]. On the other hand, it is interesting to point out that, despite we would think the singular problem (5) is harder than the non singular one (6), we obtain that, in the singular situation, $\lambda = 0$ is not the bifurcation point from infinity but $\lambda = \lambda_1$, as in the linear context (i.e., $\mu \equiv 0$).

In particular, among others, we present the following result in the second chapter.

Theorem 3 *Assume that $0 \lneqq f \in L^m(\Omega)$ with $m \geq \frac{N}{2}$. If $\lambda < \frac{\lambda_1}{1+||\mu||_{L^\infty(\Omega)}}$, then there is a solution $u \in W^{1,1}_{loc}(\Omega)$ satisfying (7) with*

$$u^\gamma \in W^{1,2}_0(\Omega)\,,\ \forall\gamma > \frac{1+||\mu||_{L^\infty(\Omega)}}{2}.$$

Other results are also shown in this chapter under less regularity requirement on the function f (i.e., $f \in L^m(\Omega)$ with $m > 1$).

So far, in Chapter 1 and 2, we have considered differential equations involving the Laplacian operator. In the remaining chapters, we address the study of singular equations including more general differential operators. We can justify the study of this operators since they appear, at least formally, in a natural way. For instance, if $r > 0$ and $a(x)$ is a measurable function satisfying $0 < \alpha \leq a(x) \leq \beta$, then we can consider, roughly speaking, the following functional

$$J(u) = \frac{1}{2}\int_\Omega (a(x) + |u|^r)\,|\nabla u|^2 - \int_\Omega f(x)u\,.$$

Without going into any further details (as for instance the domain of definition as well as the differentiability of J), we point out that the Euler Lagrange equation associated to J should be the following quasilinear equation

$$-\mathrm{div}\,[(a(x) + |u|^r)\nabla u] + \tfrac{r}{2}u|u|^{r-2}|\nabla u|^2 = f(x), \quad \text{in } \Omega. \tag{8}$$

We emphasize that in this particular model there is a huge difference between the cases $r \geq 1$ and $r < 1$. Indeed, in the first one we obtain a non singular equation, while in the second one a singular equation appears.

Similar singular problems (with the singularity on the left hand-side of the equation) have been considered in the literature. In particular, we refer to [13] where the existence of a positive solution $u \in W_0^{1,2}(\Omega)$ for

$$-\Delta u + \tfrac{|\nabla u|^2}{u^\theta} = f(x), \quad \text{in } \Omega,$$

was studied for the first time, when $0 < \theta < 1$. Further works on the subject can be found in [10, 24] and the references therein (where non linear operators with bounded coefficients are also considered).

We highlight that the differential operator associated to (8) presents more difficulties than the ones which have appeared in the works cited above. Indeed, this new differential operator not only has a quadratic growth with respect to the gradient and a singularity in $u = 0$, but moreover its principal part $\mathrm{div}\,[(a(x) + |u|^r)\,\nabla u]$ is not well defined in the Sobolev space $W_0^{1,2}(\Omega)$ and it is "unbounded with respect to u" (in both the singular and non singular cases).

Either by the above heuristic justification (coming from Calculus of Variations) or by the mathematical challenges due to these aforementioned strong difficulties, we turn our attention to the quasilinear model equation (8). As expected, we decided to start the study of (8) in the easier non singular case (i.e., $r \geq 1$). Precisely, in the third chapter we deal with the following slightly more general class of quasilinear Dirichlet problems

$$\begin{cases} -\mathrm{div}\,[(a(x) + |u|^q)\,\nabla u(x)] + b(x)\,u\,|u|^{p-1}\,|\nabla u|^2 = f(x)\,, & \text{in } \Omega, \\ u(x) = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{9}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, $p\,,q > 0$ and $a(x)\,,b(x)$ are measurable functions satisfying

$$0 < \alpha \leq a(x) \leq \beta \;\text{ and }\; 0 < \mu \leq b(x) \leq \nu \;\text{ a.e. } x \in \Omega\,. \tag{10}$$

Moreover, we only assume that $f \in L^1(\Omega)$.

Observe that, if $q = r$, $p = r - 1$ and $b(x) = r/2$, then the differential equation associated to these problems is the above model equation (8) in the non singular case.

The aim of this chapter is to study the existence of a solution belonging to the Sobolev space $W_0^{1,2}(\Omega)$ for this class of quasilinear problems. It is interesting to point out that, apart from the aforementioned difficulties, the lower order term of (9) (i.e., $b(x)\,u\,|u|^{p-1}\,|\nabla u|^2$) does not belong to $W^{-1,2}(\Omega)$ even if u is a function belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Despite, this quadratic lower order term allows us to obtain finite energy solutions, that is, solutions belonging to the Sobolev space $W_0^{1,2}(\Omega)$ even with data f belonging only to $L^1(\Omega)$. This regularizing effect is due to the fact that the lower order term $b(x)\,u\,|u|^{p-1}\,|\nabla u|^2$ satisfies the so called *absorption sign condition*, that is to say,

$$b(x)\,u\,|u|^{p-1}|\nabla u|^2 \,\cdot\, u \geq \mu|u|^{p+1}|\nabla u|^2 \geq 0\,, \text{ a.e. } x \in \Omega.$$

Boccardo and Gallouët in 1992 [36] (see also [20]) were the first to observe the regularizing effect of the lower order term.

Many papers have dealt with similar quasilinear Dirichlet problems, see for instance [23, 24, 82]. In particular, in [23], Boccardo considered the problem (9) and studied the existence of a positive solution under suitable assumptions on the summability of the data f and on the positive parameters p and q. Moreover, here it is assumed that the function f is positive. Among other results the author proved the following one.

> *"If $f \in L^1(\Omega)$, $f \geq 0$, $p \geq 2\,q$ and* (10) *holds true, then there exists a positive solution $u \in W_0^{1,2}(\Omega)$ of* (9)*, in the sense that, $b(x)\,u^p\,|\nabla u|^2 \in L^1(\Omega)$ and moreover*
>
> $$\int_\Omega (a(x) + |u|^q)\,\nabla u \nabla \varphi + \int_\Omega b(x)\,|u|^{p-1}u|\nabla u|^2\,\varphi = \int_\Omega f\,\varphi\,, \tag{11}$$
>
> *for every $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$".*

Due to the difficulties mentioned above for the study of (9) and, as it is standard in the study of singular problems, Boccardo proved the existence of a solution by an approximation scheme. In order to perform it, three steps are considered. Actually, in the first one, a family of approximate problems, whose existence of solutions u_n is known, is considered. The second step deals with *a priori* estimates on u_n. And, lastly, compactness results in some spaces are proved in order to find a solution u of (9) as a limit of the sequence $\{u_n\}$ of approximate solutions. We point out that the restrictions over the parameters in [23] are essential to address *a priori* estimates as well as the compactness properties.

Taking into account these previous results, our aim is to improve them by obtaining existence results for (9) without any sign restriction over f and without any restrictions over the parameters p and q. To avoid the restrictions imposed in [23] we need to use new tools to carry out the different steps that configure the approximation scheme in a more general setting.

In order to success, we need to give a different notion of solution that seems to be natural in this new framework. Moreover, we use a technique which has been introduced in [47] and then applied in others papers as [82] and [85]. Specifically, the main result of this chapter is the following one.

Theorem 4 *If Ω is an open, bounded set of $\mathbb{R}^N$, $p\,,q > 0$, $f \in L^1(\Omega)$ and $a(x)\,,b(x)$ are measurable functions satisfying* (10)*, then there exists a solution $u \in W_0^{1,2}(\Omega)$ of* (9) *in the following sense:*

$$(a(x) + |u|^q)\,|\nabla u| \in L^1(\Omega)\,,\; b(x)\,|u|^p\,|\nabla u|^2 \in L^1(\Omega) \text{ and}$$
$$u \text{ satisfies (11) for every } \varphi \in W_0^{1,\infty}(\Omega).$$

In addition, our method allows us to recover the existence and regularity results presented in [23] for solutions satisfying (11) with test functions belonging to the space $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Precisely, in the last section of this third chapter, we show the following result.

Theorem 5 *Let $a(x)$ and $b(x)$ be measurable functions satisfying* (10)*, p, $q > 0$ and $f \in L^m(\Omega)$ with $1 \le m \le \frac{N}{2}$. If $u \in W_0^{1,2}(\Omega)$ is the solution given by Theorem 4, then one has:*

(A) *If $m = 1$ and $p \ge 2q$, then $u \in L^{(p+2)\frac{N}{N-2}}(\Omega)$;*

(B) *If $\frac{2(q+1)N}{2N+p(N-2)+4q} \le m \le \frac{N}{2}$ and $2q \ge p \ge q-1$, then $u \in L^{(p+2)m^{**}}(\Omega)$;*

(C) *If $\frac{2N}{N+2} \le m \le \frac{N}{2}$, $q \ge 1$ and $2p \ge q-1 \ge p$, then $u \in L^{(q+1)m^{**}}(\Omega)$;*

Moreover, if (A), (B) or (C) holds true, then $\big(a(x) + |u|^q\big)\nabla u \in (L^2(\Omega))^N$ and u satisfies (11) *for every $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

Once we have studied this class of quasilinear problems (8) with quadratic growth with respect to the gradient in the non singular case, we turn our attention to the singular case in the fourth chapter. Here we consider the existence of a solution for the general problems

$$\begin{cases} -\mathrm{div}([a(x) + u^q]\nabla u) + b(x)\dfrac{1}{u^\theta}|\nabla u|^2 = f\,, & \text{in } \Omega, \\ \qquad\qquad u = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{12}$$

where $a(x)$, $b(x)$ are measurable functions satisfying (10), $f \in L^1(\Omega)$ and

$$0 < q\,, \;\; 0 < \theta < 1\,, \tag{13}$$

$$f \geq 0\,, \;\; f \not\equiv 0\,. \tag{14}$$

In relation with the model equation (8) which motivated the study of this kind of quasilinear problems, we observe that the differential equation associated to (12) is indeed a generalization of the singular equation (8). It suffices to take the values $\theta = 1 - r$, $b(x) = \frac{r}{2}$ and $q = r$ in the differential equation of (12).

It is interesting to note that, exactly as happened to the non singular class of problems (9), the lower order term $b(x)\frac{1}{u^\theta}|\nabla u|^2$ satisfies the *absorption sign condition*, that is to say

"the quadratic lower order term has the same sign of the solution"

and this condition is again the reason why the lower order term has a regularizing effect over the solutions of these quasilinear singular problems.

Here, we follow the arguments of [24] and we find a solution of (12) by approximating problem (12) with a sequence of non singular quasilinear problems with bounded data, and then proving both *a priori* estimates and convergence results on the sequence of approximating solutions. In addition, contrary to the non singular case, in this singular context it is necessary to prove that the sequence of approximate solutions is bounded below by a positive constant away from $\partial\Omega$.

We also highlight that in the study of these quasilinear singular problems (12) there exists a strong interaction between the singular term $\frac{|\nabla u|^2}{u^\theta}$ and the one $\mathrm{div}(u^q \nabla u)$ which appears in the principal part of the differential operator considered. Indeed, we find a solution of (12) which belongs to different Sobolev spaces depending on the relation we have between these two terms. Precisely, the main result presented in this fourth chapter is the following one.

Theorem 6 *Suppose that $f \in L^1$ and that* (10), (13) *and* (14) *hold true. Then there exists a solution u of* (12)*, with $u > 0$ in Ω,*

$$[a(x) + u^q]\,|\nabla u| \in L^\rho(\Omega)\,,\; \forall \rho < \frac{N}{N-1}\,, \quad b(x)|\nabla u|^2\, u^{-\theta} \in L^1(\Omega)\,,$$

and

$$\int_\Omega [a(x) + u^q]\, \nabla u \nabla \varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi = \int_\Omega f\,\varphi\,,$$

for every φ in $W_0^{1,p}(\Omega)$, $p > N$. Furthermore, we have the following summability results for u:

- *if $0 < q \leq 1 - \theta$, then u belongs to $W_0^{1,r}(\Omega)$, with $r = \frac{N(2-\theta)}{N-\theta}$;*
- *if $1 - \theta < q \leq 1$, then u belongs to $W_0^{1,r}(\Omega)$, for every $r < \frac{N(q+1)}{N+q-1}$;*

- *if $q > 1$, then u belongs to $W_0^{1,2}(\Omega)$.*

Observe that in order to obtain a solution we only need the requirement $f \in L^1(\Omega)$. Nevertheless, it is possible to improve the regularity of the solution obtained if we consider more summable data f. Indeed, we also present, in this chapter, the following regularity result.

Theorem 7 *We suppose that $f \in L^m(\Omega)$ with $1 < m < \frac{N}{2}$ and we consider $\delta := \min(\theta, 1-q)$, together with the hypothesis of Theorem 6. If u is the solution given by this theorem, then u belongs to $L^s(\Omega)$, where $s = (2-\delta)m^{**}$. Moreover,*

- *when $q < 1$,*

  1) if $1 < m < \left(\frac{2^*}{\delta}\right)'$, then $u \in W_0^{1,r}(\Omega)$, with $r = \frac{Nm(2-\delta)}{N-m\delta}$;

  2) if $m \geq \left(\frac{2^*}{\delta}\right)'$, and $m > 1$, then $u \in W_0^{1,2}(\Omega)$;

- *while if $q = 1$, then $m > 1$ implies $u \in W_0^{1,2}(\Omega)$.*

- *In the case $q > 1$, $u \in W_0^{1,2}(\Omega)$ (thanks to Theorem 6).*

We point out that, as usual, if we consider f belonging to $L^m(\Omega)$ with $m > \frac{N}{2}$, then we obtain solutions in $L^\infty(\Omega)$.

Summarizing, we have shown in the previous chapters how the approximation tools combined with *a priori* estimates are very useful in the study of singular elliptic problems. Of course, these tools are also useful to address problems of a different nature. As an example, the last chapter is devoted to the study of a quasilinear elliptic problem with Neumann boundary condition. Concretely, motivated by the results of [42], in this chapter we deal with the following model problem

$$\begin{cases} -\Delta_p u + |u|^{s-1}u = 0, & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta = \psi, & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, η is the unit outward normal on $\partial\Omega$, $1 < p < N$, $s > 0$ and $\psi \in L^m(\partial\Omega)$ with $m > 1$.

In [4], the existence of "finite energy solutions" for this problem is considered, that is to say, the authors proved that, under suitable assumptions which involve the different parameters of the problem, there exists a solution belonging to the Sobolev space $W^{1,p}(\Omega)$. In order to achieve this result, it is necessary among others requirements that $\psi \in L^m(\partial\Omega)$ with $m \geq \frac{p(N-1)}{N(p-1)}$.

Afterwards, Boccardo and Mazón [42] extended the results proved in [4] in two different directions. On the one hand, supposing that $\psi \in L^m(\partial\Omega)$ with $m \geq \frac{p(N-1)}{N(p-1)}$, they improved the regularity of the solution $u \in W^{1,p}(\Omega)$ of (15), thanks to the regularizing effect of the lower order term $|u|^{s-1}u$ over the solutions. On the other hand, they also proved some results on the Calderón-Zigmund theory for "infinite energy solutions" ($m < \frac{p(N-1)}{N(p-1)}$). Precisely, if $q := \frac{Nm(p-1)}{N-1}$ and s is small [1] ($0 < s < \frac{(N+m-1)(p-1)}{N-1-m(p-1)}$), Boccardo and Mazón proved the existence of a distributional solution of (15) with $u \in W^{1,q}(\Omega)$ provided that

- $m \in \left(1, \frac{p(N-1)}{N(p-1)}\right)$, in the case $p > 2 - \frac{1}{N}$, while
- $m \in \left(\frac{N-1}{N(p-1)}, \frac{p(N-1)}{N(p-1)}\right)$, in the case $1 < p < 2 - \frac{1}{N}$.

We point out that in the case $p > 2 - \frac{1}{N}$, the exponent $q > \frac{N(p-1)}{N-1} > 1$ for every $m \in \left(1, \frac{p(N-1)}{N(p-1)}\right)$. However, in the other case $p < 2 - \frac{1}{N}$, if the parameter m tends to the value $\frac{N-1}{N(p-1)}$, then the exponent q converges to 1. This suggests that in the limit framework $m = \frac{N-1}{N(p-1)}$, we could expect solutions belonging to the space $W^{1,1}(\Omega)$. Indeed, we prove the following theorem.

Theorem 8 *Let $\Omega \subset \mathbb{R}^N$ be a smooth, open and bounded subset. Suppose that ψ belongs to $L^m(\Omega)$ with $m = \frac{N-1}{N(p-1)}$, $1 < p < 2 - \frac{1}{N}$ and $0 < s < \frac{1+N(p-1)}{N-1}$. Then there exists a solution $u \in W^{1,1}(\Omega)$ of the problem (15) in the sense*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\,\nabla v + \int_\Omega |u|^{s-1}\,uv = \int_{\partial\Omega} \psi v\,,\ \ \forall v \in C^1(\overline{\Omega})\,.$$

As before, the proof is performed by approximation using a family of easier problems. In order to prove the convergence of the sequence of solutions of these approximate problems, the study of *a priori* estimates is essential. It is convenient to underline that in this limit framework we find new difficulties due to the fact that we are working with a non reflexive space $W^{1,1}(\Omega)$. In order to overcome this, we use, among others, the Dunford-Pettis theorem. Moreover, the trace theorem plays an important role in our proof. Indeed, this allows us to connect the different parts which appear in the weak formulation of the problem, despite this requires the smoothness of the boundary of Ω.

[1]For "larger values of s", thanks again to the lower order term, the authors also improve their regularity result.

Chapter 1

Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity

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Abstract

For an open, bounded set $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 3$, $\gamma > 0$, and $1 < p < (N+2)/(N-2)$, we prove the existence of $\Lambda > 0$ such that the singular b.v.p.

$$\begin{cases} -\Delta u = \dfrac{\lambda}{u^\gamma} + u^p\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

has two positive solutions for every $\lambda \in (0, \Lambda)$. Given $f \in L^m(\Omega)$ with $m > 1$ and M a bounded elliptic matrix, we also study the Dirichlet b.v.p. associated to the related singular equation

$$-\mathrm{div}\,(M(x)\,\nabla z) = \frac{f(x)}{z^\gamma}\,, \quad x \in \Omega\,.$$

1.1 Introduction

In this paper we study the multiplicity of positive solutions of the singular semilinear problem

$$\begin{cases} -\Delta u = \dfrac{\lambda}{u^\gamma} + u^p\,, & \text{in } \Omega\,, \\ \quad u > 0\,, & \text{in } \Omega\,, \\ \quad u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{P_λ}$$

where Ω is an open, bounded set of $\mathbb{R}^N$, $N \geq 3$, and $\lambda, \gamma, p > 0$.

Since the right-hand side is singular, it is necessary to explain the meaning of "solution of the problem". For a solution of the differential equation in (P_λ) we consider $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ such that $\frac{\phi}{u^\gamma} \in L^1(\Omega)$ for every $\phi \in W^{1,2}_0(\omega)$ and satisfying

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega \frac{\phi}{u^\gamma} + \int_\Omega u^p \phi\,, \quad \forall \phi \in W^{1,2}_0(\omega)\,, \tag{1.1}$$

for every open subset ω of Ω, such that $\omega \subset\subset \Omega$. With respect to the boundary condition $u = 0$ on the boundary $\partial\Omega$ of Ω, in contrast with [75], where this condition is understood as $(u-\varepsilon)^+ \in W^{1,2}_0(\Omega)$ for every $\varepsilon > 0$, we follow [21] and we consider a stronger requirement based on the fact that a suitable positive power of u belongs to the space $W^{1,2}_0(\Omega)$. Indeed, the author shows in [21] the existence of a positive number Λ such that for all $\lambda \in (0, \Lambda)$ the problem (P_λ) has one positive solution $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ satisfying (1.1) with

- $u \in W^{1,2}_0(\Omega)$, if $0 < \gamma \leq 1$;
- $u^{\frac{\gamma+1}{2}} \in W^{1,2}_0(\Omega)$, if $\gamma > 1$.

Recently, in [8] it is proved, using variational methods, the existence of at least two different solutions of (P_λ) belonging to $W^{1,2}_0(\Omega)$ when the parameters γ and p satisfy $\gamma < 1$, and $2 < p+1 < 2^* := \frac{2N}{N-2}$. Our aim is to extend this previous result studying the multiplicity of solutions of the problem (P_λ) for every $\gamma > 0$. Specifically, we prove the following result:

Theorem 1.1 *Let Ω be an open, bounded set of $\mathbb{R}^N$ with C^2 boundary. If $\gamma > 0$ and $2 < p + 1 < 2^*$ holds, then there is $\Lambda > 0$ such that for every $\lambda \in (0, \Lambda)$ the problem (P_λ) has two different strictly positive solutions u and v in $W^{1,2}_{loc}(\Omega) \cap L^\infty(\Omega)$ satisfying* (1.1) *with*

$$u^\alpha, v^\alpha \in W^{1,2}_0(\Omega)\,, \quad \forall \alpha > \frac{\gamma+1}{4}\,.$$

We remark that in the case $1 < \gamma < 3$, we have $\frac{\gamma+1}{4} < 1$ and thus we obtain that they are in $W^{1,2}_0(\Omega)$ too, improving the result about the existence of solution in this space proved in [21] for the case $0 < \gamma \leq 1$.

Furthermore, since $\frac{\gamma+1}{4} < \frac{\gamma+1}{2}$, the theorem also improves the regularity proved in [8, 21] for the case $\gamma > 1$.

Taking into account the ideas of [21], the proof of Theorem 1.1 consists in finding two solutions u and v of (P_λ) as limit of two different sequences u_n and v_n of solutions of suitable approximate problems (P_λ^n) (see Section 2 below). One of the key points is to show for every fixed $\lambda > 0$ the existence of an uniform *a priori* estimate for the $L^\infty(\Omega)$-norm of all solutions of (P_λ^n). To prove this result we combine ideas of [66] and [72]. Indeed, by [66] there exists an open subset $\omega_0 \subset\subset \Omega$ and a positive constant M such that

$$u(x) \leq M \max_{\overline{\omega}_0} u, \quad \forall x \in \Omega \setminus \omega_0 \,,$$

for every positive solution u of (P_λ^n); while by [72], we also deduce the existence of an upper bound independent on n for $\max_{\overline{\omega}_0} u$. By this *a priori* estimate and the continuation theorem of Leray-Schauder we deduce for each fixed $n \in \mathbb{N}$, the existence of a (uniformly bounded in n) continuum S_n in $[0, +\infty) \times C(\overline{\Omega})$ of positive solutions of (P_λ^n) emanating from $(0, 0)$ and crossing $(0, v_0)$ where $v_0 > 0$ is a solution of (P_0). Moreover, using the result of [69, Theorem 2.2], we can guide the branch S_n below a suitable family of supersolutions $U(\lambda) = w_{n,\lambda^*}$ with λ in a bounded interval in $[0, \infty)$ containing $\lambda = 0$ and $w_{n,\lambda^*} > 0$. This allows us to obtain, for small positive λ, two different sequences $\{u_n\}$ and $\{v_n\}$ of different solutions u_n and v_n of problem (P_λ^n), being $u_n \leq U(\lambda)$ and $v_n \not\leq U(\lambda)$. Moreover, both sequences are uniformly bounded in $L^\infty(\Omega)$ and, in the spirit of [78], are greater than a positive multiple of a power of φ_1, where φ_1 is an eigenfunction associated to the first eigenvalue λ_1 of the Laplacian operator. To conclude, we pass to the limit and we obtain two distinct solutions u and v in the sense of the Theorem 1.1.

The tools developed in the proof of Theorem 1.1, also allow us to handle a related singular problem studied in [51], namely

$$\begin{cases} -\operatorname{div}(M(x)\nabla z) = \dfrac{f(x)}{z^\gamma}\,, & \text{in } \Omega\,, \\ z > 0\,, & \text{in } \Omega\,, \\ z = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

where $f \in L^m(\Omega)$ with $m > 1$ and M is a bounded elliptic matrix, i.e., there exists positive constants $0 < \mu \leq \nu$ such that

$$\mu\,|\xi|^2 \leq M(x)\xi\,\xi\,, \quad |M(x)| \leq \nu\,, \quad \forall \xi \in \mathbb{R}^N\,, \quad \text{a.e. } x \in \Omega\,. \tag{1.3}$$

Under more slightly restrictive assumptions (see Theorem 1.3 below), we improve the existence results obtained in that paper for the case $\gamma > 1$. Indeed, if $1 < \gamma < \frac{3m-1}{m+1}$, we prove the existence of a solution $z \in W^{1,2}_{\mathrm{loc}}(\Omega)$ of (1.2) such that $z^\alpha \in W^{1,2}_0(\Omega)$ for all $\alpha \in \left(\frac{(m+1)(\gamma+1)}{4m}, \frac{\gamma+1}{2}\right]$. (The case $\alpha = \frac{\gamma+1}{2}$ was proved in [51]). In particular, we obtain solutions belonging to $W^{1,2}_0(\Omega)$, and thus the meaning of the boundary condition is improved.

1.2 Approximate Problems

We consider for $n \in \mathbb{N}$ the approximate problems

$$\begin{cases} -\Delta u = \lambda f_n(u) + g(u)\,, & \text{in } \Omega\,, \\ u > 0\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{P_λ^n}$$

where f_n and g are the continuous functions given by

$$f_n(s) = \frac{1}{\left(s + \frac{1}{n}\right)^\gamma}\,, \quad g(s) = s^p\,, \quad \text{for } s \geq 0\,.$$

Notice that f_n and g satisfy the following properties:

$$g(0) = 0 \quad \text{and} \quad \lim_{s \to 0^+} \frac{g(s)}{s} = 0\,, \tag{1.4}$$

$$\lim_{s \to +\infty} \frac{g(s)}{s^p} = c_1 > 0\,, \tag{1.5}$$

$$\lim_{s \to 0^+} \frac{f_n(s)}{s} = +\infty\,, \tag{1.6}$$

$$\lim_{s \to +\infty} \frac{f_n(s)}{s^p} = 0\,, \quad \text{uniformly in } n\,, \tag{1.7}$$

$$\frac{\lambda\, f_n(s) + g(s)}{s^\sigma} \quad \text{is nonincreasing for } s \geq 0\,, \text{ with } \sigma = \frac{N+2}{N-2}\,. \tag{1.8}$$

Observe, in particular, that if $\lambda = 0$ and $n \in \mathbb{N}$, then the problem (P_0^n) is but the problem (P_0).

The fact that g satisfies (1.4) implies that there exists $\delta_0 > 0$ such that

$$||u||_{L^\infty(\Omega)} > \delta_0, \text{ for every solution } u \text{ of } (P_0). \tag{1.9}$$

Indeed, by (1.4), we can pick $\delta_0 > 0$ such that $g(s) < \lambda_1 s$ for all $s \in [0, \delta_0]$. Thus, if u is a solution of (P_0), taking φ_1 as test function we get

$$\int_\Omega (\lambda_1 u - g(u))\varphi_1 = 0$$

and, consequently, $||u||_{L^\infty(\Omega)} > \delta_0$.

We consider the solution (see [59]) $w_{n,\lambda}$ of the problem

$$\begin{cases} -\Delta w = \dfrac{\lambda}{(w + \frac{1}{n})^\gamma}\,, & \text{in } \Omega\,, \\ w > 0\,, & \text{in } \Omega\,, \\ w = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.10}$$

Observe that by [51, Lemma 2.2] the sequence $w_{n,\lambda}$ is increasing with respect to n, $w_{n,\lambda} > 0$ in Ω and for every $\omega \subset\subset \Omega$ there exists a constant $c_{\lambda,\omega}$, independent of n, such that

$$w_{n,\lambda}(x) \geq c_{\lambda,\omega} > 0 \quad \text{for every } x \in \omega\,, \ \text{for every } n \in \mathbb{N}\,. \tag{1.11}$$

In addition, $z = w_{n,\lambda}$ satisfies[1]

$$\int_\Omega \nabla z \nabla \phi \leq \int_\Omega \frac{\lambda}{z^\gamma}\, \phi\,, \ \ \forall \phi \in W_0^{1,2}(\Omega)\,, \phi > 0\,, \tag{1.12}$$

which implies an *a priori* estimate of $\|w_{n,\lambda}\|_{L^\infty(\Omega)}$. We include here the proof by convenience of the reader.

Lemma 1.1 ([51]) *For each $\gamma > 0$ there exists a positive constant $T > 0$ such that every $z \in W_0^{1,2}(\Omega)$, $z > 0$ satisfying* (1.12) *belongs to $L^\infty(\Omega)$ with $||z||_{L^\infty(\Omega)} \leq T\,\lambda^{\frac{1}{\gamma+1}}$ for all $\lambda > 0$.*

Proof of Lemma 1. The proof is divided in two steps. In the first one, we show the result for $\lambda = 1$, while the second step is devoted to the general case.
Step 1. The case $\lambda = 1$. Let w be a positive function in $W_0^{1,2}(\Omega)$ which satisfies (1.12) with $\lambda = 1$. For $k \geq 1$ we take $\phi = G_k(w)$ in (1.12) to get

$$\int_\Omega |\nabla G_k(w)|^2 \leq \int_\Omega \frac{G_k(w)}{w^\gamma}\,.$$

Since $w > k \geq 1$ on the set $\{w > k\}$ where $G_k(w) \neq 0$, we obtain that

$$\int_\Omega |\nabla G_k(w)|^2 = \int_{\{w>k\}} \frac{G_k(w)}{w^\gamma} \leq \int_{\{w>k\}} G_k(w)\,.$$

Starting from this last inequality and using the Stampacchia method [87, Lemma 4.1], there exists a positive constant $T > 0$ such that $||w||_{L^\infty(\Omega)} \leq T$.
Step 2. The general case. Now, we consider a positive number $\lambda > 0$ and we take $z \in W_0^{1,2}(\Omega)\,, z > 0$ satisfying (1.12). Clearly, if we consider the scaling $w := \left(\frac{1}{\lambda}\right)^{\frac{1}{\gamma+1}} z$, then $w \in W_0^{1,2}(\Omega)$, $w > 0$ and, using (1.12),

$$\int_\Omega \nabla w \nabla \phi = \int_\Omega \left(\frac{1}{\lambda}\right)^{\frac{1}{\gamma+1}} \nabla z \nabla \phi \leq \int_\Omega \left(\frac{1}{\lambda}\right)^{\frac{1}{\gamma+1}} \frac{\lambda}{z^\gamma}\, \phi = \int_\Omega \frac{1}{w^\gamma}\,,$$

for all $\phi \in W_0^{1,2}(\Omega)\,, \phi > 0$. By Step 1, $||w||_{L^\infty(\Omega)} \leq T$, i.e. $||z||_{L^\infty(\Omega)} \leq T\,\lambda^{\frac{1}{\gamma+1}}$. □

To conclude this section, we obtain in the following lemma an *a priori* uniform estimate for the solutions of (P_λ^n) which will allow us to pass to the limit to obtain solutions of (P_λ).

[1] We note that $\int_\Omega \frac{\lambda}{z^\gamma}\, \phi$ may be equal to $+\infty$ for some functions $\phi \in W_0^{1,2}(\Omega)$.

Lemma 1.2 *For every $\lambda > 0$, there exists $R > 0$ such that $||u||_{L^\infty(\Omega)} < R$ for any positive solution u of (P_λ^n).*

Proof. Fix $\lambda > 0$. It is enough to prove that there exist an open subset $\omega_0 \subset\subset \Omega$ and positive constants R_{ω_0} and M such that

$$u(x) \leq M \max_{\overline{\omega}_0} u \leq M\, R_{\omega_0}, \quad \forall x \in \Omega \setminus \omega_0\,,$$

for every positive solution u of (P_λ^n). We proceed in two steps.
Step 1: We prove that there exists a positive constant M and an open subset $\omega_0 \subset\subset \Omega$ such that

$$u(x) \leq M \max_{\overline{\omega}_0} u\,, \quad \forall x \in \Omega \setminus \omega_0\,,$$

for every positive solution u of (P_λ^n).

Since Ω satisfies the exterior sphere condition, given an arbitrary point x on $\partial\Omega$, there exists B a ball of radius $r = r(x)$ and center $a = a(x)$ whose closure intersects $\overline{\Omega}$ only at x. We consider $J_x : \overline{\Omega} \longrightarrow \mathbb{R}^N$ the inversion on the ∂B given by

$$J_x(y) = a + r^2 \frac{y-a}{|y-a|^2}\,.$$

We observe that J_x verifies the following properties:

1. $J_x(x) = x$
2. $J_x(\overline{\Omega}) \subset \overline{B}(a; r)$
3. $J_x^{-1} = J_x$

Let $n(x)$ be the exterior unit normal at x to $\partial\Omega$ and we denote by $T_x\, J_x(\overline{\Omega})$ the tangent hyperplane in x to $J_x(\overline{\Omega})$. This hyperplane divides the space $\mathbb{R}^N$ in two disjoint half-spaces and $J_x(\Omega)$ belongs to one of them. So, we can consider another hyperplane T parallel to $T_x\, J_x(\overline{\Omega})$ which cuts off $J_x(\overline{\Omega})$ with the following property: if we reflect the region of $J_x(\overline{\Omega})$ which is between both hyperplanes along T, then the reflection is inside $J_x(\Omega)$.

Following the notation of [71], we denote by Σ_x the region of $J_x(\Omega)$ that is between both hyperplanes, in other word, there exists an open cap Σ_x in the direction of $n(x)$ (see [66, pag 51-52]).

Let t_x and ε_x be two positive numbers such that if $y \in \partial J_x(\overline{\Omega}) \cap \overline{B}(x, \varepsilon_x)$ then, $y + s\, n(x) \in J_x(\Omega) \cap \Sigma_x$ for all $s \in (0, t_x]$. We define the sets

$$\widetilde{V}_x = \{y + s\, n(x) : y \in \partial J_x(\overline{\Omega}) \cap B(x, \varepsilon_x)\,, 0 \leq s < t_x\} \subset J_x(\overline{\Omega})\,,$$

$$\widetilde{W}_x = \{y + t_x\, n(x) : y \in \partial J_x(\overline{\Omega}) \cap \overline{B}(x, \varepsilon_x)\} \subset J_x(\Omega) \cap \Sigma_x\,.$$

Since $\widetilde{V}_x$ is open and $\widetilde{W}_x$ is compact, the sets $V_x = J_x^{-1}(\widetilde{V}_x)$ is an open of $\overline{\Omega}$ and $W_x = J_x^{-1}(\widetilde{W}_x)$ is a compact subset in Ω. Consequently $\delta_x = \text{dist}\,(W_x, \partial\Omega) > 0$.

As $\partial\Omega$ is a compact subset, from the covering $\cup_{x\in\partial\Omega}V_x$, we can extract a finite one, that is to say, there exists $x_1, x_2, ..., x_n \in \partial\Omega$ such that $\partial\Omega \subset \cup_{i=1}^n V_i$, where we denote by V_i the set V_{x_i} with $\widetilde{V_i} = \widetilde{V}_{x_i}$, $t_i = t_{x_i}$, $\varepsilon_i = \varepsilon_{x_i}$, $J_i = J_{x_i}$ and $\Sigma_i = \Sigma_{x_i}$. We also denote by $\widetilde{W_i} = \widetilde{W}_{x_i}$, $W_i = W_{x_i}$ and $\delta_i = \delta_{x_i}$. Since V_i is an open subset of $\overline{\Omega}$ that contained $\partial\Omega \cap J_i^{-1}(B(x_i, \varepsilon_i))$, the set $V := \cup_i^n V_i$ is an open neighbourhood of $\partial\Omega$ in $\overline{\Omega}$. Therefore, $\text{dist}(\partial\Omega, \overline{\Omega}\setminus V) = d > 0$. Otherwise, there would exist $\{w_n\} \subset \partial\Omega$ and $\{z_n\} \subset \overline{\Omega} \setminus V$ such that $\text{d}(w_n, z_n) \to 0$. We can suppose $w_n \to w \in \partial\Omega$. Then $z_n \to w \in \partial\Omega$ contradicting that V is a neighborhood of $\partial\Omega$.

Now, we consider $k = \min\{d, \delta_1, ..., \delta_n\} > 0$ and we define the following open set $\omega_0 := \{y \in \overline{\Omega} : \text{d}(y, \partial\Omega) > \frac{k}{2}\}$. Note that $\Omega \setminus \omega_0 \subset V$ and $W_i \subset \omega_0$ for all $i = 1, ..., n$.

Let u be a positive solution of (P_λ^n) and we fix $x \in \Omega \setminus \omega_0 \subset V$. There exists $i \in \{1, ..., n\}$ such that $x \in V_i = J_i^{-1}(\widetilde{V_i})$, that is to say, there exists $s \in [0, t_i)$ and $\xi_i \in \partial J_i(\overline{\Omega}) \cap B(x_i, \varepsilon_i)$ such that $x = J_i^{-1}(\xi_i + s\, n(x_i))$. Next, we consider the Kelvin's transformation (with center a_i and radius r_i) of the function u; i.e.,

$$w(\xi) = \left(\frac{r_i}{|\xi - a_i|}\right)^{N-2} u\left(J_i(\xi)\right), \quad \text{for all } \xi \in J_i(\overline{\Omega}).$$

A computation shows that $w \in C^2(J_i(\Omega))$ satisfies

$$\begin{cases} -\Delta w(\xi) = \left(\dfrac{r_i}{|\xi - a_i|}\right)^{N+2} h_n\left(\left(\dfrac{|\xi - a_i|}{r_i}\right)^{N-2} w(\xi)\right), & \text{in } J_i(\Omega), \\ w > 0, & \text{in } J_i(\Omega), \\ w = 0, & \text{on } \partial J_i(\Omega), \end{cases}$$

where $h_n(s) = \lambda f_n(s) + g(s)$. We remark that, by (1.8), h_n is a locally Lipschitz function verifying

$$\frac{h_n(s)}{s^\sigma} \text{ is nonincreasing for } s \geq 0, \quad \text{with } \sigma = \frac{N+2}{N-2}.$$

Then the function

$$\widetilde{h}_n(\xi, t) := \left(\frac{r_i}{|\xi - a_i|}\right)^{N+2} h_n\left(\left(\frac{|\xi - a_i|}{r_i}\right)^{N-2} t\right)$$

is nonincreasing for any $t \geq 0$. Hence all the conditions of [71, Corollary 1] hold true and we deduce that $w(\xi_i + s\, n(x_i)) \leq w(\xi_i + t_i\, n(x_i))$ and thus

$$u(x) \leq \left(\frac{|\xi_i + s\, n(x_i) - a_i|}{r_i}\right)^{N-2} w(\xi_i + t_i\, n(x_i)).$$

As a consequence of this inequality, if we define

$$z_x := J_i(\xi_i + t_i\, n(x_i)) \in W_i \quad \text{and} \quad m_i := \frac{\max\{|\xi - a_i|^{N-2} \,:\, \xi \in \overline{\Sigma}_i \cap J_i(\overline{\Omega})\}}{\min\{|\xi - a_i|^{N-2} \,:\, \xi \in \overline{\Sigma}_i \cap J_i(\overline{\Omega})\}},$$

we get,

$$u(x) \leq m_i \left(\frac{|\xi_i + t_i\, n(x_i) - a_i|}{r_i} \right)^{N-2} w(\xi_i + t_i\, n(x_i)) = m_i\, u(z_x)$$

for every $x \in \Omega \setminus \omega_0$ with $z_x \in W_i \subset \omega_0$. Therefore, if $M = \max\{m_i \,:\, i = 1, ..., n\}$,

$$u(x) \leq M \max_{\overline{\omega}_0} u\,,$$

where $x \in \Omega \setminus \omega_0$ was arbitrary and the proof of Step 1 is finished.

