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Rank-one numerical index and Bishop-Phelps-Bollobás moduli of a Banach space

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RANK-ONE NUMERICAL INDEX AND BISHOP-PHELPS-BOLLOBÁS MODULI OF A BANACH SPACE

Memoria realizada por **Mario Chica Rivas** en el Departamento de Análisis Matemático de la Universidad de Granada, bajo la dirección de los doctores **D. Miguel Martín Suárez** y **D. Francisco Javier Merí de la Maza**, Profesores Titulares del Departamento de Análisis Matemático de la Universidad de Granada, para optar al grado de *Doctor Internacional en Ciencias Matemáticas*.

Candidato al grado de Doctor Internacional en Ciencias Matemáticas

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Granada, febrero de 2016

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Garantizamos:

Al firmar esta tesis doctoral, *Rank-one numerical index and Bishop-Phelps-Bollobás moduli of a Banach space*, que ésta ha sido realizada por el doctorando bajo la dirección de los directores de la tesis y hasta donde nuestro conocimiento alcanza, en la realización del trabajo, se han respetado los derechos de otros autores a ser citados, cuando se han utilizado sus resultados o publicaciones.

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A mi familia y a Esperanza

Contents

Introduction

This dissertation is devoted to study some aspects of the geometry of Banach spaces through the development of two different tools: *rank-one numerical index* of a Banach space and the *Bishop-Phelps-Bollobás moduli* of a Banach space. These tools are completely independent and that is the reason for this study to be divided into two chapters which are developed individually.

We devote the first chapter of this dissertation to study the concept of *rank-one numerical index* of a Banach space, which appeared recently in [33], to relate the numerical radius and the usual norm of rank-one operators on L_p-spaces. This concept is an analogue to the classical numerical index of a Banach space, that was introduced by G. Lumer in 1968.

We fix some notation that will be used all along the introduction. Given a Banach space X over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), X^* stands for its topological dual, B_X and S_X are, respectively, the closed unit ball and the unit sphere of X. Besides, $L(X)$ will denote the Banach algebra of all bounded linear operators on X. Let X be a Banach space. Given $T \in L(X)$, the *numerical radius* of T is

$$v(T) = \sup\{|x^*(Tx)| \,:\, (x, x^*) \in \Pi(X)\}$$

where

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} \,:\, x^*(x) = 1\}.$$

It is easy to check that v defines a semi-norm on $L(X)$ which clearly satisfies $v(T) \leqslant \|T\|$. In fact, very often v is actually a norm that is equivalent to the usual operator norm. To study if this is so, one may consider the *numerical index* of X:

$$n(X) = \inf\{v(T) \,:\, T \in L(X),\, \|T\| = 1\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. It is well know that the set of values of the numerical index of real Banach spaces fills the whole interval $[0,1]$, while for complex Banach spaces it fills the interval $[1/\mathrm{e}, 1]$ [16].

This way it is defined the *rank-one numerical index* of a Banach space X as the constant given by

$$\begin{aligned} n_1(X) &:= \max\{k \geqslant 0 \,:\, k\|T\| \geqslant v(T)\ \forall T \in L(X) \text{ with } \dim(T(X)) \leqslant 1\} \\ &= \inf\{v(T) \,:\, T \in L(X),\, \|T\| = 1,\, \dim(T(X)) \leqslant 1\}. \end{aligned}$$

We start the chapter recalling briefly the most important results about the numerical index as well as the previously known results about rank-one numerical index. After that, in section 1.2 we prove that there exists a lower bound for the rank-one numerical index which is also valid in real case, more concretely, we prove that $n_1(X) \geqslant 1/\mathrm{e}$ for every real Banach space X. Besides, we provide an example of a Banach space whose rank-one numerical index is $1/\mathrm{e}$, showing that our result is sharp.

We devote section 1.3 to the study of some stability properties of the rank-one numerical index concerning suitable sums of Banach spaces, as well as the continuity of the index with respect to the Banach-Mazur distance. In both cases, we get that it behaves similarly to the classical numerical index. We continue studying the rank-one numerical index of vector valued spaces. Obtaining that this time the behaviour differs from that of the classical numerical index. All these results can be found in a joint work with M. Martín and J. Merí [13].

In section 1.4 we present some examples involving the rank-one numerical index of a space and the one of its dual. More concretely, we show that there is a Banach space

X satisfying $n_1(X^*) < n_1(X)$. Besides, we construct some examples which relate the rank-one numerical index with some other natural indices that one may consider.

In section 1.5 we compute the rank-one numerical index for some families of polyhedral norms on the plane. More precisely, we obtain explicit formulae for the rank-one numerical index of three families of norms on $\mathbb{R}^2$: a family of hexagonal norms and two families of octagonal norms. These computations appear in a joint work with J. Merí [14]. In the last part of the section we present some small advances about the computation of the rank-one numerical index of L_p spaces. It was shown in [33] that $k_p \geqslant n_1(L_p(\mu)) \geqslant k_p^2$ for every atomless measure μ, where

$$k_p := \frac{1}{p^{1/p}q^{1/q}} = \sup_{t\geqslant 0} \frac{t^{p-1}}{1+t^p} = \sup_{t\geqslant 0} \frac{t}{1+t^p}.$$

It is easy to check that $n_1(\ell_p^2) \leqslant k_p$, so it is a natural question whether this is an equality or not. Unfortunately we were not able to solve the problem but we provide some partial results. We finish the chapter recalling the problems which remained unsolved after our work.

We devote the second chapter to study two functions that can be defined for every Banach space which, roughly speaking, give a measure of what is the best possible Bishop-Phelps-Bollobás Theorem that can be achieved in a fixed Banach space. As it is widely known the classical Bishop-Phelps theorem [5] states that the set of norm attaining functionals on a Banach space is norm dense in the dual space. Some years later, B. Bollobás [6] gave a sharper version of this theorem allowing to approximate at the same time a functional and a vector in which it almost attains the norm. This result is known nowadays as the Bishop-Phelps-Bollobás Theorem:

Let X be a Banach space. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|1 - x^*(x)| \leqslant \varepsilon^2/2$ for some $0 < \varepsilon < 1/2$. Then there exists $(y, y^*) \in \Pi(X)$ such that $\|x - y\| < \varepsilon + \varepsilon^2$ and $\|x^* - y^*\| \leqslant \varepsilon$.

This classical result is the starting point of our study. The main tools that we use are two functions which we call *Bishop-Phelps-Bollobás moduli* of a Banach space:

Let X be a Banach space. The *Bishop-Phelps-Bollobás modulus* of X is the function $\Phi_X : (0,2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1-\delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x-y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

The *spherical Bishop-Phelps-Bollobás modulus* of X is the function $\Phi_X^S : (0,2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X^S(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $\operatorname{Re} x^*(x) > 1-\delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x-y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

That is a way to measure how close can be y to x and y^* to x^* in the above result depending on how close is $x^*(x)$ to 1. The *Bishop-Phelps-Bollobás moduli* can be seen as the Hausdorff distance from a suitable set to $\Pi(X)$:

$$\Phi_X(\delta) = d_H\big(\{(x,x^*) \in B_X \times B_{X^*} \ : \ \operatorname{Re} x^*(x) > 1-\delta\}, \Pi(X)\big)$$
$$\Phi_X^S(\delta) = d_H\big(\{(x,x^*) \in S_X \times S_{X^*} \ : \ \operatorname{Re} x^*(x) > 1-\delta\}, \Pi(X)\big);$$

where $d_H(A,B)$ is the Hausdorff distance between $A, B \subset X \times X^*$ associated to the ℓ_∞-distance $\operatorname{dist}_\infty$ in $X \times X^*$, that is,

$$\operatorname{dist}_\infty\big((x,x^*),(y,y^*)\big) = \max\{\|x-y\|, \|x^*-y^*\|\}$$

for every $(x,x^*), (y,y^*) \in X \times X^*$.

In section 2.2 we prove that there is a common upper bound for the Bishop-Phelps-Bollobás moduli which is in fact sharp. Actually, what we show is somewhat more general: we calculate the best possible upper bound for $d_\infty\big((x,x^*),\Pi(X)\big)$ in any Banach space as a function of $\|x\|$ and $\|x^*\|$. To this end we consider the function

$$\Phi_X(\mu,\theta,\delta) := \sup\{d_\infty\big((x,x^*),\Pi(X)\big) \ : x \in X, x^* \in X^*, \\ \|x\| = \mu, \|x^*\| = \theta, \operatorname{Re} x^*(x) \geqslant 1-\delta\},$$

where $\delta \in (0,2)$ and $\theta, \mu \in [0,1]$ satisfy $\mu\theta \geqslant 1-\delta$. Observe that one has

$$\Phi_X^S(\delta) = \Phi_X(1,1,\delta) \qquad \text{and} \qquad \Phi_X(\delta) = \sup_{\substack{\mu\theta \in [0,1] \\ \mu\theta \geqslant 1-\delta}} \Phi_X(\mu,\theta,\delta).$$

The main result of this section tells us that for every Banach space X, every $\delta \in (0,2)$, and every $\mu, \theta \in [0,1]$ satisfying $\mu\theta > 1-\delta$, one has that

$$\Phi_X(\mu,\theta,\delta) \leqslant \min\left\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\right\}, \tag{1}$$

where

$$\Psi(\mu,\theta,\delta) := \frac{2-\mu-\theta+\sqrt{(\mu-\theta)^2+8\left(\mu\theta-1+\delta\right)}}{2}.$$

As a consequence of this, we get a sharp version of the Bishop-Phepls-Bollobás Theorem:

Let $0 < \varepsilon < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy

$$\operatorname{Re} x^*(x) > 1 - \varepsilon^2/2.$$

Then, there exists $(y, y^*) \in \Pi(X)$ such that

$$\|x-y\| < \varepsilon \quad \text{and} \quad \|x^*-y^*\| < \varepsilon.$$

In other words, we prove that $\Phi_X^S(\delta) \leqslant \sqrt{2\delta}$ for $\delta \in (0,2)$. We also provide an example for which the inequality given in (1) becomes an equality.

In section 2.3 we study some more properties of the two moduli. Namely, we stablish the relationship of the moduli of a Banach space X and those of its dual space:

$$\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta) \qquad \text{and} \qquad \Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$$

for every $\delta \in (0,2)$. Besides, we prove the continuity of $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$. We also show further that the Bishop-Phelps-Bollobás moduli are continuous with respect to the Banach-Mazur distance.

We devote section 2.4 to calculate both moduli for some classical Banach spaces. We give the exact value of the moduli for Hilbert spaces and we present many examples for which the moduli reach the maximum possible value for small δ's. Among these examples we can find $L_1(\mu)$ and $C_0(L)$ where μ is a measure and L is a locally compact Hausdorff topological space with at least two points.

In section 2.5 we show that a Banach space X satisfying $\Phi_X(\delta) = \sqrt{2\delta}$ for some $\delta \in (0,2)$ must contain almost isometric copies of the real space $\ell_\infty^{(2)}$. However, this condition is not enough: we provide an example of a three-dimensional Banach space X containing ℓ_∞^2 isometrically but such that $\Phi_X(\delta) < \sqrt{2\delta}$. To finish the chapter, in section 2.6 we gather the questions that we were not able to solve in our study.

All the results in this chapter can be found in [10, 11, 12]. The beginning of this topic was developed in a joint work with V. Kadets, M. Martín, S. Moreno-Pulido, and F. Rambla-Barreno [12]. We studied the refined versions of the moduli in collaboration with V. Kadets, M. Martín, J. Merí, and M. Soloviova [11]. Finally, the shaper results and further properties of the Bishop-Phelps-Bollobás moduli appear in [10], which is a joint work with V. Kadets, M. Martín, and J. Merí.

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Notation

All along this study we consider $\mathbb{K}$ as the field of the real numbers $\mathbb{R}$ or to the field of the complex numbers $\mathbb{C}$. We also write $\mathbb{D}$ for the set $\{\lambda \in \mathbb{K} : \|\lambda\| = 1\}$.

We use in $\mathbb{K}$ the function real part $\operatorname{Re}(\cdot)$ that naturally is the identity when $\mathbb{K} = \mathbb{R}$. Given a set $A \subseteq \mathbb{K}$, we consider the following notation:

$$\operatorname{Re} A := \{\operatorname{Re} \lambda \ : \ \lambda \in A\}.$$

Let be a Banach space X with norm $\|\cdot\|$, B_X and S_X we write respectively its unit ball and its unit sphere:

$$B_X := \{x \in X \ : \ \|x\| \leqslant 1\}, \qquad\qquad S_X := \{x \in X \ : \ \|x\| = 1\}.$$

We denote X^* to the topological dual space of X, where its natural norm is

$$\|x^*\| := \sup\{|x^*(x)| \ : \ x \in B_X\} \qquad (x^* \in X^*)$$

and for another Banach space Y, $L(X, Y)$ will be the space of bounded linear operators from a Banach space X into a Banach space Y, endowed with the usual operator norm

$$\|T\| := \sup\{\|Tx\| \ : \ x \in B_X\} \qquad (T \in L(X, Y)).$$

In the case $X = Y$ we write $L(X) := L(X, X)$ and Id for the identity operator in X. On the other hand, if X is a complex Banach space, we will write $X_{\mathbb{R}}$ for its real subjacent space.

For a A convex set, ext$A$ will be the set of its extreme points.

Given $p \in [1, +\infty)$ y $m \in \mathbb{N}$, we denote ℓ_p^m to the space $(\mathbb{K}^m, \|\cdot\|_p)$ where

$$\|(x_1, \ldots, x_m)\|_p := \left(\sum_{i=1}^{m} |x_i|^p \right)^{\frac{1}{p}} \qquad (x_1, \ldots, x_m) \in \mathbb{K}^m,$$

and as usual for $p = +\infty$:

$$\|(x_1, \ldots, x_m)\|_\infty := \max\{|x_i| \ : \ i = 1, \ldots, m\} \qquad \text{y} \qquad \ell_\infty^m = (\mathbb{K}^m, \|\cdot\|_\infty).$$

Given an arbitrary family of Banach spaces $\{X_\lambda \ : \ \lambda \in \Lambda\}$, we call ℓ_∞-*sum* of the family, and we write $[\oplus_{\lambda \in \Lambda} X_\lambda]_{\ell_\infty}$ to the subspace of the cartesian product $\prod_{\lambda \in \Lambda} X_\lambda$ of all the families $(x_\lambda)_{\lambda \in \Lambda}$ where the set $\{\|x_\lambda\| \ : \ \lambda \in \Lambda\}$ is bounded. That is a Banach space with the norm

$$\|(x_\lambda)_{\lambda \in \Lambda}\| = \sup\{\|x_\lambda\| \ : \ \lambda \in \Lambda\}.$$

As a subspace of this ℓ_∞-*sum* we find the c_0-*sum* that is written as $[\oplus_{\lambda \in \Lambda} X_\lambda]_{c_0}$. Finally, for $1 \leqslant p < \infty$, the ℓ_p-*sum*, which we write $[\oplus_{\lambda \in \Lambda} X_\lambda]_{\ell_p}$ consist of all the families $(x_\lambda)_{\lambda \in \Lambda}$ such that $(\|x_\lambda\|^p)_{\lambda \in \Lambda}$ is summable, and it is a Banach space with the norm

$$\|(x_\lambda)_{\lambda \in \Lambda}\| = \left(\sum_{\lambda \in \Lambda} \|x_\lambda\|^p \right)^{\frac{1}{p}}.$$

Chapter 1

Rank-one numerical index of a Banach space

In this chapter we study the concept of *rank-one numerical index* of a Banach space, which has been recently introduced in [33] to relate the numerical range and the usual norm of rank-one operators on L_p-spaces as an analogue to the deeply studied numerical index of a Banach space. We will recall briefly some results about numerical index which will be relevant to our discussion, as well as the results about rank-one numerical index which were known before we started our study. We analyse the basic properties of the rank-one numerical index and its behaviour with respect to some usual operations with Banach spaces. This will give some differences with the classical numerical index. Finally, we compute the rank-one numerical index for some families of polyhedral Banach spaces.

1.1 Introduction

At the beginning of the 1960's F. Bauer and G. Lumer gave two different definitions of the *numerical range* of a linear operator on a Banach space. Since they are equivalent concerning applications, we pick Bauer's one which is easier to handle. So, given $T \in L(X)$ its *numerical range* is defined as the scalar set

$$V(T) = \{x^*(Tx) \,:\, (x, x^*) \in \Pi(X)\}$$

where

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} \,:\, x^*(x) = 1\}.$$

The *numerical radius* of T is

$$v(T) = \sup\{|\lambda| \,:\, \lambda \in V(T)\}$$

and clearly satisfies $v(T) \leqslant \|T\|$. It is easy to check that v defines a semi-norm on $L(X)$. In fact, very often v is actually a norm which is equivalent to the usual operator norm. To study if this is the case, G. Lumer defined in 1968 the *numerical index* of X as the constant

$$n(X) = \inf\{v(T) \,:\, T \in L(X),\ \|T\| = 1\}.$$

Equivalently, $n(X)$ is the greatest constant $k \geqslant 0$ such that $k\|T\| \leqslant v(T)$ for every $T \in L(X)$. Note that $0 \leqslant n(X) \leqslant 1$, and $n(X) > 0$ if and only if v and $\|\cdot\|$ are equivalent norms on $L(X)$.

Even before the notion of numerical index had appeared, it was known that for a Hilbert space H of dimension greater than one, it holds that $n(H) = 1/2$ in the complex case and $n(H) = 0$ in the real case [23, p. 114]. In the first work about numerical index [16], J. Duncan, C. M. McGregor, J. D. Pryce, and A. J. White proved that $L_1(\mu)$ spaces and their isometric preduals have numerical index 1 so, in particular, $n(C(K)) = 1$ for every compact topological space K. This property is shared by the disk algebra $A(\mathbb{D})$ [8, Theorem 32.9] and, more generally, by every function algebra [43]. The exact value of the numerical indices of $L_p(\mu)$ spaces is still unknown when $1 < p < \infty$ and $p \neq 2$, but in a series of papers A. Aksoy, E. Ed-dari, and M. Khamsi [17, 18, 19] show that all infinite-dimensional $L_p(\mu)$ spaces have the same numerical index, which actually coincides with the infimum of the numerical indices of finite-dimensional $L_p(\mu)$ spaces, and the result extends to vector-valued L_p spaces. It was shown in [34] that every real

$L_p(\mu)$ space has positive numerical index for $p \neq 2$. Finally, the numerical index of some families of polyhedral norms on the plane was computed in [32].

In the following lines we present briefly some of the known results about numerical index which will be useful in our discussion. First of all, it is known since the work of J. Duncan, C. M. McGregor, J. D. Pryce, and A. J. White [16] that $v(T) = v(T^*)$ for every $T \in L(X)$, where T^* is the adjoint operator of T. So it follows that $n(X^*) \leqslant n(X)$ for every Banach space X. This inequality may be strict, as shows the example given by K. Boyko, V. Kadets, M Martín, and D. Werner [9].

It is also known that real and complex spaces behave differently with respect to the numerical index. Indeed, the set of values of the numerical index of real Banach spaces fills the whole interval $[0,1]$, while for complex Banach spaces it fills the interval $[1/\mathrm{e}, 1]$ [16]. The fact that $n(X) \geqslant 1/\mathrm{e}$ in the complex case, known as the Bonehnblust-Karlin theorem and first pointed out by B. Glickfeld [22], has important consequences in the theory of Banach algebras.

The numerical index is continuous with respect to the Banach-Mazur distance between equivalent norms and this gives that the set of values of the numerical index of a Banach space up to renorming is a non-trivial interval [21]. The numerical index of the c_0-, ℓ_1-, or ℓ_∞-sum of a family of spaces is equal to the infimum of the numerical index of the spaces and the numerical indices of the vector-valued function spaces $C(K,X)$, $L_1(\mu, X)$, and $L_\infty(\mu, X)$ are equal to the numerical index of the range space [38].

The *rank-one numerical index* of a Banach space was introduced in [33].

1.1.1 Definition. Let X be a Banach space. The *rank-one numerical index* of X is

$$\begin{aligned} n_1(X) &:= \max\{k \geqslant 0 \,:\, k\|T\| \geqslant v(T)\ \forall T \in L(X) \text{ with } \dim(T(X)) \leqslant 1\} \\ &= \inf\{v(T) \,:\, T \in L(X),\ \|T\| = 1,\ \dim(T(X)) \leqslant 1\}. \end{aligned}$$

In the aforementioned work [33] it is proved that

$$n_1(L_p(\mu)) \geqslant p^{-\frac{2}{p}} q^{-\frac{2}{q}}$$

for every $1 < p < \infty$ and every atomless measure μ, where $q = p/(p-1)$ is the conjugate exponent to p; it is also shown that $n_1(H) = 1/2$ for every real or complex Hilbert space H of dimension greater than one.

There are several motivations to study rank-one operators and the rank-one numerical index. First, in an arbitrary Banach space, it is possible to give a formula for the operator norm only for rank-one operators and so, generally, the knowledge of the numerical index does not give more information than the knowledge of the rank-one numerical index. Moreover, the study of rank-one operators and the rank-one numerical index is sometimes enough to get important consequences on the geometry of the space (this is the case of the results of [2, 29] on Banach spaces with numerical index 1, see the comment in the next paragraph). As rank-one operators are contained in any operator ideal, the rank-one numerical index gives an upper bound to any numerical index one may define associated to any operator ideal (as, for instance, the compact numerical index that we consider in section 1.4). Finally, estimations of the rank-one numerical index in concrete Banach spaces give non-trivial inequalities relating the geometry of the space and its dual, as happens with the result for atomless L_p-spaces commented earlier.

While the definition of rank-one numerical index was first given in [33], the study of numerical radius of rank-one operators was initiated much earlier. For instance, in the 1999 paper [29], the authors proved a number of results for Banach spaces with numerical index one, but they claimed that all of them are also true for Banach spaces with rank-one numerical index equal to one, since in all the proofs only rank-one operators are used. Actually, in [37] the so-called alternative Daugavet property is introduced. A Banach space X has the *alternative Daugavet property* if the norm equality

$$\max_{|\theta|=1} \|\mathrm{Id} + \theta T\| = 1 + \|T\|$$

holds for all rank-one operators on the space and, in such a case, all compact operators also satisfy that equation (actually, this is true for all operators not fixing a copy of ℓ_1, as has been recently proved in [2, Corollary 5.6]). The relation of this property with the rank-one numerical index comes from the fact known from the 1970's [16], that for

$T \in L(X)$,

$$v(T) = \|T\| \quad \Longleftrightarrow \quad \max_{|\theta|=1} \|\mathrm{Id} + \theta T\| = 1 + \|T\|.$$

Therefore, a Banach space X has the alternative Daugavet property if and only if $n_1(X) = 1$ and, in such a case, we actually have $v(T) = \|T\|$ for every operator $T \in L(X)$ which does not fix a copy of ℓ_1 (in particular, for compact operators). It also follows that for a finite-dimensional space X, if $n_1(X) = 1$, then $n(X) = 1$. This result is false in the infinite-dimensional setting, an example being $C([0,1], \ell_2)$. To finish the review about the alternative Daugavet property, let us mention that $C(K, X)$ has the alternative Daugavet property if and only if X does or K is perfect; the spaces $L_1(\mu, X)$ and $L_\infty(\mu, X)$ have the alternative Daugavet property if and only if X does or μ has no atoms [37].

1.2 A lower bound for the rank-one numerical index

The classical result of B. Glickfeld saying that $n(X) \geqslant 1/\mathrm{e}$ for every complex Banach space X tells us that $n_1(X) \geqslant 1/\mathrm{e}$ in this case. Our goal in this section is to prove that this lower bound for the rank-one numerical index is also valid in the real case and to show that it is the best possible one.

We recall that given a real or complex Banach space X, one can define the exponential function on $L(X)$ by

$$\exp(T) = \mathrm{Id} + \sum_{k=1}^{\infty} \frac{T^k}{k!} \qquad \big(T \in L(X)\big)$$

and that it follows from [7, Theorem 3.4] that

$$\|\exp(\alpha T)\| \leqslant \mathrm{e}^{|\alpha| v(T)} \qquad \big(T \in L(X),\ \alpha \in \mathbb{K}\big). \tag{1.1}$$

We are ready to state and prove the promised result.

1.2.1 Theorem. *Let X be a real Banach space. Then,*

$$n_1(X) \geqslant \frac{1}{\mathrm{e}}.$$

Proof. Let us fix a rank-one operator $T \in L(X)$. We find $x_0^* \in X^*$, $x_0 \in X$ such that

$$Tx = x_0^*(x)\, x_0 \quad (x \in X)$$

and we write $\lambda = x_0^*(x_0)$. Note that for each $\alpha \in \mathbb{R}$ one has

$$\exp(\alpha T) = \begin{cases} \mathrm{Id} + \alpha T & \text{if } \lambda = 0 \\ \mathrm{Id} + \frac{\mathrm{e}^{\lambda\alpha}-1}{\lambda}\, T & \text{if } \lambda \neq 0. \end{cases} \tag{1.2}$$

Indeed, if $\lambda = 0$ then $T^2 = 0$ and hence $\exp(\alpha T) = \mathrm{Id} + \alpha T$. If otherwise $\lambda \neq 0$, taking into account that $(\alpha T)^k = \lambda^{k-1}\alpha^k T$ for every $k \in \mathbb{N}$, we can write

$$\exp(\alpha T) = \mathrm{Id} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}\alpha^k T}{k!} = \mathrm{Id} + \left(\frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k\alpha^k}{k!}\right) T = \mathrm{Id} + \frac{\mathrm{e}^{\lambda\alpha}-1}{\lambda}\, T.$$

Now, if $v(T) = 0$ then $\lambda = 0$ (indeed, if $x_0 = 0$ the result is clear; otherwise, just pick $y^* \in S_{X^*}$ such that $y^*(x_0) = \|x_0\|$, write $y = x_0/\|x_0\| \in S_X$ and observe that $y^*(y) = 1$ and $\lambda = y^*(Ty)$). Therefore, equations (1.1) and (1.2) give in this case that

$$\|\mathrm{Id} + \alpha T\| = \|\exp(\alpha T)\| \leqslant 1 \qquad (\alpha \in \mathbb{R}).$$

This obviously implies that $T = 0$ and thus $\|T\| \leqslant \mathrm{e}v(T)$.

If otherwise $v(T) \neq 0$, we can assume without loss of generality that $v(T) = 1$ and so we have to show that $\|T\| \leqslant \mathrm{e}$. We distinguish two cases depending on wether $\lambda = 0$ or not. Suppose first that $\lambda = 0$. Then, using equations (1.1) and (1.2) for $\alpha = 1$ and $\alpha = -1$, we obtain

$$\|\mathrm{Id} + T\| \leqslant \mathrm{e} \qquad \text{and} \qquad \|\mathrm{Id} - T\| \leqslant \mathrm{e}$$

which gives

$$\|T\| = \left\|\tfrac{1}{2}(\mathrm{Id} + T) - \tfrac{1}{2}(\mathrm{Id} - T)\right\| \leqslant \frac{\mathrm{e}}{2} + \frac{\mathrm{e}}{2} = \mathrm{e},$$

as desired. Finally, when $\lambda \neq 0$ one can use (1.1) and (1.2) to obtain

$$\left\|\mathrm{Id} + \frac{e^{\lambda\alpha} - 1}{\lambda} T\right\| = \|\exp(\alpha T)\| \leqslant e^{|\alpha|} \qquad (\alpha \in \mathbb{R}).$$

Using this for $\alpha = 1$ and $\alpha = -1$ one gets

$$\left\|\mathrm{Id} + \frac{e^{\lambda} - 1}{\lambda} T\right\| \leqslant e \qquad \text{and} \qquad \left\|\mathrm{Id} + \frac{e^{-\lambda} - 1}{\lambda} T\right\| \leqslant e$$

and, therefore, one has

$$\left|\frac{e^{\lambda} - e^{-\lambda}}{2\lambda}\right| \|T\| \leqslant \frac{1}{2}\left\|\mathrm{Id} + \frac{e^{\lambda} - 1}{\lambda} T\right\| + \frac{1}{2}\left\|\mathrm{Id} + \frac{e^{-\lambda} - 1}{\lambda} T\right\| \leqslant e.$$

The desired inequality follows now from the fact that $\inf_{\lambda \neq 0}\left|\frac{e^{\lambda} - e^{-\lambda}}{2\lambda}\right| = 1.$ □

The following example shows that the inequality above is sharp. Let us comment that it is the real version of the space given in [22] of a complex two-dimensional space with numerical index equal to $1/e$.

1.2.2 Example. *There is a real two-dimensional Banach space X with $n_1(X) = 1/e$.* Indeed, consider the function $\Phi : [0, +\infty[\longrightarrow \mathbb{R}$ given by

$$\Phi(t) = \begin{cases} e^{t/e} & \text{if } t \in [0, e] \\ t & \text{if } t \geqslant e. \end{cases} \qquad (t \in [0, +\infty[)$$

Then, by Proposition 3.1 in [16] the mapping $\|\cdot\| : \mathbb{R}^2 \longrightarrow [0, +\infty[$ given by

$$\|(\alpha, \beta)\| = \begin{cases} |\alpha|\Phi(\frac{|\beta|}{|\alpha|}) & \text{if } \alpha \neq 0 \\ |\beta| & \text{if } \alpha = 0 \end{cases} \qquad ((\alpha, \beta) \in \mathbb{R}^2)$$

defines a norm on $\mathbb{R}^2$. Now, denote $X = (\mathbb{R}^2, \|\cdot\|)$ and consider the shift operator $S \in L(X)$ given by $S(\alpha, \beta) = (0, \alpha)$. Using Lemma 3.3 in [16] one obtains that

$$\|S\| = 1 \quad \text{and} \quad v(S) = \sup \frac{\Phi'}{\Phi} = 1/e,$$

which give $n_1(X) \leqslant 1/e$, as desired.

1.3 Some properties of the rank-one numerical index

This section is devoted to study the behaviour of the rank-one numerical index with respect to some natural operations with Banach spaces. In many cases, it behaves similarly to the classical numerical index. This is the case of the stability properties concerning suitable sums of Banach spaces and the continuity with respect to the Banach-Mazur distance. We will also see that the behaviour with respect to vector valued spaces differs from that of the classical numerical index.

Our first goal is to study the stability properties of the rank-one numerical index when one considers sums of Banach spaces. Given an arbitrary family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, we denote by $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}$ the c_0-sum of the family and $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_p}$ denotes the ℓ_p-sum of the family for a given p with $1 \leqslant p \leqslant \infty$.

It is shown in [38, Proposition 1] that the numerical index of a c_0-, ℓ_1-, or ℓ_∞-sum is equal to the infimum of the numerical index of the addends. As we will see later, the proof of this result can be easily adapted to the rank-one numerical index. In the already cited paper [38] it is also commented that the numerical index of an ℓ_p-sum is less or equal than the numerical index of the addends. This result has been generalized to absolute sums of Banach spaces in [35, §2]. Again, all the proofs can be adapted to the rank-one numerical index since when one considers a rank-one operator, then all the operators involved are rank-one. But, actually, we are now presenting a more general result which is new even for the classical numerical index.

1.3.1 Proposition. *Let X be a Banach space and Y, Z closed subspaces of X such that $X = Y \oplus Z$ and $\|y_1 + z\| = \|y_2 + z\|$ for $z \in Z$ and $y_1, y_2 \in Y$ with $\|y_1\| = \|y_2\|$. Then,*

$$n(X) \leqslant n(Y) \qquad \text{and} \qquad n_1(X) \leqslant n_1(Y).$$

We need a lemma which gives, in the hypotheses of the above result, the possibility of extending an operator from Y to X with the same norm and numerical radius.

1.3.2 Lemma. *Let X be a Banach space and Y, Z nontrivial closed subspaces of X such that $X = Y \oplus Z$ and $\|y_1 + z\| = \|y_2 + z\|$ for every $z \in Z$ and every $y_1, y_2 \in Y$ with $\|y_1\| = \|y_2\|$. Then, given an operator $T \in L(Y)$, the operator $\widetilde{T} \in L(X)$ defined by*

$$\widetilde{T}(y+z) = Ty \qquad (y \in Y, z \in Z),$$

satisfies $\|\widetilde{T}\| = \|T\|$ and $v(\widetilde{T}) = v(T)$.

Proof. We start with two easy observations. First, the hypothesis gives us that the projections to Y and Z given by the decomposition $X = Y \oplus Z$ have norm one. Indeed, given $y \in Y$ and $z \in Z$, one has $y = \frac{1}{2}(y+z) + \frac{1}{2}(y-z)$ which, using the fact that $\|y - z\| = \| - y + z\| = \|y + z\|$, gives

$$\|y\| \leqslant \frac{1}{2}\|y+z\| + \frac{1}{2}\|y-z\| = \|y+z\|$$

and, analogously, we get $\|z\| \leqslant \|y + z\|$.

Secondly, we show that X^* is isometrically isomorphic to $Y^* \oplus Z^*$. To do so, recall that $X^* = Z^\perp \oplus Y^\perp$ and observe that $Z^\perp \equiv Y^*$ and $Y^\perp \equiv Z^*$. Indeed, consider the mapping $J : Z^\perp \longrightarrow Y^*$ given by $Jz^\perp = z^\perp|_Y$ for $z^\perp \in Z^\perp$. Taking into account that $z^\perp(y+z) = z^\perp(y)$ and $\|y+z\| \geqslant \|y\|$ for $z^\perp \in Z^\perp, y \in Y, z \in Z$, it is clear that $\|Jz^\perp\| = \|z^\perp\|$. To see that J is onto, fix $y^* \in Y^*$, take $x^* \in X^*$ a Hahn-Banach extension of y^*, and write $x^* = z^\perp + y^\perp$ for some $z^\perp \in Z^\perp$ and $y^\perp \in Y^\perp$. Then one has $Jz^\perp = z^\perp|_Y = x^*|_Y = y^*$. Analogous arguments show that $Y^\perp \equiv Z^*$. Summarizing, we have proved that $X^* \equiv Y^* \oplus Z^*$ and that the action on X is given by

$$[y^* + z^*](y+z) = y^*(y) + z^*(z) \qquad \big(y + z \in X,\ y^* + z^* \in X^*\big).$$

Now, since Y is 1-complemented in X, it is clear that $\|\widetilde{T}\| \leqslant \|T\|$ and the reversed inequality is always true. To show that $v\big(\widetilde{T}\big) \geqslant v(T)$, fixed $(y, y^*) \in \Pi(Y)$, take a Hahn-Banach extension $x^* \in S_{X^*}$ of y^* and observe that $x^*(y) = 1$ and $x^*(\widetilde{T}(y)) = y^*(Ty)$. Therefore,

$$|y^*(Ty)| = \big|x^*(\widetilde{T}(y))\big| \leqslant v(\widetilde{T})$$

and we get the desired inequality taking supremum on $(y, y^*) \in \Pi(Y)$.

Finally, to prove the inequality $v(\widetilde{T}) \leqslant v(T)$, fixed $(x, x^*) \in \Pi(X)$, there are $y \in Y, z \in Z, y^* \in Y^*, z^* \in Z^*$ such that

$$x = y + z, \quad x^* = y^* + z^*, \quad \text{and} \quad \operatorname{Re} x^*(x) = \operatorname{Re}\big(y^*(y) + z^*(z)\big) = 1.$$

Moreover, it holds that $\|y\| \leqslant \|x\|$ and $\|y^*\| \leqslant \|x^*\|$. Hence, if we show that $\operatorname{Re} y^*(y) = \|y^*\|\|y\|$, then

$$|x^*(\widetilde{T}x)| = |y^*(Ty)| \leqslant v(T)\|y^*\|\|y\| \leqslant v(T)\|x^*\|\|x\| = v(T)$$

and the proof will be finished. To do so, given $\varepsilon > 0$, take $y_\varepsilon \in B_Y$ with $\|y_\varepsilon\| = \|y\|$ such that

$$\operatorname{Re} y^*(y_\varepsilon) > \|y^*\|\|y_\varepsilon\| - \varepsilon.$$

By hypothesis, we have that $\|y_\varepsilon + z\| = \|y + z\| = 1$ and, therefore,

$$\begin{aligned}
\operatorname{Re} y^*(y) + \operatorname{Re} z^*(z) &= \operatorname{Re}[y^* + z^*](y + z) = 1 \\
&\geqslant \operatorname{Re}[y^* + z^*](y_\varepsilon + z) = \operatorname{Re} y^*(y_\varepsilon) + \operatorname{Re} z^*(z) \\
&> \|y^*\|\|y_\varepsilon\| - \varepsilon + \operatorname{Re} z^*(z)
\end{aligned}$$

which gives $\operatorname{Re} y^*(y) > \|y^*\|\|y\| - \varepsilon$. Finally, the arbitrariness of ε tells us that $\operatorname{Re} y^*(y) \geqslant \|y^*\|\|y\|$. □

Proof of Proposition 1.3.1. For the numerical index the result is an obvious consequence of the above lemma, since for every $T \in L(Y)$ with $T \neq 0$, one has

$$n(X) \leqslant \frac{v(\widetilde{T})}{\|\widetilde{T}\|} = \frac{v(T)}{\|T\|}.$$

Taking infimum on $T \in L(Y)$ with $T \neq 0$, we get $n(X) \leqslant n(Y)$ as desired. The result for the rank-one numerical index follows using the same ideas and taking into account that when T is a rank-one operator, $\widetilde{T}$ is also a rank-one operator. □

As we already commented, in the case of c_0, ℓ_1, and ℓ_∞-sums we have the following expected result.

1.3.3 Proposition. *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of Banach spaces. Then*

$$n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big) = n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_1}\Big) = n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_\infty}\Big) = \inf\{n_1(X_\lambda) \ : \ \lambda \in \Lambda\}.$$

The proof of this result, which we include for shake of completeness, is just an adaptation of the one given in [38, Proposition 1] for the numerical index. As we mentioned above, it is enough to check that when in this proof one starts with rank-one operators, all the operators involved are also rank-one.

Proof. Let X be the c_0, ℓ_1 or the ℓ_∞-sum of the family $\{X_\lambda : \lambda \in \Lambda\}$. Then, Proposition 1.3.1 tells us that for every $\lambda \in \Lambda$ we have $n_1(X) \leqslant n_1(X_\lambda)$ and, therefore, we obtain that $n_1(X) \leqslant \inf\{n_1(X_\lambda) : \lambda \in \Lambda\}$.

