# A Relativistic Diffusion Model in Kinetic Theory

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Fdo.:

Doctorando

Fdo.:

*A Julia, Peter y a mi nueva familia*
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# Contents

# Resumen

El objetivo principal de este trabajo es analizar las soluciones de una familia de ecuaciones diferenciales parciales (EDPs) lineales y no lineales de tipo parabólico en $\mathbb{R}^{2N}$ para $N \geq 3$. En particular, estas ecuaciones pueden usarse para describir la dinámica difusiva en un marco relativista. La realización de este estudio está motivado tanto por la amplia gama de aplicaciones que tienen los modelos difusivos, así como por el poco entendimiento que se tiene de las técnicas matemáticas involucradas dentro del contexto relativista de los fenómenos de difusión.

De hecho, el término difusivo en los modelos a considerar no es uniformemente elíptico y es espacialmente degenerado, es decir, hay ausencia de algunas derivadas espaciales en el operador difusivo. Más aún, los coeficientes de algunos de los modelos estudiados dentro de este trabajo dependen de la variable temporal. Estas propiedades, y algunas otras que serán explicadas con mayor detalle en lo subsecuente, hacen distinguir a los modelos en cuestión con respecto a otros modelos difusivos estudiados en la literatura. Por lo tanto, advertimos al lector que las técnicas que se usan comúnmente para las EDPs de tipo parabólico puede que no sean aplicables dentro de nuestro contexto.

El primer modelo que consideraremos, y también el más fundamental, es la ecuación de Vlasov-Fokker-Planck relativista que fue introducida en [2]. La solución de esta ecuación describe a la función de distribución de una partícula de prueba sometida a colisiones aleatorias con un medio de fondo en equilibrio térmico (movimiento Browniano). Esta ecuación es una generalización relativista de la ecuación de Vlasov-Fokker-Planck (VFP) en el marco clásico (no relativista). Una razón para considerar la ecuación VFP relativista es porque hay aplicaciones en las cuales no se pueden omitir los efectos relativistas presentes, por ejemplo en astrofísica y en la física de los plasmas. Vamos a describir estas aplicaciones con más detalle en el capítulo 1.

A continuación damos un resumen de nuestros resultados principales en esta tesis. En nuestro primer resultado, el teorema 3.2.1, mostramos que las soluciones positivas $f_c$ de la ecuación VFP relativista convergen a las soluciones $f$ de la ecuación VFP en $L^\infty_{\text{loc}}([0,\infty), L^1(\mathbb{R}^{2N}))$ cuando la velocidad de la luz $c \to \infty$. Este resultado confirma el hecho de que la ecuación relativista

es una generalización viable de la ecuación VFP en este contexto. La sección 3.5 está dedicada a la prueba de este resultado. El argumento principal está basado en el uso de cotas a priori de los momentos de las soluciones a la ecuación relativista y estimaciones en $L^1$ de la diferencia entre una solución relativista con una no relativista $\delta f = f_c - f$. La evolución de esta diferencia está descrita por una ecuación VFP no homogénea en la cual su parte principal coincide precisamente con el operador clásico de Fokker-Planck. Ésto es conveniente porque tenemos a nuestra disposición la solución fundamental de este operador. Usaremos el principio de Duhamel y algunas propiedades de esta solución fundamental para obtener una cota apropiada en $L^1(\mathbb{R}^{2N})$ de la diferencia $\delta f$ para conseguir el resultado deseado.

El siguiente resultado de importancia es el teorema 4.3.1, el cual está enunciado y probado en la sección 4.3. En esta parte del trabajo mostramos que las soluciones espacialmente homogéneas de la ecuación VFP relativista convergen exponencialmente a su solución de equilibrio no trivial en $L^1(\mathbb{R}^N)$. Este resultado es válido solamente para valores pequeños de la temperatura del medio. La demostración se hace a partir de un argumento tipo Lyapunov —la entropia de la ecuación actúa como una función de Lyapunov— combinado con la condición de curvatura de Bakry-Emery, un criterio que asegura la validez de una desigualdad de Sobolev logarítmica. Más aún, probamos en la sección 4.4, teorema 4.4.1, que la convergencia exponencial en $L^2$ se da sin ningún tipo de restricción en la temperatura del baño térmico. En este caso usamos un argumento análogo al anterior con un criterio distinto. Mostramos que el operador elíptico de difusión tiene un gap espectral. Esta condición implica la validez de una desigualdad de Poincaré la cual nos permite probar el decaimiento exponencial deseado en tiempo de la norma $L^2$ de las soluciones.

Los dos resultados antes mencionados son una extensión del argumento que usamos originalmente en la referencia [3]:

Alcántara, J.A., Calogero, S.: *Newtonian limit and trend to equilibrium for the relativistic Fokker-Planck equation.* J. Math. Phys. **54**, 031502 (2013).

En la presente tesis, la existencia del límite Newtoniano es probado en dimensión arbitraria para condiciones iniciales que no tienen necesariamente soporte compacto, mientras que en [3] se usaron condiciones iniciales con esa propiedad en dimensión seis para simplificar el argumento. De hecho, en la sección 3.6 damos parte del argumento original de la prueba de la existencia del límite Newtoniano para mostrar como la propiedad de propagación con velocidad finita de las soluciones relativista ayuda a la simplificación antes mencionada.

Los resultados que presentaremos a continuación conciernen al sistema Vlasov-Nordström-Fokker-Planck (VNFP), un sistema no lineal de EDPs que se obtiene a partir del acoplamiento entre la ecuación VFP relativista con una ecuación de onda. El sistema VNFP es un modelo de partículas autogravitantes que considera la presencia de difusión el cual fue introducido en [2]. Los resultados presentados en este trabajo son una extensión de los que están contenidos en la referencia [4]:

Alcántara, J.A., Calogero, S., Pankavich, S.: *Spatially homogeneous solutions of the Vlasov-Nordström-Fokker-Planck system.* J. Differential. Eqs. **257**, 3700–3729 (2014).

En el artículo anterior consideramos al sistema VNFP sin fricción, mientras que en esta tesis estudiamos una ecuación más general con un término de fricción. Nuestro primer resultado para este sistema es el teorema 5.2.1 y utilizamos toda la sección 5.2 para demostrarlo. Veremos que el problema de Cauchy para soluciones espacialmente homogéneas del sistema VNFP tiene una solución fuerte que es global en tiempo cuando todas las derivades hasta orden dos tienen momentos finitos en $L^2$ de orden $\frac{\delta+1}{2}$, para alguna $\delta > 1/2$. Más aún, mostramos que el sistema exhibe un comportamiento asintótico no trivial. La función de densidad no se desvanece con o sin un término de fricción, y el campo gravitacional $\phi$ diverge a $-\infty$ conforme $t \to \infty$. Los hechos anteriores se siguen de la acotación uniforme en tiempo de los momentos de $f$ en $L^1$. Para probar la existencia de soluciones, introducimos un esquema iterativo usando la ecuación lineal de Fokker-Planck y la ecuación de Nordström asociadas al sistema y mostramos que la sucesión resultante de este procedimiento converge a una solución del sistema VNFP. Para conseguir el resultado anterior, consideramos los problemas de Cauchy para ambas ecuaciones cuando la función de densidad y el campo gravitacional están dados, respectivamente. Es interesante mencionar que el problema de Cauchy para el problema de la ecuación lineal de Fokker-Planck es estudiado mediante técnicas de la teoría de las ecuaciones estocásticas. En la sección 5.3, nuestros resultados principales son las proposiciones 5.3.1–5.3.2 y el corolario 5.3.1. Aquí consideramos la ecuación de Fokker-Planck ultra-relativista asociada a la ecuación VFP espacialmente homogénea. La razón para hacer lo anterior se debe al comportamiento asintótico formal de la ecuación bajo un reescalamiento adecuado. Este comportamiento sugiere que el perfil asintótico (no trivial) de la densidad de las soluciones relativistas y ultra-relativistas debería ser el mismo, aún cuando no hemos sido capaces de demostrarlo. Más aún, el perfil asintótico de la densidad para la ecuación ultra-relativista se puede obtener explícitamente. Para lograr lo anterior solamente es necesario observar la relación que existe entre las soluciones ultra-relativistas con las soluciones radiales de la ecuación del calor en seis dimensiones.

Ahora queremos hablar acerca de algunos problemas abiertos referentes a la ecuación VFP relativista. Ciertamente el resultado más interesante para obtener, y al mismo tiempo el más importante y desafiante, sería encontrar la expresión explícita de la solución fundamental de este modelo. Este hecho no solamente ayudaría a mejorar varios de nuestros resultados, sino también permitiría que otros resultados dentro del marco clásico pudiesen ser extendidos al relativista. Más aún, el tener a nuestra disposición la solución fundamental de la ecuación FP relativista nos permitiría abordar el problema de Cauchy para el sistema Vlasov-Nordström-Fokker-Planck sin tener que usar métodos estocásticos, ya que uno puede eliminar el campo gravitacional en el operador de colisiones de la ecuación VFP usando un conveniente cambio de variables. De hecho, el término difusivo resultante coincide con el operador elíptico sin dependencia del tiempo de la ecuación VFP lineal. Un problema menos ambicioso y más asequible sería obtener cotas precisas de esta solución fundamental. Ésto serviría en particular para quitar la hipótesis sobre el valor de la temperatura en nuestra prueba de convergencia exponencial en tiempo al equilibrio en $L^1$ de las soluciones a la ecuación VFP relativista. Si el problema anterior fuese resuelto, entonces podríamos considerar el problema de la convergencia para una familia de ecuaciones VFP en el marco general de variedades Riemannianas. Con referecia a el sistema Vlasov-Nordström-Fokker-Planck, hay varias direcciones de investigación que se pueden seguir. Por ejemplo, determinar el perfil asintótico de la función de densidad cuando las soluciones son espacialmente homogéneas permanece como una pregunta importante a responder. La justificación de lo anterior es muy simple. El modelo presenta evidencia de que los fenómenos relativistas pueden alcanzar un marco ultra-relativista a partir de la evolución del sistema. Uno podría empezar abordando este problema desde un contexto más simple, es decir, estudiando soluciones radiales del sistema como primer intento. Adoptar esta hipótesis es natural, ya que se espera que la dependencia angular de las soluciones espacialmente homogéneas desaparezca cuando $t \to \infty$. Aún en este caso, el problema sigue siendo bastante difícil de resolver. Más aún, sería muy interesante analizar el problema de Cauchy para los sistemas VNFP relativista y ultra-relativista con condiciones iniciales espacialmente no homogéneas. Probablemente si se pudiese resolver este problema, uno encontraría una manera sistemática de abordar modelos relativistas de difusión más generales. En particular, uno desea considerar las ecuaciones de Einstein en lugar de la ecuación de Nordström. En este caso, la primer dificultad a superar es la condición de elipticidad no uniforme combinada con la dependencia en tiempo de la métrica Lorentziana asociada a las ecuaciones de Einstein. Por lo tanto, uno debe considerar modelos VFP con diferentes geometrías. Ésto es una dirección interesante para adoptar por si misma. Otros problemas a resolver serían el del límite Newtoniano para el sistema VNFP y el de encontrar una tasa de convergencia a su equilibrio. Finalmente, el sistema Vlasov-Maxwell-Fokker-Planck (VMFP) es un modelo clásico que es usado para describir un plasma con colisiones aleatorias. Uno quisiera analizar varios de los problemas mencionados anteriormente para la versión relativista de este sistema. En esta dirección se puede consultar [2, 113], donde se encuentran disponibles todos los resultados probados hasta la fecha.

Concluimos este resumen con el siguiente esbozo del trabajo. La tesis está dividida en cinco capítulos y su contenido está organizado de la siguiente forma. En el capítulo 1, hacemos una reseña del concepto de difusión y de algunos elementos básicos de la teoría de la relatividad. En particular, hacemos un breve compendio de algunos de los primeros resultados más relevantes que se pueden encontrar en la teoría de difusión relativista. El capítulo 2 está dedicado a recordar resultados importantes relacionados con las soluciones de la ecuación VFP relativista, así como algunas propiedades de estas soluciones que son consideradas en lo subsecuente. Estos resultados fueron publicados en el artículo [2]. En esta parte del trabajo también introducimos los sistemas VNFP y VMFP. En el capítulo 3, el problema del límite Newtoniano para la ecuación relativista VFP es resuelto. En el capítulo 4, analizamos el comportamiento asintótico de las soluciones espacialmente homogéneas en los espacios $L^1$ y $L^2$. En el capítulo 5, recopilamos nuestros resultados referentes al sistema VNFP para soluciones espacialmente homogéneas e introducimos la ecuacón de FP ultra-relativista asociada al sistema.

# Introduction

The main objective of this dissertation is the analysis of solutions to a class of linear and non-linear parabolic partial differential equations (PDEs) in $\mathbb{R}^{2N}$ for $N \geq 3$. In particular, these equations can be used to describe diffusion dynamics in a relativistic setting. This study is motivated not only by the vast range of applications of diffusion models, but also by the still poor understanding of the mathematical techniques involved in this study. In fact, the diffusion term in the models to be considered is non-uniformly elliptic and spatially degenerate, i.e., some spatial derivatives are absent in the diffusion operator. Moreover, for some of the models studied in this thesis, the coefficients of the diffusion equation depend on the time variable. These properties, and other which will be explained in more details in the sequel, distinguish the models under discussion from the other diffusion models studied in the literature and warn that the standard techniques for parabolic PDEs might not apply to our framework.

The first and most fundamental diffusion model that we consider is the relativistic Vlasov-Fokker-Planck equation introduced in [2]. The solution of this equation describes the distribution function of a test particle undergoing random collisions with a background medium in thermodynamical equilibrium (Brownian motion). This equation is a relativistic generalization of the Vlasov-Fokker-Planck (VFP) equation in the classical (non-relativistic) setting. One reason for considering the relativistic VFP equation is that there are applications in which relativistic effects cannot be neglected, for instance in astrophysics and in plasma physics. We shall describe these applications in some details in Chapter 1 below.

This introduction continues with a summary of the main new results of this thesis. In our first result, Theorem 3.2.1, we show that positive solutions $f_c$ of the relativistic VFP equation converge in $L^\infty_{\mathrm{loc}}([0,\infty), L^1(\mathbb{R}^{2N}))$ to solutions $f$ of the VFP equation as the speed of light $c \to \infty$. The latter result confirms that the relativistic equation is indeed a viable generalization of the VFP equation in this context. Section 3.5 is dedicated to the proof of this result. The main argument relies on the use of a priori bounds on the moments of solutions to the relativistic equation and direct estimates in $L^1$ on the difference $\delta f = f_c - f$ between relativistic and non-relativistic solutions. The evolution of this difference is described by a non-homogeneous VFP equation, whose principal part coincides with the classical Fokker-Planck operator. This is quite convenient because the fundamental solution of this operator is available. Duhamel's principle and some properties of this fundamental solution are employed to obtain the appropriate bound in $L^1(\mathbb{R}^{2N})$ on the difference $\delta f$ to achieve the desired result.

Our next important result is Theorem 4.3.1, which is stated and proved in section 4.3. Here we show that spatially homogeneous solutions of the relativistic VFP equation converge exponentially fast towards a non-trivial equilibrium in $L^1(\mathbb{R}^N)$. The result is proved only for small values of the temperature of the background medium. This is obtained by using a Lyapunov type argument—the entropy of the equation acts as a Lyapunov function—combined with the Bakry-Emery curvature bound condition, a criterion that ensures the validity of a logarithmic Sobolev inequality. Moreover, we prove in section 4.4, Theorem 4.4.1, that the exponential convergence holds in $L^2$ without any restrictions on the temperature of the thermal bath. In this case we use a similar approach but with a different criterion. We show that the elliptic diffusion operator possesses a spectral gap. The latter condition implies the validity of a Poincaré inequality which allows to prove the desired exponential decay in time of the $L^2$ norm of solutions.

The two results mentioned above extend the original ones which can be found in the reference [3]:

Alcántara, J.A., Calogero, S.: *Newtonian limit and trend to equilibrium for the relativistic Fokker-Planck equation*. J. Math. Phys. **54**, 031502 (2013).

In the present thesis, the Newtonian limit result is proved in any dimension and for initial data which do not necessarily have compact support, while the latter property in dimension six was assumed for simplicity in [3]. In fact, we briefly recall part of the original proof of the Newtonian limit in section 3.6 as an example of how the finite propagation speed of relativistic solutions can be used.

The next results presented in this thesis concern the Vlasov-Nordström-Fokker-Planck system (VNFP), a non-linear system of PDEs obtained by coupling the relativistic VFP equation with a scalar wave equation. The VNFP system is a toy model for the diffusion dynamics of self-gravitating particles in the presence of diffusion which was introduced in [2]. The results presented in this thesis extend those contained in the reference [4]:

Alcántara, J.A., Calogero, S., Pankavich, S.: *Spatially homogeneous solutions of the Vlasov-Nordström-Fokker-Planck system*. J. Differential. Eqs. **257**, 3700–3729 (2014).

In this article we consider the VNFP system without friction, while the present thesis contains a study of the more general equation with a friction term. The first result for this system is Theorem 5.2.1 and its proof is carried out through all section 5.2. We show that the Cauchy problem for spatially homogeneous solutions of the VNFP system has a unique global in time strong solution when the derivatives up to order two of the initial datum have finite moments in $L^2$ of order $\frac{\delta+1}{2}$, for some $\delta > 1/2$. Moreover, we show that the system exhibits a non-trivial asymptotic behavior. The density function $f$ does not vanish with or without a friction term, and the gravitational scalar field $\phi$ diverges to $-\infty$ as $t \to \infty$. The latter facts follow from the uniform in time boundedness of the moments of $f$ in $L^1$. To prove the existence of solutions, we introduce an iterative scheme using the linear Fokker-Planck equation and the Nordström equation associated to the system and show that the resulting sequence converges to a solution of the VNFP system. In order to achieve the previous result, we consider the Cauchy problems for both equations when the density function and the gravitational field are given, respectively. It is interesting to mention that the Cauchy problem for the linear Fokker-Planck equation is studied using techniques from the theory of stochastic differential equations. In Section 5.3, our main results are Propositions 5.3.1–5.3.2 and Corollary 5.3.1. Here, the ultra-relativistic Fokker-Planck associated to the spatially homogeneous relativistic VFP equation is considered. The reason to do so is the formal limiting behavior of the equation under rescaling. This behavior suggests that the (non-trivial) asymptotic density profile of solutions to the relativistic and ultra-relativistic system should be the same, although we are not able to rigorously prove it. Moreover, the asymptotic density profile for the ultra-relativistic equation can be computed explicitly. The key observation for the latter result is the direct relation between solutions of the ultra-relativistic equation in three dimensions and solutions of the radial heat equation in six dimensions.

Now we would like to discuss some important open questions concerning the relativistic VFP. Certainly the most interesting result to obtain, and at the same time the most important and challenging one, would be to find the exact form of the fundamental solution associated to this relativistic model. The latter not only might imply an improvement on several of our results, but also would allow to extend some other known results from the classical to the relativistic setting. Moreover, knowing the fundamental solution of the relativistic FP equation would allow to treat the Cauchy problem for the Vlasov-Nordström-Fokker-Planck system without using stochastic methods, since one can avoid to deal with the explicit presence of the gravitational field in the collision operator of the VFP equation by a suitable change of variables. In fact, the resulting diffusion term coincides with the time independent elliptic operator from the linear VFP equation. A less ambitious and more achievable problem would be to obtain adequate bounds on this fundamental solution. This could help in particular to remove the small temperature assumption in our proof of exponential time convergence towards the equilibrium of $L^1$ solutions to the relativistic VFP equation. More generally, if the latter problem was solved, then one could consider the convergence problem for a wider class of VFP equations on Riemannian manifolds. Concerning the Vlasov-Nordström-Fokker-Planck system, there are several possible future research directions that one can undertake. For instance, deriving the asymptotic profile for the density when solutions are spatially homogeneous remains an important question to answer. The reason that justifies the latter is simple. The model presents evidence that relativistic phenomena can reach an ultra-relativistic regime from the evolution of the system. One could approach this problem in a simpler context, i.e., studying radial solutions of the system as a first step. This assumption seems natural, since it is expected that the dependence on the angular variables of the spatially homogeneous solutions will disappear as $t \to \infty$. Even in this case, the problem is still quite difficult to solve. Moreover, it would be interesting to analyze the Cauchy problem for the relativistic (and the ultra-relativistic) VNFP system with spatially inhomogeneous initial data. Solving this problem would likely lead to find a systematic approach for more general relativistic diffusion models. In particular, one wishes to consider the Einstein equations instead of the Nordström field equation. Then, the first difficulty to overcome is the non-uniform ellipticity condition combined with the time dependency of the Lorentzian metric associated to the Einstein equations. Therefore one must deal with VFP models with different geometries. The latter is by itself an interesting direction to pursue. The next problems to solve would be to obtain the Newtonian limit for the VNFP system and to find a rate of convergence towards its equilibrium. Finally, the Vlasov-Maxwell-Fokker-Planck (VMFP) system is a classical model that is used to describe a plasma with random collisions. Clearly, one would like to analyze several of the above mentioned problems for the relativistic version of the system. All the available results for this system are contained in [2, 113].

We conclude this introduction with a brief outline of this thesis. It is divided in five chapters and its content is organized as follows. In chapter 1, we review the concept of diffusion and some basic elements of relativity. In particular, we summarize some relevant results on relativistic diffusion theory that can be found in the earlier literature. Chapter 2 is devoted to recall some important results and properties of solutions of the relativistic VFP equation that are considered through the dissertation. These results have been published in the article [2]. We also introduce the VNFP and the VMFP systems in this part of the work. In chapter 3, the Newtonian limit problem for the relativistic VFP equation is solved. In chapter 4, the analysis of the asymptotic behavior of spatially homogeneous relativistic solutions is performed in the spaces $L^1$ and $L^2$. In chapter 5, we gather all our new results concerning the VNFP system in the spatially homogeneous regime and we introduce the ultra-relativistic FP equation associated to this system.

# Chapter 1

# An Overview on Mathematical Diffusion

The aim of the present chapter is to review some relevant aspects of diffusion theory from the relativistic and classical (non-relativistic) perspectives. To achieve this task, we need to recall some basic concepts and fundamental ideas that have been developed from the appearance of the theory. One of the main reasons to do so is that there is no systematic scheme available to introduce diffusion in a relativistic context, since some of the well-known features in the classical setting can not be adapted directly to this framework. Therefore, the relativistic diffusion theory have not yet experienced as much research progress as its classical counterpart. It is important to remark that all of the material contained in this chapter will be informally approached not only to seek as much clarity and insight on the subject as possible, but also to cover most of the essential topics in this theory. The chapter is divided in three sections. The first section is devoted to the foundations of classical diffusion. The second section contains a brief exposition of the basic elements in relativity theory. The final section is used to review some of the current progress in relativistic diffusion.