Step 2. Given an arbitrary open subset ω of Ω such that $\omega \subset\subset \Omega$, there exists $R_\omega > 0$ such that $||u||_{L^\infty(\Omega)} < R_\omega$ for every positive solution u of (P_λ^n).

To prove it, following the ideas of [73, Theorem 1.1], we argue by contradiction assuming that there exist a sequence $\{u_n\}$ of positive solutions of (P_λ^n) and a sequence of points $P_n \in \Omega$ such that

$$M_n = u_n(P_n) = \max\{u_n(x) \,:\, x \in \overline{\omega}\} \to +\infty \quad \text{as} \quad n \to +\infty\,.$$

Passing to a subsequence, if it is necessary, we may assume $P_n \to P \in \overline{\omega}$ as $n \to +\infty$. Let $2d$ be the distance of ω to $\partial\Omega$, $B_{\tilde{R}}(a)$ the ball of radius $\tilde{R}$ and center $a \in \mathbb{R}^N$ and let ω_d be the following set: $\omega_d = \{x \in \Omega \,:\, \mathrm{dist}(x, \omega) \leq d\}$. Let $\{\mu_n\}$ be a sequence of positive numbers such that $\mu_n^{\frac{2}{p-1}} M_n = 1$. Since $M_n \to +\infty$, we have $\mu_n \to 0$ as $n \to +\infty$. Fixing $\tilde{R} > 0$, we choose n_0 such that $B_{\tilde{R}}(0) \subset B_{\frac{d}{\mu_n}}(0)$. We can define the scaled function

$$v_n(y) = \mu_n^{\frac{2}{p-1}} u_n(P_n + \mu_n y)\,, \quad \forall y \in B_{\frac{d}{\mu_n}}(0)\,,$$

which satisfies

$$\sup\{v_n(y) \,:\, y \in B_{\frac{d}{\mu_n}}(0)\} = v_n(0) = 1 \tag{1.13}$$

and

$$\begin{cases} -\Delta v_n(y) = \mu_n^{\frac{2p}{p-1}} \left(\lambda f_n(u_n(P_n + \mu_n y)) + (\mu_n^{\frac{-2}{p-1}} v_n(y))^p \right), \; y \in B_{\tilde{R}}(0), \\ v_n(0) = 1. \end{cases} \tag{1.14}$$

Observing that every solution u of (P_λ^n) is a supersolution of (1.10), by [59], we have $u \geq w_{n,\lambda}$, and hence

$$\lambda f_n(u_n(P_n + \mu_n y)) \leq \frac{\lambda}{(w_\lambda(P_n + \mu_n y))^\gamma}.$$

Since the points $P_n + \mu_n y \in \overline{\omega}_d \subset \Omega$ for all $y \in B_{\tilde{R}}(0)$, by (1.11), there exists $C(\omega) > 0$ such that

$$\mu_n^{\frac{2p}{p-1}} \left(\lambda f_n(u_n(P_n + \mu_n y)) + (\mu_n^{\frac{-2}{p-1}} v_n(y))^p \right) \leq C(\omega)\;, \quad y \in B_{\tilde{R}}(0), \quad \forall n \geq n_0\,.$$

We deduce by elliptic estimates [74, Theorem 9.11] that v_n is bounded in $W^{2,q}(B_{\tilde{R}}(0))$ for every $q > 1$. The compact embedding of $W^{2,q}(B_{\tilde{R}}(0))$ into $C^{1,\beta}(B_{\tilde{R}}(0))$ ($\beta \in (0,1)$) for $q > N$ implies that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ which converges in $C^{1,\beta}(B_{\tilde{R}}(0))$ to some v. The arbitrariness of $\tilde{R}$ implies that $v \in C^{1,\beta}(\mathbb{R}^N)$ and (1.13), that $v(0) = 1$ and v is not zero. By (1.7), we can pass to the limit in (1.14) to deduce that v is a nontrivial solution of

$$-\Delta v = v^p\,, \quad \text{in } \mathbb{R}^N\,,$$

with $1 < p < \frac{N+2}{N-2}$. By [72, Theorem 1.1] this a contradiction proving the Step 2 and thus the lemma. □

1.3 Proof of Theorem 1.1.

Let δ_0 and T be given by (1.9) and Lemma 1.1. For $\delta_1 := (2p-1)^{\frac{1}{1-p}}\, T^{\frac{\gamma+1}{1-p}}$, we take $\delta_2 \in (0, \min\{\delta_0\,, \delta_1\})$ and define:

$$\Lambda := \max_{0 \le s \le \delta_2} q(s)\,, \quad \text{where } q(s) = \frac{1}{2}\left(\left(\frac{s}{T}\right)^{\gamma+1} - s^{\gamma+p}\right).$$

Fix $\lambda_0 \in (0,\Lambda)$. Notice that the function q is strictly positive in $(0,\delta_2]$ and that, by the intermediate value theorem, there exists $\delta \in (0,\delta_2]$ such that

$$\lambda_0 = q(\delta)\,. \tag{1.15}$$

Moreover, since $\delta < \delta_1$, we remark also that

$$\frac{p-1}{\gamma+1}\,\delta^{p+\gamma} < (p-1)\delta^{p+\gamma} < \lambda_0\,. \tag{1.16}$$

The proof of the Theorem 1.1 is divided in several steps:

Step 1. Construction of a family of super-solutions of (P^n_λ). Let T be given by the Lemma 1.1 and take $\lambda^* := (\delta/T)^{\gamma+1}$. Clearly,

$$\lambda^* > \lambda_0 + \left(T(\lambda^*)^{\frac{1}{\gamma+1}}\right)^p \left(T(\lambda^*)^{\frac{1}{\gamma+1}}\right)^\gamma$$

and we can choose $n_0 \in \mathbb{N}$ such that

$$\lambda^* \ge \lambda + \left(T(\lambda^*)^{\frac{1}{\gamma+1}}\right)^p \left(T(\lambda^*)^{\frac{1}{\gamma+1}} + \frac{1}{n}\right)^\gamma\,, \;\; \forall\, n \ge n_0\,, \;\; \forall \lambda \in [0,\lambda_0]\,.$$

By Lemma 1.1, the solution $w_{n,\lambda^*} \in C^2_0(\overline{\Omega})$ of the problem (1.10) with $\lambda = \lambda^*$ satisfies $||w_{n,\lambda^*}||_{L^\infty(\Omega)} \le T\,(\lambda^*)^{\frac{1}{\gamma+1}} = \delta$. Therefore,

$$\lambda^* \ge \lambda + ||w_{n,\lambda^*}||^p_{L^\infty(\Omega)} \left(||w_{n,\lambda^*}||_{L^\infty(\Omega)} + \frac{1}{n}\right)^\gamma \ge \lambda + (w_{n,\lambda^*})^p \left(w_{n,\lambda^*} + \frac{1}{n}\right)^\gamma\,,$$

from which

$$-\Delta w_{n,\lambda^*} = \frac{\lambda^*}{(w_{n,\lambda^*}+\frac{1}{n})^\gamma} \geq \frac{\lambda}{(w_{n,\lambda^*}+\frac{1}{n})^\gamma} + (w_{n,\lambda^*})^p\,,\ \forall\ n \geq n_0\,, \forall \lambda \in [0,\lambda_0]\,,$$

i.e., $w_{n,\lambda^*} \in C_0^2(\overline{\Omega})$ is a super-solution of the problem (P_λ^n) for all $n \geq n_0$ and all $\lambda \in [0,\lambda_0]$. Moreover, we remark that $||w_{n,\lambda^*}||_{L^\infty(\Omega)} \leq \delta$.

Step 2. Uniqueness of solution of the problem (P_λ^n) with small norm. Due to the convexity of the function $(p-1)\,s^p\,(s+\frac{1}{n})^{\gamma+1}$, there exists a unique positive number $M_n = M_n(\lambda) > 0$, increasing with respect to the parameter λ, such that

$$(p-1)\,M_n^p\left(M_n+\frac{1}{n}\right)^{\gamma+1} = \lambda\left(M_n\,(\gamma+1)+\frac{1}{n}\right).$$

Furthermore,

$$(p-1)\,s^p\left(s+\frac{1}{n}\right)^{\gamma+1} \leq \lambda\left(s\,(\gamma+1)+\frac{1}{n}\right)\,,\ \ \forall s \in [0,M_n]\,.$$

As a consequence, the function $\dfrac{\lambda f_n(s)+g(s)}{s}$ is decreasing in $[0,M_n]$ and thus, due to [59, Theorem 1], there exists at most one solution u_n of the problem (P_λ^n) with $||u_n||_{L^\infty(\Omega)} \leq M_n$.

In addition, by (1.16), there exists a positive number ϵ such that

$$\frac{(p-1)}{\gamma+1}(\delta+\epsilon)^{p+\gamma} < (p-1)(\delta+\epsilon)^{p+\gamma} < \lambda_0\,.$$

Notice that there exists $n_1 \in \mathbb{N}$ such that

$$\lambda_n := \frac{(p-1)\,(\delta+\epsilon)^p\,\left((\delta+\epsilon)+\frac{1}{n}\right)^{\gamma+1}}{(\delta+\epsilon)\,(\gamma+1)+\frac{1}{n}} < \lambda_0 \ \text{ for all } n \geq n_1\,,$$

and that $M_n(\lambda_n) = \delta+\epsilon$. Since M_n is increasing with respect to λ, we obtain

$$M_n(\lambda_0) \geq \delta+\epsilon\,,\ \ \forall n \geq n_1\,. \tag{1.17}$$

Step 3. There exists $\overline{\Lambda} > 0$, independent of $n \in \mathbb{N}$, such that (P_λ^n) has no solutions if $\lambda \geq \overline{\Lambda}$. Indeed, by (1.4) and (1.6), $\lim_{s\to 0^+}\frac{\lambda_1 s-g(s)}{f_1(s)} = 0$, while, by (1.5) and (1.7), $\lim_{s\to+\infty}\frac{\lambda_1 s-g(s)}{f_1(s)} = -\infty$. Hence $\overline{\Lambda} := \max\limits_{s>0}\dfrac{\lambda_1 s-g(s)}{f_1(s)} > 0$. If there exists a positive function $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \nabla\phi = \int_\Omega \big(\lambda f_n(u)+g(u)\big)\phi, \quad \text{for all } \phi \in W_0^{1,2}(\Omega)\,,$$

taking $\phi = \varphi_1$ as test function, we deduce that

$$\int_\Omega \big(\lambda_1 u - \lambda f_n(u) - g(u)\big)\varphi_1 = 0\,. \tag{1.18}$$

Since $f_1(s) \le f_n(s)$ for all $s > 0$, from this last inequality we conclude

$$\lambda < \max_{s>0} \frac{\lambda_1 s - g(s)}{f_n(s)} \le \overline{\Lambda}\,,$$

and the step is proved.

Step 4. There exists two different sequences $\{u_n\}$ and $\{v_n\}$ of solutions u_n and v_n of the problem $(P^n_{\lambda_0})$. We fix $n \ge \max\{n_0\,, n_1\}$, where n_0 and n_1 are given by Step 1 and Step 2 respectively. Define the operator $K_\lambda : C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$ by

$$K_\lambda(u) = \big(-\Delta\big)^{-1}\big(\lambda f_n(u) + g(u)\big)\,, \;\; u \in C(\overline{\Omega}),$$

where we recall that the inverse $(\Delta)^{-1}$ of the Laplacian operator $(-\Delta)$ is compact from $C(\overline{\Omega})$ into $C(\overline{\Omega})$. Thus, K_λ is also a compact operator for every λ. Observe that we can rewrite the problem (P^n_λ) as the fixed point equation $u = K_\lambda(u)$.

Notice that hypotheses (1.5) and (1.7) imply that there exists R_n (depending on n) such that every solution u of (P^n_λ) satisfies $||u||_{L^\infty(\Omega)} < R_n$ ([73, Theorem 1.1]). Thus, we can consider the Leray-Schauder topological degree of $I - K_0$, that is to say, $d\,(I - K_0, B_{R_n}, 0)$. In addition, taking into account that the number δ given by (1.15) is smaller than δ_0, by (1.9), the problem (P_0) has not solution on the boundary of the ball B_δ. As a consequence, we can also consider the Leray-Schauder topological degree $d\,(I - K_0, B_\delta, 0)$. By [66, Proposition 2.1], we have $d\,(I - K_0, B_{R_n}, 0) = 0$ and $d\,(I - K_0, B_\delta, 0) = 1$. Therefore, recalling Step 3, we can apply [2, Theorem 4.4.2] to conclude the existence of a continuum (connected and closed)

$$S_n \subset \Sigma_n = \{(\lambda, u_n) \in [0, +\infty) \times C(\overline{\Omega}) : u_n \text{ is solution of } (P^n_\lambda)\}$$

such that

$$(0,0) \in S_n \;\text{ and }\; S_n \cap (\{0\} \times (C(\overline{\Omega}) \setminus B_\delta)) \ne \varnothing\,, \quad \forall n \in \mathbb{N}. \tag{1.19}$$

In addition, if we define the continuous map $U : [0, \lambda_0] \longrightarrow C^2_0(\overline{\Omega})$ by $U(\lambda) = w_{n,\lambda^*}$, for every $\lambda \in [0, \lambda_0]$, then, by Step 1, $U(\lambda)$ is a positive super-solution and not a solution of the problem (P^n_λ) for all $\lambda \in [0, \lambda_0]$. As a consequence, since Ω satisfies the interior sphere condition, we can apply [69, Theorem 2.2] to deduce that every pair (λ, u_n) belonging to the connected component of $S_n \cap \big([0, \lambda_0] \times C(\overline{\Omega})\big)$ which contains the point $(0, 0)$ satisfies $u_n \le U(\lambda)$ in Ω. Roughly speaking, this means that the branch S_n of solutions of (P^n_λ), which emanates from $(0, 0)$, lies pointwise below the branch $\{\big(\lambda, \overline{U}(\lambda)\big)\;/\,0 \le \lambda \le \lambda_0\}$ at least until it crosses $\lambda = \lambda_0$. In particular, there exists u_n in the slice $S^{\lambda_0}_n = \{u \in C(\overline{\Omega}) \,:\, (\lambda_0, u) \in S_n\}$ which satisfies that $0 < u_n < w_{n,\lambda^*}$. Recalling that $||w_{n,\lambda^*}||_{L^\infty(\Omega)} \le \delta$, we have $||u_n||_{L^\infty(\Omega)} \le ||w_{n,\lambda^*}||_{L^\infty(\Omega)} \le \delta$. Clearly, by Step 2 and (1.17), u_n is the unique solution of $(P^n_{\lambda_0})$ with norm less or equal than $\delta+\epsilon$. By (1.19), $S_n\cap(\{0\}\times(C(\overline{\Omega})\setminus B_{\delta+\varepsilon}))\neq\varnothing$ and thus we conclude also that there exists v_n in $S_n^{\lambda_0}$ with $||v_n||_{L^\infty(\Omega)}\geq\delta+\epsilon$ and Step 4 is proved.

Step 5. Passing to the limit. Due to the arbitrariness of $\lambda_0\in(0,\Lambda)$, to conclude the proof of the theorem it suffices to see that the solutions u_n and v_n of the problem $(P^n_{\lambda_0})$ are converging toward two different solutions of problem (P_{λ_0}). To show this, we easily check that there exists a positive constant C (depending only on λ_0) such that the function

$$\underline{u}=\left(C\,\varphi_1+\frac{1}{n^{(\gamma+1)/2}}\right)^{2/(\gamma+1)}-\frac{1}{n}$$

is a subsolution of (1.10) with $\lambda=\lambda_0$. Moreover, since $\dfrac{\lambda}{(s+\frac{1}{n})^\gamma}\leq\dfrac{\lambda}{(s+\frac{1}{n})^\gamma}+s^p$ for any $s\geq 0$, each solution u of $(P^n_{\lambda_0})$ is a supersolution of (1.10) with $\lambda=\lambda_0$. Hence, using the comparison principle proved in [59], $\underline{u}\leq w_{n,\lambda_0}\leq u$. In particular, the sequences u_n and v_n obtained in the Step 4 satisfy:

$$\left(C\,\varphi_1+\frac{1}{n^{(\gamma+1)/2}}\right)^{2/(\gamma+1)}-\frac{1}{n}\leq w_{n,\lambda_0}\leq u_n\leq\delta\,, \tag{1.20}$$

and

$$\left(C\,\varphi_1+\frac{1}{n^{(\gamma+1)/2}}\right)^{2/(\gamma+1)}-\frac{1}{n}\leq w_{n,\lambda_0}\leq v_n\,,\ ||v_n||_{L^\infty(\Omega)}\geq\delta+\varepsilon>\delta\,.$$

By (1.20), the sequence u_n is uniformly bounded. By Lemma 1.2 we also obtain an uniform bound for the sequence $\{v_n\}$. In order to simplify the notation, we will denote by z_n either u_n or v_n. In any case, $z_n\in W^{1,2}_0(\Omega)$ solves $(P^n_{\lambda_0})$ and

$$\left(C\,\varphi_1+\frac{1}{n^{(\gamma+1)/2}}\right)^{2/(\gamma+1)}-\frac{1}{n}\leq z_n\leq\widetilde{C}\,, \tag{1.21}$$

where $\widetilde{C}$ is a positive constant. In addition, by (1.11), for every $\omega\subset\subset\Omega$ there exists c_ω (recall that λ_0 is fixed) such that

$$z_n(x)\geq c_\omega>0\quad\text{for every } x\in\omega\,,\ \text{for every } n\in\mathbb{N}\,. \tag{1.22}$$

Firstly, we prove that z_n is a bounded sequence in $W^{1,2}_{\text{loc}}(\Omega)$. Let ϕ be a function in $C^1_{\text{c}}(\Omega)$ and take $z_n\phi^2$ as a test function in (P^n_λ) to obtain that

$$\begin{aligned}\int_\Omega|\nabla z_n|^2\,\phi^2&=-2\int_\Omega\nabla z_n\,\nabla\phi\,z_n\,\phi+\int_\Omega\frac{\lambda_0\,z_n\,\phi^2}{(z_n+\frac{1}{n})^\gamma}+\int_\Omega z_n^{p+1}\,\phi^2\\&\leq-2\int_\Omega\nabla z_n\,\nabla\phi\,z_n\,\phi+\int_\Omega\frac{\lambda_0}{z_n^{\gamma-1}}\phi^2+\int_\Omega z_n^{p+1}\phi^2.\end{aligned}$$

By Young inequality,

$$2\left|\int_\Omega \nabla z_n \nabla\phi\, z_n\phi\right| \le \frac{1}{2}\int_\Omega |\nabla z_n|^2\phi^2 + 2\int_\Omega |\nabla\phi|^2 z_n^2\,,$$

we deduce then that

$$\frac{1}{2}\int_\Omega |\nabla z_n|^2\phi^2 \le 2\int_\Omega |\nabla\phi|^2 z_n^2 + \int_\Omega \frac{\lambda_0}{z_n^{\gamma-1}}\phi^2 + \int_\Omega z_n^{p+1}\phi^2\,.$$

Using (1.21) and (1.22) (in the case $\gamma > 1$), we get

$$\int_\Omega |\nabla z_n|^2\phi^2 \;\le\; C_\phi\,, \quad \forall n\in\mathbb{N}\,,$$

where C_ϕ is a positive constant depending on the function ϕ. In consequence, z_n is bounded in $W^{1,2}_{\rm loc}(\Omega)$. Thus, there exists $z \in W^{1,2}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$ such that, up to a subsequence, z_n converges a.e. in Ω and weakly in $W^{1,2}(\omega)$ to z for every $\omega \subset\subset \Omega$. Thanks again to (1.21) and (1.22), we can apply the Lebesgue theorem to deduce that

$$\lim_{n\to\infty}\left(\int_\Omega \frac{\lambda_0\,\phi}{(z_n+\frac{1}{n})^\gamma} + \int_\Omega \phi\, z_n^p\right) \;=\; \lambda_0\int_\Omega \frac{\phi}{z^\gamma} + \int_\Omega \phi\, z^p\,,$$

and then z satisfies (1.1). Now, we fix $\alpha > \frac{\gamma+1}{4}$ and for $\theta := 2\alpha - 1 > \frac{\gamma-1}{2}$ we take $\phi = (z_n+\frac{1}{n})^\theta - \left(\frac{1}{n}\right)^\theta$ as a test function in $(P^n_{\lambda_0})$ to obtain

$$\begin{aligned}
\frac{4\theta}{(\theta+1)^2}\int_\Omega \left|\nabla\left(\left(z_n+\frac{1}{n}\right)^\alpha - \left(\frac{1}{n}\right)^\alpha\right)\right|^2 &= \theta\int_\Omega \left(z_n+\frac{1}{n}\right)^{\theta-1}|\nabla z_n|^2\\
&= \lambda_0\int_\Omega \frac{(z_n+\frac{1}{n})^\theta - \left(\frac{1}{n}\right)^\theta}{\left(z_n+\frac{1}{n}\right)^\gamma} + \int_\Omega \left(\left(z_n+\frac{1}{n}\right)^\theta - \left(\frac{1}{n}\right)^\theta\right) z_n^p\\
&\le \lambda_0\int_\Omega \left(z_n+\frac{1}{n}\right)^{\theta-\gamma} + \int_\Omega \left(z_n+\frac{1}{n}\right)^\theta z_n^p\\
&\le \lambda_0\int_\Omega \left(z_n+\frac{1}{n}\right)^{\theta-\gamma} + \int_\Omega (z_n+1)^\theta\, z_n^p\,. \qquad (1.23)
\end{aligned}$$

By (1.21), this means that $\{(z_n+\frac{1}{n})^\alpha - \left(\frac{1}{n}\right)^\alpha\}$ is bounded in $W^{1,2}_0(\Omega)$. Indeed, this is clearly deduced in the case $\theta \ge \gamma$, while if $\theta < \gamma$, then it suffices to observe that

$$\left(z_n+\frac{1}{n}\right)^{\theta-\gamma} \le \left(C\,\varphi_1 + \frac{1}{n^{(\gamma+1)/2}}\right)^{\frac{2(\theta-\gamma)}{1+\gamma}} \le (C\,\varphi_1)^{\frac{2(\theta-\gamma)}{1+\gamma}}\,,$$

with the function $\varphi_1^{\frac{2(\theta-\gamma)}{1+\gamma}} \in L^1(\Omega)$ (since $\theta > \frac{\gamma-1}{2}$). Consequently, a subsequence of $\{(z_n+\frac{1}{n})^\alpha - \left(\frac{1}{n}\right)^\alpha\}$ is weakly convergent in $W^{1,2}_0(\Omega)$. Since z_n converges to z a.e.

in Ω, the weak limit of the above subsequence is z^α and therefore $z^\alpha \in W_0^{1,2}(\Omega)$. In conclusion, we have shown that both u and v are solutions of (P_λ).

Finally, recalling that $||u_n||_{L^\infty(\Omega)} \leq \delta$ and $||v_n||_{L^\infty(\Omega)} \geq \delta + \epsilon > \delta$ and the almost everywhere convergence of u_n and v_n to u and v, we deduce that u and v are two different solutions of (P_λ). □

Remark 1.1 *The arguments used in [51, Theorem 5.6] imply a regularity result. Observe that when $\gamma < 3$, Theorem 1.1 shows that the solutions are in $W_0^{1,2}(\Omega)$ and, in particular, belong to $W_0^{1,q}(\Omega)$ for every $q < 2$. We prove that in the other case $\gamma \geq 3$, both solutions of (P_λ) continue belonging to $W_0^{1,q}(\Omega)$ for some $q < 2$. Indeed, using the notation of the previous proof, for every $\theta > \frac{\gamma-1}{2}$, we have proved in* (1.23) *that the sequence $\left\{ \int_\Omega \left(z_n + \frac{1}{n} \right)^{\theta-1} |\nabla z_n|^2 \right\}$ is bounded with respect to n. Thus, by Hölder inequality with exponent $2/q > 1$, we have*

$$\int_\Omega |\nabla z_n|^q \leq M \left(\int_\Omega \left(z_n + \frac{1}{n} \right)^{\frac{q(1-\theta)}{2-q}} \right)^{(2-q)/2}, \tag{1.24}$$

where M is a positive constant. Since $\theta > 1$, we take $q < 2\left(\frac{2(\theta-1)}{\gamma+1} + 1\right)^{-1} < 2$ and, by (1.21), *we conclude that $\left(z_n + \frac{1}{n}\right)^{\frac{q(1-\theta)}{2-q}} \leq \left(C\,\varphi_1^{2/(\gamma+1)}\right)^{\frac{q(1-\theta)}{2-q}} \in L^1(\Omega)$ which, by* (1.24), *completes the proof of the boundedness of z_n in $W_0^{1,q}(\Omega)$ and consequently that $z \in W_0^{1,q}(\Omega)$.*

Remark 1.2 *In order to show a relation with [75], we observe that if $\alpha \geq 1$ and u^α is a positive function belonging to $W_0^{1,2}(\Omega)$, then $(u-\epsilon)^+$ belongs to $W_0^{1,2}(\Omega)$ for any $\epsilon > 0$. Indeed, given $\alpha \geq 1$ and $\epsilon > 0$, we deduce easily that if $u^\alpha \in W^{1,2}(\Omega)$, then $(u-\epsilon)^+ \in W^{1,2}(\Omega)$. In addition, we can choose a positive constant k such that $(u-\epsilon)^+ \leq k\,u^\alpha$ and thus $(u-\epsilon)^+ \in W_0^{1,2}(\Omega)$.*

Remark 1.3 *Castro and Pardo [62] have very recently improved the results of [66] dropping a technical condition. The new arguments used in [62] allow us to prove the Step 2 of Lemma 1.2 under assumptions more general than* (1.5) *and* (1.7), *namely, the hypothesis*

$$\lim_{s\to+\infty} \frac{g(s)}{s} > \lambda_1$$

and the following uniform condition

$$\forall s_0 > 0\,,\, \exists M = M(s_0) \,:\, \lambda f_n(s) + g(s) \leq M(s^\sigma + 1)\,,\ \forall s \geq s_0\,, n \in \mathbb{N}\,.$$

1.4 A related problem

We apply the tools used in Step 5 of the proof of Theorem 1.1 to obtain an existence result for the problem (1.2) with $f \in L^m(\Omega)$ for $m > 1$ and M a bounded elliptic matrix, i.e., satisfying (1.3). Following [51], a solution of the problem (1.2) is a function $z \in W^{1,2}_{\mathrm{loc}}(\Omega)$ such that $\inf_\omega z > 0$ for every $\omega \subset\subset \Omega$ which satisfies that

$$\int_\Omega M(x)\nabla z\,\nabla\varphi = \int_\Omega \frac{f\varphi}{z^\gamma}\,,\quad \forall\varphi \in C^1_{\mathrm{c}}(\Omega)\,, \tag{1.25}$$

and for which there exists a positive number $\alpha > 0$ such that $z^\alpha \in W^{1,2}_0(\Omega)$. In particular, the authors proved the following theorem.

Theorem 1.2 ([51]) *Assume that $f \in L^m(\Omega)$, with $m \geq 1$. The following assertions hold:*

1. *if $\gamma < 1$ and $m \geq \left(\frac{2^*}{1-\gamma}\right)'$, then there exists a solution z in $W^{1,2}_0(\Omega)$ of (1.2);*
2. *if $\gamma = 1$ and $m = 1$, then there exists a solution z in $W^{1,2}_0(\Omega)$ of (1.2);*
3. *if $\gamma > 1$ and $m = 1$, then there exists $z \in W^{1,2}_{loc}(\Omega)$ such that $z^{\frac{\gamma+1}{2}} \in W^{1,2}_0(\Omega)$ is a solution of (1.2).* □

The following result improves item 3. of Theorem 1.2 when $m > 1$ and $1 < \gamma < \frac{3m-1}{m+1}$.

Theorem 1.3 *Assume that the bounded, open subset Ω in $\mathbb{R}^N$ satisfies the interior sphere condition, that $f \in L^m(\Omega)$ with $m > 1$ and that there exists a positive constant f_0 such that $f(x) \geq f_0 > 0$ a.e. $x \in \Omega$.*

If $1 < \gamma < \frac{3m-1}{m+1}$, then there exists a function $z \in W^{1,2}_{loc}(\Omega)$ satisfying (1.25) such that $z^\alpha \in W^{1,2}_0(\Omega)$ for all $\alpha \in \left(\frac{(m+1)(\gamma+1)}{4m}, \frac{\gamma+1}{2}\right]$.

Remark 1.4 *Observe that when $1 < \gamma < \frac{3m-1}{m+1}$, we have $\frac{(m+1)(\gamma+1)}{4m} < 1 < \frac{\gamma+1}{2}$ and $\alpha = 1$ can be chosen in the previous theorem to deduce that $z \in W^{1,2}_0(\Omega)$. Consequently, we should highlight that the hypotheses about the boundary of Ω and $f(x) \geq f_0 > 0$ a.e. in Ω allow us to obtain a solution z of the problem (1.2) belonging to $W^{1,2}_0(\Omega)$ for all $m > 1$ and $1 < \gamma < \frac{3m-1}{m+1}$ (compare with the result of Theorem 1.2 which, under less restrictive hypotheses, shows the existence of solution in $W^{1,2}_0(\Omega)$ only for $\gamma \leq 1$).*

Proof of Theorem 1.3. Following the ideas of [51], let z_n be a solution of the problem

$$\begin{cases} -\mathrm{div}\,(M(x)\,\nabla z) = \dfrac{f_n(x)}{(z+\frac{1}{n})^\gamma}\,, & \text{in } \Omega\,,\\ z = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{1.26}$$

where $f_n(x) = \min\{f(x), n\}$. Let us now denote by λ_1 the first eigenvalue of the operator $L(v) = -\text{div}\,(M(x)\nabla v)$ on Ω with Dirichlet boundary conditions and by φ_1 the associated eigenfunction. We claim that if we denote by $w(x) = C\,\varphi_1(x) + \frac{1}{n^{(\gamma+1)/2}}$, then there exists a positive constant C (depending only on c_0, M and γ) such that the function

$$\underline{z}(x) = w(x)^{2/(\gamma+1)} - \frac{1}{n}$$

is a subsolution to (1.26). Indeed, we check that

$$\nabla\underline{z}(x) = \frac{2\,C}{1+\gamma} w(x)^{(1-\gamma)/(1+\gamma)} \nabla\varphi_1(x)\,,$$

and, if $c_0 := \min\{1, f_0\}$, that

$$\begin{aligned} -\text{div}\,(M(x)\nabla\underline{z}(x)) &= \frac{C}{w(x)^{2\gamma/(1+\gamma)}} \left\{ \frac{2\,C\,(\gamma-1)}{(1+\gamma)^2} M(x)\nabla\varphi_1\,\nabla\varphi_1 + \frac{2\lambda_1\varphi_1(x)}{1+\gamma}\, w(x) \right\} \\ &\le \frac{c_0}{w(x)^{2\gamma/(1+\gamma)}}\,, \end{aligned}$$

for $C > 0$ small enough. Using that $f_n(x) \ge c_0$ a.e. $x \in \Omega$, then we deduce that $\underline{z}$ is a subsolution of (1.26) and the claim is proved. Thus, using the comparison principle of [59],

$$z_n(x) \ge \left(C\,\varphi_1(x) + \frac{1}{n^{(\gamma+1)/2}} \right)^{2/(\gamma+1)} - \frac{1}{n}\,. \tag{1.27}$$

Now, we fix $\alpha > \frac{(m+1)(\gamma+1)}{4m}$ and for $\theta := 2\alpha - 1 > \frac{(m+1)\gamma+1-m}{2m}$ we take

$$\phi = \left(z_n + \frac{1}{n} \right)^{\theta} - \frac{1}{n^{\theta}}$$

as a test function in (1.26) to obtain

$$\mu\,\theta \int_\Omega \left(z_n + \frac{1}{n} \right)^{\theta-1} |\nabla z_n|^2 \le \int_\Omega f_n \frac{(z_n + \frac{1}{n})^{\theta} - \left(\frac{1}{n}\right)^{\theta}}{\left(z_n + \frac{1}{n}\right)^{\gamma}}\,,$$

and therefore, by Hölder inequality,

$$\begin{aligned} \frac{4\,\theta\,\mu}{(\theta+1)^2} \int_\Omega \left| \nabla \left(\left(z_n + \frac{1}{n} \right)^{\alpha} - \left(\frac{1}{n} \right)^{\alpha} \right) \right|^2 &\le \int_\Omega f_n \left(z_n + \frac{1}{n} \right)^{\theta-\gamma} \\ &\le ||f||_{L^m} \left(\int_\Omega \left(z_n + \frac{1}{n} \right)^{m'(\theta-\gamma)} \right)^{1/m'}\,. \end{aligned} \tag{1.28}$$

If we assume also that $\alpha \le \frac{\gamma+1}{2}$, then $\theta - \gamma \le 0$ and, by (1.27),

$$\left(z_n(x) + \frac{1}{n} \right)^{m'(\theta-\gamma)} \le \left(C\,\varphi_1(x) + \frac{1}{n^{(\gamma+1)/2}} \right)^{\frac{2m'(\theta-\gamma)}{1+\gamma}}\,,$$

with the function $\varphi_1^{\frac{2m'(\theta-\gamma)}{1+\gamma}} \in L^1(\Omega)$ (since $\theta > \frac{(m+1)\gamma+1-m}{2m}$ and Ω satisfies the interior sphere condition). Consequently, we deduce that $\{(z_n + \frac{1}{n})^\alpha - \left(\frac{1}{n}\right)^\alpha\}$ is bounded in $W_0^{1,2}(\Omega)$ and thus, up to a subsequence, we can assume that it weakly converges in $W_0^{1,2}(\Omega)$. In addition, since z_n is bounded in $W_{\text{loc}}^{1,2}(\Omega)$ (see the proof of Lemma 4.1 of [51]), we can also assume that z_n converges to z a.e. in Ω. Therefore, the weak limit of $\{(z_n + \frac{1}{n})^\alpha - \left(\frac{1}{n}\right)^\alpha\}$ has to be equal to z^α, proving that it belongs to $W_0^{1,2}(\Omega)$. $\square$

Chapter 2

The effect of a singular term in a quadratic quasilinear problem

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Abstract

For an open bounded set $\Omega \subset \mathbb{R}^N$, $N \geq 3$, $0 \lneqq f \in L^m(\Omega)$ with $m > 1$, $0 \leq \mu(x) \in L^\infty(\Omega)$ and assuming in addition $\|\mu\|_{L^\infty(\Omega)} < \frac{N(m-1)}{N-2m}$ if $m < \frac{N}{2}$, we prove the existence of a positive solution for the singular b.v.p

$$\begin{cases} -\Delta u = \lambda u + \mu(x) \dfrac{|\nabla u|^2}{u} + f(x), & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \end{cases}$$

provided that $\lambda < \lambda_1/(1 + \|\mu\|_{L^\infty(\Omega)})$ (extending the previous results in [9] for $\lambda = 0$). The model case $\mu(x) \equiv B < 1$ is studied in more detail obtaining in addition the uniqueness (resp. nonexistence) of positive solution if the parameter $\lambda < \frac{\lambda_1}{B+1}$ (resp. $\lambda \geq \frac{\lambda_1}{B+1}$). Even more, the solutions constitute a continuum of solutions bifurcating from infinity at $\lambda = \frac{\lambda_1}{B+1}$. This is in contrast with [12] where the multiplicity of solutions of the nonsingular problem ($\frac{1}{u}$ does not appear in the equation) is deduced due to the bifurcation from infinity at $\lambda = 0$.

2.1 Introduction

In this paper we continue the study of singular problems started in [9]. Here, we consider singular Dirichlet problems whose model is

$$-\Delta u = \lambda u + \mu(x)\,\frac{|\nabla u|^2}{u} + f(x),\ \text{in } \Omega\,, \tag{2.1}$$

$$u = 0,\ \text{on } \partial\Omega\,, \tag{2.2}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, $N \geq 3$, $\lambda \in \mathbb{R}$, $0 \lneqq f \in L^m(\Omega)$ with $m > \frac{N}{2}$ and $0 \leq \mu(x) \in L^\infty(\Omega)$. In this case, the singularity of the lower order term $\mu(x)\,\frac{|\nabla u|^2}{u}$ makes necessary to fix the meaning of solution of the Dirichlet problem (2.1)-(2.2). For a (positive) solution of the differential equation (2.1) we consider a function u belonging to $W^{1,1}_{\text{loc}}(\Omega)$ such that $u > 0$ a.e. in Ω, $\frac{|\nabla u|^2}{u} \in L^1_{\text{loc}}(\Omega)$ and

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \mu(x)\,\frac{|\nabla u|^2}{u}\,\varphi + \lambda \int_\Omega u\,\varphi + \int_\Omega f\,\varphi \tag{2.3}$$

holds true for all $\varphi \in C^1_c(\Omega)$, where $C^1_c(\Omega)$ denotes the space of C^1-functions in Ω with compact support. With respect to the boundary condition (2.2), we follow [9, 70] and we require that a suitable positive power (which depends on m) of the solution u belongs to the space $W^{1,2}_0(\Omega)$.

The existence of positive solutions for the nonsingular boundary value problem

$$\begin{aligned} -\Delta u = \lambda u + \mu(x)\,|\nabla u|^2 + f(x), \quad &\text{in } \Omega,\\ u = 0, \quad &\text{on } \partial\Omega, \end{aligned}$$

has been widely studied. For instance, the existence of a solution for $\lambda < 0$ is proved in [49] for every data f, while for $\lambda = 0$ is proved in [68] provide f is small enough. See also [1] for a more general sufficient condition when $\lambda = 0$. Uniqueness of solution u_λ is proved in [17] for $\lambda < 0$ and in [16] for $\lambda = 0$, see also [11] for more general results. If $\lambda > 0$ and $0 < \mu_0 \leq \mu(x)$, the existence and multiplicity of positive solutions has recently been proved in [12] (see also [76] for the case $\mu(x)$ constant) by applying the continuation theorem of Leray-Schauder on the parameter λ. This multiplicity phenomenon is due to the fact that $\lambda = 0$ is a bifurcation point from infinity, in contrast with the trivial case $\mu(x) \equiv 0$. Actually, if $\mu(x) \equiv 0$ (non singular and linear Dirichlet problem) and λ_1 denotes the eigenvalue associated to the first eigenfunction of the Laplacian operator, then there is a positive solution u_λ if and only if $\lambda < \lambda_1$ and the solution u_λ blows up as λ tends to λ_1, i.e. $\lambda = \lambda_1$ is a bifurcation point from infinity. On the contrary, when $0 < \mu_0 \leq \mu(x)$, the behavior of the continuum of solutions which contains the pairs (λ, u_λ), with $\lambda < 0$, has two possibilities: either $||u_\lambda||_{L^\infty(\Omega)}$ diverges when the negative parameter λ tends to $\lambda = 0$ (i.e., in this case the non singular problem does not have a solution with $\lambda = 0$), or it contains a point $(0, u_0)$, with u_0 a solution of the non singular problem for $\lambda = 0$. In this second case,

$\lambda = 0$ is a bifurcation point from infinity to the right and, consequently, there is $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, the non singular problem admits at least two different bounded solutions. Observe that, in this context, the assumption $\mu_0 \le \mu(x) \in L^\infty(\Omega)$ implies that the lower order term of the nonsingular problem satisfies

$$\mu_0 |\nabla u|^2 \le \mu(x)|\nabla u|^2 \le \|\mu\|_{L^\infty(\Omega)} |\nabla u|^2 .$$

Recently, as part of the results of [89] for space of dimensions up to N = 5, Souplet has proved that if $N \le 2$, then the assumption $\mu(x) \ge \mu_0 > 0$ in Ω can be strongly weaken to $\mu(x) \ge \mu_0 > 0$ in B, for some ball $B \subset \Omega$.