We turn to prove the reversed inequality for the c_0 and ℓ_∞ sums. If Z denotes the any of these sums, a rank-one operator $T \in L(Z)$ can be seen as a family $(T_\lambda)_{\lambda\in\Lambda}$ of rank-one operators where $T_\lambda \in L(Z, X_\lambda)$ for every λ, and $\|T\| = \sup_\lambda \|T_\lambda\|$. Given $\varepsilon > 0$, we find $\lambda_0 \in \Lambda$ such that $\|T_{\lambda_0}\| > \|T\| - \varepsilon$, and we write $X = X_{\lambda_0} \oplus_\infty Y$ where $Y = \big[\oplus_{\lambda\neq\lambda_0} X_\lambda\big]_{c_0}$ or $Y = \big[\oplus_{\lambda\neq\lambda_0} X_\lambda\big]_{l_\infty}$. Since B_Z is the convex hull of $S_{X_{\lambda_0}} \times S_Y$, we may find $x_0 \in S_{X_{\lambda_0}}$ and $y_0 \in S_Y$ such that

$$\|T_{\lambda_0}(x_0, y_0)\| > \|T\| - \varepsilon.$$

Now fix $x_0^* \in X_{\lambda_0}^*$ with $\|x_0^*\| = x_0^*(x_0) = 1$ and consider the rank-one operator $S \in L(X_{\lambda_0})$ defined by

$$Sx = T_{\lambda_0}(x, x_0^*(x)y_0) \qquad (x \in X_{\lambda_0}).$$

We clearly have

$$\|S\| \geqslant \|Sx_0\| = \|T_{\lambda_0}(x_0, y_0)\| > \|T\| - \varepsilon,$$

so we may find $(x, x^*) \in \Pi(X_{\lambda_0})$ such that

$$|x^*(Sx)| \geqslant n(X_{\lambda_0})[\|T\| - \varepsilon].$$

Now $z = (x, x_0^*(x)y_0) \in S_Z$, $z^* = (x^*, 0) \in S_{Z^*}$ satisfy $z^*(z) = 1$ and

$$|z^*(Tz)| = |x^*(T_{\lambda_0}(x, x_0^*(x)y_0))| = |x^*(Sx)| \geqslant n(X_{\lambda_0})[\|T\| - \varepsilon].$$

It follows that

$$v(T) \geqslant \inf_{\lambda} n(X_\lambda)\|T\|$$

and so $n(Z) \geqslant \inf_\lambda n(X_\lambda)$ as required.

The proof for the ℓ_1-sum is somehow the dual of the above argument. If $Z = [\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}$, a rank-one operator $T \in L(Z)$ may also be seen as a family $(T_\lambda)_{\lambda\in\Lambda}$ of rank-one operators where now $T_\lambda \in L(X_\lambda, Z)$ for all λ, and again $\|T\| = \sup_\lambda \|T_\lambda\|$. Given $\varepsilon > 0$, find $\lambda_0 \in \Lambda$ such that $\|T_{\lambda_0}\| > \|T\| - \varepsilon$, and write $Z = X_{\lambda_0} \oplus_1 Y$, $T_{\lambda_0} = (A, B)$ where $A \in L(X_{\lambda_0})$ and $B \in L(X_{\lambda_0}, Y)$. Now we choose $x_0 \in S_{X_{\lambda_0}}$ such that

$$\|T_{\lambda_0} x_0\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon,$$

we find $a_0 \in S_{X_{\lambda_0}}$, $y^* \in S_{Y^*}$ satisfying

$$\|Ax_0\| a_0 = Ax_0 \qquad \text{and} \qquad y^*(Bx_0) = \|Bx_0\|,$$

and define a rank-one operator $S \in L(X_{\lambda_0})$ by

$$Sx = Ax + y^*(Bx)a_0 \qquad (x \in X_{\lambda_0}).$$

Then

$$\|S\| \geqslant \|Sx_0\| = \Big\|Ax_0 + \|Bx_0\| a_0\Big\| = \|Ax_0\| + \|Bx_0\| > \|T\| - \varepsilon,$$

so we may find $(x, x^*) \in \Pi(X_{\lambda_0})$ such that

$$|x^*(Sx)| \geqslant n(X_{\lambda_0})[\|T\| - \varepsilon].$$

For $z = (x, 0) \in S_Z$ and $z^* = (x^*, x^*(a_0)y^*) \in S_{Z^*}$ we clearly have $z^*(z) = 1$ and

$$|z^*(Tz)| = |x^*(Ax) + x^*(a_0)y^*(Bx)| = |x^*(Sx)| \geqslant n(X_{\lambda_0})[\|T\| - \varepsilon].$$

The desired inequality $n(Z) \geqslant \inf_\lambda n(X_\lambda)$ follows. □

As a particular case of Proposition 1.3.1 we have that the rank-one numerical index of an absolute sum of Banach spaces is less or equal than the rank-one numerical index of each one of the addends (in [35, §2] the reader will find a different proof for the case of the classical numerical index). We will only give here two particular cases.

1.3.4 Corollary. *Let Λ be a nonempty set, let $\{X_\lambda \,:\, \lambda \in \Lambda\}$ be a family of Banach spaces and $1 < p < \infty$. Then*

$$n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big) \leqslant \inf\{n_1(X_\lambda) \,:\, \lambda \in \Lambda\}.$$

To present the second result we recall the notions of absolute norm and absolute sum of a family of Banach spaces. Let Λ be a nonempty set and let E be a linear subspace of $\mathbb{R}^\Lambda$. An *absolute norm* on E is a complete norm $\|\cdot\|_E$ satisfying

(a) Given $(a_\lambda), (b_\lambda) \in \mathbb{R}^\Lambda$ with $|a_\lambda| = |b_\lambda|$ for every $\lambda \in \Lambda$, if $(a_\lambda) \in E$, then $(b_\lambda) \in E$ with $\|(a_\lambda)\|_E = \|(b_\lambda)\|_E$.

(b) For every $\lambda \in \Lambda$, $\chi_{\{\lambda\}} \in E$ with $\|\chi_{\{\lambda\}}\|_E = 1$, where $\chi_{\{\lambda\}}$ is the characteristic function of the singleton $\{\lambda\}$.

The following results can be deduced from the definition above:

(c) Given $(x_\lambda), (y_\lambda) \in \mathbb{R}^\Lambda$ with $|y_\lambda| \leqslant |x_\lambda|$ for every $\lambda \in \Lambda$, if $(x_\lambda) \in E$, then $(y_\lambda) \in E$ with $\|(y_\lambda)\|_E \leqslant \|(x_\lambda)\|_E$.

(d) $\ell_1(\Lambda) \subseteq E \subseteq \ell_\infty(\Lambda)$ with contractive inclusions.

Given an arbitrary family $\{X_\lambda \,:\, \lambda \in \Lambda\}$ of Banach spaces, the *E-sum* of the family is the space

$$\begin{aligned}\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_E &:= \Big\{(x_\lambda) \,:\, x_\lambda \in X_\lambda \ \forall\lambda \in \Lambda,\ (\|x_\lambda\|) \in E\Big\} \\ &= \Big\{(a_\lambda x_\lambda) \,:\, a_\lambda \in \mathbb{R}_0^+,\ x_\lambda \in S_{X_\lambda} \ \forall\lambda \in \Lambda,\ (a_\lambda) \in E\Big\}\end{aligned}$$

endowed with the complete norm $\|(x_\lambda)\| = \|(\|x_\lambda\|)\|_E$. Examples of absolute sums are c_0-sums, ℓ_p-sums for $1 \leqslant p \leqslant \infty$, i.e. given a nonempty set Λ, we are considering $E = c_0(\Lambda)$ or $E = \ell_p(\Lambda)$. Other interesting examples are the absolute sums produced using a Banach space E with a one-unconditional basis, finite (i.e. E is $\mathbb{R}^m$ endowed with an absolute norm) or infinite (i.e. E is a Banach space with an one-unconditional basis viewed as a linear subspace of $\mathbb{R}^{\mathbb{N}}$ via the basis). With this notation we can write the promised result.

1.3.5 Corollary.

(a) *Let E be $\mathbb{R}^m$ endowed with an absolute norm, let $X_1, \dots, X_m$ be Banach spaces and write X to denote their E-sum. Then*

$$n_1(X) \leqslant \min\{n_1(X_1), \dots, n_1(X_m)\}.$$

(b) *Let E be a Banach space with a one-unconditional (infinite) basis, let $\{X_j : j \in \mathbb{N}\}$ be a sequence of Banach spaces and let X denote their E-sum. Then*

$$n_1(X) \leqslant \inf\{n_1(X_j) : j \in \mathbb{N}\}.$$

We turn now to deal with the study of the rank-one numerical index of vector valued spaces. This time its behavior differs from that of the classical numerical index.

1.3.6 Proposition. *Let K be a compact Hausdorff space, μ a positive measure, and X a Banach space. Then, the following hold:*

$$n_1\big(C(K,X)\big) = \begin{cases} 1 & \text{if } K \text{ is perfect,} \\ n_1(X) & \text{if } K \text{ is not perfect,} \end{cases}$$

$$n_1\big(L_1(\mu,X)\big) = \begin{cases} 1 & \text{if } \mu \text{ has no atoms,} \\ n_1(X) & \text{if } \mu \text{ has atomic part,} \end{cases}$$

$$n_1\big(L_\infty(\mu,X)\big) = \begin{cases} 1 & \text{if } \mu \text{ has no atoms,} \\ n_1(X) & \text{if } \mu \text{ has atomic part.} \end{cases}$$

Proof. The proof for $L_1(\mu,X)$ and $L_\infty(\mu,X)$ follows the same lines, so we only give the one for the L_1-case. Indeed, it is known that $L_1(\mu,X)$ is isometrically isomorphic to a space of the form

$$\ell_1(\Gamma, X) \oplus_1 L_1(\nu, X)$$

for suitable set Γ and atomless measure ν [15, pp. 501], being Γ empty when μ is actually atomless. Now, the result follows from Proposition 1.3.3 and the fact that $L_1(\nu, X)$ has the alternative Daugavet property [37, Theorem 3.4] and so, $n_1\big(L_1(\nu, X)\big) = 1$.

Let us now prove the result for $C(K, X)$. If K is perfect, $C(K, X)$ has the alternative Daugavet property [37, Theorem 3.4] and so $n_1\big(C(K, X)\big) = 1$. If K has an isolated point, then X is an ℓ_∞-summand of $C(K, X)$ and so Proposition 1.3.3 gives us that $n_1\big(C(K, X)\big) \leqslant n_1(X)$. For the reversed inequality, we just need to follow the first part of the proof of [38, Theorem 5] but considering rank-one operators. Indeed, we fix a rank-one operator $T \in L(C(K, X))$ with $\|T\| = 1$ and we prove that $v(T) \geqslant n_1(X)$. Given $\varepsilon > 0$, we may find $f_0 \in C(K, X)$ with $\|f_0\| = 1$ and $t_0 \in K$ such that

$$\|[Tf_0](t_0)\| > 1 - \varepsilon. \tag{1.3}$$

Denote $y_0 = f_0(t_0)$ and find a continuous function $\varphi : K \longrightarrow [0, 1]$ such that $\varphi(t_0) = 1$ and $\varphi(t) = 0$ if $\|f_0(t) - y_0\| \geqslant \varepsilon$. Now write $y_0 = \lambda x_1 + (1 - \lambda)x_2$ with $0 \leqslant \lambda \leqslant 1$, $x_1, x_2 \in S_X$, and consider the functions

$$f_j = (1 - \varphi)f_0 + \varphi x_j \in C(K, X) \qquad (j = 1, 2).$$

Then $\|\varphi f_0 - \varphi y_0\| < \varepsilon$ meaning that

$$\|f_0 - (\lambda f_1 + (1 - \lambda)f_2)\| < \varepsilon,$$

and, using (1.3), we must have

$$\|[Tf_1](t_0)\| > 1 - 2\varepsilon \qquad \text{or} \qquad \|[Tf_2](t_0)\| > 1 - 2\varepsilon.$$

By making the right choice of $x_0 = x_1$ or $x_0 = x_2$ we get $x_0 \in S_X$ such that

$$\|\left[T\left((1 - \varphi)f_0 + \varphi x_0\right)\right](t_0)\| > 1 - 2\varepsilon. \tag{1.4}$$

Next we fix $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$, denote

$$\Phi(x) = x_0^*(x)(1 - \varphi)f_0 + \varphi x \in C(K, X) \qquad (x \in X),$$

and consider the rank-one operator $S \in L(X)$ given by

$$Sx = [T(\Phi(x))](t_0) \qquad (x \in X).$$

Since, by (1.4),

$$\|S\| \geqslant \|Sx_0\| > 1 - 2\varepsilon,$$

we may find $(x, x^*) \in \Pi(X)$ such that

$$|x^*(Sx)| \geqslant n_1(X)[1 - 2\varepsilon].$$

Now, define $g \in S_{C(K,X)}$ by $g = \Phi(x)$, for this x, and consider the functional $g^* \in S_{C(K,X)^*}$ given by

$$g^*(h) = x^*(h(t_0)) \qquad (h \in C(K, X)).$$

Since $g(t_0) = x$, we have $g^*(g) = 1$ and

$$|g^*(Tg)| = |x^*(Sx)| \geqslant n_1(X)[1 - 2\varepsilon].$$

Hence $v(T) \geqslant n_1(X)$, as required. □

We may use the above result to give an example of a Banach space X such that $n_1(X^*) < n_1(X)$. Indeed, the space $X = C([0, 1], \ell_2)$ satisfies $n_1(X) = 1$ but $X^* \equiv L_1(\mu, \ell_2)$ for some measure μ which clearly contains atoms, so $n_1(X^*) = n_1(\ell_2) = 1/2$. Let us comment that this kind of example has appeared previously in the literature (see [37, Example 4.4]) using a characterization of the alternative Daugavet property for C^*-algebras and von Neumann preduals. On the other hand, for a von Neumann algebra A, it is shown in [37, Theorem 4.2] that A has the alternative Daugavet property if and only if its predual A_* does (equivalently, $n_1(A) = 1$ if and only if $n_1(A_*) = 1$) and this result was generalized to L-embedded spaces in [31, Proposition 2.3]. Actually, we may give a more general result covering any value of the rank-one numerical index. We recall that a Banach space X is said to be *L-embedded* if $X^{**} = X \oplus_1 X_s$ for some closed subspace X_s of X^{**}.

The proof of the following result is just an adaptation of the one given in [31, Theorem 2.1] for the case of the classical numerical index.

1.3.7 Proposition. *Let X be an L-embedded space. Then, $n_1(X) = n_1(X^*)$.*

Proof. We write $X^{**} = X \oplus_1 X_s$ and $P_X : X^{**} \longrightarrow X$ for the associated projection. Given a rank-one operator $T \in L(X^*)$, we consider the rank-one operators $A \in L(X)$ and $B \in L(X, X_s)$ given by $A = P_X \circ T^* \circ i_X$ and $B = (\mathrm{Id} - P_X) \circ T^* \circ i_X$ (observe that $T^* \circ i_X \equiv A \oplus B$). Given $\varepsilon > 0$, since B_X is w^*-dense in $B_{X^{**}}$ and T^* is w^*-w^*-continuous, we may find $x_0 \in S_X$ such that

$$\|T^*(x_0)\| = \|A(x_0)\| + \|B(x_0)\| > \|T^*\| - \varepsilon.$$

Now, we find $a_0 \in S_X$, $y_0^* \in S_{X_s^*}$ satisfying

$$\|A(x_0)\| a_0 = A(x_0) \qquad \text{and} \qquad y_0^*(B(x_0)) = \|Bx_0\|,$$

and we define an operator $S \in L(X)$ by

$$S(x) = A(x) + y_0^*(B(x))\, a_0 \qquad (x \in X).$$

Then

$$\|S\| \geqslant \|S(x_0)\| = \big\|A(x_0) + \|B(x_0)\| a_0\big\| = \Big(\|Ax_0\| + \|B(x_0)\|\Big)\|a_0\| > \|T^*\| - \varepsilon,$$

so we may find $(x, x^*) \in \Pi(X)$ such that

$$|x^*(S(x))| \geqslant n(X)[\|T\| - \varepsilon].$$

For $z = (x, 0) \in S_{X^{**}}$ and $z^* = (x^*,\, x^*(a_0)\, y_0^*) \in S_{X^{***}}$ we clearly have $z^*(z) = 1$ and

$$|z^*(T^*(z))| = \big|x^*(A(x)) + x^*(a_0)\, y_0^*(B(x))\big| = |x^*(S(x))| \geqslant n(X)\big[\|T^*\| - \varepsilon\big].$$

Now, letting $\varepsilon \downarrow 0$, we get

$$v(T) = v(T^*) \geqslant n(X)\, \|T^*\| = n(X)\|T\|.$$

Therefore, $n(X^*) \geqslant n(X)$, and the other inequality is always true. □

As a corollary, we get the following result for von Neumann algebras.

1.3.8 Corollary. *Let A be a von Neumann algebra and A_* its unique predual. Then,* $n_1(A) = n_1(A_*)$.

We finish this section with a result on the continuity of the rank-one numerical index of Banach spaces, analogous to the one given in [21] for the classical numerical index. Actually, the proofs are just adaptations to the new index of the ones given there. However, it is interesting to gather the results and the proofs which are valid for the rank-one numerical index. We need some definitions and notation used in the cited paper [21] which were actually taken from [8, §18].

Given a Banach space X, we denote by $\mathcal{E}(X)$ the set of all equivalent norms on X. This is an arcwise connected metric space when provided with the distance

$$d(p,q) = \log\left(\min\{k \geqslant 1 \,:\, p \leqslant kq,\, q \leqslant kp\}\right) \qquad (p,q \in \mathcal{E}(X)).$$

If $p \in \mathcal{E}(X)$ we will use the same symbol to denote the dual norm in X^* and we will use the set

$$\Pi(X,p) = \{(x,x^*) \in X \times X^* \,:\, p(x) = p(x^*) = x^*(x) = 1\}.$$

Given $T \in L(X)$, $v_p(T)$ denotes the numerical radius of T in the space (X,p):

$$v_p(T) = \sup\{|x^*(Tx)| \,:\, (x,x^*) \in \Pi(X,p)\}.$$

Besides, $n_1(X,p)$ will be the rank-one numerical index of the Banach space (X,p). Finally, we consider the set

$$\mathcal{N}_1(X) = \{n_1(X,p) \,:\, p \in \mathcal{E}(X)\}$$

which represents all the values of the rank-one numerical index that X may have up to equivalent renorming.

1.3.9 Proposition. *Let X be a Banach space.*

(a) The mapping $(p,T) \longmapsto v_p(T)$ from $\mathcal{E}(X) \times L(X)$ to $\mathbb{R}$ is uniformly continuous on bounded sets.

(b) As a consequence, the mapping $p \longmapsto n_1(X,p)$ *from* $\mathcal{E}(X)$ *to* $\mathbb{R}$ *is continuous.*

(c) Hence, $\mathcal{N}_1(X)$ *is an interval.*

(d) If $\dim(X) > 1$*, then* $1/\mathrm{e} \in \mathcal{N}_1(X)$.

Proof. Item (a) follows from an easy refinement of the proof of [8, Corollary 18.4], as it was commented in [21]: we start observing that since neither boundedness nor uniform continuity depend on the norm we fix in X (equivalent norms in X correspond equivalent norms in $L(X)$), we can use in $\mathcal{E}(X) \times \{L \in L(X) : L \text{ rank-one}\}$ any of the natural distances in a product of metric spaces. For instance, we may and do choose the maximum. Next, fix $p_0 \in \mathcal{E}(X)$, $k \geqslant 1$, and

$$G(p_0,k) = \{p \in \mathcal{E}(X), d(p,p_0) < \log k\}$$

a neighbourhood of p_0 in $\mathcal{E}(X)$. We observe that for every bounded set A in $\mathcal{E}(X) \times \{L \in L(X) \,:\, L \,\text{rank-one}\}$, there exist $k > 1$ and $N > 0$ such that

$$A \subseteq G(p_0,k) \times \{T \in L(X) \,\text{rank-one} : p_0(T) \leqslant N\}.$$

Therefore, it is enough to prove the result for sets of the form

$$G(p_0,k) \times \{T \in L(X) \,\text{rank-one} \,:\, p_0(T) \leqslant N\}.$$

Fix $p,q \in G(p_0,k)$ with $d(p,q) < \log(1+\delta)$, where $\delta > 0$ is still to be determined, and rank-one operators T, S satisfying:

$$p_0(T) \leqslant N, \quad p_0(S) \leqslant N, \quad \text{and} \quad p_0(T-S) < \delta. \tag{1.5}$$

We need to show that if $\varepsilon > 0$ then $|v_p(T) - v_q(S)| < \varepsilon$. To do so, take $(x,x^*) \in \Pi(X,p)$ and define $z = \frac{x}{q(x)}$ and $z^* = \frac{x^*}{q(x^*)}$. Then, it follows that

$$q(z) = q(z^*) = 1, \qquad d(p,q) < \log(1+\delta), \qquad \text{and} \qquad z^*(z) = \frac{1}{q(x)q(x^*)}.$$

Therefore, we get $z^*(z) \geqslant \frac{1}{(1+\delta)^2}$. So, if $\gamma > 0$ satisfies

$$\frac{1}{(1+\delta)^2} = 1 - \frac{\gamma^2}{4},$$

the Bishop-Phelps-Bollobás Theorem (see 2.1.1) applied in (X, q) provides us with $(y, y^*) \in \Pi(X, q)$ such that $q(z-y) < \gamma$ and $q(z^* - y^*) < \gamma$. Since

$$1 - \delta \leqslant \frac{1}{1+\delta} \leqslant q(x) \leqslant 1 + \delta,$$

we also obtain $q(x - z) = |1 - q(x)| \leqslant \delta$ and, analogously, $q(x^* - z^*) \leqslant \delta$. So we can write

$$q(x-y) \leqslant q(x-z) + q(z-y) \leqslant \delta + \gamma \qquad \text{and} \qquad q(x^* - y^*) \leqslant \delta + \gamma.$$

Moreover, since $q \in G(p_0, k)$ we have

$$p_0(x-y) \leqslant k(\delta+\gamma) \qquad \text{and} \qquad p_0(x^* - y^*) \leqslant k(\delta+\gamma).$$

Using this, (1.5), and taking into account that $p, q \in G(p_0, k)$ gives us

$$p_0(x^*) \leqslant kp(x^*) = k \quad \text{and} \quad p_0(y) \leqslant kq(y) = k;$$

we can estimate $|x^*(Tx) - y^*(Sy)|$ as follows:

$$\begin{aligned} |x^*(Tx) - y^*(Sy)| &\leqslant |x^*(T(x-y))| + |x^*(T-S)(y)| + |(x^*-y^*)(Sy)| \\ &\leqslant p_0(x^*)p_0(T)p_0(x-y) + p_0(x^*)p_0(T-S)p_0(y) \\ &\qquad\qquad + p_0(x^*-y^*)p_0(S)p_0(y) \\ &\leqslant k^2(2N(\delta+\gamma)+\delta). \end{aligned}$$

Since γ only depends on δ, and it tends to zero with δ, one can choose $\delta > 0$ such that the last expression is as small as we wish. So, fixed $\varepsilon > 0$ and

$$G(p_0, k) \times \{T \in L(X) \text{ rank-one} : p_0(T) \leqslant N\}$$

we have showed that there exists $\delta > 0$ satisfying that if $d(p,q) < \log(1+\delta)$ and $p_0(T-S) < \delta$, then there are $(x,x^*) \in \Pi(X,p)$ and $(y,y^*) \in \Pi(X,q)$ such that

$$|x^*(Tx) - y^*(Sy)| < \varepsilon.$$

Finally, we obtain

$$|x^*(Tx)| \leqslant \varepsilon + |y^*(Sy)| \leqslant \varepsilon + v_q(S)$$

and taking supremum in $(x,x^*) \in \Pi(X,p)$, one has

$$v_p(T) \leqslant v_q(S) + \varepsilon.$$

Now, exchanging the roles of p and q and those of $T$and S, we get the desired inequality

$$|v_p(T) - v_q(S)| \leqslant \varepsilon.$$

Item (b) follows from (a) in the same manner as the continuity of the classical numerical index is deduced in [21]. Indeed, fix $p_0 \in \mathcal{E}(X)$, let B be an open ball centered at p_0 and $S = \{T \in L(X) \ : \ p_0(T) = 1, \dim(T(X)) = 1\}$. It follows from (a) that the mapping $\Psi : B \times S \longrightarrow \mathbb{R}$ given by

$$\Psi(p,T) = \frac{v_p(T)}{p(T)} \qquad (p \in B, \ T \in S)$$

is uniformly continuous, which implies that the mapping

$$p \longmapsto \inf\{\Psi(p,T) \ : \ T \in S\} = n_1(X,p)$$

is continuous on B. (c) is an obvious consequence of (b). Finally, to prove (d) we take a two-dimensional subspace Y of X, and we write $X = Y \oplus Z$ for suitable Z. Now, let W be a two-dimensional space with $n_1(W) = 1/e$ (Example 1.2.2 for the real case and [22] for the complex case). Then, we have $X \simeq W \oplus_1 Z$, and Proposition 1.3.3 tells us that $n_1(W \oplus_1 Z) = \min\{n_1(W), n_1(Z)\} = n_1(W)$. □

1.4 Some examples and remarks

Our goal in this section is to provide some interesting examples concerning the rank-one numerical index. As we already commented, for every Banach space one has $n_1(X^*) \leqslant n_1(X)$ and this inequality may be strict. In the first example we show that this inequality can be strict in the strongest possible way.

1.4.1 Example. *There is a Banach space X such that $n_1(X) = 1$ and $n_1(X^*) = 1/\mathrm{e}$.* Indeed, let E be a two-dimensional Banach space with $n_1(E) = 1/\mathrm{e}$. Then, by [30, Theorem 3.3] there is a Banach space $X = X(E)$ satisfying $n(X) = 1$ and $X^* = E^* \oplus_1 Z$ for suitable Z. Therefore, $n_1(X) \geqslant n(X) = 1$ and $1/\mathrm{e} \leqslant n_1(X^*) \leqslant n_1(E^*) = n_1(E) = 1/\mathrm{e}$ by Theorem 1.2.1 and Proposition 1.3.3.

In the following we construct some examples which relate the rank-one numerical index with some other natural indices that one may consider, let us start by defining them. Given a Banach space X, for every $r \in \mathbb{N}$ we define the *rank-r numerical index* by

$$n_r(X) = \inf\{v(T) \ : \ T \in S_{L(X)}, \dim(T(X)) \leqslant r\}$$

and the *compact numerical index* by

$$n_{\mathrm{comp}}(X) = \inf\{v(T) \ : \ T \in S_{L(X)}, T \text{ compact}\}.$$

It is immediate that $n_r(X) \geqslant n_{r+1}(X) \geqslant n_{\mathrm{comp}}(X) \geqslant n(X)$ for every $r \in \mathbb{N}$.

We start providing a real Banach space whose compact numerical index is strictly between the classical numerical index and the rank-one numerical index. Let us recall that when $n_1(X) = 1$ for a Banach space X, then $n_{\mathrm{comp}}(X) = 1$ [37, Theorem 2.2].

1.4.2 Example. *There exists a real Banach space X such that*

$$n(X) < n_{\mathrm{comp}}(X) < n_1(X).$$

Indeed, fix a sufficiently large even number k such that $\tan\left(\frac{\pi}{2k}\right) < 1/\mathrm{e}$ and take X_k to be the two-dimensional real Banach space whose unit ball is the $2k$-sided regular polygon centered at the origin, having one of its vertices on the point $(1,0)$. Now, consider the space

$$X = C([0,1],\ell_2) \oplus_1 X_k.$$

Then, we have that $n(X) = n(\ell_2) = 0$ by [38, Proposition 1 and Theorem 5], that $n_1(X) \geqslant 1/e$ by Theorem 1.2.1, and that $n_{\mathrm{comp}}(X) = n_{\mathrm{comp}}(X_k) = n(X_k) = \tan\left(\frac{\pi}{2k}\right)$ by [32, Theorem 5] and Proposition 1.3.3.

Let us comment that we do not know if the equality $n_{\mathrm{comp}}(X) = n_1(X)$ holds for every complex Banach space X.

The next result we present is that for finite-dimensional spaces, the values of the rank-one and the rank-two numerical indices are sometimes related. We start with a lemma which does not require the space to be finite-dimensional.

1.4.3 Lemma. *Let X be a Banach space. If there is $T \in L(X)$ with $\dim(T(X)) = 2$ and $v(T) = 0$, then $n_1(X) \leqslant \frac{1}{2}$.*

Proof. By [42, Thorem 2.1], $Y = T(X)$ is a two-dimensional well-embedded Hilbert subspace of X. That is (see [42, p. 430] and [42, Proposition 1.11]), there exists a subspace Z of X such that $X = Y \oplus Z$ and $\|y_1 + z\| = \|y_2 + z\|$ for every $z \in Z$ and every $y_1, y_2 \in Y$ with $\|y_1\| = \|y_2\|$. Now, Proposition 1.3.1 gives that $n_1(X) \leqslant n_1(Y)$. Finally, we have that $n_1(Y) = \frac{1}{2}$ by [33, Proposition 3.3] since Y is a Hilbert space with dimension greater than one. $\square$

As an immediate consequence, we obtain that the numerical indices of rank-one and rank-two operators are linked for finite-dimensional Banach spaces.

1.4.4 Corollary. *Let X be a finite-dimensional (real) space with $n_2(X) = 0$. Then, $n_1(X) \leqslant \frac{1}{2}$.*

For two-dimensional spaces, the result actually deals with the classical numerical index.

1.4.5 Corollary. *Let X be a two-dimensional (real) space. If $n_1(X) > \frac{1}{2}$, then $n(X) > 0$.*

We do not know whether the above result is true for arbitrary Banach spaces.

When X is a two-dimensional real Hilbert space, one has that $n_2(X) = 0$ and $n_1(X) = 1/2$. In the next example we show that something similar can happen for the numerical indices of higher rank.

1.4.6 Example. *For every even number r there is a Banach space X_r of dimension r with $n_r(X_r) = 0$ and $n_s(X_r) > 0$ for every $s < r$.* Indeed, fixed an even number r, [41, Theorem 3.10] provides us with an r-dimensional real Banach space X_r and an onto operator $T_0 \in L(X_r)$ such that

$$\{T \in L(X_r) \, : \, v(T) = 0\} = \{\lambda T_0 \, : \, \lambda \in \mathbb{R}\}.$$

It follows that $n_r(X_r) = 0$ since $T_0 \neq 0$ and that $n_s(X_r) > 0$ for every $s < r$ since the only non-null operators with numerical radius zero are the non-null multiples of T_0 which have rank r.

We may use the above example to produce an analogue to Example 1.4.2 for the numerical indices of higher rank.

1.4.7 Example. *For every $r \in \mathbb{N}$, there exists a real Banach space X such that*

$$n(X) < n_{\text{comp}}(X) < n_r(X).$$

Indeed, write $Y = X_{r+2}$ or $Y = X_{r+1}$ of the above example depending on whether r is even or odd. We have then $n_r(Y) > 0$ and $n_{\text{comp}}(Y) = 0$ since $n_{r+1}(Y) = 0$ or $n_{r+2}(Y) = 0$ depending on our choice of Y. From Proposition 1.3.9.a (just the analogous proof to (b) there, replacing rank-one operators by rank-r operators or compact operators), we deduce that both n_{comp} and n_r are continuous with respect to equivalent norm. Therefore, as we may find polyhedral norms arbitrarily close to the norm of Y, there exists a polyhedral norm such that, calling W to the space Y endowed with this norm, we still have $n_{\text{comp}}(W) < n_r(W)$. Moreover, as W is polyhedral it cannot contain an isometric copy of $\mathbb{C}$, so Theorem 2.4 in [36] tells us that $n_{\text{comp}}(W) \neq 0$. Now, we may follow the lines of the proof of Example 1.4.2 and consider the space

$$X = C([0,1], \ell_2) \oplus_1 W$$

which satisfies $n(X) = n(\ell_2) = 0$ by [38, Proposition 1 and Theorem 5], $n_r(X) = n_r(W)$ and $n_{\text{comp}}(X) = n_{\text{comp}}(W)$ since $C([0,1], \ell_2)$ has the alternative Daugavet property and Proposition 1.3.3 is also true for the rank-r and the compact numerical indices.

We do not know if there is a Banach space X such that $n_{\text{comp}}(X) \neq \inf\limits_{r\in\mathbb{N}} n_r(X)$. If such an example exists, it cannot have the approximation property since, in that case, compact operators can be approximated in norm, and hence in numerical radius, by finite-rank operators.

1.5 Computation of the rank-one numerical index

In this section we compute the rank-one numerical index of some families of polyhedral norms on the plane. More precisely, we obtain explicit formulae for the rank-one numerical index of three families of norms on $\mathbb{R}^2$: a family of hexagonal norms and two families of octagonal norms. The numerical index of these families of norms was computed in [32].

We start recalling some facts that will be useful to our discussion. Let X and Y be Banach spaces such that there exists a surjective isometry $S : X \longrightarrow Y$. It is immediate to check that

$$v(T) = v(S^{-1}TS) \qquad \text{and} \qquad \|T\| = \|S^{-1}TS\|.$$

for every operator $T \in L(Y)$.

For a convex set A, $\text{ext}(A)$ will denote the set of its extreme points. Since a nonempty compact convex subset of $\mathbb{R}^n$ is equal to the convex hull of its extreme points, we get the following equalities for every operator T on a finite-dimensional real Banach space X (see [37, Lemma 2.5] for the numerical radius):

$$\|T\| = \sup\{\|Tx\| \ : \ x \in \text{ext}(B_X)\} \tag{1.6}$$

$$v(T) = \sup\{|x^*(Tx)| \ : \ x \in \text{ext}(B_X), \ x^* \in \text{ext}(B_{X^*}), \ x^*(x) = 1\}. \tag{1.7}$$

1.5.1 Hexagonal norms

The first family for which we compute the rank-one numerical index consists of spaces whose unit ball is hexagonal. For each $\gamma \in [0,1]$, let $X_\gamma = (\mathbb{R}^2, \|\cdot\|_\gamma)$ where the norm $\|\cdot\|_\gamma$ is given by

$$\|(x,y)\|_\gamma = \max\{|y|, |x| + (1-\gamma)|y|\} \qquad \forall\ (x,y) \in \mathbb{R}^2.$$

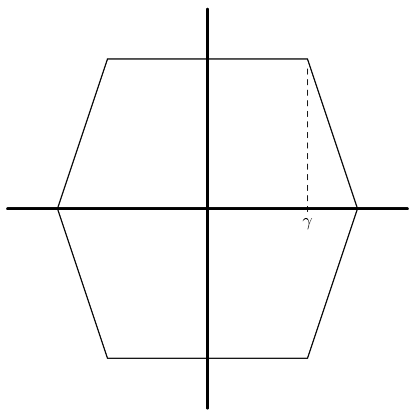


Figure 1.1: Hexagonal norms

For $\gamma = 0$ and $\gamma = 1$ we have $X_0 = (\mathbb{R}^2, \|\cdot\|_1)$ and $X_1 = (\mathbb{R}^2, \|\cdot\|_\infty)$ respectively, hence $n_1(X_0) = n_1(X_1) = 1$. If $0 < \gamma < 1$ the unit ball associated is a hexagon, see Figure 1.1.

1.5.1 Theorem. *For $\gamma \in [0,1]$, let X_γ be defined as above. Then,*

$$n_1(X_\gamma) = \begin{cases} \frac{1}{1+2\gamma} & \text{if } 0 \leqslant \gamma \leqslant \frac{1}{2}, \\ \frac{1}{3-2\gamma} & \text{if } \frac{1}{2} \leqslant \gamma \leqslant 1. \end{cases}$$

Proof. As we commented before, when $\gamma \in \{0,1\}$ we have $n_1(X_\gamma) = 1$. For $0 < \gamma < 1$, bearing the equalities (1.6) and (1.7) in mind, we have to determine the sets $\operatorname{ext}(B_{X_\gamma})$ and $\operatorname{ext}(B_{X_\gamma^*})$ to compute the rank-one numerical index. It is easy to check that

$$\operatorname{ext}(B_{X_\gamma}) = \{\pm(\gamma,1),\ \pm(1,0),\ \pm(\gamma,-1)\}.$$

Therefore, we get the following formula for the dual norm of $\|\cdot\|_\gamma$:

$$\|(x,y)\|_\gamma^* = \max\{|x|, |y| + \gamma|x|\} \qquad (x,y) \in X_\gamma^*,$$

and

$$\operatorname{ext}(B_{X_\gamma^*}) = \{\pm(1,1-\gamma),\ \pm(0,1), \pm(-1,1-\gamma)\}.$$

From this we deduce that, for each $0 < \gamma < 1$, X_γ^* is isometrically isomorphic to $X_{1-\gamma}$ which allows us to write

$$n_1(X_{1-\gamma}) = n_1(X_\gamma^*) = n_1(X_\gamma).$$

Hence, it suffices to prove that $n_1(X_\gamma) = \frac{1}{3-2\gamma}$ for $\frac{1}{2} \leqslant \gamma < 1$. So, fixed $\frac{1}{2} \leqslant \gamma < 1$, we consider the rank-one operator given by the matrix $T = \begin{pmatrix} a & b \\ a\lambda & b\lambda \end{pmatrix}$ where

$$a = 1, \qquad b = 3 - 3\gamma, \qquad \text{and} \qquad \lambda = -\frac{1}{2-\gamma}.$$

We claim that $\|T\|_\gamma = \frac{(3-2\gamma)^2}{2-\gamma}$ and $v_\gamma(T) = \frac{3-2\gamma}{2-\gamma}$. Indeed, the fact $\frac{1}{2} \leqslant \gamma < 1$ obviously gives $3 - 2\gamma > 1$ and $|4\gamma - 3| \leqslant 1$. So, using equations (1.6) and (1.7), one obtains

$$\|T\|_\gamma = \max\left\{\frac{3-2\gamma}{2-\gamma}, \frac{(3-2\gamma)^2}{2-\gamma}, \frac{1}{2-\gamma}, \frac{|4\gamma-3|(3-2\gamma)}{2-\gamma}\right\} = \frac{(3-2\gamma)^2}{2-\gamma}$$

$$v_\gamma(T) = \max\left\{\frac{3-2\gamma}{2-\gamma}, \frac{1}{2-\gamma}, \frac{|4\gamma-3|}{2-\gamma}, \frac{|4\gamma-3|(3-2\gamma)}{2-\gamma}\right\} = \frac{3-2\gamma}{2-\gamma}$$

and, therefore, $n_1(X_\gamma) \leqslant \frac{1}{3-2\gamma}$. To get the reversed inequality observe that, by [32, Theorem 1], one has $n_1(X_\gamma) \geqslant n(X_\gamma) = \frac{1}{3-2\gamma}$. $\square$

1.5.2 Octagonal norms

Our next goal is to compute the rank-one numerical index of a family of spaces with octagonal unit ball. For each $\xi \in [0,1]$ we consider the normed space $X_\xi = (\mathbb{R}^2, \|\cdot\|_\xi)$ where the norm $\|\cdot\|_\xi$ is given by

$$\|(x,y)\|_\xi = \max\left\{|x|,\ |y|,\ \frac{|x|+|y|}{1+\xi}\right\} \qquad \forall\ (x,y) \in \mathbb{R}^2.$$

If $\xi = 0$ we have $(\mathbb{R}^2, \|\cdot\|_1)$, and when $\xi = 1$ we are dealing with $(\mathbb{R}^2, \|\cdot\|_\infty)$, so we get $n_1(X_0) = n_1(X_1) = 1$. For $0 < \xi < 1$ the unit ball of X_ξ is an octagon as it is shown in Figure 1.2.