## 1.1 Classical Diffusion and Related Topics

In this section we present some of the elements that constitute the classical theory of diffusion. Our discussion will include certain historical facts and specific fundamental contributions that characterize the theory and made it possible. Since this subject is vast and covers various perspectives and aspects, we are forced to exclude several essential topics, which are beyond the scope of the current work. Therefore, we will only focus our attention on those particular matters that are directly involved in the current progress of diffusion theory when relativistic effects are accounted for. In the first part of the section we discuss the intuitive notion of diffusion. Moreover, we recall how this phenomenon is formulated in the simplest case and review some of the ideas that led to model diffusion in this manner. Then, we approach the concept of Brownian motion and look at the impact and consequences that it had on the theory.

Etymologically speaking, the word "diffusion" comes from the Latin word *diffundere* which means to spread out. The latter implies that this concept must be related to a particular kind of physical movement. In order to illustrate this kind of motion, it is convenient to present a typical situation in which this phenomenon is encountered:

*"Consider a cup of hot coffee and pour some milk into it."*

After the milk was poured, we immediately observe how milk starts spreading to regions where it has been absent. The specific mechanism in which the milk spreads in the coffee allows to distinguish this phenomenon from other types of motion, i.e., car traffic motion, the free fall of a body, etc. Other common examples where diffusion phenomena are present are given by the heat transfer problem, chemical reactions, the price behavior of stocks in the financial market, propagation of ideas, migratory behavior of some species, competition between species, etc. As a matter of fact, all the previous examples can be generally treated in the following manner. Consider a region which contains an ensemble of particles, then diffusion can be defined as:

> *"The physical movement of this ensemble from areas of higher concentration to areas of lower concentration of particles."*

One can notice that some of the examples given before apparently do not belong to this description, which partially reflects how complex diffusion actually is. This complexity can also be explained and justified through a chronological review of the concept, since there are several historical factors that influenced on the development of the diffusion theory. The main ideas to model this collective motion are based on the assumptions that no particles are lost while the motion occurs and the existence of a flux that is generated by the concentration mechanism. This last aspect is the foundation of what is known nowadays as Fick's first and second laws of diffusion. These laws state the following:

1. The flux generated by the motion is proportional to the negative of the concentration gradient, i.e., $J(t,x) = -D\nabla_x\phi(t,x)$.

2. The rate change of the density of particles in time is equal to the rate change of the flux, i.e., $\partial_t\phi(t,x) = \nabla_x \cdot (D\nabla_x\phi(t,x))$.

In the above laws, the vector $(t,x) \in (0,\infty) \times \mathbb{R}^3$ represents the particle position at time $t$, while the function $\phi(t,x)$ describes the particle density on a region. The symbols $\partial_t$, $\nabla_x$ and $\nabla_x \cdot$ denote the time derivative and the gradient and divergence operators, respectively. The constant $D > 0$ is the diffusion coefficient. Since the value of $D$ rates how particles move along a region and depends on the properties of the latter, this fact might imply that $D = D(t, x)$. Moreover, we observe that only the collective behavior of the particles is accounted for within these laws. This main feature of the model will be clarified in the forthcoming. The equation derived in point 2 is known as *the diffusion equation* and in particular, it is the prototype of a parabolic partial differential equation (PDE). The "*parabolic*" term comes from the analogous procedure to classify linear second order PDEs just as in the case of conic curves. In fact, the same transformations that are used in the previous case can be applied to a linear parabolic PDE and as a result, any second order parabolic equation with constant coefficients can always be expressed as a diffusion one by the use of an integral factor.

Before presenting more details on the deduction of these laws, it is worth to mention some previous work performed by Thomas Graham. Graham is well-known from developing the dialysis technique, a method to separate the components of a liquid through a membrane by diffusion in 1854. Another of his notorious contributions comes from his research on diffusion in gases and liquids, which was performed from 1828 to 1833. He realized that two gases of different nature mix "equally" through each other and remain in this state. Also, his experiment procedures led to a method that determines the diffusion rates on gases. His main assumption was that volumes of gas exchange were inversely proportional to the square root of their masses. Moreover, he noted the difference between the diffusion rates on gases and other mixtures. Unfortunately, his ideas were not adaptable to describe those situations as well. Despite of this, Graham's results have an additional importance for the theory, Adolf Fick based his celebrated laws of diffusion on his work in 1855. On the one hand, he considered Graham was neglecting part of "*the true nature*" of the phenomenon, which led him to unsuccessful experiments to describe diffusion of salt in its solvent. On the other hand, Fick perceived diffusion as an analogue of heat conduction, which was a remarkable and revolutionary approach for this phenomenon in that time. The same analogy also helped Georg Simon Ohm for the case of electrical conduction in 1827. Moreover, he defined what a flux is, a concept developed for heat conduction, and used the same law as the one proposed by Jean Fourier for this theory in 1822. That is the main reason why the simplest diffusion equation is also known as *the heat equation.* Probably another transcendent fact in his developments was the inclusion of a proportional constant factor $D$, *the diffusion constant.* Basically, he assumed that this factor only depends on the properties of the substance. Since this conception of diffusion was completely different, Fick experienced rejection from his ideas mainly due to two things:

1. The lack of a strong basis from a theoretical and an experimental perspective of his developments.

2. The sole idea of perceiving diffusion on liquids as transfer of heat in solids was neither conceivable nor acceptable in those times.

For the previous point, we must consider the fact that the concept of atoms and the current theory of particles were in their first development stages. Despite of theses circumstances, Fick was able to take advantage from the former atomic perspective to propose a diffusion model in which only the collective behavior of particles was described. For the remaining point, it is relevant to mention one fundamental matter in his procedures. He obtained his experimental results in a stationary regime while trying to explain a time dependent phenomenon. The fact by itself is a very good example on the importance to study the existence of *stationary* solutions for the corresponding evolution models not only for experimental purposes, but also for theoretical and practical reasons.

Until now, we have only discussed some developments performed in the theory from a "*macroscopic*" perspective, since the first achievements from a "*microscopic*" perspective started to appear after Albert Einstein published his celebrated works on the *Brownian movement* [62]. Before proceeding, it is convenient to recall these concepts. The term *microscopic* comes from the Greek words *mikrós* and *skopéō* which mean small and look, respectively. Then, we define a microscopic scale as the size in which objects or events are too small to be perceived at simple sight. Therefore, a microscope or any other device is required to amplify what it is seen. On the opposite side, events can be recorded at simple sight from a macroscopic scale. In our previous discussion, all the research in chemical reactions and heat conduction was made by naked eye or by microscopes not accurate enough to have a better perception of the phenomenon. Also, the concepts of atoms and molecules were on their first phases, as previously mentioned. This explains why the community did not adopt a microscopic description of diffusion nor conceive it at this scale. The introduction of the concept known as Brownian motion not only helped to generate new relevant fields of study in mathematics and physics, but also served as a definitive confirmation that atoms and molecules actually exist. This was further verified experimentally by Jean Perrin in [114]. The first evidence of this singular motion was found by botanist Robert Brown while studying particles of plant-pollen suspended in liquid. He observed through a microscope that these particles moved in a certain random way. Unfortunately, he could not explain nor determine the mechanism by which these pollen particles were driven. After several years, Albert Einstein gave an argument to guarantee the existence of atoms in [93], which was based on a probabilistic approach. Afterwards, he realized this discovery might also explain the "Brownian molecular movement", [62, pags. 1, 19]. In the following, we present an edited version of the argument given by Einstein in [62, pags. 13-15] to derive the diffusion equation in one dimension due to its historical importance.

Consider $n$ particles suspended in a liquid and assume that their movement is mutually independent for any given interval of time. For a small time $\tau$, the proportion of particles experiencing a displacement between $\delta$ and $\delta + d\delta$ can be expressed as $dn = n\phi(\delta)d\delta$, where $\phi(\delta)$ is a probability law that only differs from zero for very small values of $\delta$ and satisfies the following properties

$$\phi(\delta) = \phi(-\delta), \quad \int_{-\infty}^{\infty} \phi(\delta)d\delta = 1, \quad D = \frac{1}{\tau}\int_{-\infty}^{\infty} \frac{\delta^2}{2}\phi(\delta)d\delta.$$

Let $\nu = f(t,x)$ be the number of particles per unit of volume at position $x$ and time $t$. Since $\tau$ is very small, we can write $f(x, t+\tau) = f(x,t) + \tau\partial_t f$. Also, we estimate the distribution of the particles in the space variable at time $t+\tau$ from the distribution at time $t$ and expand $f(x+\delta, t)$ in powers of $\delta$. Then, we obtain the number of particles which are located at the time $t+\tau$ as follows

$$\begin{aligned} f(x, t+\tau) &= \int_{-\infty}^{\infty} f(x+\delta, t)\phi(\delta)d\delta \\ &= f + \partial_x f \int_{-\infty}^{\infty} \delta\phi(\delta)d\delta + \partial_x^2 f \int_{-\infty}^{\infty} \frac{\delta^2}{2}\phi(\delta)d\delta\,, \end{aligned}$$

where the remaining terms containing odd powers of $\delta$ vanish. The even terms are neglected due to their small contribution compared with the preceding quantity. Finally, the above identity combined with the estimate in time for $f(x, t+\tau)$ lead us to the diffusion equation

$$\partial_t f = D\partial_x^2 f,$$

from which one can easily conclude that

$$f(x,t) = \frac{n}{4\pi Dt}e^{-\frac{x^2}{4Dt}}$$

is a solution. Also, von Smoluchowski [64] and Sutherland [133] gave an explanation of the Brownian motion by using the idea of random interaction among particles. Further contributions on the matter were performed by Langevin [104, 105], Fokker [68], Planck [117], Kramers [101] and Uhlenbeck and Ornstein [135]. See [36, 144] for a review on the previous work. We remark that this was not the first deduction of the heat equation by probabilistic arguments, see [12, pags. 19-21]. The previous reference was the doctoral dissertation *Théorie de la spéculation* by Louis Bachelier. This work is considered the first accomplishment in the Option Pricing Theory and gave some of the mathematical foundations for the modern point of view. Then, it should not be a surprise that a big part of the former and current research concerning mathematical models in Finance and Economy involves the use of several techniques based on diffusion. In fact, a Nobel price in Economy was awarded to Robert C. Merton and Myron S. Scholes for their contributions. In collaboration with Fischer Black, they obtained the celebrated Black-Scholes model, a mathematical description of the price variation of stock options in a financial market. For an introduction to some of the basic and more advanced topics on the matter, see for instance [140]. For a full review and a more detailed discussion on all the previous and further historical facts can be found in [116] and the references therein.

From the previous deduction of the heat equation some tools and concepts of the modern theory can be recognized such as the use of *Itô's formula*, which relates a diffusion process with its infinitesimal generator, and the assumption of independent increments in time, from which the continuity of a Gaussian process could be obtained. Also, this argument provided the main idea to study the collective dynamics of particles when the amount of those is large, and in consequence, this behavior becomes very difficult to be deterministically approached. Moreover, this new mathematical view of diffusion phenomena led to rigorously develop the theory of stochastic processes. The latter was possibly motivated from the fact that Einstein only required to establish the existence of the transition probabilities governing the trajectories of the particles, but he never proved that a Brownian motion actually existed. As a matter of fact, Norbert Wiener was the first to construct the mathematical model for this motion, which is also known as "*the Wiener process*". Wiener approached the problem by defining an appropriate measure from subsets of the space of continuous functions on $[0,1]$, vanishing at 0. His main idea was based on recent progress in measure theory due to Lebesgue and Borel, but Wiener used the Daniell integral, which is equivalent to the integral in the sense of Lebesgue. This achievement is outstanding since current notions of the Wiener process are formulated and perceived as a stochastic object while Andrei Kolmogorov had not published at this point his proposal on the foundations of probability theory. In fact, Kolmogorov also defined probability through a measure on appropriate subsets of $\mathbb{R}^d$ by using this integral. In addition, Wiener analyzed some of the properties associated to the paths generated by the process, in which his most remarkable result is the one concerning the nowhere differentiability of the Brownian motion, and he also started to use the concept of stochastic integration with respect to the Brownian motion only for time depending integrands. See for instance [48, 93] for a more detailed discussion on the matter and for an extended review on further contributions performed by

Wiener.

Certainly Einstein exhibited that a connection between stochastic and deterministic phenomena was possible, while Wiener obtained the necessary results to develop a theory for the Brownian motion. But it was not until Kolmogorov created a precise analytical scheme for the treatment of this connection, when differential equations and stochastic processes started to share a common framework. In particular, the foundations of the theory of continuous *Markov processes* arose as a consequence from the previous investigations. Recall that a Markov process evolves without dependence from its past behavior if a precise knowledge of the present is given, since the past and future of the process are conditionally independent. A Brownian motion is the prototype of a continuous Markov process. Kolmogorov was able to achieve the latter purpose by exploiting a fundamental relation satisfied by the transition probabilities of a Markov process, the Chapman-Kolmogorov equations. Then, he used the previous feature to define the differential characteristics associated to the density of the process through a limiting procedure. If these characteristics exist, the limits will be the corresponding coefficients of two equations known as the *backward and forward differential equations*. Moreover, the probability density of the process is a solution of these equations. The previous fact allows to study the existence of a probability density for a Markov process without using its sample paths, or equivalently, the existence of a fundamental solution for the differential equation. For a Brownian motion, the forward equation corresponds to the diffusion equation and the backward equation is the formal adjoint of the latter. The introduction of these new methods to the theory of parabolic PDEs was crucial for future developments, since it motivated the problem of the existence of a fundamental solution for these equations. Later on, Kolmogorov realized that a Markov processes can be seen as a semigroup, where his former notion of differential characteristics is required to obtain the *infinitesimal generator* of the semigroup. In the one dimensional case and under appropriate growth and smoothness conditions on the coefficients, William Feller thoroughly studied existence and uniqueness of solutions for the backward and forward equations in the continuous and discontinuous cases. Also, he applied methods of semigroup theory when the transition probabilities of the process are stationary. See references [130, 132], where a thorough memoir of the life and work of Kolmogorov was performed by Shiryaev and a broad selection of essential papers due to Kolmogorov are gathered, including those directly involved on the foundations of Markov processes. See also [110, 115, 124], where the authors discuss Feller's life and achievements, and [65, 67, 147], where the authors cover the necessary material of semigroup theory that is used for diffusion models.

Within the study of diffusion processes, Itô calculus is one of the most remarkable accomplishments in the theory. This new notion of calculus was developed by Kiyoshi Itô and extends the classical calculus for the associated trajectories of a diffusion process. To overcome the nowhere differentiable issue of these trajectories, Itô created the theory of stochastic integration and introduced the concept of stochastic differential equation (SDE) as a tool to represent the infinitesimal behavior of the sample paths generated by the process. This concept also connects the evolution of the process with the parabolic PDE that governs it. As a first step, Itô proposed a large class of integrands in $L^2$ and proceeded by approximating the stochastic integral with respect to a Brownian motion as a Riemann sum in this space. The main argument involves using an isometry identity in $L^2$ and the evaluation of the integrand at the beginning of each interval. This is essential to obtain the stochastic integral. Although this idea was not completely new, Itô provided the remaining elements to the concept so it could become into a useful tool. Actually, a SDE is an integral equation, since we only have the notion of stochastic integration at our disposal, but it is always written in differential notation as

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where $W_t$ represents a Brownian motion and $\sigma(t,x), b(t,x)$, are measurable functions. The second term in the right hand side of the above equality is the stochastic integral while the first one is a deterministic integral. The idea to define a SDE as before may have been motivated from a previous attempt to construct the stochastic integral through an integration by parts formula for deterministic integrands due to Wiener. Under similar conditions as in the case of an ODE, Itô proved existence and uniqueness of solutions for a SDE and furthermore, that any solution is a Markov process. Also, Itô realized that a slight modification of the usual Leibniz rule applies in general. More precisely, if $X_t$ is a solution of an SDE and $f \in C^2$, then

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t\,,$$

where $\langle X\rangle_t$ is the quadratic variation of the process. The above expression, which is known as *Itô's formula*, has several important consequences in the theory of diffusion process. For instance, the infinitesimal generator associated to the process can be obtained from a simple argument, which is based on basic properties satisfied by the process. Moreover, solutions to the backward and forward equations can be explicitly expressed in terms of the process by using the expected values of its paths. In general, this formula and all the ideas behind it not only provided a new tool and a renewed perspective to the theory of Markov processes for further progress, but also led to develop the modern probabilistic methods that allow to solve some of the related problems in the theory. See [69] for a historical review on

Itô's work. Nowadays there are several textbooks available devoted to the matter, for instance see [9, 11, 96, 111, 123] and [66, 67], where in the last references some applications are included.

We conclude this section by recalling a very important aspect on models based on the Brownian motion. The associated Fokker-Planck equations exhibit infinite speeds of propagation which is an unphysical feature. To overcome this unpleasant feature, Andreu, Caselles and Mazón have studied a nonlinear model in [7], which is included in a family known as limited flux diffusion equations. This model was obtained by Brenier in [19], where he used optimal transportation arguments to deduce the model. The main idea of the model consists on replacing the classical flux term in Fick's laws by

$$J = \frac{mcv}{\sqrt{m^2c^2 + |v|^2}},$$

the relativistic velocity field with $m = |\phi|$ and $v = \nabla\phi$ . Unfortunately, the resulting equation does not possess other desired features to be considered in a relativistic framework. Moreover, it is not clear if one could possibly extend this model in the latter realm due to its nonlinear character. Finally, there is another proposal which will be briefly discussed in the final section of chapter.

## 1.2 Relativity

After the appearance of the celebrated work [63] by Albert Einstein in 1905, the perception of space and time drastically changed in order to explain phenomena where Newtonian physics might fail to apply. Those situations occur when objects approach the speed of light or in the presence of strong gravitational fields. Moreover, this new theory was developed in two phases: in the *special* case where two objects move at constant speed in a straight line, which corresponds to the theory of special relativity, and the *general* one. In particular, a gravitational field can only be treated in the latter case. Remarkably, this branch of physics arose from theoretical ideas in contrast to other available physical theories. Roughly speaking, relativity is based on two fundamental principles: the laws of physics are the same for "*all*" the observers and the speed of light does not change. These principles lead to one of the main features in the theory, *time is a relative quantity*. From a classical perspective, time has an absolute character. This means that if two objects move at different speeds, then the duration of the events can be recorded using *the same clock*. The latter statement is true provided that none of the objects moves with speed close to the speed of light. Otherwise, time elapses differently for each motion. Fortunately, Lorentz transformations allow to describe how these differences are related. Table 1.1 compares

| Lorentz trans. | Galilean trans. |
|---|---|
| $t' = \gamma(t - vx/c^2)$ | $t' = t$ |
| $x' = \gamma(x - vt)$ | $x' = x - vt$ |
| $y' = y$ | $y' = y$ |
| $z' = z$ | $z' = z$ |

**Table 1.1:** Lorentz and Galilean transformations in the $x$ direction.

the explicit form of these transformations in the special relativistic case with their classical analogues, the *Galilean transformations*. Here, $c$ represents the speed of light, $v$ is the velocity in which the object is moving in the $x$ direction and $\gamma = 1/\sqrt{1 - v^2/c^2}$ is Lorentz factor.

Another transcendent feature in relativity is that no body can move faster than light. Intuitively, the body becomes heavier while approaching the speed $c$. In fact, the latter could be deduced from the energy identity

$$E^2 = mc^2 = \gamma m_0 c^2,$$

which was derived in [61]. In the previous equation, $m_0$ corresponds to the rest mass of the body. In case there is no motion, the energy of the body is given by $E^2 = m_0 c^2$, since $\gamma = 1$ at rest. Another relevant quantity is the relativistic momentum of a body, which is defined as $p = mv$, since energy can also be expressed in terms of this vector as $E^2 = (m_0 c^2)^2 + (c|p|)^2$ and $(E/c, p)$ constitutes a *four vector*. The previous property not only makes reference to a dimensional aspect, but also remarks an invariance attribute under Lorentz transformations from its length. Since the notion of time in relativity is different, it also means other notions, such as length, are required to be adapted in this framework. In the usual sense, the square length of a vector can be written as $w^T g w = w \cdot w = \|w\|^2$, where $g$ denotes the identity matrix, $w$ is a vector in $\mathbb{R}^4$ and $w^T$ is the transpose of $w$. In fact, all the components of $g$ are given by diag$\{1, 1, 1, 1\}$. In the special theory of relativity, the matrix $g$ is called the *Minkowski metric* (or simply metric for short [1]) and is given by diag$\{-1, 1, 1, 1\}$[2]. The minus component is reserved for the time variable. The previous metric defines a new geometry in four dimensions with different mathematical and physical properties. For instance, there are three types of vectors that are characterized by the sign of $w^T g w$, negative for timelike vectors and positive for spacelike vectors, and the case $w^T g w = 0$ for null vectors. The origin is the point of intersection between the future light cone and the past light cone generated by the set of null vectors. Then, a vector is called future-directed if it belongs to the

[1] This is an extension of the concept in the sense of a pseudo Riemannian metric.
[2] Alternatively, one can use the convention diag$\{1, -1, -1, -1\}$.

interior of future light cone, which corresponds to a timelike vector, or lies in this cone. Similarly as before, one can defined past-directed vectors and in both cases, we can also refer to those vectors as causal.

All the previous notions can be extended for an arbitrary dimension by using a metric $g$ with components $\mathrm{diag}\{-1, 1, \cdots, 1\}$, where the number of positive entries are $N \geq 4$. Even though the most relevant case is $N = 3$, studying the previous situation in an arbitrary dimension can bring further mathematical and physical useful information for different purposes. In fact, it is necessary to consider other spaces rather than $\mathbb{R}^{N+1}$ to treat the case when a gravitational field is present. Einstein approached this problem in [60] for more general spaces that share similar properties with $\mathbb{R}^{N+1}$ in the special relativistic case. His arguments relied on an existing extension for smooth surfaces (or Riemannian manifolds in general). Now, we give a brief description on how to develop this extension by restricting the argument to the previous situation, since a similar procedure applies for the case of a Riemannian manifold. Let $S \subset \mathbb{R}^N$ be a smooth surface. Since $S$ is differentiable at each $x \in S$, there exists a tangent plane in an open neighborhood of the point which locally inherits the vector space structure of $\mathbb{R}^N$. This also means that a metric is available and is induced locally by the one from $\mathbb{R}^N$. Then, we can completely cover $S$ by a family of these neighborhoods in order to obtain a vector space structure (called the tangent bundle) for the whole surface. The latter construction is justified by two facts: the geometry of the surface can be studied in an adequate manner and a new differential calculus is obtained. Even though the previous argument is not completely accurate, it contains the main ideas that are required to extend this implementation to the case of a smooth manifolds. In fact, a smooth manifold is a topological set, which is completely covered by a family of open sets, equipped with a differential structure assigned by smooth homeomorphisms between open sets from the manifold to $\mathbb{R}^N$.