Our aim is to analyze the effect of the singularity introduced in (2.1) for the existence and multiplicity of positive solutions when $\lambda > 0$. We highlight that in this situation, the lower order term $\mu(x)\frac{|\nabla u|^2}{u}$ "is not bounded from below by $\mu_0|\nabla u|^2$" for any $\mu_0 > 0$ and we wonder if $\lambda = 0$ is still a bifurcation point from infinity. Surprisingly, despite the new difficulty due to the presence of the singular term and, in contrast with the previous papers, the existence results obtained here will be comparable with the results for the linear Dirichlet problem ($\mu(x) \equiv 0$). Indeed, by [9] we observe that the problem (2.1)-(2.2) has always a solution for every data f when $\lambda = 0$. This result is extended here to cover the case $\lambda > 0$ by proving the existence of solution for all $\lambda < \frac{\lambda_1}{1+\|\mu\|_{L^\infty(\Omega)}}$. Specifically, we show the following theorem.

Theorem 2.1 *Suppose that* $0 \lneqq f \in L^m(\Omega)$ *with* $1 < m < \frac{N}{2}$ *and that* μ *satisfies* $\|\mu\|_{L^\infty(\Omega)} < \frac{N(m-1)}{N-2m}$. *If* $\lambda < \frac{\lambda_1}{1+\|\mu\|_{L^\infty(\Omega)}}$, *then there exists a solution* $u \in W^{1,1}_{loc}(\Omega)$ *of* (2.1) *with* $u^\gamma \in W^{1,2}_0(\Omega)$ *for every* $\frac{1+\|\mu\|_{L^\infty(\Omega)}}{2} < \gamma \le \frac{m(N-2)}{2(N-2m)}$.

As a direct consequence, if $f \in L^m(\Omega)$ with $m \ge \frac{N}{2}$, then there exists a solution without any restriction on the parameter $B := \|\mu\|_{L^\infty(\Omega)}$. Indeed, given $f \in L^m(\Omega)$ with $m \ge \frac{N}{2}$ and $B > 0$, if $\gamma > \frac{B+1}{2}$, then we can choose $\frac{N(B+1)}{N+2B} < \overline{m} < \frac{N}{2}$ such that $\frac{B+1}{2} < \gamma \le \frac{\overline{m}(N-2)}{2(N-2\overline{m})}$. Thus, Theorem 2.1 implies the existence of a solution $u \in W^{1,1}_{\text{loc}}(\Omega)$ of (2.1) with $u^\gamma \in W^{1,2}_0(\Omega)$. In particular, $u \in L^q(\Omega)$ for every $q \ge 1$. Even more, we also prove (Lemma 2.2) that if $m > N/2$, then $u \in L^\infty(\Omega)$. Summing up, we have the following result.

Corollary 2.1 *Suppose that* $0 \lneqq f \in L^m(\Omega)$ *with* $m \ge \frac{N}{2}$. *If* $\lambda < \frac{\lambda_1}{1+\|\mu\|_{L^\infty(\Omega)}}$, *then there exists a solution* $u \in W^{1,1}_{loc}(\Omega)$ *satisfying* (2.3) *with* $u^\gamma \in W^{1,2}_0(\Omega)$, *for every* $\gamma > \frac{1+\|\mu\|_\infty}{2}$. *In addition, if* $m > N/2$, *then* $u \in L^\infty(\Omega)$.

We emphasize that we prove these results for a more general class of singularities. Actually, we can replace the singularity $\mu(x)/u$ in the term $\mu(x)\frac{|\nabla u|^2}{u}$ by a Carathéodory function $g : \Omega \times (0, +\infty) \longrightarrow \mathbb{R}$ satisfying that

$$0 < g(x, s) \le \frac{B}{s}, \quad \text{a.e. } x \in \Omega, \ \forall s > 0, \tag{2.4}$$

(see Theorem 2.2 and Corollary 2.2 below).

Moreover, we also remark explicitly that by convenience of the reader we have considered the linear Laplacian operator as the main differential operator in (2.1). Following the arguments of [9, Section 4] we can replace it by a nonlinear elliptic differential operator of Leray-Lions type (see Remark 2.3).

The paper is concluded by proving the optimality of $\frac{\lambda_1}{1+\|\mu\|_{L^\infty(\Omega)}}$ in the above results. It is shown that in general no (positive) solution of (2.1)-(2.2) exists for $\lambda \geq \frac{\lambda_1}{1+\|\mu\|_{L^\infty(\Omega)}}$ and that no multiplicity of solutions of (2.1)-(2.2) can be expected when $\lambda < \frac{\lambda_1}{1+\|\mu\|_{L^\infty(\Omega)}}$.

Following the arguments of [9], the proofs of the main results relies on a standard approximation procedure. Actually, in Section 2 we approximate the problem (2.1)-(2.2) by nonsingular quadratic problems which have a solution u_n. The strong maximum principle allow us to obtain a strictly positive lower bound of the sequence u_n in Ω. In Section 3 we prove the general versions of Theorem 2.1 and Corollary 2.1 by getting a solution u of (2.1)-(2.2) as limit, in a suitable sense, of the sequence u_n. Finally, we devote the last section to study our problem (2.1)-(2.2) when $\mu(x)$ is a positive constant $B < 1$ and Ω is smooth. Specifically, in this particular case we prove existence and uniqueness of solution u_λ if and only if $\lambda < \lambda_1/(1+B)$. Furthermore, we apply the continuation theorem of Leray-Schauder on the parameter λ to deduce that the set of solutions (λ, u_λ) is an unbounded continuum of solutions bifurcating from $\lambda_1/(1+B)$.

Notation. In what follows, we make frequent use of the truncation function defined by $T_k(s) = \max(-k, \min(s,k))$, for $k > 0$ and s in $\mathbb{R}$, and its "companion" function $G_k(s) = s - T_k(s)$. Moreover, we denote by $h^+ = \max\{h, 0\}$, $h^- = \min\{h, 0\}$.

2.2 Approximate Problems

Given a positive number B and a Carathéodory function g in $\Omega \times (0, +\infty)$ satisfying (2.4) , we consider the following Dirichlet problem

$$-\Delta u = \lambda u + g(x,u)|\nabla u|^2 + f(x), \text{ in } \Omega\,, \tag{2.5}$$

$$u = 0, \text{ on } \partial\Omega\,. \tag{2.6}$$

In order to study this problem, for each $n \in \mathbb{N}$, we approximate the Carathéodory function g by the function $g_n : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$g_n(x,s) := g\big(x, s + \frac{1}{n}\big)\,, \quad \forall s > 0\,, \text{ a.e. } x \in \Omega\,,$$

and is even in the variable s. Observe that g_n is a Carathéodory function such that

$$0 \leq g_n(x,s) \leq \frac{B}{|s| + \frac{1}{n}}\,, \quad \forall s \in \mathbb{R}\,, \text{ a.e. } x \in \Omega. \tag{2.7}$$

By the results of [49] (see also [83, Theorem 2.1]), there exists a bounded solution $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the following approximate problems

$$\begin{cases} -\Delta w = g_n(x,w)\,|\nabla w|^2 + \lambda T_n(w^+) + T_n(f), & \text{in } \Omega\,, \\ w = 0, & \text{on } \partial\Omega\,, \end{cases}$$

in the sense that $g_n(x,u_n)|\nabla u_n|^2 \in L^1(\Omega)$ and

$$\int_\Omega \nabla u_n \,\nabla\varphi = \int_\Omega g_n(x,u_n)|\nabla u_n|^2\,\varphi + \lambda\int_\Omega T_n(u_n^+)\,\varphi + \int_\Omega T_n(f)\,\varphi$$

holds for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Moreover by the maximum principle, $u_n \geq z > 0$ a.e. $x \in \Omega$, where z is the unique solution of

$$\begin{cases} -\Delta z = \lambda^-\, z + T_1(f), & \text{in } \Omega, \\ z = 0, & \text{on } \partial\Omega. \end{cases}$$

Consequently, for every $\omega \subset\subset \Omega$ there exists a positive constant $c_{\omega,\lambda}$ such that

$$u_n(x) \geq c_{\omega,\lambda} > 0 \;\text{ a.e. } x \in \omega\,,\; \forall n \in \mathbb{N}\,, \tag{2.8}$$

and then $0 < u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ solves the problem

$$\begin{cases} -\Delta u_n = g_n(x,u_n)\,|\nabla u_n|^2 + \lambda T_n(u_n) + T_n(f), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Lemma 2.1 *If $\lambda \in \mathbb{R}$, $0 \lneqq f \in L^m(\Omega)$ with $m > 1$ and $B > 0$, then the sequence $\{u_n\}$ of solutions of* (2.9) *satisfies*

$$\int_\Omega \nabla u_n^{B+1}\,\nabla\phi \,\leq\, (B+1)\int_\Omega \lambda^+\,u_n^{B+1}\,\phi + (B+1)\int_\Omega T_n(f)\,u_n^B\,\phi\;, \tag{2.10}$$

for all $0 \leq\, \phi \in W_0^{1,2}(\Omega)$.

Proof. Given a function $0 \leq\, \phi \in C_c^1(\Omega)$ we take $\varphi = u_n^B\,\phi$ as a test function in the weak formulation of (2.9) (thanks to (2.8) in the case $B < 1$) to deduce that

$$\begin{aligned} \int_\Omega B\,u_n^{B-1}|\nabla u_n|^2\,\phi + \int_\Omega u_n^B\,\nabla u_n \nabla\phi &= \int_\Omega g_n(x,u_n)|\nabla u_n|^2\,u_n^B\phi \\ &\quad + \int_\Omega \lambda\,T_n(u_n)\,u_n^B\,\phi + \int_\Omega T_n(f)\,u_n^B\,\phi\,, \end{aligned}$$

which gives us, thanks to (2.7) that

$$\begin{aligned} \int_\Omega \nabla u_n^{B+1}\,\nabla\phi &= (B+1)\int_\Omega u_n^B\,\nabla u_n \nabla\phi \\ &\leq\, (B+1)\int_\Omega \lambda^+\,u_n^{B+1}\,\phi + (B+1)\int_\Omega T_n(f)\,u_n^B\,\phi\,. \end{aligned}$$

Since $u_n \in L^\infty(\Omega)$, we have $u_n^{B+1} \in W_0^{1,2}(\Omega)$ and then we deduce the result by density.
□

Lemma 2.2 *i) If $1 < m < \frac{N}{2}$, $0 < B < \frac{N(m-1)}{N-2m}$ and $\lambda < \lambda_1/(B+1)$, then the sequence $\{u_n\}$ of solutions of (2.9) is bounded in $L^m(\Omega)$.*

ii) If $m > \frac{N}{2}$, $B > 0$ and $\lambda < \lambda_1/(B+1)$, then the sequence $\{u_n\}$ of solutions of (2.9) is bounded in $L^\infty(\Omega)$.

Proof. i) We argue by contradiction assuming that there exists a subsequence, not relabeled, $\{u_n\}$ of solutions of (2.9) such that

$$||u_n||_{L^m(\Omega)} \longrightarrow +\infty \ \text{ as } n \to +\infty\,.$$

Note that the function

$$0 \le z_n := \frac{u_n^{B+1}}{||u_n||^{B+1}_{L^m(\Omega)}} \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$$

satisfies, by (2.10), that

$$\int_\Omega \nabla z_n \nabla \phi \le \int_\Omega \left(\lambda^+ (B+1)\, z_n^{\frac{1}{B+1}} + (B+1) \frac{T_n(f)}{||u_n||_{L^m(\Omega)}} \right) z_n^{\frac{B}{B+1}}\, \phi\,, \tag{2.11}$$

for all $0 \le \phi \in W_0^{1,2}(\Omega)$. Hence, we obtain

$$\begin{cases} -\Delta z_n \le \rho_n\, z_n^{\frac{B}{B+1}} & \text{in } \Omega, \\ z_n \ge 0\,, & \text{in } \Omega, \\ z_n = 0, & \text{on } \partial\Omega, \end{cases}$$

where the sequence of functions

$$\rho_n = \lambda^+ (B+1)\, z_n^{\frac{1}{B+1}} + (B+1) \frac{T_n(f)}{||u_n||_{L^m(\Omega)}}$$

is bounded in $L^m(\Omega)$ (since $||z_n^{\frac{1}{B+1}}||_{L^m(\Omega)} = 1$ and $||u_n||_{L^m(\Omega)} \longrightarrow \infty$ as n tends to ∞). Consequently, the arguments of [50, Theorem 3.1, Theorem 4.1] implies that the sequence z_n is bounded in $L^{\frac{m^{**}}{B+1}}(\Omega)$ and in $W_0^{1,q}(\Omega)$ for some $\frac{N}{N-1} < q \le 2$ (recall that $m^* = \frac{Nm}{N-m}$ and $m^{**} = (m^*)^* = \frac{Nm}{N-2m}$).

Hence, thanks to Sobolev embedding there exists a function z and a subsequence, not relabeled, $\{z_n\}$ such that $z_n \longrightarrow z$ a.e $x \in \Omega$. Furthermore, using the boundedness of z_n in $L^{\frac{m^{**}}{B+1}}(\Omega)$ and Lebesgue dominated convergence theorem, we have

$$z_n \longrightarrow z \ \text{ in } L^r(\Omega)\,, \forall 1 \le r < \frac{m^{**}}{B+1}\,. \tag{2.12}$$

Thus, $z_n^{\frac{1}{B+1}}$ strongly converges in $L^m(\Omega)$ to $z^{\frac{1}{B+1}}$ and, recalling that $||z_n^{\frac{1}{B+1}}||_{L^m(\Omega)} = 1$, we get

$$||z^{\frac{1}{B+1}}||_{L^m(\Omega)} = 1\,,$$

that is, z is a nonnegative and not identically zero function.

Since $m > \left(\frac{2^*(B+1)}{2B}\right)'$ we have $m' < \frac{m^{**}}{B}$ and then $z_n^{\frac{B}{B+1}}$ strongly converges to $z^{\frac{B}{B+1}}$ in $L^{m'}(\Omega)$ (thanks to (2.12)) which implies

$$z_n^{\frac{B}{B+1}} \frac{T_n(f)}{||u_n||_{L^m(\Omega)}} \longrightarrow 0 \text{ in } L^1(\Omega)\,.$$

Therefore, taking $\phi = \varphi_1$ in (2.11) and passing to the limit as n goes to ∞ we conclude

$$\lambda_1 \int_\Omega z\,\varphi_1 \leq \lambda^+ (B+1) \int_\Omega z\,\varphi_1\,,$$

which is a contradiction since $\lambda < \frac{\lambda_1}{B+1}$.

ii) It is enough to show that the sequence u_n^{B+1} is bounded in $L^\infty(\Omega)$. We split the proof into two steps. In the first one, we prove that $||u_n^{B+1}||_{L^\infty(\Omega)}$ is bounded by a constant depending on $||u_n^{B+1}||_{L^1(\Omega)}$. In the second one, we conclude the proof showing that u_n^{B+1} is bounded in $W_0^{1,2}(\Omega)$ (and thus in $L^1(\Omega)$).

Step 1. There is a positive constant C_1 (depending only on B, λ and f) such that

$$||u_n^{B+1}||_{L^\infty(\Omega)} \leq C_1\, ||u_n^{B+1}||_{L^1(\Omega)}\,,\ \forall n \in \mathbb{N}\,.$$

In order to prove it, we define $\gamma_0 := 2B+1$ and for $k \geq 1$we take $G_k(u_n^{\gamma_0})$ as a test function in the weak formulation of (2.9). Defining

$$A_k := \{x \in \Omega\, :\, u_n^{\gamma_0}(x) > k \geq 1\},$$

we have, by (2.7), that

$$\gamma_0 \int_{A_k} u_n^{\gamma_0-1}\, |\nabla u_n|^2 \leq\ B \int_{A_k} u_n^{\gamma_0-1}\, |\nabla u_n|^2 + \int_{A_k} \lambda^+\, u_n^{\gamma_0+1} + \int_{A_k} f\, u_n^{\gamma_0}.$$

Since $k \geq 1$, we deduce by Hölder inequality with exponent m that,

$$\begin{aligned} \frac{4\,(\gamma_0 - B)}{(\gamma_0+1)^2} \int_{A_k} \left|\nabla u_n^{\frac{\gamma_0+1}{2}}\right|^2 =& (\gamma_0 - B) \int_{A_k} u^{\gamma_0-1}\, |\nabla u_n|^2 \\ \leq& \int_{A_k} \left(\lambda^+ + f\right)\, u_n^{\gamma_0+1} \\ \leq& ||\lambda^+ + f||_{L^m(\Omega)} \left(\int_{A_k} u_n^{(\gamma_0+1)m'}\right)^{\frac{1}{m'}}. \end{aligned}$$

If we define $v_n := u_n^{\frac{\gamma_0+1}{2}} \in W_0^{1,2}(\Omega)$, we can rewrite the above inequality as follows

$$\frac{4\,(\gamma_0 - B)}{(\gamma_0+1)^2} \int_{A_k} \left|\nabla v_n\right|^2 \leq ||\lambda^+ + f||_{L^m(\Omega)} \left(\int_{A_k} v_n^{2m'}\right)^{\frac{1}{m'}},$$

where we observe that $A_k = \{x \in \Omega \,:\, v_n(x) > k^{\frac{\gamma_0+1}{2\gamma_0}} \geq 1\}$. Therefore, using that $m > N/2$ and $0 < B < \gamma_0$, we can follow the arguments of Ladyzhenskaya - Ural'tseva [77] to deduce that there exists a positive constant C_1 such that

$$||u_n^{B+1}||_{L^\infty(\Omega)} = ||v_n||_{L^\infty(\Omega)} \leq C_1\, ||v_n||_{L^1(\Omega)} = C_1\, ||u_n^{B+1}||_{L^1(\Omega)}\,.$$

Step 2. There exists a positive constant C_2 (depending only on B, λ and f) such that

$$||u_n^{B+1}||_{W_0^{1,2}(\Omega)} \leq C_2\,,\ \ \forall n \in \mathbb{N}\,.$$

Indeed, given a solution u_n of (2.9) we observe that we can take u_n^{B+1} as a test function in (2.10). Hence,

$$\int_\Omega |\nabla(u_n^{B+1})|^2 \leq \lambda^+\,(B+1)\int_\Omega u_n^{2(B+1)} + (B+1)\int_\Omega T_n(f)\, u_n^{2B+1}\,.$$

Thus, using Sobolev and Hölder inequalities, we have

$$\left(1 - \frac{\lambda^+(B+1)}{\lambda_1}\right)\int_\Omega |\nabla u_n^{B+1}|^2 \leq (B+1)\,||f||_{L^m(\Omega)}\left(\int_\Omega u_n^{(2B+1)m'}\right)^{1/m'}.$$

Since $m > \frac{N}{2}$ we observe that $\frac{2B+1}{B+1}m' < 2^*$ and Hölder's inequality implies

$$\left(1 - \frac{\lambda^+(B+1)}{\lambda_1}\right)\int_\Omega |\nabla u_n^{B+1}|^2 \leq (B+1)\,||f||_{L^m(\Omega)}\,[\text{meas}(\Omega)]^{1-\frac{(2B+1)m'}{2^*(B+1)}}\left(\int_\Omega u_n^{2^*(B+1)}\right)^{\frac{2B+1}{2^*(B+1)}}.$$

By Sobolev inequality, we obtain

$$\left(1 - \frac{\lambda^+(B+1)}{\lambda_1}\right)\int_\Omega |\nabla u_n^{B+1}|^2 \leq (B+1)\mathcal{S}\,||f||_{L^m(\Omega)}\,[\text{meas}(\Omega)]^{1-\frac{(2B+1)m'}{2^*(B+1)}}\left(\int_\Omega |\nabla u_n^{B+1}|^2\right)^{\frac{2B+1}{2(B+1)}},$$

where $\mathcal{S}$ denotes the Sobolev constant. Since $\frac{2B+1}{2(B+1)} < 1$, we conclude the boundedness of $\{u_n^{B+1}\}$ in $W_0^{1,2}(\Omega)$, as we desired. □

2.3 The main result

We devote this section to prove existence results for the general problem (2.5)-(2.6). Precisely, we deduce the following theorem.

Theorem 2.2 *Suppose that* $0 \lneqq f \in L^m(\Omega)$ *with* $1 < m < \frac{N}{2}$ *and that the Carathéodory function* g *satisfies* (2.4) *with* $B < \frac{N(m-1)}{N-2m}$. *If* $\lambda < \frac{\lambda_1}{1+B}$, *then there exists a solution* $u \in W^{1,1}_{loc}(\Omega)$ *of* (2.5) *with* $u^\gamma \in W^{1,2}_0(\Omega)$ *for every* $\frac{1+B}{2} < \gamma \leq \frac{m(N-2)}{2(N-2m)}$.

Remark 2.1 *Observe that* $\frac{1+B}{2} < \frac{m(N-2)}{2(N-2m)}$ *because* $B < \frac{N(m-1)}{N-2m}$.

Remark 2.2 *Observe that if* $g(x,s) = \mu(x)/u$, *with* $\mu \in L^\infty(\Omega)$, *then* g *satisfies condition* (2.4) *for* $B = \|\mu\|_{L^\infty(\Omega)}$. *Therefore, Theorem 2.2 is more general than Theorem 2.1 presented in the Introduction.*

Proof of Theorem 2.2. Let $\{u_n\}$ be the sequence of solutions of (2.9) satisfying (2.8). First of all we prove some a priori estimates for this sequence $\{u_n\}$. For each $n \in \mathbb{N}$, we fix $0 < \delta < \frac{1}{n}$ and we take

$$\varphi = (u_n + \delta)^{2\gamma-1} - \delta^{2\gamma-1}, \qquad \frac{B+1}{2} < \gamma \leq \frac{m^{**}}{2^*}$$

as a test function in the weak formulation of (2.9). Observe that

$$\frac{m^{**}}{2^*} = \frac{m(N-2)}{2(N-2m)}$$

and that $2\gamma - 1 > B > 0$. Hence, by (2.7),

$$(2\gamma - 1 - B)\int_\Omega |\nabla u_n|^2 (u_n + \delta)^{2\gamma-2} \leq \int_\Omega \left(\lambda^+ u_n + f\right)\,(u_n+\delta)^{2\gamma-1} - \int_\Omega \delta^{2\gamma-1}\lambda^- u_n\,.$$

By Lemma 2.2, the term $\lambda^+ u_n + f$ is bounded in $L^m(\Omega)$ by a positive constant M. Moreover, if δ tends to zero we can use Fatou Lemma (recall that $\gamma > \frac{B+1}{2}$), Lebesgue dominated convergence theorem and Hölder inequality to get

$$(2\gamma - 1 - B)\int_\Omega |\nabla u_n|^2\, u_n^{2(\gamma-1)} \leq M\left(\int_\Omega u_n^{(2\gamma-1)m'}\right)^{\frac{1}{m'}}.$$

Now, by Sobolev inequality we have

$$\begin{aligned}\frac{\mathcal{S}^2(2\gamma-1-B)}{\gamma^2}\left(\int_\Omega u_n^{2^*\gamma}\right)^{\frac{2}{2^*}} &\leq \frac{2\gamma-1-B}{\gamma^2}\int_\Omega |\nabla u_n^\gamma|^2 \\ &= (2\gamma-1-B)\int_\Omega |\nabla u_n|^2\, u_n^{2(\gamma-1)} \\ &\leq M\left(\int_\Omega u_n^{(2\gamma-1)m'}\right)^{\frac{1}{m'}}.\end{aligned}$$

Notice that the choice of γ implies that $2^*\gamma \geq (2\gamma-1)\,m'$ and $\frac{2\gamma-1}{2^*\gamma} < \frac{2}{2^*}$. Thus, by the previous inequality, the sequence $\{u_n\}$ is bounded in $L^{m^{**}}(\Omega)$ as well as the sequence $\{u_n^\gamma\}$ is bounded in $W_0^{1,2}(\Omega)$ for every $\frac{B+1}{2} < \gamma \leq \frac{m^{**}}{2^*}$. In particular, there exists a positive constant C such that

$$\int_\Omega |\nabla u_n|^2\, u_n^{2(\gamma-1)} \leq C\,. \tag{2.13}$$

In addition, if for every $\omega \subset\subset \Omega$ we consider the constant c_ω given by (2.8) (since λ is fixed, we do not denote the dependence of the constant c_ω on λ), we have by (2.7)

$$c_\omega^{2\gamma-1}\int_\omega g_n(x,u_n)\,|\nabla u_n|^2 \leq c_\omega^{2\gamma-1}\int_\omega \frac{|\nabla u_n|^2}{u_n+\frac{1}{n}} \leq \int_\Omega |\nabla u_n|^2\, u_n^{2(\gamma-1)}\,,$$

and by (2.13) the sequence $\{g_n(x,u_n)|\nabla u_n|^2\}$ is bounded in $L^1(\omega)$. This implies that $-\Delta u_n$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$.

Furthermore, inequality (2.13) also implies the following estimates:

i) If $\frac{2N}{N+2} \leq m < \frac{N}{2}$, observe that $\frac{m^{**}}{2^*} \geq 1$ and then we can choose $\gamma \geq 1$. In this way, using again (2.8), we have

$$c_\omega^{2\gamma-2}\int_\omega |\nabla u_n|^2 \leq \int_\Omega |\nabla u_n|^2\, u_n^{2(\gamma-1)} < C\,,$$

and the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\omega)$ for every $\omega \subset\subset \Omega$.

ii) If $1 < m < \frac{2N}{N+2}$, then $\frac{m^{**}}{2^*} < 1$ and so $2(\gamma-1) < 0$. Now, for every $k>0$ we deduce from (2.13) that

$$k^{2(\gamma-1)}\int_\Omega |\nabla T_k(u_n)|^2 \leq \int_{\{0<u_n<k\}} u_n^{2(\gamma-1)}|\nabla u_n|^2 \leq C.$$

This implies that the sequence $\{T_k(u_n)\}$ is bounded in $W_0^{1,2}(\Omega)$ for every $k>0$. Moreover, in this case, following the ideas of [35] we prove that the sequence $\{u_n\}$ is bounded in $W_0^{1,m^*}(\Omega)$ too. Indeed, since $m^* < 2$ we obtain by Hölder inequality

$$\int_\Omega |\nabla u_n|^{m^*} \leq \left(\int_\Omega |\nabla u_n|^2 u_n^{2(\frac{m^{**}}{2^*}-1)}\right)^{\frac{m^*}{2}} \left(\int_\Omega u_n^{m^{**}}\right)^{\frac{2-m^*}{2}}.$$

Therefore, using (2.13) with $\gamma = \frac{m^{**}}{2^*}$ and the boundedness of $\{u_n\}$ in $L^{m^{**}}(\Omega)$, we deduce that the sequence $\{u_n\}$ is bounded in $W_0^{1,m^*}(\Omega)$.

Summing up, by all the above estimates, we can assume that there exists a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that

$$u^\gamma \in W_0^{1,2}(\Omega), \qquad \frac{B+1}{2} < \gamma \leq \frac{m^{**}}{2^*}, \qquad g(x,u)|\nabla u|^2 \in L^1_{\mathrm{loc}}(\Omega),$$

and a subsequence (not relabeled) such that u_n, respectively ∇u_n, converges a.e. in Ω to u, respectively ∇u. In addition,

i) If $\frac{2N}{N+2} \le m < \frac{N}{2}$, then u_n converges weakly in $W_0^{1,2}(\omega)$ to u, for every $\omega \subset\subset \Omega$.

ii) If $1 < m < \frac{2N}{N+2}$, then $T_k(u_n)$ converges weakly in $W_0^{1,2}(\Omega)$ to $T_k(u)$ for every $k > 0$ and u_n converges weakly in $W_0^{1,m^*}(\Omega)$ to u.

Furthermore, again by (2.8), $u(x) \ge c_\omega > 0$ a.e. $x \in \omega$ and, in particular, $u > 0$ in Ω.

The proof is concluded using these convergences to pass to the limit in the weak formulation of (2.9):

$$\int_\Omega \nabla u_n \, \nabla \varphi = \int_\Omega g_n(x, u_n) |\nabla u_n|^2 \, \varphi + \lambda \int_\Omega T_n(u_n) \, \varphi + \int_\Omega T_n(f) \, \varphi,$$

for all function $\varphi \in C_c^1(\Omega)$. Notice that without loss of generality, we can assume that $\varphi \ge 0$. We follow the technique of [9].

Given $0 \le \varphi \in C_c^1(\Omega)$, we firstly observe that the convergences almost everywhere in Ω of ∇u_n (respectively, u_n, $g_n(x, u_n)$) to ∇u (respectively u, $g(x, u)$) and the weak convergence of ∇u_n to ∇u in $(L^{m^*}(\Omega))^N$, if $1 < m < \frac{2N}{N+2}$ or in $L^2(\omega)$, if $\frac{2N}{N+2} \le m < \frac{N}{2}$, with $\omega = \text{supp } \varphi$, imply by Fatou lemma that

$$\int_\Omega \nabla u \nabla \varphi \ge \int_\Omega g(x, u) \, |\nabla u|^2 \, \varphi + \lambda \int_\Omega u \, \varphi + \int_\Omega f \, \varphi.$$

To prove the reversed inequality (and conclude the proof) we define the function

$$H(s) = \begin{cases} 1, & \text{if } 0 \le s \le 1, \\ 2 - s, & \text{if } 1 \le s \le 2, \\ 0, & \text{if } s \ge 2. \end{cases}$$

Since $u \ge c_\omega > 0$ in $\omega = \text{supp } \varphi$, we take

$$\frac{T_k(u_n)^{2\gamma-1}}{T_k(u)^{2\gamma-1}} \, H\big(\frac{u_n}{k}\big) \, \varphi \qquad \text{for } \gamma = \frac{m^{**}}{2^*}$$

as a test function in the weak formulation of (2.9). Following the arguments of [9, Theorem 1.3] we pass to the limit in n to deduce that

$$\int_\Omega \nabla u \nabla \varphi \, H\big(\frac{u}{k}\big) - \int_\Omega g(x, u) \, |\nabla T_k(u)|^2 \, H\big(\frac{u}{k}\big) \, \varphi \\ -\lambda \int_\Omega u \, H\big(\frac{u}{k}\big) \varphi \le \int_\Omega f \, H\big(\frac{u}{k}\big) \varphi + \epsilon(k) \, ,$$

with $\epsilon(k)$ tending to zero as k goes to infinite. Finally, since $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $u > 0$ in Ω, we can pass to the limit as k tends to infinite to conclude the reversed inequality

$$\int_\Omega \nabla u \nabla \varphi \leq \int_\Omega g(x,u)\,|\nabla u|^2\,\varphi + \lambda \int_\Omega u\,\varphi + \int_\Omega f\,\varphi\,,$$

and thus the proof. □

As we noticed in the introduction, we can also state the following consequence of Theorem 2.2 for the problem (2.5)-(2.6).

Corollary 2.2 *Suppose that* $0 \lneqq f \in L^m(\Omega)$ *with* $m \geq \frac{N}{2}$ *and that* g *satisfies* (2.4). *If* $\lambda < \frac{\lambda_1}{1+B}$, *then there exists a solution* $u \in W^{1,1}_{loc}(\Omega)$ *of* (2.5)-(2.6) *with* $u^\gamma \in W^{1,2}_0(\Omega)$, *for every* $\gamma > \frac{1+B}{2}$. *In addition, if* $m > N/2$, *then* $u \in L^\infty(\Omega)$.

Remark 2.3 *We emphasize that all the results presented here can be also obtained if we deal with more general differential operators. Specifically and similarly to [9, Section 4], it is possible to handle the following Dirichlet problem*

$$-div(A(x,u\nabla u)) = \lambda u + g(x,u)|\nabla u|^2 + f(x), \quad in\ \Omega\,,$$
$$u = 0, \quad on\ \partial\Omega\,,$$

where $g(x,s)$ *is a Carathéodory function satisfying* (2.4) *and*

$$A(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is a Carathéodory function such that for a.e. $x \in \Omega$ *and for any* $s \in \mathbb{R}$ *and* ξ, $\xi' \in \mathbb{R}^N$ *we have*

$$\exists \alpha > 0 : \quad A(x,s,\xi)\xi \geq \alpha|\xi|^2\,,$$
$$\exists \beta > 0 : \quad |A(x,s,\xi)| \leq \beta|\xi|\,,$$
$$\big(A(x,s,\xi) - A(x,s,\xi')\big)\cdot(\xi - \xi') > 0\,, \quad \xi \neq \xi'\,.$$

2.4 Optimality for the model problem

In this section, we assume that Ω is smooth, $0 < B < 1$ and that $0 \lneqq f \in L^m(\Omega)$ with $m > \frac{N}{2}$. We consider the following model problem

$$\begin{cases} -\Delta u = B\,\dfrac{|\nabla u|^2}{u} + \lambda u + f(x), & \text{in } \Omega, \\ \qquad\quad u = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{P_λ}$$

Since $\frac{1+B}{2} < 1$, we observe that it is possible to choose $\gamma = 1$ in Theorem 2.1 and thus the solutions u of (P_λ) belong to $W^{1,2}_0(\Omega)$. Consequently, our definition of solution in this particular case is the following one.

Definition 2.1 *We say that u is a solution of the problem (P_λ) if $u \in W_0^{1,2}(\Omega)$ is such that, $u > 0$ a.e. in Ω, $\dfrac{|\nabla u|^2}{u} \in L^1_{loc}(\Omega)$ and it satisfies*

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \frac{|\nabla u|^2}{u}\,\varphi + \lambda \int_\Omega u\,\varphi + \int_\Omega f\,\varphi\,, \quad \forall \varphi \in C_c^1(\Omega).$$

Remark 2.4 *In the formulation of the solution the test function in $C_c^1(\Omega)$ can be replaced by functions $\varphi \in W_0^{1,2}(\Omega)$ such that $\dfrac{|\nabla u|^2}{u}\,\varphi \in L^1(\Omega)$.*

We observe that, under the assumption $B < 1$ and $m > N/2$, we can follow the arguments of [77] (taking $G_k(u)$ as a test function) to deduce that every solution of (P_λ) belongs to $L^\infty(\Omega)$.

Theorem 2.3 *Assume that Ω is smooth. If $0 \lneqq f \in L^m(\Omega)$ with $m > \frac{N}{2}$ and $0 < B < 1$, then (P_λ) has a solution if and only if $\lambda < \frac{\lambda_1}{B+1}$. Moreover, for every $\lambda < \frac{\lambda_1}{B+1}$, there is a unique solution $u_\lambda \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and*

$$S = \left\{ (\lambda, u_\lambda) \,:\, \lambda < \frac{\lambda_1}{B+1} \right\},$$

is an unbounded continuum (connected and closed) in $\mathbb{R} \times L^\infty(\Omega)$ satisfying

$$\|u_\lambda\|_{L^\infty(\Omega)} \to +\infty\,, \ \text{as } \lambda \nearrow \frac{\lambda_1}{B+1}.$$

Proof. We split the proof into several steps.
Step 1. For every $\omega \subset\subset \Omega$ there exists a constant $c_{\omega,\lambda}$ such that $u \geq c_{\omega,\lambda} > 0$ a.e. $x \in \omega$ for every solution u of (P_λ). Indeed, by the maximum principle, $u \geq z > 0$ a.e. $x \in \Omega$, where z is the unique solution of

$$z \in W_0^{1,2}(\Omega) \,:\, -\Delta z = \lambda^- z + f\,, \quad \text{in } \Omega\,.$$

Step 2. If u is a solution of (P_λ), then

$$\int_\Omega \nabla(u^{B+1})\,\nabla\phi = (B+1)\int_\Omega \lambda u^{B+1}\,\phi + (B+1)\int_\Omega f\,u^B\,\phi\,, \tag{2.14}$$

for all $\phi \in W_0^{1,2}(\Omega)$. Indeed, as in Lemma 2.1, given a solution u of (P_λ) and a function $\phi \in C_c^1(\Omega)$ we take $\varphi = u^B\,\phi$ as a test function in the weak formulation of (P_λ) to show that (2.14) holds for $\phi \in C_c^1(\Omega)$. Since $u^{B+1} \in W_0^{1,2}(\Omega)$, by density, we conclude the step.
Step 3. If $\lambda \geq \lambda_1/(B+1)$ then (P_λ) has no solution. Suppose that u is a solution of (P_λ). We take $\phi = \varphi_1 \geq 0$ in (2.14), where φ_1 is the first eigenfunction of the Laplacian operator, to obtain

$$\lambda_1 \int_\Omega u^{B+1}\,\varphi_1 \;=\; (B+1)\lambda \int_\Omega u^{B+1}\,\varphi_1 + (B+1)\int_\Omega f\,u^B\,\varphi_1.$$

By Step 1,

$$0 \geq \left(\lambda_1 - (B+1)\lambda\right) \int_\Omega u^{B+1}\,\varphi_1 = (B+1)\int_\Omega f\,u^B\,\varphi_1 > 0$$

which implies $\lambda > \lambda_1/(B+1)$ as we desired.
Step 4. Let λ_0 be a fix number such that $0 \leq \lambda_0 < \frac{\lambda_1}{B+1}$. Then there exists a positive constants C_{λ_0} such that for every $\lambda \leq \lambda_0$ one has $||u||_{L^\infty(\Omega)} \leq C_{\lambda_0}$, for every solution u of (P_λ). The proof is achieved following the same arguments of Lemma 2.2-ii).
Step 5. For every fixed $\lambda < \frac{\lambda_1}{B+1}$ there exists at most one solution u of (P_λ). Indeed, if $u_i \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $i = 1,2$, are solutions of (P_λ), then $v_i := u_i^{B+1}$, $i = 1,2$, solve the sublinear problem

$$\begin{cases} -\Delta v - \lambda(B+1)v = (B+1)\,f\,v^\theta\,, \text{ in } \Omega \\ \qquad 0 \leq v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,, \end{cases}$$

with $\theta := \frac{B}{B+1}$. In particular, we have

$$\int_\Omega \nabla v_1 \nabla\varphi - (B+1)\int_\Omega \lambda v_1\,\varphi = (B+1)\int_\Omega f\,v_1^\theta\,\varphi\,, \tag{2.15}$$

$$\int_\Omega \nabla v_2 \nabla\varphi - (B+1)\int_\Omega \lambda v_2\,\varphi = (B+1)\int_\Omega f\,v_2^\theta\,\varphi\,, \tag{2.16}$$

for every $\varphi \in W_0^{1,2}(\Omega)$. Now, we consider a smooth nondecreasing function Θ such that

$$\Theta(s) = \begin{cases} 0, & \text{if } s \leq 0\,, \\ 1, & \text{if } s \geq 1\,. \end{cases}$$

If for $\varepsilon > 0$ we define

$$\Theta_\varepsilon(s) = \Theta\left(\frac{s}{\varepsilon}\right)\,,\; s \in \mathbb{R}\,,$$

then $v_1\,\Theta_\varepsilon(v_1 - v_2)$ and $v_2\,\Theta_\varepsilon(v_1 - v_2)$ belong to $W_0^{1,2}(\Omega)$ and thus it is possible to take $\varphi = v_2\,\Theta_\varepsilon(v_1 - v_2)$ as a test function in (2.15) and $\varphi = v_1\,\Theta_\varepsilon(v_1 - v_2)$ as a test function in (2.16). Subtracting the deduced identities, we can repeat the arguments of [57, Appendix II, Method II] to get

$$\int_{\{v_1 > v_2\}} f\,v_1\,v_2\,(v_2^{\theta-1} - v_1^{\theta-1}) = 0\,,$$

which implies that

$$v_1 \leq v_2 \quad \text{a.e. } \Omega^* := \{x \in \Omega \,:\, f(x) > 0\}.$$

Hence, we have

$$\begin{aligned} -\Delta v_1 - \lambda(B+1)v_1 &= (B+1)\int_{\Omega^*} f v_1^\theta \\ &\leq (B+1)\int_{\Omega^*} f v_2^\theta = -\Delta v_2 - \lambda(B+1)v_2, \quad x \in \Omega\,, \end{aligned}$$

with $v_1, v_2 \in W_0^{1,2}(\Omega)$. Since $\lambda(B+1) < \lambda_1$, the maximum principle yields $v_1 \le v_2$ in Ω. Similarly, interchanging v_1 and v_2, we prove that $v_1 \ge v_2$ in Ω and we conclude that $v_1 = v_2$. Therefore, $u_1 = u_2$ and the uniqueness of solution when $\lambda < \lambda_1/(B+1)$ has been proved.