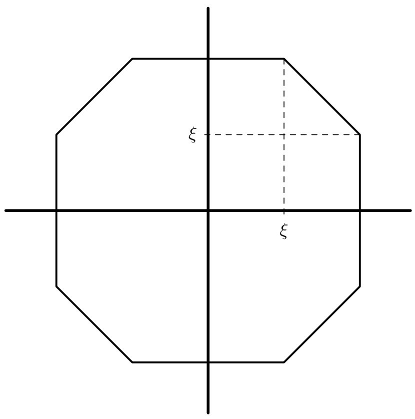


Figure 1.2: Octagonal norms

1.5.2 Theorem. *For $\xi \in [0,1]$, let X_ξ defined as above. Then,*

$$n_1(X_\xi) = \begin{cases} \frac{1-\xi}{1+\xi} & \text{if } 0 \leqslant \xi \leqslant \sqrt{5}-2, \\ \frac{2+2\xi^2}{2+\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}} & \text{if } \sqrt{5}-2 \leqslant \xi \leqslant \sqrt{2}-1, \\ \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}} & \text{if } \sqrt{2}-1 \leqslant \xi \leqslant \frac{\sqrt{5}-1}{2}, \\ \xi & \text{if } \frac{\sqrt{5}-1}{2} \leqslant \xi \leqslant 1. \end{cases}$$

Proof. Taking into account the comments before the statement of the theorem, we only have to deal with $0 < \xi < 1$. In this case it is readily checked that

$$\operatorname{ext}(B_{X_\xi}) = \{\pm(1,\xi), \pm(1,-\xi), \pm(\xi,1), \pm(\xi,-1)\}.$$

This way, one obtains that the dual norm of $\|\cdot\|_\xi$ is given by

$$\|(x,y)\|_\xi^* = \max\{|x| + \xi|y|, |y| + \xi|x|\} \qquad (x,y) \in X_\xi^*.$$

Hence the extreme points of $B_{X_\xi^*}$ are

$$\operatorname{ext}(B_{X_\xi^*}) = \left\{\pm(1,0), \pm(0,1), \pm\left(\frac{1}{1+\xi}, \frac{1}{1+\xi}\right), \pm\left(\frac{1}{1+\xi}, \frac{-1}{1+\xi}\right)\right\}.$$

Let $T \in L(X_\xi)$ be a rank-one operator represented by the matrix $\begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}$. Using equations (1.6) and (1.7) one obtains

$$\|T\|_\xi = \max\Big\{|a| + |b|\xi,\ |\lambda|(|a| + |b|\xi),\ |a|\xi + |b|,\ |\lambda|(|a|\xi + |b|), \qquad (1.8)$$
$$\frac{1+|\lambda|}{1+\xi}(|a| + |b|\xi),\ \frac{1+|\lambda|}{1+\xi}(|a|\xi + |b|)\Big\}$$

and

$$v_\xi(T) = \max\Big\{|a| + |b|\xi,\ |\lambda|(|a|\xi + |b|), \qquad (1.9)$$
$$\frac{|a\xi + b\lambda| + |b + a\lambda\xi|}{1+\xi},\ \frac{|a + b\lambda\xi| + |b\xi + a\lambda|}{1+\xi}\Big\}.$$

We divide the proof into three different cases depending on some of the values of ξ.

Case 1 : $\frac{\sqrt{5}-1}{2} \leqslant \xi \leqslant 1$. It is immediate to check that the rank-one operator $T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $\|T_1\|_\xi = 1$ and $v_\xi(T_1) = \max\left\{\xi, \frac{1}{1+\xi}\right\} = \xi$. Hence, one has $n_1(X_\xi) \leqslant \xi$. On the other hand, Theorem 2 in [32] tells us that $n(X_\xi) = \max\left\{\xi, \frac{1-\xi}{1+\xi}\right\}$ and so $n_1(X_\xi) \geqslant n(X_\xi) \geqslant \xi$.

Case 2 : $\sqrt{2}-1 \leqslant \xi \leqslant \frac{\sqrt{5}-1}{2}$. For these values of ξ it follows that $1-\xi-\xi^2 \geqslant 0$. We will use this fact throughout the proof of this case without explicit mention.

In order to prove $n_1(X_\xi) \geqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}$, fixed a rank-one operator $T = \begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}$ we have to show that $\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}$. We start observing that there is no loss of generality in assuming $|\lambda| \leqslant 1$. Indeed, since the operator S represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an onto isometry satisfying $S^2 = \mathrm{Id}$, we have that

$$v_\xi(T) = v_\xi(STS) \quad \text{and} \quad \|T\|_\xi = \|STS\|_\xi.$$

Therefore, if $|\lambda| > 1$ it is enough to consider $\widetilde{\lambda} = \frac{1}{\lambda}$ and the operator $\widetilde{T} = \widetilde{\lambda} STS = \begin{pmatrix} b & a \\ \widetilde{\lambda} b & \widetilde{\lambda} a \end{pmatrix}$ which satisfies

$$\frac{v_\xi(\widetilde{T})}{\|\widetilde{T}\|_\xi} = \frac{v_\xi(T)}{\|T\|_\xi}.$$

Hence we can set $|\lambda| \leqslant 1$ for the rest of the proof of Case 2.

Next we observe that if $|a| \geqslant |b|$ then we are done. Indeed, in such a case using equations (1.8) and (1.9) one has

$$\|T\|_\xi = \max\left\{|a| + |b|\xi, \frac{1+|\lambda|}{1+\xi}(|a| + |b|\xi)\right\} \qquad \text{and} \qquad v_\xi(T) \geqslant |a| + |b|\xi.$$

This, together with the facts $\xi \geqslant \sqrt{2}-1$ and $1-\xi-\xi^2 \geqslant 0$, tells us that

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \min\left\{1, \frac{1+\xi}{1+|\lambda|}\right\} \geqslant \frac{1+\xi}{2}$$
$$\geqslant \frac{2+2\xi^2}{1+3\xi+(1+\xi)} \geqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}.$$

Hence we can assume that $|a| < |b|$ from now on. This gives the following value for $\|T\|_\xi$:

$$\|T\|_\xi = \max\left\{|a|\xi+|b|,\ \frac{1+|\lambda|}{1+\xi}(|a|\xi+|b|)\right\}. \tag{1.10}$$

We divide the rest of the argument into two cases depending on $|\lambda|$.

• If $|\lambda| \leqslant \xi$, using equations (1.9) and (1.10) one obtains $\|T\|_\xi = |a|\xi+|b|$ and

$$v_\xi(T) \geqslant \max\left\{|a|+|b|\xi, \frac{|b|-|a||\lambda|\xi+\big||a|\xi-|b||\lambda|\big|}{1+\xi}\right\}$$
$$\geqslant \max\left\{|a|+|b|\xi, \frac{1-|\lambda|}{1+\xi}(|a|\xi+|b|), \frac{1+|\lambda|}{1+\xi}(|b|-|a|\xi)\right\}$$

which give

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \max\left\{\frac{|a|+|b|\xi}{|a|\xi+|b|}, \frac{1-|\lambda|}{1+\xi}, \frac{1+|\lambda|}{1+\xi}\frac{|b|-|a|\xi}{|a|\xi+|b|}\right\}$$
$$= \max\left\{\frac{\frac{|a|}{|b|}+\xi}{\frac{|a|}{|b|}\xi+1}, \frac{1-|\lambda|}{1+\xi}, \frac{1+|\lambda|}{1+\xi}\frac{|b|-|a|\xi}{|a|\xi+|b|}\right\}.$$

We claim that $\max\left\{\frac{1-|\lambda|}{1+\xi}, \frac{1+|\lambda|}{1+\xi}\frac{|b|-|a|\xi}{|a|\xi+|b|}\right\} \geqslant \frac{1-\frac{|a|}{|b|}\xi}{1+\xi}$. Indeed, if $0 \leqslant |\lambda| \leqslant \frac{|a|}{|b|}\xi$ then $\frac{1-|\lambda|}{1+\xi} \geqslant \frac{1-\frac{|a|}{|b|}\xi}{1+\xi}$. On the other hand, if $\frac{|a|}{|b|}\xi \leqslant |\lambda|$ then we have

$$\frac{1+|\lambda|}{1+\xi}\frac{|b|-|a|\xi}{|a|\xi+|b|} \geqslant \frac{1+\frac{|a|}{|b|}\xi}{1+\xi}\frac{|b|-|a|\xi}{|a|\xi+|b|} = \frac{1-\frac{|a|}{|b|}\xi}{1+\xi}.$$

Hence, we can write

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \max\left\{\frac{\frac{|a|}{|b|}+\xi}{\frac{|a|}{|b|}\xi+1}, \frac{1-\frac{|a|}{|b|}\xi}{1+\xi}\right\} \geqslant \min_{0\leqslant t\leqslant 1}\max\left\{\frac{t+\xi}{t\xi+1}, \frac{1-t\xi}{1+\xi}\right\}.$$

Since $\frac{t+\xi}{t\xi+1}$ increases with t and $\frac{1-t\xi}{1+\xi}$ is decreasing as a function of t, the minimum above is attained at $t_0 = \frac{-1-\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}{2\xi^2}$ which satisfies $0 \leqslant t_0 \leqslant 1$ and $\frac{t_0+\xi}{t_0\xi+1} = \frac{1-t_0\xi}{1+\xi}$. So we have that

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \frac{1-t_0\xi}{1+\xi} = \frac{1+3\xi-\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}{2\xi(1+\xi)}$$
$$= \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}.$$

• If $\xi \leqslant |\lambda| \leqslant 1$, using equations (1.9) and (1.10) we have $\|T\|_\xi = \frac{1+|\lambda|}{1+\xi}(|a|\xi+|b|)$ and

$$v_\xi(T) \geqslant \max\left\{|a|+|b|\xi, |\lambda|(|a|\xi+|b|), \frac{|b||\lambda|-|a|\xi+|b|-|a||\lambda|\xi}{1+\xi}\right\}$$
$$= \max\left\{|a|+|b|\xi, |\lambda|(|a|\xi+|b|), \frac{1+|\lambda|}{1+\xi}(|b|-|a|\xi)\right\}.$$

So we can write

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \max\left\{\frac{1+\xi}{1+|\lambda|}\frac{|a|+|b|\xi}{|a|\xi+|b|}, \frac{|\lambda|(1+\xi)}{1+|\lambda|}, \frac{|b|-|a|\xi}{|a|\xi+|b|}\right\}$$
$$= \max\left\{\frac{1+\xi}{1+|\lambda|}\frac{\frac{|a|}{|b|}+\xi}{\frac{|a|}{|b|}\xi+1}, \frac{|\lambda|(1+\xi)}{1+|\lambda|}, \frac{1-\frac{|a|}{|b|}\xi}{1+\frac{|a|}{|b|}\xi}\right\}.$$

Since $\frac{1+\xi}{1+|\lambda|}\frac{\frac{|a|}{|b|}+\xi}{\frac{|a|}{|b|}\xi+1}$ is decreasing as a function of $|\lambda|$, $\frac{|\lambda|(1+\xi)}{1+|\lambda|}$ increases with $|\lambda|$, and both expressions take the value $\frac{\frac{|a|}{|b|}+\xi}{\frac{|a|}{|b|}+1}$ when $|\lambda| = \frac{\frac{|a|}{|b|}+\xi}{\frac{|a|}{|b|}\xi+1}$, we have that

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \max\left\{\frac{\frac{|a|}{|b|}+\xi}{\frac{|a|}{|b|}+1}, \frac{1-\frac{|a|}{|b|}\xi}{1+\frac{|a|}{|b|}\xi}\right\} \geqslant \min_{0\leqslant t\leqslant 1}\max\left\{\frac{t+\xi}{t+1}, \frac{1-t\xi}{1+t\xi}\right\}.$$

Using the facts that $\frac{t+\xi}{t+1}$ increases with t, $\frac{1-t\xi}{1+t\xi}$ is a decreasing function of t, and that the number $t_1 = \frac{-\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}}{4\xi}$ satisfies $\frac{t_1+\xi}{t_1+1} = \frac{1-t_1\xi}{1+t_1\xi}$ and $0 \leqslant t_1 \leqslant 1$, we can continue the estimation as follows

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \frac{t_1+\xi}{t_1+1} = \frac{-\xi+3\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}}{3\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}}$$
$$= \frac{2+2\xi^2}{2+\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}}.$$

Finally, we claim that for $\sqrt{2}-1 \leqslant \xi \leqslant \frac{\sqrt{5}-1}{2}$ the following holds:

$$\frac{2+2\xi^2}{2+\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}} \geqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}.$$

Therefore,

$$\frac{v_\xi(T)}{\|T\|_\xi} \geqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}$$

which finishes the proof of the inequality $n_1(X_\xi) \geqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}$. To prove the claim it suffices to show that

$$2+\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4} \leqslant 1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}$$

or, equivalently

$$\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4} - \sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)} \leqslant 2\xi+\xi^2-1.$$

To do so, observe that for $\alpha > \beta > 0$ one has $\sqrt{\alpha}-\sqrt{\beta} = \frac{\alpha-\beta}{\sqrt{\alpha}+\sqrt{\beta}} \leqslant \frac{\alpha-\beta}{2\sqrt{\beta}}$. Besides, for $\xi \in [\sqrt{2}-1, \frac{\sqrt{5}-1}{2}]$, we have

$$8\xi-7\xi^2+2\xi^3+\xi^4-\big((1+\xi)^2+4\xi^2(1-\xi-\xi^2)\big) = (2\xi+\xi^2-1)(1-4\xi+5\xi^2) \geqslant 0.$$

So, calling

$$\alpha = 8\xi-7\xi^2+2\xi^3+\xi^4 \qquad \text{and} \qquad \beta = (1+\xi)^2+4\xi^2(1-\xi-\xi^2)$$

we can write

$$\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4} - \sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)} \leqslant \frac{(2\xi+\xi^2-1)(1-4\xi+5\xi^2)}{2\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}.$$

This gives the desired inequality just taking into account that

$$\frac{1-4\xi+5\xi^2}{2\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}} \leqslant \frac{1-4\xi+5\xi^2}{2(1+\xi)} \leqslant 1.$$

Now we turn to prove the inequality $n_1(X_\xi) \leqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}$. To this end, we consider the rank-one operator $T_2 = \begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}$ where $b = 2\xi^2$,

$$a = -1-\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}, \quad \text{and} \quad \lambda = -\frac{-1-\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}{2\xi}.$$

Since $\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)} \geqslant 1+\xi$, it follows that $a \geqslant 0$ and $\lambda \leqslant 0$. Moreover, it is clear that $|\lambda| = \xi\frac{a}{b}$ and it is routine to check that $a \leqslant b$, which gives $|\lambda| \leqslant \xi$. Hence, using equations (1.8) and (1.9) we obtain

$$\|T_2\|_\xi = a\xi + b$$

and

$$v_\xi(T_2) = \max\left\{a+b\xi, \frac{b-\frac{a^2}{b}\xi^2}{1+\xi}, \frac{a(1-\xi^2)+(b-\frac{a^2}{b})\xi}{1+\xi}\right\} = \max\left\{a+b\xi, \frac{b-\frac{a^2}{b}\xi^2}{1+\xi}\right\}.$$

Our choice of a and b gives that $a+b\xi = \frac{b-\frac{a^2}{b}\xi^2}{1+\xi}$ and, therefore, we can write

$$\frac{v_\xi(T_2)}{\|T_2\|_\xi} = \frac{a+b\xi}{a\xi+b} = \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}$$

which tells us that $n_1(X_\xi) \leqslant \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}}$.

Case 3 : $0 \leqslant \xi \leqslant \sqrt{2}-1$. This case can be obtained from the preceding ones. We start proving that the operator $J : X_\xi \longrightarrow X_{\frac{1-\xi}{1+\xi}}$ given by the matrix

$$J = \frac{1}{1+\xi}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is an onto isometry. To do so, given $(x,y) \in X_\xi$ observe that

$$\begin{aligned}
\|J(x,y)\|_{\frac{1-\xi}{1+\xi}} &= \left\| \tfrac{1}{1+\xi}(x+y, y-x) \right\|_{\frac{1-\xi}{1+\xi}} \\
&= \max\left\{ \frac{|x+y|}{1+\xi}, \frac{|y-x|}{1+\xi}, \frac{1}{1+\xi}\,\frac{|x+y|+|y-x|}{1+\frac{1-\xi}{1+\xi}} \right\} \\
&= \max\left\{ \frac{|x+y|}{1+\xi}, \frac{|y-x|}{1+\xi}, \frac{|x+y|+|y-x|}{2} \right\} \\
&= \max\left\{ \frac{|x|+|y|}{1+\xi}, \max\{|x|,|y|\} \right\} = \|(x,y)\|_\xi .
\end{aligned}$$

Therefore, X_ξ and $X_{\frac{1-\xi}{1+\xi}}$ are isometrically isomorphic and $n_1(X_\xi) = n_1\left(X_{\frac{1-\xi}{1+\xi}}\right)$.

Besides, observe that the function $f : [0,1] \longrightarrow [0,1]$ given by $f(\xi) = \frac{1-\xi}{1+\xi}$ satisfies

$$f^{-1} = f, \quad f\big([0,\sqrt{5}-2]\big) = \left[\tfrac{\sqrt{5}-1}{2}, 1\right], \text{ and } f\big([\sqrt{5}-2, \sqrt{2}-1]\big) = \left[\sqrt{2}-1, \tfrac{\sqrt{5}-1}{2}\right].$$

So, for $\xi \in [0, \sqrt{5}-2]$ we have that $f(\xi) \in [\frac{\sqrt{5}-1}{2}, 1]$, and using Case 1 we obtain

$$n_1(X_\xi) = n_1(X_{f(\xi)}) = f(\xi) = \frac{1-\xi}{1+\xi}.$$

For $\xi \in [\sqrt{5}-2, \sqrt{2}-1]$, calling

$$g(\xi) = \frac{2+2\xi^2}{2+\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}} \qquad \text{and}$$
$$h(\xi) = \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}},$$

it is routinely checked that $h(f(\xi)) = g(\xi)$. Hence, taking into account that $f(\xi) \in [\sqrt{2}-1, \frac{\sqrt{5}-1}{2}]$, we can use Case 2 to write

$$n_1(X_\xi) = n_1(X_{f(\xi)}) = h(f(\xi)) = g(\xi)$$

which finishes the proof. □

Since the rank-one numerical index of a finite-dimensional normed linear space and the one of its dual is the same, the last theorem allows us to compute the rank-one numerical index of a new family of normed spaces. Concretely, for $\varrho \in [\frac{1}{2}, 1]$ we consider the space $Y_\varrho = (\mathbb{R}^2, \|\cdot\|_\varrho)$ where the norm is given by

$$\|(x,y)\|_\varrho = \max\left\{|x| + \frac{1-\varrho}{\varrho}|y|, |y| + \frac{1-\varrho}{\varrho}|x|\right\} \qquad \forall\ (x,y) \in \mathbb{R}^2.$$

For $\varrho \in]\frac{1}{2}, 1[$, the unit ball is an octagon (see Figure 1.3) with

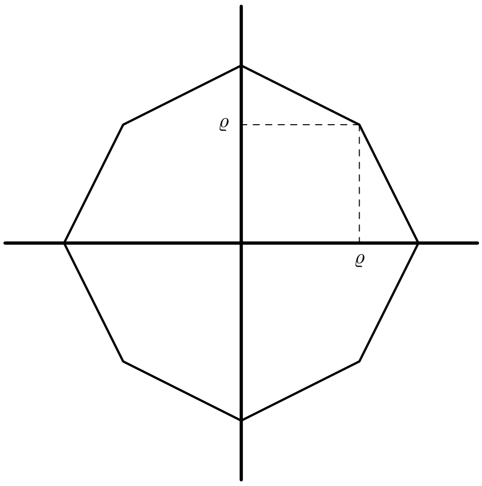


Figure 1.3: More octagonal norms

$$\text{ext}(B_{Y_\varrho}) = \{\pm(1,0), \pm(0,1), \pm(\varrho,\varrho), \pm(\varrho,-\varrho)\}.$$

Therefore, one has that $X_\xi^* = Y_\varrho$ where $\varrho = \frac{1}{1+\xi}$. Equivalently, $Y_\varrho^* = X_\xi$ where $\xi = \frac{1-\varrho}{\varrho}$. Hence, the following corollary is an immediate consequence of Theorem 1.5.2.

1.5.3 Corollary. *For every $\varrho \in [\frac{1}{2}, 1]$, let Y_ϱ be defined as above. Then*

$$n_1(Y_\varrho) = \begin{cases} \frac{1-\varrho}{\varrho} & \text{if } \frac{1}{2} \leqslant \varrho \leqslant \frac{2}{\sqrt{5}+1}, \\ \frac{2\varrho^2+2(1-\varrho)^2}{3\varrho-2\varrho^2+\sqrt{\varrho^2+4(1-\varrho)^2(\varrho^2+\varrho-1)}} & \text{if } \frac{2}{\sqrt{5}+1} \leqslant \varrho \leqslant \frac{1}{\sqrt{2}}, \\ \frac{2\varrho^2+2(1-\varrho)^2}{3\varrho-1+\sqrt{1-2\varrho-7\varrho^2+24\varrho^3-16\varrho^4}} & \text{if } \frac{1}{\sqrt{2}} \leqslant \varrho \leqslant \frac{1}{\sqrt{5}-1}, \\ 2\varrho - 1 & \text{if } \frac{1}{\sqrt{5}-1} \leqslant \varrho \leqslant 1. \end{cases}$$

1.5.3 The rank-one numerical index of L_p spaces

Computing the classical numerical index of L_p spaces has remained as an open problem since the beginning of the theory in the early 70's. Although there have been some recent advances, there is still a long way to go in order to fully solve the problem. It is therefore natural to ask about the computation of the rank-one numerical index of L_p spaces. We devote the following lines to present the small advances we were able to achieve.

We start recalling the results known before we began our work. It was shown in [33, Theorem 3.1] that $k_p \geqslant n_1(L_p(\mu)) \geqslant k_p^2$ for every atomless measure μ, where

$$k_p := \frac{1}{p^{1/p}q^{1/q}} = \sup_{t\geqslant 0} \frac{t^{p-1}}{1+t^p} = \sup_{t\geqslant 0} \frac{t}{1+t^p}.$$

The number k_p is the numerical radius of both operators $T_1(x,y) = (y,0)$ and $T_2(x,y) = (0,x)$ defined on the real or complex space ℓ_p^2. Since k_p tends to 1 when $p \to \infty$ or $p \to 1$, the estimation given above is sharp in those cases. However, one has $n_1(H) = 1/2$ for every Hilbert space H of dimension greater than one [33, Proposition 3.3]. So, the estimation from below is far from being sharp when p is close to 2 since $k_2 = 1/2$.

One may wonder if there is a similar estimation for $n_1(L_p(\mu))$ when μ has atoms. A reasonable starting point is the real space ℓ_p^2. Since $\|T_1\| = 1$ and $v(T_1) = k_p$ it follows that $n_1(\ell_p^2) \leqslant k_p$. We conjecture that this is, in fact, an equality. Unfortunately, we have not been able to prove it.

Let us fix $p \neq 1, 2, \infty$ and observe that a rank-one operator T on ℓ_p^2 can be represented by the matrix

$$T = \begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}$$

where $a, b, \lambda \in \mathbb{R}$. On the one hand, we have that

$$\begin{aligned} \|T\| &= \sup_{(x,y)\in S_{\ell_p^2}} \left\| \big(ax+by, \lambda(ax+by)\big) \right\|_p \\ &= (1+|\lambda|^p)^{1/p} \sup_{(x,y)\in S_{\ell_p^2}} |ax+by| = (1+|\lambda|^p)^{1/p}(|a|^q+|b|^q)^{1/q}. \end{aligned}$$

On the other hand, using Lemma 3.2 in [16] one can compute the numerical radius of T as follows:

$$v(T) = \sup_{t\geqslant 0} \frac{|a+b\lambda t^p| + |bt + a\lambda t^{p-1}|}{1+t^p}.$$

We would like to get a lower bound for $\frac{v(T)}{\|T\|}$ which does not depend on a, b, and λ. The following observation tells us that the minimum value for $\frac{v(T)}{\|T\|}$ occurs for a precise configuration of signs of a, b, λ. Indeed, consider the operator

$$S = \begin{pmatrix} |a| & |b| \\ -|\lambda||a| & -|\lambda||b| \end{pmatrix}$$

and observe that $\|S\| = \|T\|$ and

$$v(S) = \sup_{t\geqslant 0} \frac{\big||a| - |b||\lambda|t^p\big| + \big||b|t - |a||\lambda|t^{p-1}\big|}{1+t^p} \leqslant v(T).$$

Hence from now on we may and do assume that

$$T = \begin{pmatrix} a & b \\ -\lambda a & -\lambda b \end{pmatrix}$$

where $a, b, \lambda \geqslant 0$.

We claim that when $b = 0$ one obtains $\frac{v(T)}{\|T\|} \geqslant k_p$. Indeed, in this case we have

$$T = \begin{pmatrix} a & 0 \\ -\lambda a & 0 \end{pmatrix}$$

and, therefore, $v(T) = \sup_{t\geqslant 0} \frac{a(1+\lambda t^{p-1})}{1+t^p}$ and $\|T\| = a(1+\lambda^p)^{1/p}$. We distinguish two different cases depending on the values of λ. If $\lambda \leqslant (pq^{p/q}-1)^{1/p}$ one can use the value $t=0$ to estimate $v(T)$ and so

$$\frac{v(T)}{||T||} \geqslant \frac{1}{(1+\lambda^p)^{1/p}} \geqslant k_p.$$

If otherwise $\lambda \geqslant (pq^{p/q}-1)^{1/p}$, we use the value $t=(p-1)^{1/p}$ (which satisfies $\frac{t^{p-1}}{1+t^p} = k_p$) to estimate $v(T)$:

$$\frac{v(T)}{||T||} \geqslant \frac{1/p+\lambda k_p}{(1+\lambda^p)^{1/p}}.$$

Therefore, the claim will be proved if we show that $g(\lambda) = \frac{(1/p+\lambda k_p)^p}{1+\lambda^p} \geqslant k_p^p$ for $\lambda \geqslant (pq^{p/q}-1)^{1/p}$. Since we have that $\lim_{\lambda\to\infty} g(\lambda) = k_p^p$, the desired inequality will hold if we show that $g(\lambda)$ is decreasing for $\lambda \geqslant (pq^{p/q}-1)^{1/p}$. To do so, observe that

$$\begin{aligned} g'(\lambda) &= \frac{(1/p+\lambda k_p)^{p-1}(1+\lambda^p)pk_p - p\lambda^{p-1}(1/p+\lambda k_p)^p}{(1+\lambda^p)^2} \\ &= \frac{(1/p+\lambda k_p)^{p-1}(pk_p - \lambda^{p-1})}{(1+\lambda^p)^2} \end{aligned}$$

which tells us that $g(\lambda)$ is decreasing for $\lambda \geqslant (pk_p)^{1/(p-1)}$. Observe finally that

$$(pk_p)^{\frac{1}{p-1}} = \left(p\frac{1}{p^{1/p}} \left(\frac{p-1}{p}\right)^{\frac{p-1}{p}} \right)^{\frac{1}{p-1}} = (p-1)^{\frac{1}{p}} \leqslant (pq^{p/q}-1)^{\frac{1}{p}}$$

and so the claim holds.

Now we can suppose that $b \neq 0$ and use this to get rid of one variable in our problem. Indeed, call $\mu = \frac{a}{b}$ and observe that the operator

$$\widetilde{T} = \begin{pmatrix} \mu & 1 \\ -\lambda\mu & -\lambda \end{pmatrix}$$

satisfies $\frac{v(\widetilde{T})}{\|\widetilde{T}\|} = \frac{v(T)}{\|T\|}$. For this kind of operators we have that

$$||\widetilde{T}|| = (1+\mu^q)^{1/q}(1+\lambda^p)^{1/p} \qquad \text{and}$$

$$v(\widetilde{T}) = \sup_{t\geqslant 0} \frac{|\mu - \lambda t^p| + |t - \lambda\mu t^{p-1}|}{1+t^p}.$$

On the one hand, using the value $t = 0$ one obtains $v(\widetilde{T}) \geqslant \mu$. On the other hand, it is immediate to check that

$$v(\widetilde{T}) \geqslant \lim_{t\to\infty} \frac{|\mu - \lambda t^p| + |t - \lambda\mu t^{p-1}|}{1+t^p} = \lambda$$

and so

$$\frac{v(\widetilde{T})}{\|\widetilde{T}\|} \geqslant \frac{\max\{\lambda,\mu\}}{(1+\mu^q)^{1/q}(1+\lambda^p)^{1/p}}.$$

Using this we can obtain the desired estimate for some values of λ and μ: if $\lambda \geqslant \frac{1}{(p-1)^{1/p}}$ and $\mu \leqslant (q-1)^{1/q}$ or $\mu \geqslant \frac{1}{(q-1)^{1/q}}$ and $\lambda \leqslant (p-1)^{1/p}$, then $\frac{v(\widetilde{T})}{\|\widetilde{T}\|} \geqslant k_p$. Indeed, suppose first that $\lambda \geqslant \frac{1}{(p-1)^{1/p}}$ and $\mu \leqslant (q-1)^{1/q}$. Since $\frac{1}{(1+\mu^q)^{1/q}}$ is a decreasing function and $\frac{\lambda}{(1+\lambda^p)^{1/p}} = \left(\frac{\lambda^p}{1+\lambda^p}\right)^{1/p}$ is increasing we can write

$$\frac{v(\widetilde{T})}{\|\widetilde{T}\|} \geqslant \frac{1}{(1+\mu^q)^{1/q}}\frac{\lambda}{(1+\lambda^p)^{1/p}} \geqslant \frac{1}{q^{\frac{1}{q}}}\frac{1}{p^{\frac{1}{p}}} = k_p.$$

If otherwise $\mu \geqslant \frac{1}{(q-1)^{1/q}}$ and $\lambda \leqslant (p-1)^{1/p}$, one can proceed analogously using this time that $\frac{\mu}{(1+\mu^q)^{1/q}}$ is increasing and that $\frac{1}{(1+\lambda^p)^{1/p}}$ is decreasing:

$$\frac{v(\widetilde{T})}{\|\widetilde{T}\|} \geqslant \frac{\mu}{(1+\mu^q)^{1/q}}\frac{1}{(1+\lambda^p)^{1/p}} \geqslant \frac{1}{q^{\frac{1}{q}}}\frac{1}{p^{\frac{1}{p}}} = k_p.$$

However, with this kind of reasoning, we have been not able to get further useful information. There is still a big range of the values of $\mu \geqslant 0$ and $\lambda \geqslant 0$ for which the problem is not solved.

1.6 Open problems

We devote this brief section to recall some of the questions which raised during our study of the rank-one numerical index and that remain unsolved.

It is clear that the rank-one numerical index may differ from the compact numerical index for real Banach spaces (see Example 1.4.2). But as far as we know, they might coincide in the complex case.

1.6.1 Problem. Let X be a complex Banach space. Is it true that $n_1(X) = n_{\text{comp}}(X)$?

To answer this question it could be useful to compute the numerical index and the rank-one numerical index of some families of finite-dimensional complex spaces.

The compact numerical index is obviously bounded above by the infimum of the finite-rank numerical indices. We do not know if the inequality can be strict.

1.6.2 Problem. Is there a Banach space X such that $n_{\text{comp}}(X) \neq \inf_{r\in\mathbb{N}} n_r(X)$?

Of course, if such a Banach space exists it must fail the approximation property.

To finish our list we recall that the computation of the rank-one numerical numerical of ℓ_p spaces is far from being achieved, as we discussed in Section 1.5.

1.6.3 Problem. Compute $n_1(\ell_p)$ for $1 < p < \infty$ with $p \neq 2$.

Chapter 2

Bishop-Phelps-Bollobás moduli of a Banach space

This chapter is devoted to study two functions that can be defined for every Banach space which, roughly speaking, give a measure of what is the best possible Bishop-Phelps-Bollobás Theorem that can be achieved in a fixed Banach space. These functions are what we call the Bishop-Phelps-Bollobás moduli of a Banach space. We show that these moduli have natural continuity properties and that they are well related to duality. The main result of the chapter gives an upper bound for the moduli that produces a sharp version of the Bishop-Phelps-Bollobás Theorem. We give the exact value of the moduli for Hilbert spaces and we show that for several classical examples of Banach spaces they take their maximum value. Finally, we obtain a necessary condition for that maximal behaviour to occur: the containment of almost isometric copies of the real space $\ell_\infty^{(2)}$.

2.1 Introduction

As is widely known, given a Banach space X and a functional $x^* \in X^*$ it is said that x^* attains its norm if

$$\|x^*\| = \max\{|x^*(x)| : x \in B_X\}.$$

Obviously not every functional attains its norm in every Banach space. On the other hand, for every element $x^* \in X^*$ there exists $x_0^{**} \in B_{X^{**}}$ such that $\|x^*\| = x_0^{**}(x^*)$ by Hahn-Banach theorem. Hence, if X is reflexive then every functional attains its norm. In 1957 R. C. James [25] proved that this condition is enough to ensure reflexivity. A few years later E. Bishop and R. Phelps proved that the set of norm attaining functionals on a Banach space is norm dense in the dual space. In 1970, B. Bollobás [6] gave a sharper version of this result allowing to approximate at the same time a functional and a vector at which it almost attains the norm. Our aim in this chapter is to study the best possible approximation of this kind that is valid in every Banach space.

We first present the original result by Bollobás, nowadays known as the *Bishop-Phelps-Bollobás theorem*, which is the leitmotiv of our work.

2.1.1 Theorem (Bishop-Phelps-Bollobás theorem [6]).
Let X be a Banach space. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $|1 - x^*(x)| \leqslant \varepsilon^2/2$ for some $0 < \varepsilon < 1/2$. Then there exists $(y, y^*) \in \Pi(X)$ such that $\|x - y\| < \varepsilon + \varepsilon^2$ and $\|x^* - y^*\| \leqslant \varepsilon$.

The idea is that given $(x, x^*) \in S_X \times S_{X^*}$ such that $x^*(x) \sim 1$, there exist $y \in S_X$ close to x and $y^* \in S_{X^*}$ close to x^* for which $y^*(y) = 1$. This result has many applications, especially for the theory of numerical ranges, see [7, 8].

Our objective is to introduce two moduli which measure, for a given Banach space, what is the best possible Bishop-Phelps-Bollobás theorem in this space, that is, how close can be y to x and y^* to x^* in the result above depending on how close is $x^*(x)$ to 1. In the first modulus, we allow the vector and the functional to have norm less than or equal to one, whereas in the second modulus we only consider norm-one vectors and functionals.

2.1.2 Definitions (Bishop-Phelps-Bollobás moduli)**.** Let X be a Banach space.

(a) The *Bishop-Phelps-Bollobás modulus* of X is the function $\Phi_X : (0,2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in B_X \times B_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

(b) The *spherical Bishop-Phelps-Bollobás modulus* of X is the function $\Phi_X^S : (0,2) \longrightarrow \mathbb{R}^+$ such that given $\delta \in (0,2)$, $\Phi_X^S(\delta)$ is the infimum of those $\varepsilon > 0$ satisfying that for every $(x, x^*) \in S_X \times S_{X^*}$ with $\operatorname{Re} x^*(x) > 1 - \delta$, there is $(y, y^*) \in \Pi(X)$ with $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.

Evidently, $\Phi_X^S(\delta) \leqslant \Phi_X(\delta)$, so any estimation from above for $\Phi_X(\delta)$ is also valid for $\Phi_X^S(\delta)$ and, viceversa, any estimation from below for $\Phi_X^S(\delta)$ is also valid for $\Phi_X(\delta)$.

Recall that the dual of a complex Banach space X is isometric (taking real parts) to the dual of the real subjacent space $X_{\mathbb{R}}$. Also, $\Pi(X)$ does not change if we consider X as a real Banach space (indeed, if $(x, x^*) \in \Pi(X)$ then $x^* \in S_{X^*}$ and $x \in S_X$ satisfies $x^*(x) = 1$ so, obviously, $\operatorname{Re} x^*(x) = 1$ and $(x, \operatorname{Re} x^*) \in \Pi(X_{\mathbb{R}})$). Therefore, only the real structure of the space is playing a role in the above definitions. Nevertheless, we prefer to develop the theory for real and complex spaces which, actually, does not suppose much more effort. This is mainly because for classical sequence or function spaces, the real space underlying the complex version of the space is not equal, in general, to the real version of the space. Unless otherwise is stated, the (arbitrary or concrete) spaces we are dealing with will be real or complex and the results work in both cases.