At this moment, there are several things to remark. First, considering the case of a surface is restrictive, but it suffices for our purposes since several features of this case can be incorporated in the general framework. Moreover, the algebraic structure of the vector spaces generated by the tangent planes at each point of the surface (also referred as tangent spaces) will allow to define certain geometrical quantities that have a fundamental role in the required formulation of the main problem. Moreover, the notion of a metric in a broader sense can be introduced in the previous situation. This follows by replacing the usual metric in these tangent spaces with different ones depending on the situation under study. For instance, this new metric can be positive definite, which defines the metric of a Riemannian manifold in a loose sense. The only possibility that concern us is when the metric shares similar properties with the Minkowski metric. More precisely, this metric must have a signature $\{-,+,\cdots,+\}$, i.e., it has $N$ positive eigenvalues and a negative one. Metrics possessing the previous property are known as *Lorentzian* metrics and smooth manifolds equipped with this sort of metric are referred as Lorentzian manifolds. The main idea in the theory of general relativity is that the geometry of space-time changes in the presence of gravitational forces. This phenomenon can be described with the same elements and tools as in the case of a surface.

Before proceeding, it is convenient to introduce some notation. We use $x^\alpha = (t, x^1, \cdots, x^N)$ and $x^i = (x^1, \cdots, x^N)$ to denote a vector in $\mathbb{R}^{N+1}$ and its associated spacelike vector, respectively. The latter is done to make distinction between Greek and Latin indices. Also, the components of a vector will be referred with the same notation for each $\alpha = 0, 1, \cdots, N$, for instance. If $g$ stands for a metric in $\mathbb{R}^{N+1}$, then we write $x \cdot y = g_{\alpha\beta}x^\alpha y^\beta$ for each $x^\alpha, y^\beta \in \mathbb{R}^{N+1}$, i.e., the sum is understood over repeated indices. This notation is known as the *Einstein summation convention* and it helps to simplify computations and expressions. An element of the dual space of $\mathbb{R}^{N+1}$ (a covector), or in general from any vector space of dimension $N+1$, will be denoted by $x_\alpha$. This will be congruent with our previous definitions and it serves a purpose, which is known as *lowering indices*. Notice that any covector has a representation in terms of $x_\beta = g_{\alpha\beta}x^\beta$, since $g_{\alpha\beta}$ is a linear functional. Similarly, we can also raise indices to obtain a vector from a covector as $x^\alpha = g^{\alpha\beta}x_\beta$, where $g^{\alpha\beta} = g^{-1}$ is the inverse of the metric $g$, i.e., $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$ with $\delta^\alpha_\gamma$ is the Kronecker delta function. The previous statement always holds true since all the components in the corresponding diagonal matrix to obtain the signature are nonzero, i.e., the metric is non-degenerate. It is important to remark that the previous elements not only allow to introduce some of the required geometrical concepts in relativity, but also make algebraic computations more efficient and simpler to perform than with the standard notational conventions. In fact, all the geometric objects in the theory depend on maps (also called tensors) that act on the tangent spaces and their dual spaces. In addition, one can keep track on which space the tensor is acting from the position of each index and express it differently by simple operations, just as in the case of a vector, a covector or the delta function.

Now that we have developed some of the elements required to work with Lorentzian manifolds, we are ready to introduce the notions of relativity in its general form. In order to include gravitational forces, Einstein extended relativity based on the idea that the geometry of space-time was deformed by the presence of matter generating these forces [60]. To achieve the latter, he proposed a relation between the energy-momentum tensor $T_{\alpha\beta}$, a quantity depending on the matter content, and the changes of curvature in space-time, which are given in terms of the Ricci curvature tensor $R_{\alpha\beta}$. This relation is described by *the Einstein field equations*, which is a nonlinear PDE system. In four dimensions [59], these equations can be expressed as

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where $g$ is the metric, $\Lambda$ is the cosmological constant and $R = g^{\alpha\beta}R_{\alpha\beta}$ is the scalar curvature. The speed of light and the gravitational constant have been set equal to one. This system constitutes the cornerstone of the general theory of relativity. The explicit PDE character of the system comes from the definition of the Ricci tensor, it depends on derivative terms of the metric up to order two, and the metric becomes the unknown part. Therefore, one of the main problems in general relativity consists in finding an appropriate metric that solves the Einstein equations. In the above expression, $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$ is the Einstein tensor and measures the curvature changes in the manifold. In particular for $\Lambda = 0$, the Minkowski metric is solution of this system in the absence of matter, since $G_{\alpha\beta} = 0$ for this metric, i.e., Minkowski space-time is flat. Mathematically speaking, this argument explains why a strong gravitational field can not be consider in the special relativistic case. Another interesting feature of the system concerns its conservation laws. Einstein equations must be divergence free in the sense of Lorentzian manifolds when matter is present. This is a consistency property that follows from the divergence free property for the case $T \equiv 0$ and implies conservation of energy-momentum. See for instance [121], where Rendall systematically discusses the topic and other related ones in the theory.

## 1.3 Relativistic effects in Diffusion Phenomena

As previously reviewed, classical diffusion has been intensively researched over the past decades leading to a considerable amount of progress. In contrast, the progress in relativistic diffusion is still in its early stages. There are several factors that have made this situation a difficult task and might explain why there is still no consistent and systematic approach of this phenomenon. For instance, experiments to test any proposal are hard to perform and as a consequence, all the available research on the area relies on theoretical intuition. Since accepting infinite speeds of propagation in a relativistic model is no longer an option, any attempt to mimic previous progress on diffusion becomes challenging. Although the incorporation of Lorentz invariance is fundamental, it also produces nontrivial technical and conceptual issues which might cause the loss of other desired properties. Fortunately, the information available at the moment can give us some enlightenment on certain aspects to be considered and the possible directions that could be pursued. Probably the first accomplishment in the theory was achieved by Jüttner in [95]. He introduced the first generalization of the Maxwell distribution function that accounts relativistic effects for a gas in equilibrium. His deduction relied on a thermodynamical argument where the Boltzmann entropy function was used. For a discussion on the matter in a more general setting, Dunkel, Talkner and Hänggi considered relative entropies depending on different invariant measures in [58]. In particular, the analysis included the Jüttner distribution and a modified version of the latter. Their approach allowed them to find desired symmetry properties with respect to each associated measure, but the information was inconclusive to decide which distribution function is more convenient. See also [40, 74, 78] for a further discussion on the matter.

Concerning the invariance under Lorentz transformations of the one particle distribution function in phase space, Van Kampen started the rigorous treatment of the problem in [136]. Surprisingly, his approach only required the motion of the particles and as a consequence, the distribution function does not necessarily satisfy an equation of motion. In particular, the ideal gas and the free particles[3] cases were considered. Also, Van Kampen showed that the current-density constitute a four-vector that satisfies the continuity equation. See also [45, 134], where the authors discussed some of the issues involved in the Lorentz invariance property for the one particle distribution, reviewed different proposals in the literature and analyzed some inconsistencies in previous arguments to show this invariance property. In fact, there is still no agreement on how to demand Lorentz invariance in some of the most common scenarios.

The appearance of stochastic processes as a theory to describe diffusion at a particle level motivates to approach the relativistic analogue in this sense. In particular, one wonders if it is possible to construct a relativistic process to model this phenomenon satisfying the Markov property, since the latter is desirable due to its theoretical and practical implications. For the special theory of relativity, Dudley started a systematic study of Lorentz invariant processes in [51]. He constructed these processes by defining a probability measure on sets of trajectories that are invariant under the action of the Lorentz group with speed less or equal than the speed of light. Moreover, he obtained a detailed characterization of these processes, i.e., properties with respect to the measure and the proper time[4], the semigroups generated in phase space (the hyperboloid) and the diffusion processes in phase-spacetime. The latter topic was already treated in [127] by Schay. Probably the most remarkable result in this reference (Thm.11.3), and so far, states that if one wants to work with Lorentz invariant Markov processes, then the

[3]Particles are not confined and do not interact.
[4]The time parameter or clock used along the path.

process must have states in phase-spacetime. Therefore, the class of Lorentz invariant processes in spacetime does not posses the Markov property. The positive and negative implications are quite immediate and explain the slow growth of the theory: there are at least two ways to define a relativistic diffusion process without any certainty if one of those could lead to an appropriate definition of the phenomenon. Independently from the previous work, Hakim was also able to arrive at the same conclusions in [86, Prop. 2]. His arguments were based on previous ideas by Łopuszański in [107]. Both authors noticed that the associated Fokker-Planck equation in spacetime reduced to a conservation identity which automatically excluded the possibility of having a Markov process. Also, Hakim approached the problem of defining a relativistic stochastic process in phase space from two different equivalent perspectives, one of those corresponds to the use of densities depending on the proper time. See also [85]. It is important to remark that there were no more attempts to extend the previous work nor to explore several related questions concerning the topic for a very long time. At least from a mathematical perspective, this could be explained by the fact that even at this moment some of those problems are still difficult to solve. In recent years, relativistic diffusion has attracted again the attention of some researchers. In the following, we will review some of the recent progress in the context of special and general relativity from a probabilistic perspective.

In the literature, stochastic differential equations are also referred as *Langevin equations* and are useful to describe the microscopic behavior of a phenomenon when random effects are assumed, see [72, 123]. Since the microscopic behavior is directly connected with a Fokker-Planck equation (this represents the collective or macroscopic dynamics), it is equivalent to consider either case. The first relativistic generalization of the Langevin equation was introduced in [44] by Debbasch, Mallick and Rivet in order to extend the Ornstein-Uhlenbeck process in the special relativistic context. Moreover, the Fokker-Planck equation associated to this process was found and some numerical simulations were performed. The purpose of the latter was to test the convergence of the evolution towards its equilibrium and to verify their predictions on the specific form of this state under the assumption of spatial homogeneity (no dependence on the position). Debbasch extended the previous notion to the case of a curved space-time in [42]. In particular, he analyzed in the spatially flat Friedmann-Robertson-Walker metric and showed that a modified version of the Jüttner distribution function is an equilibrium state for the corresponding Fokker-Plank equation. Later on in [43], Debbasch and Chevalier reviewed some of the main relativistic diffusion models that could be found in the literature at that moment which motivated the work performed in [38]. In the latter reference, they introduced a family of relativistic stochastic processes in which all of the models previously discussed in [43] belong to this class.

A similar program as the one described before was carried out by Dunkel and Hänggi. In [55], the authors proposed a Langevin equation parametrized by proper time and included effects of external forces. They made special emphasis on the stochastic integration rule to derive three Fokker-Planck equations with their corresponding steady states. Among these Fokker-Planck models, there is only one in which the Jüttner distribution function is the steady state of the system. In addition, some numerical simulations were performed to compare each Fokker-Planck model in stationary regime. A previous treatment of the problem was performed in [54] with a dimension reduction. In [52], the aim of authors was to identify invariant stationary momentum distributions for the classical and relativistic Brownian motions by using an integral criterion in one dimension. This criterion allowed the authors to obtain the transition probability distribution function for a Brownian particle after collision. The purpose to do so was to recover further information from the collision process among particles. In fact, their results in the non-relativistic framework corroborated the fact that the Maxwellian is the invariant distribution after particles had collided while in the relativistic case, a modified version of the Jüttner distribution function was the one to remain invariant. The previous situation might not be a coincidence as their following work [56] showed. The authors in collaboration with Weber studied how the relativistic Langevin equation changes under different time parameters. In particular, they presented the relativistic Langevin equation under the action of Lorentz transformations. Also, they showed that the modified Jüttner distribution function arises as a steady state when this Langevin equation is parametrized by the proper time. See for instance [41], where further justification of this matter was achieved. See also the treatment performed by Herrmann in [90] and by Haba in [79, 80, 81, 82]. A thorough review from Dunkel and Hänggi on relativistic diffusion in special relativity can be found in [53].

Accounting diffusion phenomena in the general relativity framework is a very complicated task since the geometry of space and time is completely determined by the Einstein field equations. As a consequence, the coupling between these equations and the relativistic diffusion equation will generally result to be inconsistent. In order to handle this issue, one must confront some important aspects that are very difficult to experimentally verify. For instance in [20], Calogero proposed a Lorentz invariant model in which the addition of an appropriate cosmological scalar field in the Einstein field equations was required. Also, the qualitative behavior of solutions for the simplest cosmological model, the flat Robertson-Walker spacetime, was studied. The main advantage of the approach used in [20] consists in the simplicity of the argument to define diffusion, which is basically adding the Laplace-Beltrami operator in the corresponding Vlasov equation (the free transport of particles case). See also [5, 21, 30] for further treatment of the problem. In [77], Haba treated a similar situation as the previous one for the ultra-relativistic problem on the hyperboloid with a modified version of the Einstein field equations. The extra term in the previous system is justified to compensate the coupling between models since there is an undetectable part in the phenomenon, which is also the reason to consider a random interaction. His analysis consisted on deriving ultra-relativistic solutions of the model at finite temperature. The limiting behavior of solutions as the temperature approached to zero was also accounted for in order to explicitly calculate the extra term in the Einstein field equations. The main motivation to pursue this program was to describe the expansion of the universe in different stages by means of diffusion. From a microscopical perspective, Franchi and Le Jan proposed a generalization of the relativistic Markov process constructed in [51] for the case of Lorentzian manifolds in [70]. The main idea to obtain this relativistic process is very similar to the one used to prove the existence of a Brownian motion on a Riemannian manifold. Also, they gave the exact form of the infinitesimal generator of the process which corresponds to the geodesic flow plus a Laplace operator. In particular, the authors considered the Schwarzschild space and gave a detailed analysis of the relativistic stochastic process in this case. See also [71, 84, 91].

Another interesting approach of the problem without the use of stochastic arguments was performed by Chacón-Acosta and Kremer in [35]. They were able to obtain two relativistic Fokker-Planck equations from a relativistic version of the Boltzmann equation. As a first step, the authors performed an approximation of the collision integral in the Boltzmann equation for a relativistic gas and as a result, the values of the friction and diffusion coefficients for the Fokker-Planck operator followed from integral factors that account the collisions experienced by the particles. Moreover, they considered the case where classical Brownian particles are mixed with a relativistic gas in equilibrium. Then, they used the relativistic Boltzmann equation satisfied by the Brownian particles distribution in terms of the relativistic gas and proceeded similarly as in the previous situation.

As mentioned above, there is a second alternative to construct a model for relativistic diffusion in space-time which involves to treat non-Markovian processes. Unfortunately, this direction is less explored than the Markovian case, which is also more natural to consider since diffusion depends on previous states, i.e., the process should have memory of its past. Even though classical diffusion models do not possess the desired finite propagation speed of particles and their associated stochastic processes only keep record of their immediate past, these have proven to reasonably approximate the phenomenon under appropriate restrictions. Moreover, this could explain and justify why the research on the matter still focuses on models based in parabolic equations. See for instance [100], where the authors analyzed some of the involved issues from these models. The previous reference generated a controversy and in response, the following works [89, 99] were published.

To our knowledge, there are two proposals in the literature as plausible options to model relativistic diffusion from a non-Markovian perspective. One of those involves using the telegraph equation [109] since its structure not only resembles the simplest diffusion equation with an extra second order term in time, but also the model predicts finite propagation speed from its hyperbolic nature. As a consequence, solutions of this equation will show similar behavior and properties from solutions of parabolic and hyperbolic equations. Unfortunately, the latter also means that singular propagation fronts will appear, which are not a physical part of the phenomenon. In addition, the classical diffusion and wave equations can be recovered from two different limiting procedures. Some further facts are also discussed in [109] such as four derivations of the model, a method to find solutions, etc. See also [97, 98]. The second approach of the problem was given in [57] by Dunkel, Talkner and Hänggi. They used the fact that the probability transition densities in the non-relativistic case can be written as an integral in terms of the total action per mass under appropriate boundary conditions. Then, the extension to the relativistic case is straightforward and the resulting process is not Markovian. Moreover, the new process has continuous paths which is a main advantage in comparison with the telegraph equation. Also, this perspective might open the possibility to characterize other kinds of diffusion phenomena.

# Chapter 2

# The relativistic Fokker-Planck equation

In this chapter we review some previous results obtained for a relativistic generalization of a kinetic Fokker-Planck equation. The latter work is not only our main motivation for the current presentation, but it is also the foundation to study new mathematical challenges. One reason to consider and analyze this generalization relies on the fact that essential relativistic features are captured by the model. Moreover, the latter also allows to introduce two relativistic systems of interest in gravitational and plasma physics. This follows from coupling the Fokker-Planck dynamics to a non-linear scalar mean-field in the gravitational case, the Nordström theory, and to the Maxwell equations of electrodynamics in the case of plasmas.

## 2.1 Introduction

Fokker-Planck equations provide a continuous description for the stochastic collective dynamics of a large amount of particles. The prototype model and most basic example to consider is when the movement of each particle is assumed to be governed by the Brownian motion, the random motion of a test particle immersed in a fluid in thermodynamical equilibrium. If the test particle is heavier than the molecules of the fluid, then it is possible to approximate the microscopic forces acting on the test particle by two driving mechanisms: diffusion and friction. Assuming that the mass of each particle equals to one, the kinetic equation describing the evolution of the distribution function for test particles is the following linear Fokker-Planck (or Kramers) equation [123]:

$$\partial_t f + p \cdot \nabla_x f = \nabla_p \cdot (\sigma \nabla_p f + \beta p f) \,. \tag{2.1}$$

Here the non-negative function $f$ denotes the distribution of particles and depends on the variables $(t, x, p)$, where $t > 0$ is the time variable and

$(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$ are the phase-space coordinates that represent position $x$ and momentum $p$. The positive constants $\beta$ and $\sigma$ are the friction and diffusion parameters, respectively. Equations like (2.1), or variants thereof, have several applications in different fields of physics and engineering. In astrophysics, for example, they model the effect of interstellar nebulae in a galaxy [128] or even dark matter [108]. In plasma physics, these equations take into account the effect of grazing close encounters among the ions (the heavy particles) and the electrons.

A common strategy to derive Fokker-Planck type equations is to start from a system of stochastic differential equations (SDEs) that represents the microscopic behavior of each single particle. Then, it can be shown that the probability law of the process solving the SDEs satisfies a parabolic partial differential equation (PDE) of this kind. The latter easily follows by applying standard results from the theory of stochastic calculus. For instance, the system of SDEs associated to the kinetic equation (2.1) are given by

$$\dot{x}(t) = p(t), \quad \dot{p}(t) = -\beta p(t) + \sqrt{2}\,\sigma B(t), \tag{2.2}$$

where $B(t)$ is a standard Brownian motion in $\mathbb{R}^N$, i.e., a Gaussian process with covariance $\langle B(t), B(t')\rangle = \delta(t-t')$ and centered at the origin. Then, well-known results from Itô calculus allow to obtain equation (2.1) from system (2.2), see [36, 96, 123] for details.

Despite of the fact that equation (2.1) provides a reasonable model for describing diffusion in many situations, it also features an incompatible property with the well-established physical law that prevents particles from moving faster than light. In other words, the transport-diffusion term $-p \cdot \nabla_x f + \sigma \Delta_p f$ in (2.1) operates with infinite velocity: if particles are initially distributed in a compact region of space, the probability to find these particles everywhere will be instantaneously non-zero, i.e., the initial distribution $f(0,x,p)$ is compactly supported in the variable $x$ and $f > 0$ for any $(t,x,p) \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$. The latter statement is equivalent to say that some particles will travel faster than the speed of light. Recent works in the mathematical and physical literature put forward two possible solutions to eliminate this undesirable feature. One consists in replacing the classical linear diffusive (Laplace) operator with a non-linear diffusion term as performed in [7] for the so-called " relativistic" heat equation. Another mathematically simpler solution is obtained by replacing (2.1) with a model that is still linear and at the same time, it will turn out to be consistent with the relativistic mechanics of particles, where the property of finite propagation speed enters in a natural fashion. For the later case, the physics literature abounds of proposals for what should represent the correct relativistic generalization of (2.1), see for instance [35, 55]. Thus the first problem to confront is the choice of the relativistic Fokker-Planck equation to consider. In this work, the following equation is analyzed:

$$\partial_t f + \hat{p} \cdot \nabla_x f = \nabla_p \cdot (\sigma D \nabla_p f + \beta f p)\,, \tag{2.3}$$

where $\hat{p}$ is the relativistic velocity and $D$ is the *relativistic diffusion matrix*. These quantities are given by

$$\hat{p} = \frac{p}{\sqrt{1+|p|^2}}, \qquad D = \frac{I + p \otimes p}{\sqrt{1+|p|^2}}.$$

The previous model coincides with one of the equations proposed in [55], namely [55, Eq. (47)], and it is the subject of a recent series of papers by Haba [79, 81, 78, 83]. In these references, several generalizations of (2.3) are introduced, including models for massless particles, for particles with spin and models with more general friction terms.

For applications in astrophysics (resp. plasma physics), it is necessary to add the interaction of the particles with the self-generated gravitational (resp. electric) field. In the non-relativistic case this leads to the non-linear Vlasov-Poisson-Fokker-Planck system:

$$\partial_t f + p \cdot \nabla_x f - \nabla_x U \cdot \nabla_p f = \nabla_p \cdot (pf + \nabla_p f)\,, \tag{2.4a}$$

$$\Delta_x U = \lambda \rho, \quad \rho(t,x) = \int_{\mathbb{R}^N} f(t,x,p)\,\mathrm{d}p, \tag{2.4b}$$

where we have set all the physical constants equal to one. When $\lambda = 1$, the model corresponds to the gravitational case, while $\lambda = -1$ is used in the plasma physics context. Sections 2.4–2.5 will be devoted to introduce the relativistic generalizations for these models. In the gravitational case we will couple the Fokker-Planck dynamics to a relativistic scalar theory of gravity, the Nordström theory, which has already been used as a toy model for Einstein's theory of general relativity, see for instance [24, 28, 29, 129]. Unfortunately, there are fundamental difficulties in formulating a Fokker-Planck theory in general relativity that are briefly recalled at the beginning of Section 2.4. In order to avoid this issue, we consider instead a simpler model, which we name the Vlasov-Nordström-Fokker-Planck system. In the plasma physics case we couple the relativistic Fokker-Planck equation (2.3) to the Maxwell equations of electrodynamics. The resulting model is named the Vlasov-Maxwell-Fokker-Planck system. We remark that this model is different from the one considered in [16, 103, 146], which uses the non-relativistic Fokker-Planck equation (2.1). The main result for both systems (with an external confining potential) is the existence of steady states solutions for *all* possible values of the mass. We do so by variational techniques inspired by [47]. Note that in the gravitational case the result can be improved in comparison with the one available for the Vlasov-Poisson-Fokker-Planck system $(2.4)_{\lambda=1}$, for which the existence of steady states is only known for a properly small mass [18]. The main advantage of the Vlasov-Nordström-Fokker-Planck system is that its energy is positive definite while in the non-relativistic framework is indefinite.