Step 6. Existence of a continuum of solutions of (P_λ) *and its properties.* It is clear that the set

$$S = \left\{ (\lambda, u_\lambda) \, : \, \lambda < \frac{\lambda_1}{B+1} \right\},$$

is a continuum (connected and closed) in $\mathbb{R} \times L^\infty(\Omega)$ (since the function $\lambda \to u_\lambda$ is continuous). To conclude our result, we are going to show that $\|u_\lambda\|_{L^\infty(\Omega)}$ blows-up when the parameter λ tends to $\frac{\lambda_1}{B+1}$. In order to do it, we use the Leray-Schauder topological degree. In this sense, we consider the operator

$$K : [0, +\infty) \times L^\infty(\Omega) \longrightarrow L^\infty(\Omega)$$

by defining, for every $\lambda \ge 0$ and for every $w \in L^\infty(\Omega)$, $K(\lambda, w) \equiv K_\lambda(w)$ as the unique solution u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the problem

$$\begin{cases} -\Delta u = B \, \dfrac{|\nabla u|^2}{u} + \lambda w^+ + f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega \, . \end{cases}$$

Observe that the operator K is well defined since the existence is due to [9] and the uniqueness has been proved in Step 5. Using this notation, our problem (P_λ) can be rewritten as a fixed point equation, namely

$$u = K_\lambda(u) \, , \; u \in L^\infty(\Omega).$$

In addition K is compact (see Lemma 2.3 in the appendix) and, by Step 4, for every fixed $\lambda_0 < \frac{\lambda_1}{B+1}$, there is R such that

$$||u||_{L^\infty(\Omega)} < R \, ,$$

for all solution u of (P_λ), with $\lambda \le \lambda_0$.

As a consequence, if B_R denotes the ball of radius R and center 0, the Leray-Schauder topological degree $d(I - K_\lambda, B_R, 0)$ is well defined and it is constant for all $\lambda \in [0, \lambda_0]$ (by the homotopy property).

Furthermore, we easily check that $d(I - K_0, B_R, 0) = 1$ by considering for $t \in [0,1]$ the homotopically equivalent problems

$$\begin{cases} -\Delta u = tB \, \dfrac{|\nabla u|^2}{u} + f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega \, . \end{cases}$$

Therefore,

$$d(I - K_\lambda, B_R, 0) = d(I - K_0, B_R, 0) = 1 \, .$$

By the continuation theorem of Leray-Schauder [2], we obtain the existence of an unbounded continuum of solutions of (P_λ) in $[0, \frac{\lambda_1}{B+1}) \times L^\infty(\Omega)$. Clearly, by the uniqueness of (P_λ), this set is the one

$$S = \left\{ (\lambda, u_\lambda) \,:\, \lambda < \frac{\lambda_1}{B+1} \right\}.$$

To conclude the step and the proof, we observe that $\|u\|_{L^\infty(\Omega)}$ blows-up as λ tends to $\frac{\lambda_1}{B+1}$ since the continuum S is unbounded in $[0, \frac{\lambda_1}{B+1}) \times L^\infty(\Omega)$. $\square$

2.5 Appendix

Lemma 2.3 *If $0 \le \lambda_n$ are convergent to λ and w_n is L^∞-weakly convergent to w, then the sequence of the unique solution u_n of*

$$\begin{cases} -\Delta u_n = B\,\dfrac{|\nabla u_n|^2}{u_n} + \lambda_n w_n^+ + f(x), & \textit{in } \Omega, \\ u_n > 0\,, & \textit{in } \Omega, \\ u_n = 0, & \textit{on } \partial\Omega\,, \end{cases} \tag{2.17}$$

is strongly convergent in $L^\infty(\Omega)$ to the unique solution u of

$$\begin{cases} -\Delta u = B\,\dfrac{|\nabla u|^2}{u} + \lambda w^+ + f(x), & \textit{in } \Omega, \\ u > 0\,, & \textit{in } \Omega, \\ u = 0, & \textit{on } \partial\Omega\,. \end{cases} \tag{2.18}$$

Proof. Firstly, we observe that the sequence $\{\lambda_n\, w_n^+\}$ is bounded in $L^\infty(\Omega)$ and thus there exists a positive constant C such that

$$0 \le |\lambda_n|\, w_n^+ + f \le C + f\,. \tag{2.19}$$

Next, we divide the proof into several steps.
Step 1. The sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Indeed, thanks to Remark 2.4, it is possible to take u_n as a test function in the weak formulation of (2.17) to obtain, using (2.19), that

$$(1-B)\int_\Omega |\nabla u_n|^2 \le \int_\Omega (C+f) u_n\,,$$

which gives us the result after applying Sobolev and Hölder inequalities.

Consequently, there exists a function $u \in W_0^{1,2}(\Omega)$ such that, up to a subsequence, u_n converges to u weakly in $W_0^{1,2}(\Omega)$.
Step 2. The sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Indeed, one can take $G_k(u_n)$ as a test function (with $k \ge 1$) in the weak formulation of (2.17) to obtain, using $|G_k(u_n)| \le |u_n|$ and (2.19), that

$$(1-B)\int_\Omega |\nabla G_k(u_n)|^2 \le \int_{\{u_n > k\}} (C+f)\, G_k(u_n)\,.$$

Since $m > N/2$ and $0 < B < 1$, one can follow the arguments of Ladyzhenskaya - Ural'tseva ([77]) to deduce the result thanks to Step 1 .

As a consequence, the function u given by Step 1 belongs to $L^\infty(\Omega)$.

Step 3. The sequence $\{u_n\}$ *is bounded in* $C^{0,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$. In order to do it, let ξ be a function belonging to $C_c^\infty(\Omega)$ with $0 \leq \xi(x) \leq 1$ for all $x \in \Omega$ and compact support in a ball B_ρ of radious ρ. Given a solution u_n of (2.17), we take $\varphi = \xi^2 G_k(u_n)$ as a test function in the weak formulation of (2.17). Thanks to the smoothness of Ω, we follow the same arguments of Theorem 1.1 of Chapter 4 in [77] to deduce $u_n \in C^{0,\alpha}(\bar{\Omega})$ and $||u_n||_{C^{0,\alpha}(\bar{\Omega})} \leq C(||u_n||_\infty, f, B, \Omega)$. Therefore, using Step 2, we deduce the boundedness of $\{u_n\}$ in $C^{0,\alpha}(\bar{\Omega})$. Since, by Ascoli-Arzela's theorem, the embedding $C^{0,\alpha}(\bar{\Omega}) \hookrightarrow C^0(\bar{\Omega})$ is compact, then the sequence $\{u_n\}$ strongly converges in $L^\infty(\Omega)$ to the function u given by Step 1.

Step 4. The function u given by Step 1 is a solution of (2.18). Folowing the arguments of [9] and using the above a priori estimates, we can pass to the limit in the weak formulation of the problem (2.17) to prove that u is a solution of (2.18). □

Chapter 3

A quasilinear Dirichlet problem with quadratic growth respect to the gradient and L^1 data

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Abstract

For an open, bounded set $\Omega \subset \mathbb{R}^N$, measurable bounded functions $a(x)\,, b(x)$ which are strictly positive and $p, q > 0$, we prove the existence of a weak solution of the quasilinear b.v.p

$$\begin{cases} -\mathrm{div}\,[(a(x) + |u|^q)\,\nabla u(x)] + b(x)\,u\,|u|^{p-1}\,|\nabla u|^2 = f(x)\,, & \text{in } \Omega\,, \\ u(x) = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

The datum f is assumed to be in $L^1(\Omega)$ and does not satisfy any sign assumption.

3.1 Introduction

We study the existence of solutions of a quasilinear Dirichlet problem with quadratic growth respect to the gradient and irregular data. The problem is the following one:

$$\begin{cases} -\mathrm{div}\,[(a(x)+|u|^q)\,\nabla u(x)] + b(x)\,u\,|u|^{p-1}\,|\nabla u|^2 = f(x)\,, & \text{in } \Omega\,, \\ u(x)=0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, $p\,,q>0$, $f\in L^1(\Omega)$ and $a(x)\,,b(x)$ are measurable functions satisfying

$$0<\alpha\le a(x)\le\beta \;\text{ and }\; 0<\mu\le b(x)\le\nu \;\text{ a.e. } x\in\Omega\,. \tag{3.2}$$

The existence of solutions of quasilinear Dirichlet problem having quadratic growth with respect to the gradient has been widely studied. A motivation for the study of these problems relies on the fact that they are strongly related with the study of integral functionals. For example, the differential equation of (3.1) is, at least formally, a generalized version of the Euler-Lagrange equation for a functional of the type

$$J(u)=\frac{1}{2}\int_\Omega (1+|u|^r)|\nabla u|^2-\int_\Omega f\,u\,,\;\; u\in W_0^{1,2}(\Omega)\,,\; r>1\,,$$

when $a(x)=1\,,q=r\,,b(x)=r/2$ and $p=r-1$.

Many papers (for example [23], [24], [36], [82], [83]) have dealt with the Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}\,(A(x,u,\nabla u))+g(x,u)\,|\nabla u|^2=f(x)\,, & \text{in } \Omega\,, \\ u(x)=0\,, & \text{on } \partial\Omega\,, \end{cases}$$

where $A:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function subject to certain structural inequalities and $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the sign condition

$$s\,g(x,s)\ge 0\,,\;\;\forall s\in\mathbb{R}\,. \tag{3.3}$$

In particular, in (3.1), we have that $A(x,u,\nabla u)=(a(x)+|u|^q)\,\nabla u$ satisfies

$$A(x,s,\xi)\,\xi\ge\alpha|\xi|^2 \quad\text{and}\quad |A(x,s,\xi)|\le(\beta+|s|^q)\,|\xi|\,, \tag{3.4}$$

a.e. $x\in\Omega$, for every $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$. Moreover the lower order term

$$g(x,u)\,|\nabla u|^2=b(x)\,u\,|u|^{p-1}\,|\nabla u|^2$$

satisfies (3.3) and

$$g(x,s)\,s\ge\mu|s|^{p+1} \quad\text{and}\quad |g(x,s)|\le\nu|s|^p\,. \tag{3.5}$$

In this context, we refer to [24] where existence results are proved under similar growth assumptions but with sign restriction over the data f. In our case, we do not impose to f to satisfy any sign assumption. Moreover, in contrast with the result of [82], where it is assumed that $A(x, s, .)$ is coercive, monotone and satisfying the growth condition $|A(x, s, \xi)| \le C\ (d(x) + |s| + |\xi|)$ with C a positive constant and $d \in L^2(\Omega)$, we highlight that in our case the required growth of A is more general, handling growths greater than the above linear one. We also refer to the classical paper [36], where $-\mathrm{div}\,(A(x, u, \nabla u))$ is a Leray-Lions operator and $f \in L^1(\Omega)$. In that paper, it is proved existence results of unbounded solutions under the sign condition on the lower order term. We point out here that in [36] the authors firstly observed that, due to the assumption (3.3), the lower order term has a regularizing effect over the solutions even if f belongs only to $L^1(\Omega)$.

In addition, the existence of weak solutions of (3.1) is studied in [23] when $f \ge 0$ and under suitable assumptions on the summability of the data f and on the positive parameters p and q. In particular, it is proved that if $f \in L^1(\Omega)$, $f \ge 0$, $p \ge 2\,q$ and (3.2) holds true, then there exists a positive weak solution $u \in W_0^{1,2}(\Omega)$ of (3.1), in the sense that, $b(x)\, u^p\, |\nabla u|^2 \in L^1(\Omega)$ and

$$\int_\Omega (a(x) + u^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, u^p |\nabla u|^2\, \varphi = \int_\Omega f\, \varphi\,,\ \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

Moreover, in this case, it is also shown that u belongs to $L^{\frac{N(p+2)}{N-2}}(\Omega)$. Similar results about existence and regularity are also given depending on different assumptions on p and q.

The aim of this paper is to improve [23], by obtaining existence results for (3.1) without any sign restriction over f and without any restrictions over the parameters p and q. In order to deduce such a result, we need to give a different notion of solution that seems to be natural in this framework. Specifically, we prove the following theorem.

Theorem 3.1 *If Ω is an open, bounded set of $\mathbb{R}^N$, $p\,, q > 0$, $f \in L^1(\Omega)$ and $a(x)\,, b(x)$ are measurable functions satisfying* (3.2)*, then there exists $u \in W_0^{1,2}(\Omega)$, solution of* (3.1) *in the following sense:*

$$(a(x) + |u|^q)\, |\nabla u| \in L^1(\Omega)\,,\ b(x)\, |u|^p\, |\nabla u|^2 \in L^1(\Omega)\,, \tag{3.6}$$

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, |u|^{p-1} u\, |\nabla u|^2\, \varphi = \int_\Omega f\, \varphi\,,\ \forall \varphi \in W_0^{1,\infty}(\Omega)\,. \tag{3.7}$$

With respect to the previous results, we note that Theorem 3.1 does not impose any restrictions on the parameters p and q and the datum f does not satisfy any sign assumption. However, we remark that we are taking test function in $W_0^{1,\infty}(\Omega)$ instead of $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. We also study the problem (3.1) with some restrictions on the parameters p and q obtaining, in this case, some regularity results.

The proof of Theorem 3.1 consists in finding a solution of (3.1) as limit of a sequence of solutions of approximate problems. Keeping this in mind, we organize the paper in three different parts. Section 2 contains a brief summary about the properties of the approximate problems. In Section 3 we pass to the limit in the approximate problems to obtain a solution of (3.1) using a technique which has been introduced in [47] and then applied in other papers (see [82] and [85]). We point out here that we do not need strong convergence in $W_0^{1,2}(\Omega)$ of the truncations of the approximating solution to pass to the limit and to obtain a solution. Finally, in the last section, we obtain some regularity results under some restrictions on the parameters p and q and on the summability of f.

Remark 3.1 *Notice that all the results proved in this paper deal with problem* (3.1)*. Nevertheless, we stress that the assumptions* (3.3)*,* (3.4) *and* (3.5) *on the functions* A *and* g *are the key to prove existence results, under the same assumptions on* f*.*

Notation. In what follows, we denote by $h^+ = \max\{h, 0\}$, $h^- = -\min\{h, 0\}$.

3.2 Approximate Problems.

Firstly, we fix a function $f \in L^\infty(\Omega)$ and study the existence of a solution v of the problem

$$\begin{cases} -\mathrm{div}\,[(a(x) + |v|^q)\,\nabla v] + b(x)\,v\,|v|^{p-1}\,|\nabla v|^2 = f(x)\,, & \text{in } \Omega\,, \\ v = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.8}$$

Lemma 3.1 *(See [49]) If* $f \in L^\infty(\Omega)$, $p\,, q > 0$ *and* $a(x)\,, b(x)$ *are measurable functions satisfying* (3.2)*, then there exists a solution* v *of* (3.8)*, in the sense that* v *belongs to* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *and*

$$\int_\Omega (a(x) + |v|^q)\,\nabla v \nabla\varphi + \int_\Omega b(x)\,v\,|v|^{p-1}\,|\nabla v|^2\,\varphi = \int_\Omega f\,\varphi\,,$$

for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

We include here the proof by convenience of the reader
Proof. The idea of the proof relies on a standard approximation procedure. Let us define, for $n \in \mathbb{N}$, the operator

$$A_n(v) = -\mathrm{div}\,[(a(x) + |T_n(v)|^q)\,\nabla v] + b(x)\,T_n(v)\,|T_n(v)|^{p-1}\,|\nabla v|^2$$

where $T_n(s) := \max\{\min\{n, s\}, -n\}$, and let us consider the following sequence of approximating problems

$$\begin{cases} A_n(v_n) = f(x)\,, & \text{in } \Omega\,, \\ v_n = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

From the results of [49], it follows the existence of a solution $v_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ in the sense that v_n satisfies

$$\int_\Omega (a(x)+|T_n(v_n)|^q)\,\nabla v_n \nabla\varphi + \int_\Omega b(x)\,T_n(v_n)\,|T_n(v_n)|^{p-1}\,|\nabla v_n|^2\,\varphi = \int_\Omega f\,\varphi\,,$$

for every $\varphi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$. The use of $G_k(v_n)$ as a test function implies, using (3.2) and dropping the positive terms, that

$$\alpha\int_\Omega |G_k(v_n)|^2 \le \int_\Omega |f|\,|G_k(v_n)|\,,$$

and then, applying [87, Lemma 4.1], the sequence $\{v_n\}$ is bounded in $L^\infty(\Omega)$. If we suppose that $\{v_n\}$ is bounded by r in $L^\infty(\Omega)$, we deduce that the function $v := v_{r+1}$ belonging to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ is a solution of (3.8). □

Now, we can consider for $n\in\mathbb{N}$ the approximate problems

$$\begin{cases} -\mathrm{div}\,[(a(x)+|u_n|^q)\,\nabla u_n] + b(x)\,u_n\,|u_n|^{p-1}\,|\nabla u_n|^2 = f_n(x)\,, & \text{in } \Omega\,,\\ u_n = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{3.9}$$

where $f_n \in L^\infty(\Omega)$, $|f_n| \le |f|$ and $||f_n - f||_{L^1(\Omega)} \to 0$, (for example, we can take $f_n = T_n(f)$).

By Lemma 3.1, it follows the existence of a solution $u_n \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ of (3.9) and we study its properties in the following lemma.

Lemma 3.2 *Let $a(x)\,,b(x)$ be measurable functions satisfying* (3.2) *and $p,q>0$. If $\{u_n\}$ is a sequence of solutions u_n of* (3.9) *given by Lemma 3.1, then*

1. *There exists $R>0$ such that $||u_n||_{W_0^{1,2}(\Omega)} \le R$, for all $n\in\mathbb{N}$.*
2. *There exists $M>0$ such that $\int_\Omega b(x)|u_n|^p|\nabla u_n|^2 \le M$, for all $n\in\mathbb{N}$.*
3. *The solutions u_n satisfy*

$$\int_\Omega \frac{(a(x)+|u_n|^q)\,|\nabla T_k(u_n)|^2}{k} \le \frac{1}{\sqrt{k}}\,||f||_{L^1(\Omega)} + \int_{\{|u_n|>\sqrt{k}\}} |f|\,,\quad \forall k>0\,.$$

Remark 3.2 *Let $\{u_n\}$ be a sequence of solutions u_n of* (3.9) *given by Lemma 3.1. As a consequence of 1. of Lemma 3.2, there exists $u\in W_0^{1,2}(\Omega)$ such that, up to a subsequence, u_n converges weakly to u in $W_0^{1,2}(\Omega)$.*

Proof of Lemma 3.2. 1.- We easily check that the use of $T_k(u_n)$ as a test function implies that

$$\alpha\int_{\{|u_n|<k\}} |\nabla u_n|^2 + k\int_{\{|u_n|\ge k\}} b(x)\,|u_n|^p|\nabla u_n|^2 \le k\,||f||_{L^1(\Omega)}\,,$$

and thus

$$\int_\Omega |\nabla u_n|^2 = \int_{\{|u_n|\le k\}} |\nabla u_n|^2 + \int_{\{|u_n|>k\}} |\nabla u_n|^2 \le \left(\frac{k}{\alpha} + \frac{1}{k^p\,\mu}\right) ||f||_{L^1(\Omega)}\,,$$

where $\alpha\,,\mu$ are given by (3.2). In consequence 1. is proved.
2.- In the same manner, using $T_1(u_n)$ as a test function, we can see that

$$\int_{\{|u_n|\ge 1\}} b(x)\,|u_n|^p|\nabla u_n|^2 \le ||f||_{L^1(\Omega)}\,,$$

and then, using 1., we conclude that

$$\int_\Omega b(x)\,|u_n|^p|\nabla u_n|^2 \le M\,,$$

which is 2.
3.- We fix $k > 0$ and observe that

$$\begin{aligned}\int_\Omega |f_n||T_k(u_n)| &\le \sqrt{k}\int_{\{|u_n|\le\sqrt{k}\}} |f_n| + k\int_{\{|u_n|>\sqrt{k}\}} |f_n| \\ &\le \sqrt{k}\,||f||_{L^1(\Omega)} + k\int_{\{|u_n|>\sqrt{k}\}} |f|\,.\end{aligned}$$

Then, the use of $\frac{T_k(u_n)}{k}$ as a test function in (3.9) implies, dropping the positive integral coming from the lower order term and using the above inequality, that

$$\int_\Omega \frac{(a(x)+|u_n|^q)\,|\nabla T_k(u_n)|^2}{k} \le \int_\Omega \frac{|f_n|\,|T_k(u_n)|}{k} \le \frac{1}{\sqrt{k}}||f||_{L^1(\Omega)} + \int_{\{|u_n|>\sqrt{k}\}} |f|\,,$$

which completes the proof. □

Lemma 3.3 *Let $a(x)\,,b(x)$ be measurable functions satisfying* (3.2) *and $p,q > 0$. If $\{u_n\}$ is a sequence of solutions of* (3.9) *given by Lemma 3.1 and u is given by Remark 3.2, then*

1. *There exists a subsequence (not relabelled) such that $\{\nabla u_n(x)\}$ converges to $\nabla u(x)$ a.e. in Ω.*

2. *For a.e. $k > 0$, u satisfies the following inequality*

$$\int_\Omega \frac{(a(x)+|u|^q)\,|\nabla T_k(u)|^2}{k} \le \frac{1}{\sqrt{k}}||f||_{L^1(\Omega)} + \int_{\{|u|>\sqrt{k}\}} |f|\,.$$

3. *u is such that $b(x)\,|u|^p\,|\nabla u|^2 \in L^1(\Omega)$.*

4. *u is such that $(a(x)+|u|^q)\,|\nabla u| \in L^1(\Omega)$.*

Proof. 1.- We prove the a.e. convergence of the sequence $\{\nabla u_n(x)\}$ in the spirit of [23, Lemma 2.6] (see also [22] and [45]). We fix $h, k > 0$ and take $T_h(u_n - T_k(u))$ as a test function in (3.9) to obtain, thanks to 2. of Lemma 3.2, that

$$\alpha \limsup_{n\to\infty} \int_\Omega |\nabla T_h(u_n - T_k(u))|^2 \le h\left(M + ||f||_{L^1(\Omega)}\right) . \tag{3.10}$$

Now, we fix $r < 2$. By 1. of Lemma 3.2, we have

$$\begin{aligned}\int_\Omega |\nabla(u_n - u)|^r = & \int\limits_{\{|u_n-u|\le h,\, |u|\le k\}} |\nabla(u_n - u)|^r + \int\limits_{\{|u_n-u|\le h,\, |u|>k\}} |\nabla(u_n - u)|^r \\ & + \int\limits_{\{|u_n-u|>h\}} |\nabla(u_n - u)|^r \\ & \le \int_\Omega |\nabla T_h(u_n - T_k(u))|^r + 2^{r-1}R^r \ \mathrm{meas}\{|u| > k\}^{1-\frac{r}{2}} \\ & + 2^{r-1}R^r \ \mathrm{meas}\{|u_n - u| > h\}^{1-\frac{r}{2}} .\end{aligned}$$

Since u_n converges to u in measure, using Hölder's inequality and (3.10), we deduce for every $h > 0$ and $k > 0$, that

$$\limsup_{n\to\infty} \int_\Omega |\nabla(u_n - u)|^r \le h^{\frac{r}{2}} \left(\frac{M + ||f||_{L^1(\Omega)}}{\alpha}\right)^{\frac{r}{2}} |\Omega|^{1-\frac{r}{2}} + 2^{r-1}R^r \ \mathrm{meas}\{|u| > k\}^{1-\frac{r}{2}} .$$

That is, letting $h \to 0$ and then $k \to +\infty$, that

$$\int_\Omega |\nabla(u_n - u)|^r \longrightarrow 0 .$$

In consequence, we conclude that (up to a subsequence) $\nabla u_n(x)$ converges a.e. to $\nabla u(x)$ in Ω and 1. is proved.

2.- We observe that

$$\lim_{n\to\infty} \int_{\{|u_n|>\sqrt{k}\}} |f| = \int_{\{|u|>\sqrt{k}\}} |f| , \quad \text{a.e. } k > 0 .$$

Then, using 3. of Lemma 3.2, the a.e. convergence of the sequence $\{\nabla u_n\}$ given by the above item and Fatou's Lemma, we deduce

$$\int_\Omega \frac{(a(x) + |u|^q)\,|\nabla T_k(u)|^2}{k} \le \frac{1}{\sqrt{k}}||f||_{L^1(\Omega)} + \int_{\{|u|>\sqrt{k}\}} |f| , \quad \text{a.e. } k > 0 ,$$

which gives us the result.

3.- The fact that ∇u_n converges to ∇u a.e. in Ω (thanks to 1.), 2. of Lemma 3.2 and Fatou's Lemma imply

$$\int_\Omega b(x)|u|^p|\nabla u|^2 \le M ,$$

which is precisely the assertion of 3.

4.- In order to prove that $(a(x)+|u|^q)\,|\nabla u| \in L^1(\Omega)$, we claim that there exist positive constants c_0 and $\widetilde{c}_0$ such that

$$\frac{1}{\widetilde{c}_0}\,(a(x)+|u_n|^q) \le (a(x)+|u_n|)^q \le c_0\,(a(x)+|u_n|^q)\,. \tag{3.11}$$

Indeed, by assumption (3.2)

$$\frac{(\alpha+|s|)^q}{\beta+|s|^q} \le \frac{(a(x)+|s|)^q}{a(x)+|s|^q} \le \frac{(\beta+|s|)^q}{\alpha+|s|^q}\,,\ \forall s\in\mathbb{R}\,,$$

which easily gives (3.11). Next, we divide the proof in three steps:

Step 1: For every $\lambda>1$, $(a(x)+|u|)^{q-\lambda}\,|\nabla u|^2 \in L^1(\Omega)$. Indeed, given $\lambda>1$, we take $\left(1-\frac{\alpha}{(\alpha+|u_n|)^{\lambda-1}}\right)\operatorname{sign}(u_n)$ as a test function in (3.9) to obtain, dropping the positive term $\int_\Omega b(x)|u_n|^p|\nabla u_n|^2\left(1-\frac{\alpha}{(\alpha+|u_n|)^{\lambda-1}}\right)$, that

$$\alpha(\lambda-1)\int_\Omega \frac{(a(x)+|u_n|^q)\ |\nabla u_n|^2}{(\alpha+|u_n|)^\lambda} \le \int_\Omega |f|\,.$$

Hence, using (3.11), it follows that

$$\int_\Omega \frac{(a(x)+|u_n|)^q\ |\nabla u_n|^2}{(\alpha+|u_n|)^\lambda} \le \frac{c_0}{\alpha(\lambda-1)}\,||f||_{L^1(\Omega)}\,,$$

which gives us, applying the Fatou Lemma, the proof of Step 1.

Step 2: $u\in L^{\frac{2^*(q-\lambda+2)}{2}}(\Omega)$, for every $1<\lambda\le q+2-2/2^*$. By Step 1., we have

$$\begin{aligned}\left(\frac{2}{q-\lambda+2}\right)^2\int_\Omega\left|\nabla\left((\alpha+|u|)^{\frac{q-\lambda+2}{2}}-\alpha^{\frac{q-\lambda+2}{2}}\right)\right|^2 &= \int_\Omega(\alpha+|u|)^{q-\lambda}\ |\nabla u_n|^2\\ &\le \frac{c_0}{\alpha(\lambda-1)}\,||f||_{L^1(\Omega)}\end{aligned}$$

and, by Sobolev inequality, we deduce that

$$\left(\frac{2}{q-\lambda+2}\right)^2\left(\int_\Omega\left|(\alpha+|u|)^{\frac{q-\lambda+2}{2}}-\alpha^{\frac{q-\lambda+2}{2}}\right|^{2^*}\right)^{2/2^*} \le \frac{S^2\,c_0}{\alpha(\lambda-1)}\,||f||_{L^1(\Omega)}\,,$$

where S is the Sobolev constant. This proves the Step 2.

Step 3: $(a(x) + |u|^q)\,|\nabla u|$ *belongs to* $\in L^1(\Omega)$. From (3.11), using Hölder's inequality with exponent 2, Step 1. and (3.2), it follows that

$$\begin{aligned}\int_\Omega (a(x) + |u|^q)\,|\nabla u| &\le \widetilde{c}_0 \int_\Omega (a(x) + |u|)^q\,|\nabla u| \\ &= \widetilde{c}_0 \int_\Omega (a(x) + |u|)^{\frac{q-\lambda}{2}}\,|\nabla u|\,(a(x) + |u|)^{\frac{q+\lambda}{2}} \\ &\le \widetilde{c}_0 \left(\frac{S^2\,c_0}{\alpha(\lambda-1)}\,||f||_{L^1(\Omega)}\right)^{\frac12} \left(\int_\Omega (a(x) + |u|)^{q+\lambda}\right)^{\frac12} \\ &\le \widetilde{c}_0 \left(\frac{S^2\,c_0}{\alpha(\lambda-1)}\,||f||_{L^1(\Omega)}\right)^{\frac12} \left(\int_\Omega (\beta + |u|)^{q+\lambda}\right)^{\frac12}.\end{aligned}$$

Since $q + 1 < \frac{2^*(q+1)}{2}$, we can take $\lambda > 1$ such that $q + \lambda \le \dfrac{2^*\,(q - \lambda + 2)}{2}$ and then, by Step 2., $|u|^{q+\lambda} \in L^1(\Omega)$. This completes the proof. □

3.3 Proof of Theorem 3.1.

Our aim is to prove that the weak limit u (given by Remark 3.2) of u_n is a solution of the problem (3.1). Recall that, $b(x)\,|u|^p\,|\nabla u|^2 \in L^1(\Omega)$ and $(a(x) + |u|^q)\;|\nabla u| \in L^1(\Omega)$ by 3. and 4. of Lemma 3.3 respectively, and then (3.6) holds. To conclude the proof, we only have to show that the identity (3.7) holds. To this aim, let B be a function in $C^1(\mathbb{R})$ such that

$$B(s) = \begin{cases} 1, & \text{if } |s| \le \frac12\,, \\ \in [0,1], & \text{if } \frac12 \le |s| \le 1\,, \\ 0, & \text{if } |s| \ge 1\,, \end{cases}$$

and define the function

$$H(s) = \frac{s\,|s|^p}{p+1}\,, \quad \forall s \in \mathbb{R}\,.$$

The proof of (3.7) will be divided into three steps.

Step 1. For almost every positive k, u *satisfies the following inequality:*

$$\begin{aligned}&\int_\Omega \left\{(a(x) + |u|^q)\,\nabla u \left[\nabla\psi - \frac{\nu}{\alpha}\nabla u^-\,|u^-|^p\psi\right] + b(x)\,|\nabla u|^2\,|u|^{p-1}\,u\psi\right\} e^{\frac{-\nu H(u^-)}{\alpha}}\,B\left(\frac{u}{k}\right) \\ &\le \int_\Omega f\,\psi\,e^{\frac{-\nu H(u^-)}{\alpha}}\,B\left(\frac{u}{k}\right) + ||B'||_{L^\infty(\Omega)}\,||\psi||_{L^\infty(\Omega)} \left(\frac{||f||_{L^1(\Omega)}}{\sqrt{k}} + \int_{\{|u|>\sqrt{k}\}} |f|\right) \qquad (3.12)\end{aligned}$$

for all $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *with* $\psi \ge 0$.

In order to prove it, given $k > 0$, we fix $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $\psi \geq 0$ and take

$$\phi = \psi\, e^{\dfrac{-\nu H(u_n^-)}{\alpha}} B\left(\frac{u_n}{k}\right)$$

as a test function in (3.9) to deduce, using 3. of Lemma 3.2, that

$$\int_\Omega \Big\{\big(a(x)+|u_n|^q\big)\nabla u_n\Big[\nabla\psi - \frac{\nu}{\alpha}\nabla u_n^- |u_n^-|^p\psi\Big] + b(x)|\nabla u_n|^2|u_n|^{p-1}u_n\psi\Big\} e^{\frac{-\nu H(u_n^-)}{\alpha}} B\left(\frac{u_n}{k}\right)$$

$$\leq \int_\Omega f_n\psi e^{\frac{-\nu H(u_n^-)}{\alpha}} B\left(\frac{u_n}{k}\right) + ||B'||_{L^\infty(\Omega)}||\psi||_{L^\infty(\Omega)}\left(\frac{||f||_{L^1(\Omega)}}{\sqrt{k}} + \int_{\{|u_n|>\sqrt{k}\}} |f|\right). \quad (3.13)$$

In order to pass to the limit in (3.13) as n tends to ∞, we first observe that, by Remark 3.2

$$\nabla u_n \longrightarrow \nabla u \ \text{ weakly in } (L^2(\Omega))^N .$$

Moreover, since $B(\frac{u_n}{k})$ is equal to zero if $|u_n| > k$, we have

$$(a(x) + |u_n|^q)\ \nabla\psi\, e^{\frac{-\nu H(u_n^-)}{\alpha}} B\left(\frac{u_n}{k}\right) \longrightarrow (a(x) + |u|^q)\ \nabla\psi\, e^{\frac{-\nu H(u^-)}{\alpha}} B\left(\frac{u}{k}\right),$$

strongly in $(L^2(\Omega))^N$, and then

$$\lim_{n\to\infty}\int_\Omega (a(x) + |u_n|^q)\ \nabla u_n\nabla\psi\, e^{\frac{-\nu H(u_n^-)}{\alpha}} B\left(\frac{u_n}{k}\right)$$
$$= \int_\Omega (a(x) + |u|^q)\ \nabla u\nabla\psi\, e^{\frac{-\nu H(u^-)}{\alpha}} B\left(\frac{u}{k}\right).$$

Now, we study the convergence of the second and third term of the left hand side of (3.13). Firstly, we observe that, by (3.2), $\dfrac{\nu}{\alpha}\, a(x) \geq b(x)$ and thus

$$b(x)\,|\nabla u_n|^2\,|u_n|^{p-1}\,u_n - \frac{\nu}{\alpha}\,(a(x) + |u_n|^q)\ \nabla u_n\nabla u_n^-\,|u_n^-|^p =$$
$$b(x)\,|\nabla u_n|^2\,|u_n|^{p-1}\,u_n\,\chi_{\{u_n\geq 0\}} +$$
$$\left(\frac{\nu}{\alpha}\,(a(x) + |u_n|^q)\ |\nabla u_n|^2\,|u_n|^p - b(x)|\nabla u_n|^2\,|u_n|^p\right)\chi_{\{u_n<0\}} \geq 0\,.$$

As a consequence, recalling that u_n and ∇u_n converge to u and ∇u a.e. in Ω respectively, and applying Fatou's Lemma, we deduce that

$$\liminf_{n\to\infty}\left\{\int_\Omega \Big[-\frac{\nu}{\alpha}\,(a(x) + |u_n|^q)\ \nabla u_n\nabla u_n^-\,|u_n^-|^p\right.$$
$$\left. + b(x)\,|\nabla u_n|^2\,|u_n|^{p-1}\,u_n\Big]\,\psi\, e^{\frac{-\nu H(u_n^-)}{\alpha}} B\left(\frac{u_n}{k}\right)\right\} \geq$$

$$\geq \int_\Omega \Big(-\frac{\nu}{\alpha}\,(a(x)+|u|^q)\ \nabla u\nabla u^-\,|u^-|^p + b(x)\,|\nabla u|^2\,|u|^{p-1}\,u\Big)\ \psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\ B\left(\frac{u}{k}\right).$$

On the other hand,

$$\lim_{n\to\infty}\int_\Omega f_n\,\psi\, e^{\frac{-\nu H(u_n^-)}{\alpha}}\ B\left(\frac{u_n}{k}\right) = \int_\Omega f\,\psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\ B\left(\frac{u}{k}\right),$$

$$\lim_{n\to\infty}\int_{\{|u_n|>\sqrt{k}\}} |f| = \int_{\{|u|>\sqrt{k}\}} |f|\,,\quad \text{a.e. } k>0\,.$$

Finally, passing to the inferior limit in (3.13) as n goes to infinity, we conclude for almost every positive k, that

$$\int_\Omega \Big\{\ (a(x)+|u|^q)\,\nabla u\big[\nabla\psi - \frac{\nu}{\alpha}\nabla u^-\,|u^-|^p\,\psi\big] + b(x)\,|\nabla u|^2\,|u|^{p-1}\,u\,\psi\Big\} e^{\frac{-\nu H(u^-)}{\alpha}}\ B\left(\frac{u}{k}\right)$$
$$\leq \int_\Omega f\,\psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\ B\left(\frac{u}{k}\right) + ||B'||_{L^\infty(\Omega)}\,||\psi||_{L^\infty(\Omega)}\left(\frac{||f||_{L^1(\Omega)}}{\sqrt{k}} + \int_{\{|u|>\sqrt{k}\}}|f|\right),$$

for all $\psi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $\psi\geq 0$ which gives us Step 1.