The following notations will help to the understanding and further use of Definitions 2.1.2. Let X be a Banach space and fix $0 < \delta < 2$. Write

$$A_X(\delta) := \{(x, x^*) \in B_X \times B_{X^*} \ : \ \operatorname{Re} x^*(x) > 1 - \delta\},$$
$$A_X^S(\delta) := \{(x, x^*) \in S_X \times S_{X^*} \ : \ \operatorname{Re} x^*(x) > 1 - \delta\}.$$

It is clear that

$$\Phi_X(\delta) = \sup_{(x,x^*)\in A_X(\delta)} \ \inf_{(y,y^*)\in\Pi(X)} \max\{\|x-y\|, \|x^*-y^*\|\},$$
$$\Phi_X^S(\delta) = \sup_{(x,x^*)\in A_X^S(\delta)} \ \inf_{(y,y^*)\in\Pi(X)} \max\{\|x-y\|, \|x^*-y^*\|\}.$$

We denote $d_H(A, B)$ the Hausdorff distance between $A, B \subset X \times X^*$ associated to the ℓ_∞-distance $\operatorname{dist}_\infty$ in $X \times X^*$, that is,

$$\operatorname{dist}_\infty\big((x, x^*), (y, y^*)\big) = \max\{\|x-y\|, \|x^*-y^*\|\}$$

for $(x, x^*), (y, y^*) \in X \times X^*$, and

$$d_H(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} \operatorname{dist}_\infty(a, b)\, , \sup_{b\in B} \inf_{a\in A} \operatorname{dist}_\infty(a, b)\right\}$$

for $A, B \subset X \times X^*$. We clearly have that

$$\Phi_X(\delta) = d_H\big(A_X(\delta), \Pi(X)\big) \qquad \text{and} \qquad \Phi_X^S(\delta) = d_H\big(A_X^S(\delta), \Pi(X)\big)$$

for every $0 < \delta < 2$ (observe that $\Pi(X) \subset A_X(\delta)$ and $\Pi(X) \subset A_X^S(\delta)$ for every δ).

The following result is immediate.

2.1.3 Remark. *Let X be a Banach space. Given $\delta_1, \delta_2 \in (0, 2)$ with $\delta_1 < \delta_2$, one has*

$$A_X(\delta_1) \subset A_X(\delta_2) \quad \textit{and} \quad A_X^S(\delta_1) \subset A_X^S(\delta_2).$$

Therefore, the functions $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$ are increasing.

Routine computations and the fact that the Hausdorff distance does not change if we take closure in one of the sets, provide the following observations.

2.1.4 Remark. *Let X be a Banach space. Then, for every $\delta \in (0,2)$, one has*

$$\begin{aligned}\Phi_X(\delta) := \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\big\}\\ = \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\big\}\\ = \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\big\}\\ = \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in B_X \times B_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\big\},\end{aligned}$$

and

$$\begin{aligned}\Phi_X^S(\delta) := \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\big\}\\ = \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) < \varepsilon\big\}\\ = \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) > 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\big\}\\ = \inf\big\{\varepsilon > 0 \,:\, &\forall (x,x^*) \in S_X \times S_{X^*} \text{ with } \operatorname{Re} x^*(x) \geqslant 1-\delta,\\ &\exists (y,y^*) \in \Pi(X) \text{ with } \operatorname{dist}_\infty((x,x^*),(y,y^*)) \leqslant \varepsilon\big\}.\end{aligned}$$

2.2 The upper bound of the moduli

Observe that *the smaller are the functions $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$, the better is the approximation on the space*. It can be deduced from the Bishop-Phelps-Bollobás theorem that there is a common upper bound for $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$ for all Banach spaces X. Our main goal here is to obtain the best possible upper bound for the moduli which is valid for every Banach space which is, in fact, sharp. To do so, we consider a somewhat more general problem: to calculate the best possible upper bound for $d_\infty\big((x,x^*),\Pi(X)\big)$ in any Banach space X as a function of $\|x\|$, $\|x^*\|$, and $x^*(x)$. More precisely, given a Banach space X and fixed $\delta\in(0,2)$ and $\mu,\theta\in[0,1]$ satisfying $\mu\theta\geqslant 1-\delta$, we consider the function

$$\Phi_X(\mu,\theta,\delta):=\sup\{d_\infty\big((x,x^*),\Pi(X)\big)\ :\ x\in X,\ x^*\in X^*,\ \|x\|=\mu,\ \|x^*\|=\theta,\ \operatorname{Re}x^*(x)\geqslant 1-\delta\}.$$

Observe that, with this notation, we have that

$$\Phi_X^S(\delta)=\Phi_X(1,1,\delta)\qquad\text{and}\qquad \Phi_X(\delta)=\sup_{\substack{\mu\theta\in[0,1]\\ \mu\theta\geqslant 1-\delta}}\Phi_X(\mu,\theta,\delta).$$

In order to present the promised result, we introduce the following notation: for $\delta\in(0,2)$ and $\mu,\theta\in[0,1]$ with $\mu\theta>1-\delta$, we define the function

$$\Psi(\mu,\theta,\delta):=\frac{2-\mu-\theta+\sqrt{(\mu-\theta)^2+8\,(\mu\theta-1+\delta)}}{2}.$$

2.2.1 Theorem. *Let X be a Banach space, $\delta\in(0,2)$, and $\mu,\theta\in[0,1]$ satisfying $\mu\theta>1-\delta$. Then,*

$$\Phi_X(\mu,\theta,\delta)\leqslant\min\left\{\Psi(\mu,\theta,\delta),1+\mu,1+\theta\right\}.$$

Let us provide some preliminary results needed in the proof of this theorem. The first one gives an easy inequality and also covers the trivial case in which $\mu\theta=1-\delta$.

2.2.2 Remark. *Let X be a Banach space, $\delta\in(0,2)$, and $\mu,\theta\in[0,1]$ satisfying $\mu\theta\geqslant 1-\delta$. Then, the inequality $\Phi_X(\mu,\theta,\delta)\geqslant 1-\min\{\mu,\theta\}$ holds. Moreover, if $\mu\theta=1-\delta$, in fact one has $\Phi_X(\mu,\theta,\delta)=1-\min\{\mu,\theta\}$.*
Indeed, fix a pair $(x_0,x_0^*)\in\Pi(X)$ and write $x=\mu x_0$ and $x^*=\theta x_0^*$. Then, it is clear

that $x^*(x) \geqslant 1-\delta$ and

$$\Phi_X(\mu,\theta,\delta) \geqslant d_\infty\big((x,x^*),\Pi(X)\big) = \inf_{(y,y^*)\in\Pi(X)} \max\{\|x-y\|,\|x^*-y^*\|\}$$
$$\geqslant \inf_{(y,y^*)\in\Pi(X)} \max\{1-\mu,1-\theta\} = 1-\min\{\mu,\theta\}.$$

To prove the moreover part, suppose that $\mu\theta = 1-\delta$ and observe that given any pair $(x,x^*)\in X\times X^*$ satisfying $\|x\|=\mu$, $\|x^*\|=\theta$, and $\operatorname{Re} x^*(x)\geqslant 1-\delta$ we get that $\operatorname{Re} x^*(x) = 1-\delta$. Now, if $\mu\theta>0$ we take $y=\frac{x}{\mu}$ and $y^*=\frac{x^*}{\theta}$ which satisfy $\operatorname{Re} y^*(y)=1$ and

$$d_\infty\big((x,x^*),\Pi(X)\big) \leqslant \max\{\|x-y\|,\|x^*-y^*\|\} = 1-\min\{\mu,\theta\}.$$

Taking supremum in (x,x^*), we get $\Phi_X(\mu,\theta,\delta)\leqslant 1-\min\{\mu,\theta\}$. If $\mu\theta=0$, an analogous argument with obvious simplifications gives the desired inequality.

Next, we present some elementary observations on the function Ψ which will be useful in our arguments.

2.2.3 Lemma. *For $\delta\in(0,2)$ and $\mu,\theta\in[0,1]$ with $\mu\theta>1-\delta$, we have*

a) $\Psi(\mu,\theta,\cdot)$ *is an increasing function.*

b) $\Psi(\mu,\theta,\delta)=\Psi(\theta,\mu,\delta)$.

c) $\Psi(\mu,\theta,1+\mu^2)=1+\mu$.

d) If $\delta\leqslant 1$, then $\Psi(\mu,\theta,\delta)\leqslant 1+\mu$ and $\Psi(\mu,\theta,\delta)\leqslant 1+\theta$.

Proof. Only item c) needs an explanation: observe that

$$\begin{aligned}\Psi(\mu,\theta,1+\mu^2) &= \frac{2-\mu-\theta+\sqrt{(\mu-\theta)^2+8\,(\mu\theta+\mu^2)}}{2}\\ &= \frac{2-\mu-\theta+\sqrt{(3\mu+\theta)^2}}{2}\\ &= 1+\mu.\end{aligned}$$

□

Finally, we will need the following result from [40].

2.2.4 Lemma ([40, Corollary 2.2]). *Suppose C is a closed convex subset of the Banach space X, that $z^* \in S_{X^*}$ and that $\eta > 0$ and $z \in C$ are such that*

$$\operatorname{Re}\sup z^*(C) \leqslant \operatorname{Re} z^*(z) + \eta.$$

Then, for any $k \in (0,1)$ there exist $\tilde{y}^* \in X^*$ and $\tilde{y} \in C$ such that

$$\sup \operatorname{Re} \tilde{y}^*(C) = \tilde{y}^*(\tilde{y}), \qquad \|z - \tilde{y}\| \leqslant \frac{\eta}{k}, \qquad \|z^* - \tilde{y}^*\| \leqslant k.$$

We are now ready to prove the main result of the section.

Proof of Theorem 2.2.1. Fixed $(x, x^*) \in X \times X^*$ satisfying $\|x\| = \mu$, $\|x^*\| = \theta$, and $\operatorname{Re} x^*(x) \geqslant 1 - \delta$, we take $y_1 \in S_X$ satisfying $\|x - y_1\| \leqslant 1$, $y_1^* \in S_{X^*}$ such that $y_1^*(y_1) = 1$ and observe that

$$\max\{\|x - y_1\|, \|x^* - y_1^*\|\} \leqslant 1 + \theta.$$

We can produce a dual argument by means of the Bishop-Phelps theorem: find a norm attaining functional $y_2^* \in S_{X^*}$ with $\|x^* - y_2^*\| \leqslant 1$ and a point $y_2 \in S_X$ satisfying $y_2^*(y_2) = 1$. Then, we have that

$$\max\{\|x - y_2\|, \|x^* - y_2^*\|\} \leqslant 1 + \mu$$

and, therefore,

$$d_\infty\big((x, x^*), \Pi(X)\big) \leqslant \min\{1 + \mu, 1 + \theta\}.$$

Now, since $\Psi(\mu, \theta, 1 + \mu^2) = 1 + \mu$, $\Psi(\mu, \theta, 1 + \theta^2) = 1 + \theta$, and $\Psi(\mu, \theta, \cdot)$ is an increasing function, the proof will be finished if we show that

$$d_\infty\big((x, x^*), \Pi(X)\big) \leqslant \Psi(\mu, \theta, \delta)$$

for $\delta < \min\{1 + \mu^2, 1 + \theta^2\}$. In this case $\mu\theta > 1 - \delta > -\theta^2$ which implies that $\theta > 0$. Thus we can define

$$\eta = \frac{\mu\theta - 1 + \delta}{\theta} > 0, \qquad z = x, \qquad \text{and} \qquad z^* = \frac{x^*}{\theta}$$

which satisfy $\operatorname{Re} z^*(z) + \eta \geqslant \mu$. Besides, consider

$$k = \frac{\theta - \mu + \sqrt{(\mu - \theta)^2 + 8(\mu\theta - 1 + \delta)}}{4\theta}$$

It is clear that $k > 0$ and, using the fact that $\delta < 1 + \theta^2$, it is not difficult to verify that $k < 1$:

$$k < \frac{\theta - \mu + \sqrt{(\mu - \theta)^2 + 8(\mu\theta + \theta^2)}}{4\theta} = \frac{\theta - \mu + \sqrt{(\mu + 3\theta)^2}}{4\theta} = 1.$$

Therefore, we may apply Lemma 2.2.4 for $C = \mu B_X$, $z^* \in S_{X^*}$, $z \in \mu B_X = C$, $\eta > 0$, and $0 < k < 1$ to obtain $\widetilde{y}^* \in X^*$ and $\widetilde{y} \in C$ satisfying

$$\widetilde{y}^*(\widetilde{y}) = \sup \operatorname{Re} \widetilde{y}^*(C) = \mu\|\widetilde{y}^*\|, \quad \|z - \widetilde{y}\| \leqslant \frac{\eta}{k}, \quad \text{and} \quad \|z^* - \widetilde{y}^*\| = \left\| \frac{x^*}{\theta} - \widetilde{y}^* \right\| \leqslant k.$$

As $k < 1$, we get $\widetilde{y}^* \neq 0$ and we can write $y^* = \frac{\widetilde{y}^*}{\|\widetilde{y}^*\|}$, $y = \frac{\widetilde{y}}{\mu}$, to obtain that $(y, y^*) \in \Pi(X)$. This way, we have that

$$\|x - y\| = \left\| z - \frac{\widetilde{y}}{\mu} \right\| \leqslant \|z - \widetilde{y}\| + \left\| \widetilde{y} - \frac{\widetilde{y}}{\mu} \right\| \leqslant \frac{\eta}{k} + 1 - \mu. \tag{2.1}$$

On the other hand we can estimate $\|x^* - y^*\|$ as follows:

$$\begin{aligned}\|x^* - y^*\| &= \left\| x^* - \frac{\widetilde{y}^*}{\|\widetilde{y}^*\|} \right\| \leqslant \|x^* - \theta\widetilde{y}^*\| + \left\| \theta\widetilde{y}^* - \frac{\widetilde{y}^*}{\|\widetilde{y}^*\|} \right\| \qquad (2.2)\\ &\leqslant \theta \left\| \frac{x^*}{\theta} - \widetilde{y}^* \right\| + \big|\theta\|\widetilde{y}^*\| - 1\big| \leqslant \theta \left\| \frac{x^*}{\theta} - \widetilde{y}^* \right\| + \big|\theta\|\widetilde{y}^*\| - \theta\big| + |1 - \theta| \\ &\leqslant \theta \left(\left\| \frac{x^*}{\theta} - \widetilde{y}^* \right\| + \big|\|\widetilde{y}^*\| - 1\big| \right) + 1 - \theta \leqslant 2\theta \left\| \frac{x^*}{\theta} - \widetilde{y}^* \right\| + 1 - \theta \\ &\leqslant 2k\theta + 1 - \theta.\end{aligned}$$

Finally, it is routine to check that $\frac{\eta}{k} + 1 - \mu = 2k\theta + 1 - \theta = \Psi(\mu, \theta, \delta)$. Therefore,

$$d_\infty\big((x, x^*), \Pi(X)\big) \leqslant \max\{\|x - y\|, \|x^* - y^*\|\} \leqslant \Psi(\mu, \theta, \delta)$$

which finishes the proof. □

Since $\Phi_X^S(\delta) = \Phi_X(1,1,\delta) \leqslant \Psi(1,1,\delta) = \sqrt{2\delta}$, we obtain the following estimate for the spherical modulus as an immediate consequence of the above result.

2.2.5 Corollary. *Let X be a Banach space and $\delta \in (0,2)$. Then $\Phi_X^S(\delta) \leqslant \sqrt{2\delta}$.*

With just a bit of effort, we can obtain the analogue result for the general modulus.

2.2.6 Remark. *Let $\delta \in (0,2)$ and $\mu, \theta \in [0,1]$ be such that $\mu\theta \geqslant 1-\delta$. If $\mu\theta < 1$ then*

$$\min\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\} < \sqrt{2\delta}.$$

Proof. Since $\min\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\}$ is symmetric in μ and θ we can suppose without loss of generality that $\mu = \min\{\mu,\theta\} < 1$.

If $\delta < 1+\mu^2$, using that $\Psi(\mu,\theta,\cdot)$ is an increasing function and $\Psi(\mu,\theta,1+\mu^2) = 1+\mu$ we deduce that $\min\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\} = \Psi(\mu,\theta,\delta)$. Besides, we observe that

$$\begin{aligned}
\frac{\partial\Psi}{\partial\mu}(\mu,\theta,\delta) &= \frac{1}{2}\left(-1+\frac{\mu+3\theta}{\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}\right) \\
&= \frac{1}{2}\left(\frac{\mu+3\theta-\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}\right) \\
&= \frac{1}{2}\left(\frac{8(1+\theta^2-\delta)}{\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}\left(\mu+3\theta+\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}\right)}\right) \\
&> 0.
\end{aligned}$$

So $\Psi(\cdot,\theta,\delta)$ is strictly increasing in the interval $[\mu,1]$ and, therefore, we can write

$$\Psi(\mu,\theta,\delta) < \Psi(1,\theta,\delta). \tag{2.3}$$

In case $\theta = 1$ we obtain $\Psi(\mu,\theta,\delta) < \Psi(1,1,\delta) = \sqrt{2\delta}$ as desired. If otherwise $\theta < 1$, using that Ψ is symmetric in μ and θ, one can easily check that

$$\frac{\partial \Psi}{\partial \theta}(1,\theta,\delta) = \frac{1}{2}\left(\frac{8(2-\delta)}{\sqrt{(1-\theta)^2+8(\theta-1+\delta)}\left(\theta+3+\sqrt{(1-\theta)^2+8(\theta-1+\delta)}\right)}\right) > 0.$$

Hence, we can continue the estimation in (2.3) as follows:

$$\Psi(\mu,\theta,\delta) < \Psi(1,\theta,\delta) < \Psi(1,1,\delta) = \sqrt{2\delta}.$$

Finally, when $\delta \geqslant 1+\mu^2$ it follows that

$$\min\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\} = 1+\mu \leqslant 1+\sqrt{\delta-1}.$$

This, together with the fact that $1+\sqrt{\delta-1} < \sqrt{2\delta}$ for $\delta \in [1,2)$, finishes the proof. □

2.2.7 Corollary. *Let X be a Banach space and $\delta \in (0,2)$. Then $\Phi_X(\delta) \leqslant \sqrt{2\delta}$.*

As a consequence of all of this, we obtain an improvement of the Bishop-Phelps-Bollobás theorem which we write in two equivalent ways.

2.2.8 Corollary. *Let X be a Banach space.*

(a) *Let $0 < \varepsilon < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy*

$$\operatorname{Re} x^*(x) > 1 - \varepsilon^2/2.$$

Then, there exists $(y, y^*) \in \Pi(X)$ such that

$$\|x-y\| < \varepsilon \quad \text{and} \quad \|x^*-y^*\| < \varepsilon.$$

(b) *Let $0 < \delta < 2$ and suppose that $x \in B_X$ and $x^* \in B_{X^*}$ satisfy*

$$\operatorname{Re} x^*(x) > 1 - \delta.$$

Then, there exists $(y, y^*) \in \Pi(X)$ such that

$$\|x-y\| < \sqrt{2\delta} \quad \text{and} \quad \|x^*-y^*\| < \sqrt{2\delta}.$$

We conclude the section presenting an example for which the estimation given in Theorem 2.2.1 is sharp.

2.2.9 Example. Let X be the real space ℓ_∞^2, $\delta \in (0,2)$, and μ, $\theta \in [0,1]$ satisfying $\mu\theta > 1-\delta$. Then, there exists a pair $(x,x^*) \in X \times X^*$ with $\|x\| = \mu$, $\|x^*\| = \theta$, $x^*(x) \geqslant 1-\delta$, and such that

$$d_\infty\big((x,x^*),\Pi(X)\big) = \min\left\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\right\}.$$

Therefore, $\Phi_X(\mu,\theta,\delta) = \min\left\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\right\}$ for all possible values of δ, μ, θ.

Proof. We divide the proof into three cases depending on the expression at which the minimum is attained.

Case 1: $\min\left\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\right\} = \Psi(\mu,\theta,\delta)$.

Since $\Psi(\mu,\theta,\cdot)$ is an increasing function and $\Psi(\mu,\theta,1+\theta^2) = 1+\theta \geqslant \Psi(\mu,\theta,\delta)$ we have that $\delta \leqslant 1+\theta^2$. Thus, we can write $\mu\theta > 1-\delta \geqslant -\theta^2$ which implies $\theta > 0$, so we can define

$$k = \tfrac{\theta-\mu+\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{4\theta}, \quad x = (\mu, 1-\Psi(\mu,\theta,\delta)) \quad \text{and} \quad x^* = (\theta(1-k), \theta k).$$

As we observed in the proof of Theorem 2.2.1, $k \in (0,1)$ and so $\|x^*\| = \theta$. Besides, we can estimate as follows

$$\Psi(\mu,\theta,\delta)) \geqslant \frac{2-\mu-\theta+\sqrt{(\mu-\theta)^2}}{2} \geqslant 1-\mu.$$

This, together with the fact that $\Psi(\mu,\theta,\delta)) \leqslant 1+\mu$, gives us $\|x\| = \mu$. Moreover, we have that

$$\begin{aligned} x^*(x) &= \mu\theta(1-k) + (1-\Psi(\mu,\theta,\delta))\theta k = \mu\theta + (1-\mu-\Psi(\mu,\theta,\delta))\theta k \\ &= \mu\theta + \tfrac{\theta-\mu-\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{2} \quad \tfrac{\theta-\mu+\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{4} \\ &= \mu\theta - (\mu\theta-1+\delta) = 1-\delta. \end{aligned}$$

In view of Theorem 2.2.1, to finish the proof in this case we only need to show that

$$d_\infty\big((x,x^*),\Pi(X)\big) \geqslant \Psi(\mu,\theta,\delta).$$

Fixed $(y,y^*) \in \Pi(X)$ there are $a,b,c,d \in \mathbb{R}$ such that $y=(a,b)$, $y^*=(c,d)$ and

$$\max\{|a|,|b|\}=1, \qquad |c|+|d|=1, \qquad \text{and} \qquad ac+bd=1.$$

We distinguish two cases depending on the values of d. Suppose first that $d \leqslant 0$ and recall that $k \geqslant 0$ to write

$$\begin{aligned}\|x^*-y^*\| &= \big|c-\theta(1-k)\big| + \big|d-\theta k\big| \\ &\geqslant |c|-\theta(1-k)+|d|+\theta k = 2\theta k+1-\theta = \Psi(\mu,\theta,\delta).\end{aligned}$$

If otherwise $d>0$, then the inequality

$$|c|+|d| = 1 = ac+bd \leqslant |c|+bd$$

yields that $b=1$ and we can write

$$\|x-y\| = \max\{|\mu-a|,|\Psi(\mu,\theta,\delta)|\} \geqslant \Psi(\mu,\theta,\delta),$$

which finishes the proof for Case 1.

Case 2: $\min\{\Psi(\mu,\theta,\delta),1+\mu,1+\theta\} = 1+\theta$.

In this case we have that $\delta \geqslant 1+\theta^2$ and $\mu \geqslant \theta$. So defining

$$x=(\mu,-\theta) \qquad \text{and} \qquad x^*=(0,\theta),$$

it is clear that $\|x\|=\mu$, $\|x^*\|=\theta$, and $x^*(x) = -\theta^2 \geqslant 1-\delta$. To verify the inequality $d_\infty\big((x,x^*),\Pi(X)\big) \geqslant 1+\|x^*\|$ one can proceed analogously to the previous case.

Case 3: $\min\{\Psi(\mu,\theta,\delta),1+\mu,1+\theta\} = 1+\mu$.

In this case one has that $\delta \geqslant 1+\mu^2$ and $\theta \geqslant \mu$. So we can consider

$$x=(\mu,-\mu) \qquad \text{and} \qquad x^* = \left(\frac{\theta-\mu}{2},\frac{\theta+\mu}{2}\right)$$

which satisfy $\|x\|=\mu$, $\|x^*\|=\theta$, and $x^*(x) = -\mu^2 \geqslant 1-\delta$. Again, verifying the inequality $d_\infty\big((x,x^*),\Pi(X)\big) \geqslant 1+\|x\|$ can be done analogously to Case 1. □

2.3 Basic properties of the moduli

In this section we present some of the basic properties that the Bishop-Phelps-Bollobás moduli have. Among other results, we will show that the functions $\Phi_X(\cdot)$ and $\Phi_X^S(\cdot)$ are continuous, as well as the continuity of these functions with respect to the Banach-Mazur distance. We also study the relationship of the Bishop-Phelps-Bollobás moduli of a Banach space and those of its dual space.

2.3.1 Continuity with respect to the parameter.

In order to study the continuity of the Bishop-Phelps-Bollobás moduli, the only thing we have to understand is *how close* to each other the sets $A_X(\delta)$ and $A_X(\delta_0)$ are $\big($or $A_X^S(\delta)$ and $A_X^S(\delta_0)\big)$ when $\delta < \delta_0 \in (0,2)$ are close. To do so, we need the following three lemmas which could be of independent interest.

2.3.1 Lemma. *For every pair* $(x_0, x_0^*) \in B_X \times B_{X^*}$ *there is a pair* $(y, y^*) \in \Pi(X)$ *with*

$$\operatorname{Re}\big[y^*(x_0) + x_0^*(y)\big] \geqslant 0.$$

Moreover, if actually $\operatorname{Re} x_0^*(x_0) > 0$ *then* $(y, y^*) \in \Pi(X)$ *can be selected to satisfy*

$$\operatorname{Re}\big[y^*(x_0) + x_0^*(y)\big] \geqslant 2\sqrt{\operatorname{Re}\big(x_0^*(x_0)\big)}.$$

Proof. 1. Take $y_0 \in S_X \cap \ker x_0^*$ and let y_0^* be a supporting functional at y_0. Then

$$\operatorname{Re}\big[y_0^*(x_0) + x_0^*(y_0)\big] = \operatorname{Re} y_0^*(x_0)$$

If the right hand side is positive we can take $y = y_0$, $y^* = y_0^*$, in the opposite case take $y = -y_0$, $y^* = -y_0^*$.

2. For the second statement, take $y = \frac{x_0}{\|x_0\|}$ and let y^* be a supporting functional at y. Then, since for a fixed $a > 0$ the minimum of $f(t) := t + \frac{a}{t}$ for $t > 0$ equals $2\sqrt{a}$, we get

$$\operatorname{Re}\big[y^*(x_0) + x_0^*(y)\big] = \|x_0\| + \frac{1}{\|x_0\|}\operatorname{Re} x_0^*(x_0) \geqslant 2\sqrt{\operatorname{Re} x_0^*(x_0)},$$

as desired. □

The above lemma allows us to prove the following result which we will use to show the continuity of the Bishop-Phelps-Bollobás modulus.

2.3.2 Lemma. *Let X be a Banach space and let $(x_0, x_0^*) \in A_X(\delta_0)$ with $0 < \delta < \delta_0 < 2$. Then:*

Case 1: If $\delta, \delta_0 \in]0,1]$ then

$$\operatorname{dist}\big((x_0, x_0^*), A_X(\delta)\big) \leqslant 2\frac{\sqrt{1-\delta} - \sqrt{1-\delta_0}}{1-\sqrt{1-\delta_0}}.$$

Case 2: If $\delta, \delta_0 \in [1,2)$ then

$$\operatorname{dist}\big((x_0, x_0^*), A_X(\delta)\big) \leqslant 2\frac{2-\delta_0}{\delta_0} \cdot \frac{\delta_0 - \delta}{\delta_0 - 1 + \sqrt{1-2\delta+\delta\delta_0}}.$$

Proof. Denote $t = \operatorname{Re} x_0^*(x_0)$. Let $(y, y^*) \in \Pi(X)$ be from the previous lemma (in case 1 we use part 2 of the lemma, in case 2 we use part 1). For every $\lambda \in [0,1]$ we define $x_\lambda = (1-\lambda)x_0 + \lambda y$ and $x_\lambda^* = (1-\lambda)x_0^* + \lambda y^*$. Both x_λ and x_λ^* belong to corresponding balls, and $\operatorname{dist}_\infty\big((x_0, x_0^*), (x_\lambda, x_\lambda^*)\big) \leqslant 2\lambda$. We have

$$\operatorname{Re} x_\lambda^*(x_\lambda) = (1-\lambda)^2 t + \lambda(1-\lambda)\operatorname{Re}\big[y^*(x_0) + x_0^*(y)\big] + \lambda^2, \tag{2.4}$$

so in case 1

$$\operatorname{Re} x_\lambda^*(x_\lambda) \geqslant (1-\lambda)^2 t + 2\lambda(1-\lambda)\sqrt{t} + \lambda^2 = \Big((1-\lambda)\sqrt{t} + \lambda\Big)^2.$$

Now we are looking for a possibly small value of λ, for which $(x_\lambda, x_\lambda^*) \in A_X(\delta)$. If $\delta \geqslant 1-t$, the value $\lambda = 0$ satisfies the required condition and $\operatorname{dist}_\infty\big((x_0, x_0^*), A_X(\delta)\big) = 0$. If $0 < \delta < 1-t$ then the positive solution in λ of the equation $\Big((1-\lambda)\sqrt{t} + \lambda\Big)^2 = 1-\delta$ is

$$\lambda_t = \frac{\sqrt{1-\delta} - \sqrt{t}}{1-\sqrt{t}}.$$

Evidently, $\lambda_t \in [0,1]$, so $(x_{\lambda_t}, x^*_{\lambda_t}) \in A_X(\delta)$. Since λ_t decreases in t,

$$\operatorname{dist}_\infty\left((x_0, x_0^*), A_X(\delta)\right) \leqslant 2\lambda_t \leqslant 2\lambda_{1-\delta_0} = 2\frac{\sqrt{1-\delta}-\sqrt{1-\delta_0}}{1-\sqrt{1-\delta_0}}.$$

This completes the proof of case 1.

In the case 2 we may assume that $t \leqslant 1-\delta$ (otherwise the corresponding distance is 0 and the job is done), so $t \leqslant 0$. By part 1 of the previous lemma and (2.4)

$$\operatorname{Re} x^*_\lambda(x_\lambda) \geqslant (1-\lambda)^2 t + \lambda^2,$$

so we are solving in λ the equation

$$(1-\lambda)^2 t + \lambda^2 - 1 + \delta = 0, \quad \text{i.e.} \quad (1+t)\lambda^2 - 2t\lambda + (t-1+\delta) = 0.$$

The discriminant of this equation is $D = -t\delta - \delta + 1$. Note that $D \geqslant -(1-\delta)\delta - \delta + 1 = (1-\delta)^2 \geqslant 0$ and $t - 1 + \delta \leqslant 0$, so there is a positive solution of our equation given by

$$\lambda_t = \frac{1}{1+t}(t+\sqrt{D}) = \frac{1}{1+t}(t+\sqrt{1-t\delta-\delta}).$$

This λ_t decreases in t, so

$$\lambda_t \leqslant \lambda_{1-\delta_0} = \frac{1}{\delta_0}(1-\delta_0+\sqrt{1-2\delta+\delta\delta_0}) = \frac{2+\delta_0}{\delta_0}\cdot\frac{\delta_0-\delta}{\delta_0-1+\sqrt{1-2\delta+\delta\delta_0}}$$

which finishes the proof. □

To show the continuity of the spherical modulus we need the following result.

2.3.3 Lemma. *Let X be a Banach space and let $(x_0, x_0^*) \in A_X^S(\delta_0)$ with $0 < \delta < \delta_0 < 2$.*

Case 1: If $\delta < 1$, then

$$\operatorname{dist}_\infty\left((x_0, x_0^*), A_X^S(\delta)\right) \leqslant \frac{4(\delta_0-\delta)}{\delta_0}.$$

Case 2: If $\delta \in [1,2)$ and $2-\sqrt{2-\delta_0} < \delta < \delta_0$, then

$$\operatorname{dist}_\infty\left((x_0, x_0^*), A_X^S(\delta)\right) \leqslant \frac{2(\delta_0-\delta)}{2-\delta}.$$

Proof. Let us start with case 1. Fix $\xi \in (0, \delta)$. As $\|x_0^*\| = 1$, we may find $y_\xi \in S_X$ satisfying $x_0^*(y_\xi) > 1 - \xi$. For every $\lambda \in [0,1]$ we define

$$x(\lambda, \xi) = \lambda x_0 + (1-\lambda) y_\xi.$$

Consider $\lambda_\xi = \frac{\delta - \xi}{\delta_0 - \xi} \in [0,1]$ and write $x_\xi = x(\lambda_\xi, \xi)$. A straightforward verification shows that

$$\operatorname{Re} x_0^*(x_\xi) > 1 - \delta$$

and so, as $1 - \delta \geqslant 0$, we have that $x_\xi \neq 0$ and also that

$$\operatorname{Re} x_0^*\left(\frac{x_\xi}{\|x_\xi\|}\right) > 1 - \delta.$$

Therefore, $\left(\frac{x_\xi}{\|x_\xi\|}, x_0^*\right) \in A_X^S(\delta)$. We have

$$\begin{aligned}\left\|x_0 - \frac{x_\xi}{\|x_\xi\|}\right\| &\leqslant \|x_0 - x_\xi\| + \left\|x_\xi - \frac{x_\xi}{\|x_\xi\|}\right\| = \|x_0 - x_\xi\| + \big|\|x_\xi\| - 1\big| \\ &\leqslant 2\|x_\xi - x_0\| = 2(1-\lambda_\xi)\|x_0 - y_\xi\| \leqslant 4\left(\frac{\delta_0 - \delta}{\delta_0 - \xi}\right).\end{aligned}$$

We get the result by just letting $\xi \longrightarrow 0$.

Let us prove case 2. If $\operatorname{Re} x_0^*(x_0) > 1 - \delta$, then the proof is done. Suppose that

$$1 - \delta \geqslant \operatorname{Re} x_0^*(x_0) > 1 - \delta_0.$$

Fix $\xi \in \left(0, \min\{2 - \delta_0, \frac{4\delta - 2 - \delta_0 - \delta^2}{\delta - 1}\}\right)$ (observe that $\frac{4\delta - 2 - \delta_0 - \delta^2}{\delta - 1} > 0$ by the conditions on δ). As $\|x_0^*\| = 1$, we may find $y_\xi \in S_X$ satisfying $x_0^*(y_\xi) > 1 - \xi$. Now, we consider

$$\lambda_\xi = \frac{\delta_0 - \delta}{2 - \delta - \xi} \qquad \text{and} \qquad x_\xi = x_0 + \lambda_\xi y_\xi.$$

Notice that $\lambda_\xi \in (0,1)$ (since $\delta < \delta_0$ and $\xi < 2 - \delta_0$) and

$$\|x_\xi\| \geqslant \|x_0\| - \lambda \|y_\xi\| = 1 - \lambda_\xi > 0.$$

Also, observe that

$$\operatorname{Re} x_0^*(x_\xi) \leqslant 1 - \delta + \lambda_\xi = \frac{(1-\delta)(2-\delta-\xi) + \delta_0 - \delta}{2 - \delta - \xi}$$

so, $\operatorname{Re} x_0^*(x_\xi) \leqslant 0$ since $\xi \leqslant \frac{4\delta-2-\delta_0-\delta^2}{\delta-1}$. Now,

$$\operatorname{Re} x_0^*\left(\frac{x_\xi}{\|x_\xi\|}\right) \geqslant \operatorname{Re} x_0^*\left(\frac{x_\xi}{1-\lambda_\xi}\right) > \frac{1-\delta_0+\lambda_\xi(1-\xi)}{1-\lambda_\xi} = 1-\delta.$$

Therefore, $\left(\frac{x_\xi}{\|x_\xi\|}, x_0^*\right) \in A_X^S(\delta)$. We have

$$\left\|x_0 - \frac{x_\xi}{\|x_\xi\|}\right\| \leqslant 2\|x_0 - x_\xi\| = 2\lambda_\xi = 2\left(\frac{\delta_0-\delta}{2-\delta-\xi}\right).$$

Consequently, letting $\xi \longrightarrow 0$, we get

$$\operatorname{dist}_\infty\big((x_0, x_0^*), A_X^S(\delta)\big) \leqslant \frac{2(\delta_0-\delta)}{2-\delta}$$

as we desired. □

We are ready to state and prove the promised result.

2.3.4 Proposition. *Let X be a Banach space. Then, the functions*

$$\delta \longmapsto \Phi_X(\delta) \qquad \textit{and} \qquad \delta \longmapsto \Phi_X^S(\delta)$$

are continuous in $(0,2)$.

Proof. Let us give the proof for $\Phi_X(\delta)$. Observe that for $\delta_1, \delta_2 \in (0,2)$ with $\delta_1 < \delta_2$, one has

$$\begin{aligned} 0 < \Phi_X(\delta_2) - \Phi_X(\delta_1) &= d_H\left(A_X(\delta_2), \Pi(X)\right) - d_H\left(A_X(\delta_1), \Pi(X)\right) \\ &\leqslant d_H\left(A_X(\delta_2), A_X(\delta_1)\right). \end{aligned}$$

Now, the continuity follows routinely from Lemma 2.3.2. An analogous argument allows to prove the continuity of $\Phi_X^S(\delta)$ from Lemma 2.3.3. □

We finish this subsection showing that when the general modulus attains its maximum value is, in fact, because the spherical modulus does.

2.3.5 Proposition. *Let X be a Banach space. For every $\delta \in (0,2)$, the condition $\Phi_X(\delta) = \sqrt{2\delta}$ is equivalent to the condition $\Phi_X^S(\delta) = \sqrt{2\delta}$.*

Proof. Since $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}$, the implication $\left[\Phi_X^S(\delta) = \sqrt{2\delta}\right] \Rightarrow \left[\Phi_X(\delta) = \sqrt{2\delta}\right]$ is evident. Let us prove the inverse implication. Let $\Phi_X(\delta) = \sqrt{2\delta}$. Then there is a sequence of pairs $(x_n, x_n^*) \in B_X \times B_{X^*}$ such that $\operatorname{Re} x_n^*(x_n) > 1 - \delta$ but for every $(y, y^*) \in \Pi(X)$ we have

$$\|x_n - y\| \geqslant \sqrt{2\delta} - \frac{1}{n} \quad \text{or} \quad \|x_n^* - y^*\| \geqslant \sqrt{2\delta} - \frac{1}{n}.$$

This, together with Theorem 2.2.1, gives

$$\sqrt{2\delta} - \frac{1}{n} \leqslant \Phi_X(\|x_n\|, \|x_n^*\|, \delta) \leqslant \min\{\Psi(\|x_n\|, \|x_n^*\|, \delta), 1 + \|x_n\|, 1 + \|x_n^*\|\}.$$

By passing to a subsequence we may and do assume that $\|x_n\| \longrightarrow \mu$ and $\|x_n^*\| \longrightarrow \theta$ for some $\mu, \theta \in [0,1]$ as $n \to \infty$. So, letting $n \to \infty$ in the preceding inequality we get

$$\sqrt{2\delta} \leqslant \min\{\Psi(\mu, \theta, \delta), 1 + \mu, 1 + \theta\}.$$

Now, Remark 2.2.6 gives us that $\mu = 1$ and $\theta = 1$. Denote $\tilde{x}_n = \frac{x_n}{\|x_n\|}$, $\tilde{x}_n^* = \frac{x_n^*}{\|x_n^*\|}$. In the case when $\delta \in (0,1]$, we have $\operatorname{Re} \tilde{x}_n^*(\tilde{x}_n) > 1 - \delta$ but for every $(y, y^*) \in \Pi(X)$

$$\|\tilde{x}_n - y\| \geqslant \sqrt{2\delta} - \frac{1}{n} - \|x_n - \tilde{x}_n\| \quad \text{or} \quad \|\tilde{x}_n^* - y^*\| \geqslant \sqrt{2\delta} - \frac{1}{n} - \|\tilde{x}_n^* - x_n^*\|.$$

Since the right-hand sides of the above inequalities go to $\sqrt{2\delta}$, we get the condition $\Phi_X^S(\delta) = \sqrt{2\delta}$.