In Section 2.2, we derive the relativistic Fokker-Planck equation (2.3) by showing that certain important physical properties satisfied by the non-relativistic model (2.1) are maintained. For instance, the non-relativistic equation (2.1) is invariant by Galilean transformations when $\beta = 0$, i.e., in the absence of friction. In the case of equation (2.3), the Lorentz invariance property, which is the relativistic analogue, will be automatically fulfilled from the arguments that we will use. Note that in both cases, the friction term breaks the equivalence of inertial reference systems. In Section 2.3, we prove that solutions of equation (2.3) enjoy some other mathematically and physically desirable features, in particular that they behave consistently with the finite propagation speed of particles.

## 2.2 Derivation of the relativistic model

The purpose of this section is to justify the reason to choose equation (2.3) as a relativistic generalization for equation (2.1). In particular, we will show that it is possible to derive this model by merely demanding that certain analogous physical properties of the non-relativistic case are maintained. Otherwise, following a stochastic approach for this relativistic version as in the case of equation (2.1) is problematic for at least two reasons. It is not very clear how to define a "standard" relativistic Brownian motion and; there are multiple ways to obtain a Fokker-Planck equation from a system of SDEs when the diffusion matrix is not constant. For instance, the latter situation already occurs in (2.3) while this is not an issue for equation (2.1). It is equivalent to obtained the previous model from (2.2) by either using Itô or Stratonovich calculus, since $\sigma$ is constant. As a direct consequence of these "ambiguities", there exist different models in the literature which are named "relativistic Fokker-Planck equation", see [43, 53] for a review. For the rest of the chapter, we will not refer to the system of SDEs associated to the (relativistic) stochastic process since this will not be required. Despite of this situation, it can be observed that our equation coincides with one of the models derived in [53, 55] by stochastic calculus methods.

Now, we require to recall some properties that the non-relativistic Fokker-Planck equation (2.1) possesses. In fact, we focus on the following two:

(NR1) In the absence of friction, i.e., when $\beta = 0$, equation (2.1) is Galilean invariant[1]. The latter means that under the change of variables

$$\tilde{t} = t\,,\;\; \tilde{x} = x - ut,\;\; \tilde{p} = p - u, \qquad \tilde{f}(\tilde{t},\tilde{x},\tilde{p}) = f(t,x,p),$$

$\tilde{f}$ is a solution of $(2.1)_{\beta=0}$ if and only if $f$ is a solution, $\forall u \in \mathbb{R}^N$.

(NR2) The Maxwellian distribution function

$$\mathscr{M}(p) = e^{-\beta|p|^2/2\sigma}$$

is a non-trivial static solution of (2.1). In fact, up to a multiplicative constant, it is the only global equilibrium of the equation.

Now, we propose a relativistic generalization of (2.1) by requiring that the relativistic analogues of the properties (NR1) and (NR2) hold. More precisely, the relativistic Fokker-Planck equation should satisfy:

(R1) Invariance under Lorentz transformations in the absence of friction, i.e., under the change of variables[2]

$$u_0 = \sqrt{1+|u|^2}, \quad \tilde{t} = u_0 t - u\cdot x\,, \quad \tilde{x} = x - ut + \frac{u_0 - 1}{|u|^2}u(u\cdot x),$$
$$\tilde{p} = p - u\sqrt{1+|p|^2} + \frac{u_0-1}{|u|^2}u(u\cdot p), \quad \tilde{f}(\tilde{t},\tilde{x},\tilde{p}) = f(t,x,p),$$

$\tilde{f}$ is a solution of the frictionless equation if and only if $f$ is a solution, $\forall u \in \mathbb{R}^N$.

(R2) The function $\mathscr{J}$ defined by

$$\mathscr{J}(p) = e^{-\gamma\sqrt{1+|p|^2}},$$

must be a static solution, for some constant $\gamma > 0$. $\mathscr{J}$ is known as the Jüttner distribution (or relativistic Maxwellian).

In our opinion, the simplest and most natural way to obtain (R1) is the following. Firstly we replace the transport term in the left hand side of (2.1) by its relativistic counterpart

$$\sqrt{1+|p|^2}\,\partial_t + p\cdot\nabla_x = \sum_{\mu=0}^{N} p^\mu\partial_\mu = p^\mu\partial_\mu,$$

with $p^0 = \sqrt{1+|p|^2}$, $p = (p^1,\cdots,p^N)$, $\partial_0 = \partial_t$ and $\partial_i = \partial_{x^i}$. Secondly the diffusive operator $\Delta_p = \nabla_p\cdot\nabla_p$ on the right side of (2.1) is replaced

[1] The friction term $\nabla_p\cdot(\beta p f)$ breaks the Galilean invariance of (2.1), since it corresponds to the microscopic velocity-dependent force $F = -\beta p(t)$ in (2.2).

[2] We fix $c = 1$, where $c$ is the speed of light.

by the Laplace-Beltrami (LB) operator $\Delta_p^h$ over the Riemannian manifold $(\mathbb{R}^N, h)$, where $h$ is the hyperbolic metric, the Riemannian metric induced by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$ over the hyperboloid $\mathfrak{H} = \{(p^0, p) : p^0 = \sqrt{1+|p|^2}\}$. The components of the metric in the base $\partial_{p^i} \otimes \partial_{p^j}$ of the linear space of second order covariant tensor fields on $\mathfrak{H}$ are given by

$$h_{ij} = \delta_{ij} - \hat{p}_i \hat{p}_j, \quad \hat{p} = \frac{p}{p_0},$$

where $p_k = \delta_{kl} p^l$ [3] and $p_0 = p^0$. The Lorentz invariance property of the operator $\Delta_p^h$ comes from the fact that a Lorentz transformation in the momentum variable corresponds to a translation over the hyperboloid $\mathfrak{H}$. Now, let $(h^{-1})^{ij} = \delta^{ij} + p^i p^j$ denote the inverse matrix of $h_{ij}$, i.e., $(h^{-1})^{ik} h_{kj} = \delta^i_j$, with $|h| = \det(h_{ij}) = (1+|p|^2)^{-1}$. The action of the LB operator $\Delta_p^h$ on scalar functions is given by

$$\Delta_p^h f = \frac{1}{\sqrt{|h|}} \partial_{p^i} \left( \sqrt{|h|} (h^{-1})^{ij} \partial_{p^j} f \right). \tag{2.5}$$

Therefore the frictionless relativistic Fokker-Planck equation is

$$\partial_t f + \hat{p} \cdot \nabla_x f = \sigma \partial_{p^i} \left( \frac{\delta^{ij} + p^i p^j}{\sqrt{1+|p|^2}} \partial_{p^j} f \right), \tag{2.6}$$

where $\sigma > 0$ is the diffusion constant.

To achieve (R2), it is sufficient to add a friction term to the right hand side of (2.6) of the form $\partial_{p^i}(q^i(p) f)$ and such that the current

$$A^i = \sigma \frac{\delta^{ij} + p^i p^j}{\sqrt{1+|p|^2}} \partial_{p^j} f + q^i f$$

vanishes for $f = \mathscr{J}$, since $\nabla_x \mathscr{J} = 0$. Using the following computations

$$\begin{aligned} &\partial_{p^l} \mathscr{J} = -\gamma \mathscr{J} \frac{p^l}{p^0}, \quad \frac{\delta^{ij} + p^i p^j}{\sqrt{1+|p|^2}} p^j = p^0 p^i, \\ &\frac{\delta^{ij} + p^i p^j}{\sqrt{1+|p|^2}} \partial_{p^j} \mathscr{J} = -\gamma p^i \mathscr{J}, \end{aligned} \tag{2.7}$$

we immediately see that $A^i = 0$ if and only if $q^i(p) = \gamma \sigma p^i$. Then, the relativistic Fokker-Planck equation with friction is:

$$\partial_t f + \hat{p} \cdot \nabla_x f = \partial_{p^i} \left( \sigma \frac{\delta^{ij} + p^i p^j}{\sqrt{1+|p|^2}} \partial_{p^j} f + \beta f p^i \right), \tag{2.8}$$

[3] The position of the indexes (above or below) is changed using the Euclidean metric.

where $\beta = \gamma\sigma$ is the friction parameter. Since all the results involving equation (2.8) are independent from the value of the physical constants, we set $\beta = \sigma = \gamma = 1$. Moreover, in order to guarantee the existence of an equilibrium with finite mass in the whole space, we assume that the system is subject to the action[4] of an external confining potential $V = V(x)$, and write the equation in the following final form

$$\partial_t f + \hat{p} \cdot \nabla_x f - \nabla_x V \cdot \nabla_p f = \nabla_p (D\nabla_p f + pf), \tag{2.9}$$

where $(t, x, p) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N$, $\hat{p} = p/p^0$ and $D$ is the diffusion matrix

$$D^{ij} = \frac{\delta^{ij} + p^i p^j}{\sqrt{1 + |p|^2}} = (\delta^{ij} + p^i p^j)(p^0)^{-1}, \tag{2.10}$$

with $p^0 = \sqrt{1 + |p|^2}$. Throughout the chapter we assume $V \in C^1$ and

$$e^{-V} \in L^1(\mathbb{R}^N).$$

To conclude this section, we remark that (2.8) coincides with one of the equations proposed in [55], namely [55, Eq. (47)]. In this reference, the authors derive three different relativistic Fokker-Planck equations starting from a particular relativistic Langevin dynamics and using the pre-, mid- and post-point rule of discretization for stochastic integrals, see also [53]. Equation (2.8) is the only one, among the equations introduced in [55], that satisfies the properties (R1)-(R2) above. Our purpose in the following section is to review previous results concerning this equation.

## 2.3 Properties of relativistic solutions

In this section, we prove some fundamental properties that solutions of equation (2.9) possess. Our first task is to establish an existence and uniqueness result associated to the initial value problem of equation (2.9). To achieve the latter, we will consider the Cauchy problem for the equation

$$\partial_t h(t, x, p) + Ah(t, x, p) = 0, \qquad A = T - L, \tag{2.11}$$

with $(t, x, p) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N$ and the operators $T$, $L$ are given by

$$T = v(p) \cdot \nabla_x - \nabla_x V \cdot \nabla_p, \quad L = \Delta_p^g + W, \tag{2.12}$$

[4] The action of the external potential is equivalent to that of a spatially dependent friction term, which can be seen by writing (2.9) in the form

$$\partial_t f + \hat{p} \cdot \nabla_x f = \nabla_p \cdot (D\nabla_p f + f\,(p + \nabla_x V)).$$

where $\Delta_p^g$ is the LB operator with respect to the Riemannian metric $g$ on $\mathbb{R}^N$, see (2.5), and $v, W$ are the vector fields

$$Wh = g^{-1}\nabla_p \log u \cdot \nabla_p h, \quad u = \sqrt{\det g^{-1}}e^{-E}, \quad v = \nabla_p E, \tag{2.13}$$

for some non-negative function $E = E(p)$. The matrix $g^{-1}$ stands for the inverse matrix of $g$, i.e., $g^{ij}g_{jk} = \delta_k^i$[5]. We assume that $g, E, V \in C^\infty$ and $e^{-E}, e^{-V} \in L^1(\mathbb{R}^N)$. The latter property allows to define the probability measure $\mu$ as

$$\mathrm{d}\mu = \Theta^{-1}e^{-E-V}\,\mathrm{d}p\,\mathrm{d}x, \quad \Theta = \int_{\mathbb{R}^{2N}} e^{-E(p)-V(x)}\,\mathrm{d}p\,\mathrm{d}x.$$

The previous measure is used to establish our results on weighted Sobolev spaces which will enable us to equivalently interpret those results in the usual Sobolev ones by a simple transformation in terms of $h$ in (2.11).

The main reasons to consider equation (2.11) as a first step to prove the existence and uniqueness result for equation (2.9) are very simple. First, equation (2.9) can be expressed as a particular case of equation (2.11). Also, the main argument to deal with the degeneracy in $p$ can not be applied directly for any function $V$ with low regularity. In fact, we require the corresponding result for (2.11) when $V \in C^\infty$ and the initial data are $C^1$ with compact support. From here, this result easily extends for the case of equation (2.9). Before we state and prove the corresponding Cauchy problem for (2.11), we gather some basic properties of the operators $A$, $T$ and $L$ with respect to the measure $\mu$.

**Lemma 2.3.1.** *Assume that $h, h_1, h_2 \in C^\infty$, then the following holds*

(a) $\int_{\mathbb{R}^{2N}} h\,Th\,\mathrm{d}\mu = 0$;

(b) $\int_{\mathbb{R}^N} h\,Lh\,e^{-E}\,\mathrm{d}p = -\int_{\mathbb{R}^N} g^{ij}\,\partial_{p^i}h\,\partial_{p^j}h\,e^{-E}\,\mathrm{d}p$;

(c) $A(h_1h_2) = h_1Ah_2 + h_2Ah_1 - 2g^{ij}\partial_{p^i}h_1\,\partial_{p^j}h_2$.

*Proof.* To proof *(a)*, we use definition (2.12) and perform an integration by parts to see that

$$\int_{\mathbb{R}^{2N}} h(v\cdot\nabla_x h - \nabla_x V\cdot\nabla_p h)\mathrm{d}\mu = \int_{\mathbb{R}^{2N}} h^2(v\cdot\nabla_x V - \nabla_x V\cdot\nabla_p E)\mathrm{d}\mu + \int_{\mathbb{R}^{2N}} (\nabla_p h\cdot\nabla_x V - v\cdot\nabla_x h)h\mathrm{d}\mu,$$

[5] For notational convenience, we simply write $(g^{-1})^{ij} = g^{ij}$ in this case.

and the assertion follows by the previous identity and the fact $v(p) = \nabla_p E$. For the second statement, we account the definition of the LB operator (2.5) and apply an integration by parts to the right hand side of *(b)* as follows

$$\begin{aligned}\int_{\mathbb{R}^N} h\Delta_p^g h\, e^{-E}\mathrm{d}p &= \int_{\mathbb{R}^N} hg^{ij}\partial_{p^j} h\left(\partial_{p^i}E + \frac{1}{2|g|}\partial_{p^i}|g|\right)e^{-E}\mathrm{d}p \\ &\quad - \int_{\mathbb{R}^N} g^{ij}\partial_{p^i}h\,\partial_{p^j}h\, e^{-E}\,\mathrm{d}p \\ &= -\int_{\mathbb{R}^N} g^{ij}\partial_{p^i}h\,\partial_{p^j}h\, e^{-E}\,\mathrm{d}p - \int_{\mathbb{R}^N} hWh\, e^{-E}\,\mathrm{d}p\,,\end{aligned}$$

where we used that

$$\frac{1}{2|g|}\partial_{p^i}|g| = -\partial_{p_i}\left(\log\frac{1}{\sqrt{|g|}}\right), \qquad \nabla_p E = -\nabla_p \log e^{-E},$$

and $\det g^{-1} = 1/|g|$. The proof of *(c)* follows directly by Leibniz's rule. □

Now we are ready to present one of the main results of this chapter. The proof is based on the arguments performed in [87, Prop. 5.5] for the non-relativistic Fokker-Planck equation (2.1) (with external potential), and in [22, App. A], which studies the Cauchy problem for (2.11) when $x \in \mathbb{T}^d$ (the $d-$dimensional torus) without external potential.

**Theorem 2.3.1.** *Assume that $h_{\mathrm{in}} \in C_c^1(\mathbb{R}^N \times \mathbb{R}^N)$ and for all $p \in \mathbb{R}^N$, the following conditions hold*

$$\det(\partial_{p^i}v_j) \neq 0, \quad \partial_{p^i}(g^{ij}\partial_{p^j}E) \leq \omega, \ \textit{for some}\ \omega > 0, \tag{2.14}$$

$$g^{ij}\partial_{p^i}E\,\partial_{p^j}E \geq \theta|\nabla_p E|^2, \ \textit{for some}\ \theta > 0, \tag{2.15}$$

$$\frac{g^{ij}(p)}{|p|^2} \to 0, \ \textit{as}\ |p| \to \infty \quad \forall\, i,j = 1,\dots d. \tag{2.16}$$

*Then, there exists a unique solution for the Cauchy problem associated to equation (2.11) with initial datum given by $h(0,x,p) = h_{\mathrm{in}}$ and*

$$h \in C([0,\infty), L^2(\mathrm{d}\mu)).$$

*Proof.* Consider the operator $A$ defined by (2.11)–(2.12) on the domain $D(A) = C_c^\infty(\mathbb{R}^{2N})$ and denote $\mathcal{H} = L^2(\mathrm{d}\mu)$. Our main goal is to show that the closure of the operator $A$ generates a contraction semigroup on $\mathcal{H}$. To achieve this purpose, it suffices to prove that $A$ is accretive and the range of $A + \lambda I$ is dense in $\mathcal{H}$ for some $\lambda > 0$, see for instance [87, Sec. 5.2]. As a direct consequence of (a)–(b) in Lemma 2.3.1, we see that $A$ is accretive

$$\langle h \mid Ah\rangle_{\mathcal{H}} = \langle h \mid Th\rangle_{\mathcal{H}} - \langle h \mid Lh\rangle_{\mathcal{H}} = \int_{\mathbb{R}^{2N}} g^{ij}\partial_{p^i}h\,\partial_{p^j}h\,\mathrm{d}\mu \geq 0.$$

In order to prove the remaining property, we will require to show that the operator $A$ is hypoelliptic. Let $a = \sqrt{g^{-1}}$, the positive definite matrix such that $a^2 = g^{-1}$. A direct computation shows that

$$-A = \sum_{i=1}^{N} Y_i^2 + Y_0,$$

where $Y_0$ and $Y_i$ denote the vector fields

$$\begin{aligned} Y_0 h &= (\mathrm{div}_p a) \cdot a \nabla_p h - g^{ij} \partial_{p^i} E \, \partial_{p^j} h - Th, \\ Y_i h &= a_i^k \partial_{p^k} h. \end{aligned}$$

Then $A$ is hypoelliptic if $-A$ satisfies a rank 2 Hormander's condition, i.e., the vector fields $\{Y_i, Z_j\}$ is a basis of $\mathbb{R}^{2N}$ with $Z_i := [Y_0, Y_i]$[6], see [92]. Observe that

$$Z_i = B_i^k \partial_{p^k} + C_i^j \partial_{x^j},$$

where $B$ is a $N \times N$ matrix whose exact form is irrelevant for what follows and $C_i^j = a_i^k \partial_{p^k} v^j$, since $v = \nabla_p E$. Thus we are able to represent the linear transformation $\{\partial_{x^i}, \partial_{p^j}\} \to \{Y_k, Z_l\}$ by

$$F = \begin{pmatrix} 0 & a \\ C & B \end{pmatrix},$$

whose determinant is $|\det F| = \det a |\det C| = \det g |\det(\partial_{p^k} v^j)|$, which is positive because $\det(\partial_{p^i} v_j)$ is non-zero by assumption and the claim follows.