Step 2. u satisfies the following inequality :

$$\int_\Omega (a(x)+|u|^q)\,\nabla u\nabla\varphi^+ + \int_\Omega b(x)\,|u|^{p-1}\,u|\nabla u|^2\varphi^+ \leq \int_\Omega f\,\varphi^+\,,\ \forall\varphi\in W_0^{1,\infty}(\Omega)\,.$$

The idea is to take a particular function ψ in (3.12) and pass to the limit as k tends to infinity. Keeping this in mind, we fix $\varphi\in W_0^{1,\infty}(\Omega)$ and define $\sigma(k)$ such that

$$e^{\frac{\nu\, H(\sigma(k))}{\alpha}} = \frac{1}{\sqrt{\frac{1}{\sqrt{k}}\,||f||_{L^1(\Omega)} + \int_{\{|u|>\sqrt{k}\}}|f|}}\,. \tag{3.14}$$

Observe that $\sigma(k)\longrightarrow +\infty$, as k tends to $+\infty$. Next, we choose

$$\psi = e^{\frac{\nu H(u^-)}{\alpha}}\ B\left(\frac{u}{\sigma(k)}\right)\ \varphi^+$$

in (3.12) to deduce that

$$\int_\Omega \big[(a(x)+|u|^q)\,\nabla u\nabla\varphi^+ + b(x)\,|u|^{p-1}\,u|\nabla u|^2\varphi^+\big]\ B\left(\frac{u}{k}\right)\ B\left(\frac{u}{\sigma(k)}\right)$$
$$\leq \int_\Omega f\,\varphi^+\ B\left(\frac{u}{k}\right)\ B\left(\frac{u}{\sigma(k)}\right) + ||B'||_{L^\infty(\Omega)}\,||\varphi||_{L^\infty(\Omega)}\sqrt{\frac{1}{\sqrt{k}}\,||f||_{L^1(\Omega)} + \int_{\{|u|>\sqrt{k}\}}|f|}$$
$$+\ \frac{||B'||_{L^\infty(\Omega)}\,||\varphi||_{L^\infty(\Omega)}}{\sigma(k)}\int_\Omega (a(x)+|u|^q)\ |\nabla T_{\sigma(k)}(u)|^2\,.$$

Finally, using 2. of Lemma 3.3 in the last term of the above inequality, we deduce, for almost every $k > 0$, that

$$\begin{aligned}
&\int_\Omega \left[(a(x)+|u|^q)\,\nabla u\nabla\varphi^+ + b(x)\,|u|^{p-1}\,u|\nabla u|^2\varphi^+\right] B\left(\frac{u}{k}\right)\, B\left(\frac{u}{\sigma(k)}\right)\\
&\le \int_\Omega f\,\varphi^+\, B\left(\frac{u}{k}\right)\, B\left(\frac{u}{\sigma(k)}\right) + ||B'||_{L^\infty(\Omega)}\,||\varphi||_{L^\infty(\Omega)}\sqrt{\frac{1}{\sqrt{k}}\,||f||_{L^1(\Omega)} + \int_{\{|u|>\sqrt{k}\}}|f|}\\
&\qquad + ||B'||_{L^\infty(\Omega)}\,||\varphi||_{L^\infty(\Omega)}\left(\frac{||f||_{L^1(\Omega)}}{\sqrt{\sigma(k)}} + \int_{\left\{|u|>\sqrt{\sigma(k)}\right\}}|f|\right). \qquad (3.15)
\end{aligned}$$

In order to pass to the limit as k tends to infinity in the last inequality, we recall that $b(x)\,|u|^p\,|\nabla u|^2 \in L^1(\Omega)$ and $(a(x)+|u|^q)\;|\nabla u| \in L^1(\Omega)$ by 3. and 4. of Lemma 3.3. Moreover,

$$\lim_{k\to\infty} B\left(\frac{u}{k}\right)\, B\left(\frac{u}{\sigma(k)}\right) = 1\,,$$

$$\lim_{j\to\infty}\left(\frac{1}{\sqrt{j}}\,||f||_{L^1(\Omega)} + \int_{\left\{|u|>\sqrt{j}\right\}}|f|\right) = 0\,.$$

Consequently, passing to the limit ($k\to+\infty$) in (3.15), we conclude that

$$\int_\Omega (a(x)+|u|^q)\,\nabla u\nabla\varphi^+ + \int_\Omega b(x)\,|u|^{p-1}\,u|\nabla u|^2\varphi^+ \le \int_\Omega f\,\varphi^+\,,\ \forall\varphi\in W_0^{1,\infty}(\Omega)$$

and Step 2. is proved.

Step 3. To obtain the reverse inequality, we repeat Step 1. and Step 2. in the following way: firstly, we take

$$\phi = \psi\, e^{\frac{-\nu H(u_n^+)}{\alpha}}\, B\left(\frac{u_n}{k}\right),$$

with $\psi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $\psi\le 0$ as a test function and, following the same ideas of Step 1., we obtain for almost every positive k that

$$\begin{aligned}
&\int_\Omega \left\{\,(a(x)+|u|^q)\,\nabla u\left[\nabla\psi - \frac{\nu}{\alpha}\nabla u^-\,|u^+|^p\,\psi\right] + b(x)\,|\nabla u|^2\,|u|^{p-1}\,u\,\psi\right\} e^{\frac{-\nu H(u^+)}{\alpha}}\, B\left(\frac{u}{k}\right)\\
&\quad\le \int_\Omega f\,\psi\, e^{\frac{-\nu H(u^+)}{\alpha}}\, B\left(\frac{u}{k}\right) + ||B'||_{L^\infty(\Omega)}\,||\psi||_{L^\infty(\Omega)}\left(\frac{||f||_{L^1(\Omega)}}{\sqrt{k}} + \int_{\{|u|>\sqrt{k}\}}|f|\right).
\end{aligned}$$

Next, we choose

$$\psi = e^{\frac{\nu H(u^+)}{\alpha}}\, B\left(\frac{u}{\sigma(k)}\right)\,(-\varphi^-)\,,$$

where $\sigma(k)$ is given by (3.14) and with $\varphi \in W_0^{1,\infty}(\Omega)$. Applying the same argument of Step 2. we deduce that

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla(-\varphi^-) + \int_\Omega b(x)\, |u|^{p-1}\, u|\nabla u|^2(-\varphi^-) \le \int_\Omega f\,(-\varphi^-)\,,$$

for all $\varphi \in W_0^{1,\infty}(\Omega)$.

As a consequence, summarizing Step 2. and Step 3., we have

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, |u|^{p-1}\, u|\nabla u|^2\varphi \le \int_\Omega f\,\varphi\,, \quad \forall \varphi \in W_0^{1,\infty}(\Omega)\,.$$

Interchanging φ and $-\varphi$ we conclude that

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, |u|^{p-1}\, u|\nabla u|^2\varphi = \int_\Omega f\,\varphi\,, \quad \forall \varphi \in W_0^{1,\infty}(\Omega)\,,$$

which completes the proof of Theorem 3.1. □

3.4 Restrictions on the parameters and consequences

In this section we study the regularity of the solutions of (3.1) under some restriction on the positive parameters p and q. Specifically, we prove the following:

Theorem 3.2 *Let $a(x)$ and $b(x)$ be measurable functions satisfying (3.2), $p\,,q > 0$ and $f \in L^m(\Omega)$ with $1 \le m \le \frac{N}{2}$. If $u \in W_0^{1,2}(\Omega)$ is given by Theorem 3.1, then*

(A) *If $m = 1$, $p \ge 2q$, then u belongs to $L^{(p+2)\frac{N}{N-2}}(\Omega)$;*

(B) *If $\frac{2(q+1)N}{2N+p(N-2)+4q} \le m \le \frac{N}{2}$, $2q \ge p \ge q-1$, then u belongs to $L^{(p+2)m^{**}}(\Omega)$;*

(C) *If $\frac{2N}{N+2} \le m \le \frac{N}{2}$, $q \ge 1$, $2p \ge q-1 \ge p$, then u belongs to $L^{(q+1)m^{**}}(\Omega)$;*

Moreover, if (A), (B) or (C) holds true, then $\big(a(x) + |u|^q\big)\nabla u \in (L^2(\Omega))^N$ and u satisfies

$$\int_\Omega \big(a(x) + |u|^q\big)\nabla u \nabla \varphi + \int_\Omega b(x)\, u\, |u|^{p-1}|\nabla u|^2\varphi = \int_\Omega f\varphi\,,$$

for every $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Remark 3.3 *In contrast with the results of [23, Theorem 2.2]), we remark that we have obtained the same regularity results in spite of the fact that f does not satisfy any sign assumption.*

Proof of Theorem 3.2. As the arguments used are the same that in [23, Lemma 2.9], we only give a sketch of the proof. The key point is to prove an a priori estimate on the sequence u_n of solutions of (3.9) given by Lemma 3.1 (since the compactness has been proved in Theorem 3.1). Let $\{u_n\}$ be a sequence of solutions of (3.9).
(A).- Using 2. of Lemma 3.2 and Sobolev inequality, we deduce immediately that the sequence $\{u_n\}$ is bounded in $L^{(p+2)\frac{N}{N-2}}(\Omega)$. As a consequence u belongs to $L^{(p+2)\frac{N}{N-2}}(\Omega)$ and case (A) is obtained.

Moreover, if (A) holds true, the fact that $2q \le p$ implies

$$\begin{aligned}\int_\Omega \big(a(x)+|u_n|^q\big)^2|\nabla u_n|^2 &\le \int\limits_{\{1\le|u_n|\}} 2\big(\beta^2+|u_n|^p\big)|\nabla u_n|^2 + \int\limits_{\{|u_n|<1\}} 2\big(\beta^2+1\big)|\nabla u_n|^2 \\ &= \int\limits_{\{1\le|u_n|\}} 2\beta^2|\nabla u_n|^2 + \int\limits_{\{|u_n|<1\}} 2\big(\beta^2+1\big)|\nabla u_n|^2 + \int\limits_{\{1\le|u_n|\}} 2|u_n|^p|\nabla u_n|^2 \\ &\le 2(\beta^2+1)\int_\Omega |\nabla u_n|^2 + 2\int_\Omega |u_n|^p|\nabla u_n|^2 .\end{aligned}$$

Therefore, using Lemma 3.2 and (3.2), we obtain

$$\begin{aligned}\int_\Omega \big(a(x)+|u_n|^q\big)^2|\nabla u_n|^2 &\le 2(\beta^2+1)\int_\Omega |\nabla u_n|^2 + 2\int_\Omega |u_n|^p|\nabla u_n|^2 \\ &\le 2(\beta^2+1)R + 2\frac{M}{\mu}\end{aligned}$$

and, taking into account that $\big(a(x)+|u_n|^q\big)\nabla u_n \to \big(a(x)+|u|^q\big)\nabla u$ a.e. in Ω, we deduce that $\big(a(x)+|u|^q\big)\nabla u \in (L^2(\Omega))^N$.

(B).- We define $r = \frac{(p+2)m^{**}}{2^*}$ and observe that $r2^* = (2r-2-p)m' = (p+2)m^{**}$ and that $2r-2-p \ge 0$. Now, we fix a positive number ϵ and use

$$[(\epsilon+|u_n|)^{2r-2-p} - \epsilon^{2r-2-p}]\text{sign}\,(u_n)$$

as a test function in (3.9) to obtain, after several steps, that the sequence $\{u_n\}$ is bounded in $L^{(p+2)m^{**}}(\Omega)$. As a consequence, we conclude that u belongs to $L^{(p+2)m^{**}}(\Omega)$ and case (B) is obtained.

Moreover, the use of $[(\epsilon+|u_n|)^{2r-2-p} - \epsilon^{2r-2-p}]\text{sign}\,(u_n)$ as a test function in (3.9) also implies that the sequence $\displaystyle\int_\Omega u_n^{2(r-1)}|\nabla u_n|^2$ is bounded. In addition, if case (B) holds, we also have that $q \le r-1$ (since $m \ge \frac{2(q+1)N}{2N+p(N-2)+4q}$) and then

$$\begin{aligned}\int_\Omega \big(a(x)+|u_n|^q\big)^2|\nabla u_n|^2 &\le \int\limits_{\{1\le|u_n|\}} 2\big(\beta^2+|u_n|^{2q}\big)|\nabla u_n|^2 + \int\limits_{\{|u_n|<1\}} 2\big(\beta^2+1\big)|\nabla u_n|^2\\ &\le 2(\beta^2+1)\int_\Omega |\nabla u_n|^2 + 2\int_{\{|u_n|\ge 1\}} |u_n|^{2q}|\nabla u_n|^2\\ &\le 2(\beta^2+1)\int_\Omega |\nabla u_n|^2 + 2\int_\Omega |u_n|^{2(r-1)}|\nabla u_n|^2\,,\end{aligned}$$

which implies (since $\big(a(x)+|u_n|^q\big)\nabla u_n \to \big(a(x)+|u|^q\big)\nabla u$ a.e. in Ω) again that $\big(a(x)+|u|^q\big)\nabla u \in (L^2(\Omega))^N$.

(C).- We define $r=\frac{(q+1)m^{**}}{2^*}$ and note that $r2^*=(2r-1-q)m'=(q+1)m^{**}$ and that $2r-1-q\ge 0$. Similarly to the above case, we fix a positive number ϵ and use

$$[(\epsilon+|u_n|)^{2r-1-q}-\epsilon^{2r-1-q}]\text{sign}\,(u_n)$$

as test function in (3.9) to obtain that the sequence $\{u_n\}$ is bounded in $L^{(q+1)m^{**}}(\Omega)$. As a consequence, u belongs to $L^{(q+1)m^{**}}(\Omega)$ which proves case (C).

Moreover, the use of $[(\epsilon+|u_n|)^{2r-1-q}-\epsilon^{2r-1-q}]\text{sign}\,(u_n)$ as test function in (3.9) also implies that the sequence $\int_\Omega |u_n|^{2\left(\frac{(q+1)m^{**}}{2^*}-1\right)}|\nabla u_n|^2$ is bounded. In addition, if we assume case (C), we also have that $q\le \frac{(q+1)m^{**}}{2*}-1$ (since $\frac{2N}{N+2}\le m$) and then

$$\begin{aligned}\int_\Omega \big(a(x)+|u_n|^q\big)^2|\nabla u_n|^2 &\le \int\limits_{\{1\le|u_n|\}} 2\big(\beta^2+|u_n|^{2q}\big)|\nabla u_n|^2 + \int\limits_{\{|u_n|<1\}} 2\big(\beta^2+1\big)|\nabla u_n|^2\\ &\le 2(\beta^2+1)\int_\Omega |\nabla u_n|^2 + \int_\Omega |u_n|^{2\left(\frac{(q+1)m^{**}}{2^*}-1\right)}|\nabla u_n|^2\,,\end{aligned}$$

which implies that $\big(a(x)+|u|^q\big)\nabla u \in (L^2(\Omega))^N$.

To finish, we prove that we can take the test function in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ instead of $W_0^{1,\infty}(\Omega)$. Indeed, given $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, we consider $\varphi_n\in W_0^{1,\infty}(\Omega)$ such that φ_n converges to φ strongly in $W_0^{1,2}(\Omega)$ and *-weakly in $L^\infty(\Omega)$. Firstly, we observe that

$$\lim_{n\to\infty}\int_\Omega b(x)\,u\,|u|^{p-1}|\nabla u|^2\varphi_n = \int_\Omega b(x)\,u\,|u|^{p-1}|\nabla u|^2\varphi\,.$$

Moreover, since if (A), (B) or (C) holds true, then $\big(a(x)+|u|^q\big)\nabla u\in (L^2(\Omega))^N$, we obtain that

$$\lim_{n\to\infty}\int_\Omega \big(a(x)+|u|^q\big)\nabla u\nabla\varphi_n = \int_\Omega \big(a(x)+|u|^q\big)\nabla u\nabla\varphi\,.$$

As a consequence,

$$\int_\Omega \big(a(x) + |u|^q\big)\nabla u \nabla \varphi + \int_\Omega b(x)\, u\, |u|^{p-1} |\nabla u|^2 \varphi = \int_\Omega f \varphi\,,\ \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$$

which concludes the proof. □

Chapter 4

A class of quasilinear Dirichlet problems with unbounded coefficients and singular quadratic lower order terms

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Abstract

We study existence and regularity of positive solutions of problems like

$$\begin{cases} -\mathrm{div}([a(x)+u^q]\nabla u) + b(x)\dfrac{1}{u^\theta}|\nabla u|^2 = f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

depending on the values of $q > 0$, $0 < \theta < 1$, and on the summability of the datum $f \geq 0$ in Lebesgue spaces.

4.1 Introduction

In this paper we are going to study the existence of solutions for the problem

$$\begin{cases} -\mathrm{div}([a(x)+u^q]\nabla u) + b(x)\dfrac{1}{u^\theta}|\nabla u|^2 = f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

We assume that Ω is a bounded, open set of $\mathbb{R}^N$ $(N > 2)$, that

$$q > 0\,, \ \ 0 < \theta < 1\,, \tag{4.2}$$

$$f \geq 0\,, \ f \not\equiv 0\,, \qquad f \in L^m(\Omega)\,, \ m \geq 1\,, \tag{4.3}$$

and that $a(x)$ and $b(x)$ are measurable functions such that

$$0 < \alpha \leq a(x) \leq \beta, \tag{4.4}$$

$$0 < \mu \leq b(x) \leq \nu. \tag{4.5}$$

The boundary value problem (4.1) is a quasilinear elliptic problem having a lower order term with quadratic growth with respect to the gradient. The interest in the study of this kind of problems arises naturally since the Euler-Lagrange equations of some integral functionals of the Calculus of Variations are of this form. This is one of the reasons why the quadratic growth is also called "natural". If the principal part is like a p-Laplace operator, the natural growth of the lower order term is of order p. A general theory of the existence and the motivation of the study in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ can be found in [49]. Furthermore, simple examples of integral functionals show that the assumption

"the quadratic lower order term has the same sign of the solution"

is natural, and it allows (see [47]) to prove existence of unbounded solutions (always in $W_0^{1,p}(\Omega)$). Such assumption was also used in [36] to prove the regularizing effect of the lower order term: i.e., existence of finite energy solutions even if the right hand side is only a summable function (if the datum is a measure, nonexistence results can be found in [40]). A complete study of these problems can be found in [23], [81] (see also the papers cited therein).

Recently, a problem introduced by D. Arcoya (see [13] and [7]) gave a strong impulse to the study of quasilinear problems having a quadratic lower order term which becomes singular where the solution is zero, since it depends on a negative power of the solution. This is the case of problem (4.1), which has the added difficulty of having an unbounded elliptic operator. Problems like (4.1) can be seen (at least formally) as Euler-Lagrange equations of functional integrals of the Calculus of Variations. For example, if f belongs to $L^2(\Omega)$, and

$$q = 1 - \theta\,, \quad \text{and} \quad b(x) \equiv \frac{1-\theta}{2}\,,$$

then solutions of (4.1) are minima of the functional

$$J(v) = \frac{1}{2}\int_\Omega [a(x) + |v|^{1-\theta}]|\nabla v|^2 - \int_\Omega f\,v\,, \tag{4.6}$$

defined on a suitable subset of $W_0^{1,2}(\Omega)$.

The study of problems with these features was developed in several recent papers (see [24], [10], and the references therein); here we follow the approach of [24] (see Section 2 for the details).

Our main result is the following.

Theorem 4.1 *Suppose that f belongs to $L^1(\Omega)$, and that (4.2), (4.4) and (4.5) hold true. Then there exists a solution u of (4.1), with $u > 0$ in Ω,*

$$[a(x) + u^q]|\nabla u| \in L^\rho(\Omega)\,,\ \forall \rho < \frac{N}{N-1}\,, \qquad b(x)|\nabla u|^2\,u^{-\theta} \in L^1(\Omega)\,,$$

and

$$\int_\Omega [a(x) + u^q]\,\nabla u \nabla \varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi = \int_\Omega f\,\varphi\,, \tag{4.7}$$

for every φ in $W_0^{1,p}(\Omega)$, $p > N$. Furthermore, we have the following summability results for u:

$$\begin{cases} \bullet\ \textit{if } 0 < q \le 1-\theta,\ u \textit{ belongs to } W_0^{1,r}(\Omega), \textit{ with } r = \frac{N(2-\theta)}{N-\theta};\\ \bullet\ \textit{if } 1-\theta < q \le 1,\ u \textit{ belongs to } W_0^{1,r}(\Omega), \textit{ for every } r < \frac{N(q+1)}{N+q-1};\\ \bullet\ \textit{if } q > 1, \textit{ then } u \textit{ belongs to } W_0^{1,2}(\Omega). \end{cases} \tag{4.8}$$

Remark 4.1 *Remark that $\frac{N(2-\theta)}{N-\theta} < \frac{N(q+1)}{N+q-1}$ if $q > 1-\theta$.*

We will prove Theorem 4.1 by approximating problem (4.1) with a sequence of nonsingular quasilinear quadratic problems with bounded data, and then proving both *a priori* estimates and convergence results on the sequence of approximating solutions (see Lemma 4.1 and Lemma 4.3). We will then prove regularity results on the solutions, depending on the summability of the datum f and on the possible values of q and θ. In the final sections, we will study the minimization of functionals like (4.6) (and the connections to (4.1)), and the correct assumptions on the datum f in order to have test functions in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ instead of Lipschitz function as in (4.7).

We will make frequent use, in what follows, of the truncation function defined by $T_k(s) = \max(-k, \min(s,k))$, for $k > 0$ and s in $\mathbb{R}$, and of its "companion" function $G_k(s) = s - T_k(s)$.

4.2 Proof of the main result

As stated in the Introduction, we approximate problem (4.1) by a sequence of non-singular, quadratic quasilinear problems with bounded data.

Take $0 < \varepsilon < 1$ belonging to a sequence converging to zero, and consider the following problems

$$\begin{cases} -\mathrm{div}([a(x) + |u_\varepsilon|^q]\nabla u_\varepsilon) + b(x)\dfrac{u_\varepsilon|\nabla u_\varepsilon|^2}{(|u_\varepsilon| + \varepsilon)^{\theta+1}} = \dfrac{f}{1+\varepsilon f} & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

From the results of [36], [47], it follows the existence of a solution u_ε belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Moreover $u_\varepsilon \geq 0$ since the right hand side is positive (by the assumptions on f) and since the quadratic lower order term has the same sign of the solution. Therefore u_ε solves

$$\begin{cases} -\mathrm{div}([a(x) + u_\varepsilon^q]\nabla u_\varepsilon) + b(x)\dfrac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon + \varepsilon)^{\theta+1}} = \dfrac{f}{1+\varepsilon f} & \text{in } \Omega, \\ u_\varepsilon \geq 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

in the sense that u_ε satisfies

$$\int_\Omega [a(x) + u_\varepsilon^q]\nabla u_\varepsilon \nabla\Phi + \int_\Omega \frac{b(x)\, u_\varepsilon\, |\nabla u_\varepsilon|^2}{(u_\varepsilon + \varepsilon)^{\theta+1}}\,\Phi = \int_\Omega \frac{f}{1+\varepsilon f}\Phi \tag{4.10}$$

for every test function Φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

We are going to study the properties of the sequence $\{u_\varepsilon\}$ of solutions of (4.9) in the following lemmas, with the aim of passing to the limit in order to obtain a solution of (4.1). Note that *a priori* estimates are not enough due to the nonlinear nature of the equation, so that strong convergence results (in suitable spaces) will be necessary.

Our first result yields some *a priori* estimates on $\{u_\varepsilon\}$.

Lemma 4.1 *Suppose that* (4.2), (4.3), (4.4), *and* (4.5) *hold true. Then the sequence* $\{u_\varepsilon\}$ *satisfies the following estimates for every* $\varepsilon > 0$, *and for every* $k > 0$:

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon + \varepsilon)^{\theta+1}} \leq \int_\Omega f\,, \tag{4.11}$$

$$\frac{1}{k}\int_\Omega [a(x) + u_\varepsilon^q]|\nabla T_k(u_\varepsilon)|^2 \leq \int_\Omega f\,\frac{T_k(u_\varepsilon)}{k}\,. \tag{4.12}$$

Furthermore, the sequence $\{T_k(u_\varepsilon)\}$ *is bounded in* $W_0^{1,2}(\Omega)$, *the sequence* $\{u_\varepsilon\}$ *is bounded in* $W_0^{1,r}(\Omega)$, *with* r *as in the statement of Theorem 4.1, and the sequence* $u_\varepsilon^q|\nabla u_\varepsilon|$ *is bounded in* $L^\rho(\Omega)$, *for every* $\rho < \frac{N}{N-1}$.

Remark 4.2 *As a consequence of Lemma 4.1, there exists a subsequence (not relabeled) and a function $u \in W_0^{1,r}(\Omega)$ (with r as in the statement of Theorem 4.1) such that u_ε almost everywhere converges to u, and $T_k(u_\varepsilon)$ weakly converges to $T_k(u)$ in $W_0^{1,2}(\Omega)$ for every $k > 0$.*

Proof of Lemma 4.1. Take $k > 0$, and choose $\frac{T_k(u_\varepsilon)}{k}$ as test function in (4.9). We obtain

$$\frac{1}{k}\int_\Omega [a(x) + u_\varepsilon^q]|\nabla T_k(u_\varepsilon)|^2 + \int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_k(u_\varepsilon)}{k} \le \int_\Omega \frac{f}{1+\varepsilon f}\frac{T_k(u_\varepsilon)}{k}. \quad (4.13)$$

Dropping the nonnegative first term, we obtain

$$\int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\frac{T_k(u_\varepsilon)}{k} \le \int_\Omega \frac{f}{1+\varepsilon f}\frac{T_k(u_\varepsilon)}{k} \le \int_\Omega f\,.$$

Letting k tend to 0 we deduce (4.11) by Fatou's Lemma.

On the other hand, dropping the nonnegative second term of (4.13), we have

$$\frac{1}{k}\int_\Omega [a(x) + u_\varepsilon^q]|\nabla T_k(u_\varepsilon)|^2 \le \int_\Omega \frac{f}{1+\varepsilon f}\frac{T_k(u_\varepsilon)}{k} \le \int_\Omega f\,\frac{T_k(u_\varepsilon)}{k}\,, \quad (4.14)$$

i.e., (4.12) holds true. As a consequence of (4.14) and using (4.4) it easily follows the boundedness (with respect to ε) of the sequence $\{T_k(u_\varepsilon)\}$ in $W_0^{1,2}(\Omega)$.

Next, we study the estimates of the sequence $\{u_\varepsilon\}$ in $W_0^{1,r}(\Omega)$. We split the proof in three parts according to the values of q and θ.

If $0 < q \le 1-\theta$, starting from (4.11), and using (4.5), we have

$$\frac{\mu}{2^{\theta+1}}\int_{\{u_\varepsilon \ge 1\}} \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^\theta} \le \mu \int_{\{u_\varepsilon\ge 1\}} \frac{u_\varepsilon}{(u_\varepsilon+\varepsilon)^{\theta+1}}\,|\nabla u_\varepsilon|^2 \le \int_\Omega f\,.$$

Then, if $r < 2$, and thanks to Hölder inequality,

$$\int_\Omega |\nabla G_1(u_\varepsilon)|^r = \int_\Omega \frac{|\nabla G_1(u_\varepsilon)|^r}{u_\varepsilon^{\theta\frac{r}{2}}}\,u_\varepsilon^{\theta\frac{r}{2}} \le C_1\bigg(\int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^{\frac{\theta r}{2-r}}\bigg)^{\frac{2-r}{2}}. \quad (4.15)$$

Choosing r such that $r^* = \frac{\theta r}{2-r}$, we obtain $r = \frac{N(2-\theta)}{N-\theta} < 2$ so that, by Sobolev inequality,

$$\bigg(\int_\Omega G_1(u_\varepsilon)^{r^*}\bigg)^{\frac{r}{r^*}} \le C_2\bigg(\int_{\{u_\varepsilon\ge 1\}} u_\varepsilon^{r^*}\bigg)^{\frac{\theta}{r^*}} \le C_3\bigg(\int_\Omega G_1(u_\varepsilon)^{r^*}\bigg)^{\frac{\theta}{r^*}} + C_3\,.$$

Since $\theta < r$ (as it is easily seen), the last estimate implies that $G_1(u_\varepsilon)$, hence u_ε, is bounded in $L^{r^*}(\Omega)$. Using (4.15), we then have that $G_1(u_\varepsilon)$ is bounded in $W_0^{1,r}(\Omega)$. Since $T_1(u_\varepsilon)$ is bounded in $W_0^{1,2}(\Omega)$, hence in $W_0^{1,r}(\Omega)$, we have that u_ε is bounded in $W_0^{1,r}(\Omega)$, as desired.

If $1-\theta<q\leq 1$, choose as test function $1-(1+u_\varepsilon)^{1-\lambda}$, with $\lambda>1$. Dropping positive terms, and using (4.4), we obtain,

$$\int_\Omega \frac{\alpha+u_\varepsilon^q}{(1+u_\varepsilon)^\lambda}|\nabla u_\varepsilon|^2 \leq \int_\Omega f\,,$$

which then implies (since $q\leq 1$)

$$\min(\alpha,1)\int_\Omega \frac{(1+u_\varepsilon)^q}{(1+u_\varepsilon)^\lambda}|\nabla u_\varepsilon|^2 \leq \int_\Omega f\,.$$

If $r<2$ we then have, as before,

$$\int_\Omega |\nabla u_\varepsilon|^r = \int_\Omega \frac{|\nabla u_\varepsilon|^r}{(1+u_\varepsilon)^{\frac{r(\lambda-q)}{2}}}(1+u_\varepsilon)^{\frac{r(\lambda-q)}{2}} \leq \Big(C_4\int_\Omega f\Big)^{\frac{2}{r}}\Big(\int_\Omega (1+u_\varepsilon)^{\frac{r(\lambda-q)}{2-r}}\Big)^{\frac{2-r}{r}}.$$

Choosing r such that $r^*=\frac{r(\lambda-q)}{2-r}$, we have $r=\frac{N(2+q-\lambda)}{N+q-\lambda}$; since $\lambda>1$, we have $r<\frac{N(q+1)}{N+q-1}<2$. Thus,

$$\Big(\int_\Omega u_\varepsilon^{r^*}\Big)^{\frac{r}{r^*}} \leq C_5\Big(\int_\Omega (1+u_\varepsilon)^{r^*}\Big)^{\frac{\lambda-q}{r^*}}\,,$$

which, since $\lambda-q<r$, implies the boundedness of u_ε in $L^{r^*}(\Omega)$. This boundedness then implies the boundedness of u_ε in $W_0^{1,r}(\Omega)$, as desired.

If $q>1$, we choose as test function $1-(1+u_\varepsilon)^{1-q}$, which yields

$$\frac{\min(\alpha,1)}{2^{q-1}}\int_\Omega |\nabla u_\varepsilon|^2 \leq \min(\alpha,1)\int_\Omega \frac{1+u_\varepsilon^q}{(1+u_\varepsilon)^q}|\nabla u_\varepsilon|^2 \leq \int_\Omega f\,,$$

from which the boundness of $\{u_\varepsilon\}$ in $W_0^{1,2}(\Omega)$ follows.

Finally, starting again from

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(1+u_\varepsilon)^{\lambda-q}} \leq \frac{1}{\min(\alpha,1)}\int_\Omega f\,,$$

which holds for every $\lambda>1$, we have

$$\begin{aligned}\int_\Omega u_\varepsilon^{q\rho}|\nabla u_\varepsilon|^\rho &\leq \int_\Omega \frac{|\nabla u_\varepsilon|^\rho}{(1+u_\varepsilon)^{\frac{\rho(\lambda-q)}{2}}}(1+u_\varepsilon)^{\frac{\rho(\lambda+q)}{2}} \\ &\leq \Big(\frac{1}{\min(\alpha,1)}\int_\Omega f\Big)^{\frac{\rho}{2}}\Big(\int_\Omega (1+u_\varepsilon)^{\frac{\rho(\lambda+q)}{2-\rho}}\Big)^{\frac{2-\rho}{2}}\,,\end{aligned}$$

which then implies

$$\Big(\int_\Omega u_\varepsilon^{(q+1)\rho^*}\Big)^{\frac{\rho}{\rho^*}} \leq C_6\Big(\int_\Omega u_\varepsilon^{\frac{\rho(\lambda+q)}{2-\rho}}\Big)^{\frac{2-\rho}{2}}.$$

Choosing ρ so that $(q+1)\rho^* = \frac{\rho(\lambda+q)}{2-\rho}$ yields $\rho = \frac{N(2+q-\lambda)}{N(q+1)-(\lambda+q)}$. Since $\lambda > 1$, we have an estimate on $u_\varepsilon^q|\nabla u_\varepsilon|$ in $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, as desired. $\square$

The next result yields the strict positivity of u. Before starting it, let us define for $t \geq 0$ the functions

$$H_\varepsilon(t) = \frac{(t+\varepsilon)^{1-\theta} - \varepsilon^{1-\theta}}{1-\theta}\,, \qquad H_0(t) = \frac{t^{1-\theta}}{1-\theta}\,, \tag{4.16}$$

and

$$\Phi_\varepsilon(t) = \mathrm{e}^{-\nu\frac{H_\varepsilon(t)}{\alpha}}\,, \qquad \Phi_0(t) = \mathrm{e}^{-\nu\frac{H_0(t)}{\alpha}}\,. \tag{4.17}$$

Lemma 4.2 *Suppose that* (4.2), (4.3) (4.4) *and* (4.5) *hold true. If u is given by Remark 4.2, then $u > 0$ in Ω.*

Proof. Let v be fixed in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, with $v \geq 0$, and choose $v\,\Phi_\varepsilon(u_\varepsilon)$ as a test function in (4.9), which can be done since it belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Hence, using that

$$\Phi_\varepsilon'(t) = -\frac{\nu}{\alpha}\frac{1}{(\varepsilon+t)^\theta}\Phi_\varepsilon(t)\,,$$

we obtain

$$\int_\Omega [a(x)+u_\varepsilon^q]\nabla u_\varepsilon \nabla v\,\Phi_\varepsilon(u_\varepsilon) - \frac{\nu}{\alpha}\int_\Omega [a(x)+u_\varepsilon^q]\frac{|\nabla u_\varepsilon|^2}{(\varepsilon+u_\varepsilon)^\theta}\,\Phi_\varepsilon(u_\varepsilon)\,v + \int_\Omega b(x)\frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\Phi_\varepsilon(u_\varepsilon)\,v = \int_\Omega \frac{f}{1+\varepsilon f}v\,\Phi_\varepsilon(u_\varepsilon)\,.$$

Since $v \geq 0$, using (4.4) and (4.5), we have

$$\int_\Omega [a(x)+u_\varepsilon^q]\nabla u_\varepsilon \nabla v\,\Phi_\varepsilon(u_\varepsilon) - \frac{\nu}{\alpha}\int_\Omega \alpha\frac{|\nabla u_\varepsilon|^2}{(\varepsilon+u_\varepsilon)^\theta}\Phi_\varepsilon(u_\varepsilon)\,v + \int_\Omega \nu\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^\theta}\Phi_\varepsilon(u_\varepsilon)\,v \geq \int_\Omega \frac{f}{1+f}v\,\Phi_\varepsilon(u_\varepsilon)\,.$$

Hence,

$$\int_\Omega \{\Phi_\varepsilon(u_\varepsilon)[a(x)+u_\varepsilon^q]\}\nabla u_\varepsilon \nabla v \geq \int_\Omega \frac{f}{1+f}v\,\Phi_\varepsilon(u_\varepsilon)\,, \tag{4.18}$$

for all v in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$.

Now, given $\delta > 0$, define the function

$$\psi_\delta(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1,\\ -\frac{1}{\delta}(t-1-\delta), & \text{if } 1 \leq t < \delta+1,\\ 0, & \text{if } \delta+1 \leq t, \end{cases}$$

and fix a function φ in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\geq 0$. Taking $v=\psi_\delta(u_\varepsilon)\varphi$ in (4.18) we have

$$\int_\Omega \psi_\delta(u_\varepsilon)\,\Phi_\varepsilon(u_\varepsilon)\,[a(x)+u_\varepsilon^q]\nabla u_\varepsilon\nabla\varphi \geq \int_\Omega \frac{f}{1+f}\Phi_\varepsilon(u_\varepsilon)\,\psi_\delta(u_\varepsilon)\,\varphi + \frac{1}{\delta}\int_{\{1\leq u_\varepsilon(x)<\delta+1\}} \Phi_\varepsilon(u_\varepsilon)[a(x)+u_\varepsilon^q]|\nabla u_\varepsilon|^2\,\varphi\,,$$

and thus, dropping the positive term,

$$\int_\Omega \psi_\delta(u_\varepsilon)\,\Phi_\varepsilon(u_\varepsilon)\,[a(x)+u_\varepsilon^q]\nabla u_\varepsilon\nabla\varphi \geq \int_\Omega \frac{f\,\Phi_\varepsilon(u_\varepsilon)}{1+f}\psi_\delta(u_\varepsilon)\,\varphi\,.$$

Then, passing to the limit as δ tends to zero, we obtain

$$\int_\Omega \Phi_\varepsilon(T_1(u_\varepsilon))[a(x)+T_1(u_\varepsilon)^q]\nabla T_1(u_\varepsilon)\nabla\varphi \geq \int_{\{0\leq u_\varepsilon<1\}} \frac{f\,\Phi_\varepsilon(T_1(u_\varepsilon))}{1+f}\,\varphi\,.$$

Since, by Remark 4.2, $\nabla T_1(u_\varepsilon)$ weakly converges in $(L^2(\Omega))^N$, we can pass to the limit in ε even if our original problem is nonlinear, to obtain

$$\int_\Omega \Phi_0(T_1(u))[a(x)+T_1(u)^q]\nabla T_1(u)\nabla\varphi \geq \int_{\{0\leq u\leq 1\}} \frac{f\,\Phi_0(T_1(u))}{1+f}\varphi\,,$$

for all $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $\varphi\geq 0$.

If we define

$$w(x)=\int_0^{T_1(u(x))}\Phi_0(t)\,dt\,,$$

we have that w belongs to $W_0^{1,2}(\Omega)$; furthermore, since

$$\Phi_0(T_1(u))\geq\Phi_0(1)=\ \mathrm{e}^{-\frac{\nu}{\alpha(1-\theta)}}>0\,,$$

we deduce from the last inequality that

$$\int_\Omega [a(x)+T_1(u)^q]\nabla w\,\nabla\varphi \geq \int_\Omega \left[\frac{T_1(f)\ \mathrm{e}^{-\frac{\nu}{\alpha(1-\theta)}}}{1+f}\chi_{\{0\leq u\leq 1\}}\right]\varphi\,, \tag{4.19}$$

for all $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $\varphi\geq 0$, and then, by density, for every nonnegative φ in $W_0^{1,2}(\Omega)$. Hence, w is a supersolution of a linear Dirichlet problem with a strictly positive and bounded, measurable coefficient, since

$$\alpha\leq a(x)+T_1(u)^q\leq\beta+1\,,$$

and with right hand side a nonnegative function, not indentically zero. The strong maximum principle (see [74]) then implies that $w>0$ in Ω. Since $T_1(u)\geq w$ (due to the fact that $\Phi_0(t)\leq 1$), we conclude that $T_1(u)>0$ in Ω, which then implies that $u>0$ in Ω, since $u\geq T_1(u)$. □

Remark 4.3 *The conclusion of Lemma 4.2 is a consequence of the strong maximum principle. Moreover, Harnack's inequality gives the stronger conclusion: if* $\omega \subset\subset \Omega$, *then there exists* $c_\omega > 0$ *such that* $u \geq c_\omega > 0$.