In the case of $\delta \in (1,2)$, we no longer know that $\operatorname{Re} \tilde{x}_n^*(\tilde{x}_n) > 1 - \delta$, but what we do know is that $\liminf \operatorname{Re} \tilde{x}_n^*(\tilde{x}_n) \geqslant 1 - \delta$, and that gives us the desired condition $\Phi_X^S(\delta) = \sqrt{2\delta}$ thanks to the continuity of the spherical modulus (Proposition 2.3.4). □

2.3.2 Duality.

Our next aim is to compare the moduli of a Banach space and the ones of its dual space. The following lemma will be the key to prove the result. It is actually an easy application of the Principle of Local Reflexivity, which we remind for the sake of clearness.

2.3.6 Lemma. *For $\varepsilon > 0$, let $(x, x^*) \in B_X \times B_{X^*}$ and let $(\tilde{y}^*, \tilde{y}^{**}) \in \Pi(X^*)$ such that*

$$\|x^* - \tilde{y}^*\| < \varepsilon \quad \text{and} \quad \|x - \tilde{y}^{**}\| < \varepsilon.$$

Then there is a pair $(y, y^*) \in \Pi(X)$ such that

$$\|x - y\| < \varepsilon \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

2.3.7 Theorem (Principle of Local Reflexivity (see Theorem 11.2.4 [1]))**.** *Let X be a Banach space. Suppose that F is a finite-dimensional subspace of X^{**} and G is a finite-dimensional subspace of X^*. Then given $\varepsilon > 0$ there is a subspace E of X containing $F \cap X$ with $\dim E = \dim F$, and a linear isomorphism $T : F \longrightarrow E$ with $\|T\|\|T^{-1}\| < 1 + \varepsilon$ such that*

$$Tx = x, \quad x \in F \cap X$$

and

$$x^*(Tx^{**}) = x^{**}(x^*).$$

Proof of Lemma 2.3.6: First chose $\varepsilon' < \varepsilon$ such that still

$$\|x^* - \tilde{y}^*\| < \varepsilon' \quad \text{and} \quad \|x - \tilde{y}^{**}\| < \varepsilon'.$$

Now, we consider $\xi > 0$ such that

$$(1 + \xi)\varepsilon' + \xi + \sqrt{\frac{2\xi}{1 + \xi}} < \varepsilon,$$

and use the Principle of Local Reflexivity to get an operator $T : \text{lin}\{x, \tilde{y}^{**}\} \longrightarrow X$ satisfying

$$\|T\|\,\|T^{-1}\| \leqslant 1 + \xi, \quad T(x) = x, \quad \tilde{y}^*(T(\tilde{y}^{**})) = y^{**}(\tilde{y}^*) = 1.$$

Next, we consider $\tilde{x} = \dfrac{T(\tilde{y}^{**})}{\|T(\tilde{y}^{**})\|} \in S_X$ and $\tilde{x}^* = \tilde{y}^* \in S_{X^*}$, observe that

$$\operatorname{Re} \tilde{x}^*(\tilde{x}) > \frac{1}{1 + \xi} = 1 - \frac{\xi}{1 + \xi},$$

and we use Corollary 2.2.8 to get $(y, y^*) \in \Pi(X)$ satisfying that

$$\|\tilde{x} - y\| < \sqrt{\frac{2\xi}{1+\xi}} \quad \text{and} \quad \|\tilde{x}^* - y^*\| < \sqrt{\frac{2\xi}{1+\xi}}.$$

Let us show that $(y, y^*) \in \Pi(X)$ fulfill our requirements.

$$\begin{aligned}\|x - y\| &\leqslant \|T(x) - T(\tilde{y}^{**})\| + \|T(\tilde{y}^{**}) - \tilde{x}\| + \|\tilde{x} - y\| \\ &< (1+\xi)\varepsilon' + \xi + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon\end{aligned}$$

and, analogously,

$$\|x^* - y^*\| \leqslant \|x^* - \tilde{y}^*\| + \|\tilde{y}^* - y^*\| < \varepsilon' + \sqrt{\frac{2\xi}{1+\xi}} < \varepsilon,$$

getting the desired result. □

We can now present the promised result about duality.

2.3.8 Proposition. *Let X be a Banach space. Then*

$$\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta) \quad \textit{and} \quad \Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$$

for every $\delta \in (0, 2)$.

Proof. The proof is the same for both moduli, so we are only presenting it for $\Phi_X(\delta)$. Fix $\delta \in (0, 2)$. We consider any $\varepsilon > 0$ such that $\Phi_{X^*}(\delta) < \varepsilon$ and for a given $(x, x^*) \in A_X(\delta)$ consider $(x^*, x) \in A_{X^*}(\delta)$ (we identify X as a subspace of X^{**}) and so we may find $(\tilde{y}^*, \tilde{y}^{**}) \in \Pi(Y^*)$ such that

$$\|x^* - \tilde{y}^*\| < \varepsilon \quad \text{and} \quad \|x - \tilde{y}^{**}\| < \varepsilon.$$

From Lemma 2.3.6, we find $(y, y^*) \in \Pi(X)$ such that

$$\|x - y\| < \varepsilon \quad \text{and} \quad \|x^* - y^*\| < \varepsilon.$$

This means that $\Phi_X(\delta) \leqslant \varepsilon$ and, therefore, $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$, as desired. □

We do not know whether the inequalities in Proposition 2.3.8 can be strict. Of course, this cannot be the case when the space is reflexive.

2.3.9 Corollary. *For every reflexive Banach space X, one has $\Phi_X(\delta) = \Phi_{X^*}(\delta)$ and $\Phi^S_X(\delta) = \Phi^S_{X^*}(\delta)$ for every $0 < \delta < 2$.*

2.3.3 Continuity with respect to the Banach space.

Our next goal is to show that the Bishop-Phelps-Bollobás modulus of a Banach space is continuous with respect to the Banach-Mazur distance. To do so we will follow the notation of section 1.3 in Chapter 1. Given a Banach space X, recall that $\mathcal{E}(X)$ denotes the set of all equivalent norms to the natural norm in X endowed with the distance:

$$d(p,q) = \log\left(\min\left\{k \geqslant 1 : \frac{1}{k}p \leqslant q \leqslant kp\right\}\right) \qquad (p,q \in \mathcal{E}(X)).$$

For $p_0 \in \mathcal{E}(X)$ and $k > 1$ we will consider the open set in $\mathcal{E}(X)$ given by

$$G(p_0,k) = \{p \in \mathcal{E}(X) : d(p,p_0) < \log k\}.$$

For $\delta \in (0,2)$, we write $\Phi_{(X,p)}$ to denote the Bishop-Phelps-Bollobás modulus of X when it is endowed with the norm p. Besides, we consider the set

$$A_p(\delta) = \{(x,x^*) \in X \times X^* \ : \ p(x) \leqslant 1,\ p(x^*) \leqslant 1,\ x^*(x) > 1-\delta\}.$$

Finally, we write $d_p(A,B)$ to denote the Hausdorff distance between $A, B \subset X \times X^*$ associated to the ℓ_∞-distance $d_{\infty,p}$ in $X \times X^*$ when X and X^* are endowed with the norm p. That is,

$$d_{\infty,p}\big((x,x^*),(y,y^*)\big) = \max\{p(x-y), p(x^*-y^*)\} \qquad \big((x,x^*),(y,y^*) \in X \times X^*\big)$$

and

$$d_p(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d_{\infty,p}(a,b), \sup_{b\in B}\inf_{a\in A} d_{\infty,p}(a,b)\right\}.$$

Observe that with this notation one has that $\Phi_{(X,p)}(\delta) = d_p(A_p(\delta), \Pi(X,p))$. Finally we need a result from [8] which tells us that $\Pi(X,p)$ and $\Pi(X,q)$ are close when p and q are close.

2.3.10 Lemma (Theorem 18.3 in [8]). *Let X be a Banach space. Then, the function $p \longmapsto \Pi(X,p)$ is continuous from $\mathcal{E}(X)$ to the set of non-void closed bounded subsets of $X \times X^*$ endowed with the Hausdorff distance, and it is uniformly continuous on each bounded subset of $\mathcal{E}(X)$.*

2.3.11 Theorem. *Let X be a Banach space and $\delta \in (0,2)$. The functions*

$$\Phi_{(X,\cdot)}(\delta) : \mathcal{E}(X) \longrightarrow \mathbb{R} \qquad \text{and} \qquad \Phi^S_{(X,\cdot)}(\delta) : \mathcal{E}(X) \longrightarrow \mathbb{R}$$

are continuous.

To prove this result we need two lemmas which may be of independent interest.

2.3.12 Lemma. *Let X be a Banach space, $\delta \in (0,2)$, $p_0 \in \mathcal{E}(X)$, and $k > 1$. Let $\eta > 0$ and $p, q \in G(p_0, k)$ satisfying $d(p,q) < \log(1+\eta)$.*

Case 1: If $\delta \in (0,1]$, then

$$d_{p_0}\left(A_p(\delta), A_q(\delta)\right) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}.$$

Case 2: If $\delta \in (1,2)$ and $(\delta-1)(1+\eta)^2 < 1$, then

$$d_{p_0}\left(A_p(\delta), A_q(\delta)\right) < k\eta + 2k\frac{\eta(2+\eta)}{(1+\eta)^2}.$$

Proof. We suppose first that $\delta \in (0,1]$ and we write $\delta_0 = 1 - \frac{1-\delta}{(1+\eta)^2}$. Given $(x,x^*) \in A_p(\delta)$, define $x_0 = \frac{p(x)}{q(x)}x$ and $x_0^* = \frac{p(x^*)}{q(x^*)}x^*$ which obviously satisfy $q(x_0) \leqslant 1$ and $q(x_0^*) \leqslant 1$. Besides, it is immediate to check that

$$\operatorname{Re} x_0^*(x_0) = \operatorname{Re} x^*(x)\frac{p(x)p(x^*)}{q(x)q(x^*)} \geqslant \frac{\operatorname{Re} x^*(x)}{(1+\eta)^2} > \frac{1-\delta}{(1+\eta)^2} = 1-\delta_0,$$

and so $(x_0, x_0^*) \in A_q(\delta_0)$. Observe that if $\delta < 1$ then $\delta_0 > \delta$, and we can use Case 1 of Lemma 2.3.2 for X endowed with the norm q to obtain $(y, y^*) \in A_q(\delta)$ satisfying

$$\max\{q(x_0 - y), q(x_0^* - y^*)\} < 2\frac{\sqrt{1-\delta} - \sqrt{1-\delta_0}}{1 - \sqrt{1-\delta_0}} = \frac{2\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}};$$

where we used that $\sqrt{1-\delta_0} = \frac{\sqrt{1-\delta}}{1+\eta}$. Now we can estimate as follows

$$\begin{aligned} p_0(x-y) &\leqslant p_0(x-x_0) + p_0(x_0-y) \\ &\leqslant p_0(x)\left|1 - \frac{p(x)}{q(x)}\right| + kq(x_0-y) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}} \end{aligned}$$

and an analogous argument gives us the same inequality for the number $p_0(x^* - y^*)$. Therefore, we have that $d_{p_0}\big((x,x^*), A_q(\delta)\big) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}$ for every $(x,x^*) \in A_p(\delta)$. Exchanging the roles of p and q one obtains $d_{p_0}\big((z,z^*), A_p(\delta)\big) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}$ for every $(z,z^*) \in A_q(\delta)$ and hence

$$d_{p_0}\big(A_p(\delta), A_q(\delta)\big) < k\eta + \frac{2k\eta\sqrt{1-\delta}}{1+\eta-\sqrt{1-\delta}}.$$

In the particular case in which $\delta = 1$ it suffices to observe that $\operatorname{Re} x_0^*(x_0) > 0$ and so (x_0, x_0^*) belongs to $A_q(\delta)$. Therefore one obtains the estimation $d_{p_0}\big((x,x^*), A_q(\delta)\big) < k\eta$.

Suppose now that $\delta \in (1,2)$ and define this time $\delta_0 = 1 + (\delta-1)(1+\eta)^2$. Given $(x,x^*) \in A_p(\delta)$ we consider as in the previous case $x_0 = \frac{p(x)}{q(x)}x$ and $x_0^* = \frac{p(x^*)}{q(x^*)}x^*$ which satisfy $q(x_0) \leqslant 1$ and $q(x_0^*) \leqslant 1$. Using the facts that $p(x)/q(x) < 1+\eta$, $p(x^*)/q(x^*) < 1+\eta$ and $1-\delta < 0$, we can write

$$\operatorname{Re} x_0^*(x_0) = \operatorname{Re} x^*(x)\frac{p(x)p(x^*)}{q(x)q(x^*)} \geqslant (1-\delta)\frac{p(x)p(x^*)}{q(x)q(x^*)} > (1-\delta)(1+\eta)^2 = 1-\delta_0,$$

and so $(x_0, x_0^*) \in A_q(\delta_0)$. Since $2 > \delta_0 > \delta$, we can use Case 2 of Lemma 2.3.2 for X endowed with the norm q to obtain $(y, y^*) \in A_q(\delta)$ satisfying

$$\begin{aligned} \max\{q(x_0-y), q(x_0^*-y^*)\} &< 2\frac{2-\delta_0}{\delta_0}\frac{\delta_0-\delta}{\delta_0 - 1 + \sqrt{1-2\delta+\delta\delta_0}} \\ &\leqslant 2\frac{2-\delta_0}{\delta_0}\frac{\delta_0-\delta}{\delta_0-1} \leqslant 2\frac{\delta_0-\delta}{\delta_0-1} = 2\frac{\eta(2+\eta)}{(1+\eta)^2}. \end{aligned}$$

From this point one can proceed as in the previous case to obtain

$$d_{p_0}\left(A_p(\delta), A_q(\delta)\right) < k\eta + 2k\frac{\eta(2+\eta)}{(1+\eta)^2},$$

which finishes the proof. □

One can obtain an analogous result for the spherical modulus using the same proof.

2.3.13 Lemma. *Let X be a Banach space, $\delta \in (0,2)$, $p_0 \in \mathcal{E}(X)$, and $k > 1$. Let $\eta > 0$ and $p, q \in G(p_0, k)$ satisfying $d(p,q) < \log(1+\eta)$.*

Case 1: If $\delta \in (0,1]$, then

$$d_{p_0}\left(A_p^S(\delta), A_q^S(\delta)\right) < k\eta + \frac{4k(1-\delta)(2\eta+\eta^2)}{\delta + 2\eta + \eta^2}.$$

Case 2: If $\delta \in (1,2)$, suppose that $(\delta-1)(1+\eta)^2 < 1$ and $2 - \sqrt{1-(\delta-1)(1+\eta)^2} < \delta$, then

$$d_{p_0}\left(A_p^S(\delta), A_q^S(\delta)\right) < k\eta + 2k(2\eta+\eta^2)\frac{\delta-1}{2-\delta}.$$

Proof. The proof follows exactly the same lines as the proof of Lemma 2.3.12, using Lemma 2.3.3 instead of Lemma 2.3.2 in the corresponding cases. We observe that when $\delta = 1$, Lemma 2.3.3 cannot be used. In this case it suffices to take into account that the element (x_0, x_0^*) lies in $A_q^S(\delta)$ if (x, x^*) is in $A_p^S(\delta)$ so the estimation $d_{p_0}\left((x,x^*), A_q^S(\delta)\right) < k\eta$ follows as in the proof of Lemma 2.3.12. □

We are ready to show that the Bishop-Phelps-Bollobás moduli are continuous in the metric space $\mathcal{E}(X)$.

Proof of Theorem 2.3.11. Fixed $p_0 \in \mathcal{E}(X)$ and $k > 1$, we consider the open set in $\mathcal{E}(X)$ given by $G(p_0,k) = \{p \in \mathcal{E}(X) : d(p,p_0) < \log k\}$. Let $\eta > 0$ be such that

$$(\delta-1)(1+\eta)^2 < 1$$

and $p, q \in G(p_0, k)$ satisfying

$$d(p,q) < \log(1+\eta).$$

Then we can estimate as follows

$$\begin{aligned}
\Phi_{(X,p)}(\delta) - \Phi_{(X,q)}(\delta) &= d_p\left(A_p(\delta), \Pi(X,p)\right) - d_q\left(A_q(\delta), \Pi(X,q)\right) \\
&\leqslant d_p\left(A_p(\delta), A_q(\delta)\right) + d_p\left(A_q(\delta), \Pi(X,p)\right) \\
&\qquad - d_q\left(A_q(\delta), \Pi(X,p)\right) + d_q\left(\Pi(X,p), \Pi(X,q)\right) \\
&\leqslant kd_{p_0}\left(A_p(\delta), A_q(\delta)\right) + (1+\eta)d_q\left(A_q(\delta), \Pi(X,p)\right) \\
&\qquad - d_q\left(A_q(\delta), \Pi(X,p)\right) + kd_{p_0}\left(\Pi(X,p), \Pi(X,q)\right) \\
&\leqslant kd_{p_0}\left(A_p(\delta), A_q(\delta)\right) + k\eta d_{p_0}\left(A_q(\delta), \Pi(X,p)\right) \\
&\qquad + kd_{p_0}\left(\Pi(X,p), \Pi(X,q)\right) \\
&\leqslant kd_{p_0}\left(A_p(\delta), A_q(\delta)\right) + 2k\eta + kd_{p_0}\left(\Pi(X,p), \Pi(X,q)\right).
\end{aligned}$$

Exchanging the roles of p and q, we get

$$\left|\Phi_{(X,p)}(\delta) - \Phi_{(X,q)}(\delta)\right| \leqslant kd_{p_0}\left(A_p(\delta), A_q(\delta)\right) + 2k\eta + kd_{p_0}\left(\Pi(X,p), \Pi(X,q)\right).$$

This, together with Lemma 2.3.10 and Lemma 2.3.12, gives the continuity of $\Phi_{(X,\cdot)}(\delta)$.

A completely analogous argument allows to prove the continuity of $\Phi^S_{(X,\cdot)}$ from Lemma 2.3.13. □

There is a classical way to measure when two Banach spaces are close, the so-called Banach-Mazur distance, and which is related to our approach using the distance between equivalent norms. Given two isomorphic Banach spaces X and Y, the *Banach-Mazur distance* between X and Y is defined by

$$d_{BM}(X,Y) = \log\inf\{\|T\|\|T^{-1}\| \,:\, T \text{ an isomorphism of } X \text{ onto } Y\}.$$

Note that $d_{BM}(X,Y) \geqslant 0$ and $d_{BM}(X,Z) \leqslant d_{BM}(X,Y) + d_{BM}(Y,Z)$. Given a Banach space X, we write $\mathcal{I}(X)$ to denote the set of all Banach spaces isomorphic to X, which is semimetric space when endowed with the Banach-Mazur distance. Then, the result above about the continuity of the Bishop-Phelps-Bollobás moduli on $\mathcal{E}(X)$ can be easily translated to the new setting.

2.3.14 Corollary. *Let X be a Banach space and $\delta \in (0,2)$. The functions from $\mathcal{I}(X)$ to $\mathbb{R}$ given by*

$$Y \longmapsto \Phi_Y(\delta) \quad \text{and} \quad Y \longmapsto \Phi_Y^S(\delta) \qquad \big(Y \in \mathcal{I}(X)\big)$$

are continuous.

The way to deduce the above result from Theorem 2.3.11 is given by the next lemma, which is well-known (see [20, Exercise 1.75], for instance) and relates $\mathcal{E}(X)$ and $\mathcal{I}(X)$. We include an easy proof for the sake of completeness.

2.3.15 Lemma. *Let X_0, X_1 be Banach spaces. If $T : X_1 \longrightarrow X_0$ is an isomorphism, there exists a norm $p_1 \in \mathcal{E}(X_0)$ such that (X_0, p_1) is isometrically isomorphic to $(X_1, \|\cdot\|_{X_1})$ and satisfying that*

$$\|x\|_{X_0} \leqslant p_1(x) \leqslant \|T\|\|T^{-1}\| \, \|x\|_{X_0}$$

for all $x \in X_0$.

Proof. Define $p_1(x) = \|T\|\|T^{-1}(x)\|_{X_1}$ for every $x \in X_0$. Then, it is clear that (X_0, p_1) is isometrically isomorphic to $(X_1, \|\cdot\|_{X_1})$. Also, for each $x \in X_0$ we have

$$p_1(x) = \|T\|\|T^{-1}(x)\|_{X_1} \leqslant \|T\|\|T^{-1}\| \, \|x\|_{X_0}$$

and, on the other hand,

$$\|x\|_{X_0} = \|T(T^{-1}(x))\|_{X_0} \leqslant \|T\|\|T^{-1}(x)\|_{X_1} = p_1(x). \qquad \square$$

An easy consequence of the continuity of the Bishop-Phelps-Bollobás moduli is that they coincide for Banach spaces which are almost isometric.

2.3.16 Corollary. *Let X and Y be almost isometric Banach spaces (i.e. $d_{BM}(X,Y)=0$). Then $\Phi_X(\delta)=\Phi_Y(\delta)$ and $\Phi^S_X(\delta)=\Phi^S_Y(\delta)$ for every $\delta\in(0,2)$.*

2.4 Computation of the Bishop-Phelps-Bollobás moduli

In this section we calculate the two moduli for some classical Banach spaces. Among other results, the moduli of every real or complex Hilbert space are calculated. Besides that, we present a number of spaces for which both moduli reach the maximal possible value for small δ's. This is the case of c_0, ℓ_1 and, more in general, $L_1(\mu)$, $C_0(L)$, and unital C^*-algebras with non-trivial centralizer.

We start observing that there is a universal lower bound for the Bishop-Phelps-Bollobás modulus for small values of δ.

2.4.1 Remark. *Let X be a Banach space. Then, $\Phi_X(\delta)\geqslant\delta$ for every $\delta\in(0,1]$.*

Indeed, using Remark 2.2.2 for $\mu=1$ and $\theta=1-\delta$, we obtain

$$\Phi_X(\delta)\geqslant\Phi_X(1,1-\delta,\delta)\geqslant 1-\min\{1,1-\delta\}=\delta$$

as desired.

Let us remark that we do not know a result giving a lower bound for $\Phi_X(\delta)$ when $\delta>1$, outside of the trivial one $\Phi_X(\delta)\geqslant 1$.

The first example we give consists in the simplest Banach space $X=\mathbb{R}$. In this case we can obtain the distance of a fixed pair $(x,x^*)\in B_X\times B_{X^*}$ to $\Pi(X)$ in terms of $|x|$ and $|x^*|$.

2.4.2 Example. Let $\delta\in(0,2)$, $x,x^*\in\mathbb{R}$ such that $|x|,|x^*|\leqslant 1$ with $x^*x>1-\delta$, then

$$d_\infty\big((x,x^*),\Pi(\mathbb{R})\big)=\begin{cases}1-\min\{|x|,|x^*|\} & \text{if } x^*x\geqslant 0\\ 1+\min\{|x|,|x^*|\} & \text{if } x^*x<0.\end{cases}$$

Proof. We take $y = y^* \in \{-1, 1\}$ to be the sign of the number in $\{x, x^*\}$ which has bigger modulus (in case $|x| = |x^*|$ any choice will do). If $x^*x \geqslant 0$ then $|x - y| = 1 - |x|$ and $|x^* - y^*| = 1 - |x^*|$. So

$$\begin{aligned} d_\infty\big((x,x^*),\Pi(\mathbb{R})\big) &= \min\big\{d_\infty\big((x,x^*),(1,1)\big), d_\infty\big((x,x^*),(-1,-1)\big)\big\} \\ &= d_\infty\big((x,x^*),(y,y^*)\big) = 1 - \min\{|x|,|x^*|\}. \end{aligned}$$

If otherwise $x^*x < 0$ we have that

$$\begin{aligned} d_\infty\big((x,x^*),\Pi(\mathbb{R})\big) &= \min\big\{d_\infty\big((x,x^*),(1,1)\big), d_\infty\big((x,x^*),(-1,-1)\big)\big\} \\ &= d_\infty\big((x,x^*),(y,y^*)\big) = 1 + \min\{|x|,|x^*|\} \end{aligned}$$

which finishes the proof. □

From this result we can deduce the value of both moduli for the real line.

2.4.3 Example. For every $\delta \in (0, 2)$ the following hold:

$$\Phi^S_{\mathbb{R}}(\delta) = 0 \qquad \text{and} \qquad \Phi_{\mathbb{R}}(\delta) = \begin{cases} \delta & \text{if } 0 < \delta \leqslant 1, \\ 1 + \sqrt{\delta - 1} & \text{if } 1 < \delta < 2. \end{cases}$$

Proof. The result for $\Phi^S_{\mathbb{R}}$ is an obvious consequence of the fact $S_{\mathbb{R}} = \{-1, 1\}$. To compute $\Phi_{\mathbb{R}}(\delta)$, we fix first $\delta \in (0, 1]$ and observe that taking $x = 1 - \delta$, $x^* = 1$ it is clear that $\Phi_{\mathbb{R}}(\delta) \geqslant d_\infty\big((x,x^*),\Pi(\mathbb{R})\big) = \delta$. To prove the reversed inequality, fixed $x, x^* \in [-1, 1]$ satisying $x^*x \geqslant 1 - \delta \geqslant 0$, observe that $\min\{|x|,|x^*|\} \geqslant 1 - \delta$. This, together with Example 2.4.2, allows us to write

$$d_\infty\big((x,x^*),\Pi(\mathbb{R})\big) = 1 - \min\{|x|,|x^*|\} \leqslant \delta$$

which, taking supremum on (x, x^*), implies that $\Phi_{\mathbb{R}}(\delta) \leqslant \delta$.

Fix now $\delta \in (1, 2)$, and observe that taking $x = \sqrt{\delta - 1}$ and $x^* = -\sqrt{\delta - 1}$ one has

$$x^*x = 1 - \delta, \qquad |x + 1| = \sqrt{\delta - 1} + 1, \qquad \text{and} \qquad |x^* - 1| = \sqrt{\delta - 1} + 1$$

which give that $\Phi_{\mathbb{R}}(\delta) \geqslant 1+\sqrt{\delta-1}$. To get the reversed inequality, we fix $x, x^* \in [-1,1]$ satisying $x^*x \geqslant 1-\delta$. If $x^*x \geqslant 0$ then $\min\{|x|,|x^*|\} \geqslant 1-\delta$ which, together with Example 2.4.2, tells us that

$$d_\infty\big((x,x^*),\Pi(\mathbb{R})\big) = 1-\min\{|x|,|x^*|\} \leqslant \delta \leqslant 1+\sqrt{\delta-1};$$

where we used that $\delta \leqslant 1+\sqrt{\delta-1}$ for $\delta \in (1,2)$. If otherwise $x^*x < 0$, then $|x^*||x| = -x^*x \leqslant \delta-1$ and so $\min\{|x|,|x^*|\} \leqslant \sqrt{\delta-1}$. Now using Example 2.4.2 we can write

$$d_\infty\big((x,x^*),\Pi(\mathbb{R})\big) = 1+\min\{|x|,|x^*|\} \leqslant 1+\sqrt{\delta-1}.$$

Taking supremum on (x,x^*) we obtain the desired inequality. □

We now turn to compute the moduli of a Hilbert space of (real) dimension greater than one. As we commented in the introduction, both Φ_H and Φ_H^S only depend on the real structure of the space, so we may and do suppose that H is a real Hilbert space of dimension greater than or equal to 2. Let us also recall that H^* identifies with H and that the action of a vector $y \in H$ on a vector $x \in H$ is nothing but their inner product denoted by $\langle x,y\rangle$. In particular, it is clear that

$$\Pi(H) = \big\{(z,z) \in S_H \times S_H\big\}.$$

In the next result, fixed a pair $(x,y) \in B_H \times B_H$, we obtain the distance from (x,y) to $\Pi(H)$ in terms of $\|x\|$, $\|y\|$, and $\langle x,y\rangle$.

2.4.4 Theorem. *Let H be a real Hilbert space with* $\dim(H) \geqslant 2$ *and let x,y be different points in B_H with $\|x\| \geqslant \|y\|$. We call*

$$A = \left\{(x,y) \in H\times H : \langle x,y\rangle \geqslant \|y\|^2 + \|y\|\frac{\|x\|^2-\|y\|^2}{2}\right\}.$$

Then,

$$d_\infty\big((x,y),\Pi(H)\big) = \begin{cases} 1-\|y\| & \text{if } (x,y)\in A,\\ \sqrt{1-\langle x,y\rangle - 2\lambda\sqrt{\|x\|^2\|y\|^2-\langle x,y\rangle^2}} & \text{if } (x,y)\notin A,\end{cases}$$

where

$$\lambda = \frac{-2\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2} + \sqrt{4\big(\|x\|^2+\|y\|^2-2\langle x,y\rangle\big) - (\|x\|^2-\|y\|^2)^2}}{2\big(\|x\|^2+\|y\|^2-2\langle x,y\rangle\big)}.$$

To prove this result we need two easy observations concerning the distance between points in a Hilbert space.

2.4.5 Lemma. *Let* $\alpha_0 \in]-\pi,\pi]$, $a \geqslant 0$, $b \geqslant 0$, *and let* $f : [\alpha_0-\pi, \alpha_0+\pi] \longrightarrow \mathbb{R}$ *be defined by*

$$f(\alpha) = \|(a\cos(\alpha_0), a\sin(\alpha_0)) - (b\cos(\alpha), b\sin(\alpha))\|_2.$$

If $ab > 0$, *then* f *decreases in* $[\alpha_0-\pi,\alpha_0]$ *and increases in* $[\alpha_0,\alpha_0+\pi]$. *If* $ab=0$, *then* f *is constant.*

Proof. Only the case $ab > 0$ needs an explanation. Taking into account that $f^2(\alpha) = a^2+b^2-2ab\cos(\alpha_0-\alpha)$, it suffices to observe that $ab\cos(\alpha_0-\alpha)$ increases in the interval $[\alpha_0-\pi,\alpha_0]$ and decreases in $[\alpha_0,\alpha_0+\pi]$. □

2.4.6 Remark. Lemma 2.4.5 is telling us, in particular, that given a circumference C and a point x in the same plane which is not the center of C, the minimum distance from x to C is attained at the intersection point of C and the half-line starting at the center of C which passes through x.

Proof of Theorem 2.4.4. If $y=0$, we have to show that $d_\infty\big((x,0),\Pi(H)\big) = 1$, but this is clear since $d_\infty((x,0),(\frac{x}{\|x\|},\frac{x}{\|x\|})) \leqslant 1$ and every $h \in S_H$ satisfies $d_\infty((x,0),(h,h)) \geqslant \|h\| = 1$. So we can set $y \neq 0$ for the rest of the proof.

In the next step we show that we can reduce the problem to the 2-dimensional case. Let X be the 2-dimensional subspace of H containing x and y. We claim that $d_\infty\big((x,y),\Pi(H)\big) = d_\infty\big((x,y),\Pi(X)\big)$. Indeed, since $\Pi(X) \subset \Pi(H)$, the inequality $d_\infty\big((x,y),\Pi(H)\big) \leqslant d_\infty\big((x,y),\Pi(X)\big)$ is evident. To prove the reversed inequality, fixed $h \in S_H$, consider the plane P which contains h, intersects X in a line and which is orthogonal to the line containing x and y. Set $h_X \in X$ to be the intersection point of P and the line containing x and y. We observe that $P \cap S_H$ is a circle which contains h and we write $\widetilde{h}_X$ to denote the intersection point of $P \cap S_H$ and the half-line starting at the centre of $P \cap S_H$ and containing h_X. If h_X happens to be the centre of $P \cap S_H$, any of the two points in $P \cap S_H \cap X$ can be taken as $\widetilde{h}_X$. By Remark 2.4.6 we have that $\|h - h_X\| \geqslant \|\widetilde{h}_X - h_X\|$. Finally, using the orthogonality between P and the line containing x and y, we can write

$$\|x - h\| = (\|x - h_X\|^2 + \|h_X - h\|^2)^{1/2} \geqslant (\|x - h_X\|^2 + \|h_X - \widetilde{h}_X\|^2)^{1/2} = \|x - \widetilde{h}_X\|$$

and, similarly $\|y - h\| \geqslant \|y - \widetilde{h}_X\|$. Hence, we get $d_\infty((x,y),(h,h)) \geqslant d_\infty\big((x,y),\Pi(X)\big)$ and taking infimum for $h \in S_H$ we obtain the desired inequality. Thus, we can suppose that H is 2-dimensional and we can identify it with $(\mathbb{R}^2, \|\cdot\|_2)$.

Set $\widetilde{x} = \frac{x}{\|x\|}$ and $\widetilde{y} = \frac{y}{\|y\|}$. Of the two points in S_H whose distances to x and y are equal, let m be the one that minimizes that distance. We claim that $d_\infty\big((x,y),\Pi(X)\big)$ is attained at one of the three pairs $(\widetilde{x},\widetilde{x}), (\widetilde{y},\widetilde{y})$ or (m,m). Indeed, for $h \in S_H$ denote $f_x(h) = \|x - h\|$, $f_y(h) = \|y - h\|$, and $f(h) = \max\{f_x(h), f_y(h)\}$. It is clear that f attains its minimum, say that it does at $h_0 \in S_H$. Then h_0 must be either a point of local minimum of f_x, or a point of local minimum of f_y, or it satisfies $f_x(h_0) = f_y(h_0)$. Lemma 2.4.5 tells us that the only local minimum for f_x is $\widetilde{x}$ and the only local minimum for f_y is $\widetilde{y}$. So h_0 must one of the following four points: $\widetilde{x}$, $\widetilde{y}$, m and the remaining point p of S_H whose distances to x and y are equal, but for sure $f(p)$ is not the minimal value, so we omit this possibility.

To obtain the value of $d_\infty\big((x,y),\Pi(X)\big)$, we have to determine which is the suitable pair among $(\widetilde{x},\widetilde{x})$, $(\widetilde{y},\widetilde{y})$, and (m,m). We distinguish two cases depending on the value of $\langle x,y\rangle$:

If $\langle x,y\rangle \geqslant \|y\|^2 + \|y\|\frac{\|x\|^2 - \|y\|^2}{2}$ then

$$\|y - \widetilde{y}\|^2 = (1 - \|y\|)^2 \geqslant \|x\|^2 + 1 - \frac{2}{\|y\|}\langle x,y\rangle = \|x - \widetilde{y}\|^2$$

which gives us that $\|y-\widetilde{y}\| \geqslant \|x-\widetilde{y}\|$, and so $d_\infty\big((x,y),(\widetilde{y},\widetilde{y})\big) = \|y-\widetilde{y}\| = 1-\|y\|$. On the other hand, Remark 2.4.6 tells us that $\|y-\widetilde{y}\| = \operatorname{dist}(y,S_H)$. Therefore, we can write

$$d_\infty\big((x,y),(\widetilde{y},\widetilde{y})\big) = \|y-\widetilde{y}\| = \operatorname{dist}(y,S_H) \leqslant d_\infty((x,y),\Pi(H)) \leqslant d_\infty\big((x,y),(\widetilde{y},\widetilde{y})\big),$$

finishing the proof in this case.

Suppose otherwise that $\langle x,y\rangle < \|y\|^2 + \|y\|\frac{\|x\|^2-\|y\|^2}{2}$. Then we obtain $\|y-\widetilde{y}\| < \|x-\widetilde{y}\|$, and thus $d_\infty\big((x,y),(\widetilde{y},\widetilde{y})\big) = \|x-\widetilde{y}\|$. We observe that since $\|y\| \leqslant \|x\|$ and $\|x\|+\|y\| \leqslant 2$ we also have

$$\langle x,y\rangle < \|y\|^2 + \|y\|\frac{\|x\|^2-\|y\|^2}{2} \leqslant \|x\|^2 + \|x\|\frac{\|y\|^2-\|x\|^2}{2}.$$

Hence, we can deduce analogously that $\|x-\widetilde{x}\| < \|y-\widetilde{x}\|$ and so $d_\infty\big((x,y),(\widetilde{x},\widetilde{x})\big) = \|y-\widetilde{x}\|$.