Finally, we prove that the range of $\lambda + A$ is dense in $\mathcal{H}$ for some $\lambda > 0$. We must show that if $h \in \mathcal{H}$ is such that

$$\langle h | (\lambda + A) f \rangle_{\mathcal{H}} = 0, \quad \text{for all } f \in D(A), \tag{2.17}$$

then $h = 0$. Equation (2.17) is equivalent to state that $h$ is a distributional solution of

$$(\lambda + T - L) h = 0.$$

Since the operator $\lambda + T - L$ is hypoelliptic, this property implies that $h \in C^\infty$ by definition. Now setting $h_1 = \phi$, $h_2 = \phi h$ in *(c)* of Lemma 2.3.1, multiplying the previous identity by $h$, integrating over the whole domain and using that $\left\langle h | (\lambda + A)(\phi^2 h) \right\rangle_{\mathcal{H}} = 0$, by (2.17), we obtain

$$\begin{aligned} \lambda \int \phi^2 h^2 \, \mathrm{d}\mu + \int g^{ij} \partial_{p^i} (\phi h) \, \partial_{p^j} (\phi h) \, \mathrm{d}\mu = \int & h^2 g^{ij} \partial_{p^i} \phi \, \partial_{p^j} \phi \, \mathrm{d}\mu \\ & - \int h^2 \phi T \phi \, \mathrm{d}\mu. \end{aligned} \tag{2.18}$$

[6] Here, $[\cdot, \cdot]$ denotes the usual Lie bracket and it is defined by $[Y_0, Y_i] = Y_0(Y_i) - Y_i(Y_0)$

Let $f = he^{-E/2-V/2}$. Then, we see that the second term in (2.18) can be written in terms of $f$ as follows

$$\begin{aligned}\int g^{ij}\partial_{p^i}(\phi h)\,\partial_{p^j}(\phi h)\,\mathrm{d}\mu = &\int g^{ij}\partial_{p^i}(\phi f)\,\partial_{p^j}(\phi f)\Theta^{-1}\,\mathrm{d}p\,\mathrm{d}x\\ &+\frac{1}{4}\int \phi^2 h^2 g^{ij}\partial_{p^i}E\,\partial_{p^j}E\,\mathrm{d}\mu\\ &+\frac{1}{2}\int g^{ij}\partial_{p^i}(\phi^2 f^2)\,\partial_{p^j}E\,\Theta^{-1}\,\mathrm{d}p\,\mathrm{d}x.\end{aligned}$$

Also, performing an integration by parts in the last term allows to find that

$$\begin{aligned}\frac{1}{2}\int g^{ij}\partial_{p^i}(\phi^2 f^2)\,\partial_{p^j}E\,\Theta^{-1}\,\mathrm{d}p\,\mathrm{d}x &= -\frac{1}{2}\int \phi^2 h^2\,\partial_{p^i}(g^{ij}\partial_{p^j}E)\,\mathrm{d}\mu\\ &\geq -\omega\int \phi^2 h^2\,\mathrm{d}\mu,\end{aligned}$$

where we used the condition $\partial_{p^i}(g^{ij}\partial_{p^j}E)\leq\omega$. Therefore, the identity (2.18) leads to the inequality

$$\begin{aligned}(\lambda-\omega)\int\phi^2h^2\,\mathrm{d}\mu+\frac{1}{4}\int\phi^2h^2g^{ij}\partial_{p^i}E\,\partial_{p^j}E\,d\mu \leq &\int h^2 g^{ij}\partial_{p^i}\phi\,\partial_{p^j}\phi\,\mathrm{d}\mu\\ &-\int h^2\phi T\phi\,\mathrm{d}\mu.\end{aligned}$$

Now, we introduce the sequence $\phi=\phi_k(x,p)=\psi(x/k_1)\psi(p/k_2)$, with $k=(k_1,k_2)\in\mathbb{N}^2$ and $\psi\in C_c^\infty$, $0\leq\psi\leq 1$, $\psi=1$ on $B(0,1/2)$ and $\operatorname{supp}\psi\subset B(0,1)$. From the previous identity, we obtain the bound

$$\begin{aligned}(\lambda-\omega)\int\phi_k^2h^2\,\mathrm{d}\mu+\frac{1}{4}\int\phi_k^2h^2g^{ij}\partial_{p^i}E\,\partial_{p^j}E\,\mathrm{d}\mu \leq &\frac{C}{k_2^2}\int h^2\sup_{i,j}|g^{ij}|\chi_{|p|<k_2}\,\mathrm{d}\mu\\ &+|\,\langle\phi_k h\,\nabla_pE\cdot\nabla_x\phi_k,h\rangle_{\mathcal{H}}\,|\\ &+|\,\langle\phi_k h\nabla_xV\cdot\nabla_p\phi_k,h\rangle_{\mathcal{H}}\,|,\end{aligned}\tag{2.19}$$

for some $C>0$. Using Young's inequality, we can estimate the last two terms of (2.19) as follows

$$\begin{aligned}|\,\langle\phi_k h\,\nabla_pE\cdot\nabla_x\phi_k,h\rangle\,| &\leq\frac{C}{k_1}\left(\frac{1}{4\epsilon_1}\int\phi_k^2h^2|\nabla_pE|^2\mathrm{d}\mu+\epsilon_1\int h^2\mathrm{d}\mu\right),\\ |\,\langle\phi_k h\nabla_xV\cdot\nabla_p\phi_k,h\rangle\,| &\leq\frac{C\zeta_{k_1}}{k_2}\left(\frac{1}{4\epsilon_2}\int\phi_k^2h^2\mathrm{d}\mu+\epsilon_2\int h^2\mathrm{d}\mu\right),\end{aligned}$$

which are valid for all $\epsilon_1,\epsilon_2>0$, with $\zeta_{k_1}=\sup_{|x|\leq k_1}\{|\nabla_xV|\}$. Now, we use the values $\epsilon_1=C/(\theta k_1)$ in the first line, $\epsilon_2=C\zeta_{k_1}/(4k_2)$ in the second line and invoke property (2.15) to get

$$\frac{\lambda-\omega-1}{C}\int\phi_k^2h^2\mathrm{d}\mu\leq\frac{1}{k_2^2}\int h^2\sup_{i,j}|g^{ij}|\,\chi_{|p|<k_2}\mathrm{d}\mu+\left(\frac{1}{k_1^2}+\frac{\zeta_{k_1}^2}{k_2^2}\right)\int h^2\mathrm{d}\mu.$$

Taking first the limit $k_2 \to \infty$ and then $k_1 \to \infty$, we see that $h = 0$ for $\lambda > \omega + 1$. This concludes the proof of the theorem. $\square$

As a consequence of the previous result, we are able to establish the existence and uniqueness of solutions for the corresponding initial value problem of equation (2.9) under weaker assumptions. We will only include some details for simplicity.

**Theorem 2.3.2.** *Given $0 \leq f_{\text{in}} \in L^1$, there exists a unique global solution of equation (2.9) with initial datum $f(0,x,p) = f_{\text{in}}(x,p)$ and*

$$0 \leq f \in C([0,\infty), L^1(\mathbb{R}^N \times \mathbb{R}^N)).$$

*Proof.* First, we verify that equation (2.9) can be written in the form (2.11). Let $f = e^{-E-V}h$, $E(p) = \sqrt{1+|p|^2} = p^0$ and $g = D^{-1}$, which is given by (2.10). Then, we notice that

$$\begin{aligned}\partial_t f + \hat{p}\cdot\nabla_x f - \nabla_x V\cdot\nabla_p f =& \, e^{-E-V}\left[\partial_t h + \hat{p}\cdot\nabla_x h - \nabla_x V\cdot\nabla_p h\right] \\ &- e^{-E-V} f \left[\hat{p}\cdot\nabla_x V - \nabla_x V\cdot\nabla_p E\right] \\ =& \, e^{-E-V}\left[\partial_t h + \hat{p}\cdot\nabla_x h - \nabla_x V\cdot\nabla_p h\right],\end{aligned}$$

with $\hat{p} = p/p^0 = \nabla_p\sqrt{1+|p|^2} = v$, see (2.13), and also

$$\begin{aligned}\nabla_p(D\nabla_p f + pf) &= \partial_{p^i}\left(D^{ij}\partial_{p^j}(he^{-E-V}) + p^i f\right) \\ &= \partial_{p^i}\left(e^{-E-V}D^{ij}\partial_{p^j}h\right) \\ &= e^{-E-V}\partial_{p^i}\left(D^{ij}\partial_{p^j}h\right) - e^{-E-V}D^{ij}\frac{p_i}{p^0}\partial_{p^j}h\,,\end{aligned}$$

by using (2.7). Thus, $h$ solves the equation

$$\partial_t h + \hat{p}\cdot\nabla_x h - \nabla_x V\cdot\nabla_p h = \partial_{p^i}\left(D^{ij}\partial_{p^j}h\right) + D^{ij}\partial_{p^i}\log e^{-p^0}\partial_{p^j}h. \tag{2.20}$$

Next, we consider the metric $g$ given by

$$g_{ij} = p^0\delta_{ij} - \frac{p_i p_j}{p^0},$$

which satisfies

$$\begin{aligned}D^{kj}g_{jl} &= \left(\frac{\delta^{kj} + p^k p^j}{\sqrt{1+|p|^2}}\right)\left(p^0\delta_{jl} - \frac{p_j p_l}{p^0}\right) \\ &= \delta^k_l + p^k p_l - \frac{p^k p_l + p^k p_l |p|^2}{(p^0)^2} = \delta^k_l,\end{aligned}$$

i.e., $D^{ij}$ is the inverse matrix of the metric $g$ with $\det g = |g| = p^0$. Using the definition of the LB operator (2.5), we perform the following calculation

$$\begin{aligned}\partial_{p^i}\left(\frac{\sqrt{|g|}}{\sqrt{|g|}}D^{ij}\partial_{p^j}h\right) &= \frac{1}{\sqrt{|g|}}\partial_{p^i}\left(\sqrt{|g|}D^{ij}\partial_{p^j}h\right) + \partial_{p^i}\left(\frac{1}{\sqrt{|g|}}\right)\sqrt{|g|}D^{ij}\partial_{p^j}h \\ &= \Delta_p^g h - D^{ij}\frac{p_i}{2(p^0)^2}\partial_{p^j}h \\ &= \Delta_p^g h + D^{ij}\partial_{p_i}\left(\log\sqrt{|g|^{-1}}\right)\partial_{p^j}h\,.\end{aligned}$$

Combining equation (2.20) with the previous identity and bearing in mind the fact $|g|^{-1} = \det g^{-1}$, we see that (2.9) takes the form (2.11) as stated.

Next, we approximate the external potential $V(x)$ by a smooth function and the initial datum by a sequence $h_{\text{in},m} \geq 0$ of smooth functions with compact support. By Theorem 2.3.1, we have that for each fixed $m \in \mathbb{N}$ there exists a unique solution of equation (2.9) with $f_m \in C([0,\infty), L^2)$. The previous result can be applied since it is easy to verify that conditions (2.14)–(2.16) are fulfilled from the following quantities

$$\begin{aligned}D^{ij}\partial_{p^i}E\,\partial_{p^j}E &= \frac{\delta^{ij}+p^ip^j}{\sqrt{1+|p|^2}}\frac{p^j}{p^0}\frac{p^i}{p^0} = |\nabla_p p^0|^2 p^0 \geq |\nabla_p E|^2, \\ D^{ij}(p) &\approx 1+|p|, \qquad \partial_{p^i}(D^{ij}\partial_{p^j}p^0) = \partial_{p^i}p^i = N, \\ \partial_{p^i}\left(\frac{p^j}{p^0}\right) &= \partial_{p^i}v^j = \frac{1}{(p^0)^2}\left(p^0\delta^{ij} - \frac{p^ip^j}{p^0}\right) = \frac{g_{ij}}{(p^0)^2},\end{aligned}$$

where we used that $E = p^0 \geq 1$ and $\partial_{p^k}p^0 = p^k/p^0 = v^k$. By standard methods (see [31, 72, 137] for instance), one can prove the $L^1$-contraction property

$$\|f_k - f_m\|_{L^1} \leq \|f_{\text{in},k} - f_{\text{in},m}\|_{L^1}.$$

Thus the sequence $f_m$ converges in $L^1$ to a solution. The uniqueness is also a direct consequence of the $L^1$-contraction property. The non-negativity of regular solutions can be proved by studying the evolution of a suitable regularization of $\text{sign}(f)$ (see again [31, 72, 137]). □

It is important to remark that the global existence and uniqueness of solutions for equation (2.9) can be proven under lower regularity conditions. For instance, see [139] for the case of the non-relativistic equation.

As previously mentioned, one of the main reasons to consider equation (2.9) comes from the fact that this model is compatible with the desired finite propagation speed of particles in relativity. In general, any kinetic equation of the form

$$\partial_t f + v(p)\cdot\nabla_x f = \mathcal{C}[f],$$

will possess this propagation property if the vector field $v$ and the operator $\mathcal{C}$ (possibly non-linear) satisfy $|v| \leq 1$ and

$$\int_{\mathbb{R}^N} \mathcal{C}[f](t,x,p)\mathrm{d}p = 0.$$

In fact, for the proof of the result is required an intrinsic property that solutions of the above equation satisfy. Define the functions

$$\rho(t,x) = \int_{\mathbb{R}^N} f\,\mathrm{d}p, \quad j(t,x) = \int_{\mathbb{R}^N} v(p)\,f\,\mathrm{d}p,$$

and notice that $\rho$ and $j$ are related by the continuity equation

$$\partial_t\rho + \nabla_x \cdot j = 0, \tag{2.21}$$

for all $t \geq 0$ and $x \in \mathbb{R}^N$. Moreover, since $|v| \leq 1$, then $|j| \leq \rho$ for $f \geq 0$. The latter fact provides the key point in the argument to obtain the result, which is based on the celebrated uniqueness theorem for non-linear wave equations due to Fritz John [94], see also [131]. Now, we are in position to show that all the relevant relativistic kinetic equations have the finite propagation speed property, if these equations can be expressed in terms of the continuity equation (2.21).

**Lemma 2.3.2.** *Assume that $\rho, j \in C^1$ verify equation (2.21), $|j| \leq \rho$ and $\rho(0,x) = 0$ for $|x - x_0| \leq t_0$. Then $\rho(t,x) = 0$, for $(t,x) \in \Lambda(t_0,x_0)$, where*

$$\Lambda(t_0,x_0) = \{(t,x) \in [0,t_0] \times \mathbb{R}^N : |x - x_0| \leq t_0 - t\}$$

*is the past light cone with vertex on $(t_0,x_0)$ and base on $t = 0$.*

*Proof.* Consider the function

$$\Phi(s,x) = t_0 - [(t_0 - s)^2 + t_0^{-2}(2t_0 s - s^2)|x - x_0|^2]^{1/2},$$

and the set $R_s(t_0,x_0) = \{(t,x) : t \leq \Phi(s,x),\ |x - x_0| \leq t_0\}$. Notice that

$$\Phi(0,x) = 0, \quad \lim_{s\to t_0} \Phi(s,x) = t_0 - |x - x_0|, \quad \Phi_{||x-x_0|=t_0} = 0, \tag{2.22}$$

and

$$\Lambda(t_0,x_0) = \bigcup_{0\leq s<t_0} R_s(t_0,x_0).$$

Also, we define

$$\rho_\cap(s,x) = \rho(\Phi(s,x),x), \quad j_\cap(s,x) = j(\Phi(s,x),x).$$

Since $\rho, j$ satisfy (2.21), then $\rho_\cap$, $j_\cap$ verify the identity

$$\partial_s\rho_\cap = -\nabla \cdot j_\cap \partial_s\Phi + \partial_s j_\cap \cdot \nabla_x\Phi.$$

Next, we use (2.22) and perform some integrations by parts to see that

$$\begin{aligned}\int\limits_{|x-x_0|<t_0} \rho_\cap(s,x)\,\mathrm{d}x &= \int\limits_{|x-x_0|<t_0} \int_0^s \partial_\tau \rho_\cap(\tau,x)\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int\limits_{|x-x_0|<t_0} \int_0^s (-\nabla\cdot j_\cap \partial_\tau \Phi + \partial_\tau j_\cap \cdot \nabla_x \Phi)\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \int\limits_{|x-x_0|<t_0} j_\cap \cdot \nabla_x \Phi(s,x)\,\mathrm{d}x \le \theta(s) \int\limits_{|x-x_0|<t_0} \rho_\cap(s,x)\,\mathrm{d}x,\end{aligned}$$

where the latter bound was obtained by using $|j| \le \rho$ and the fact that for all $0 \le s < t_0$

$$|\nabla_x \Phi(s,x)| = \frac{t_0^{-2}(2t_0 s - s^2)|x-x_0|}{[(t_0-s)^2 + t_0^{-2}(2t_0 s - s^2)|x-x_0|^2]^{1/2}} \le \frac{\sqrt{2t_0 s - s^2}}{t_0} = \theta(s).$$

Since $\theta(s) < 1$ and $\rho \ge 0$ is continuous, we conclude that

$$\int\limits_{|x-x_0|<t_0} \rho_\cap\,\mathrm{d}x = 0 \Rightarrow \rho = 0 \text{ on } \Lambda(t_0,x_0).$$

□

Notice that as a consequence of the previous result and the wave nature of the transport equation (2.21), we obtain a similar result when the initial datum $\rho(0,x)$ is compactly supported. This observation leads to the finite speed property of particles mentioned before.

In the remainder of the section, we assume that solutions of equation (2.9) are smooth and decays rapidly at infinity. This allows to simplify some of the computations in the following results. The generalization to the actual regularity of solutions is achieved by introducing a suitable smooth positive approximation $f_\varepsilon$, for which the following calculations hold up to error terms that vanish in the limit toward a solution (i.e., $\varepsilon \to 0$). We refer to [18] for the details of this procedure in the non-relativistic case.

Now, we prove the finite speed property of particles for the relativistic Fokker-Planck model:

**Proposition 2.3.1.** *If the initial datum for equation (2.9) satisfies $f_{\text{in}} = 0$ for $|x| > R$ and some $R > 0$, then $f = 0$ for $|x| > R + t$ and all $t > 0$.*

*Proof.* Introduce the density and the current density

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,p)\,\mathrm{d}p, \quad j(t,x) = \int_{\mathbb{R}^N} \hat{p} f(t,x,p)\,\mathrm{d}p.$$

Clearly, $|j| \leq \rho$ since $\hat{p} = p/p^0 \leq 1$. Also, the continuity equation (2.21) holds. This is a direct consequence from the facts that equation (2.9) can be written in the form

$$\partial_t f + \hat{p} \cdot \nabla_x f = \nabla_p \cdot (D \nabla_p f + f\,(p + \nabla_x V)),$$

and an application of the divergence theorem after an integration in the $p$ variable in the previous identity. Then, the result follows by Lemma 2.3.2.

□

In the end of the next chapter, we will use the previous property for an alternative approach to obtain the boundedness control on the behavior of relativistic solutions in the *Newtonian limit* problem for equation (2.9).

Now, we want to focus on two other relevant and fundamental aspects of equation (2.9). The first one is concerned to the fact that the model does not exhibit particles loss, which is meaningful from the physics perspective. The second aspect is directly related to the convergence of solutions towards the non-trivial equilibrium state of the equation, which is one of the main subjects in this work. The latter problem will be treated in Chapter 4. In order to obtain part of the information previously mentioned, we need to recall some basic concepts. The positive part of a real-valued function $\sigma$ is $\sigma^+ = \max(0, \sigma)$. For a non-negative density $f$, its mass is defined by

$$M[f](t) = \int_{\mathbb{R}^{2N}} f(t,x,p)\,\mathrm{d}p\,\mathrm{d}x, \tag{2.23}$$

and its free energy, or (relative) entropy functional is given by

$$\mathcal{Q}[f](t) = \int_{\mathbb{R}^{2N}} f(t,x,p)\left(\sqrt{1+|p|^2} + V(x) + \log f(t,x,p)\right)\mathrm{d}p\,\mathrm{d}x. \tag{2.24}$$

The next proposition studies the evolution of the functionals $M$ and $\mathcal{Q}$.

**Proposition 2.3.2.** *For any non-negative solution of equation (2.9), the following properties are valid:*

*(i) The mass is constant:* $M[f(t)] = M[f_{\rm in}]$.

*(ii) If* $\mathcal{Q}_+[f_{\rm in}] < \infty$, *where*

$$\mathcal{Q}_+[f] = \int_{\mathbb{R}^{2N}} f\left(\sqrt{1+|p|^2} + V(x) + \log^+ f\right)\mathrm{d}p\,\mathrm{d}x, \tag{2.25}$$

*then* $f \log f \in C([0,\infty), L^1(\mathbb{R}^{2N}))$ *and the entropy identity holds*

$$\frac{d\mathcal{Q}}{dt} = -4 \int_{\mathbb{R}^{2N}} D^{ij}(p)\partial_{p^i}\left(\sqrt{f/\mathscr{J}}\right)\partial_{p^j}\left(\sqrt{f/\mathscr{J}}\right)\mathscr{J}\,\mathrm{d}p\,\mathrm{d}x, \tag{2.26}$$

*where* $\mathscr{J}(p) = e^{-\sqrt{1+|p|^2}}$. *In particular, we have that for all* $t > 0$

$$\int_0^t \int_{\mathbb{R}^{2N}} D^{ij}(p)\partial_{p^i}\left(\sqrt{f/\mathscr{J}}\right)\partial_{p^j}\left(\sqrt{f/\mathscr{J}}\right) \mathscr{J}\, \mathrm{d}p\, \mathrm{d}x\, \mathrm{d}s < \infty.$$

*Proof.* Proving the mass conservation property follows by a straightforward application of the divergence theorem. As to the entropy identity (2.26), we begin by computing

$$\frac{d\mathcal{Q}}{dt} = \int_{\mathbb{R}^{2N}} \partial_t f \left(\sqrt{1+|p|^2} + V + \log f\right) \mathrm{d}p\, \mathrm{d}x\, .$$

Next, we define $\partial_t f = FP[f] - T[f]$, which is justified by equation (2.9) and where the operator $T$ is given by (2.12) and the vector field by $v(p) = \hat{p} = p/p^0$. First, we see that

$$\begin{aligned}\int_{\mathbb{R}^{2N}} T[f]\sqrt{1+|p|^2}\, \mathrm{d}p\, \mathrm{d}x &= \int_{\mathbb{R}^{2N}} \left(p\cdot\nabla_x f - \sqrt{1+|p|^2}\,\nabla_x V\cdot\nabla_p f\, \mathrm{d}p\right) \mathrm{d}x\\ &= \int_{\mathbb{R}^{2N}} \nabla_x\cdot(pf)\, \mathrm{d}p\, \mathrm{d}x + \int_{\mathbb{R}^{2N}} \hat{p}\cdot\nabla_x V f\, \mathrm{d}p\, \mathrm{d}x\\ &= \int_{\mathbb{R}^{2N}} \hat{p}\cdot\nabla_x V f\, \mathrm{d}p\, \mathrm{d}x, \qquad (2.27)\end{aligned}$$

where we used the divergence theorem. For the integral of $T[f]V$, we have

$$\begin{aligned}\int_{\mathbb{R}^{2N}} T[f]V\, \mathrm{d}p\, \mathrm{d}x &= \int_{\mathbb{R}^{2N}} (V\hat{p}\cdot\nabla_x f - V\nabla_x V\cdot\nabla_p f)\, \mathrm{d}p\, \mathrm{d}x\\ &= -\int_{\mathbb{R}^{2N}} \nabla_x V\cdot\hat{p} f\, \mathrm{d}p\, \mathrm{d}x - \int_{\mathbb{R}^d} \nabla_p\cdot(V\nabla_x V f)\, \mathrm{d}p\, \mathrm{d}x\\ &= -\int_{\mathbb{R}^{2N}} \hat{p}\cdot\nabla_x V f\, \mathrm{d}p\, \mathrm{d}x. \qquad (2.28)\end{aligned}$$

For the integral of $T[f]\log f$, we use that for $z = (x,p)$ and a vector field $A$ such that $\nabla_z\cdot A = 0$, the following holds

$$\nabla_z\cdot[A(f\log f - f)] = A\log f\cdot\nabla_z f$$

and therefore, taking $A = (\hat{p}, -\nabla_x V)$, we get

$$\begin{aligned}\int_{\mathbb{R}^{2N}} T[f]\,(\log f)\, \mathrm{d}p\, \mathrm{d}x &= \int_{\mathbb{R}^{2N}} A\log f\cdot\nabla_z f\, \mathrm{d}p\, \mathrm{d}x\\ &= \int_{\mathbb{R}^{2N}} \nabla_z\cdot[A(f\log f - f)]\, \mathrm{d}p\, \mathrm{d}x = 0. \qquad (2.29)\end{aligned}$$

Adding (2.27)–(2.29), we see that

$$\int_{\mathbb{R}^{2N}} T[f]\left(\sqrt{1+|p|^2} + V + \log f\right) \mathrm{d}p\, \mathrm{d}x = 0. \qquad (2.30)$$

Now, the contribution of the term $FP[\cdot]$ is integrated by parts to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} FP[f]\left(\sqrt{1+|p|^2}+V+\log f\right)\,\mathrm{d}p\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^{2N}} \left(\sqrt{1+|p|^2}+V+\log f\right)\nabla_p\cdot(fp+D\nabla_p f)\,\mathrm{d}p\,\mathrm{d}x \\
&\quad = -\int_{\mathbb{R}^{2N}} \left(\hat{p}_i+\frac{1}{f}\partial_{p^i} f\right)\left(fp^i+D^{ij}\partial_{p^j} f\right)\,\mathrm{d}p\,\mathrm{d}x \\
&\quad = -\int_{\mathbb{R}^{2N}} \frac{1}{f}D^{ij}\left(f\hat{p}_i+\partial_{p^i} f\right)\left(f\hat{p}_j+\partial_{p^j} f\right)\,\mathrm{d}p\,\mathrm{d}x,
\end{aligned}$$

where we used the identity

$$D^{ij}\hat{p}_i = \frac{\delta^{ij}p_i+p^j|p|^2}{1+|p|^2} = p^j.$$

Accounting that

$$\partial_{p^k}\left(\sqrt{f/\mathscr{J}}\right) = \frac{1}{2\sqrt{f/\mathscr{J}}}(\mathscr{J}^{-1}f\hat{p}_k+\mathscr{J}^{-1}\partial_{p^k} f),$$

we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} FP[f]\left(\sqrt{1+|p|^2}+V+\log f\right)\,\mathrm{d}p\,\mathrm{d}x \\
&\quad = -4\int_{\mathbb{R}^{2N}} D^{ij}(p)\partial_{p^i}\left(\sqrt{f/\mathscr{J}}\right)\partial_{p^j}\left(\sqrt{f/\mathscr{J}}\right)\mathscr{J}\,\mathrm{d}p\,\mathrm{d}x. \qquad (2.31)
\end{aligned}$$

Adding (2.30) and (2.31) concludes the proof. □

From the previous result, the conservation law (i) assures that there is no loss of mass from the particles. Also, this mass conservation property of $f$ was essential to derive (2.26). This identity allows to interpret equation (2.9) as a gradient flow for $\mathcal{Q}$, or equivalently, the free energy functional acts as a Lyapunov function for the dynamics generated by this equation. In general, any $f=\alpha(x)\mathscr{J}(p)$ is a critical point of $\mathcal{Q}$ by (2.26) when $f$ solves (2.9). This is a nice variational characterization that can be made more precise. Recall that $e^{-V}\in L^1$. It is clear that, for each $M>0$, there exists a unique regular stationary solution with mass $M$ of equation (2.9), which is given by

$$f_0(x,p) = m_M(x,p) = \frac{M}{\Theta}\mathscr{J}_V(x,p), \qquad (2.32a)$$

where

$$\mathscr{J}_V(x,p) = e^{-\sqrt{1+|p|^2}-V}, \qquad \Theta = \int_{\mathbb{R}^{2N}} \mathscr{J}_V(x,p)\,\mathrm{d}p\,\mathrm{d}x. \qquad (2.32b)$$

The regularity assertion on the solution is a consequence of the argument used in Theorem 2.3.1. We proved that the operator $L$, defined by writing equation (2.9) in the form $\partial_t f = Lf$, is hypoelliptic provided that the external potential $V \in C^\infty$. From this property, one obtains that the equilibria of equation (2.9), which solve $Lf = 0$ , are automatically smooth. Moreover, as in the non-relativistic case, one can prove that the equilibrium solution is a minimizer of the entropy functional. To see this, we need first to recall the following general result proved in [47, Lemma 1.1], which will also play a crucial role in the following sections.