Now we prove that the gradients of the approximating solutions u_ε almost everywhere converge in Ω. Due to the nonlinearity of the equation, this result will be crucial in order to pass to the limit in the approximate equations. Related results can be found in [22] and [45].

Lemma 4.3 *Suppose that* (4.2), (4.3), (4.4), *and* (4.5) *hold true. If* u *is given by Remark 4.2, then there exists a subsequence (not relabelled) such that* $\{\nabla u_\varepsilon\}$ *converges to* ∇u *almost everywhere in* Ω. *Furthermore,* u *is such that* $b(x)|\nabla u|^2 u^{-\theta}$ *belongs to* $L^1(\Omega)$, $[a(x)+u^q]|\nabla u|$ *belongs to* $L^\rho(\Omega)$, *for every* $\rho < \frac{N}{N-1}$, *and for every* $k > 0$ *we have*

$$\frac{1}{k}\int_\Omega [a(x)+u^q]|\nabla T_k(u)|^2 \leq \int_\Omega f\,\frac{T_k(u)}{k}\,. \tag{4.20}$$

Proof. Given h, $k > 0$, we choose $T_h[u_\varepsilon - T_k(u)]$ as a test function in (4.9) to obtain, using (4.11), that

$$\alpha \int_\Omega |\nabla T_h[u_\varepsilon - T_k(u)]|^2 \leq 2h\,\|f\|_{L^1(\Omega)} - \int_\Omega [a(x)+u_\varepsilon^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)]\,.$$

Setting $M = h+k$, we remark that $\nabla T_h[u_\varepsilon - T_k(u)] \neq 0$ implies $u_\varepsilon \leq M$. Hence,

$$\int_\Omega [a(x)+u_\varepsilon^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)] = \int_\Omega [a(x)+T_M(u_\varepsilon)^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)]\,.$$

Since $\nabla T_h[u_\varepsilon - T_k(u)]$ weakly converges to $\nabla T_h[u - T_k(u)]$ in $(L^2(\Omega))^N$ and morever $[a(x)+T_M(u_\varepsilon)^q]\nabla T_k(u)$ strongly converges to $[a(x)+T_M(u)^q]\nabla T_k(u)$ in the same space, we have

$$\lim_{\varepsilon\to 0^+}\int_\Omega [a(x)+u_\varepsilon^q]\nabla T_k(u)\nabla T_h[u_\varepsilon - T_k(u)] = 0\,,$$

due to the fact that $\nabla T_k(u)\nabla T_h[u - T_k(u)] \equiv 0$. Therefore,

$$\alpha \limsup_{\varepsilon\to 0^+}\int_\Omega |\nabla T_h[u_\varepsilon - T_k(u)]|^2 \leq 2h\,\|f\|_{L^1(\Omega)}\,. \tag{4.21}$$

Now, let $s < r \leq 2$, where r is as in the statement of Theorem 4.1. If R is such that the norm of $\{u_\varepsilon\}$ in $W_0^{1,r}(\Omega)$ is bounded by R (see Lemma 4.1), we have

$$\begin{aligned}\int_\Omega |\nabla(u_\varepsilon - u)|^s &= \int_{\{|u_\varepsilon - u|\leq h,\, u\leq k\}} |\nabla(u_\varepsilon - u)|^s \\ &\quad + \int_{\{|u_\varepsilon - u|\leq h,\, u> k\}} |\nabla(u_\varepsilon - u)|^s + \int_{\{|u_\varepsilon - u|> h\}} |\nabla(u_\varepsilon - u)|^s \\ &\leq \int_\Omega |\nabla T_h[u_\varepsilon - T_k(u)]|^s + 2^s R^s\ \mathrm{meas}(\{u>k\})^{1-\frac{s}{r}} \\ &\quad + 2^s R^s\ \mathrm{meas}(\{|u_\varepsilon - u|>h\})^{1-\frac{s}{r}}\,.\end{aligned}$$

Since u_ε converges to u in measure, using Hölder's inequality and (4.21), we deduce for every $h > 0$ and $k > 0$, that

$$\limsup_{\varepsilon \to 0^+} \int_\Omega |\nabla(u_\varepsilon - u)|^s \le \left(\frac{2\,h\,\|f\|_{L^1(\Omega)}}{\alpha} \right)^{\frac{s}{2}} \text{meas}(\Omega)^{1-\frac{s}{2}} + 2^{s-1} R^s \,\text{meas}(\{u > k\})^{1-\frac{s}{r}} .$$

Letting h tend to zero, and then k tends to infinity, we obtain

$$\lim_{\varepsilon \to 0^+} \int_\Omega |\nabla(u_\varepsilon - u)|^s = 0 \,, \tag{4.22}$$

which then implies that (up to a subsequence) ∇u_ε almost everywhere converges to ∇u in Ω.

Using the almost everywhere convergence of both ∇u_ε and u_ε, Fatou lemma and Lebesgue theorem, we can pass to the limit in (4.12) to have that

$$\frac{1}{k} \int_\Omega [a(x) + u^q] |\nabla T_k(u)|^2 \le \int_\Omega f \, \frac{T_k(u)}{k} \,,$$

which is exactly (4.20).

Furthermore, the fact that ∇u_ε converges to ∇u almost everywhere in Ω, (4.11), and Fatou Lemma imply

$$\int_\Omega b(x) \frac{|\nabla u|^2}{u^\theta} \le \int_\Omega f \,,$$

which is what we wanted to prove.

Finally, using the almost everywhere convergence of the sequence ∇u_ε, the boundedness of $u_\varepsilon^q |\nabla u_\varepsilon|$ in $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, and Fatou Lemma we obtain that $u^q |\nabla u|$ belongs to $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, as desired. □

We are now ready to prove the main result of this paper.

Proof of Theorem 4.1. We are going to prove that the weak limit u given by Remark 4.2 is a solution of the singular problem (4.1). By Remark 4.2 and Remark 4.3 we recall that u belongs to $W_0^{1,r}(\Omega)$, and is such that $u > 0$ in Ω. Moreover, $[a(x) + u^q]|\nabla u|$ and $b(x) u^{-\theta} |\nabla u|^2$ both belong to $L^1(\Omega)$ by Lemma 4.3.

In order to prove the result, we have to pass to the limit in (4.10). To this aim, let $0 \le B(s) \le 1$ be a function in $C^1(\mathbb{R})$ such that

$$B(s) = \begin{cases} 1, & \text{if } 0 \le s \le \frac{1}{2}, \\ 0, & \text{if } s \ge 1. \end{cases}$$

Furthermore, if $k > 0$, and u as in Remark 4.2, we define

$$Q(k) = \int_\Omega f \, \frac{T_k(u)}{k} \,.$$

Notice that by Lebesgue theorem, and the assumptions on f, one has

$$\lim_{k\to+\infty} Q(k) = 0\,. \tag{4.23}$$

The proof of the result will be achieved in two steps.
Step 1. The first inequality. We fix $\psi \in W_0^{1,p}(\Omega)$, $p > N$, with $\psi \geq 0$ and take

$$\Phi = \psi\, B\Big(\frac{u_\varepsilon}{k}\Big)$$

as a test function in (4.10). Since

$$\nabla\Phi = B\Big(\frac{u_\varepsilon}{k}\Big)\nabla\psi + \frac{\psi}{k}B'\Big(\frac{u_\varepsilon}{k}\Big)\nabla u_\varepsilon = \nabla\psi B\Big(\frac{u_\varepsilon}{k}\Big) + \frac{\psi}{k}B'\Big(\frac{u_\varepsilon}{k}\Big)\nabla T_k(u_\varepsilon)\,,$$

by the assumptions on B, we have

$$\begin{aligned}\int_\Omega [a(x) + T_k(u_\varepsilon)^q]\nabla T_k(u_\varepsilon)\nabla\psi B\Big(\frac{u_\varepsilon}{k}\Big) + \frac{1}{k}\int_\Omega [a(x) + T_k(u_\varepsilon)^q]|\nabla T_k(u_\varepsilon)|^2\psi\, B'\Big(\frac{u_\varepsilon}{k}\Big)\\ + \int_\Omega \frac{b(x)\, T_k(u_\varepsilon)\, |\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^{\theta+1}}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big) = \int_\Omega \frac{f}{1+\varepsilon f}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big)\,.\end{aligned}$$

Hence, using (4.12), we have

$$\begin{aligned}\int_\Omega [a(x) + T_k(u_\varepsilon)^q]\nabla T_k(u_\varepsilon)\nabla\psi B\Big(\frac{u_\varepsilon}{k}\Big) + \int_\Omega \frac{b(x)\, T_k(u_\varepsilon)\, |\nabla T_k(u_\varepsilon)|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big)\\ \leq \int_\Omega \frac{f}{1+\varepsilon f}\,\psi\, B\Big(\frac{u_\varepsilon}{k}\Big) + \|B'\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)}\int_\Omega f\,\frac{T_k(u_\varepsilon)}{k}\,.\end{aligned}$$

Using the weak convergence of the truncates in $W_0^{1,2}(\Omega)$, and the almost everywhere convergence of both ∇u_ε and u_ε, we can use Fatou lemma and Lebesgue theorem to pass to the limit in the above inequality as ε tends to zero to obtain

$$\begin{aligned}\int_\Omega [a(x) + T_k(u)^q]\nabla T_k(u)\nabla\psi B\Big(\frac{u}{k}\Big) + \int_\Omega \frac{b(x)\, |\nabla T_k(u)|^2}{T_k(u)^\theta}\,\psi\, B\Big(\frac{u}{k}\Big)\\ \leq \int_\Omega f\,\psi\, B\Big(\frac{u}{k}\Big) + \|B'\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)}\, Q(k)\,,\end{aligned}$$

for all $\psi \in W_0^{1,p}(\Omega)$, $p > N$, with $\psi \geq 0$.

Now we let k tend to infinity; using the fact that B is bounded, that $[a(x)+u^q]|\nabla u|$ belongs to $L^\rho(\Omega)$, for every $\rho < \frac{N}{N-1}$, and that $b(x)u^{-\theta}|\nabla u|^2$ belongs to $L^1(\Omega)$ by Lemma 4.3, and (4.23), we obtain

$$\int_\Omega [a(x) + u^q]\,\nabla u\nabla\psi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\psi \leq \int_\Omega f\,\psi\,,$$

for every $\psi \in W_0^{1,p}(\Omega)$, $p > N$, with $\psi \geq 0$; i.e., u is a subsolution of problem (4.1).
Step 2. The second inequality. Let ψ be in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, with $\psi \leq 0$, let H_ε be given by (4.16), and choose

$$\phi = \psi \ \mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}} \, B\Big(\frac{u_\varepsilon}{k}\Big)$$

as test function in (4.10). Thus, recalling that both $B(\frac{u_\varepsilon}{k})$ and $B'(\frac{u_\varepsilon}{k})$ are zero on the set $\{u_\varepsilon > k\}$, we obtain

$$\begin{aligned}\int_\Omega [a(x) + T_k(u_\varepsilon)^q]\Big(\nabla T_k(u_\varepsilon)\nabla\psi - \frac{\nu}{\alpha}\frac{|\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^\theta}\,\psi\Big)\mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}} \, B\Big(\frac{u_\varepsilon}{k}\Big) \\ + \int_\Omega [a(x) + T_k(u_\varepsilon)^q]\frac{|\nabla T_k(u_\varepsilon)|^2}{k}\psi \mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}} \, B'\Big(\frac{u_\varepsilon}{k}\Big) \\ + \int_\Omega \frac{b(x)\,T_k(u_\varepsilon)\,|\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^{\theta+1}}\,\psi \ \mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}} \, B\Big(\frac{u_\varepsilon}{k}\Big) = \int_\Omega \frac{f}{1+\varepsilon f}\,\psi \ \mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}} \, B\Big(\frac{u_\varepsilon}{k}\Big).\end{aligned}$$

Remark now that, by the assumptions on a and b, and since $\psi \leq 0$, we have

$$\frac{|\nabla T_k(u_\varepsilon)|^2}{(T_k(u_\varepsilon)+\varepsilon)^\theta}\psi \ \mathrm{e}^{-\frac{\nu H_\varepsilon(u_\varepsilon)}{\alpha}} \, B\Big(\frac{u_\varepsilon}{k}\Big)\Big[\frac{b(x)T_k(u_\varepsilon)}{T_k(u_\varepsilon)+\varepsilon} - \frac{\nu}{\alpha}[a(x) + u_\varepsilon^q]\Big] \geq 0\,.$$

Therefore, using the almost everywhere convergence of both ∇u_ε and u_ε, the weak convergence of $T_k(u_\varepsilon)$ in $W_0^{1,2}(\Omega)$, Fatou lemma and both (4.12) and (4.20), we obtain, letting ε tend to zero,

$$\begin{aligned}\int_\Omega \Big\{[a(x){+}T_k(u)^q]\Big(\nabla T_k(u)\nabla\psi - \frac{\nu}{\alpha}\frac{|\nabla T_k(u)|^2}{T_k(u)^\theta}\,\psi\Big) + \frac{b(x)\,|\nabla T_k(u)|^2}{T_k(u)^\theta}\,\psi\Big\}\mathrm{e}^{-\frac{\nu H_0(u)}{\alpha}} \, B\Big(\frac{u}{k}\Big) \\ \leq \int_\Omega f\,\psi \ \mathrm{e}^{-\frac{\nu H_0(u)}{\alpha}} \, B\Big(\frac{u}{k}\Big) + \|B'\|_{L^\infty(\Omega)}\|\psi\|_{L^\infty(\Omega)}\,Q(k)\,. \quad (4.24)\end{aligned}$$

The idea now is to take a particular function ψ and pass to the limit as k tends to infinity. Let $k > 0$ be large enough such that

$$\sigma(k) = \Big(- \frac{\alpha(1-\theta)}{2\nu}\log(Q(k))\Big)^{\frac{1}{1-\theta}}$$

is well defined (see (4.23)), and note that

$$\lim_{k\to+\infty} \sigma(k) = +\infty\,,$$

since the argument of the logarithm tends to zero as k diverges. Note also that, by definition,

$$\mathrm{e}^{\frac{\nu\,H_0(\sigma(k))}{\alpha}} = \frac{1}{\sqrt{Q(k)}}\,. \qquad (4.25)$$

Let φ belong to $C^1_c(\Omega)$, with $\varphi \le 0$; since u is strictly positive on compact subsets of Ω (see Remark 4.3) we have that $u^{-\theta}\varphi$ belongs to $L^\infty(\Omega)$, so that the negative function

$$\psi = \mathrm{e}^{\frac{\nu H_0(u)}{\alpha}} B\Big(\frac{u}{\sigma(k)}\Big)\varphi$$

belongs to $W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$ (also because both B and B' have compact support in $\mathbb{R}$). Hence, it can be chosen as test function in (4.24) to obtain, after cancelling equal terms, and using (4.20) and (4.25),

$$\begin{aligned}
\int_\Omega \Big\{ [a(x) + T_k(u)^q]\nabla T_k(u)\nabla\varphi + \frac{b(x)\,|\nabla T_k(u)|^2}{T_k(u)^\theta}\,\varphi \Big\} B\Big(\frac{u}{k}\Big) B\Big(\frac{u}{\sigma(k)}\Big) \\
\le \int_\Omega f\,\varphi\, B\Big(\frac{u}{k}\Big) B\Big(\frac{u}{\sigma(k)}\Big) + \|B'\|_{L^\infty(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \sqrt{Q(k)} \\
+ \frac{1}{\sigma(k)}\|B'\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(\Omega)} \int_\Omega [a(x) + T_{\sigma(k)}(u)^q]|\nabla T_{\sigma(k)}(u)|^2 \\
\le \int_\Omega f\,\varphi\, B\Big(\frac{u}{k}\Big) B\Big(\frac{u}{\sigma(k)}\Big) + \|B'\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(\Omega)} \Big(\sqrt{Q(k)} + Q(\sigma(k))\Big)\,.
\end{aligned}$$

To finish, we pass to the limit as k tends to infinity. Using once again that B is bounded, that both $[a(x)+u^q]|\nabla u|$ and $b(x)u^{-\theta}|\nabla u|^2$ belong to $L^1(\Omega)$ by Lemma 4.3, and (4.23), we have

$$\int_\Omega [a(x) + u^q]\nabla u\nabla\varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi \le \int_\Omega f\,\varphi\,,$$

for all φ in $C^1_c(\Omega)$, with $\varphi \le 0$. Using the results of Lemma 4.3, we conclude by density that

$$\int_\Omega [a(x) + u^q]\nabla u\nabla\varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi \le \int_\Omega f\,\varphi$$

for all φ in $W^{1,p}_0(\Omega)$, $p > N$, with $\varphi \le 0$.

Putting together the results of both steps we conclude that

$$\int_\Omega [a(x) + u^q]\nabla u\nabla\varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi \le \int_\Omega f\,\varphi\,,$$

for all $\varphi \in W^{1,p}_0(\Omega)$, $p > N$ and then (exchanging φ with $-\varphi$)

$$\int_\Omega [a(x) + u^q]\nabla u\nabla\varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi = \int_\Omega f\,\varphi$$

for every φ in $W^{1,p}_0(\Omega)$, $p > N$. □

4.3 Summability results

As stated in the Introduction, in this Section we prove some regularity results on the solution u given by Theorem 4.1, depending on summability assumptions on f, and on the values of both q and θ.

Theorem 4.2 *Let $\delta = \min(\theta, 1-q)$, and let $1 < m < \frac{N}{2}$. Then the solution u belongs to $L^s(\Omega)$, where $s = m^{**}(2-\delta)$. Furthermore, if $q < 1$, then*

1) if $1 < m < \left(\frac{2^*}{\delta}\right)'$, then u belongs to $W_0^{1,r}(\Omega)$, with $r = \frac{Nm(2-\delta)}{N-m\delta}$;

2) if $m \geq \left(\frac{2^*}{\delta}\right)'$, and $m > 1$, the u belongs to $W_0^{1,2}(\Omega)$.

If $q = 1$, then $m > 1$ implies u in $W_0^{1,2}(\Omega)$, while if $q > 1$ then u belongs to $W_0^{1,2}(\Omega)$ by the results of Theorem 4.1.

Remark 4.4 *Note that, by definition, $\delta < 1$.*

Before giving the proof of Theorem 4.2, we need a lemma.

Lemma 4.4 *Let $\delta = \min(\theta, 1-q)$, and let $\gamma > 0$; then there exists $C_0 > 0$ such that*

$$\gamma(t+\varepsilon)^{\gamma-1}(\alpha+t^q) + \mu t(t+\varepsilon)^{\gamma-1-\theta} \geq C_0(t+\varepsilon)^{\gamma-\delta}\,, \tag{4.26}$$

for every $t \geq 0$.

Proof. Multiplying (4.26) by $(t+\varepsilon)^{\delta-\gamma}$, we have to prove that

$$\gamma(t+\varepsilon)^{\delta-1}(\alpha+t^q) + \mu t(t+\varepsilon)^{\delta-1-\theta} \geq C_0 > 0\,.$$

If $\delta = \theta$, we have to prove that

$$\gamma\frac{\alpha+t^q}{(t+\varepsilon)^{1-\theta}} + \mu\frac{t}{t+\varepsilon} \geq C_0 > 0\,.$$

Clearly, if $t \geq \varepsilon$ we have $\frac{t}{t+\varepsilon} \geq \frac{1}{2}$, while if $t < \varepsilon$ we have $\frac{\alpha+t^q}{(t+\varepsilon)^{1-\theta}} \geq \frac{\alpha}{(2\varepsilon)^{1-\theta}} \geq \frac{\alpha}{2^{1-\theta}}$, since $\varepsilon < 1$; therefore, the claim is proved.

If, instead, $\delta = 1-q$, we have to prove that

$$\gamma\frac{\alpha+t^q}{(t+\varepsilon)^q} + \mu\frac{t}{(t+\varepsilon)^{q+\theta}} \geq C_0 > 0\,,$$

which is true since the first term is greater than $\frac{\gamma}{2^q}$ if $t \geq \varepsilon$, and is greater than $\frac{\gamma\alpha}{2^q}$ if $t \leq \varepsilon$. □

Proof of Theorem 4.2. The key point is to prove an *a priori* estimate on the sequence $\{u_\varepsilon\}$ since the compactness has been proved in Theorem 4.1.

Let $\gamma > 0$, and choose, following [35], $(u_\varepsilon+\varepsilon)^\gamma - \varepsilon^\gamma$ as test function in (4.9); we obtain, using the assumptions on a and b, and dropping a negative term,

$$\gamma\int_\Omega (\alpha + u_\varepsilon^q)(u_\varepsilon+\varepsilon)^{\gamma-1}|\nabla u_\varepsilon|^2 + \mu\int_\Omega \frac{u_\varepsilon(u_\varepsilon+\varepsilon)^\gamma}{(u_\varepsilon+\varepsilon)^{\theta+1}}|\nabla u_\varepsilon|^2$$
$$\leq \int_\Omega f\,(u_\varepsilon+\varepsilon)^\gamma + \varepsilon^\gamma\int_\Omega \frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} \leq \int_\Omega f\,(u_\varepsilon+\varepsilon)^\gamma + C_1\varepsilon^\gamma\,,$$

where in the last passage we have used (4.11). In the left hand side we have

$$\int_\Omega |\nabla u_\varepsilon|^2[\gamma(\alpha+u_\varepsilon^q)(u_\varepsilon+\varepsilon)^{\gamma-1} + \mu\, u_\varepsilon(u_\varepsilon+\varepsilon)^{\gamma-1-\theta}]\,.$$

Recalling Lemma 4.4, we have, if $\delta = \min(\theta, 1-q)$,

$$\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma-\delta}|\nabla u_\varepsilon|^2 \leq C_2\int_\Omega f\,(u_\varepsilon+\varepsilon)^\gamma + C_2\varepsilon^\gamma\,. \tag{4.27}$$

Now we rewrite

$$\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma-\delta}|\nabla u_\varepsilon|^2 = \frac{4}{(\gamma-\delta+2)^2}\int_\Omega \left|\nabla[(u_\varepsilon+\varepsilon)^{\frac{\gamma-\delta+2}{2}} - \varepsilon^{\frac{\gamma-\delta+2}{2}}]\right|^2,$$

and use Sobolev and Hölder inequalities to obtain

$$\left(\int_\Omega [(u_\varepsilon+\varepsilon)^{\frac{(\gamma-\delta+2)}{2}} - \varepsilon^{\frac{(\gamma-\delta+2)}{2}}]^{2^*}\right)^{\frac{2}{2^*}} \leq C_3\|f\|_{L^m(\Omega)}\left(\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma m'}\right)^{\frac{1}{m'}} + C_3\,.$$

Since $[(t+\varepsilon)^\beta - \varepsilon^\beta]^{2^*} \geq C_4(t+\varepsilon)^{2^*\beta} - C_4$, for every $t > 0$ (and for a suitable C_4 independent on ε) we then have

$$\left(\int_\Omega [C_4(u_\varepsilon+\varepsilon)^{\frac{2^*(\gamma-\delta+2)}{2}} - C_4]\right)^{\frac{2}{2^*}} \leq C_3\|f\|_{L^m(\Omega)}\left(\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma m'}\right)^{\frac{1}{m'}} + C_3\,.$$

Choosing γ such that $\frac{2^*}{2}(\gamma-\delta+2) = \gamma m'$ yields

$$\gamma = \frac{N(m-1)(2-\delta)}{N-2m}\,,$$

and then $\gamma > 0$ (since $m > 1$), and $\gamma m' = m^{**}(2-\delta) = s$. Therefore, we have

$$\left(\int_\Omega [C_4(u_\varepsilon+\varepsilon)^s - C_4]\right)^{\frac{2}{2^*}} \leq C_3\|f\|_{L^m(\Omega)}\left(\int_\Omega (u_\varepsilon+\varepsilon)^s\right)^{\frac{1}{m'}} + C_3\,,$$

which then yields, since $\frac{2}{2^*} > \frac{1}{m'}$ being $m < \frac{N}{2}$, that

$$\|u_\varepsilon\|_{L^s(\Omega)}^{2-\delta} \leq C_5\|f\|_{L^m(\Omega)}\,.$$

By Fatou lemma, and the almost everywhere convergence of u_ε to u, we obtain that u belongs to $L^s(\Omega)$, as desired.

Remark that once γ is chosen, (4.27) becomes

$$\int_\Omega (u_\varepsilon+\varepsilon)^{\gamma-\delta}|\nabla u_\varepsilon|^2 \le C_2\int_\Omega f(u_\varepsilon+\varepsilon)^\gamma + C_2\varepsilon^\gamma \le C_6\,. \tag{4.28}$$

Now we turn to gradient estimates. If $q<1$ and $\gamma\ge\delta$, that is if $m\ge\left(\frac{2^*}{\delta}\right)'$, from (4.28) we obtain

$$\int_{\{u_\varepsilon>1\}}|\nabla u_\varepsilon|^2\le C_7\,,$$

which, together with the boundedness of $T_1(u_\varepsilon)$ in $W_0^{1,2}(\Omega)$, yields that u_ε is bounded in the same space. Therefore, u belongs to $W_0^{1,2}(\Omega)$.

If, instead, $1<m<\left(\frac{2^*}{\delta}\right)'$, i.e., if $\gamma<\delta$, let $r<2$ and write

$$\int_\Omega|\nabla u_\varepsilon|^r=\int_\Omega\frac{|\nabla u_\varepsilon|^r}{(u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2}}}(u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2}}\,.$$

Then, by Hölder inequality, and (4.28)

$$\int_\Omega|\nabla u_\varepsilon|^r\le\left(\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\delta-\gamma}}\right)^{\frac{r}{2}}\left(\int_\Omega(u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2-r}}\right)^{\frac{2-r}{2}}\le C_7\left(\int_\Omega(u_\varepsilon+\varepsilon)^{\frac{(\delta-\gamma)r}{2-r}}\right)^{\frac{2-r}{2}}.$$

We now choose r such that $\frac{(\delta-\gamma)r}{2-r}=\frac{Nm(2-\delta)}{N-2m}$, with $\gamma=\frac{N(m-1)(2-\delta)}{N-2m}$. This yields $r=\frac{Nm(2-\delta)}{N-\delta m}$. Therefore, u_ε is bounded in $W_0^{1,r}(\Omega)$, so that u belongs to the same space.

If $q=1$, and $m>1$, we choose $\log(1+u_\varepsilon)$ as test function in (4.9), to obtain, after dropping nonnegative terms, that

$$\min(\alpha,1)\int_\Omega|\nabla u_\varepsilon|^2\le\int_\Omega\frac{\alpha+u_\varepsilon}{1+u_\varepsilon}|\nabla u_\varepsilon|^2\le\int_\Omega f\,\log(1+u_\varepsilon)\,,$$

and this gives an *a priori* estimate of u_ε in $W_0^{1,2}(\Omega)$ since $\log(1+u_\varepsilon)$ is bounded in $L^{m'}(\Omega)$. □

Remark 4.5 *If we assume that f belongs to $L^m(\Omega)$ with $m>\frac{N}{2}$, we can prove that the sequence $\{u_\varepsilon\}$ is bounded in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ (so that $u\in L^\infty(\Omega)$ as well). Indeed, taking $G_k(u_\varepsilon)$ as a test function in (4.9), using the sign condition on the quadratic lower order term and dropping the positive terms we obtain that*

$$\alpha\int_\Omega|\nabla G_k(u_\varepsilon)|^2\le\int_\Omega f\,G_k(u_\varepsilon)$$

which implies the result by the classical Stampacchia boundedness theorem (see [87]). Once we have proved that boundedness of u_ε in $L^\infty(\Omega)$, the boundedness of u_ε in $W_0^{1,2}(\Omega)$ easily follows (choosing for example u_ε as test function).

Remark 4.6 *If $q = 1$, it is enough to assume that $f \log(1+f)$ belongs to $L^1(\Omega)$ to obtain that u belongs to $W_0^{1,2}(\Omega)$.*

4.4 Minimization

In this Section we deal with the minimization problem for a functional of the Calculus of Variations whose Euler-Lagrange equation is of the type of (4.1), with $q = 1 - \theta$; note that this case is the "dividing range" in every result on the solution u of (4.1) proved so far (see (4.8) in Theorem 4.1 and Theorem 4.2).

Let us define the functional

$$J(v) = \frac{1}{2} \int_\Omega \Big[a(x) + |v|^{1-\theta} \Big] |\nabla v|^2 - \int_\Omega f\, v\,, \qquad v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

We have the following result.

Theorem 4.3 *Let $f \geq 0$, f in $L^m(\Omega)$, with $m \geq \frac{2N+(1-\theta)N}{N+2+(1-\theta)N}$. Then there exists a function u in $W_0^{1,2}(\Omega) \cap L^{(2-\theta)m^{**}}(\Omega)$, with $u \geq 0$, such that*

$$\frac{1}{2} \int_\Omega \Big[a(x) + u^{1-\theta} \Big] |\nabla u|^2 - \int_\Omega f\, u \leq J(v)\,, \qquad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,. \tag{4.29}$$

Furthermore, u is a solution of the equation

$$\begin{cases} -div([a(x) + u^{1-\theta}]\nabla u) + \dfrac{1-\theta}{2} \dfrac{|\nabla u|^2}{u^\theta} = f & \text{in } \Omega, \\ \hfill u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.30}$$

Remark 4.7 *We point out that the result is interesting if*

$$\frac{2N + (1-\theta)N}{N + 2 + (1-\theta)N} \leq m < \frac{2N}{N+2}\,,$$

since in this case the functional cannot be defined on $W_0^{1,2}(\Omega)$ (both terms may be unbounded).

Proof. Let $\varepsilon > 0$ and define

$$g_\varepsilon(t) = (1-\theta) \int_0^t \frac{s}{(|s| + \varepsilon)^{\theta+1}}\, ds\,,$$

and note that, for $t \geq 0$, we have

$$0 \leq g_\varepsilon(t) \leq t^{1-\theta}\,. \tag{4.31}$$

Define, for v in $W_0^{1,2}(\Omega)$, the functional

$$J_\varepsilon(v) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega [a(x) + g_\varepsilon(v)]|\nabla v|^2 - \int_\Omega \frac{f\,v}{1+\varepsilon\,f}\,, & \text{if } |v|^{1-\theta}|\nabla v|^2 \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that, thanks to (4.31), the first integral in the definition of J_ε is finite if $|v|^{1-\theta}|\nabla v|^2$ belongs to $L^1(\Omega)$.

We claim that there exists u_ε in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $u_\varepsilon \geq 0$, minimum of J_ε on $W_0^{1,2}(\Omega)$. Indeed, it is easy to see that the functional is coercive, since (recalling (4.4))

$$J_\varepsilon(v) \geq \frac{\alpha}{2}\int_\Omega |\nabla v|^2 - \frac{1}{\varepsilon}\int_\Omega v\,,$$

while weak lower semicontinuity in $W_0^{1,2}(\Omega)$ follows from a classical result by De Giorgi (see [67]). Thus the functional has a minimum u_ε in $W_0^{1,2}(\Omega)$, and one can prove that u_ε belongs to $L^\infty(\Omega)$ using standard techniques by Stampacchia (see [87]), and starting from the inequalities $J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(T_k(u_\varepsilon))$, $k \geq 0$. The fact that $u_\varepsilon \geq 0$ easily follows from the assumption $f \geq 0$, using that $J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(u_\varepsilon^+)$. Furthermore, starting from the inequality $J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(u_\varepsilon + t\varphi)$, with φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, one can prove that u_ε is a solution of

$$\begin{cases} -\text{div}([a(x) + g_\varepsilon(u_\varepsilon)]\nabla u_\varepsilon) + \dfrac{1-\theta}{2}\dfrac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}} = \dfrac{f}{1+\varepsilon\,f} & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.32}$$

in the sense that

$$\int_\Omega [a(x) + g_\varepsilon(u_\varepsilon)]\nabla u_\varepsilon \nabla\varphi + \frac{1-\theta}{2}\int_\Omega \frac{u_\varepsilon|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{\theta+1}}\varphi = \int_\Omega \frac{f\,\varphi}{1+\varepsilon\,f}\,,$$

for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Note now that problem (4.32) is essentially problem (4.9), thanks to inequality (4.31). Therefore, starting from (4.32) and using the assumptions on m, one has that u_ε is bounded in $W_0^{1,2}(\Omega)$ (since $m > (\frac{2^*}{\theta})'$) and in $L^s(\Omega)$, with $s = m^{**}(2-\theta)$ (see Theorem 4.2). Therefore, and up to subsequences, it converges, weakly in $W_0^{1,2}(\Omega)$ and weakly in $L^s(\Omega)$, to a function u. Furthermore, ∇u_ε almost everywhere converges to ∇u in Ω (see Lemma 4.3), and u is a solution of (4.30) (see Theorem 4.1).

Since u_ε is a minimum of J_ε, and

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega [a(x) + g_\varepsilon(v)]|\nabla v|^2 - \int_\Omega \frac{f\,v}{1+\varepsilon\,f}\,,$$

if v belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, we have

$$\frac{1}{2}\int_\Omega [a(x) + g_\varepsilon(u_\varepsilon)]|\nabla u_\varepsilon|^2 - \int_\Omega \frac{f\,u_\varepsilon}{1+\varepsilon\,f} \leq \frac{1}{2}\int_\Omega [a(x) + g_\varepsilon(v)]|\nabla v|^2 - \int_\Omega \frac{f\,v}{1+\varepsilon\,f}\,,$$

for every v in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. The weak convergence of u_ε to u in $L^s(\Omega)$, with $s = m^{**}(2-\theta)$, and the assumptions on m imply that

$$\frac{1}{m} + \frac{1}{m^{**}(2-\theta)} \le 1\,,$$

so that

$$\lim_{\varepsilon\to 0^+} \int_\Omega \frac{f\,u_\varepsilon}{1+\varepsilon\, f} = \int_\Omega f\,u\,.$$

Furthermore, the almost everywhere convergence of u_ε and ∇u_ε, and Fatou lemma, imply that

$$\int_\Omega [a(x) + u^q]|\nabla u|^2 \le \liminf_{\varepsilon\to 0^+} \int_\Omega [a(x) + g_\varepsilon(u_\varepsilon)]|\nabla u_\varepsilon|^2\,.$$

Thus,

$$J(u) \le \liminf_{\varepsilon\to 0^+} J_\varepsilon(u_\varepsilon)\,.$$

On the other hand, if v belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, one also has

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(v) = J(v)\,,$$

and so, passing to the limit in the inequalities $J_\varepsilon(u_\varepsilon) \le J_\varepsilon(v)$ one obtains that

$$\frac{1}{2}\int_\Omega [a(x) + u^q]|\nabla u|^2 - \int_\Omega f\,u \le \frac{1}{2}\int_\Omega [a(x) + v^q]|\nabla v|^2 - \int_\Omega f\,v\,,$$

for every v in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Hence, (4.29) holds. □

4.5 "Finite energy" solutions

In this Section we give the precise assumptions on the datum f (depending on the values of the parameters q and θ) that allow to widen the class of test function from $W_0^{1,p}(\Omega)$, $p > N$, to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, which is the "standard" set of test functions for quadratic quasilinear equations. In order to do that, we only need to have $u^q|\nabla u|$ in $L^2(\Omega)$, since this assumption (together with the fact that $T_k(u)$ belongs to $W_0^{1,2}(\Omega)$ for every k) yields that u belongs to $W_0^{1,2}(\Omega)$, and since the lower order term $b(x)\,u^{-\theta}|\nabla u|^2$ always belongs to $L^1(\Omega)$ for any value of q and θ, and for every f in $L^1(\Omega)$. In analogy with the "standard" quasilinear case, we will call these functions "finite energy" solutions.

In order to have $u^q|\nabla u|$ in $L^2(\Omega)$, we can either choose u^{q+1} as test function and use the higher order part of the equation, or choose $u^{2q+\theta}$ as test function and use the lower order term. Clearly, in order to do that one has to work on the approximate equations (4.9), choosing either u_ε^{q+1} or $u_\varepsilon^{2q+\theta}$, and proving *a priori* estimates which then pass to the limit thanks to the results proved in Section 2. Since it is better to choose the power having the lower exponent, if we define $\sigma = \min(2q+\theta, q+1)$, the choice of u_ε^σ yields, after dropping nonnegative terms,

$$\int_\Omega u_\varepsilon^{2q}|\nabla u_\varepsilon|^2 \le C \int_\Omega f\, u_\varepsilon^\sigma + C\,.$$

Therefore, if we assume that f belongs to $L^m(\Omega)$, an *a priori* estimate on $u_\varepsilon^q|\nabla u_\varepsilon|$ in $L^2(\Omega)$ will follow if the summability of u_ε^σ is larger than m', the Hölder conjugate of m.

We now recall that, setting $\delta = \min(\theta, 1-q)$, one has by Theorem 4.2 that u_ε is bounded in $L^s(\Omega)$, with $s = m^{**}(2-\delta)$. Therefore, the desired *a priori* estimate will hold true if

$$\frac{\sigma}{m^{**}(2-\delta)} \le 1 - \frac{1}{m}\,.$$

We now remark that $\sigma = \min(2q+\theta, q+1) = 2q + \min(\theta, 1-q) = 2q+\delta$. Therefore, the previous inequality can be rewritten as

$$\frac{2q+\delta}{m^{**}(2-\delta)} \le 1 - \frac{1}{m}\,.$$

Recalling that $\frac{1}{m^{**}} = \frac{1}{m} - \frac{2}{N}$, the previous inequality becomes

$$m \ge \frac{2N(q+1)}{N(2-\delta)+4q+2\delta} = \frac{2N(q+1)}{(N+2)(q+1)+(N-2)(1-q-\delta)}\,.$$

If $\delta = 1-q$, the above inequality is

$$m \ge \frac{2N}{N+2}\,;$$

in other words, the "standard" assumption on the datum which yields finite energy solutions for uniformly elliptic and bounded operators, yields solutions such that $u^q|\nabla u|$ belongs to $L^2(\Omega)$. Since $\delta = 1-q$ implies that the principal part of the equation gives a better estimate than the lower order term, this was somehow to be expected.

If $\delta = \theta$, the situation is different: in this case, the lower order term is "dominant" with respect to the differential operator, and the assumption on m becomes

$$m \ge \frac{2N(q+1)}{N(2-\theta)+4q+2\theta} = \frac{2N(q+1)}{(N+2)(q+1)+(N-2)(1-q-\theta)}\,,$$

with $\theta < 1-q$. Note that this assumption implies that

$$\frac{2N(q+1)}{(N+2)(q+1)+(N-2)(1-q-\theta)} < \frac{2N}{N+2}\,,$$

so that if the lower order term is "dominant", one needs less summability on f in order to have "finite energy" solutions. Note that, in this case, we have a condition depending on both q and θ since we want to use the lower order term (where $u^{-\theta}$ appears) to obtain an estimate on u^{2q}. Furthermore, since

$$\frac{2N(q+1)}{N(2-\theta)+4q+2\theta} > 1 \qquad \Longleftrightarrow \qquad (2q+\theta)(N-2) > 0\,,$$

which is always true, the lower bound on m is always strictly larger than 1. In other words, if the lower order term is "dominant", one never has finite energy solutions in the case of $L^1(\Omega)$ data: a fact which is in contrast with well-known results on quasilinear equations having a quadratic lower order term which does not vanish as the solution u tends to infinity.