Let us check that this time one has $d_\infty\big((x,y),(m,m)\big) \leqslant \min\big\{\|x-\widetilde{y}\|, \|y-\widetilde{x}\|\big\}$ and so (m,m) is the suitable pair. We start observing that, up to a rotation, we can assume without loss of generality that

$$x = (a_x\cos(\alpha_x), a_x\sin(\alpha_x)) \qquad \text{and} \qquad y = (a_y\cos(\alpha_y), a_y\sin(\alpha_y)),$$

where $a_x > 0$, $a_y > 0$, $\alpha_x, \alpha_y \in [0,\pi]$, and $\alpha_x \leqslant \alpha_y$. Then, by Lemma 2.4.5, the function $f_x : [\alpha_x, \alpha_y] \longrightarrow \mathbb{R}$ given by

$$f_x(\alpha) = \|(a_x\cos(\alpha_x), a_x\sin(\alpha_x)) - (\cos(\alpha),\sin(\alpha))\|$$

is increasing and the function $f_y : [\alpha_x, \alpha_y] \longrightarrow \mathbb{R}$ given by

$$f_y(\alpha) = \|(a_y\cos(\alpha_y), a_y\sin(\alpha_y)) - (\cos(\alpha),\sin(\alpha))\|$$

is decreasing. Besides, we have that

$$\begin{aligned} f_x(\alpha_x) &= \|x-\widetilde{x}\| < \|y-\widetilde{x}\| = f_y(\alpha_x) \quad \text{and} \\ f_y(\alpha_y) &= \|y-\widetilde{y}\| < \|x-\widetilde{y}\| = f_x(\alpha_y). \end{aligned}$$

So there is $\alpha_1 \in (\alpha_x, \alpha_y)$ satisfying $f_x(\alpha_1) = f_y(\alpha_1)$. Obviously one has that $m = (\cos(\alpha_1), \sin(\alpha_1))$,

$$\|x - m\| = f_x(\alpha_1) = f_y(\alpha_1) < f_y(\alpha_x) = \|y - \widetilde{x}\|, \quad \text{and}$$
$$\|y - m\| = f_x(\alpha_1) < f_x(\alpha_y) = \|x - \widetilde{y}\|.$$

We finish the proof computing $d_\infty\big((x,y),(m,m)\big)$. To this end, we write $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (y_2 - x_2, x_1 - y_1)$ which is orthogonal to $x - y$ and obviously satisfies $\|z\| = \|x - y\|$. We can assume without loss of generality (exchanging z by $-z$ if necessary) that $\langle x - y, z\rangle \geqslant 0$. With this notation we can write $m = \frac{x+y}{2} + \lambda z$ for suitable λ that we have to compute. Since m must be in S_H we obtain the following equation for λ :

$$1 = \|m\|^2 = \frac{\|x+y\|^2}{4} + \lambda\langle x+y, z\rangle + \lambda^2\|x-y\|^2.$$

Besides, observe that

$$\langle x+y, z\rangle^2 = 4(x_1y_2 - x_2y_1)^2 = 4(\|x\|^2\|y\|^2 - \langle x,y\rangle^2)$$

and, therefore,

$$1 = \frac{\|x+y\|^2}{4} + 2\lambda\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2} + \lambda^2\|x-y\|^2. \tag{2.5}$$

Observe further that

$$\|y-m\|^2 = \|x-m\|^2 = \left\|\frac{x-y}{2} + \lambda z\right\|^2 = \frac{\|x-y\|^2}{4} + \lambda^2\|x-y\|^2.$$

Hence, we have to pick λ to be the solution of (2.5) which has smaller modulus, that is:

$$\lambda = \frac{-2\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2} + \sqrt{4\big(\|x\|^2\|y\|^2 - \langle x,y\rangle^2\big) + 4\|x-y\|^2 - \|x+y\|^2\|x-y\|^2}}{2\|x-y\|^2}.$$

Taking into account that

$$\begin{aligned}\|x+y\|^2\|x-y\|^2 &= \big(\|x\|^2 + \|y\|^2 + 2\langle x,y\rangle\big)\big(\|x\|^2 + \|y\|^2 - 2\langle x,y\rangle\big)\\ &= \big(\|x\|^2 + \|y\|^2\big)^2 - 4\langle x,y\rangle^2\end{aligned}$$

we get

$$4\big(\|x\|^2\|y\|^2 - \langle x,y\rangle^2\big) - \|x+y\|^2\|x-y\|^2 = -\big(\|x\|^2 - \|y\|^2\big)^2.$$

This, together with $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle$, gives the expected value for λ. Finally, using (2.5) we obtain

$$\begin{aligned} d_\infty\big((x,y),(m,m)\big) = \|x-m\| &= \sqrt{\frac{\|x-y\|^2}{4} + \lambda^2\|x-y\|^2} \\ &= \sqrt{\tfrac{\|x-y\|^2}{4} + 1 - \tfrac{\|x+y\|^2}{4} - 2\lambda\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}} \\ &= \sqrt{1 - \langle x,y\rangle - 2\lambda\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}} \end{aligned}$$

which finishes the proof. □

2.4.7 Corollary. *Let H be a real Hilbert space with* $\dim(H) \geqslant 2$*, $\delta \in (0,2)$, and $\mu,\theta \in [0,1]$ satisfying $\mu \geqslant \theta$ and $\mu\theta > 1-\delta$. Then,*

$$\Phi_H(\mu,\theta,\delta) = \begin{cases} 1-\theta & \text{if } 1-\delta \geqslant \theta^2 + \theta\frac{\mu^2-\theta^2}{2}, \\ \max\left\{1-\theta, \sqrt{\delta - 2\lambda_\delta\sqrt{\mu^2\theta^2-(1-\delta)^2}}\right\} & \text{if } 1-\delta < \theta^2 + \theta\frac{\mu^2-\theta^2}{2}, \end{cases}$$

where

$$\lambda_\delta = \frac{-2\sqrt{\mu^2\theta^2-(1-\delta)^2} + \sqrt{4\big(\mu^2+\theta^2-2+2\delta\big) - (\mu^2-\theta^2)^2}}{2\big(\mu^2+\theta^2-2+2\delta\big)}.$$

Proof. Suppose first that $1-\delta \geqslant \theta^2 + \theta\frac{\mu^2-\theta^2}{2}$ and fix an arbitrary pair $(x,y) \in H\times H$ with $\|x\| = \mu$, $\|y\| = \theta$ and $\langle x,y\rangle \geqslant 1-\delta$. Then, $\langle x,y\rangle \geqslant \theta^2 + \theta\frac{\mu^2-\theta^2}{2}$ and Theorem 2.4.4 gives $d_\infty\big((x,y),\Pi(H)\big) = 1-\theta$, taking supremum in (x,y) we obtain $\Phi_H(\mu,\theta,\delta) \leqslant 1-\theta$. The reversed inequality always holds by Remark 2.2.2.

Suppose now that $1-\delta < \theta^2 + \theta\frac{\mu^2-\theta^2}{2}$. As we observed at the beginning of the proof of Theorem 2.4.4, we can suppose that $\dim(H) = 2$ and so we can assume the identification $H = (\mathbb{R}^2, \|\cdot\|_2)$. Fix an arbitrary pair $(x,y) \in H\times H$ with $\|x\| = \mu$, $\|y\| = \theta$ and $\langle x,y\rangle \geqslant 1-\delta$. Renaming x and y if necessary and using a suitable rotation, we can suppose without loss of generality that $x = (\mu, 0)$ and $y = \theta(\cos(\alpha), \sin(\alpha))$ with $\alpha \in [0, \pi]$. Let $\alpha_1 \in [0, \pi]$ be so that the point $z = \theta(\cos(\alpha_1), \sin(\alpha_1))$ satisfies $\langle x, z\rangle = \mu\theta\cos(\alpha_1) = 1 - \delta$. Observe that, in fact, one has $\alpha \in [0, \alpha_1]$.

Next, we write $\varepsilon = \max\left\{1-\theta, \sqrt{\delta - 2\lambda_\delta\sqrt{\mu^2\theta^2 - (1-\delta)^2}}\right\}$ and we use Theorem 2.4.4 for x and z to obtain

$$d_\infty\big((x,z), \Pi(H)\big) = \sqrt{\delta - 2\lambda_\delta\sqrt{\mu^2\theta^2 - (1-\delta)^2}}.$$

Let $\alpha_2 \in [0, \pi]$ be so that the point $m = (\cos(\alpha_2), \sin(\alpha_2))$ satisfies

$$\|x - m\| = \|z - m\| = \sqrt{\delta - 2\lambda\sqrt{\mu^2\theta^2 - (1-\delta)^2}}\,.$$

If $\alpha \in [0, \alpha_2]$ then we can use Lemma 2.4.5 with $\alpha_0 = 0$, $a = \mu$, and $b = 1$ to obtain that $\widetilde{y} = (\cos(\alpha), \sin(\alpha))$ satisfies

$$\|x - \widetilde{y}\| = \big\|(\mu, 0) - (\cos(\alpha), \sin(\alpha))\big\| \leqslant \big\|(\mu, 0) - (\cos(\alpha_2), \sin(\alpha_2))\big\| = \|x - m\|$$

and, therefore

$$d_\infty\big((x,y), \Pi(H)\big) \leqslant \max\big\{\|x - \widetilde{y}\|, \|y - \widetilde{y}\|\big\} \leqslant \max\big\{\|x - m\|, 1 - \theta\big\} = \varepsilon.$$

If $\alpha \in [\alpha_2, \alpha_1]$ (obviously this case does not occur when $\alpha_2 > \alpha_1$), we use Lemma 2.4.5 with $\alpha_0 = \alpha_2$, $a = 1$, and $b = \theta$ to obtain that

$$\begin{aligned}\|m - y\| &= \big\|(\cos(\alpha_2), \sin(\alpha_2)) - (\theta\cos(\alpha), \theta\sin(\alpha))\big\| \\ &\leqslant \big\|(\cos(\alpha_2), \sin(\alpha_2)) - (\theta\cos(\alpha_1), \theta\sin(\alpha_1))\big\| = \|m - z\|.\end{aligned}$$

This allows us to write

$$d_\infty\big((x,y), \Pi(H)\big) \leqslant \max\big\{\|x - m\|, \|y - m\|\big\} \leqslant \max\big\{\|x - m\|, \|z - m\|\big\} \leqslant \varepsilon.$$

So, for every $(x, y) \in H \times H$ with $\|x\| = \mu$, $\|y\| = \theta$ and $\langle x, y\rangle \geqslant 1 - \delta$ we have $d_\infty\big((x,y), \Pi(H)\big) \leqslant \varepsilon$ and, therefore, $\Phi_H(\mu, \theta, \delta) \leqslant \varepsilon$. To prove the reversed inequality, it suffices to recall that $\Phi_H(\mu, \theta, \delta) \geqslant 1 - \theta$ always holds and that $\Phi_H(\mu, \theta, \delta) \geqslant d_\infty\big((x,z), \Pi(H)\big) = \sqrt{\delta - 2\lambda_\delta\sqrt{\mu^2\theta^2 - (1-\delta)^2}}$. □

With some more work, we can compute the Bishop-Phelps-Bollobás moduli of a Hilbert space as a consequence of the above results.

2.4.8 Example. Let H be a Hilbert space with dimension over $\mathbb{R}$ greater than or equal to two. Then:

(a) $\Phi_H^S(\delta) = \sqrt{2-\sqrt{4-2\delta}}$ for every $\delta \in (0,2)$.

(b) For $\delta \in (0,1]$, $\Phi_H(\delta) = \max\left\{\delta, \sqrt{2-\sqrt{4-2\delta}}\right\}$.
For $\delta \in (1,2)$, $\Phi_H(\delta) = \sqrt{\delta}$.

Proof. (a). Using Corollary 2.6.4 for $\mu = \theta = 1$, we get $\Phi_H(1,1,\delta) = \sqrt{\delta - 2\lambda_\delta\sqrt{2\delta-\delta^2}}$ where $\lambda_\delta = \frac{\sqrt{2\delta}-\sqrt{2\delta-\delta^2}}{2\delta}$ and, therefore,

$$\Phi_H^S(\delta) = \Phi_H(1,1,\delta) = \sqrt{2-\sqrt{4-2\delta}}.$$

We will use the following claim in the proof of (b).

Claim: Given $x, y \in S_H$ with $x+y \neq 0$, write $z = \frac{x+y}{\|x+y\|}$ to denote the normalized midpoint. Then

$$\|x-z\| = \|y-z\| = \sqrt{2-\sqrt{2+2\langle x,y\rangle}}.$$

Indeed, we have $\|x-z\|^2 = 2 - 2\langle x,z\rangle$ and

$$2\langle x,z\rangle = \frac{2\langle x, x+y\rangle}{\|x+y\|} = \frac{2+2\langle x,y\rangle}{\sqrt{2+2\langle x,y\rangle}},$$

giving $\|x-z\| = \sqrt{2-\sqrt{2+2\langle x,y\rangle}}$, being the other equality true by symmetry.

(b). We first fix $\delta \in (0,1)$ and write $\varepsilon_0 = \max\left\{\delta, \sqrt{2-\sqrt{4-2\delta}}\right\}$. The inequality $\Phi_H(\delta) \geqslant \varepsilon_0$ follows from the fact that $\Phi_H(\delta) \geqslant \Phi_H^S(\delta)$, the result in item (a) and Remark 2.4.1. To get the reversed inequality, we could try to optimize the function $\Phi_H(\mu,\theta,\delta)$ in $\mu,\theta \in [0,1]$ but the difficulties in this optimization are relevant. For that reason, we give an alternative proof. We first observe that

$$\Phi_H(\delta) \leqslant \Phi_{\operatorname{lin}\{x,y\}}(\delta) \quad \forall x,y \in B_H \text{ with } \langle x,y\rangle = 1-\delta. \tag{2.6}$$

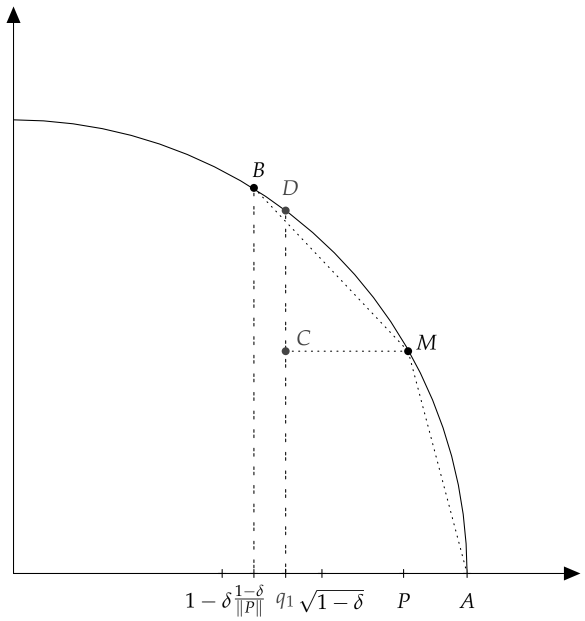


Figure 2.1: Calculating $\Phi_H(\delta)$ for $\delta \in (0,1)$

This follows from the obvious fact that $\Phi_\cdot(\delta)$ increases when we restrict to subspaces. This implies that it is enough to show that for

$$P = (p_1, 0), Q = (q_1, q_2) \in B_{\ell_2^2}$$

such that $p_1, q_2 > 0$, $\|P\| > \|Q\|$, and $\langle P, Q\rangle \geqslant 1-\delta$ where ℓ_2^2 is the 2-dimensional Hilbert space, there exists $z \in S_{\ell_2^2}$ so that $\|P - z\| \leqslant \varepsilon_0$ and $\|Q - z\| \leqslant \varepsilon_0$. Now, it is straightforward to check that we have

$$\|P\| \in \left[\sqrt{1-\delta}, 1\right] \text{ and } q_1 = \frac{1-\delta}{\|P\|} \in \left[1-\delta, \sqrt{1-\delta}\right].$$

Figure 2.1 helps to the better understanding of the rest of the proof.

Consider $M = \left(\sqrt{\frac{1-\delta+\|P\|}{2\|P\|}}, \sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}\right)$, which is the normalized midpoint between $A = (1,0)$ and $B = \left(\frac{1-\delta}{\|P\|}, \sqrt{1 - (\frac{1-\delta}{\|P\|})^2}\right)$ and write Δ to denote the arc of the unit sphere of H between A and M. We claim that $Q \in \bigcup_{z\in\Delta} B(z, \varepsilon_0)$ and $P \in \bigcap_{z\in\Delta} B(z, \varepsilon_0)$.

Observe that this gives that there is $z \in \Delta \subset S_H$ whose distance to P and Q is less than or equal to ε_0, finishing the proof. Let us prove the claim. First, we show that $Q = (q_1, q_2) \in \bigcup_{z\in\Delta} B(z, \varepsilon_0)$. If $q_2 \leqslant \sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}$, the ball of radius ε_0 centered at the point of Δ with second coordinate equal to q_2 contains the point Q since $\varepsilon_0 \geqslant \operatorname{dist}((q_1, 0), A) \geqslant \operatorname{dist}(Q, \Delta)$. For greater values of q_2, write first $C = \left(q_1, \sqrt{\frac{\|P\|-(1-\delta)}{2\|P\|}}\right)$, which belongs to $B(M, \varepsilon_0)$ by the previous argument. Also, as M is the normalized mid point between A and B, we have by the claim at the beginning of this proof that

$$\|M - B\| = \sqrt{2 - \sqrt{2 + 2\langle A, B\rangle}} = \sqrt{2 - \sqrt{2 + 2\frac{1-\delta}{\|P\|}}} \leqslant \sqrt{2 - \sqrt{4 - 2\delta}} \leqslant \varepsilon_0$$

so, also, $\|M - D\| \leqslant \varepsilon_0$. Therefore, both points C and D belong to $B(M, \varepsilon_0)$, so also the whole segment $[C, D]$ is contained there, and this proves the first part of the claim. To show the second part of the claim, that $P \in \bigcap_{z\in\Delta} B(z, \varepsilon_0)$, we consider the function

$$f(p) := 1 + p^2 - \sqrt{2p(p + 1 - \delta)} \qquad (p \in [\sqrt{1-\delta}, 1])$$

and observe that it is a convex function, so

$$f(p) \leqslant \max\{f(1), f(\sqrt{1-\delta}\} \leqslant \varepsilon_0^2.$$

It follows that

$$\|P - M\| = \sqrt{1 + \|P\|^2 - \sqrt{2\|P\|(\|P\| + 1 - \delta)}} \leqslant \varepsilon_0,$$

hence $M \in B(P, \varepsilon_0)$. As also $A \in B(P, \varepsilon_0)$, it follows that the whole circular arc Δ is contained in $B(P, \varepsilon_0)$ or, equivalently, that $P \in \bigcap_{z\in\Delta} B(z, \varepsilon_0)$.

Now, we fix $\delta \in (1, 2)$. Analogously to what we did before in equation (2.6), to show that $\Phi_H(\delta) \leqslant \sqrt{\delta}$, it is enough to consider the two-dimensional case and, given $P = (\|P\|, 0) \in B_H$, $Q = (q_1, q_2) \in B_H$ with $q_2 \geqslant 0$, to find $z \in S_H$ such that

$$\|z - P\| \leqslant \sqrt{\delta}, \qquad \|z - Q\| \leqslant \sqrt{\delta}.$$

Routine computations show that

$$z = \left(\frac{\|P\| + q_1}{2}, \sqrt{1 - (\frac{\|P\| + q_1}{2})^2}\right) \in S_H$$

does the job. For the other inequality, we fix an orthonormal basis $\{e_1, e_2, \dots\}$ of H, consider

$$P = \sqrt{\delta - 1}\, e_1 \in B_H, \qquad Q = -\sqrt{\delta - 1}\, e_1 \in B_H$$

and observe that $\langle P, Q\rangle = 1 - \delta$. For any $z \in S_H$, we write $z_1 = \langle z, e_1\rangle$ and we compute

$$\begin{aligned}
\max\{\|z-P\|^2, \|z-Q\|^2\} = &\\
&= \max\left\{|z_1 - \sqrt{\delta-1}|^2 + 1 - |z_1|^2,\ |z_1 + \sqrt{\delta-1}|^2 + 1 - |z_1|^2\right\}\\
&= \max_{\pm} |z_1 \pm \sqrt{\delta-1}|^2 + 1 - |z_1|^2 = (|z_1| + \sqrt{\delta-1})^2 + 1 - |z_1|^2\\
&= \delta + 2\sqrt{\delta-1}|z_1| \geqslant \delta.
\end{aligned}$$

It follows that $\Phi_H(\delta) \geqslant \sqrt{\delta}$, as desired. □

Our next aim is to present a number of examples for which the values of the Bishop-Phelps-Bollobás moduli are the maximum possible, namely $\Phi_X^S(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ for small δ's. As we always have $\Phi_X^S(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}$, it is enough if we prove the formally stronger result that $\Phi_X^S(\delta) = \sqrt{2\delta}$ for small δ's. Actually, the two facts are equivalent as we proved in Proposition 2.3.5. Moreover, we will actually show that $\Phi_X(\mu, \theta, \delta)$ is maximum for many values of the parameters.

The first result is about Banach spaces admitting an L-decomposition. As a consequence we will obtain the moduli of the $L_1(\mu)$ spaces.

2.4.9 Proposition. *Let X be a Banach space. Suppose that there are two (non-trivial) subspaces Y and Z such that $X = Y \oplus_1 Z$. Let $\delta \in (0,1)$ and $\mu, \theta \in [0,1]$ with $1 - \delta < \mu\theta \leqslant 2(1-\delta)$. Then, there exists a pair $(x_0, x_0^*) \in X \times X^*$ with $\|x_0\| = \mu$, $\|x_0^*\| = \theta$ and $\operatorname{Re} x_0^*(x_0) \geqslant 1 - \delta$ satisfying*

$$d_\infty\big((x_0, x_0^*), \Pi(X)\big) = \Psi(\mu, \theta, \delta).$$

Therefore, $\Phi_X(\mu, \theta, \delta) = \Psi(\mu, \theta, \delta)$ for the cited values of δ, μ, θ.

Proof. Since $0 < 1-\delta < \mu\theta$ we get that $\mu > 0$, so we can take

$$k = \frac{\mu - \theta + \sqrt{(\mu-\theta)^2 + 8(\mu\theta - 1 + \delta)}}{4\mu}$$

which satisfies $0 \leqslant k \leqslant 1$ because $\delta < 1$. Next, we fix pairs $(y_0, y_0^*) \in \Pi(Y)$ and $(z_0, z_0^*) \in \Pi(Z)$, and we define

$$x_0 = (\mu k y_0, \mu(1-k)z_0) \qquad \text{and} \qquad x_0^* = ((1-\Psi(\mu,\theta,\delta))y_0^*, \theta z_0^*)\,.$$

The facts $|1-\Psi(\mu,\theta,\delta)| \leqslant \theta$ and $0 \leqslant k \leqslant 1$ imply that $\|x_0\| = \mu$, $\|x_0^*\| = \theta$. Moreover, we can write

$$\begin{aligned}\operatorname{Re} x_0^*(x_0) &= \mu\theta + (1-\theta-\Psi(\mu,\theta,\delta))\mu k \\ &= \mu\theta + \tfrac{\mu-\theta-\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{2}\ \tfrac{\mu-\theta+\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{4} = 1-\delta.\end{aligned}$$

Given $(x,x^*) \in \Pi(X)$, write $x = (y,z) \in Y \oplus_1 Z$, $x^* = (y^*, z^*) \in Y \oplus_\infty Z$ and observe that

$$\|y\| + \|z\| = 1 = \operatorname{Re} x^*(x) = \operatorname{Re} y^*(y) + \operatorname{Re} z^*(z) \leqslant \|y^*\|\|y\| + \|z^*\|\|z\|. \tag{2.7}$$

If it holds $\|(1-\Psi(\mu,\theta,\delta))y_0^* - y^*\| \geqslant \Psi(\mu,\theta,\delta)$ then $\|x_0^* - x^*\| \geqslant \Psi(\mu,\theta,\delta)$ and we are done. If otherwise we have that $\|(1-\Psi(\mu,\theta,\delta))y_0^* - y^*\| < \Psi(\mu,\theta,\delta)$ then we can write

$$\big||1-\Psi(\mu,\theta,\delta)| - \|y^*\|\big| \leqslant \|(1-\Psi(\mu,\theta,\delta))y_0^* - y^*\| < \Psi(\mu,\theta,\delta).$$

Now the hypothesis $\mu\theta \leqslant 2(1-\delta)$ gives us that $1-\Psi(\mu,\theta,\delta) \geqslant 0$ and hence

$$\big|1-\Psi(\mu,\theta,\delta) - \|y^*\|\big| < \Psi(\mu,\theta,\delta).$$

From this it follows that $\|y^*\| < 1$ and so, $y = 0$ by (2.7), giving $\|z\| = 1$. But then

$$\begin{aligned}\|x_0 - x\| &= k\mu\|y_0\| + \|\mu(1-k)z_0 - z\| \\ &\geqslant k\mu + \big|\mu(1-k) - \|z\|\big| = (2k-1)\mu + 1 = \Psi(\mu,\theta,\delta)\end{aligned}$$

which finishes the proof. □

In the hypotheses of the above result we have that

$$\Phi_X^S(\delta) = \Phi_X(1,1,\delta) = \Psi(1,1,\delta) = \sqrt{2\delta}$$

for every $\delta \in (0,1/2]$, which gives the value of the Bishop-Phelps-Bollobás moduli for every space admitting an L-decomposition.

2.4.10 Corollary. *Let X be a Banach space. Suppose that there are two (non-trivial) subspaces Y and Z such that $X = Y \oplus_1 Z$. Then $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.*

As a consequence we obtain the value of the moduli for $L_1(\mu)$ spaces.

2.4.11 Example. *Let (Ω,Σ,ν) be a measure space such that $L_1(\nu)$ has dimension greater than one and let E be any non-zero Banach space. Then, $\Phi_{L_1(\nu,E)}(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_{L_1(\nu,E)}(\delta) = \Phi^S_{L_1(\nu,E)}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.*
Indeed, we may find two measurable sets $A,B \subset \Omega$ with empty intersection such that $\Omega = A \cup B$. Then $Y = L_1(\nu|_A, E)$ and $Z = L_1(\nu|_B, E)$ are non-null, $L_1(\nu,E) = Y \oplus_1 Z$ and so the result follows from Proposition 2.4.9.

As it may be expected, a dual argument to the one given in Proposition 2.4.9 allows us to deduce an analogue result for a Banach space which decomposes as an ℓ_∞-sum. Actually, in this case we will get a better result using ideals instead of subspaces. Given a Banach space X we will write w^* to denote the weak*-topology $\sigma(X^*,X)$ of X^*.

2.4.12 Proposition. *Let X be a Banach space. Suppose that $X^* = Y \oplus_1 Z$ where Y and Z are (non-trivial) subspaces of X^* such that $\overline{Y}^{w^*} \neq X^*$ and $\overline{Z}^{w^*} \neq X^*$. Let $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. Then, there exists a pair $(x_0,x_0^*) \in X \times X^*$ with $\|x_0\| = \mu$, $\|x_0^*\| = \theta$ and $\operatorname{Re} x_0^*(x_0) \geqslant 1-\delta$ satisfying*

$$d_\infty\big((x_0,x_0^*),\Pi(X)\big) = \Psi(\mu,\theta,\delta).$$

Therefore, $\Phi_X(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ *for the cited values of* δ, μ, θ. *In particular,* $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ *for every* $\delta \in (0,1/2]$.

Proof. Since $0 < 1-\delta < \mu\theta$ we get that $\theta > 0$, so we can take

$$k = \frac{\theta - \mu + \sqrt{(\mu-\theta)^2 + 8(\mu\theta - 1 + \delta)}}{4\theta}$$

which satisfies $0 \leqslant k \leqslant 1$ because $\delta < 1$.

We claim that there are $y_0, z_0 \in S_X$, $y_0^* \in S_Y$, and $z_0^* \in S_Z$ such that

$$\operatorname{Re} y_0^*(y_0) = 1, \quad \operatorname{Re} z_0^*(z_0) = 1, \quad y^*(z_0) = 0 \ \forall y^* \in Y, \quad z^*(y_0) = 0 \ \forall z^* \in Z.$$

Indeed, we define y_0 and y_0^*, being z_0 and z_0^* analogous. By assumption there is $y_0 \in S_X$ such that $z^*(y_0) = 0$ for every $z^* \in Z$ and we may choose $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(y_0) = 1$ and we only have to prove that $x^* \in Y$ and then write $y_0^* = x^*$. But we have $x^* = y^* + z^*$ with $y^* \in Y$, $z^* \in Z$ and

$$1 = \operatorname{Re} x^*(y_0) = \operatorname{Re} y^*(y_0) \leqslant \|y^*\| \leqslant \|y^*\| + \|z^*\| = 1,$$

so $z^* = 0$ and $x^* \in Y$.

We now define

$$x_0^* = (k\theta y_0^*, (1-k)\theta z_0^*) \in X^*, \qquad x_0 = (1 - \Psi(\mu,\theta,\delta))y_0 + \mu z_0 \in X$$

and first we observe that

$$\begin{aligned}\operatorname{Re} x_0^*(x_0) &= \mu\theta + (1 - \mu - \Psi(\mu,\theta,\delta))\theta k \\ &= \mu\theta + \tfrac{\theta-\mu-\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{2}\ \tfrac{\theta-\mu+\sqrt{(\mu-\theta)^2+8(\mu\theta-1+\delta)}}{4} = 1-\delta.\end{aligned}$$

Besides, since $0 \leqslant k \leqslant 1$, it is clear that $\|x_0^*\| = \theta$. Let us check that $\|x_0\| = \mu$. Indeed, using the fact that $|1 - \Psi(\mu,\theta,\delta)| \leqslant \mu$, for every $x^* = y^* + z^* \in S_{X^*}$ one has

$$\begin{aligned}|x^*(x_0)| &= |(y^* + z^*)((1 - \Psi(\mu,\theta,\delta))y_0 + \mu z_0)| \\ &\leqslant |1 - \Psi(\mu,\theta,\delta)|\|y^*\| + \mu\|z^*\| \leqslant \mu(\|y^*\| + \|z^*\|) = \mu.\end{aligned}$$

This, together with $|z_0^*(x_0)| = \mu$, gives $\|x_0\| = \mu$.

Let $(x, x^*) \in \Pi(X)$. We consider the semi-norm $\|\cdot\|_Y$ defined on X by $\|x\|_Y := \sup\{|y^*(x)| \,:\, y^* \in S_{Y^*}\}$ which is smaller than or equal to the original norm, write $x^* = y^* + z^*$ with $y^* \in Y$ and $z^* \in Z$, and observe that

$$\|y^*\| + \|z^*\| = 1 = \operatorname{Re} x^*(x) = \operatorname{Re} y^*(x) + \operatorname{Re} z^*(x) \leqslant \|y^*\|\|x\|_Y + \|z^*\|\|x\|. \tag{2.8}$$

If $\|x_0 - x\|_Y \geqslant \Psi(\mu, \theta, \delta)$ we obviously have $\operatorname{dist}\big((x_0, x_0^*), \Pi(X)\big) \geqslant \Psi(\mu, \theta, \delta)$.

Otherwise $\|x_0 - x\|_Y < \Psi(\mu, \theta, \delta)$, and we can write

$$\big||1 - \Psi(\mu, \theta, \delta)| - \|x\|_Y\big| \leqslant \|(1 - \Psi(\mu, \theta, \delta))y_0 - x\|_Y = \|x_0 - x\|_Y < \Psi(\mu, \theta, \delta).$$

Now, the hypothesis $\mu\theta \leqslant 2(1 - \delta)$ gives us that $1 - \Psi(\mu, \theta, \delta) \geqslant 0$ and hence

$$|1 - \Psi(\mu, \theta, \delta) - \|x\|_Y| < \Psi(\mu, \theta, \delta).$$

From this it follows that $\|x\|_Y < 1$ and so, $y^* = 0$ by (2.8), giving $\|z^*\| = 1$. But then

$$\begin{aligned} \|x_0^* - x^*\| &= k\theta\|y_0^*\| + \|\theta(1 - k)z_0^* - z^*\| \\ &\geqslant k\theta + \big|\theta(1 - k) - \|z^*\|\big| = (2k - 1)\theta + 1 = \Psi(\mu, \theta, \delta) \end{aligned}$$

which finishes the proof. □

As an immediate consequence, we obtain the mentioned result for Banach spaces which decompose as an ℓ_∞-sum of two non-trivial subspaces.

2.4.13 Corollary. *Let X be a Banach space. Suppose that there are two (non-trivial) subspaces Y and Z such that $X = Y \oplus_\infty Z$. Let $\delta \in (0, 1)$ and $\mu, \theta \in [0, 1]$ with $1 - \delta < \mu\theta \leqslant 2(1 - \delta)$. Then, there exists a pair $(x_0, x_0^*) \in X \times X^*$ with $\|x_0\| = \mu$, $\|x_0^*\| = \theta$ and $\operatorname{Re} x_0^*(x_0) \geqslant 1 - \delta$ satisfying*

$$d_\infty\big((x_0, x_0^*), \Pi(X)\big) = \Psi(\mu, \theta, \delta).$$

Therefore, $\Phi_X(\mu, \theta, \delta) = \Psi(\mu, \theta, \delta)$ for the cited values of δ, μ, θ. In particular, $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0, 1/2]$.

This corollary applies to vector-valued L_∞ spaces and vector-valued c_0 spaces.

2.4.14 Examples.

(a) Let (Ω, Σ, ν) be a measure space containing two disjoint measurable sets with positive measure and let X be a Banach space. Then, $\Phi_{L_\infty(\nu,X)}(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_{L_\infty(\nu,X)}(\delta) = \Phi^S_{L_\infty(\nu,X)}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.

(b) Let Γ be a set with at least two points and let X be a non-trivial Banach space. Then, $\Phi_{c_0(\Gamma,X)}(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_{c_0(\Gamma,X)}(\delta) = \Phi^S_{c_0(\Gamma,X)}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.

Moreover, Proposition 2.4.12 allows to get the result for vector-valued $C_0(L)$ spaces using the concept of M-ideal which we recall (we refer the reader to [24] for background on M-ideals).

2.4.15 Definition. Let X be a Banach space. Given a subspace J of X, J is called *M-ideal* if $J^\perp$ is an L-summand in X^*. In this case, we have that $X^* = J^\perp \oplus_1 J^\sharp$ where $J^\sharp = \{x^* \in X^* \, : \, \|x^*\| = \|x^*_{|_J}\|\} \equiv J^*$.

If X contains a non-trivial M-ideal J, one has $X^* = J^\perp \oplus_1 J^\sharp$. To apply Proposition 2.4.12 we need that $J^\sharp$ is not $\sigma(X^*,X)$-dense. Actually, $J^\sharp$ is not dense in X^* if and only if there is $x_0 \in X \setminus \{0\}$ such that $\|x_0 + y\| = \max\{\|x_0\|, \|y\|\}$ for every $y \in J$ (this is easy to verify and a proof can be found in [4]). Therefore, we get the following result.

2.4.16 Corollary. *Let X be a Banach space. Suppose that there is a non-trivial M-ideal J of X and a point $x_0 \in X \setminus \{0\}$ such that $\|x_0 + y\| = \max\{\|x_0\|, \|y\|\}$ for every $y \in J$. Then, $\Phi_X(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.*

The above corollary applies to non-trivial $C_0(L)$ spaces.

2.4.17 Example. *Let L be a locally compact Hausdorff topological space with at least two points and let E be any non-zero Banach space. Then, $\Phi_{C_0(L,E)}(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_{C_0(L,E)}(\delta) = \Phi^S_{C_0(L,E)}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.*
Indeed, we may find a non-empty non-dense open subset U of L and consider the subspace

$$J = \{f \in C_0(L,E) \,:\, f|_U = 0\},$$

which is an M-ideal of $C_0(L,E)$ by [24, Corollary VI.3.4] (use the simpler [24, Example I.1.4.a] for the scalar-valued case) and it is non-zero since $L \setminus U$ has non-empty interior. As U is open and non-empty, we may find a non-null function $x_0 \in C_0(L,E)$ whose support is contained in U. It follows that $\|x_0 + y\| = \max\{\|x_0\|, \|y\|\}$ for every $y \in J$ by disjointness of the supports.

A sufficient condition to be in the hypotheses of Corollary 2.4.16 is that a Banach space X contains two non-trivial M-ideals J_1 and J_2 such that $J_1 \cap J_2 = \{0\}$. In this case, J_1 and J_2 are complementary M-summands in $J_1 + J_2$ (see [24, Proposition I.1.17]). Let us comment that this is actually what happens in $C(K)$ when K has more than one point.

2.4.18 Corollary. *Let X be a Banach space. Suppose there are two non-trivial M-ideals J_1 and J_2 such that $J_1 \cap J_2 = \{0\}$. Then, $\Phi_X(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.*

A sufficient condition for a Banach space to have two non-intersecting M-ideals is that its centralizer is non-trivial (i.e. has dimension at least two). We are not going into details, but roughly speaking, the *centralizer* $Z(X)$ of a Banach space X is a closed subalgebra of $L(X)$ isometrically isomorphic to $C(K_X)$ where K_X is a Hausdorff topological space, and it is possible to see X as a $C(K_X)$-submodule of $\prod_{k\in K_X} X_k$ for suitable X_k's. We refer to [3, §3.B] and [24, §I.3] for the details. It happens that every M-ideal of $C(K_X)$ produces an M-ideal of X in a suitable way (see [3, §4.A]) and if $Z(X)$ contains more than one point, then two non-intersecting M-ideals appear in X, so our corollary above applies.

2.4.19 Corollary. *Let X a Banach space. If $Z(X)$ has dimension greater than one, then $\Phi_X(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.*

To give some new examples coming from this corollary, we recall that the centralizer of a unital (complex) C^*-algebra identifies with its center (see [24, Theorem V.4.7] or [3, Example 3 in page 63]).

2.4.20 Example. *Let A be a unital C^*-algebra with non-trivial center. Then, $\Phi_A(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ for $\delta \in (0,1)$ and $\mu,\theta \in [0,1]$ with $1-\delta < \mu\theta \leqslant 2(1-\delta)$. In particular, $\Phi_A(\delta) = \Phi_A^S(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$.*

None of the results of this section applies to $L(H)$, where H is a Hilbert space, since its center is trivial and, even though it contains $K(H)$ as an M-ideal, there is no element $x_0 \in L(H)$ satisfying the requirements of Corollary 2.4.16 (see [4, page 538]). Let us also comment that, when H is infinite-dimensional, the bidual of $L(H)$ is a C^*-algebra with non-trivial centralizer, so $\Phi_{L(H)^{**}}(\delta) = \Phi^S_{L(H)^{**}}(\delta) = \sqrt{2\delta}$ for every $\delta \in (0,1/2]$ (indeed, $L(H)$ is a C^*-algebra which contains a non-trivial M-ideal, so $L(H)^{**}$ contains weak-star non-trivial M-ideals which came from a central projection, see [24, Proposition V.4.5] for instance). To sum up, it would be interesting to see whether the algebra $L(H)$ for an infinite-dimensional Hilbert space H has the maximum Bishop-Phelps-Bollobás moduli. If there is $\delta \in (0,1/2]$ such that $\Phi_{L(H)}(\delta) < \sqrt{2\delta}$, then this would be an example in which the inequality in Proposition 2.3.8 is strict.

We finish this section showing two pictures: one with the Bishop-Phelps-Bollobás moduli of $\mathbb{R}$, $\mathbb{C}$, and ℓ_∞^2, and another one with the corresponding values of the spherical Bishop-Phelps-Bollobás moduli.