**Lemma 2.3.3** ([47]). *Let us consider a measurable set $\Omega \subset \mathbb{R}^d$ and the functional*

$$\mathcal{H}[g] = \int_\Omega g(y) \log g(y)\, \mathrm{d}y + \int_\Omega g(y)h(y)\, \mathrm{d}y, \tag{2.33}$$

*with $g \in L^1(\Omega)$ non-negative such that $g(\log g)^+ \in L^1(\Omega)$. Assume that $h \in L^1(\Omega; g(y)dy)$ is such that $e^{-h} \in L^1(\Omega; dy)$, then $g \log g \in L^1(\Omega; dy)$ and*

$$\mathcal{H}[g] - \mathcal{H}[m_g] \geq \frac{1}{2} \int_\Omega \left( \sqrt{g(y)} - \sqrt{m_g(y)} \right)^2 \mathrm{d}y,$$

$$\textit{where} \quad m_g(y) = \frac{\int_\Omega g\, \mathrm{d}y}{\int_\Omega e^{-h}\, \mathrm{d}y} e^{-h}.$$

An immediate consequence of the previous lemma is the variational characterization of $m_g(y)$ as a minimum of $\mathcal{H}$:

**Corollary 2.3.1.** *With the same hypotheses of Lemma 2.3.3,*

$$H(M) = \inf \left\{ \mathcal{H}[g] \,:\, g \geq 0,\, g \in L^1(\Omega),\, \int_\Omega g(y)\, \mathrm{d}y = M \right\}$$

*is bounded from below for any $M > 0$ and*

$$H(M) = \mathcal{H}[\bar{g}] = M \log \left( \frac{M}{\int_\Omega e^{-h}\, \mathrm{d}y} \right) \quad \textit{with} \quad \bar{g} = M \frac{e^{-h}}{\int_\Omega e^{-h}\, \mathrm{d}y}.$$

*In fact, $\bar{g}$ is the only minimum of $\mathcal{H}(g)$.*

If we take $d = 2N$, $y = (x, p)$, $\Omega = \mathbb{R}^{2N}$, $g = f$ and $h = \sqrt{1 + |p|^2} + V$, we have that $\bar{g} = m_M$ and $\mathcal{H}(g) = \mathcal{Q}(f)$ by (2.32) and (2.33), respectively. Thus, the next result follows directly from Lemma 2.3.3 and Corollary 2.3.1:

**Corollary 2.3.2.** *Assume that $f \in L^1(\mathbb{R}^{2N})$ and $e^{-V} \in L^1(\mathbb{R}^N)$, with $f \geq 0$, are such that*

$$\mathcal{Q}_+[f] = \int_{\mathbb{R}^{2N}} f \left( \sqrt{1 + |p|^2} + V(x) + \log^+ f \right) \mathrm{d}p\, \mathrm{d}x < \infty.$$

*Then,*

$$\mathcal{Q}[f] - \mathcal{Q}[m_M] \geq \frac{1}{2} \int_{\mathbb{R}^{2N}} \left( \sqrt{f(x,p)} - \sqrt{m_M(x,p)} \right)^2 \mathrm{d}p \, \mathrm{d}x,$$

*and $m_M$ is the unique minimum of*

$$Q(M) = \inf \left\{ \mathcal{Q}[f] \, : 0 \leq f \in L^1(\mathbb{R}^{2N}), \int_{\mathbb{R}^{2N}} f(x,p) \, \mathrm{d}p \, \mathrm{d}x = M, \, \mathcal{Q}_+[f] < \infty \right\}$$
$$= M \log \left[ \frac{M}{\int_{\mathbb{R}^{2N}} e^{-(\sqrt{1+|p|^2}+V)} \, \mathrm{d}p \, \mathrm{d}x} \right].$$

The previous result justifies the reason to consider the functional $\mathcal{Q}$ as a Lyapunov function to study the trend to the equilibrium problem for solutions of equation (2.9) that will be treated in Chapter 4. Also, in the next sections we will generalize these results for the non-linear Vlasov-Nordström-Fokker-Planck and Vlasov-Maxwell-Fokker-Planck systems.

## 2.4 The Vlasov-Nordström-Fokker Planck system

In the remainder of the chapter, we will consider two non-linear mean field models built on the relativistic Fokker-Planck equation (2.9). Both models generalize the Vlasov-Poisson-Fokker-Planck system in the plasma physics and in the gravitational cases. For simplicity, we shall consider only the three dimensional case, i.e., $x, p \in \mathbb{R}^3$ (the field equations change with the dimension).

In this section, the corresponding relativistic model for the gravitational case is introduced. Unfortunately, we will not do this in the framework of general relativity (although this is the physically correct relativistic theory of gravity), since the latter would lead to face fundamental difficulties. In fact, modeling dissipative systems in general relativity is not yet understood, not even at a formal level. The main reason is that the Einstein equations by themselves imply that the mass/energy/momentum of the system must be conserved[7]. To overcome this fundamental issue, we will propose an alternative relativistic theory of gravity, the Nordström theory, which has already been used in the collisionless case (without diffusion) as a simpler model in comparison with the Einstein-Vlasov system [28, 129]. The resulting system—the Vlasov-Nordström-Fokker-Planck system—will be derived from similar arguments as the ones applied in Section 2.2 for the relativistic

[7]The situation is similar to what happens in electrodynamics, where the Maxwell equations alone imply the conservation of charge (2.52) and therefore, the dynamics of the coupled matter model must be compatible with it (which is true for the relativistic Fokker-Planck equation considered in the previous section).

Fokker-Planck equation.

In the present case, we assume that the space-time is given by the pseudo-Riemannian manifold $(\mathbb{R}^4, g)$, with Lorentzian metric

$$g = e^{2\phi}\eta,$$

where $\eta$, $\phi : \mathbb{R}^4 \to \mathbb{R}$ are the Minkowski metric and the gravitational field, respectively. Let $(t, x^1, x^2, x^3)$ be a system of coordinates which set the Minkowski metric in the canonical form $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Then

$$g = -e^{2\phi}dt^2 + e^{2\phi}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{2.34}$$

The geodesics of the metric (2.34) are the solutions of the ODE system

$$\frac{dt}{ds} = p^0, \quad \frac{dx^i}{ds} = p^i, \quad \frac{dp^\mu}{ds} = -\Gamma^\mu_{\nu\sigma}p^\nu p^\sigma, \tag{2.35}$$

where $s$ is the geodesic parameter and $\Gamma^\mu_{\nu\sigma}$ are the Christoffel symbols of $g$, which are explicitly obtained as follows

$$\begin{aligned}\Gamma^\mu_{\nu\sigma} &= \frac{1}{2}g^{\mu\gamma}\left(\partial_\nu g_{\sigma\gamma} + \partial_\sigma g_{\nu\gamma} - \partial_\gamma g_{\nu\sigma}\right)\\ &= g^{\mu\gamma}\left(g_{\sigma\gamma}\partial_\nu\phi + g_{\nu\gamma}\partial_\sigma\phi - g_{\nu\sigma}\partial_\gamma\phi\right)\\ &= \delta^\mu_\sigma\partial_\nu\phi + \delta^\mu_\nu\partial_\sigma\phi - e^{-2\phi}\eta^{\mu\gamma}\partial_\gamma\phi g_{\nu\sigma}\,. \end{aligned} \tag{2.36}$$

Here, we used that the inverse of $g$ takes the form $g^{\mu\nu} = e^{-2\phi}\eta^{\mu\nu}$. Now, consider a system of particles with unit mass that move along the geodesic curves. The geodesic motion reflects the physical property that the particles interact only through the gravitational field. If we want to interpret $p^\mu$ as a four-momentum vector for the particles, we need to impose that $p^\mu$ has length equal to $-1$, i.e., $g_{\mu\nu}p^\mu p^\nu = -1$. This entails

$$p^0 = \sqrt{e^{-2\phi} + |p|^2}, \quad |p|^2 = \delta_{ij}p^ip^j. \tag{2.37}$$

Let $f(t, x, p)$, $x = (x^1, x^2, x^3)$ and $p = (p^1, p^2, p^3)$, be the distribution function of particles in the position $x$ at time $t$ with four-momentum vector $p^\mu = (p^0, p) = (\sqrt{e^{-2\phi} + |p|^2}, p)$. Having assumed that the solutions of (2.35) are the particles trajectories, we obtain that $f$ satisfies the equation

$$p^0\partial_t f + p\cdot\nabla_x f - \Gamma^i_{\mu\nu}p^\mu p^\nu\partial_{p^i}f = 0,$$

where $p^0$ is given by (2.37). Next, we substitute (2.36) in the previous equation to obtain the Vlasov equation for collisionless particles

$$p^0\partial_t f + p\cdot\nabla_x f - \left[2\left(p_0\partial_t\phi + p\cdot\nabla_x\phi\right)p + e^{-2\phi}\nabla_x\phi\right]\cdot\nabla_p f = 0.$$

In order to obtain the corresponding Fokker-Planck equation, we need to add appropriate diffusion and friction terms in the right hand side of this equation. Motivated by the discussion in Section 2.2, the LB operator $\Delta_p^h f$ for the diffusion term appears to be adequate in this case, where $h$ is the metric induced by (2.34) over the hyperboloid $p^0 = \sqrt{e^{-2\phi} + |p|^2}$. It can be verified that

$$h_{ij} = e^{2\phi}\left(\delta_{ij} - \frac{p_i p_j}{e^{-2\phi} + |p|^2}\right),$$

with $p_i = \delta_{ij} p^j$[8], $|h| = \det h = \left(e^{-2\phi} + |p|^2\right)^{-1} e^{4\phi}$ and inverse matrix

$$(h^{-1})^{ij} = e^{-2\phi}\delta^{ij} + p^i p^j.$$

Therefore,

$$\Delta_p^h f = \frac{1}{\sqrt{|h|}}\partial_{p^i}\left(\sqrt{|h|}(h^{-1})^{ij}\partial_{p^j} f\right) = p^0 \partial_{p^i}\left(\frac{e^{-2\phi}\delta^{ij} + p^i p^j}{\sqrt{e^{-2\phi} + |p|^2}}\partial_{p^j} f\right).$$

Then, the Fokker-Planck equation in the absence of friction adopts the form

$$Sf - \left[2S\phi\, p + \frac{e^{-2\phi}\nabla_x \phi}{\sqrt{e^{-2\phi} + |p|^2}}\right]\cdot \nabla_p f = \partial_{p^i}\left(\frac{e^{-2\phi}\delta^{ij} + p^i p^j}{\sqrt{e^{-2\phi} + |p|^2}}\partial_{p^j} f\right), \tag{2.38a}$$

where

$$Su = \partial_t u + \frac{p}{\sqrt{e^{-2\phi} + |p|^2}}\cdot \nabla_x u. \tag{2.38b}$$

For the gravitational field $\phi$, we postulate the non-linear wave equation

$$\Box\phi := \partial_t^2 \phi - \Delta_x \phi = -e^{6\phi}\int_{\mathbb{R}^3} \frac{f(t,x,p)}{\sqrt{e^{-2\phi} + |p|^2}}\,\mathrm{d}p, \tag{2.38c}$$

which has been justified in [24]. Now, doing the change of variables $\tilde{f}(t,x,p) = f(t,x,e^{-2\phi}p)$, the system (2.38) takes the form

$$\begin{aligned} &\partial_t \tilde{f} + \nabla_p\left(\sqrt{e^{2\phi} + |p|^2}\right)\cdot \nabla_x \tilde{f} - \nabla_x\left(\sqrt{e^{2\phi} + |p|^2}\right)\cdot \nabla_p \tilde{f} \\ &\qquad = \partial_{p^i}\left(\Lambda_\phi^{ij}(p)\partial_{p^j}\tilde{f}\right), \end{aligned} \tag{2.39a}$$

$$\Box\phi = -e^{2\phi}\int_{\mathbb{R}^3}\frac{\tilde{f}(t,x,p)}{\sqrt{e^{2\phi} + |p|^2}}\,\mathrm{d}p, \tag{2.39b}$$

[8] Although the metric $g$ is not Euclidean, we keep using the metric $\delta_{ij}$ for moving up and down indexes.

where

$$\Lambda_\phi^{ij}(p) = \frac{e^{4\phi}\delta^{ij} + e^{2\phi}p^i p^j}{\sqrt{e^{2\phi} + |p|^2}}. \tag{2.39c}$$

The system (2.39) is the Vlasov-Nordström-Fokker-Planck system in the absence of friction. It is invariant under the Lorentz *type* transformations given in [26]. To introduce a friction term, we first notice that for any time independent scalar function $\phi_0 = \phi_0(x)$ (not necessarily a solution of equation (2.39b)), the left hand side of (2.39a) vanishes when the density $\tilde{f}$ is given by

$$\tilde{f} = \tilde{f}_0(x,p) = e^{-\sqrt{e^{2\phi_0}+|p|^2}}.$$

This suggests that a friction term of the form $\nabla_p \cdot (q\tilde{f})$ should be used on the right side of (2.39a), if it satisfies

$$\Lambda_\phi^{ij}(p)\partial_{p^j}\tilde{f} + q^i\tilde{f} = 0, \quad \text{if} \;\; \tilde{f} = e^{-\sqrt{e^{2\phi}+|p|^2}}.$$

It can be verified that $q = e^{2\phi}p$. Adding this friction term and an external potential to (2.39a), we get

$$\partial_t f + \nabla_p\left(\sqrt{e^{2\phi}+|p|^2}\right)\cdot\nabla_x f - \nabla_x\left(\sqrt{e^{2\phi}+|p|^2} + V(x)\right)\cdot\nabla_p f = \partial_{p^i}\left(\Lambda_\phi^{ij}(p)\partial_{p^j} f + e^{2\phi}p^i f\right), \tag{2.40a}$$

$$\Box\phi = -e^{2\phi}\int_{\mathbb{R}^3} \frac{f(t,x,p)}{\sqrt{e^{2\phi}+|p|^2}}\,\mathrm{d}p, \tag{2.40b}$$

where $\Lambda_\phi^{ij}(p)$ is given by (2.39c) and the tilde is removed for notational simplicity. The system (2.40) will be called the Vlasov-Nordström-Fokker-Planck (VNFP) system.

In the following, we discuss some of the properties that remain valid for this non-linear system. First, we prove the mass conservation and the entropy identity for time-dependent solutions. For the latter, it is important to remark that the existence of solutions is still an open problem in general and therefore, the analysis of the time-dependent solutions that we will perform is only formal. This matter will be approached in the final chapter for particular solutions of the form $f(t,x,p) = f(t,p)$, $\phi(t,x) = \phi(t)$ (the spatially homogeneous case). Then, we refer as "regular solution" to a pair of functions which are sufficiently regular to enable the calculations. In addition, we assume that solutions of VNFP system are such that $e^\phi$ is bounded in any finite interval of time. This is true as soon as the initial data for the field equation (2.40b) are bounded and $f \geq 0$. To see this, note that regular solutions of (2.40b) verify $\phi = \phi_{\mathrm{hom}} + \psi$, where $\psi$ solves (2.40b) with zero initial data and $\phi_{\text{hom}}$ solves the linear wave equation $\Box\phi_{\text{hom}} = 0$ with the same data as $\phi$. Since the right hand side of (2.40b) is non-positive, then $\psi \le 0$, and therefore, $e^{\phi} = e^{\phi_{\text{hom}}} e^{\psi} \le e^{\phi_{\text{hom}}}$ is bounded as claimed. Finally, we will also show that steady states for VNFP system exist.

The mass of regular solutions of VNFP is defined again by (2.23), with $f \ge 0$, and the corresponding free energy for this case is given by

$$\begin{aligned}\mathcal{K}[f,\phi,\partial_t\phi] &= \int_{\mathbb{R}^6} f\left(\sqrt{e^{2\phi}+|p|^2} + V(x) + \log f\right) \mathrm{d}p\,\mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3}\left(|\partial_t\phi|^2 + |\nabla_x\phi|^2\right)\mathrm{d}x \\ &= \mathcal{Q}[f,\phi] + \mathcal{I}[\phi,\partial_t\phi]. \end{aligned} \tag{2.41}$$

Notice that the energy part $\mathcal{I}$ of the entropy functional is non-negative, in contrast to what happens in the gravitational VPFP system, the non-relativistic analogue. We will exploit this advantage when we treat the existence of steady states for the VNFP system.

In the following result, we gather some properties of the VNFP system which are the extension for those ones proven for the linear case:

**Proposition 2.4.1.** *For regular solutions of* (2.40)*, we have:*

*(i)* $M(t) \equiv$*constant.*

*(ii) The entropy functional satisfies*

$$\frac{d\mathcal{K}}{dt} = -4\int_{\mathbb{R}^6} \Lambda_\phi^{ij}(p)\partial_{p^i}\left(\sqrt{f/\mathscr{J}^\phi}\right)\partial_{p^j}\left(\sqrt{f/\mathscr{J}^\phi}\right)\mathscr{J}^\phi\,\mathrm{d}p\,\mathrm{d}x, \tag{2.42}$$

*where* $\mathscr{J}^\phi(x,p) = e^{-\sqrt{e^{2\phi}+|p|^2}}$.