We therefore have the following result.

Theorem 4.4 *Suppose that* (4.2), (4.4) *and* (4.5) *hold true, and let*

$$m_0 = \begin{cases} \dfrac{2N}{N+2}, & \text{if } \theta \geq 1-q, \\[2ex] \dfrac{2N(q+1)}{N(2-\theta)+4q+2\theta}, & \text{if } \theta < 1-q. \end{cases}$$

If f belongs to $L^m(\Omega)$, with $m \geq m_0$, and u is a solution of (4.1) *given by Theorem 4.1, then $u^q|\nabla u|$ belongs to $L^2(\Omega)$ and one has*

$$\int_\Omega [a(x)+u^q]\nabla u \nabla\varphi + \int_\Omega \frac{b(x)\,|\nabla u|^2}{u^\theta}\,\varphi = \int_\Omega f\,\varphi\,,$$

for every φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Chapter 5

$W^{1,1}(\Omega)$ solutions of nonlinear problems with nonhomogeneous Neumann boundary conditions

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Abstract

In this paper we study the existence of $W^{1,1}(\Omega)$ distributional solutions of the nonlinear problems with Neumann boundary condition. The simplest model is

$$\begin{cases} -\Delta_p u + |u|^{s-1}u = 0, & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta = \psi, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $1 < p < N$, $s > 0$, η is the unit outward normal on $\partial\Omega$ and $\psi \in L^m(\partial\Omega)$, $m > 1$.

This paper deals with the study of elliptic boundary value problems in a bounded, open subset Ω of $\mathbb{R}^N$, $(N \geq 2)$, with nonregular data. More concretely, the new results of this paper concern the Neumann problems whose background comes from Dirichlet problems. This is the reason why we start recalling some results about these ones.

5.1 Dirichlet problems

5.1.1 Linear operators

The first problem considered is the linear boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where

$$f(x) \in L^m(\Omega), \quad 1 < m < N, \tag{5.1}$$

and $M(x)$ is a bounded elliptic matrix, that is which satisfies, for some positive constants α and β,

$$\alpha|\xi|^2 \le M(x)\,\xi\,\xi, \quad |M(x)| \le \beta\,, \quad \forall\, \xi \in \mathbb{R}^N\,. \tag{5.2}$$

Linear operators with smooth coefficients

One of the presentations of the *Calderon-Zygmund theory* is the following one.

$$\begin{array}{c} \textit{Let } M(x) \textit{ be smooth.} \text{ If we assume (5.1), (5.2) hold,} \\ \text{then the map } f \mapsto u \text{ is continuous from } L^m(\Omega) \text{ to } W_0^{1,m^*}(\Omega), \end{array} \tag{5.3}$$

where $m^* = \frac{mN}{N-m}$. Note that m^* is not well defined if $m \ge N$.

Linear operators with discontinuous coefficients

If it is assumed that $M(x)$ is only bounded and elliptic (discontinuous coefficients), then Guido Stampacchia ([87]) proved that

$$\begin{cases} \bullet \text{ if } \frac{2N}{N+2} \le m < \frac{N}{2}, \text{ then } u \in L^{m^{**}}(\Omega); \\ \\ \bullet \text{ if } m = \frac{N}{2}, \text{ then } u \text{ has exponential summability;} \\ \\ \bullet \text{ if } m > \frac{N}{2}, \text{ then } u \in L^\infty(\Omega). \end{cases} \tag{5.4}$$

Observe that $m \ge \frac{2N}{N+2}$ implies the existence and uniqueness of the weak solution $u \in W_0^{1,2}(\Omega)$, by Lax-Milgram theorem. Moreover, if $M(x)$ is smooth, then (5.4) is a consequence of (5.3) with $m \ge \frac{2N}{N+2}$ and of the Sobolev embedding.

Note that the Stampacchia summability theorem only deals with the function u (with a straight proof) and it does not concern ∇u.

Then, what is the situation of the statement (5.3), if the coefficients of $M(x)$ are discontinuous?

If we read the Theorem 5.1 below in the linear case, with $p = 2$ and $1 < m < \frac{2N}{N+2}$, then the statement (5.3) is true even if the coefficients of $M(x)$ are discontinuous.

However the uniqueness can fail (see [86]). Furthermore the Meyers theorem ([80]) asserts that the statement (5.3) is true, under the assumption $\frac{2N}{N+2} < m \leq \frac{2N}{N+2} + \epsilon$, for some $\epsilon = \epsilon(\alpha, \beta) > 0$, even if the coefficients of $M(x)$ are discontinuous.

On the other hand, we point out that recently (see [27]) is proved that the statement (5.3) is false, if $m > \frac{N}{2}$. To be more clear, the weak solution u exists by Lax-Milgram theorem and it is bounded by Stampacchia boundedness theorem, but it does not belong to $W_0^{1,m^*}(\Omega)$, if the coefficients of $M(x)$ are discontinuous and if $m > \frac{N}{2}$. The case $\frac{2N}{N+2} + \epsilon < m \leq \frac{N}{2}$ is open.

5.1.2 Nonlinear operators

The simplest example of nonlinear (and variational) boundary value problem is the problem for the p–Laplace operator

$$\begin{cases} -\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 < \alpha \leq a(x) \leq \beta$, $1 < p < N$.

The general Leray-Lions differential operator studied is

$$A(v) = -\mathrm{div}\,(a(x, v, \nabla v)), \tag{5.5}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that the following conditions holds for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \neq \eta \in \mathbb{R}^N$:

$$\begin{cases} a(x, s, \xi)\xi \geq \alpha\,|\xi|^p\,, \\ |a(x, s, \xi)| \leq \beta|\xi|^{p-1}\,, \\ [a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0\,, \end{cases}$$

where α, β are positive constants.

We recall that the above assumptions imply that A is a pseudomonotone and coercive differential operator (acting on $W_0^{1,p}(\Omega)$) and thus it is surjective (see [31], [52], [60], [79]).

Finite energy solutions

The classical theory of nonlinear elliptic equations states that $W_0^{1,p}(\Omega)$ is the natural functional spaces framework in order to find weak solutions of the general Dirichlet problem

$$u \in W_0^{1,p}(\Omega) : \; A(u) = f\,, \tag{5.6}$$

if the function f belongs to the dual space of $W_0^{1,p}(\Omega)$, i.e., if $m \geq (p^\star)' = \frac{Np}{pN+p-N}$. Moreover, in this case, the following nonlinear version of the Stampacchia regularity theorem (5.4) is proved in [33]:

$$\begin{cases} \bullet \text{ if } \frac{Np}{pN+p-N} \le m < \frac{N}{p}, \text{ then } u \in L^{[(p-1)m^*]^*}(\Omega) \ ; \\ \bullet \text{ if } m = \frac{N}{p}, \text{ then } u \text{ has exponential summability;} \\ \bullet \text{ if } m > \frac{N}{p}, \text{ then } u \in L^\infty(\Omega). \end{cases}$$

We recall that "$u \in W_0^{1,p}(\Omega)$ is a weak solution of the Dirichlet problem $A(u) = f$" means that

$$\int_\Omega a(x,u,\nabla u)\,\nabla v = \int_\Omega f\,v, \quad \forall\, v \in W_0^{1,p}(\Omega)\,.$$

Infinite energy solutions

On the other hand, the existence of $W_0^{1,p}(\Omega)$ solutions of the problem (5.6) fails if the right hand side is a function $f \in L^m(\Omega)$, $m \ge 1$, which does not belong to the dual space of $W_0^{1,p}(\Omega)$: it is possible to find distributional solutions in functional spaces "larger" than $W_0^{1,p}(\Omega)$, but contained in $W_0^{1,1}(\Omega)$ (see [34], [35]). One of the results is the following theorem.

Theorem 5.1 (Nonlinear C.-Z. theory for infinite energy solutions) *If $f \in L^m(\Omega)$, $\max\big(1, \frac{N}{N(p-1)+1}\big) < m < \frac{Np}{pN+p-N} = (p^\star)'$, then there exists a distributional solution $u \in W_0^{1,(p-1)m^\star}(\Omega)$ of (5.6).*

Recall that "$u \in W_0^{1,q}(\Omega)$ is a distributional solution of the nonlinear Dirichlet problem $A(u) = f$" means that

$$\int_\Omega a(x,u,\nabla u)\,\nabla v = \int_\Omega f\,v, \quad \forall\, v \in W_0^{1,\infty}(\Omega)\,.$$

Note that, in this case,

$$\text{it is not possible to choose } v = u \text{ as a test function.} \tag{5.7}$$

Even for the simple operator $-\Delta_p(u)$, we remark that if $m < (p^\star)'$, it is not possible to use the standard weak formulation since the two terms

$$\langle -\Delta_p(u), v\rangle, \qquad \int_\Omega f v,$$

make no sense if $v \in W_0^{1,p}(\Omega)$.

The steps of the proof (used in [35]) of Theorem 5.1 are the following ones.

1. Consider the nonlinear Dirichlet problems

$$u_n \in W_0^{1,p}(\Omega) : \; A(u_n) \; = \frac{f}{1 + \frac{1}{n}|f|} \, . \tag{5.8}$$

   The existence of a solution u_n is a consequence of the Leray-Lions theorem. Moreover every u_n is a bounded function, thanks to the Stampacchia boundedness theorem; so that it is possible to use nonlinear composition of u_n as a test function.

2. The sequence $\{u_n\}$ is bounded in $W_0^{1,(p-1)m^\star}(\Omega)$. Thus, there exists a function $u \in W_0^{1,(p-1)m^\star}(\Omega)$ and a subsequence $\{u_{n_j}\}$ such that $\{u_{n_j}\}$ converges weakly in $W_0^{1,(p-1)m^\star}(\Omega)$ to u. Observe that the assumption $m > \frac{N}{N(p-1)+1}$ implies that $(p-1)m^\star > 1$ (reflexive framework).

3. Of course the weak convergence is not enough to pass to the limit in the term $a(x, u_{n_j}, \nabla u_{n_j})$, so that the second important step is the proof that the sequence $\{\nabla u_{n_j}(x)\}$ converges a.e. (see [22], [34], [35]).

4. Then, with the use of Real Analysis tools, it is possible to pass to the limit in (5.8) and to prove that u is a distributional solution.

$W_0^{1,1}(\Omega)$ solutions

In [38], for some values of p and m (borderline cases of the Theorem 5.1), is proved the existence of solutions belonging to $W_0^{1,1}(\Omega)$ and not belonging to $W_0^{1,q}(\Omega)$, $1 < q < p$. We point out that the existence in $W_0^{1,1}(\Omega)$ is not so usual in the study of elliptic problems. In [38] are proved the following two theorems, where the main difficulty is due to the a priori estimate of the sequence of the approximate solutions $\{u_n\}$ of (5.8) in the non-reflexive space $W_0^{1,1}(\Omega)$.

Theorem 5.2 *Let $f \in L^m(\Omega)$, $m = \frac{N}{N(p-1)+1}$, $1 < p < 2 - \frac{1}{N}$. Then there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of (5.6).*

Theorem 5.3 *Assume*

$$\int_\Omega |f| \, \log(1 + |f|) < \infty,$$

and $p = 2 - \frac{1}{N}$. Then there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of (5.6).

Entropy solutions

In the case $1 \le m < \frac{N}{N(p-1)+1}$ the difficulty presented in (5.7) is even stronger, the solutions u of (5.6), in general, do not belong to $W_0^{1,p}(\Omega)$, nor to some $W_0^{1,q}(\Omega)$, but the key point is the proof that they satisfy

$$\int_\Omega |\nabla T_k(u)|^p \le \frac{\|f\|_{L^1(\Omega)}}{\alpha}\, k, \quad \forall\, k > 0,$$

where

$$T_k(s) = \begin{cases} s, & \text{if } |s| \le k, \\ k\dfrac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

In order to overcome the difficulty $u \notin W_0^{1,q}(\Omega)$, a definition involving the truncations was introduced in [18], inspired by [37]. This definition is also useful in the study of the uniqueness, if $a(x, s, \xi)$ does not depend on s.

Definition 5.1 *Let $f \in L^1(\Omega)$. A measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$ is an entropy solution of (5.6) if*

$$\int_\Omega a(x, u, \nabla u)\, \nabla T_k(u - \varphi) \le \int_\Omega f\, T_k(u - \varphi)\,,$$

for every φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

In [18], the existence of entropy solutions u is proved under the assumptions $f \in L^1(\Omega)$ and $p > 1$. Moreover it is also proved that (see [34])

$$\begin{cases} \displaystyle\int_\Omega |\nabla T_k(u)|^p \le \frac{\|f\|_{L^1(\Omega)}}{\alpha}\, k, \quad \forall\, k > 0; \\ \displaystyle\int_\Omega |\nabla \log(1 + |u|)|^p \le \frac{\|f\|_{L^1(\Omega)}}{\alpha}; \\ u \in W_0^{1,q}(\Omega),\ q < \dfrac{(p-1)N}{N-1}, \text{ if } p > 2 - \dfrac{1}{N}. \end{cases}$$

The corresponding unilateral problems are studied in [29], [30].

5.1.3 The impact of a lower order term depending on u

The presence of a lower order term can play an important role (see [19], [53], [54], [55], [56], [58]); in particular it is important in order to improve the summability of the weak solutions. For example, for the solution u of the following semilinear problem (where $\lambda > 0$ and $r > 0$)

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \lambda\, |u|^{r-1}\, u = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

we still have (5.4), but we also have

$$\lambda \|u\|^r_{L^{rm}(\Omega)} \le \|f\|_{L^m(\Omega)},$$

which is better than $u \in L^{m^{**}}(\Omega)$, if $r > \frac{N}{N-2m}$.

Even more important is the case of distributional solutions: in [63] is proved that,

$$\text{if } r \geq \tfrac{1}{m-1}, \text{ then } u \in W_0^{1,2}(\Omega), \tag{5.9}$$

even if $m < \frac{2N}{N+2}$.

Moreover, in [26] is studied the Dirichlet problem

$$u \in W_0^{1,p}(\Omega): \; A(u) + u = f\,, \tag{5.10}$$

for small values of p. It is proved that, if $f \in L^m(\Omega)$, with m such that

$$\frac{2}{p} \leq m \leq \min\Big(\frac{(2-p)N}{p}, 2\Big), \qquad 1 < p \leq 2 - \frac{2}{N},$$

then there exists a distributional solution $u \in W_0^{1,\frac{pm}{2}}(\Omega)$ of (5.10).

5.1.4 The impact of a lower order term depending on ∇u

The presence of a lower order term depending on ∇u plays a very important role; not only in order to improve the summability stated in (5.4), but also in order to have a regularizing effect as in (5.9). For example, for the solution u of the following quasi-linear problem (where $\lambda > 0$ and $r > 0$) having lower order term with natural growth with respect to the gradient

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + \lambda\, u|u|^{r-1}\, |\nabla u|^2 = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

in [36], is proved that $u \in W_0^{1,2}(\Omega)$, if $f \in L^1(\Omega)$. See also [39], where the datum is sum of an element in $W^{-1,2}(\Omega)$ and of a function in $L^1(\Omega)$.

5.2 Neumann problems

In this section, we consider the following nonlinear problem with Neumann boundary conditions

$$\begin{cases} A(u) + |u|^{s-1}u = 0, & \text{in } \Omega, \\ a(x,u,\nabla u)\; \cdot \eta = \psi, & \text{on } \partial\Omega, \end{cases} \tag{5.11}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $1 < p < N$, $s > 0$, η is the unit outward normal on $\partial\Omega$, and

$$\psi \in L^m(\partial\Omega), \; m > 1.$$

The differential operator A, acting on $W^{1,p}(\Omega)$, is defined by (5.5) and it satisfies the classical Leray-Lions assumptions. Thus the starting point of the Neumann problem again is the Leray-Lions theory; of course the coercivity depends on the different definition of norm.

Important contributions to the entropy solutions of very general nonlinear Neumann problem can be found in [3], [4], [5], [6].

In [42], a Calderon-Zygmund theory for the problem (5.11) is studied. For the sake of simplicity the authors only consider the case $p = 2$, in order to avoid more than two parameters and they prove the following theorem.

Theorem 5.4 *Assume $p = 2$ and $s > 0$, the following holds:*

(1) If $m > N - 1$ there is a bounded weak solution of problem (5.11).

(2) If $m = N - 1$, there is a weak solution of problem (5.11) *such that $e^{\lambda|u|} \in L^1(\Omega)$, for every $\lambda > 0$.*

(3) If $2 - \frac{2}{N} \leq m < N - 1$ there is weak solution u of problem (5.11) *with*

$$u \in L^{\frac{m(N-2)}{(N-1-m)}+s-1}(\Omega).$$

(4) If $1 < m < 2 - \frac{2}{N}$, there is a distributional solution u of problem (5.11) *with $u \in W^{1,q}(\Omega)$, for q satisfying*

$$\begin{cases} q = \dfrac{mN}{N-1}, \quad 1 < s < \dfrac{N-1+m}{N-1-m}; \\[2ex] q = \dfrac{2(m-1)(N-1)}{(N-1-m)(s+1)} + \dfrac{2s}{s+1}, \quad s \geq \dfrac{N-1+m}{N-1-m}. \end{cases}$$

5.2.1 The main result

Our aim is to study the borderline framework. We will prove, for some values of p, s and m, the existence of distributional solutions of (5.11) belonging to $W^{1,1}(\Omega)$.

We prove the following theorem.

Theorem 5.5 *Suppose that $\psi \in L^m(\partial\Omega)$ with $m = \frac{N-1}{N(p-1)}$, $1 < p < 2 - \frac{1}{N}$ and $0 < s < \frac{1+N(p-1)}{N-1}$. Then there exists a solution $u \in W^{1,1}(\Omega)$ of the problem* (5.11) *in the sense*

$$\int_\Omega a(x,u,\nabla u)\,\nabla v + \int_\Omega |u|^{s-1}\,uv = \int_{\partial\Omega} \psi v\,, \;\; \forall v \in C^1(\bar{\Omega})\,.$$

Remark 5.1 *Taking into account that $p < 2 - \frac{1}{N}$, we observe that $\frac{N-1}{(p-1)N} > 1$. Furthermore, $0 < p - 1 \leq \frac{1+N(p-1)}{N-1}$ and so it is possible to take $s = p - 1$.*

In order to prove our result, we approximate the problem (5.11) by a sequence of problems with $L^\infty(\partial\Omega)$ data and we prove some a priori estimates.

We define $\psi_n = T_n(\psi)$ or $\psi_n = \frac{\psi_n}{1+\frac{1}{n}|\psi|}$ and we deal with the sequence $\{u_n\}$ of functions $u_n \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ weak solutions of

$$\int_\Omega a(x,u_n,\nabla u_n)\,\nabla v + \int_\Omega |u_n|^{s-1}u_n v = \int_{\partial\Omega} \psi_n v\,, \quad \forall\, v\in W^{1,p}(\Omega)\,. \tag{5.12}$$

Observe that the existence of such sequence is a consequence of the Leray-Lions theory ([79]). We obtain a solution of (5.11) as a limit of the sequence $\{u_n\}$.

Throughout the proof of Theorem 5.5, we will use the following results.

Theorem 5.6 (Trace Theorem) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1 and with the segment property and $1\le p<N$. There is a constant C depending only on p, N such that*

$$\|v\|_{L^{\frac{p(N-1)}{N-p}}(\partial\Omega)} \le C\|v\|_{W^{1,p}(\Omega)} \quad \forall\, v\in W^{1,p}(\Omega).$$

Lemma 5.1 *The norm*

$$\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}$$

is an equivalent norm in $W^{1,p}(\Omega)$ to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$ for all $1\le q\le p$.

Proof. It is enough to prove that

$$\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}$$

is an equivalent norm in $W^{1,p}(\Omega)$ to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. In order to do it, we use the Poincaré inequality and we have that there is $C_0>0$ such that

$$||u||_{L^p(\Omega)} - (\text{meas}\{\Omega\})^{1/p}||u||_{L^1(\Omega)} \le ||\,u - ||u||_{L^1(\Omega)}||_{L^p(\Omega)} \le C_0\,||\nabla u||_{L^p(\Omega)}\,.$$

Then,

$$||u||_{L^p(\Omega)} \le (\text{meas}\{\Omega\})^{1/p}||u||_{L^1(\Omega)} + C_0\,||\nabla u||_{L^p(\Omega)}\,,$$

which implies the result. □

Lemma 5.2 *Assume that $v\in W^{1,p}(\Omega)$ and $0<\theta<1$. Then, there exists $M>0$ such that*

$$\|v\|_{W^{1,p}(\Omega)} \le M\bigg(||\nabla v||_{L^p(\Omega)} + \bigg[\int_\Omega |v|^\theta\bigg]^{\frac{1}{\theta}}\bigg).$$

Proof. Take $0<\lambda<1$ such that $1=(1-\lambda)\theta+\lambda p^*$. By Hölder and Young inequalities, we have

$$\int_\Omega |v| = \int_\Omega |v|^{(1-\lambda)\theta+\lambda p^*} \le \left[\int_\Omega |v|^\theta\right]^{1-\lambda}\left[\int_\Omega |v|^{p^*}\right]^\lambda$$
$$= \left[\int_\Omega |v|^\theta\right]^{\frac{1}{\theta}(1-\lambda)\theta} ||v||^{\lambda p^*}_{L^{p^*}(\Omega)} \le \varepsilon ||v||_{L^{p^*}(\Omega)} + C_\varepsilon \left[\int_\Omega |v|^\theta\right]^{\frac{1}{\theta}(1-\lambda)\theta\frac{1}{1-\lambda p^*}}.$$

Thus, by Sobolev's inequality

$$||\nabla v||_{L^p(\Omega)} + ||v||_{L^1(\Omega)} \le ||\nabla v||_{L^p(\Omega)} + \varepsilon ||v||_{L^{p^*}(\Omega)} + C_\varepsilon\left[\int_\Omega |v|^\theta\right]^{(1-\lambda)\frac{1}{1-\lambda p^*}}$$
$$\le ||\nabla v||_{L^p(\Omega)} + \varepsilon\, S\, ||\nabla v||_{L^p(\Omega)} + C_\varepsilon\left[\int_\Omega |v|^\theta\right]^{\frac{1}{\theta}},$$

and then

$$(1-\varepsilon S)||\nabla v||_{L^p(\Omega)} + ||v||_{L^1(\Omega)} \le ||\nabla v||_{L^p(\Omega)} + C_\varepsilon\left[\int_\Omega |v|^\theta\right]^{\frac{1}{\theta}}$$
$$\le M_\varepsilon ||\nabla v||_{L^p(\Omega)} + M_\varepsilon\left[\int_\Omega |v|^\theta\right]^{\frac{1}{\theta}}$$

which give us the result after using Lemma 5.1. □

Remark 5.2 *If* $0<\gamma\le p$ *and*

$$\int_\Omega |\nabla v|^p + \int_\Omega |v|^\gamma \le A,$$

then there is $R>0$ *such that*

$$\|v\|_{W^{1,p}(\Omega)} \le R + RA^{\frac{1}{\gamma}}.$$

Indeed, if $0<\gamma<1$*, then* $\gamma=\theta$ *and Lemma 5.2 implies*

$$\|v\|_{W^{1,p}(\Omega)} \le M\left(||\nabla v||_{L^p(\Omega)} + \left[\int_\Omega |v|^\theta\right]^{\frac{1}{\theta}}\right) \le M\left(A^{\frac{1}{p}} + A^{\frac{1}{\theta}}\right) \le R + RA^{\frac{1}{\theta}}.$$

If $1\le\gamma\le p$*, then* $\gamma=q$ *and Lemma 5.1 implies*

$$\|v\|_{W^{1,p}(\Omega)} \le C\left(||\nabla v||_{L^p(\Omega)} + \left[\int_\Omega |v|^q\right]^{\frac{1}{q}}\right) \le C\left(A^{\frac{1}{p}} + A^{\frac{1}{q}}\right) \le R + RA^{\frac{1}{q}}.$$

Proof of Theorem 5.5. We define $\theta := \frac{N-p}{p(N-1)}$ and we observe that $0 < \theta < 1$ since $p > 1$. Let ε be a strictly positive number and consider the function

$$v_n = [(\varepsilon + |u_n|)^{p\theta-p+1} - \varepsilon^{p\theta-p+1}]\operatorname{sign}(u_n)$$

which is bounded since $1 - p(1-\theta) > 0$. Taking v_n as test function in (5.12) we have

$$\alpha(p\theta - p + 1)\int_\Omega |\nabla u_n|^p(\varepsilon + |u_n|)^{p\theta-p} + \int_\Omega |u_n|^s[(\varepsilon + |u_n|)^{p\theta-p+1} - \varepsilon^{p\theta-p+1}]$$
$$\leq \|\psi\|_{L^m(\partial\Omega)}\left[\int_{\partial\Omega}(\varepsilon + |u_n|)^{(p\theta-p+1)m'}\right]^{\frac{1}{m'}}$$

and thus

$$\alpha(p\theta - p + 1)\int_\Omega |\nabla u_n|^p(\varepsilon + |u_n|)^{p\theta-p} + \int_\Omega |u_n|^{p\theta-p+1+s} - \varepsilon^{p\theta-p+1}\int_\Omega |u_n|^s$$
$$\leq \|\psi\|_{L^m(\partial\Omega)}\left[\int_{\partial\Omega}(\varepsilon + |u_n|)^{(p\theta-p+1)m'}\right]^{\frac{1}{m'}}.$$

Since every u_n is bounded, using Fatou and Lebesgue theorem, we can pass to the limit when $\varepsilon \to 0$ and we obtain

$$\int_\Omega |\nabla|u_n|^\theta|^p + \int_\Omega |u_n|^{p\theta-p+1+s} \leq \|\psi\|_{L^m(\partial\Omega)}\left[\int_{\partial\Omega}|u_n|^{(p\theta-p+1)m'}\right]^{\frac{1}{m'}}. \tag{5.13}$$

Now we need to split the difference between $s \geq p-1$ and $s < p-1$.

On the one hand, if $s \geq p-1$, the inequality

$$-\frac{1}{2} + \frac{1}{2}|s|^{p\theta} + \frac{1}{2}|s|^{p\theta-p+1+s} \leq |s|^{p\theta-p+1+s}, \ s \geq p-1,$$

and (5.13) imply

$$\int_\Omega |\nabla|u_n|^\theta|^p + \int_\Omega |u_n|^{p\theta} + \int_\Omega |u_n|^{p\theta-p+1+s} \leq C_1\left[\int_{\partial\Omega}|u_n|^{(p\theta-p+1)m'}\right]^{\frac{1}{m'}} + C_1. \tag{5.14}$$

From here, using Theorem 5.6, we obtain

$$\left[\int_{\partial\Omega}|u_n|^{\frac{p\theta(N-1)}{N-p}}\right]^{\frac{N-p}{N-1}} + \int_\Omega |u_n|^{p\theta-p+1+s} \leq C_2\left[\int_{\partial\Omega}|u_n|^{(p\theta-p+1)m'}\right]^{\frac{1}{m'}} + C_2. \tag{5.15}$$

Thanks to the choice of θ and the value of m we have

$$\theta\,\frac{p(N-1)}{N-p} = (p\theta - p + 1)m' = 1\,. \tag{5.16}$$

Moreover, $\frac{N-p}{N-1} > \frac{1}{m'}$ since $m < \frac{N-1}{p-1}$. Therefore, from (5.15), we deduce that

$$\int_{\partial\Omega} |u_n| \leq C_3.$$

Consequently, since $p^*\theta = 1^* = \frac{N}{N-1}$, from (5.14) and using Sobolev inequality it follows that

$$\int_\Omega |u_n|^{\frac{N}{N-1}} \leq C_4\,, \quad \int_\Omega |u_n|^{\frac{N-p}{N-1}-p+1+s} \leq C_4\,, \tag{5.17}$$

where in the last inequality we have used that

$$p\theta - p + 1 + s = \frac{N-p}{N-1} - p + 1 + s\,.$$

We observe that

$$\frac{N}{N-1} > \frac{N-p}{N-1} - p + s + 1 \Longleftrightarrow s < \frac{1+N(p-1)}{N-1}\,,$$

and that the choice of θ implies

$$(1-\theta)p' = 1^*\,.$$

Furthermore, by the above estimates $\displaystyle\int_\Omega \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}} \leq C_5$, and then using Hölder's inequality, we deduce

$$\int_\Omega |\nabla u_n| = \int_\Omega \frac{|\nabla u_n|}{|u_n|^{(1-\theta)}} |u_n|^{(1-\theta)} \leq C_6 \left[\int_\Omega |u_n|^{(1-\theta)p'}\right]^{\frac{1}{p'}} = C_6 \left[\int_\Omega |u_n|^{1^*}\right]^{\frac{1}{p'}}$$

which with (5.17) gives us the boundedness of the sequence $\{u_n\}$ in $W^{1,1}(\Omega)$.

On the other hand, if $0 < s < p-1$, from (5.13) we have

$$\int_\Omega |\nabla |u_n|^\theta|^p + \int_\Omega (|u_n|^\theta)^{\frac{p\theta-p+1+s}{\theta}} \leq \|\psi\|_{L^m(\partial\Omega)} \left[\int_{\partial\Omega} |u_n|^{(p\theta-p+1)m'}\right]^{\frac{1}{m'}}.$$

Since, $0 < \gamma = \frac{p\theta-p+1+s}{\theta} < p$, by Remark 5.2,

$$\||u_n|^\theta\|_{W^{1,p}(\Omega)} \leq C_7 + C_7 \left[\int_{\partial\Omega} |u_n|^{(p\theta-p+1)m'}\right]^{\frac{1}{m'\gamma}}. \tag{5.18}$$

Therefore, by Theorem 5.6, we obtain that

$$\left[\int_{\partial\Omega} (|u_n|^\theta)^{\frac{p(N-1)}{N-p}}\right]^{\frac{N-p}{p(N-1)}} \leq C_8 \left[\int_{\partial\Omega} |u_n|^{(p\theta-p+1)m'}\right]^{\frac{\theta}{m'(p\theta-p+1+s)}} + C_8.$$

Since, $\frac{N-p}{p(N-1)} > \frac{\theta}{m'(p\theta-p+1+s)}$ and recalling (5.16), we obtain

$$\int_{\partial\Omega} |u_n| \le C_9.$$

As a consequence, using (5.18) and the fact that $p^*\theta = 1^*$ we deduce again that

$$\int_\Omega |u_n|^{\frac{N}{N-1}} \le C_{10}\,,$$

and we can follow the same arguments that we did when $s \ge p-1$.

In any case ($0 < s < p-1$ or $s \ge p-1$), we have that the sequence $\{u_n\}$ is bounded in $W^{1,1}(\Omega)$. Consequently, there exists a subsequence (not relabelled) $\{u_n\}$ converging in $L^r(\Omega)$ with $1 \le r < \frac{N}{N-1}$ and almost everywhere in Ω to a function u in $L^r(\Omega)$.

Furthermore, the use of $T_k(u_n)$ as a test function yields that

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p \le k \int_{\partial\Omega} |\psi|\,, \tag{5.19}$$

which implies the boundedness of $\{T_k(u_n)\}$ in $W^{1,p}(\Omega)$.

Now, we prove that

$$u_n \rightharpoonup u\,, \quad \text{in } W^{1,1}(\Omega)\,.$$

We define the real function

$$\varphi_{k,\varepsilon}(s) = \begin{cases} 0, & \text{if } 0 \le s \le k, \\ \dfrac{s-k}{\varepsilon}, & \text{if } k < s \le \varepsilon + k, \\ +1, & \text{if } s > \varepsilon + k, \\ \varphi_{k,\varepsilon}(-s) = -\varphi_{k,\varepsilon}(s), & \end{cases}$$

and we follow the arguments of [38]. We define again $\theta := \frac{N-p}{p(N-1)}$ and we take $v_n = |u_n|^{p\theta-p+1}\,\varphi_{k,\varepsilon}(u_n)$ as a test function in (5.12). Dropping the positive term, we get

$$\begin{aligned} \alpha(p\theta-p+1)\int_\Omega |\nabla u_n|^p |u_n|^{p\theta-p}\,\varphi_{k,\varepsilon}(|u_n|) + \int_\Omega |u_n|^{s-1}|u_n|^{p\theta-p+1}\,u_n\varphi_{k,\varepsilon}(u_n) \\ \le \left[\int_{\{x\in\partial\Omega:|u_n|\ge k\}} |\psi|^m\right]^{\frac{1}{m}} \left[\int_{\{x\in\partial\Omega:|u_n|\ge k\}} |u_n|^{(p\theta-p+1)m'}\right]^{\frac{1}{m'}}. \end{aligned}$$

Passing to the limit as ε goes to zero we obtain, thanks to the boundedness of $\{u_n\}$ in $L^1(\partial\Omega)$, that

$$\int_{\{x\in\Omega:|u_n|\ge k\}} \frac{|\nabla u_n|^p}{|u_n|^{p(1-\theta)}} \le C_{11}\left[\int_{\{x\in\partial\Omega:|u_n|\ge k\}} |\psi|^m\right]^{\frac{1}{m}}.$$

Then, using Hölder inequality, the fact that $(1-\theta)p' = 1^*$ and the boundedness of $\{u_n\}$ in $L^{1^*}(\Omega)$, we deduce

$$\int_{\{x\in\Omega:|u_n|\geq k\}} |\nabla u_n| = \int_{\{x\in\Omega:|u_n|\geq k\}} \frac{|\nabla u_n|}{|u_n|^{(1-\theta)}}|u_n|^{(1-\theta)} \leq C_{12}\Big[\int_{\{x\in\partial\Omega:|u_n|\geq k\}} |\psi|^m\Big]^{\frac{1}{mp}}.$$

Therefore, if E is a measurable subset of Ω,

$$\begin{aligned}\int_E |\nabla u_n| &\leq C_{12}\Big[\int_{\{x\in\partial\Omega:|u_n|\geq k\}} |\psi|^m\Big]^{\frac{1}{mp}} + \int_E |\nabla T_k(u_n)\,| \\ &\leq C_{12}\Big[\int_{\{x\in\partial\Omega:|u_n|\geq k\}} |\psi|^m\Big]^{\frac{1}{mp}} + \Big[\int_\Omega |\nabla T_k(u_n)\,|^p\Big]^{\frac{1}{p}} (\text{meas}\{E\})^{1-\frac{1}{p}},\end{aligned}$$

and we deduce, using (5.19), that

$$\lim_{\text{meas}(E)\to 0} \int_\Omega |\nabla u_n| = 0 \tag{5.20}$$

uniformly with respect to n. As a consequence, using Dunford-Pettis Theorem and the fact that u_n converges to u strongly in $L^1(\Omega)$, we prove that $\nabla u \in L^1(\Omega)$ and $\{\nabla u_n\}$ weakly converges to ∇u in $(L^1(\Omega))^N$.

To finish, we claim that

$$\nabla u_n \to \nabla u \quad \text{in } (L^1(\Omega))^N. \tag{5.21}$$

First of all, we prove the almost everywhere convergence of $\{\nabla u_n(x)\}$ to $\nabla u(x)$. In order to do it, we use $T_k(u_n) - T_k(u)$ as a test function in (5.12). Hence,

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\,\nabla[T_k(u_n)-T_k(u)] + \int_\Omega |u_n|^{s-1}u_n[T_k(u_n)-T_k(u)] = \\ = \int_{\partial\Omega} \psi_n[T_k(u_n)-T_k(u)].\end{aligned}$$

Observe that the the last integral converges to zero. Moreover, since $0 < s < \frac{N}{N-1}$ and $\{u_n\}$ converging in $L^r(\Omega)$ for all $1 \leq r < \frac{N}{N-1}$, then the second integral also converges to zero. Thus, we deduce

$$\lim_n \int_\Omega a(x,u_n,\nabla u_n)\,\nabla[T_k(u_n)-T_k(u)] = 0. \tag{5.22}$$

In [22] is proved that the convergence (5.22) implies the almost everywhere convergence of $\{\nabla u_n(x)\}$ to $\nabla u(x)$ (see also [34], [35]).

Therefore, (5.20) and Vitali theorem imply our claim (5.21).

Using all the above estimates, we can prove that u is a $W^{1,1}(\Omega)$ distributional solution of (5.11). Indeed, since $0 < s < \frac{N}{N-1}$ and $0 < p-1 < 1$, we have that

$$\lim_n \int_\Omega a(x, u_n, \nabla u_n)\, \nabla v = \int_\Omega a(x, u, \nabla u)\, \nabla v\,,$$

$$\lim_n \int_\Omega |u_n|^{s-1} u_n v = \int_\Omega |u|^{s-1} u v\,,$$

for all $v \in C^1(\bar{\Omega})$. As a consequence, $u \in W^{1,1}(\Omega)$ satisfies

$$\int_\Omega a(x, u, \nabla u)\, \nabla v + \int_\Omega |u|^{s-1}\, uv = \int_{\partial\Omega} \psi v\,, \;\; \forall v \in C^1(\bar{\Omega})$$

which completes the proof. □

Resumen

En esta memoria de tesis doctoral se pretende estudiar varias de las cuestiones más relevantes en la teoría de las ecuaciones semilineales y casilineales de tipo elíptico. En particular, los resultados presentados en este manuscrito se concentran en el estudio de problemas que poseen una no linealidad que es singular en cero. Varias razones motivan nuestro estudio, abarcando desde las aplicaciones modeladas por este tipo de ecuaciones hasta el interés puramente matemático que surge desde el Cálculo de Variaciones.

Esta tesis está dividida en cinco capítulos, cada uno de los cuales contiene los resultados que se han obtenido. Todos los capítulos se pueden leer de forma independiente, aunque la mayoría de la terminología utilizada, así como varias de las técnicas presentadas, son compartidas por cada uno de ellos. Exceptuando algunos cambios de notación, realizados con el fin de unificar la presentación de esta memoria, así como la incorporación de la bibliografía completa al final del manuscrito, el Capítulo 1 está publicado como [14] en la revista Nonlinear Anal., el Capítulo 3 está publicado como [81] en la revista Nonlinear Anal., el Capítulo 4 está publicado como [44] en la revista Milan J. Math., el Capítulo 5 está publicado como [43] en la revista Milan J. Math., mientras que el Capítulo 2 se corresponde con el artículo de investigación [15] que aún está pendiente de ser publicado.

Si bien es cierto que cada capítulo contiene su propia introducción, referida al problema concreto que se presenta en éste, a continuación se ha considerado oportuno presentar y motivar globalmente todos los resultados que aparecen en esta memoria.