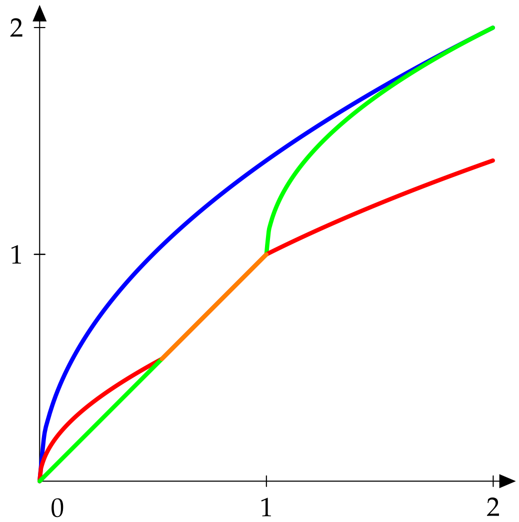


Figure 2.2: The value of $\Phi_X(\delta)$ for $\mathbb{R}$, $\mathbb{C}$, ℓ_∞^2

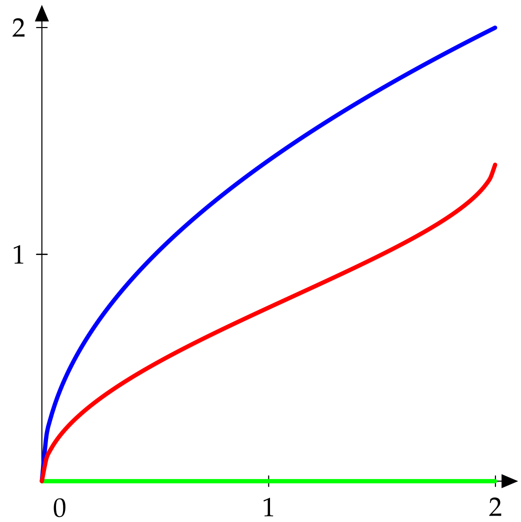


Figure 2.3: The value of $\Phi_X^S(\delta)$ for $\mathbb{R}$, $\mathbb{C}$, ℓ_∞^2

2.5 Banach spaces with the greatest possible moduli

The aim of this section is to show that a Banach space X with maximum value of $\Phi_X(\delta)$ for some $\delta \in (0,2)$ contains almost isometric copies of the **real** space $\ell_\infty^{(2)}$. For $\delta < 1/2$ this was proved, with a much more complicated proof, in [12, Theorem 5.8] and [11, Corollary 3.4] (in the last reference, the result is a consequence of a quantitative approach). We start recalling the definition of containment of almost isometric copies of a Banach space.

2.5.1 Definition. Let X, E be Banach spaces. *X contains almost isometric copies of E* if for every $\varepsilon > 0$ there exist a subspace $E_\varepsilon \subset X$ and a bijective linear operator $T : E \longrightarrow E_\varepsilon$ with $\|T\| < 1+\varepsilon$ and $\|T^{-1}\| < 1+\varepsilon$.

The unit ball of the real space $\ell_\infty^{(2)}$ is the square with the vertexes $u = (1,1)$, $v = (1,-1)$, $-u$ and $-v$, and the vertexes satisfy $\|u - v\| = \|u + v\| = 2$. The following easy result, which we state here for future use, is well known and follows from the above description of the shape of the unit ball of $\ell_\infty^{(2)}$.

2.5.2 Lemma. *Let X be a Banach space.*

(a) X contains the real space $\ell_\infty^{(2)}$ isometrically if and only if there are elements $u, v \in S_E$ such that $\|u - v\| = \|u + v\| = 2$.

(b) X contains almost isometric copies of the real space $\ell_\infty^{(2)}$ if and only if there are elements $u_n, v_n \in X$, $n \in \mathbb{N}$ such that

$$\lim \|u_n\| = \lim \|v_n\| = 1, \quad \liminf \|u_n - v_n\| \geqslant 2 \quad \text{and} \quad \liminf \|u_n + v_n\| \geqslant 2.$$

(c) X contains almost isometric copies of the real space $\ell_\infty^{(2)}$ if and only if X^* does (see [28, Corollary 2], for instance).

Our promised result can be stated as follows.

2.5.3 Theorem. *Let X be a Banach space and suppose that there is $\delta \in (0,2)$ satisfying $\Phi_X(\delta) = \sqrt{2\delta}$. Then, X contains almost isometric copies of the real space $\ell_\infty^{(2)}$.*

We need a couple of preliminary results. The first one is a sufficient condition for a Banach space to contain almost isometric copies of the real space $\ell_\infty^{(2)}$ which can be of independent interest.

2.5.4 Lemma. *Let X be a Banach space. Suppose that there exist $k \in (0,1)$ and two sequences (x_n) in S_X and (y_n) in $X \setminus \{0\}$ satisfying*

$$\limsup \|x_n - y_n\| \leqslant k \qquad \text{and} \qquad \liminf \left\| x_n - \frac{y_n}{\|y_n\|} \right\| \geqslant 2k.$$

Then X contains almost isometric copies of the real space $\ell_\infty^{(2)}$.

We will use the following result which is surely well known, but we include an elementary proof.

2.5.5 Remark. Let X be a Banach space, $k \in (0,1)$ and let (u_n), (v_n) sequences of elements of X such that

$$\limsup \|u_n\| \leqslant 1, \quad \limsup \|v_n\| \leqslant 1 \quad \text{and} \quad \liminf \|(1-k)u_n + kv_n\| \geqslant 1.$$

Then $\liminf \|u_n + v_n\| \geqslant 2$.

Proof. Write $m_n = \|(1-k)u_n + kv_n\|$, take $f_n \in S_{X^*}$ such that

$$\operatorname{Re} f_n\big((1-k)u_n + kv_n\big) = \big\|(1-k)u_n + kv_n\big\| = m_n,$$

and observe that $\limsup \operatorname{Re} f_n(u_n) \leqslant 1$ and $\limsup \operatorname{Re} f_n(v_n) \leqslant 1$. As

$$m_n = (1-k)\operatorname{Re} f_n(u_n) + k \operatorname{Re} f_n(v_n),$$

we have

$$\operatorname{Re} f_n(u_n) = \frac{1}{1-k}\big(m_n - k\operatorname{Re} f_n(v_n)\big) \quad \text{and} \quad f_n(v_n) = \frac{1}{k}\big(m_n - (1-k)\operatorname{Re} f_n(u_n)\big).$$

Now,

$$\liminf \operatorname{Re} f_n(u_n) \geqslant \frac{1}{1-k}\big(\liminf m_n - k \limsup \operatorname{Re} f_n(v_n)\big) \geqslant 1$$

and

$$\liminf \operatorname{Re} f_n(v_n) \geqslant \frac{1}{k}\big(\liminf m_n - (1-k)\limsup \operatorname{Re} f_n(u_n)\big) \geqslant 1.$$

Finally,

$$\begin{aligned}\liminf \|u_n + v_n\| &\geqslant \liminf \operatorname{Re} f_n(u_n + v_n)\\ &\geqslant \liminf \operatorname{Re} f_n(u_n) + \liminf \operatorname{Re} f_n(v_n) \geqslant 2.\end{aligned}$$ □

Proof of Lemma 2.6.5. Up to subsequences, we may and do suppose that

$$\lim \|x_n - y_n\| \leqslant k, \quad \lim \left\| x_n - \frac{y_n}{\|y_n\|} \right\| \geqslant 2k, \quad \text{and} \quad \exists \lim \|y_n\|.$$

We first observe that since (x_n) lies in S_X, using the triangle inequality we have that

$$\lim\big|1 - \|y_n\|\big| = \big|1 - \lim\|y_n\|\big| \leqslant k.$$

Now, we have

$$\begin{aligned}2k \leqslant \lim\left\|x_n - \frac{y_n}{\|y_n\|}\right\| &= \lim\frac{1}{\|y_n\|}\big\|\|y_n\|x_n - y_n\big\| \\ &= \lim\frac{1}{\|y_n\|}\big\|\|y_n\|(x_n - y_n) + (1 - \|y_n\|)y_n\big\| \\ &\leqslant \lim\|x_n - y_n\| + \lim\big|1 - \|y_n\|\big| \leqslant k + k = 2k.\end{aligned}$$

Hence, all the inequalities above are in fact equalities, and so we have

$$\lim\|x_n - y_n\| = k, \quad \lim\left\|x_n - \frac{y_n}{\|y_n\|}\right\| = 2k, \quad \text{and} \quad \big|1 - \lim\|y_n\|\big| = k. \tag{2.9}$$

Using Lemma 2.5.2, it is enough to find two sequences (u_n), (v_n) in X such that

$$\lim\|u_n\| = 1, \quad \lim\|v_n\| = 1, \quad \liminf\|u_n + v_n\| \geqslant 2, \quad \text{and} \quad \liminf\|u_n - v_n\| \geqslant 2.$$

We distinguish two cases depending on the values of $\lim\|y_n\|$. Suppose first that $\lim\|y_n\| = 1 - k$ and take

$$u_n = \frac{y_n}{1-k} \quad \text{and} \quad v_n = \frac{x_n - y_n}{k} \quad (n \in \mathbb{N}),$$

which satisfy that $\lim\|u_n\| = \lim\|v_n\| = 1$. We have $(1-k)u_n + kv_n = x_n \in S_X$, and we may apply Remark 2.5.5 to get that $\liminf\|u_n + v_n\| \geqslant 2$. On the other hand,

$$\begin{aligned}\|u_n - v_n\| &= \frac{1}{k}\|ku_n - (x_n - y_n)\| = \frac{1}{k}\|ku_n - x_n + (1-k)u_n\| \\ &= \frac{1}{k}\|u_n - x_n\| = \frac{1}{k}\left\|\frac{y_n}{1-k} - x_n\right\| \\ &\geqslant \frac{1}{k}\left(\left\|\frac{y_n}{\|y_n\|} - x_n\right\| - \left\|\frac{y_n}{1-k} - \frac{y_n}{\|y_n\|}\right\|\right) \longrightarrow 2.\end{aligned}$$

Therefore, $\liminf\|u_n - v_n\| \geqslant 2$. This finishes the proof in this case.

If, otherwise, $\lim \|y_n\| = 1 + k$, take

$$u_n = \frac{y_n}{1+k} \quad \text{and} \quad v_n = \frac{y_n - x_n}{k} \qquad (n \in \mathbb{N}),$$

which satisfy that $\lim \|u_n\| = \lim \|v_n\| = 1$. Observe that

$$\begin{aligned}\|u_n - v_n\| &= \frac{1}{k}\left\|k\frac{y_n}{1+k} + x_n - y_n\right\| = \frac{1}{k}\left\|x_n - \frac{y_n}{1+k}\right\| \\ &\geqslant \frac{1}{k}\left(\left\|x_n - \frac{y_n}{\|y_n\|}\right\| - \left\|\frac{y_n}{1+k} - \frac{y_n}{\|y_n\|}\right\|\right) \longrightarrow 2.\end{aligned}$$

Therefore, $\liminf \|u_n - v_n\| \geqslant 2$. On the other hand,

$$\begin{aligned}\big\|(1-k)u_n + kv_n\big\| &= \big\|(1-k)u_n + y_n - x_n\big\| = \big\|(1-k)u_n + (1+k)u_n - x_n\big\| \\ &= \big\|2u_n - x_n\big\| \geqslant 2\|u_n\| - \|x_n\| \longrightarrow 1,\end{aligned}$$

so $\liminf \big\|(1-k)u_n + kv_n\big\| \geqslant 1$ and we may apply Remark 2.5.5 to get the desired condition $\liminf \|u_n + v_n\| \geqslant 2$. □

Observe that if the sequences (x_n) and (y_n) in Lemma 2.6.5 are constant, what we get (with much easier proof) is an isometric copy of the real space $\ell_\infty^{(2)}$. Let us state this result.

2.5.6 Corollary. *Let X be a Banach space. Suppose that there are $x \in S_X$, $y \in X \setminus \{0\}$ and $k \in (0,1)$ satisfying*

$$\|x - y\| = k \qquad \text{and} \qquad \left\|x - \frac{y}{\|y\|}\right\| = 2k.$$

Then the real linear span of $\{x, y\}$ is isometrically isomorphic to the real space $\ell_\infty^{(2)}$.

We would like to mention that both Lemma 2.6.5 and Corollary 2.5.6 are false for $k = 0$ and $k = 1$. The case of $k = 0$ is immediate, as in every Banach space we may find unit vectors x, y satisfying the requirements of the corollary, and the corresponding constant sequences satisfy the requirements of the lemma. The case of $k = 1$ in the corollary cannot happen: if X is a Banach space, $x \in S_X$ and $y \in X \setminus \{0\}$ satisfy $\|x - y\| = 1$ and $\left\| x - \frac{y}{\|y\|} \right\| = 2$, it follows that $|1 - \|y\|| = 1$ (see equation (2.9)), so $\|y\| = 2$; but then

$$4 = \|2x - y\| \leqslant \|x\| + \|x - y\| \leqslant 2,$$

a contradiction. Finally, hypothesis of Lemma 2.6.5 for $k = 1$ are satisfied in every Banach space X. Indeed, fix $x \in S_X$ and consider $x_n = x \in S_X$ and $y_n = \frac{-1}{n} x \in X \setminus \{0\}$. Then, $\|x_n - y_n\| = 1 + \frac{1}{n}$ and $\left\| x_n - \frac{y_n}{\|y_n\|} \right\| = \|2x\| = 2$.

Proof of Theorem 2.5.3. Consider a strictly increasing sequence (ρ_n) of positive numbers with $\lim \rho_n = 1$ and such that $\frac{\sqrt{2\delta}}{2\rho_n} < 1$ for every $n \in \mathbb{N}$. By Proposition 2.3.5, we have that $\Phi_X^S(\delta) = \sqrt{2\delta}$, so for every $n \in \mathbb{N}$ there are $x_n \in S_X$ and $x_n^* \in S_{X^*}$ satisfying that

$$\operatorname{Re} x_n^*(x_n) \geqslant 1 - \delta$$

and such that

$$\max\{\|x_n - z\|, \|x_n^* - z^*\|\} \geqslant \sqrt{2\delta}\rho_{n+1} \tag{2.10}$$

for every $(z, z^*) \in \Pi(X)$. Next, we apply Lemma 2.2.4 with $x_n^* \in S_{X^*}$, $x_n \in B_X$, and $k_n = \frac{\sqrt{2\delta}}{2\rho_n} \in (0, 1)$ to obtain $y_n^* \in X^*$ and $y_n \in S_X$ satisfying

$$\|y_n^*\| = \operatorname{Re} y_n^*(y_n), \qquad \|x_n - y_n\| \leqslant \frac{\delta}{k_n} = \sqrt{2\delta}\rho_n\,, \qquad \text{and} \qquad \|x_n^* - y_n^*\| \leqslant k_n = \frac{\sqrt{2\delta}}{2\rho_n}\,.$$

As $k_n < 1$ and $\|x_n^* - y_n^*\| \leqslant k_n$, we get that $y_n^* \neq 0$ and so, $\left(y_n, \frac{y_n^*}{\|y_n^*\|} \right) \in \Pi(X)$. As we have that $\|x_n - y_n\| \leqslant \sqrt{2\delta}\rho_n < \sqrt{2\delta}\rho_{n+1}$, we get from equation (2.10) that

$$\left\| x_n^* - \frac{y_n^*}{\|y_n^*\|} \right\| \geqslant \sqrt{2\delta}\rho_{n+1}.$$

Summarizing, we have found two sequences (x_n^*) in S_{X^*} and $(y_n^*) \in X^* \setminus \{0\}$ such that

$$\limsup \|x_n^* - y_n^*\| \leqslant \frac{\sqrt{2\delta}}{2} \quad \text{and} \quad \liminf \left\| x_n^* - \frac{y_n^*}{\|y_n^*\|} \right\| \geqslant \sqrt{2\delta}\,.$$

Now, Lemma 2.6.5 gives that X^* contains almost isometric copies of the real space $\ell_\infty^{(2)}$, and so does X (Lemma 2.5.2), as desired. □

Let us remark that for complex Banach spaces, we cannot expect that Theorem 2.5.3 provides a *complex* copy of ℓ_∞^2 in the dual of the space. Namely, the two-dimensional complex space $X = \ell_1^2$ satisfies $\Phi_X(\delta) = \sqrt{2\delta}$ for $\delta \in (0,2)$ but it does not contain the complex space ℓ_∞^2 (of course, it contains the real space ℓ_∞^2 as a subspace since ℓ_1^2 and ℓ_∞^2 are isometric in the real case). We do not know whether it is true a result saying that if a complex space X satisfies $\Phi_X(\delta) = \sqrt{2\delta}$ for some $\delta \in (0,2)$, then X contains a copy of the complex space ℓ_1^2 or a copy of the complex space ℓ_∞^2.

In the following lines we present an example which tells us that the existence of an ℓ_∞^2-subspace does not imply that $\Phi_X(\delta) = \sqrt{2\delta}$, even when X has dimension 3. For every $\delta \in (0,1/2)$ we denote $\varepsilon = \sqrt{2\delta}$, so $0 < \varepsilon < 1$. We denote $B_\varepsilon^3 \subset \mathbb{R}^3$ the absolute convex hull of the following 11 points A_k, $k = 1,\dots,11$ (or, what is the same, the convex hull of 22 points $\pm A_k$, $k = 1,\dots,11$):

$$A_1 = (0,0,\frac{3}{4}),$$

$$A_2 = (1-\varepsilon,1,\frac{\varepsilon}{2}),\ A_3 = (1-\varepsilon,-1,\frac{\varepsilon}{2}),\ A_4 = (\varepsilon-1,1,\frac{\varepsilon}{2}),\ A_5 = (\varepsilon-1,-1,\frac{\varepsilon}{2}),$$

$$A_6 = (1,1-\varepsilon,\frac{\varepsilon}{2}),\ A_7 = (-1,1-\varepsilon,\frac{\varepsilon}{2}),\ A_8 = (1,\varepsilon-1,\frac{\varepsilon}{2}),\ A_9 = (-1,\varepsilon-1,\frac{\varepsilon}{2}),$$

$$A_{10} = (1,1,0),\ A_{11} = (1,-1,0).$$

Denote D_ε ("D" from "Diamond") the normed space $\big(\mathbb{R}^3, \|\cdot\|\big)$, for which B_ε^3 is its unit ball. Then D_ε^* can be viewed as $\mathbb{R}^3$ with the polar of B_ε^3 as the unit ball, and the action of $x^* \in D_\varepsilon^*$ on $x \in D_\varepsilon$ is just the standard inner product in $\mathbb{R}^3$. Let us list, without proof, some properties of D_ε whose verification is straightforward:

- The subspace of D_ε formed by vectors of the form $(x_1,x_2,0)$ is canonically isometric to ℓ_∞^2.
- There are no other isometric copies of ℓ_∞^2 in D_ε.
- The subspace of D_ε^* formed by vectors of the form $(x_1,x_2,0)$ is canonically isometric to ℓ_1^2 (and so, is isometric to ℓ_∞^2).

- There are no other isometric copies of ℓ_∞^2 in D_ε^*.
- The following operators act as isometries both on D_ε and D_ε^*:

  $$(x_1, x_2, x_3) \longmapsto (x_2, x_1, x_3) \quad \text{and} \quad (x_1, x_2, x_3) \longmapsto (x_1, -x_2, x_3).$$

  In other words, changing the sign of one coordinate or rearranging the first two coordinates do not change the norm of an element.

We state the promised result and refer the reader to [12, Theorem 6.1] for its proof.

2.5.7 Theorem. *Let $\delta \in (0, 1/2)$, $\varepsilon = \sqrt{2\delta}$, and $X = D_\varepsilon$. Then $\Phi_X(\delta) < \sqrt{2\delta}$.*

2.6 Open problems

In the following lines we gather some of the problems which remain unsolved after our study. We recall that Proposition 2.3.8 gives $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$ and $\Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$ for every Banach space and every $\delta \in (0, 2)$. It is natural to wonder if this inequalities are always equalities.

2.6.1 Problem. Is it true that $\Phi_X(\delta) = \Phi_{X^*}(\delta)$ or $\Phi_X^S(\delta) = \Phi_{X^*}^S(\delta)$ for every Banach space and every $\delta \in (0, 2)$?

As we explained at the end of Section 2.4, a candidate to get a counterexample to this conjecture is the algebra $L(H)$ for an infinite-dimensional Hilbert space H. We know by the results in Section 2.4 that the bidual of $L(H)$ has maximal Bishop-Phelps-Bollobás moduli, so if there is $\delta \in (0, 1/2]$ such that $\Phi_{L(H)}(\delta) < \sqrt{2\delta}$ then $L(H)$ would tell us that the above equality may not hold.

Self duality of Hilbert spaces together with the fact that they present sometimes "perfect" behaviour with respect to some geometric properties prompt us to pose the following question.

2.6.2 Problem. Let X be a Banach space with $\dim(X) \geqslant 2$, H a Hilbert space with dimension over $\mathbb{R}$ greater than one, and $\delta \in (0,2)$. Do the following inequalities hold?

$$\Phi_X^S(\delta) \geqslant \Phi_H^S(\delta) \quad \text{and} \quad \Phi_X(\delta) \geqslant \Phi_H(\delta).$$

We conclude with another natural problem in the theory we just developed.

2.6.3 Problem. Compute the Bishop-Phelps-Bollobás moduli for $L_p(\mu)$ spaces.

It seems hard to get a complete solution to this problem mainly because one has to deal with arbitrary points in the unit ball of $L_p(\mu)$ and its dual. However, one possible approach is to start with the spaces $L_p[0,1]$ for $1 < p < \infty$ which are almost transitive (this fact was first reported by A. Pełczynski and S. Rolewicz [39]). Recall that a Banach space X is almost transitive if for every $x \in S_X$ the orbit

$$\mathcal{O}_X(x) = \{T(x) \ : \ T \in L(X) \text{ onto isometry}\}$$

satisfies $\overline{\mathcal{O}_X(x)} = S_X$. Therefore, in order to estimate $\Phi^S_{L_p[0,1]}(\delta)$ one only needs to deal with one point in $S_{L_p[0,1]}$ and the functionals with big action on it.

Resumen

Esta memoria está dedicada al estudio de dos herramientas relacionadas con la geometría de los espacios de Banach: el *índice numérico de rango uno* y el *módulo de Bishop-Phelps-Bollobás*. Si bien está dividida en dos partes independientes, hay un hilo conductor que es el uso de los operadores más simples posibles en un espacio de Banach, los que tienen rango uno y los funcionales.

Capítulo 1: Índice numérico de rango uno de un espacio de Banach

Índice numérico de rango uno de un espacio de Banach

Dado un espacio de Banach X, se denomina *rango numérico* de un operador $T \in L(X)$ al siguiente conjunto de escalares:

$$V(T) = \{x^*(Tx) \,:\, (x, x^*) \in S_X \times S_{X^*} \,:\, x^*(x) = 1\}.$$

Este concepto fue introducido por F. Bauer en los años 1960 como una extensión del conocido rango numérico para matrices. Así, se define el *radio numérico* de un operador T como

$$v(T) = \sup\{|\lambda| \,:\, \lambda \in V(T)\}$$

que claramente satisface que $v(T) \leqslant \|T\|$, y define una seminorma en $L(X)$. A menudo esta seminorma es en realidad una norma equivalente a la norma usual de operadores.

Para estudiar este hecho, G. Lumer definió en 1968 el *índice numérico* de un espacio de Banach X como la constante

$$n(X) = \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\}.$$

Es inmediato que $n(X)$ es la constante más grande $k \geqslant 0$ de forma que $k\|T\| \leqslant v(T)$ para todo operador $T \in L(X)$. Se tienen trivialmente que $0 \leqslant n(X) \leqslant 1$; $n(X) = 1$ significa que radio numérico y norma de operadores coinciden y $n(X) >$ si, y sólo si, es una norma equivalente a la norma de operadores en $L(X)$.

Antes incluso de que la noción de índice numérico apareciera, era conocido que para un espacio de Hilbert H con dimensión mayor que 1, se tiene que $\|T\| \leqslant 2v(T)$ para todo operador $T \in L(H)$ en el caso complejo. En el caso real, siempre se puede obtener un operador con norma 1 y radio numérico igual a 0. Así pues, existe una clara diferencia de comportamiento del índice numérico frente al caso real y el caso complejo. Además, el conjunto de valores del índice numérico es diferente en cada caso, como mostramos a continuación, hecho observado por J. Duncan, C.M. McGregor, J. D. Pryce y A. J. White en su artículo pionero sobre índice numérico [16]:

$$\{n(X) : X \text{ espacio normado complejo}\} = [\mathrm{e}^{-1}, 1]$$

$$\{n(X) : X \text{ espacio normado real}\} = [0, 1].$$

Para terminar esta pequeña introducción acerca del índice numérico clásico, comentamos brevemente algunos de los resultados más relevantes. En el mencionado trabajo [16], los autores probaron que espacios de tipo $L_1(\mu)$ y sus preduales isométricos tienen índice numérico 1. En particular, se tiene $n(C(K)) = 1$ para todo espacio topológico compacto K. Asimismo, demostraron que el radio numérico de un operador y el de su adjunto coinciden, dando lugar a la desigualdad $n(X^*) \leqslant n(X)$ para todo espacio de Banach X. Desigualdad que puede ser estricta, véase el ejemplo dado por K. Boyko, V. Kadets, M. Martín y D. Werner en [9].

El valor del índice numérico para los espacios $L_p(\mu)$ con $p \neq 2$ y $1 < p < \infty$, sigue siendo todavía deconocido a pesar de que se han hecho algunos avances al respecto, ver [17, 18, 34].

El *índice numérico de rango uno* de un espacio de Banach fue introducido en [33] como una herramienta auxiliar para avanzar en el conocimiento del índice numérico de los espacios $L_p(\mu)$.

2.6.1 Definición. Sea X un espacio de Banach. El *índice numérico de rango uno* de X es la constante dada por

$$\begin{aligned} n_1(X) &:= \max\{k \geqslant 0 \,:\, k\|T\| \geqslant v(T)\ \forall T \in L(X) \text{ con } \dim(T(X)) \leqslant 1\} \\ &= \inf\{v(T) \,:\, T \in L(X),\ \|T\| = 1,\ \dim(T(X)) \leqslant 1\}, \end{aligned}$$

que obviamente verifica $n(X) \leqslant n_1(X)$.

En el mencionado artículo [33] se prueba que para todo $1 < p < \infty$ y toda medida sin átomos μ, se tiene que

$$n_1(L_p(\mu)) \geqslant p^{-\frac{2}{p}} q^{-\frac{2}{q}}$$

donde $q = p/(p-1)$ es el exponente conjugado de p. También se prueba que $n_1(H) = 1/2$ para todo espacio de Hilbert H con dimensión mayor que uno.

Aunque la definición del *índice numérico de rango uno* fue dada en [33], el estudio del radio numérico de operadores de rango uno fue iniciada mucho antes. En [29] los autores prueban una serie de resultados para espacios de Banach con índice numérico igual a uno, en los que en realidad sólo se utiliza la igualdad entre radio numérico y norma para operadores de rango uno.

Cota inferior del índice numérico de rango uno

Como consecuencia del resultado de Glickfeld [22], tenemos que $n_1(X) \geqslant n(X) \geqslant 1/e$ para todo espacio de Banach complejo X. El primer resultado que presentamos nos dice que esta desigualdad también es válida en caso real, lo que nos da la primera diferencia entre el índice de rango uno y el índice numérico.

2.6.2 Teorema. *Sea X un espacio de Banach real. Entonces,*

$$n_1(X) \geqslant \frac{1}{\mathrm{e}}.$$

Además esta desigualdad es óptima, pues el siguiente ejemplo proporciona un espacio de Banach real de dimensión dos con índice numérico de rango uno igual a $1/\mathrm{e}$.

2.6.3 Ejemplo. *Existe un espacio de Banach real de dimensión dos X con $n_1(X) = 1/\mathrm{e}$.*
Basta considerar la función $\Phi : [0, +\infty[\longrightarrow \mathbb{R}$ dada por

$$\Phi(t) = \begin{cases} \mathrm{e}^{t/\mathrm{e}} & \text{si } t \in [0, \mathrm{e}] \\ t & \text{si } t \geqslant \mathrm{e}, \end{cases} \qquad (t \in [0, +\infty[)$$

para definir la siguiente norma en $\mathbb{R}^2$:

$$\|(\alpha, \beta)\| = \begin{cases} |\alpha|\Phi\left(\frac{|\beta|}{|\alpha|}\right) & \text{si } \alpha \neq 0 \\ |\beta| & \text{si } \alpha = 0 \end{cases} \qquad \big((\alpha, \beta) \in \mathbb{R}^2\big).$$

Se tiene entonces que el espacio $X = (\mathbb{R}^2, \|\cdot\|)$ cumple lo deseado.

Propiedades del índice numérico de rango uno

Esta sección la dedicamos al estudio de algunas propiedades del índice numérico de rango uno. En primer lugar nos ocupamos del comportamiento del mismo frente a ciertas operaciones naturales con espacios de Banach. Concretamente, estudiamos el índice numérico de rango uno de algunos tipos de espacios de Banach, y la relación entre el índice numérico de rango uno de espacios con valores vectoriales y el del espacio de llegada. Dada una familia arbitraria de espacios de Banach $\{X_\lambda : \lambda \in \Lambda\}$, denotamos por $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{c_0}$ y $\left[\bigoplus_{\lambda\in\Lambda} X_\lambda\right]_{\ell_p}$ respectivamente a la c_0-suma y la ℓ_p-suma de la familia para $1 \leqslant p \leqslant \infty$. Algunos de los resultados que exponemos a continuación fueron probados en el caso del índice numérico clásico en [38]. Como veremos, el índice numérico clásico y el índice numérico de rango uno se comportan de forma similar cuando tratamos con sumas discretas, como en el caso las c_0-, ℓ_1-, o ℓ_∞- sumas. Sin embargo, hay diferencias en el caso de espacios con valores vectoriales.

Comenzamos con un resultado general que es nuevo incluso en el caso del índice numérico clásico.

2.6.4 Proposición. *Sea X un espacio de Banach e Y, Z subespacios cerrados de X tales que $X = Y \oplus Z$, $\|y_1 + z\| = \|y_2 + z\|$ para $z \in Z$ y $y_1, y_2 \in Y$ con $\|y_1\| = \|y_2\|$. Entonces,*

$$n(X) \leqslant n(Y) \qquad y \qquad n_1(X) \leqslant n_1(Y).$$

Como comentamos anteriormente, véase [38], es conocido el comportamiento del índice numérico en una de c_0-, ℓ_1-, o ℓ_∞- suma de espacios Banach es el ínfimo de los índices numéricos de los sumandos. Lo mismo ocurre para el índice numérico de rango uno.

2.6.5 Proposición. *Sea $\{X_\lambda \,:\, \lambda \in \Lambda\}$ una familia de espacios de Banach. Entonces*

$$n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big) = n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_1}\Big) = n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_\infty}\Big) = \inf\{n_1(X_\lambda) \,:\, \lambda \in \Lambda\}.$$

Como caso particular de la Proposición 2.6.4 obtenemos el siguiente resultado para las normas Hölder.

2.6.6 Corolario. *Sea Λ un conjunto no vacío, $1 < p < \infty$ y $\{X_\lambda \,:\, \lambda \in \Lambda\}$ una familia de espacios de Banach. Entonces*

$$n_1\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{\ell_p}\Big) \leqslant \inf\{n_1(X_\lambda) \,:\, \lambda \in \Lambda\}.$$

Ahora presentamos un resultado usando que involucra la noción de norma absoluta y suma absoluta de una familia de espacios de Banach, ver [7].

2.6.7 Corolario.

(a) Sea E el espacio $\mathbb{R}^m$ provisto de una norma absoluta, y $X_1, \ldots, X_m$ espacios de Banach. Denotamos por X a su E-suma. Entonces

$$n_1(X) \leqslant \min\big\{n_1(X_1), \ldots, n_1(X_m)\big\}.$$

(b) Sea E un espacio de Banach con una base uno-incondicional (infinita), y $\{X_j \,:\, j \in \mathbb{N}\}$ una sucesión de espacios Banach. Denotamos por X a su E-suma. Entonces

$$n_1(X) \leqslant \inf\big\{n_1(X_j) \,:\, j \in \mathbb{N}\big\}.$$

En el siguiente resultado presentamos el índice numérico de rango uno de los espacios con valores vectoriales más usuales. Resaltamos que el comportamiento difiere al del índice numérico clásico.

2.6.8 Proposición. *Sea K un espacio de Hausdorff compacto, μ una medida positiva y X un espacio de Banach. Entonces se tiene lo siguiente:*

$$n_1\big(C(K,X)\big) = \begin{cases} 1 & \text{si } K \text{ es perfecto,} \\ n_1(X) & \text{si } K \text{ no es perfecto,} \end{cases}$$

$$n_1\big(L_1(\mu,X)\big) = \begin{cases} 1 & \text{si } \mu \text{ no tiene átomos,} \\ n_1(X) & \text{si } \mu \text{ tiene átomos,} \end{cases}$$

$$n_1\big(L_\infty(\mu,X)\big) = \begin{cases} 1 & \text{si } \mu \text{ no tiene átomos,} \\ n_1(X) & \text{si } \mu \text{ tiene átomos.} \end{cases}$$

Usando este resultado, es fácil construir un espacio de Banach X de forma que $n_1(X^*) < n_1(X)$. Basta considerar el espacio $X = C([0,1], \ell_2)$ que satisface que $n_1(X) = 1$ pero $X^* \equiv L_1(\mu, \ell_2)$ para alguna medida μ con átomos, así $n_1(X^*) = n_1(\ell_2) = 1/2$. Este tipo de ejemplos ya aparece en [37, Ejemplo 4.4]. Comentemos que para un álgebra de von Neumann A, se prueba en [37, Teorema 4.2] que cumple $n_1(A) = 1$ si y solo si su predual A_* cumple lo mismo. Generalizamos este resultado, no sólo cubriendo el caso del índice numérico de rango uno igual a uno, si no considerando cualquier valor.

2.6.9 Proposición. *Sea X un espacio L-embebido. Entonces, $n_1(X) = n_1(X^*)$.*

Concluimos esta sección con un resultado de continuidad del índice numérico de rango uno respecto a la distacia de Banach-Mazur, análogo al que se tiene para el índice numérico clásico [21]. Como consecuencia se obtiene que el conjunto de valores que puede tomar el índice numérico de rango uno es el intervalo $[1/\mathrm{e}, 1]$.

Dado un espacio de Banach X, denotamos por $\mathcal{E}(X)$ al conjunto de las normas equivalentes en X, que es un espacio métrico cuando se considera la distancia

$$d(p,q) = \log\left(\min\{k \geqslant 1 \,:\, p \leqslant kq,\, q \leqslant kp\}\right) \qquad (p,q \in \mathcal{E}(X)).$$

Dado un operador $T \in L(X)$, $v_p(T)$ denota el radio numérico de T en el espacio (X,p):

$$v_p(T) = \sup\{|x^*(Tx)| \,:\, p(x) = p(x^*) = x^*(x) = 1\}.$$

Denotamos por $n_1(X,p)$ al índice numérico de rango uno del espacio de Banach (X,p), y el conjunto

$$\mathcal{N}_1(X) = \{n_1(X,p) \,:\, p \in \mathcal{E}(X)\}$$

contiene todos los posibles valores del índice numérico de rango uno del espacio por renormación equivalente.

2.6.10 Proposición. *Sea X un espacio de Banach.*

(a) La aplicación $(p,T) \longmapsto v_p(T)$ de $\mathcal{E}(X) \times L(X)$ a $\mathbb{R}$ es uniformemente continua en conjuntos acotados.

(b) Como consecuencia, la aplicación $p \longmapsto n_1(X,p)$ de $\mathcal{E}(X)$ a $\mathbb{R}$ es continua.

(c) Por tanto, $\mathcal{N}_1(X)$ es un intervalo.

(d) Si $\dim(X) > 1$, entonces $1/\mathrm{e} \in \mathcal{N}_1(X)$.

Alunos ejemplos y consideraciones sobre el índice numérico de rango uno

En la sección anterior expusimos un ejemplo de un espacio de Banach X de forma que $n_1(X^*) < n_1(X)$. A continuación damos un ejemplo de forma que esta desigualdad es lo más fuerte posible. Es decir, encontramos un espacio de Banach X que verifica $n_1(X^*) = 1/\mathrm{e}$ y $n_1(X) = 1$.

Basta considerar un espacio de Banach E de dimensión dos con $n_1(E) = 1/\mathrm{e}$. Entonces, usando [30, Teorema 3.3] existe un espacio $X = X(E)$ que satisface $n(X) = 1$ y $X^* = E^* \oplus_1 Z$ para un adecuado Z. Por tanto,

$$n_1(X) = n(X) = 1 \quad \text{y} \quad 1/\mathrm{e} \leqslant n_1(X^*) \leqslant n_1(E^*) = n_1(E) = 1/\mathrm{e}$$

por el Teorema 2.6.2 y la Proposición 2.6.5.

Además del índice numérico clasico y el índice numérico de rango uno, podemos considerar una gama de índices entre ellos, es decir: dado un espacio de Banach X, para cada $r \in \mathbb{N}$ podemos definir el *índice numérico de rango-r* por

$$n_r(X) = \inf\{v(T) \ : \ T \in S_{L(X)}, \dim(T(X)) \leqslant r\}$$

y el *índice numérico compacto* por

$$n_{\mathrm{comp}}(X) = \inf\{v(T) \ : \ T \in S_{L(X)}, T \text{ compact}\}.$$

Obviamente se tiene la desigualdad $n_r(X) \geqslant n_{r+1}(X) \geqslant n_{\mathrm{comp}}(X) \geqslant n(X)$ para todo $r \in \mathbb{N}$ y si un espacio de Banach tiene índice numérico de rango uno igual a 1, entonces todos los demás índices son iguales a 1. Es posible construir un espacio real en el que dos de las desigualdades anteriores son estrictas.

2.6.11 Ejemplo. *Existe un espacio de Banach real X tal que*

$$n(X) < n_{\mathrm{comp}}(X) < n_1(X).$$

Por otro lado, la siguiente proposición nos dice que el índice numérico y el índice numérico de rango dos están relacionados para espacios reales de dimensión finita.

2.6.12 Proposición. *Sea X un espacio real de dimensión finita con $n_2(X) = 0$. Entonces, $n_1(X) \leqslant \frac{1}{2}$.*

Para espacios de dimensión dos se reescribe de la siguiente manera más sugerente:

2.6.13 Corolario. *Sea X un espacio de Banach (real) de dimensión 2. Si $n_1(X) > \frac{1}{2}$, entonces $n(X) > 0$.*

En este punto hemos de comentar que no sabemos si la igualdad $n_{\text{comp}}(X) = n_1(X)$ es siempre cierta en el caso complejo. Tampoco sabemos si existe un espacio de Banach X de forma que se verifique $n_{\text{comp}}(X) \neq \inf\limits_{r\in\mathbb{N}} n_r(X)$. En caso de existir, está claro que no podría tener la propiedad de aproximación.

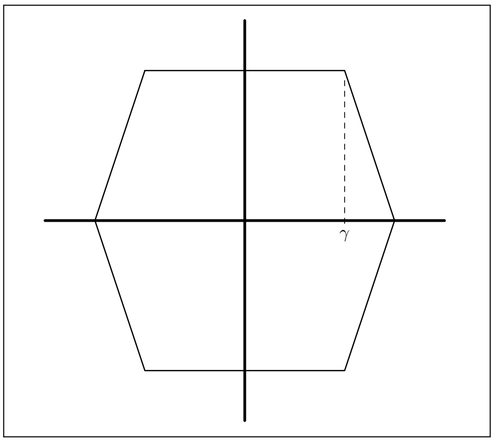
Cálculo del índice numérico de rango uno

En general calcular el índice numérico de un espacio concreto es una tarea difícil, solamente se conocen pocos espacios donde es relativamente sencillo calcular su índice numérico. Lo mismo ocurre en el caso del índice numérico de rango uno. La última sección de este capítulo sobre el índice numérico de rango uno está dedicada al cálculo del mismo para ciertas familias de normas poliédricas en el plano: tratamos el caso de espacios de Banach con normas hexagonales y octagonales. Comentemos que el cómputo del índice numérico clásico de estas mismas familias de normas poliédricas en el plano se había hecho en el trabajo de J. Merí y M. Martín en [32].