*(iii) Let* $e^{-V} \in L^1(\mathbb{R}^3)$*. Then, the static solutions of VNFP system with mass* $M > 0$ *are of the form*

$$(f_0(x,p),\phi_0(x)) = (m_M(x,p),\phi_0(x)), \tag{2.43a}$$

*where*

$$m_M = \frac{M}{\Theta}\mathscr{J}_V, \quad \mathscr{J}_V = e^{-\sqrt{e^{2\phi_0}+|p|^2}-V}, \quad \Theta = \|\mathscr{J}_V\|_{L^1}, \tag{2.43b}$$

*and* $\phi_0$ *solves*

$$\Delta\phi_0 = e^{2\phi_0}\int_{\mathbb{R}^3}\frac{m_M(x,p)}{\sqrt{e^{2\phi_0}+|p|^2}}\,\mathrm{d}p. \tag{2.43c}$$

*Proof.* The proof of (i) is straightforward. To show that identity (2.42) in (ii) holds, we first observe that

$$\begin{aligned}\frac{d\mathcal{I}}{dt} &= \int_{\mathbb{R}^3} \left(\partial_t\phi\partial_t^2\phi + \nabla_x\phi\cdot\nabla\partial_t\phi\right) \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \partial_t\phi\Box\phi \,\mathrm{d}x = -\int_{\mathbb{R}^6} \frac{fe^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+|p|^2}}\,\mathrm{d}p\,\mathrm{d}x,\end{aligned}$$

due to an integration by parts and the field equation (2.40b). On the other hand, we compute the time derivative of $\mathcal{Q}$ to see that

$$\frac{d\mathcal{Q}}{dt} = \int_{\mathbb{R}^6} \partial_t f\left(\sqrt{e^{2\phi}+|p|^2} + V + \log f\right)\mathrm{d}p\,\mathrm{d}x - \frac{d\mathcal{I}}{dt}, \tag{2.44}$$

where the specific form of $\mathcal{I}'$ and the mass conservation property were used. For the remaining term in (2.44), we use again that $\partial_t f = FP[f] - T[f]$, where $T = \nabla_p(\sqrt{e^{2\phi}+|p|^2})\cdot\nabla_x - \nabla_x(\sqrt{e^{2\phi}+|p|^2}+V)\cdot\nabla_p$. Similarly as in the proof of Proposition 2.3.2, we have that the integrals containing the transport term $T$ satisfy

$$\begin{aligned}&\int_{\mathbb{R}^6} T[f]\sqrt{e^{2\phi}+|p|^2}\,\mathrm{d}p\,\mathrm{d}x = \int_{\mathbb{R}^6}\frac{\nabla_x V\cdot pf}{\sqrt{e^{2\phi}+|p|^2}}\mathrm{d}p\,\mathrm{d}x,\\ &\int_{\mathbb{R}^6} T[f]\log f\,\mathrm{d}p\,\mathrm{d}x = 0,\\ &\int_{\mathbb{R}^6} T[f]V\,\mathrm{d}p\,\mathrm{d}x = -\int_{\mathbb{R}^6}\frac{\nabla_x V\cdot pf}{\sqrt{e^{2\phi}+|p|^2}}\mathrm{d}p\,\mathrm{d}x.\end{aligned}$$

The previous identities imply that there is no contribution from the term involving $T[f]$ in (2.44). Finally, we perform an integration by parts in the integral containing the diffusive term to obtain

$$\begin{aligned}&\int_{\mathbb{R}^6} FP[f]\left(\sqrt{e^{2\phi}+|p|^2}+V+\log f\right)\mathrm{d}p\,\mathrm{d}x\\ &\quad= -\int_{\mathbb{R}^6}\left(\hat{p}_i + \frac{1}{f}\partial_{p^i}f\right)\left(e^{2\phi}fp^i + \Lambda_\phi^{ij}(p)\partial_{p^j}f\right)\mathrm{d}p\,\mathrm{d}x\\ &\quad= -4\int_{\mathbb{R}^6}\mathscr{J}^\phi\Lambda_\phi^{ij}(p)\partial_{p^i}\left(\sqrt{f/\mathscr{J}^\phi}\right)\partial_{p^j}\left(\sqrt{f/\mathscr{J}^\phi}\right)\mathrm{d}p\,\mathrm{d}x,\end{aligned}$$

where we used that

$$e^{2\phi}p^j = \Lambda_\phi^{ij}(p)\hat{p}_i \quad\text{and}\quad \partial_{p^k}\left(\sqrt{f/\mathscr{J}^\phi}\right) = \frac{(\mathscr{J}^\phi)^{-1}}{2\sqrt{f/\mathscr{J}^\phi}}\left(f\hat{p}_k + \partial_{p^k}f\right).$$

This concludes the proof of (ii). For the last statement, we notice that $\mathscr{J}_V \in L^1(\mathbb{R}^6)$, because $e^{\phi_0}$ is bounded. Also, we see that static solutions must have the form $f_0(x,p) = \alpha(x)\mathscr{J}^{\phi_0}(x,p)$ by (ii), i.e., $\mathcal{K}' = 0$ and

$$\Lambda_\phi^{ij}(p)\partial_{p^i}\left(\sqrt{f/\mathscr{J}^\phi}\right)\partial_{p^j}\left(\sqrt{f/\mathscr{J}^\phi}\right) \geq 0,$$

since $\Lambda_\phi^{ij}(p)$ is positive definite. Substituting in (2.40a), we obtain

$$p \cdot (\nabla \alpha + \alpha \nabla_x V) = 0,$$

and therefore, $\alpha = Ce^{-V}$. □

In comparison with the linear case, it is not clear if the previous result will be useful to determine the long time asymptotic behavior of solutions from system (2.40) or any other related matter, since there are several technical issues to overcome. The first challenge arises from the lack of diffusion in the $x$ variable. A similar argument to prove existence of solutions as the one shown in the previous section does not apply for the linear equation (2.40a) when the field $\phi$ is given. This situation is already difficult for the treatment of the trend to the equilibrium for the relativistic Fokker-Planck equation. Even in a simpler regime (the spatially homogeneous case), this is a non-trivial problem for the VNFP system. The presence of the time variable in the coefficients for the latter transforms the original trend problem into one where the convergence is towards a self similar-profile (there is no steady state). Fortunately, some answers can be obtained for the latter problem as we will see in the final chapter.

In the following, we treat the existence of steady state solutions for the system (2.40). It is important to mention that the latter problem for the VPFP system in the gravitational case has not been completely solved. So far, there is only a small mass result which is proven in [18]. The authors of the previous reference use a fixed point argument inspired by the arguments contained in [49, 50]. This argument also applies *mutatis mutandis* to the VNFP system as follows.

Consider the equation for the gravitational potential of steady states, equation (2.43c), which can be written in terms of $u = -\phi_0$ as

$$\Delta u = -e^{-2u-V} \frac{M}{\Theta} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{e^{-2u}+|p|^2}}}{\sqrt{e^{-2u}+|p|^2}} \, \mathrm{d}p. \tag{2.45}$$

Next, define the solution operator $K$ of (2.45) as the convolution of the right hand side of this equation with $1/(4\pi|x|)$. By standard estimates, one can prove that for $M$ small enough, the operator $K$ is a contraction in the space $X = \{v \in L^\infty(\mathbb{R}^3) : 0 \leq v \leq 1\}$ and by the fixed point theorem, we obtain:

**Proposition 2.4.2.** *There exists $M_0 > 0$ such that equation (2.45) has a unique solution $u \in L^\infty$ for all $M < M_0$. This solution defines a steady state for the VNFP system, through (2.43), and satisfies $\lim_{|x|\to\infty} u = 0$.*

However, the latter result can be considerably improved in the relativistic case. In fact, we will show the existence of steady states for *all* masses. Let us denote

$$\Gamma_M = \{f : \mathbb{R}^6 \to \mathbb{R} \,:\, 0 \leq f \in L^1(\mathbb{R}^6)\,,\, \|f\|_{L^1(\mathbb{R}^6)} = M\,,\, \mathcal{Q}_+[f] < \infty\},$$

where $\mathcal{Q}_+$ is similarly defined as in (2.25) by using the definition of $\mathcal{Q}$ in (2.41). Additionally, we recall the definition of the space $D^1(\mathbb{R}^3)$ which is

$$D^1(\mathbb{R}^3) = \{\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^3) : \nabla\phi \in L^2 \text{ and } \phi \text{ vanishes at infinity}\},$$

where the vanishing condition at infinity on $\phi$ means that for all $a > 0$, the set $\{x \in \mathbb{R}^3 : |\phi(x)| > a\}$ has finite Lebesgue measure. Functions in the space $D^1(\mathbb{R}^3)$ satisfy the Sobolev inequality

$$\|\phi\|_{L^6} \leq \eta \|\nabla\phi\|_{L^2}, \quad \eta = \frac{2}{\sqrt{3}}\pi^{-2/3}, \tag{2.46}$$

see [106, Thm. 8.3]. Now, we are ready to prove the following:

**Theorem 2.4.1.** *There exists at least one solution*[9] $\phi_0$ *of* (2.43c) *for all* $M > 0$. *Moreover, the corresponding steady state, given by* (2.43b)*, is a minimizer of the entropy functional:*

$$K(M) = \inf\{\mathcal{K}(f,\phi,\psi)\,,\, f \in \Gamma_M\,,\, \phi \in D^1(\mathbb{R}^3)\,,\, \psi \in L^2(\mathbb{R}^3)\},$$

*where* $\mathcal{K}$ *is defined by* (2.41)*, i.e.,* $K(M) = \mathcal{K}(m_M, \phi_0, 0)$.

*Proof.* First, we notice that

$$K(M) = \inf_{\Gamma_M \times D^1} \mathcal{E}(f,\phi),$$

where $\mathcal{E}(f,\phi) = \mathcal{K}(f,\phi,0)$. We divide the proof in five steps.

*Step 1: $K(M)$ is bounded.* It is easy to see that

$$\mathcal{E}(f,\phi) \geq \int_{\mathbb{R}^6} f(|p| + V(x) + \log f)\, \mathrm{d}p\, \mathrm{d}x. \tag{2.47}$$

Using Lemma 2.3.3 with $g = f$, $h = |p| + V$, $\Omega = \mathbb{R}^6$, we have that

$$\mathcal{E}(f,\phi) \geq M \log\left(\frac{M}{\int_{\mathbb{R}^6} e^{-|p|-V}\mathrm{d}p\,\mathrm{d}x}\right).$$

*Step 2: Weak convergence of minimizing sequences.* Let $(f_n, \phi_n)$ be a minimizing sequence. Since $\phi_n$ is uniformly bounded in $D^1$, and by the Sobolev inequality (2.46), there exists a subsequence, still denoted by $\phi_n$, and $\phi_0 \in D^1$ such that

$$\phi_n \rightharpoonup \phi_0 \text{ in } L^6 \text{ and } \nabla_x\phi_n \rightharpoonup \nabla_x\phi_0 \text{ in } L^2. \tag{2.48}$$

[9] By Proposition 2.4.2, the solution is unique for $M$ small.

In order to establish the weak convergence of $f_n$ in $L^1$, we use the argument developed in [47, pag. 129]. Let us show first that $f_n$ does not concentrate. If it did, we could find $\varepsilon > 0$, a bounded sequence $x_n \in \mathbb{R}^3$ and a sequence $R_n \to \infty$ such that

$$\int_{|x_n - x| \leq R_n} f_n(x,p)\,\mathrm{d}p\,\mathrm{d}x = \varepsilon, \quad \text{for all } n \in \mathbb{N}.$$

Splitting $\mathcal{E}$ in two integrals and applying independently inequality (2.47) and Lemma 2.3.3 to each term, we obtain

$$\begin{aligned}\mathcal{E}(f_n,\phi_n) \geq &\int_{|x-x_n|>R_n} f_n(\log f_n + |p| + V(x))\,\mathrm{d}p\,\mathrm{d}x \\ &+ \int_{|x_n-x|\leq R_n} f_n(\log f_n + |p| + V(x))\,\mathrm{d}p\,\mathrm{d}x \\ \geq &(M-\varepsilon)\log\left(\frac{M-\varepsilon}{\int_{|x_n-x|>R_n} e^{-|p|-V}\,\mathrm{d}p\,\mathrm{d}x}\right) \\ &+\varepsilon\log\left(\frac{\varepsilon}{\int_{|x_n-x|\leq R_n} e^{-|p|-V}\,\mathrm{d}p\,\mathrm{d}x}\right). \end{aligned} \tag{2.49}$$

Since $e^{-|p|-V} \in L^1$, the following relations hold

$$\lim_{n\to\infty}\int_{|x_n-x|>R_n} e^{-|p|-V}\,\mathrm{d}p\,\mathrm{d}x = 0,$$
$$\text{and} \quad \lim_{n\to\infty}\int_{|x_n-x|\leq R_n} e^{-|p|-V}\,\mathrm{d}p\,\mathrm{d}x = \|e^{-|p|-V}\|_{L^1(\mathbb{R}^6)}.$$

Therefore, inequality (2.49) implies that $\mathcal{E}(f_n,\phi_n) \to \infty$ as $n \to \infty$. This contradicts the fact that $(f_n,\phi_n)$ is a minimizing sequence. Now, we prove that $f_n$ is tight. If not, for all $R_0 > 0$ we can find $\varepsilon > 0$ and $R > R_0$ with the following property

$$\lim_{n\to\infty}\int_{|x|+|p|>R} f_n\,\mathrm{d}p\,\mathrm{d}x > \varepsilon.$$

Whence, we have the estimate

$$\begin{aligned}\mathcal{E}(f_n,\phi_n) \geq \left(\int_{|x|+|p|>R} f_n\,\mathrm{d}p\,\mathrm{d}x\right)\Big[\log\int_{|x|+|p|>R} f_n\,\mathrm{d}p\,\mathrm{d}x \\ - \log\int_{|x|+|p|>R} e^{-|p|-V}\,\mathrm{d}p\,\mathrm{d}x\Big],\end{aligned}$$

again by (2.47) and Lemma 2.3.3. This implies that

$$\lim_{R\to\infty}\int_{|x|+|p|>R} e^{-|p|-V}\,\mathrm{d}p\,\mathrm{d}x \geq \varepsilon e^{-K(M)/\varepsilon} > 0,$$

and contradicts the fact that $e^{-|p|-V} \in L^1$. We conclude that there exists $f_0 \in L^1$ and a subsequence $f_n$ such that

$$f_n \rightharpoonup f_0 \text{ in } L^1.$$

*Step 3: Pointwise convergence of minimizing sequences.* As proven in [106, Cor. 8.7], the weak convergence (2.48) implies that

$$\phi_n \to \phi_0, \quad \text{pointwise a.e.,} \tag{2.50}$$

again up to the extraction of a subsequence. Moreover, by the argument used in [29, Lemma 5], we may assume that $\phi_n \le 0$ almost everywhere. Intuitively, this is a direct consequence from the fact that $(f_n, \phi_n^-)$ turns to be a minimizing sequence if $(f_n, \phi_n)$ is, where $\sigma^- = \min(0, \sigma)$. Next, we show that $f_n$ converges pointwise a.e. (up to subsequences). Given a minimizing sequence $(f_n, \phi_n)$, define

$$m_n = \frac{M}{\int_{\mathbb{R}^6} e^{-\sqrt{e^{2\phi_n}+|p|^2}-V}\,\mathrm{d}p\,\mathrm{d}x} e^{-\sqrt{e^{2\phi_n}+|p|^2}-V}.$$

By Lemma 2.3.3, we know that

$$\mathcal{E}(f_n, \phi_n) - \mathcal{E}(m_n, \phi_n) \ge \frac{1}{2}\int_{\mathbb{R}^6} (\sqrt{f_n} - \sqrt{m_n})^2\,\mathrm{d}p\,\mathrm{d}x.$$

This implies from one hand that $(m_n, \phi_n)$ is again a minimizing sequence and, on the other hand, that $\lim_{n\to\infty}(f_n - m_n) = 0$ pointwise a.e. after extracting a suitable subsequence. Moreover, since

$$e^{-\sqrt{e^{2\phi_n}+|p|^2}-V} \to e^{-\sqrt{e^{2\phi_0}+|p|^2}-V}, \quad \text{pointwise a.e.,}$$

due to (2.50), and

$$\int_{\mathbb{R}^6} e^{-\sqrt{e^{2\phi_n}+|p|^2}-V}\,\mathrm{d}p\,\mathrm{d}x \to \int_{\mathbb{R}^6} e^{-\sqrt{e^{2\phi_0}+|p|^2}-V}\,\mathrm{d}p\,\mathrm{d}x,$$

which follows by the dominated convergence theorem, i.e., $\phi_n \le 0$ and $e^{-|p|-V} \in L^1$, we obtain

$$f_n \to \frac{M}{\int_{\mathbb{R}^6} e^{-\sqrt{e^{2\phi_0}+|p|^2}-V}\,\mathrm{d}p\,\mathrm{d}x} e^{-\sqrt{e^{2\phi_0}+|p|^2}-V}, \quad \text{pointwise a.e.,}$$

since $m_n$ and $f_n$ have the same limit. In particular, we notice that $f_0$ is strictly positive and bounded.

*Step 4: $(f_0, \phi_0)$ is a minimizer.* We prove that $\mathcal{E}$ is weakly lower semi-continuous. Clearly

$$\liminf_{n\to\infty} \int |\nabla_x \phi_n|^2\,\mathrm{d}x \ge \int |\nabla_x \phi_0|^2\,\mathrm{d}x.$$

Moreover, Fatou's lemma ($f_n \geq 0$) allows to obtain

$$\liminf_{n\to\infty} \int f_n(\sqrt{e^{2\phi_n}+|p|^2}+V+\log f_n) \geq \int f_0(\sqrt{e^{2\phi_0}+|p|^2}+V+\log f_0),$$

and the claim follows: $K(M) = \mathcal{E}(f_0, \phi_0)$.

*Step 5: $(f_0, \phi_0)$ is a steady state of the VNFP system.* Since we already proved in step 3 that

$$f_0 = \frac{M}{\int_{\mathbb{R}^6} e^{-\sqrt{e^{2\phi_0}+|p|^2}-V}\,\mathrm{d}p\,\mathrm{d}x} e^{-\sqrt{e^{2\phi_0}+|p|^2}-V},$$

we only need to show that $\phi_0$ solves the non-linear elliptic equation (2.43c). To this purpose we define $\phi_h = \phi_0 + h\eta$, where $\eta = \eta(x)$ is any $C^\infty$ function with compact support and $h \in \mathbb{R}$. Using that $0 < f_0 < \infty$ and $\phi_0 \leq 0$, it is straightforward to show that $\mathcal{E}(f_0, \phi_h)$ is differentiable in $h$. The derivative at $h = 0$ must vanish and this entails that $\phi_0$ solves

$$\Delta\phi_0 = e^{2\phi_0} \int_{\mathbb{R}^3} \frac{f_0}{\sqrt{e^{2\phi_0}+|p|^2}}\,\mathrm{d}p\,\mathrm{d}x$$

in the sense of distributions. This completes the proof of the theorem. □

## 2.5 The Vlasov-Maxwell-Fokker-Planck system

In this final section we present the corresponding relativistic generalization in the plasma physics context of the Vlasov-Poisson-Fokker-Planck system. Although this system will not be treated in the forthcoming chapters, we consider relevant to review those results which also apply in this case. The model is obtained by coupling the relativistic Fokker-Planck equation

$$\partial_t f + \hat{p}\cdot\nabla_x f + F\cdot\nabla_p f = \partial_{p^i}\left(D^{ij}\partial_{p^j} f + f p^i\right), \tag{2.51a}$$

with the system of Maxwell equations given by[10]

$$\begin{array}{llll} \partial_t E = \nabla_x \wedge B - j, & (i) & \nabla_x\cdot E = \rho, & (ii) \\ \partial_t B = -\nabla_x \wedge E, & (iii) & \nabla_x \cdot B = 0, & (iv) \end{array} \tag{2.51b}$$

where the Lorentz force field $F : [0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}^3$ (with external potential $V$), the relativistic velocity field and the diffusion matrix are

$$F = E + \hat{p}\times B - \nabla_x V, \quad \hat{p} = \frac{p}{\sqrt{1+|p|^2}}, \quad D^{ij} = \frac{\delta^{ij}+p^i p^j}{\sqrt{1+|p|^2}}. \tag{2.51c}$$

[10] Up to a suitable normalization of the physical constants.

The functions $E, B : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ represent the electric and the magnetic fields, respectively, and the charge and current density that generate the previous fields are given by

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,p)\,\mathrm{d}p, \quad j = \int_{\mathbb{R}^3} \hat{p} f(t,x,p)\,\mathrm{d}p. \tag{2.51d}$$

Notice that $(\rho, j)$ satisfies the local conservation of charge

$$\partial_t \rho + \nabla_x \cdot j = 0, \tag{2.52}$$

as a direct consequence of (2.51a), which makes it consistent to couple the Maxwell equations and the Fokker-Planck equation. The system (2.51) will be called the (relativistic) Vlasov-Maxwell-Fokker-Planck system, or VMFP for short. As mentioned above, it generalizes the Vlasov-Poisson-Fokker-Plank (VPFP) system in the plasma physics case. Therefore, system (2.51) takes into account relativistic effects in a plasma, such as the propagation of electromagnetic waves. It is important to remark that there exist other models in the literature which are named "Vlasov-Maxwell-Fokker-Planck", see [16, 103, 146]. These systems couple Maxwell's equations to the non-relativistic Fokker-Planck equation (2.1) or the variant where the velocity $p$ in (2.1) is replaced by the relativistic counterpart $\hat{p}$ defined above in (2.51c). It is important to remark that in [113], the authors proved the first existence and uniqueness result of system (2.51) for the "*one and one-half dimensional*" case, i.e., when $x \in \mathbb{R}$, $p \in \mathbb{R}^2$. Unfortunately, this is the only existence result available in the literature for the evolution problem. Then, we will again present our results assuming the necessary regularity to perform the calculations.

Similarly as before, the mass for solutions of (2.51) is given by (2.23), with $f \geq 0$, and the entropy functional for this case is

$$\mathcal{K}[f, E, B] = \mathcal{Q}[f] + \mathcal{I}[E, B], \tag{2.53}$$

where the functionals $\mathcal{Q}$ and $\mathcal{I}$ are given by (2.24) and

$$\mathcal{I}[E, B] = \frac{1}{2} \int_{\mathbb{R}^3} \left( |E|^2 + |B|^2 \right) \mathrm{d}x.$$

Now, we are in position to prove the corresponding properties for solutions of the VMFP system:

**Proposition 2.5.1.** *For regular solutions of* (2.51) *we have the following:*

*(i) The mass is preserved:* $M(t) = const.$

*(ii) The entropy functional satisfies*

$$\frac{d\mathcal{K}}{dt} = -4 \int_{\mathbb{R}^6} D^{ij}(p) \partial_{p^i} \left( \sqrt{f/\mathscr{J}} \right) \partial_{p^j} \left( \sqrt{f/\mathscr{J}} \right) \mathscr{J} \,\mathrm{d}p\,\mathrm{d}x. \tag{2.54}$$

(iii) *Let $e^{-V} \in L^1(\mathbb{R}^3)$. Regular static solutions of (2.51) with mass $M$ verify*

$$(f_0(x,p), E_0(x), B_0(x)) = (m_M(x,p), -\nabla U(x), 0), \tag{2.55a}$$

*where*

$$m_M = \frac{M}{\Theta}\mathscr{J}_V e^{-U}, \quad \Theta = \int_{\mathbb{R}^6} e^{-U(x)}\mathscr{J}_V(x,p)\,\mathrm{d}p\,\mathrm{d}x, \tag{2.55b}$$

*$\mathscr{J}_V$ is given by (2.32b) and $U$ is a solution of*

$$-\Delta U = \rho, \quad \rho = \int_{\mathbb{R}^3} m_M(x,p)\,\mathrm{d}p. \tag{2.55c}$$

*Proof.* Proving (i) is straightforward. To achieve (ii), we use (2.53) and notice that

$$\frac{d\mathcal{K}}{dt} = \int_{\mathbb{R}^6} \partial_t f\left(\sqrt{1+|p|^2} + V + \log f\right)\mathrm{d}p\,\mathrm{d}x + \frac{d\mathcal{I}}{dt}.$$

Let $\partial_t f = FP[f] - T[f]$, where in this case we have to employ (2.51c) for the operator $T[\cdot]$. Therefore, it only remains to calculate the derivative of $\mathcal{I}[E,B]$ and the part of $d\mathcal{Q}/dt$ containing the term $E + \hat{p}\times B$ in $T[\cdot]$, since the other terms from $T[\cdot]$ and $FP[\cdot]$ are the same as in the linear case, cf. Proposition 2.3.2. Using (2.51bi) and (2.51biii), we have

$$\begin{aligned}\frac{d\mathcal{I}}{dt} &= \int_{\mathbb{R}^3}(E\cdot\partial_t E + B\cdot\partial_t B)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}(E\cdot(\nabla_x\wedge B - j) + B\cdot(-\nabla_x\wedge E))\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}\big[(E\cdot(\nabla_x\wedge B) - B\cdot(\nabla_x\wedge E)) - E\cdot j\big]\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3}\nabla_x\big((B\times E) - E\cdot j\big)\,\mathrm{d}x = -\int_{\mathbb{R}^3}E\cdot j\,\mathrm{d}x\,.\end{aligned}$$