En el primer capítulo, se consideran problemas cuya ecuación modelo más básica es de la forma

$$-\Delta u = g(x,u)\,, \quad \text{en } \Omega\,,$$

siendo $\Omega \subset \mathbb{R}^N$ abierto y acotado y g una función que presenta una singularidad en $u = 0$. Este tipo de problemas ha sido extensivamente estudiado en la últimas décadas, siendo sus precursores el trabajo de Stuart [88], así como el trabajo de Crandall, Rabinowitz y Tartar [65]. En el primero de ellos, el autor consideró una función $g(x,s)$ que "explota en $s = 0$" cuando x se aproxima a un punto perteneciente al borde de Ω. Sin embargo, en el segundo de ellos, los autores consideraron una función singular $g(x,s) = g(s)$ que no depende de la variable x y estudian la existencia de solución así como propiedades de continuidad sobre ésta. Posteriormente, en 1991, Lazer y McKenna [78] estudiaron la existencia de solución clásica (y la regularidad de la misma) para el problema de Dirichlet asociado a la ecuación anterior en el caso de que

$$g(x,u) = \frac{f(x)}{u^\gamma}\,,$$

siendo f una función Hölder continua y estrictamente positiva en $\overline{\Omega}$ y γ un parámetro estrictamente positivo. En particular, en este trabajo los autores probaron que

> "*si para* $0<\alpha<1$ *se verifica que* $\partial\Omega \in C^{2,\alpha}$*,* $f \in C^{0,\alpha}(\overline{\Omega})$*,* $f(x)>0$ *para todo* $x\in\overline{\Omega}$ *y* $\gamma>0$*, entonces existe una única función* $u\in C^{2,\alpha}(\Omega)\cap C(\overline{\Omega})$*,* $u>0$ *en* Ω*, solución del problema de Dirichlet*"

$$\begin{cases} -\Delta u = \dfrac{f(x)}{u^\gamma}, & en\ \Omega, \\ u = 0, & en\ \partial\Omega. \end{cases} \tag{5.23}$$

Obsérvese que la condición de borde impuesta en (5.23) dificulta aún más el estudio de este tipo de ecuaciones singulares. En efecto, la interacción entre el requisito "$u=0$ en $\partial\Omega$ " y la singularidad considerada implica que, para cualquier solución u, el término $1/u(x)^\gamma$ diverge en todo punto x de la frontera del conjunto Ω.

El estudio de la existencia de solución de los problemas elípticos también puede abordarse desde un punto de vista distribucional. Esto es, buscar soluciones distribucionales u de la ecuación diferencial asociada a (5.23) que, en algún sentido a especificar, verifiquen la condición de borde "$u=0$ en $\partial\Omega$". Más detalladamente, se buscarán soluciones $u\in W^{1,1}_{\rm loc}(\Omega)$ de la ecuación diferencial

$$-\Delta u = \frac{f(x)}{u^\gamma}, \quad \text{en } \Omega, \tag{5.24}$$

tales que, $u>0$ a.e. en Ω, $\frac{f(x)}{u^\gamma}\in L^1_{\rm loc}(\Omega)$ y además verifiquen (solución distribucional)

$$\int_\Omega \nabla u \nabla\phi = \int_\Omega \frac{f(x)}{u^\gamma}\phi\,, \quad \forall \phi\in C^1_c(\Omega)\,.$$

Con el fin de dar un sentido a la condición "$u=0$ en $\partial\Omega$", merece la pena destacar el sorprendente resultado probado por Lazer y McKenna en su trabajo de 1991. Concretamente, los autores demostraron que

> "*la única solución* u *del problema de Dirichlet* (5.23) *pertenece al espacio de Sobolev* $W^{1,2}_0(\Omega)$ *sí y sólo sí el parámetro* $\gamma<3$."

Consecuentemente, en el marco distribucional no sería esperable encontrar soluciones que pertenezcan al espacio de Sobolev $W^{1,2}_0(\Omega)$ para cualquier valor positivo del parámetro γ. Por tanto, será necesario establecer un nuevo concepto para la condición "$u=0$ en $\partial\Omega$".

Precisamente, en el año 2010, Boccardo y Orsina [51] estudiaron la existencia de solución distribucional y positiva para el problema (5.23). Con respecto a la condición de borde "$u = 0$ en $\partial\Omega$", en contraste con [61, 75] dónde esta condición se entiende mediante la imposición $(u - \varepsilon)^+ \in W_0^{1,2}(\Omega)$ para todo $\varepsilon > 0$, se sigue la línea de trabajo planteada en [9]. Es decir, se requiere una condición aún más fuerte, a saber, que determinadas potencias positivas de la solución de la ecuación diferencial (5.24) pertenezcan al espacio de Sobolev $W_0^{1,2}(\Omega)$.

En este trabajo, los autores necesitaron estudiar separadamente los casos en los que $\gamma < 1$, $\gamma = 1$ y $\gamma > 1$, relacionando cada uno de éstos con la regularidad del dato f. En particular, demostraron el siguiente resultado.

"Supongamos que $f \in L^m(\Omega)$, con $m \geq 1$.

1. *Si $\gamma < 1$ y $m \geq \left(\frac{2^*}{1-\gamma}\right)'$, entonces existe una solución positiva u de la ecuación (5.24) tal que $u \in W_0^{1,2}(\Omega)$;*
2. *Si $\gamma = 1$ y $m = 1$, entonces existe una solución positiva u de la ecuación (5.24) tal que $u \in W_0^{1,2}(\Omega)$;*
3. *Si $\gamma > 1$ y $m = 1$, entonces existe una solución positiva u de la ecuación (5.24) tal que $u^{\frac{\gamma+1}{2}} \in W_0^{1,2}(\Omega)$".*

Obsérvese que, en relación con los resultados obtenidos por Lazer y McKenna, aquí sólo se ha probado la existencia de solución perteneciente al espacio de Sobolev $W_0^{1,2}(\Omega)$ en el caso $\gamma \leq 1$.

Naturalmente, se pueden considerar otro tipo de problemas semilineales singulares que presentan otras nuevas dificultades respecto al anterior. Tal es por ejemplo el caso del siguiente modelo.

$$\begin{cases} -\Delta u = \dfrac{\lambda}{u^\gamma} + u^p, & \text{en } \Omega, \\ u = 0, & \text{en } \partial\Omega, \end{cases} \tag{5.25}$$

siendo λ, γ parámetros positivos y $p > 1$. A las dificultades que conlleva el estudio de este tipo de problemas singulares, debemos añadir que éste es un problema superlineal, esto es,

$$\lim_{s \to +\infty} \frac{1}{s} \left(\frac{\lambda}{s^\gamma} + s^p \right) = +\infty \,,$$

lo que complica, entre otros aspectos, el estudio de *cotas a priori*.

Este modelo fue considerado en 1989 por Coclite y Palmieri [64], y posteriormente ha sido extendido en trabajos tales como [21, 75].

Boccardo [21] consideró el problema anterior siguiendo la definición utilizada en [51] para el estudio distribucional del problema (5.23). Por un lado, se dice que u es una solución distribucional positiva de la ecuación diferencial asociada al problema (5.25) si $u \in W_{\mathrm{loc}}^{1,2}(\Omega) \cap L^\infty(\Omega)$ es tal que $u > 0$ a.e. Ω, $\frac{\phi}{u^\gamma} \in L^1(\Omega)$ para todo

$\phi \in W_0^{1,2}(\omega)$ y verifica

$$\int_\Omega \nabla u \nabla \phi = \lambda \int_\Omega \frac{\phi}{u^\gamma} + \int_\Omega u^p \phi \,, \quad \forall \phi \in W_0^{1,2}(\omega)\,, \tag{5.26}$$

para todo subconjunto abierto ω de Ω, tal que $\omega \subset\subset \Omega$. Por otro lado, con respecto a la condición de borde "$u = 0$ en $\partial\Omega$" se requiere que determinadas potencias positivas de la función u pertenezcan al espacio de Sobolev $W_0^{1,2}(\Omega)$. En concreto, utilizando como en [28] el método de sub y super solución, el autor probó en [21] que

> "*existe un número positivo* Λ *tal que para todo* $\lambda \in (0, \Lambda)$ *el problema* (5.25) *admite una solución* $0 < u \in W_{loc}^{1,2}(\Omega) \cap L^\infty(\Omega)$ *verificando* (5.26) *con*
>
> - $u \in W_0^{1,2}(\Omega)$, *si* $0 < \gamma \leq 1$;
> - $u^{\frac{\gamma+1}{2}} \in W_0^{1,2}(\Omega)$, *si* $\gamma > 1$."

Posteriormente, Arcoya y Boccardo [8] consideraron el estudio de la multiplicidad de solución para este problema y demostraron que

> "*para* λ *pequeño, existen al menos dos soluciones positivas distintas de* (5.25) *pertenecientes al espacio* $W_0^{1,2}(\Omega)$ *siempre y cuando los parámetros verifiquen que* $\gamma < 1$ *y* $2 < p + 1 < 2^* := \frac{2N}{N-2}$."

Los autores probaron este resultado de multiplicidad utilizando métodos variacionales. Por ello es inevitable en este trabajo la restricción impuesta $\gamma < 1$.

Teniendo en cuenta los resultados conocidos hasta el momento, el primer objetivo de nuestra investigación aborda el estudio de la multiplicidad de solución para el problema anterior para cualquier parámetro positivo γ. En concreto, el teorema principal del capítulo primero es el siguiente.

Theorem 1 *Sea* Ω *un subconjunto abierto y acotado de* $\mathbb{R}^N$ *cuya frontera* $\partial\Omega \in C^2$. *Si* $\gamma > 0$ *y* $2 < p+1 < 2^*$, *entonces existe* $\Lambda > 0$ *tal que para todo* $\lambda \in (0, \Lambda)$ *el problema* (5.25) *tiene al menos dos soluciones positivas distintas* $u\,, v \in W_{loc}^{1,2}(\Omega) \cap L^\infty(\Omega)$, *verificando* (5.26) *y tal que*

$$u^\alpha, v^\alpha \in W_0^{1,2}(\Omega)\,, \quad \forall \alpha > \frac{\gamma+1}{4}\,.$$

Obsérvese que en el caso $1 < \gamma < 3$, se tiene que $\frac{\gamma+1}{4} < 1$ y por tanto, se obtienen siempre soluciones pertenecientes al espacio $W_0^{1,2}(\Omega)$. En este sentido, se obtiene una mejora de los resultados de existencia en este espacio presentados en [21] para el caso $0 < \gamma \leq 1$.

Por otro lado, es interesante puntualizar que las técnicas y herramientas desarrolladas para la prueba de este resultado, permiten mejorar el significado de la condición de borde obtenido en [51] para el problema (5.23) en el caso en que $\gamma > 1$. En concreto, en la última sección del primer capítulo se presenta también el siguiente resultado.

Theorem 2 *Supongamos que Ω es un subconjunto abierto y acotado de $\mathbb{R}^N$ que verifica la condición de la esfera interior y que $f \in L^m(\Omega)$ con $m > 1$ es tal que existe f_0 de forma que $f(x) \geq f_0 > 0$ a.e. $x \in \Omega$.*

Si $1 < \gamma < \frac{3m-1}{m+1}$, entonces existe $u \in W^{1,1}_{loc}(\Omega)$ solución de la ecuación (5.24) tal que $u^\alpha \in W^{1,2}_0(\Omega)$ para todo $\alpha \in \left(\frac{(m+1)(\gamma+1)}{4m}, \frac{\gamma+1}{2}\right]$.

Obsérvese que cuando $1 < \gamma < \frac{3m-1}{m+1}$ se tiene que $\frac{(m+1)(\gamma+1)}{4m} < 1 < \frac{\gamma+1}{2}$ y por tanto $\alpha = 1$ puede ser elegido en este teorema. Como consecuencia, para todo $m > 1$ y $1 < \gamma < \frac{3m-1}{m+1}$, obtenemos una solución u del problema (5.23) que pertenece a $W^{1,2}_0(\Omega)$. Esto es, la hipótesis añadida sobre la regularidad del borde de Ω así como la hipótesis $f(x) \geq f_0 > 0$ a.e. en Ω, nos han permitido mejorar la condición "$u = 0$ en $\partial\Omega$" probada por Boccardo y Orsina para el problema (5.23), aproximándonos así a los límites naturales que fueron establecidos por Lazer y McKenna en su trabajo de 1991.

Continuando con el estudio de problemas singulares, se puede plantear también la existencia de solución para problemas casilineales de la forma siguiente

$$\begin{cases} -\Delta u = \mu(x)\dfrac{|\nabla u|^2}{u} + \lambda u + f(x), & \text{en } \Omega, \\ u = 0, & \text{en } \partial\Omega, \end{cases} \tag{5.27}$$

siendo Ω un subconjunto abierto y acotado de $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $0 \lneqq f \in L^m(\Omega)$ con $m > N/2$ y $0 \lneqq \mu(x) \in L^\infty(\Omega)$. Obsérvese que en este tipo de problemas el término singular tiene además un crecimiento cuadrático en el gradiente.

Merece la pena señalar que en el caso trivial $\mu(x) \equiv 0$ (problema de Dirichlet lineal no singular) es conocido que éste tiene solución positiva siempre que $\lambda < \lambda_1$, siendo λ_1 el valor propio asociado a la primera autofunción del laplaciano. Aún más, en este caso ocurre que $\lambda = \lambda_1$ es un punto de bifurcación desde infinito.

Conviene recordar que los problemas casilineales no singulares con crecimiento cuadrático en el gradiente han sido ampliamente estudiados en la literatura. Concretamente, éstos fueron extensamente desarrollados por Boccardo, Murat y Puel en toda una serie de trabajos, véanse por ejemplo [46, 47, 48, 49]. En particular, en [49] los autores probaron la existencia de solución (acotada) para el problema

$$\begin{cases} -\Delta u = \mu(x)|\nabla u|^2 + \lambda u + f(x), & \text{en } \Omega, \\ u = 0, & \text{en } \partial\Omega, \end{cases} \tag{5.28}$$

para cualquier valor $\lambda < 0$, siendo $\mu(x)$ una función acotada y positiva y $0 \lneqq f \in L^m(\Omega)$ con $m > N/2$. Mientras que, en 1995, Barles y Murat [17] demostraron la unicidad de la misma.

La existencia de solución (acotada) para $\lambda = 0$ fue estudiada por Ferone y Murat [68] en 1998, probándose ésta siempre y cuando el dato f fuese suficientemente pequeño. La condición sobre f impuesta en este trabajo fue mejorada en 2006 por

Abdellaoui, Dall'Aglio y Peral, en [1]. Es más, Barles, Blanc, Georgelin, Kobylanski [16] demostraron en 1999 la unicidad de solución en este caso ($\lambda = 0$) pero suponiendo de nuevo ciertas condiciones sobre "el tamaño" del dato f.

En 2013, Jeanjean y Sirakov [76] observaron que la unicidad de solución para el problema anterior puede fallar en el caso de que el parámetro λ sea positivo. De hecho, suponiendo que $\mu(x) \equiv \mu > 0$, los autores transformaron, mediante un conveniente cambio de variable, el problema (5.28) en uno semilineal, y así demostraron que para $\lambda > 0$, y pequeño, el problema anterior admite al menos dos soluciones distintas y acotadas.

Recientemente, Arcoya, De Coster, Jeanjean y Tanaka, en [12] (véase también [11]), observaron que dicho fenómeno de multiplicidad para el problema (5.28) es debido, en contraste con el caso lineal ($\mu \equiv 0$), al hecho de que $\lambda = 0$ es un punto de bifurcación desde infinito. En este sentido, al continuo de soluciones que contiene a los pares (λ, u_λ), siendo $\lambda < 0$ y u_λ la única solución de (5.28) asociada a este valor, le puede suceder dos fenómenos. A saber, o bien $||u_\lambda||_{L^\infty(\Omega)}$ diverge cuando el parámetro negativo λ se aproxima a $\lambda = 0$ (i.e., en este caso el problema (5.28) no tiene solución con $\lambda = 0$), o por el contrario, el continuo de soluciones contiene un punto de la forma $(0, u_0)$, siendo u_0 una solución de (5.28) con $\lambda = 0$. En este caso, $\lambda = 0$ es un punto de bifurcación por la derecha y es esto lo que da lugar a la multiplicidad correspondiente. En concreto, suponiendo que $\mu(x) \geq \mu_0 > 0$, los autores probaron que si el problema (5.28) con $\lambda = 0$ tiene solución, entonces existe $\lambda_0 > 0$ de forma que para todo $\lambda \in (0, \lambda_0)$, el problema anterior admite al menos dos soluciones distintas y acotadas.

Obsérvese que, en este contexto, la hipótesis $\mu_0 \leq \mu(x) \in L^\infty(\Omega)$ implica que el término de orden inferior del problem (5.28) satisfaga

$$0 \leq \mu_0 |\nabla u|^2 \leq \mu(x) |\nabla u|^2 \leq ||\mu||_{L^\infty(\Omega)} |\nabla u|^2 .$$

Cabe preguntarse, si la bifurcación desde infinito en $\lambda = 0$ se mantiene cuando se introduce una singularidad, tal y como se ha hecho al plantear el problema (5.27). En este sentido, este capítulo se dedica especialmente al estudio de este problema casilineal singular. Obsérvese que en este marco, el término de order inferior $\mu(x)\frac{|\nabla u|^2}{u}$ no satisface "estar por encima de $\mu_0 |\nabla u|^2$", siendo $\mu_0 > 0$.

En analogía con los problemas singulares tratados anteriormente, diremos que u es solución positiva de la ecuación diferencial asociada a (5.27), esto es, de la ecuación

$$-\Delta u = \mu(x) \frac{|\nabla u|^2}{u} + \lambda u + f(x) ,$$

si $u \in W^{1,1}_{\text{loc}}(\Omega)$ es tal que, $u > 0$ a.e. en Ω, $\frac{|\nabla u|^2}{u} \in L^1_{\text{loc}}(\Omega)$ y además verifica

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega \mu(x) \frac{|\nabla u|^2}{u} \varphi + \lambda \int_\Omega u \varphi + \int_\Omega f \varphi , \tag{5.29}$$

para todo $\varphi \in C_c^1(\Omega)$. Con respecto a la condición de borde $u = 0$ en $\partial\Omega$, inspirados en las definiciones dadas anteriormente, así como en el trabajo [70], requerimos que una potencia positiva de la función u, solución de la ecuación diferencial, pertenezca al espacio de Sobolev $W_0^{1,2}(\Omega)$.

Por un lado, la existencia de solución para el problema (5.27) en el caso particular $\lambda = 0$ ha sido demostrada por Arcoya, Boccardo, Leonori y Porretta en [9]. Por otro lado, es interesante observar que, a pesar de que se podría considerar el problema singular (5.27) más difícil que el problema no singular (5.28), probaremos que, en el marco singular, $\lambda = 0$ no es un punto de bifurcación desde infinito mientras que si lo es $\lambda = \lambda_1$, al igual que ocurre en el marco lineal, i.e., en el caso de que $\mu \equiv 0$.

En particular, entre otros resultados, en el segundo capítulo se presenta lo siguiente.

Theorem 3 *Supongamos que* $0 \lneqq f \in L^m(\Omega)$ *con* $m \geq \frac{N}{2}$. *Si* $\lambda < \frac{\lambda_1}{1+||\mu||_{L^\infty(\Omega)}}$, *entonces existe una solución* $u \in W_{loc}^{1,1}(\Omega)$ *verificando* (5.29) *con*

$$u^\gamma \in W_0^{1,2}(\Omega)\,,\ \forall \gamma > \frac{1 + ||\mu||_{L^\infty(\Omega)}}{2}.$$

En este capítulo también se estudian los casos en que la regularidad del dato f es menor. Es decir, se obtienen resultados similares al anterior para los casos $f \in L^m(\Omega)$ con $1 < m < \frac{N}{2}$.

Hasta este momento, en los capítulos primero y segundo, se han considerado ecuaciones que involucran al operador Laplaciano. Sin embargo, otros tipos de operadores diferenciales elípticos aparecen de forma natural en la literatura. Tal es el caso del operador asociado a la ecuación de Euler-Lagrange del funcional

$$J(u) = \frac{1}{2}\int_\Omega (a(x) + |u|^r)\,|\nabla u|^2 - \int_\Omega f(x)u,$$

siendo Ω un subconjunto abierto y acotado de $\mathbb{R}^N$, $a(x)$ una función medible que verifica $0 < \alpha \leq a(x) \leq \beta$ y $r > 0$. En concreto, y al menos formalmente, dicha ecuación se corresponde con la siguiente ecuación casilineal modelo.

$$-\text{div}\,[(a(x) + |u|^r)\nabla u] + \tfrac{r}{2}u|u|^{r-2}|\nabla u|^2 = f(x), \quad \text{en } \Omega. \tag{5.30}$$

Obsérvese que, en este modelo concreto, existe una gran diferencia entre los casos $r \geq 1$ (ecuación casilineal no singular) y $r < 1$ (ecuación casilineal singular).

Problemas singulares de naturaleza similar han sido considerados en la literatura. En [13] se estudió por primera vez la existencia de solución positiva de

$$-\Delta u + \tfrac{|\nabla u|^2}{u^\theta} = f(x), \quad \text{en } \Omega,$$

cuando $0 < \theta < 1$. Véanse por ejemplo [10, 24], así como las referencias citadas en éstos, para posteriores avances (en los que se tratan además operadores diferenciales no lineales pero con coeficientes acotados).

Observemos que el operador diferencial de (5.30) presenta muchas más trabas que el considerado en los trabajos apenas citados. De hecho, éste no solo tiene un crecimiento cuadrático en el gradiente y es singular en la solución sino que además su parte principal, esto es, div $[(a(x) + |u|^q)\,\nabla u]$, no está definida en el espacio de Sobolev $W_0^{1,2}(\Omega)$ y es "no acotada con respecto a u" (sea o no el problema singular). Ya de por sí esta dificultad hace conveniente empezar estudiando en el capítulo tercero la ecuación modelo (5.30) en el caso no singular, es decir, cuando $r \geq 1$. En concreto, en este capítulo se trata con la siguiente clase de problemas casilineales generales.

$$\begin{cases} -\mathrm{div}\,[(a(x) + |u|^q)\,\nabla u(x)] + b(x)\,u\,|u|^{p-1}\,|\nabla u|^2 = f(x)\,, & \text{en } \Omega, \\ u(x) = 0\,, & \text{en } \partial\Omega\,, \end{cases} \tag{5.31}$$

siendo Ω un subconjunto abierto y acotado de $\mathbb{R}^N$, $p\,,q > 0$, $f \in L^1(\Omega)$ y $a(x)\,,b(x)$ funciones medibles que verifican

$$0 < \alpha \leq a(x) \leq \beta \ \text{ y } \ 0 < \mu \leq b(x) \leq \nu \ \text{ a.e. } x \in \Omega\,. \tag{5.32}$$

Obsérvese que si $q = r$, $p = r-1$ y $b(x) = r/2$, la ecuación diferencial de este problema se corresponde con la anterior ecuación modelo (5.30) (en el caso no singular).

El objetivo particular que nos planteamos en este capítulo es la existencia de solución perteneciente al espacio de Sobolev $W_0^{1,2}(\Omega)$ para este tipo de problemas. Merece la pena destacar que, además de las dificultades comentadas anteriormente, incluso si pensamos en funciones $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, el término de orden inferior de la ecuación asociada al problema (5.31), es decir $b(x)\,u\,|u|^{p-1}\,|\nabla u|^2$, no pertenece al espacio $W^{-1,2}(\Omega)$. No obstante, a pesar del inconveniente que añade este término con crecimiento cuadrático en el gradiente, será éste el que permitirá obtener soluciones de energía finita, es decir, soluciones pertenecientes a $W_0^{1,2}(\Omega)$, aún cuando se considere un dato f perteneciente sólo al espacio $L^1(\Omega)$. Esto es debido a que este término $b(x)\,u\,|u|^{p-1}\,|\nabla u|^2$ verifica lo que se conoce como condición de signo, es decir,

$$b(x)\,u\,|u|^{p-1}|\nabla u|^2 \,\cdot\, u \geq \mu|u|^{p+1}|\nabla u|^2 \geq 0\,, \text{ a.e. } x \in \Omega.$$

Fueron Boccardo y Gallouët [36] en 1992, véase también [20], los primeros autores en observar este efecto regularizante que produce esta condición de signo sobre la regularidad de las soluciones. Es decir, probaron que, a pesar de considerar problemas con un dato f cuya sumabilidad es pequeña (i.e., $f \in L^m(\Omega)$, con $1 \leq m < \frac{2N}{N+2}$), gracias al término de orden inferior, es posible obtener soluciones pertenecientes al espacio de Sobolev clásico (i.e. $W_0^{1,2}(\Omega)$).

El estudio de problemas casilineales de este tipo ha sido considerado en diversos trabajos tales como [23, 24, 82]. Concretamente en [23], Boccardo consideró el problema (5.31) y estudió la existencia de solución positiva suponiendo ciertas hipótesis que relacionan los parámetros p y q, y además supone que el dato f es positivo (y tiene cierta regularidad). En particular, entre otros resultados, el autor probó que "*si* $f \in L^1(\Omega)$, $f \geq 0$, $p \geq 2q$ *y se verifica la condición* (5.32), *entonces existe una solución positiva* $u \in W_0^{1,2}(\Omega)$ *de* (5.31) *en el sentido que* $b(x)\, u^p\, |\nabla u|^2 \in L^1(\Omega)$ *y además verifica*

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, |u|^{p-1} u |\nabla u|^2\, \varphi = \int_\Omega f\, \varphi\,, \qquad (5.33)$$

para toda función $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$".

Otros resultados similares son presentados en este trabajo [23] al modificar las relaciones establecidas entre los diferentes parámetros del problema.

Debido, entre otros aspectos, a los inconvenientes que se han expuesto anteriormente para el estudio del problema (5.31), en [23], Boccardo demostró la existencia de solución vía la consideración de problemas aproximados sobre los que se conocen resultados de existencia. Así, el autor encontró una solución u de (5.31) como límite de una determinada sucesión $\{u_n\}$ de soluciones aproximadas. Para conseguir esto, es necesario abordar el estudio de ciertas cotas *a priori* así como algunos resultados de convergencia para la sucesión considerada. En este sentido, las restricciones que se imponen a los parámetros en este trabajo, así como la imposición de positividad sobre el dato f, resultan esenciales.

Con el objetivo de extender todos los resultados previos conocidos para este problema (5.31), el capítulo tercero generaliza éstos obteniendo la existencia de solución perteneciente al espacio de Sobolev $W_0^{1,2}(\Omega)$ para cualesquiera parámetros positivos p y q y para cualquier dato $f \in L^1(\Omega)$ (no necesariamente positivo). Con este fin, es necesario establecer una nuevo concepto de solución de (5.31), el cual es natural en este contexto. En concreto, el resultado principal presentado en este capítulo es el siguiente.

Theorem 4 *Si* $\Omega \subset \mathbb{R}^N$ *es abierto y acotado,* $p\,,q > 0$, $f \in L^1(\Omega)$ *y* $a(x)\,,b(x)$ *son funciones medibles verificando* (5.32), *entonces existe una solución* $u \in W_0^{1,2}(\Omega)$ *de* (5.31) *en el siguiente sentido:*

$$(a(x) + |u|^q)\, |\nabla u| \in L^1(\Omega)\,,\ b(x)\, |u|^p\, |\nabla u|^2 \in L^1(\Omega) \text{ y}$$
$$u \text{ verifica (5.33) para toda función } \varphi \in W_0^{1,\infty}(\Omega).$$

Siguiendo la línea de trabajo considerada en [23], la idea principal para probar este teorema consiste en encontrar una solución u de (5.31) *como límite de una sucesión* $\{u_n\}$ de soluciones u_n de ciertos problemas aproximados más sencillos. En este sentido, será necesario estudiar las propiedades de estas soluciones aproximadas, así como la convergencia de esta sucesión en determinados espacios de funciones. Es más, debido a la naturaleza del operador diferencial que se está considerando, será necesario demostrar que, en algún sentido a especificar, la sucesión $\{u_n\}$ converge fuertemente hacia una función u que será la solución buscada. Para ello, y a diferencia de los argumentos seguidos en los trabajos previos, se utilizará una de las técnicas introducidas en [47] y que posteriormente ha sido utilizada en artículos como [82] y [85].

Además, la obtención de la solución u de (5.31) como límite de la sucesión $\{u_n\}$ de soluciones aproximadas, permite recuperar los resultados de existencia y regularidad presentados en [23] para soluciones verificando (5.33) con funciones test pertenecientes a $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Concretamente, el resultado principal de la última sección de este tercer capítulo es el siguiente.

Theorem 5 *Sean $a(x)$ y $b(x)$ funciones medibles verificando (5.32), $p\,,q > 0$, $f \in L^m(\Omega)$ con $1 \leq m \leq \frac{N}{2}$. Si $u \in W_0^{1,2}(\Omega)$ es la solución obtenida en el Teorema 4, se tiene:*

(A) *Si $m = 1$ y $p \geq 2q$, entonces $u \in L^{(p+2)\frac{N}{N-2}}(\Omega)$;*

(B) *Si $\frac{2(q+1)N}{2N+p(N-2)+4q} \leq m \leq \frac{N}{2}$ y $2q \geq p \geq q-1$, entonces $u \in L^{(p+2)m^{**}}(\Omega)$;*

(C) *Si $\frac{2N}{N+2} \leq m \leq \frac{N}{2}$, $q \geq 1$ y $2p \geq q-1 \geq p$, entonces $u \in L^{(q+1)m^{**}}(\Omega)$;*

Además, en todos los casos, es decir si se satisface (A), (B) o (C), se tiene también que $\big(a(x) + |u|^q\big)\nabla u \in (L^2(\Omega))^N$ y que u verifica (5.33) para toda $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Una vez estudiado esta clase de problemas casilineales (5.31) con crecimiento cuadrático en el gradiente en el caso no singular, procedemos en el capítulo cuarto a estudiar éstos en el caso singular. En concreto, nos planteamos la existencia de solución positiva para los problemas generales

$$\begin{cases} -\mathrm{div}([a(x) + u^q]\nabla u) + b(x)\dfrac{1}{u^\theta}|\nabla u|^2 = f & \text{en } \Omega, \\ u = 0 & \text{en } \partial\Omega, \end{cases} \tag{5.34}$$

siendo $a(x)$, $b(x)$ funciones medibles verificando (5.32), $f \in L^1(\Omega)$ y

$$0 < q\,, \;\; 0 < \theta < 1\,, \tag{5.35}$$

$$f \geq 0\,, \;\; f \not\equiv 0\,. \tag{5.36}$$

En relación con la ecuación modelo (5.30) que motivó el estudio de esta clase de problemas, se observa que la ecuación diferencial del problema (5.34) no es sino una generalización de aquella ecuación (5.30). De hecho, ésta se obtiene en el caso particular $\theta = 1 - r$, $b(x) = \frac{r}{2}$ y $q = r$.

Es conveniente señalar que, al igual que ocurría para la clase de problemas no singulares (5.31), el témino de orden inferior $b(x)\frac{1}{u^\theta}|\nabla u|^2$ verifica la condición de signo, es decir,

> *"el signo del término con crecimiento cuadrático en el gradiente es el mismo que el de las soluciones del problema"*

y de nuevo, ésta tendrá un efecto regularizante sobre las soluciones de esta clase de problemas casilineales singulares.

Seguiremos la línea de trabajo desarrollada en [24], y así, la estrategia para encontrar una solución de (5.34) consistirá en buscar ésta como límite de una sucesión de soluciones de problemas casilineales no singulares aproximados. Para ello, será necesario estudiar la existencia de cotas *a priori* así como resultados de convergencia. Además, en este caso singular será esencial demostrar, a diferencia de lo que ocurría en el capítulo anterior, que la sucesión de soluciones aproximadas está inferiormente acotada "por una constante positiva".

Merece la pena señalar que en el estudio de esta clase de problemas casilineales singulares (5.34) se produce una fuerte interacción entre el término singular $\frac{|\nabla u|^2}{u^\theta}$ y el término $\mathrm{div}(u^q \nabla u)$ que aparece en la parte principal del operador diferencial considerado. De hecho, el espacio de Sobolev al que pertenecerá la solución de (5.34) dependerá directamente de las relaciones existentes entre estos dos términos. Concretamente, el resultado principal presentado en este cuarto capítulo es el siguiente.

Theorem 6 *Si las funciones medibles $a(x)$ y $b(x)$, las constantes q y θ y la función $f \in L^1(\Omega)$ verifican* (5.32), (5.35) *y* (5.36), *entonces existe una solución u de* (5.34) *en el sentido que $u > 0$ en Ω,*

$$[a(x) + u^q]\, |\nabla u| \in L^\rho(\Omega)\,, \ \forall \rho < \frac{N}{N-1}\,, \quad b(x)|\nabla u|^2\, u^{-\theta} \in L^1(\Omega)\,,$$

y

$$\int_\Omega [a(x) + u^q]\, \nabla u \nabla \varphi + \int_\Omega \frac{b(x)\, |\nabla u|^2}{u^\theta}\, \varphi = \int_\Omega f\, \varphi\,,$$

para toda φ in $W_0^{1,p}(\Omega)$, $p > N$. Además, se tiene que

- *si $0 < q \le 1 - \theta$, entonces u pertenece a $W_0^{1,r}(\Omega)$, con $r = \frac{N(2-\theta)}{N-\theta}$;*
- *si $1 - \theta < q \le 1$, entonces u pertenece a $W_0^{1,r}(\Omega)$, para todo $r < \frac{N(q+1)}{N+q-1}$;*
- *si $q > 1$, entonces u pertenece a $W_0^{1,2}(\Omega)$.*

Obsérvese que para la existencia de solución en este teorema ha sido suficiente considerar un dato f perteneciente al espacio $L^1(\Omega)$. Sin embargo, se puede mejorar la regularidad de la solución obtenida si se consideran datos más regulares. Específicamente, en este capítulo también se presenta el siguiente resultado.

Theorem 7 *Si adicionalmente a las hipótesis del Teorema 6, suponemos que $f \in L^m(\Omega)$ con $1 < m < \frac{N}{2}$ y consideramos $\delta := \min(\theta, 1-q)$ y la solución u dada por dicho teorema, entonces u pertenece a $L^s(\Omega)$, siendo $s = m^{**}(2-\delta)$. Además,*

- *cuando $q < 1$,*

1) si $1 < m < \left(\frac{2^*}{\delta}\right)'$*, entonces* $u \in W_0^{1,r}(\Omega)$*, con* $r = \frac{Nm(2-\delta)}{N-m\delta}$*;*

2) si $m \geq \left(\frac{2^*}{\delta}\right)'$ *(y* $m > 1$*), entonces* $u \in W_0^{1,2}(\Omega)$*;*

- *mientras que si* $q = 1$ *(y* $m > 1$*), entonces* $u \in W_0^{1,2}(\Omega)$*.*
- *En el caso* $q > 1$*,* $u \in W_0^{1,2}(\Omega)$ *(gracias al Teorema 6).*

Es adecuado puntualizar que, como es habitual, si se considera un dato f tal que $f \in L^m(\Omega)$ con $m > \frac{N}{2}$, entonces se obtienen soluciones acotadas, es decir, en $L^\infty(\Omega)$.

En conclusión, hemos visto en los capítulos anteriores cómo las técnicas de aproximación y obtención de *cotas a priori* son útiles en el estudio de los problemas elípticos singulares. Qué duda cabe que dichas herramientas son también válidas para abordar otros problemas de naturaleza diferente. Como una muestra de ello, dedicamos el último capítulo al estudio mediante éstas de un problema elíptico casilineal con condición de borde tipo Neumann. En concreto, motivados por los resultados de [42], durante este capítulo nos centraremos en el siguiente problema modelo

$$\begin{cases} -\Delta_p u + |u|^{s-1}u = 0, & \text{en } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta = \psi, & \text{en } \partial\Omega, \end{cases} \tag{5.37}$$

siendo Ω un subconjunto abierto y acotado de $\mathbb{R}^N$ con frontera regular, η el vector normal unitario exterior a $\partial\Omega$, $1 < p < N$, $s > 0$ y $\psi \in L^m(\partial\Omega)$ con $m > 1$.

En [4] se ha considerado el estudio de existencia de solución de "energía finita" para este problema, es decir, los autores demostraron bajo ciertas hipótesis que involucran a los parámetros del problema, que éste tiene una solución perteneciente al espacio de Sobolev $W^{1,p}(\Omega)$. Para que esto sea posible se debe imponer, entre otras restricciones, que $\psi \in L^m(\partial\Omega)$ con $m \geq \frac{p(N-1)}{N(p-1)}$.

Posteriormente, Boccardo y Mazón [42] extendieron los resultados demostrados en [4] en dos direcciones distintas. Por un lado, suponiendo que $\psi \in L^m(\partial\Omega)$ con $m \geq \frac{p(N-1)}{N(p-1)}$, mejoraron la regularidad de la solución $u \in W^{1,p}(\Omega)$ de (5.37) aprovechando el efecto regularizante del término de orden inferior $|u|^{s-1}u$ sobre ésta. Por otro lado, desarrollaron también resultados que conciernen a la teoría de Calderón-Zigmund asociada a este problema para las soluciones de energía "infinita" ($m < \frac{p(N-1)}{N(p-1)}$). Precisamente, si $q := \frac{Nm(p-1)}{N-1}$ y s es pequeño[1] ($0 < s < \frac{(N+m-1)(p-1)}{N-1-m(p-1)}$), Boccardo y Mazón prueban la existencia de una solución distribucional de (5.37) con $u \in W^{1,q}(\Omega)$ siempre que

- $m \in \left(1, \frac{p(N-1)}{N(p-1)}\right)$, en el caso $p > 2 - \frac{1}{N}$, mientras que
- $m \in \left(\frac{N-1}{N(p-1)}, \frac{p(N-1)}{N(p-1)}\right)$, en el caso $1 < p < 2 - \frac{1}{N}$.

[1] Para "valores de s mayores", gracias de nuevo al efecto regularizante del término de orden inferior, los autores también mejoran el espacio al que pertenecen las soluciones distribucionales.

Observemos que en el caso $p > 2 - \frac{1}{N}$, el exponente $q > \frac{N(p-1)}{N-1} > 1$ para todo $m \in \left(1, \frac{p(N-1)}{N(p-1)}\right)$. Por contrapartida, en el caso $p < 2 - \frac{1}{N}$, si el parámetro m tiende al valor $\frac{N-1}{N(p-1)}$, entonces el exponente q converge a 1. Esto nos puede sugerir que en el marco límite $m = \frac{N-1}{N(p-1)}$, se podría esperar tener aún soluciones en el espacio $W^{1,1}(\Omega)$. De hecho, probaremos el siguiente teorema.

Theorem 8 *Sea $\Omega \subset \mathbb{R}^N$ abierto, acotado y regular. Supongamos que ψ pertenece a $L^m(\Omega)$ con $m = \frac{N-1}{N(p-1)}$, $1 < p < 2 - \frac{1}{N}$ y $0 < s < \frac{1+N(p-1)}{N-1}$. Entonces existe una solución $u \in W^{1,1}(\Omega)$ del problema (5.37) en el siguiente sentido*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \, \nabla v + \int_\Omega |u|^{s-1} \, uv = \int_{\partial\Omega} \psi v \,, \ \ \forall v \in C^1(\overline{\Omega}) \,.$$

Como se ha comentado anteriormente, la demostración se realiza vía aproximación por una familia de problemas más sencillos. Para probar la convergencia de la sucesión de soluciones aproximadas de éstos, será esencial el estudio de cotas *a priori*. Conviene destacar que en este marco límite encontramos nuevas dificultades debido al hecho de trabajar en el espacio no reflexivo $W^{1,1}(\Omega)$. Para solventar esta dificultad se utilizará, entre otros, el Teorema de Dunford-Pettis. También será esencial en esta demostración el "Teorema de la Traza", el cual requerirá la regularidad del borde $\partial\Omega$ de Ω, pero nos permitirá relacionar los diferentes miembros que aparecen en la formulación débil del problema.

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