Normas hexagonales

La primera familia de normas poliédricas a la cual le calculamos su índice numérico de rango uno consiste en una familia de normas hexagonales. Para cada $\gamma \in [0,1]$, se considera $X_\gamma = (\mathbb{R}^2, \|\cdot\|_\gamma)$ con la norma $\|\cdot\|_\gamma$ dada por

$$\|(x,y)\|_\gamma = \max\{|y|, |x| + (1-\gamma)|y|\} \qquad \forall\, (x,y) \in \mathbb{R}^2.$$



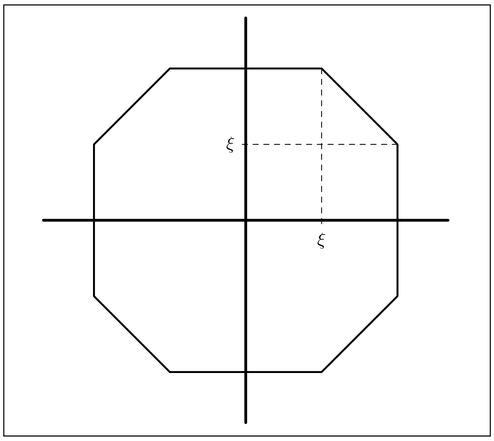
2.6.14 Teorema. *Para $\gamma \in [0,1]$, sea X_γ el espacio definido anteriormente. Entonces,*

$$n_1(X_\gamma) = \begin{cases} \frac{1}{1+2\gamma} & \text{si } 0 \leqslant \gamma \leqslant \frac{1}{2}, \\ \frac{1}{3-2\gamma} & \text{si } \frac{1}{2} \leqslant \gamma \leqslant 1. \end{cases}$$

Normas octagonales

De la misma forma, calculamos el índice numérico de rango uno para la siguiente familia cuya bola unidad tiene forma de octágono. Para ello, consideramos $\xi \in [0,1]$ y sea $X_\xi = (\mathbb{R}^2, \|\cdot\|_\xi)$ el espacio normado donde su norma tiene la expresión siguiente

$$\|(x,y)\|_\xi = \max\left\{|x|,\ |y|,\ \frac{|x|+|y|}{1+\xi}\right\} \qquad \forall\, (x,y) \in \mathbb{R}^2.$$



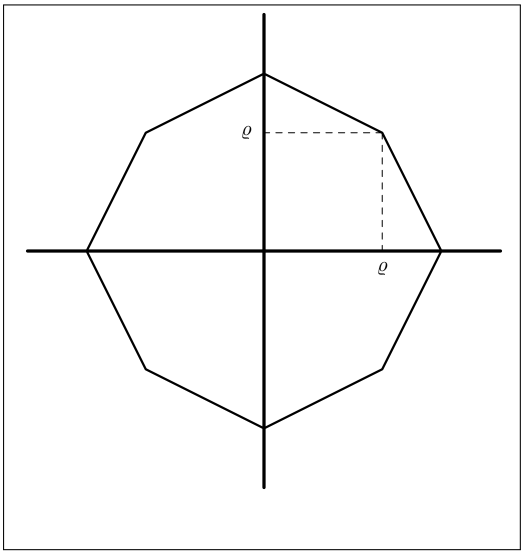
2.6.15 Teorema. *Para $\xi \in [0,1]$, sea X_ξ definido anteriormente. Entonces,*

$$n_1(X_\xi) = \begin{cases} \frac{1-\xi}{1+\xi} & \text{si } 0 \leqslant \xi \leqslant \sqrt{5}-2, \\ \frac{2+2\xi^2}{2+\xi-\xi^2+\sqrt{8\xi-7\xi^2+2\xi^3+\xi^4}} & \text{si } \sqrt{5}-2 \leqslant \xi \leqslant \sqrt{2}-1, \\ \frac{2+2\xi^2}{1+3\xi+\sqrt{(1+\xi)^2+4\xi^2(1-\xi-\xi^2)}} & \text{si } \sqrt{2}-1 \leqslant \xi \leqslant \frac{\sqrt{5}-1}{2}, \\ \xi & \text{si } \frac{\sqrt{5}-1}{2} \leqslant \xi \leqslant 1. \end{cases}$$

Como el índice numérico de rango uno de un espacio y el de su dual coinciden en el caso finito-dimensional, el resultado anterior nos permite obtener el índice numérico de rango uno de los espacios duales a los X_ξ. Más detalladamente, para $\varrho \in [\frac{1}{2}, 1]$ consideramos el espacio $Y_\varrho = (\mathbb{R}^2, \|\cdot\|_\varrho)$ con la norma dada por

$$\|(x,y)\|_\varrho = \max\left\{|x| + \frac{1-\varrho}{\varrho}|y|, |y| + \frac{1-\varrho}{\varrho}|x|\right\} \qquad \forall\, (x,y) \in \mathbb{R}^2.$$

De esta forma, se tiene que $X_\xi^* = Y_\varrho$ donde $\varrho = \frac{1}{1+\xi}$. Equivalentemente, $Y_\varrho^* = X_\xi$ donde $\xi = \frac{1-\varrho}{\varrho}$. Por tanto, tenemos la siguiente consecuencia inmediata del Teorema 2.6.15.



2.6.16 Corolario. *Para todo* $\varrho \in [\frac{1}{2}, 1]$*, sea* Y_ϱ *definido como anteriormente. Entonces*

$$n_1(Y_\varrho) = \begin{cases} \frac{1-\varrho}{\varrho} & \text{si } \frac{1}{2} \leqslant \varrho \leqslant \frac{2}{\sqrt{5}+1}, \\ \frac{2\varrho^2+2(1-\varrho)^2}{3\varrho-2\varrho^2+\sqrt{\varrho^2+4(1-\varrho)^2(\varrho^2+\varrho-1)}} & \text{si } \frac{2}{\sqrt{5}+1} \leqslant \varrho \leqslant \frac{1}{\sqrt{2}}, \\ \frac{2\varrho^2+2(1-\varrho)^2}{3\varrho-1+\sqrt{1-2\varrho-7\varrho^2+24\varrho^3-16\varrho^4}} & \text{si } \frac{1}{\sqrt{2}} \leqslant \varrho \leqslant \frac{1}{\sqrt{5}-1}, \\ 2\varrho-1 & \text{si } \frac{1}{\sqrt{5}-1} \leqslant \varrho \leqslant 1. \end{cases}$$

Problemas abiertos

Para concluir este capítulo del índice numérico de rango uno, damos un breve repaso a algunas de las cuestiones que han quedado abiertas en nuestro estudio.

2.6.17 Problema. *Sea* X *un espacio de Banach complejo. ¿Es verdad que* $n_1(X) = n_{comp}(X)$*?*

Como paso previo a resolver este problema, sería interesante calcular el índice numérico de rango uno para algunas familias de espacios de Banach complejos.

2.6.18 Problema. *¿Existe algún espacio de Banach X tal que $n_{comp}(X) \neq \inf_{r\in\mathbb{N}} n_r(X)$?*

2.6.19 Problema. *Calcular $n_1(\ell_p)$ para $1 < p < \infty$ con $p \neq 2$.*

Capítulo 2: Módulo de Bishop-Phelps-Bollobás de un espacio de Banach

La segunda parte de esta memoria está dedicada al desarrollo de una nueva herramienta en el estudio de la geometría de los espacios de Banach, el *módulo de Bishop-Phelps-Bollobás*.

Como es bien conocido, si X es un espacio de Banach y $x^* \in X^*$, se dice que x^* alcanza su norma si

$$\|x^*\| = \max\{|x^*(x)| : x \in B_X\}.$$

Es claro que no todo funcional alcanza su norma; de hecho, R. C. James establece en 1957 que esto ocurre sólo si estamos en un espacio reflexivo. Años después, E. Bishop y R. Phelps [5] probaron que el conjunto de funcionales que alcanza su norma es denso en el espacio dual. Más tarde, B. Bollobás [6], proporcionó una versión mejorada de este teorema permitiendo aproximar al mismo tiempo un funcional y un vector en el que casi alcanza la norma. Este resultado es el punto de arranque de nuestro trabajo.

2.6.20 Teorema (Teorema de Bishop-Phelps-Bollobás [6]).
Sea X un espacio de Banach. Supongamos que $x \in S_X$ y $x^* \in S_{X^*}$ satisfacen que $|1 - x^*(x)| \leqslant \varepsilon^2/2$ para algún $0 < \varepsilon < 1/2$. Entonces existe una pareja $(y, y^*) \in \Pi(X)$ verificando que $\|x - y\| < \varepsilon + \varepsilon^2$ y $\|x^* - y^*\| \leqslant \varepsilon$.

La idea fundamental de nuestro trabajo consiste en encontrar el mejor teorema posible de Bishop-Phelps-Bollobás para un espacio de Banach dado. Tiene sentido plantearse dicho problema considerando de partida vectores y funcionales con norma igual a uno, o permitiendo que tomen norma menor o igual a uno. Desarrollamos la teoría en ambos casos.

2.6.21 Definición (Módulos de Bishop-Phelps-Bollobás)**.**
Sea X un espacio de Banach.

(a) El *módulo de Bishop-Phelps-Bollobás* de X es la función $\Phi_X : (0,2) \longrightarrow \mathbb{R}^+$ tal que dado $\delta \in (0,2)$, $\Phi_X(\delta)$ es el ínfimo de los $\varepsilon > 0$ que satisfacen que para toda pareja $(x, x^*) \in B_X \times B_{X^*}$ con $\operatorname{Re} x^*(x) > 1 - \delta$, hay una pareja $(y, y^*) \in \Pi(X)$ con $\|x - y\| < \varepsilon$ y $\|x^* - y^*\| < \varepsilon$.

(b) El *módulo esférico de Bishop-Phelps-Bollobás* de X es la función $\Phi_X^S : (0,2) \longrightarrow \mathbb{R}^+$ tal que dado $\delta \in (0,2)$, $\Phi_X^S(\delta)$ es el ínfimo de los $\varepsilon > 0$ que satisfacen que para toda pareja $(x, x^*) \in S_X \times S_{X^*}$ con $\operatorname{Re} x^*(x) > 1 - \delta$, hay una pareja $(y, y^*) \in \Pi(X)$ con $\|x - y\| < \varepsilon$ y $\|x^* - y^*\| < \varepsilon$.

Resaltamos que, como el dual de un espacio de Banach complejo X es isométrico (tomando parte real) al dual del espacio real subyacente $X_\mathbb{R}$ y $\Pi(X)$ no cambia si consideramos X como espacio de Banach real, sólo la estructura real del espacio se ve involucrada en las definiciones anteriores.

Consideraremos la siguiente notación que facilita la comprensión y uso de las definiciones anteriores. Sea X un espacio de Banach y fijamos $0 < \delta < 2$. Escribimos

$$A_X(\delta) := \big\{(x, x^*) \in B_X \times B_{X^*} \,:\, \operatorname{Re} x^*(x) > 1 - \delta\big\},$$
$$A_X^S(\delta) := \big\{(x, x^*) \in S_X \times S_{X^*} \,:\, \operatorname{Re} x^*(x) > 1 - \delta\big\}.$$

De esta forma se tiene que

$$\Phi_X(\delta) = \sup_{(x,x^*)\in A_X(\delta)} \ \inf_{(y,y^*)\in\Pi(X)} \max\{\|x - y\|, \|x^* - y^*\|\},$$
$$\Phi_X^S(\delta) = \sup_{(x,x^*)\in A_X^S(\delta)} \ \inf_{(y,y^*)\in\Pi(X)} \max\{\|x - y\|, \|x^* - y^*\|\}.$$

Si además denotamos por $d_H(A,B)$ a la distancia de Hausdorff entre $A,B \subset X \times X^*$ asociada a la ℓ_∞-distancia dist_∞ en $X \times X^*$, se obtiene la siguiente expresión para los módulos para todo $0 < \delta < 2$

$$\Phi_X(\delta) = d_H\big(A_X(\delta), \Pi(X)\big) \qquad \text{y} \qquad \Phi_X^S(\delta) = d_H\big(A_X^S(\delta), \Pi(X)\big).$$

De esta forma, se observa claramente que cuanto menores son $\Phi_X(\cdot)$ y $\Phi_X^S(\cdot)$, mejor es el Teorema de de Bishop-Phelps-Bollobás que se puede obtener en el espacio.

Cota superior para el módulo de Bishop-Phelps-Bollobás

En esta sección probamos que existe una cota superior para ambos módulos de Bishop-Phelps-Bollobás, que además es óptima. Como consecuencia, se obtiene una mejora del teorema de Bishop-Phelps-Bollobás [6]. Nuestro objetivo concreto consiste en obtener la mejor cota superior para $d_\infty\big((x,x^*),\Pi(X)\big)$ en cualquier espacio de Banach X como una función de $\|x\|$, $\|x^*\|$ y $x^*(x)$. Más concretamente, fijados $\delta \in (0,2)$ y $\mu, \theta \in [0,1]$ satisfaciendo $\mu\theta \geqslant 1-\delta$, consideramos la función

$$\Phi_X(\mu,\theta,\delta) := \sup\{d_\infty\big((x,x^*),\Pi(X)\big) \,:\, x \in X, x^* \in X^*, \\ \|x\| = \mu, \|x^*\| = \theta, \operatorname{Re} x^*(x) \geqslant 1-\delta\}$$

que nos permite estudiar los dos módulos simultáneamente:

$$\Phi_X^S(\delta) = \Phi_X(1,1,\delta) \qquad \text{y} \qquad \Phi_X(\delta) = \sup_{\substack{\mu\theta\in[0,1] \\ \mu\theta\geqslant 1-\delta}} \Phi_X(\mu,\theta,\delta).$$

Para presentar el resultado anunciado, introducimos también la siguiente notación: para $\delta \in (0,2)$ y $\mu, \theta \in [0,1]$ con $\mu\theta > 1-\delta$, definimos la función

$$\Psi(\mu,\theta,\delta) := \frac{2-\mu-\theta+\sqrt{(\mu-\theta)^2+8\,(\mu\theta-1+\delta)}}{2}.$$

2.6.22 Teorema. *Sea X un espacio de Banach, $\delta \in (0,2)$, y $\mu, \theta \in [0,1]$ verificando que $\mu\theta > 1-\delta$. Entonces,*

$$\Phi_X(\mu,\theta,\delta) \leqslant \min\left\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\right\}.$$

Con un poco de esfuerzo se consigue de este resultado una cota superior para ambos módulos de Bishop-Phelps-Bollobás.

2.6.23 Corolario. *Sea X un espacio de Banach y $\delta \in (0,2)$. Entonces*

$$\Phi^S_X(\delta) \leqslant \Phi_X(\delta) \leqslant \sqrt{2\delta}.$$

También se obtiene como consecuencia del estudio de $\Phi_X(\mu,\theta,\delta)$ que el máximo valor de $\Phi_X(\delta)$ sólo puede darse cuando también $\Phi^S_X(\delta)$ es máximo.

2.6.24 Proposición. *Sea X un espacio de Banach. Para todo $\delta \in (0,2)$, la condición $\Phi_X(\delta) = \sqrt{2\delta}$ es equivalente a la condición $\Phi^S_X(\delta) = \sqrt{2\delta}$.*

Como consecuencia, obtenemos una versión mejorada del teorema de Bishop-Phelps-Bollobás.

2.6.25 Corolario. *Sea X un espacio de Banach.*

(a) Sea $0 < \varepsilon < 2$ y supongamos que $x \in B_X$ y $x^* \in B_{X^*}$ verifican que

$$\operatorname{Re} x^*(x) > 1 - \varepsilon^2/2.$$

Entonces, existe una pareja $(y,y^*) \in \Pi(X)$ tal que

$$\|x-y\| < \varepsilon \quad \text{and} \quad \|x^*-y^*\| < \varepsilon.$$

(b) Sea $0 < \delta < 2$ y supongamos que $x \in B_X$ y $x^* \in B_{X^*}$ verifican que

$$\operatorname{Re} x^*(x) > 1 - \delta.$$

Entonces, existe una pareja $(y,y^*) \in \Pi(X)$ tal que

$$\|x-y\| < \sqrt{2\delta} \quad \text{and} \quad \|x^*-y^*\| < \sqrt{2\delta}.$$

Para terminar esta sección presentamos un espacio de Banach para el que la estimación dada anteriormente en el Teorema 2.6.22 es óptima.

2.6.26 Ejemplo. Sea X el espacio de Banach real ℓ_∞^2, $\delta \in (0,2)$, y consideramos $\mu, \theta \in [0,1]$ verificando que $\mu\theta > 1-\delta$. Entonces existe una pareja $(x, x^*) \in X \times X^*$ con $\|x\| = \mu$, $\|x^*\| = \theta$, $x^*(x) \geqslant 1-\delta$ y tal que

$$d_\infty\big((x,x^*), \Pi(X)\big) = \min\left\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\right\}.$$

Por tanto, $\Phi_X(\mu,\theta,\delta) = \min\left\{\Psi(\mu,\theta,\delta), 1+\mu, 1+\theta\right\}$ para todos los posibles valores de δ, μ, θ. En particular, $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$.

Propiedades del módulo de Bishop-Phelps-Bollobás

En esta segunda sección presentamos las propiedades básicas más interesantes que tiene el módulo de Bishop-Phelps-Bollobás. Entre ellas destacamos resultados de continuidad respecto de la variable δ, así como de continuidad de estas funciones con respecto a la distancia Banach-Mazur. Además, analizamos la relación que guarda el módulo de un espacio con el de su dual.

2.6.27 Proposición. *Sea X un espacio de Banach. Entonces, las funciones*

$$\delta \longmapsto \Phi_X(\delta) \qquad y \qquad \delta \longmapsto \Phi_X^S(\delta)$$

son continuas en $(0,2)$.

Usando este resultado de continuidad y el Principio de Reflexividad Local (ver Teorema 11.2.4 [1]), obtenemos la relación entre los módulos de un espacio y de su dual, más concretamente, se demuestra que la aproximación en un espacio es mejor que en su dual.

2.6.28 Proposición. *Sea X un espacio de Banach. Entonces*

$$\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta) \quad y \quad \Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$$

para todo $\delta \in (0,2)$.

Desafortunadamente, no sabemos si la desigualdad dada anteriormente puede ser estricta, aunque está claro que en el caso reflexivo esto no puede pasar.

2.6.29 Corolario. *Sea X un espacio de Banach reflexivo, se tiene que $\Phi_X(\delta) = \Phi_{X^*}(\delta)$ y $\Phi_X^S(\delta) = \Phi_{X^*}^S(\delta)$ para cada $0 < \delta < 2$.*

Probamos además que los módulos de Bishop-Phelps-Bollobás son continuos con respecto a la distancia de Banach-Mazur.

2.6.30 Proposición. *Sea X un espacio de Banach $\delta \in (0,2)$. Entonces las funciones $\Phi_{(X,\cdot)}(\delta) : \mathcal{E}(X) \longrightarrow \mathbb{R}$ y $\Phi_{(X,\cdot)}^S(\delta) : \mathcal{E}(X) \longrightarrow \mathbb{R}$ son continuas.*

Cálculo del módulo de Bishop-Phelps-Bollobás

Esta parte se dedica al cómputo de ambos módulos para algunos espacios de Banach clásicos: $\mathbb{R}$ y los espacios de Hilbert tanto reales como complejos de dimensión mayor que uno. También, se presentan ejemplos donde ambos módulos toman el máximo valor posible para valores de δ no demasiado grandes. Este es el caso de c_0, ℓ_1 y, con un poco de más generalidad, de $L_1(\mu)$, $C_0(L)$, y C^*-algebras unitales con centralizador no trivial.

Comenzamos observando que existe una cota inferior universal para el módulo de Bishop-Phelps-Bollobás siempre y cuando δ tome valores pequeños.

2.6.31 Observación. Dado X un espacio de Banach. Entonces $\Phi_X(\delta) \geqslant \delta$ para todo $\delta \in (0,1]$.

Para valores más grandes de δ desconocemos si hay una cota inferior universal que no sea la trivial dada por $\Phi_X(\delta) \geqslant 1$.

Empezamos calculando ambos módulos en el caso más sencillo posible, $X = \mathbb{R}$.

2.6.32 Ejemplo. Para todo $\delta \in (0,2)$ se verifica:

$$\Phi^S_{\mathbb{R}}(\delta) = 0 \qquad \text{y} \qquad \Phi_{\mathbb{R}}(\delta) = \begin{cases} \delta & \text{si } 0 < \delta \leqslant 1, \\ 1 + \sqrt{\delta - 1} & \text{if } 1 < \delta < 2. \end{cases}$$

A continuación, calculamos ambos módulos para un espacio de Hilbert H de dimensión (real) mayor que uno. Al igual que el caso de $\mathbb{R}$, comenzamos con un resultado que nos da la distancia exacta de una pareja $(x,y) \in B_H \times B_H$ a $\Pi(H)$ en términos de $\|x\|$, $\|y\|$ y el producto escalar $\langle x,y\rangle$.

2.6.33 Teorema. *Sea H un espacio de Hilbert real con $\dim(H) \geqslant 2$ y sean x,y puntos diferentes en B_H con $\|x\| \geqslant \|y\|$. Llamamos*

$$A = \left\{ (x,y) \in H \times H : \langle x,y\rangle \geqslant \|y\|^2 + \|y\| \frac{\|x\|^2 - \|y\|^2}{2} \right\}.$$

Entonces,

$$d_\infty\big((x,y), \Pi(H)\big) = \begin{cases} 1 - \|y\| & \text{si } (x,y) \in A, \\ \sqrt{1 - \langle x,y\rangle - 2\lambda\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}} & \text{si } (x,y) \notin A, \end{cases}$$

donde

$$\lambda = \frac{-2\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2} + \sqrt{4\big(\|x\|^2 + \|y\|^2 - 2\langle x,y\rangle\big) - (\|x\|^2 - \|y\|^2)^2}}{2\big(\|x\|^2 + \|y\|^2 - 2\langle x,y\rangle\big)}.$$

Como consecuencia obtenemos el valor de $\Phi_H(\mu,\theta,\delta)$.

2.6.4 Corollary. *Sea H un espacio de Hilbert real con* $\dim(H) \geqslant 2$, $\delta \in (0,2)$ *y* $\mu, \theta \in [0,1]$ *tales que* $\mu \geqslant \theta$ *y* $\mu\theta > 1-\delta$. *Entonces*

$$\Phi_H(\mu,\theta,\delta) = \begin{cases} 1-\theta & \text{if } 1-\delta \geqslant \theta^2 + \theta\frac{\mu^2-\theta^2}{2}, \\ \max\left\{1-\theta, \sqrt{\delta - 2\lambda_\delta\sqrt{\mu^2\theta^2-(1-\delta)^2}}\right\} & \text{if } 1-\delta < \theta^2 + \theta\frac{\mu^2-\theta^2}{2}, \end{cases}$$

donde

$$\lambda_\delta = \frac{-2\sqrt{\mu^2\theta^2-(1-\delta)^2} + \sqrt{4\big(\mu^2+\theta^2-2+2\delta\big) - (\mu^2-\theta^2)^2}}{2\big(\mu^2+\theta^2-2+2\delta\big)}.$$

Trabajando un poco más también podemos calcular $\Phi_H^S(\delta)$ y $\Phi_H(\delta)$.

2.6.34 Ejemplo. Sea H un espacio de Hilbert con dimensión sobre $\mathbb{R}$ mayor que uno. Entonces:

(a) $\Phi_H^S(\delta) = \sqrt{2-\sqrt{4-2\delta}}$ para todo $\delta \in (0,2)$.

(b) Para $\delta \in (0,1]$, $\Phi_H(\delta) = \max\left\{\delta, \sqrt{2-\sqrt{4-2\delta}}\right\}$.
Para $\delta \in (1,2)$, $\Phi_H(\delta) = \sqrt{\delta}$.

Como adelantamos al principio de este resumen, ahora se proporcionan algunos ejemplos de espacios de Banach para los cuales su módulo de Bishop-Phelps-Bollobás es el máximo posible, es decir, $\Phi_X^S(\delta) = \Phi_X(\delta) = \sqrt{2\delta}$ para valores pequeños de δ.

Comenzamos con un resultado que asegura que si un espacio de Banach admite una L-descomposición, entonces su módulo es el máximo posible. Como consecuencia, podemos calcular el módulo de espacios de tipo $L_1(\mu)$.

2.6.35 Proposición. *Sea X un espacio de Banach. Supongamos que hay dos subespacios (no triviales) Y y Z tales que $X = Y \oplus_1 Z$. Sea $\delta \in (0,1)$ y $\mu, \theta \in [0,1]$ con $1-\delta < \mu\theta \leqslant 2(1-\delta)$. Entonces, existe una pareja $(x_0, x_0^*) \in X \times X^*$ con $\|x_0\| = \mu$,*

$\|x_0^*\| = \theta$ y $\operatorname{Re} x_0^*(x_0) \geqslant 1 - \delta$ satisfaciendo

$$d_\infty\big((x_0, x_0^*), \Pi(X)\big) = \Psi(\mu, \theta, \delta).$$

Por tanto, $\Phi_X(\mu, \theta, \delta) = \Psi(\mu, \theta, \delta)$ para los valores indicados de δ, μ, θ.

En las hipótesis del teorema anterior tenemos que

$$\Phi_X^S(\delta) = \Phi_X(1, 1, \delta) = \Psi(1, 1, \delta) = \sqrt{2\delta}$$

para todo $\delta \in (0, 1/2]$. Como consecuencia, obtenemos el módulo para espacios que admitan una L-descomposición:

2.6.36 Corolario. *Sea X un espacio de Banach. Supongamos que hay dos subespacios no triviales Y, Z tales que $X = Y \oplus_1 Z$. Entonces $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ para todo $\delta \in (0, 1/2]$.*

2.6.37 Ejemplo. *Sea (Ω, Σ, ν) un espacio de medida tal que $L_1(\nu)$ tiene dimensión mayor que uno y sea E cualquier espacio de Banach no nulo. Entonces*

$$\Phi_{L_1(\nu,E)}(\mu, \theta, \delta) = \Psi(\mu, \theta, \delta)$$

para $\delta \in (0, 1)$ y $\mu, \theta \in [0, 1]$ con $1 - \delta < \mu\theta \leqslant 2(1 - \delta)$. En particular,

$$\Phi_{L_1(\nu,E)}(\delta) = \Phi^S_{L_1(\nu,E)}(\delta) = \sqrt{2\delta}$$

para todo $\delta \in (0, 1/2]$.

Usando un argumento dual se pueden obtener los resultados análogos para espacios de Banach que se descomponen en una ℓ_∞-suma. De hecho, se puede mejorar usando ideales en vez de subespacios. Si denotamos por w^* a la topología débil* $\sigma(X^*, X)$ en X^*, tenemos el siguiente resultado.

2.6.38 Proposición. *Sea X un espacio de Banach. Supongamos que $X^* = Y \oplus_1 Z$ donde Y, Z son subespacios no triviales de X^* tales que $\overline{Y}^{w^*} \neq X^*$ y $\overline{Z}^{w^*} \neq X^*$. Sea $\delta \in (0,1)$ y $\mu, \theta \in [0,1]$ con $1-\delta < \mu\theta \leqslant 2(1-\delta)$. Entonces, existe una pareja $(x_0, x_0^*) \in X \times X^*$ con $\|x_0\| = \mu$, $\|x_0^*\| = \theta$ y $\operatorname{Re} x_0^*(x_0) \geqslant 1-\delta$ verificando*

$$d_\infty\big((x_0, x_0^*), \Pi(X)\big) = \Psi(\mu, \theta, \delta).$$

Por tanto, $\Phi_X(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ para los valores citados de δ, μ, θ. En particular, $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ para todo $\delta \in (0, 1/2]$.

Como corolario tenemos el resultado para espacios de Banach que se descomponen como una ℓ_∞-suma dos subespacios no triviales.

2.6.39 Corolario. *Sea X un espacio de Banach. Supongamos que hay dos subespacios de no triviales Y, Z tales que $X = Y \oplus_\infty Z$. Sea $\delta \in (0,1)$ y $\mu, \theta \in [0,1]$ con $1-\delta < \mu\theta \leqslant 2(1-\delta)$. Entonces, existe una pareja $(x_0, x_0^*) \in X \times X^*$ con $\|x_0\| = \mu$, $\|x_0^*\| = \theta$ y $\operatorname{Re} x_0^*(x_0) \geqslant 1-\delta$ verificando*

$$d_\infty\big((x_0, x_0^*), \Pi(X)\big) = \Psi(\mu, \theta, \delta).$$

Por tanto, $\Phi_X(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$ para los valores δ, μ, θ. En particular, $\Phi_X(\delta) = \Phi_X^S(\delta) = \sqrt{2\delta}$ para todo $\delta \in (0, 1/2]$.

Asimismo, este resultado puede aplicarse a espacios de tipo L_∞ y c_0 con valores vectoriales.

2.6.40 Ejemplos.

(a) Sea (Ω, Σ, ν) un espacio de medida que contiene dos conjuntos disjuntos medibles con medida positiva y sea X un espacio de Banach. Entonces

$$\Phi_{L_\infty(\nu,X)}(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$$

para $\delta \in (0,1)$ y $\mu, \theta \in [0,1]$ con $1-\delta < \mu\theta \leqslant 2(1-\delta)$. En particular,

$$\Phi_{L_\infty(\nu,X)}(\delta) = \Phi^S_{L_\infty(\nu,X)}(\delta) = \sqrt{2\delta}$$

para todo $\delta \in (0,1/2]$.

(b) Sea Γ con al menos dos puntos, y X un espacio de Banach no trivial. Entonces

$$\Phi_{c_0(\Gamma,X)}(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$$

para $\delta \in (0,1)$ y $\mu, \theta \in [0,1]$ con $1-\delta < \mu\theta \leqslant 2(1-\delta)$. En particular,

$$\Phi_{c_0(\Gamma,X)}(\delta) = \Phi^S_{c_0(\Gamma,X)}(\delta) = \sqrt{2\delta}$$

para todo $\delta \in (0,1/2]$.

La Proposición 2.6.38 nos proporciona un resultado análogo para espacios de tipo $C_0(L)$ con valores vectoriales, usando el concepto de M-ideal. Referimos al lector dirigirse a [24] para más detalles sobre este concepto.

2.6.41 Corolario. *Sea X un espacio de Banach. Supongamos que hay un M-ideal J no trivial de X y un punto $x_0 \in X \setminus \{0\}$ tal que $\|x_0 + y\| = \max\{\|x_0\|, \|y\|\}$ para todo $y \in J$. Entonces*

$$\Phi_X(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$$

para $\delta \in (0,1)$ y $\mu, \theta \in [0,1]$ con $1-\delta < \mu\theta \leqslant 2(1-\delta)$. En particular,

$$\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$$

para todo $\delta \in (0,1/2]$.

Como caso particular de este corolario tenemos el siguiente ejemplo.

2.6.42 Ejemplo. *Sea L un espacio topológico localmente compacto Hausdorff con al menos dos puntos, y sea E cualquier espacio de Banach no nulo. Entonces*

$$\Phi_{C_0(L,E)}(\mu,\theta,\delta) = \Psi(\mu,\theta,\delta)$$

para $\delta \in (0,1)$ y $\mu, \theta \in [0,1]$ con $1-\delta < \mu\theta \leqslant 2(1-\delta)$. En particular,

$$\Phi_{C_0(L,E)}(\delta) = \Phi^S_{C_0(L,E)}(\delta) = \sqrt{2\delta}$$

para todo $\delta \in (0,1/2]$.

A continuación, vamos a desarrollar una serie de condiciones suficientes para que se cumplan las hipótesis del Corolario 2.6.41. La primera de ellas es que el espacio de Banach en cuestión contenga dos M-ideales J_1 y J_2 con $J_1 \cap J_2 = \{0\}$. Esto ocurre en el caso de $C(K)$ cuando K tiene más de un punto. A su vez, una condición suficiente para que un espacio de Banach tenga dos M-ideales con intersección vacía, es que su centralizador tenga al menos dimensión dos. Un ejemplo de ello son las C^*-álgebras con centro no trivial.

En relación a estos últimos resultados, queremos comentar el caso especial del espacio de Banach $L(H)$, donde H es un espacio de Hilbert. Este ejemplo es bastante interesante ya que ninguno de los resultados anteriores es aplicable pues su centro es trivial. Además, no hay elementos $x_0 \in L(H)$ satisfaciendo los requisitos del Corolario 2.6.41. Por otra parte, en dimensión infinita, el bidual de $L(H)$ es una C^*-álgebra con centro no trivial, por lo que se tiene que $\Phi_{L(H)^{**}}(\delta) = \Phi^S_{L(H)^{**}}(\delta) = \sqrt{2\delta}$ para todo $\delta \in (0,1/2]$. Por tanto, sería muy interesante calcular el módulo de $L(H)$, porque en el caso de no ser el máximo posible, obtendríamos un ejemplo para el cual la desigualdad de la Proposición 2.6.28 sería estricta.

Espacios de Banach con el máximo valor posible para el módulo

Obsérvese que todos los espacios de Banach presentados en la sección anterior que satisfacen que su módulo de Bishop-Phelps-Bollobás es el máximo posible, esto es, $\Phi_X(\delta) = \Phi^S_X(\delta) = \sqrt{2\delta}$, contienen copias casi isométricas de ℓ^2_∞. El objetivo de esta sección es demostrar que este hecho no es una coincidencia, si no que se trata de una condición necesaria para que un espacio de Banach tenga máximo valor para su módulo de Bishop-Phelps-Bollobás. Sin más dilación presentamos el resultado principal de esta parte de la memoria.

2.6.43 Teorema. *Sea X un espacio de Banach. Supongamos que existe $\delta \in (0,2)$ verificando que $\Phi_X^S(\delta) = \sqrt{2\delta}$. Entonces, X contiene copias casi-isométricas del espacio de Banach real ℓ_∞^2.*

La demostración necesita este lema que puede tener utilidad por si mismo.

2.6.5 Lemma. *Sea X un espacio de Banach. Supongamos que podemos encontrar $k \in (0,1)$ y dos sucesiones (x_n) en S_X e (y_n) en $X \setminus \{0\}$ tales que*

$$\limsup \|x_n - y_n\| \leqslant k \qquad y \qquad \liminf \left\| x_n - \frac{y_n}{\|y_n\|} \right\| \geqslant 2k.$$

Entonces X contiene copias casi-isométricas del espacio real $\ell_\infty^{(2)}$.

Comentamos que un resultado como el Teorema 2.6.43 que provea de una copia compleja de ℓ_∞^2 no es posible. Basta considerar el espacio complejo de dimensión dos $X = \ell_1^2$ que satisface que $\Phi_X(\delta) = \sqrt{2\delta}$ para $\delta \in (0,2)$, pero no contiene ninguna copia compleja de ℓ_∞^2 (obviamente si contiene una copia real de ℓ_∞^2). No sabemos si es cierto que un espacio de Banach complejo X satisfaciendo que $\Phi_X(\delta) = \sqrt{2\delta}$ para algún $\delta \in (0,2)$ tiene que contener una copia del espacio complejo ℓ_1^2 o del espacio complejo ℓ_∞^2.

A raíz del resultado anterior surge la pregunta de si la contención de ℓ_∞^2 es suficiente para tener máximo módulo de Bishop-Phelps-Bollobás. Esto no es así pues existe un contraejemplo de dimensión 3.

2.6.44 Ejemplo. Para cada $\delta \in (0,1/2)$ existe un espacio de Banach real X_δ de dimensión 3 que contiene una copia isométrica de ℓ_∞^2 pero que verifica $\Phi_X(\delta) < \sqrt{2\delta}$.

Problemas abiertos

En lo siguiente, hacemos una recopilación de los problemas que no han podido ser resueltos en nuestro estudio. Empezamos recordando la Proposición 2.6.28 que nos dice que $\Phi_X(\delta) \leqslant \Phi_{X^*}(\delta)$ y $\Phi_X^S(\delta) \leqslant \Phi_{X^*}^S(\delta)$ para todo espacio de Banach y todo $\delta \in (0,2)$. Es natural preguntarse si la desigualdad puede ser en realidad una igualdad.

2.6.45 Problema. *¿ Es verdad que $\Phi_X(\delta) = \Phi_{X^*}(\delta)$ o $\Phi_X^S(\delta) = \Phi_{X^*}^S(\delta)$ para todo espacio de Banach y $\delta \in (0,2)$?*

Como se comentó en la Sección 2.4, un candidato para el contraejemplo es el álgebra $L(H)$ para un espacio de Hilber H de dimensión infinita.

2.6.46 Problema. *Sea X un espacio de Banach con $\dim(X) \geqslant 2$, H un espacio de Hilbert con dimensión sobre $\mathbb{R}$ mayor o igual que dos, y $\delta \in (0,2)$. ¿ Se verifica la siguiente desigualdad?*

$$\Phi_X^S(\delta) \geqslant \Phi_H^S(\delta) \quad y \quad \Phi_X(\delta) \geqslant \Phi_H(\delta).$$

En otras palabras, nos estamos preguntando si el espacio de Hilbert puede ser el espacio (con dimensión mayor que uno) con menor módulo de Bishop-Phelps-Bollobás posible. Por último, y como problema natural para completar este estudio, nos preguntamos acerca del módulo de Bishop-Phelps-Bollobás en espacios de tipo $L_p(\mu)$.

2.6.47 Problema. *Calcular el módulo de Bishop-Phelps-Bollobás para los espacios $L_p(\mu)$.*

Sin lugar a dudas estamos ante un problema difícil de abordar; la dificultad principal reside en que tenemos que tratar con puntos y funcionales arbitrarios de la bola de $L_p(\mu)$ y su dual. Sin embargo, un buen comienzo podría ser considerar el espacio $L_p[0,1]$ para $1 < p < \infty$, pues este espacio es casitransitivo (esto es, para cada punto de su esfera unidad, el conjunto de las imágenes de dicho punto por isomorfismos isométricos es denso en la esfera unidad) y, por tanto, bastaría trabajar con un único punto de la esfera unidad de $L_p[0,1]$.

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