Moreover, using the fact that $V$ and $E$ do not depend on $p$ and the identities $\nabla_p\cdot(\hat{p}\times B) = 0$ and

$$\nabla_p\cdot[(E+\hat{p}\times B)(f\log f - f)] = (E+\hat{p}\times B)\log f\cdot\nabla_p f,$$

we obtain

$$\int_{\mathbb{R}^6}(E+\hat{p}\times B)\cdot\nabla_p f\,(\log f + V)\,\mathrm{d}p\,\mathrm{d}x = \\ \int_{\mathbb{R}^6}\nabla_p\cdot[(E+\hat{p}\times B)(f\log f - f + Vf)]\mathrm{d}p\,\mathrm{d}x.$$

We also have

$$\int_{\mathbb{R}^6} \sqrt{1+|p|^2}\,(E+\hat{p}\times B)\cdot\nabla_p f\,\mathrm{d}p\,\mathrm{d}x = -\int_{\mathbb{R}^6} \hat{p}\cdot(E+\hat{p}\times B)\,f\,\mathrm{d}p\,\mathrm{d}x$$
$$= -\int_{\mathbb{R}^3} E\cdot j\,\mathrm{d}x.$$

The second equality in the above identity is due to the definition of the current density $j$ and the orthogonality between $\hat{p}$ and $\hat{p}\times B$. Then, the proof of (ii) follows. To show (iii), we first notice that combining the fact $\partial_t B_0 \equiv 0$ and equation (2.51biii) implies that there exists a function $U(x)$ such that $E_0 = -\nabla U(x)$. Using (2.51bii), we obtain $-\Delta U = \rho$. Moreover, observe that the distribution function adopts the form $f_0(x,p) = \alpha(x)\mathscr{J}(p)$ for some non-negative function $\alpha = \alpha(x)$ as a consequence of (2.54) applied to a static solution. In particular, $j = 0$ (since it is the integral of an odd function) and the equations for the field $B_0$ are equivalent to $\nabla\times B_0 = \nabla\cdot B_0 = 0 \Rightarrow B_0 \equiv 0$. Now replacing $f_0 = \alpha\mathscr{J}$, $E_0 = -\nabla U$ and $B_0 = 0$ in (2.51a) we obtain

$$\hat{p}\cdot\nabla\alpha + \alpha\hat{p}\cdot\nabla(U+V) = 0.$$

It is clear that the only non-trivial regular solution of the previous equation is $\alpha = Ce^{-U-V}$, where $C$ is any positive constant. The value $C = M/\Theta$ follows by the definition of $M$. □

Now, we prove the existence of (regular) static solutions for system (2.51). In particular, we show that the free energy functional (2.53) subject to

$$\nabla\cdot E = \rho, \quad \nabla\cdot B = 0, \quad \int_{\mathbb{R}^6} f\,\mathrm{d}p\,\mathrm{d}x = M,$$

attains its minimum exactly in the static solution of (2.51) with mass $M$. The following proof generalizes the one given in [47, Prop. 2.2] for the VPFP system. Note that the variational problem for VMFP differs from that of VPFP studied in [47] in two aspects. Firstly, the electromagnetic field appears as an independent variable in the entropy functional, while for VPFP the electric field is given by the convolution product of $\rho$ with $1/(4\pi|x|)$. Secondly, the local constraints $\nabla\cdot E = \rho$, $\nabla\cdot B = 0$ are required in the variational problem for VMFP. Nevertheless, we will be able to reduce the problem at hand to the equivalent one for the VPFP system considered in [47]. In particular, we will show that the above minimization problem is equivalent to minimizing a reduced entropy functional $\mathcal{K}_{\mathrm{red}}$ that resembles the free energy in the non-relativistic case. To this purpose we use the following simple result.

**Lemma 2.5.1.** *The solutions of the variational problem*

$$\inf_{h\in\mathfrak{D}} \mathcal{R}(h) = \inf_{h\in\mathfrak{D}} \int_{\mathbb{R}^3} |h|^2\,\mathrm{d}x,$$

*where* $\mathfrak{D} = \{h \in L^2(\mathbb{R}^3) \, : \, \nabla h = g\}$, $g \in L^1(\mathbb{R}^3)$, *are of the form* $h = -\nabla U$, *where* $-\Delta U = g$.

*Proof.* Let $\phi$ be a test function. The first variation of $\mathcal{R}$ evaluated on a critical point has to vanish, which implies

$$\frac{d}{dt}\mathcal{R}(h+t\phi)\,|_{t=0} = \int_{\mathbb{R}^3} \frac{d}{dt}|h+t\phi|^2\,|_{t=0}\; \mathrm{d}x = \int_{\mathbb{R}^3} 2h\cdot\phi\,\mathrm{d}x = 0.$$

In particular, consider test functions of the form $\phi = \nabla \wedge v$, which entails

$$0 = \int_{\mathbb{R}^3} h\cdot\nabla\wedge v\,\mathrm{d}x = -\int_{\mathbb{R}^3} \nabla\wedge h\cdot v\,\mathrm{d}x,$$

for all $v \in C_c^\infty(\mathbb{R}^3)$. From here we infer that $\nabla\wedge h = 0$ and as a consequence, there exists $U$ such that $h = -\nabla U$. Substituting this value in $\nabla\cdot h = g$ allows to conclude the proof. ☐

Next, let $X = X_1 \times X_2$, where the sets $X_1$, $X_2$ are defined as follows

$$X_1 = \left\{ f \in L^1(\mathbb{R}^6) \, : \, f \geq 0,\, \|f\|_{L^1(\mathbb{R}^6)} = M,\, \mathcal{Q}_+[f] < \infty \right\},$$
$$X_2 = \left\{ (E,B) \in L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3) \, : \, \nabla\cdot E = \rho,\, \nabla\cdot B = 0 \right\},$$

with $\mathcal{Q}_+[f]$ given by (2.25). Also, we consider the functional

$$\mathcal{K}_{\mathrm{red}}(f) = \int_{\mathbb{R}^6} f\left(\sqrt{1+|p|^2} + \frac{1}{2}U + V + \log f\right)\mathrm{d}p\,\mathrm{d}x,$$

and the restrictions $\rho = \int_{\mathbb{R}^3} f\,\mathrm{d}p$ and $-\Delta U = \rho$. Now, we are able to establish the following result:

**Proposition 2.5.2.** *Let* $e^{-V} \in L^1(\mathbb{R}^3)$ *and*

$$K(M) = \inf_X \left\{ \mathcal{K}[f,E,B] \right\}.$$

*Then, the following conditions hold:*

*(i)* $K(M) = \inf_{X_1}\{\mathcal{K}_{\mathrm{red}}(f)\}$;

*(ii)* $K(M)$ *is bounded from below for any* $M > 0$;

*(iii) The minimizer is unique and* $K(M) = \mathcal{K}_{\mathrm{red}}(m_M)$, *where* $m_M$ *is given by* (2.55).

*Proof.* To show (i), we see that the minimum (if it exists) verifies

$$\begin{aligned} K(M) = \inf_X\{\mathcal{K}(f,E,B)\} &= \inf_{X_1}\left\{\inf_{X_2}\{\mathcal{I}(E,B)\} + \mathcal{Q}(f)\right\}_{\rho=\int f\,\mathrm{d}p} \\ &= \inf_{X_1}\left\{\frac{1}{2}\int_{\mathbb{R}^3}|\nabla U|^2\,\mathrm{d}x + \mathcal{Q}(f)\right\}_{\rho=\int f\,\mathrm{d}p} \\ &= \inf_{X_1}\{\mathcal{K}_{\mathrm{red}}(f)\}, \end{aligned}$$

since by Lemma 2.5.1, for $g_1 = \rho$ and $g_2 = 0$, we have $E = -\nabla U$, $-\Delta U = \rho$ and $B = -\nabla \tilde{U}$, $-\Delta \tilde{U} = 0$, which implies $\tilde{U} \equiv 0$. Also, we have that

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla U|^2 \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^3} U\Delta U \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3} \rho U \,\mathrm{d}x = \int_{\mathbb{R}^6} \frac{1}{2} f U \,\mathrm{d}p \,\mathrm{d}x,$$

and the original problem is therefore reduced to minimize the functional $\mathcal{K}_{\mathrm{red}}(f)$, which, up to substituting $\sqrt{1+|p|^2}$ with $|p|^2/2$, coincides with the free energy in the non-relativistic case. Thus the claims (ii) and (iii) can be established as in [47, Prop. 2.2]. □

To conclude this section, it is important to remark that the existence of steady states for the VMFP system can also be established by studying directly equation (2.55c), as done in [76] for the non-relativistic case. The non-existence results proved there when $e^{-V} \notin L^1$ (see also [47]) remain valid in the relativistic case as well.

# Chapter 3

# The Newtonian Limit

The aim of this chapter consists on relating solutions of the relativistic Fokker-Planck model that was presented in the previous chapter with those solutions of its associated classical counterpart, the Vlasov-Fokker-Planck equation (VFP). In order to achieve this task, we consider the relativistic equation in terms of a parameter $c$, which could be though as the speed of light. Then, we show that these relativistic solutions converge to solutions of the VFP equation in the $L^1$ sense as $c \to \infty$. As a consequence of this procedure, we obtain the Newtonian limit of this equation which validates the model as a genuine relativistic generalization of the VFP equation.

## 3.1 Review of some previous results

When relativistic effects become a relevant part to describe particle motion, one should use a model where at least some essential relativistic features are present. For instance, particles must exhibit finite propagation speed without exceeding the speed of light $c$ and time can no longer be treated as an absolute quantity, since these considerations are part of the main foundations of the theory. If a classical counterpart is available, it would be desirable to recover this model from the relativistic one in some sort of sense. This requirement comes from the fact that classical mechanics can be though as a limiting case of relativity [122]. Roughly speaking, this should follow by allowing particles to have unbounded speeds. More precisely, if we consider the speed of light $c$ as a parameter and let $c \to \infty$, at least formally, the resulting model from the limiting process, if one exists, must be the classical one. This is the central point of the chapter with respect to the relativistic generalization of the Fokker-Planck equation considered in the previous chapter with an explicit dependance on the parameter $c$. The previous procedure is known as the Newtonian limit of a relativistic model [121]. There are some works in the literature within the framework of the gravitational theory where the authors have accomplished to prove the existence of a Newtonian limit in their corresponding cases. In [119], Rein and Rendall considered spherically symmetric classical solutions of the Vlasov-Einstein system (VE). They proved that these solutions converge uniformly to solutions of the Vlasov-Poisson system (VP), locally in time, for arbitrary initial data with compact support on the position variable and requiring one moment on the particle density. Also, the previous situation still holds globally in time for small initial data. This result is quite remarkable since the latter was achieved for the most representative model in relativity. In [120], Rendall improved the previous result in a more general setting. He showed that there exist families of asymptotically flat solutions of the VE system depending on the speed of light which converge to the VP system. A more related work in this direction was developed by Frittelli and Reula in [73] where they showed that given a Newtonian solution, it is possible to construct a family of solutions for the Einstein field equations. In [118], Reimold extends some results concerning the Newtonian limit problem in general relativity by using the frame theory of Ehlers. In the context of plasma physics, Schaeffer was able to show that solutions of the relativistic Vlasov-Maxwell system converge to solutions of the VP system in [126] under certain regularity conditions on the initial data. In order to prove this convergence, he used a representation formula for the Maxwell's equations that allowed him to have the required control on the limiting behavior of solutions. A similar argument was applied by Calogero and Lee for the Nordström-Vlasov system to obtain its non-relativistic limit in [27], another generalization of the VP system in the astrophysics case. Finally, we would like to mention that the Newtonian limit for a relativistic version of the Boltzmann equation was obtained by one of the previous authors in [25]. The importance of the latter result is due to the fact that the Boltzmann equation [75] is one of the most representative models in kinetic theory to account collisions among particles.

## 3.2 Statement of the result

In this section we present the main result of this chapter, Theorem 3.2.1, which not only relates solutions of equations (3.1) and (3.2), but also states some sufficient assumptions to obtain this connection. Also, we make some comments on the result and summarize the strategy to prove it.

Now, we proceed by recalling that for non-relativistic particles with mass $m > 0$ and in suitable physical units, the Vlasov-Fokker-Planck equation adopts the following form

$$\partial_t f + p \cdot \nabla_x f = \Delta_p f + \frac{\theta}{m} \nabla_p \cdot (pf), \quad \theta = \frac{1}{kT}. \tag{3.1}$$

Here, $f = f(t, x, p) \geq 0$ stands for the one-particle distribution function in phase space while the independent variables are the time $t \geq 0$, the position of the particle $x \in \mathbb{R}^N$ and its momentum $p \in \mathbb{R}^N$. Notice that in the definition of the dimensional constant $\theta$, $T$ represents the temperature of the thermal bath and $k$ is the Boltzmann's constant. The main reason to present equation (3.1) in this particular form is motivated by our interest in kinetic theory. We interpret its left hand side as transport phenomena while its right one is a collision operator depending only in the momentum variable. We emphasize that this particular choice to measure collisions is made for simplicity since a different approach can be carried out by using a Boltzmann type operator [25, 75].

As we have seen in the previous chapter, the main interest to study the *Newtonian* model (3.1) comes from astrophysics as well as from the plasma physics case. Unfortunately, this model does not possess relevant relativistic properties as the ones mentioned before, In fact, there are not several models available in the literature to account relativistic phenomena when random collisions are not neglected. As justified in the previous chapter, a relativistic generalization of equation (3.1) in the same physical units is written as

$$\partial_t f + mc\frac{p}{p^0}\cdot\nabla_x f = \partial_{p^i}\left(D^{ij}\partial_{p^j} f + \frac{\theta}{m}p^i f\right), \tag{3.2}$$

where $D$ is the diffusion matrix given by

$$D^{ij} = \frac{mc}{p^0}\left(\delta^{ij} + \frac{p^i p^j}{m^2c^2}\right), \quad p^0 = \sqrt{m^2c^2 + |p|^2}\,, \tag{3.3}$$

and $c > 0$. We interpret the parameter $c$ as the speed of light when its corresponding value is substituted. In this case, the Jüttner distribution function reads as $\mathscr{J} = e^{-\theta c p^0}$ which corresponds to the nontrivial equilibrium state of equation (3.2). As we have seen in the previous chapter, equation (3.2) seems to be a reasonable relativistic generalization of (3.1) since the model gathers several desirable relativistic properties. Fortunately, we can provide further justification of this matter. In fact, we will prove that (3.1) is indeed the correct Newtonian limit of (3.2). More precisely, we will show that solutions of (3.2) converge in $L^1$ to solutions of (3.1) as $c \to \infty$. Finally, the forthcoming analysis for Newtonian limit problem will be made in the following sections.

Before proceeding, we establish some conventions for the rest of the chapter. For notational convenience, we set $(x,p) = z$ and $\mathbb{R}^{2N} = \mathbb{R}^d$. We hope that all the related quantities are clear from the context. For instance,

$$|z| = |(x,p)|,\ dz = dp\,dx,\ \nabla_z = (\nabla_x, \nabla_p), \ldots,\ \text{etc.}$$

Now, we are in position to state the main result of this chapter:

**Theorem 3.2.1.** *Let $f, f_c \in C^2((0,\infty)\times\mathbb{R}^d)$ be positive solutions of equations (3.1) and (3.2), respectively, with non negative initial data $f^{in}, f_c^{in}$. Assume that the initial datum $f_c^{in}$ satisfies*

$$\|f_c^{in} - f^{in}\|_{L^1} \to 0 \text{ as } c \to \infty\,,$$

$$\Gamma_{\omega,\gamma}[f_c^{in}] := \int_{\mathbb{R}^d} \left[|\nabla_z f_c^{in}|^2 + |z|^\omega |\nabla_x f_c^{in}|^2 + |z|^\gamma |\nabla_p f_c^{in}|^2\right] dz < \infty, \tag{3.4}$$

*for some $\gamma, \omega$ such that $\gamma > d+2$ and $\omega > d+4$. Then $f_c(t) \to f(t)$ in $L^1$ as $c \to \infty$, uniformly on every compact interval of time.*

As one could expect, a similar result as the previous one should hold with less restrictive assumptions. For instance, if the exact form of fundamental solution of (3.2) was known, we might be able to dispense of condition (3.4). Instead, we need to use the fundamental solution of equation (3.1) and condition (3.4) turns out to be sufficient to control the required estimates. In fact, any strategy in a similar direction would involve a more careful and detailed analysis.

From a physical perspective, the previous theorem is a very natural consequence for any valid relativistic model. The $L^1$ framework is one of the most physically relevant scenarios for these models since particle loss is not expected as $c \to \infty$, i.e., mass is a conserved quantity. Also, Theorem 3.2.1 is the most rigorous result available for this relativistic model.

In order to prove Theorem 3.2.1, we will follow the next strategy:

- First, we show that solutions of equation (3.2) will inherit the bound (3.4). This boundedness property is very important since it enables to control the behavior of the solution in the limit.
- Next, we consider the difference $\delta f$ between solutions of equations (3.1)–(3.2). This leads to analyze the time evolution of the nonhomogeneous FP equation (3.8). At this point, Duhamel's principle will become crucial since we can exploit the available representation formula for this equation which essentially is given in terms of the initial condition of $\delta f$ and derivative terms of $f_c$.
- Finally, we estimate the $L^1$ norm of $\delta f$ using property (3.4), the representation formula derived in the previous step and a simple interpolation argument. The previous estimation allows to take the limit $c \to \infty$ and conclude the result.

We will divide the previous analysis in three sections: one in which we will obtain the required a priori estimates mentioned in the first step, another one in which we will derive the nonhomogeneous FP equation and recall some useful properties of the fundamental solution associated to equation (3.1), and the following one where the convergence will be proved.

In order to avoid some technical difficulties, which could possibly be dealt with, we will work with smooth solutions of (3.1)–(3.2) throughout the chapter. In the final section of the chapter, we will sketch the argument developed in [2] when a compact support assumption on the initial datum is considered. We will perform this because we can encounter an improvement on the estimations for this case.

## 3.3 A Priori Bounds

In this section we begin our analysis on the Newtonian limit problem for the relativistic equation (3.2). In order to obtain the desired $L^1$ convergence, we will show that $\|f_c - f\|_{L^1}$ can be bounded in terms of $\Gamma_{\omega,\gamma}[f_c]$. Although this condition seems to be too strong, the main strategy to prove Theorem 3.2.1 (Section 3.4) will clarify this assumption. Therefore, we require to show that any solution of the relativistic Fokker-Planck equation inherits the boundedness property of the initial data (3.4). The following result has been adapted from one we proved in [2] for our present situation:

**Lemma 3.3.1.** *Let $f_c$ be a solution of equation (3.2) with initial datum satisfying property (3.4) for all $\gamma \geq 2$ and $\gamma < \omega$. Then, $\Gamma_{\omega,\gamma}[f_c(t)] < \infty$ holds for all $\gamma, \omega \geq 0$ and $t > 0$.*

*Proof.* We start the proof by defining the vector functions $v = \nabla_p f$ and $u = \nabla_x f$. Now we proceed to prove the assertion. The latter will be performed in two steps. First, we exploit the fact that the corresponding bound for $u$ does not require to estimate terms containing $v$. Then, the estimate for the integral containing the term $|(x,p)|^\gamma |v|^2$ will follow by using the bound from the previous step and a similar reasoning.

1. Observe that each component of $u$ satisfies equation (3.2) since the coefficients of this equation do not depend explicitly on the $x$ variable, or equivalently,

$$\partial_t u + mc\frac{p}{p^0} \cdot \nabla_x u = \partial_{p^i}\left(D^{ij}\partial_{p^j} u + \frac{\theta}{m} p^i u\right) . \tag{3.5}$$

Then, we can multiply (3.5) by $|(x,p)|^\omega u$ and integrate the resulting expression over $\mathbb{R}^d$ to obtain the next identity

$$\begin{aligned}\partial_t \int_{\mathbb{R}^d} |(x,p)|^\omega |u|^2 \, dz = &- mc \int_{\mathbb{R}^d} |z|^\omega \frac{p}{p^0} \cdot \nabla_x |u|^2 \, dz \\ &+ 2\int_{\mathbb{R}^d} |z|^\omega \partial_{p^i}(D^{ij}\partial_{p^j} u + \beta p^i u) \cdot u \, dz \, .\end{aligned} \tag{3.6}$$

Notice that the right hand side of equality (3.6) can be integrated by parts. As a consequence, the first term can be bounded as follows

$$-mc\int_{\mathbb{R}^d} |(x,p)|^{\omega}\frac{p}{p^0}\cdot\nabla_x|u|^2\,dp\,dx = \omega mc\int_{\mathbb{R}^d}|z|^{\omega-2}\frac{x\cdot p}{p^0}|u|^2\,dz \leq \omega\int_{\mathbb{R}^d}|z|^{\omega}|u|^2\,dz\,,$$

where we used that $|p|,|x|\leq|z|$ and $mc\leq p^0$. Proceeding similarly as before and using that $\partial_{p^i}(p^i|(x,p)|^{\omega}) = (N|(x,p)|^{\omega}+\omega|(x,p)|^{\omega-2}|p|^2)$, the third term in (3.6) is estimated as follows

$$\begin{aligned}2\beta\int_{\mathbb{R}^d}|z|^{\omega}\partial_{p^i}(p^iu)\cdot u\,dp\,dx &= \beta\int_{\mathbb{R}^d}\left[d|z|^{\omega}-\partial_{p^i}(p^i|z|^{\omega})\right]|u|^2\,dz\\ &= \beta\int_{\mathbb{R}^d}\left(N|z|^{\omega}-\omega|p|^2|z|^{\omega-2}\right)|u|^2\,dz\\ &\leq \beta N\int_{\mathbb{R}^d}|z|^{\omega}|u|^2\,dz\,.\end{aligned}$$

The last inequality holds since $d=2N$ and the term containing the $\omega$ factor in the second line is negative. In order to treat the remaining term in (3.6), we need to account the fact that $D^{ij}\partial_{p^i}u\,\partial_{p^j}u\geq 0$ and the inequality

$$\begin{aligned}\partial_{p^j}(|(x,p)|^{\omega-2}p_iD^{ij}) &= \partial_{p^j}\left(|z|^{\omega-2}p_i\frac{mc}{p^0}\left[\delta^{ij}+\frac{p^ip^j}{m^2c^2}\right]\right)\\ &= (mc)^{-1}\partial_{p^j}(|z|^{\omega-2}p^jp^0)\\ &\leq (mc)^{-1}[(\omega-2+N)p^0|z|^{\omega-2}+|z|^{\omega-1}]\\ &\lesssim (1+|z|)|z|^{\omega-2}\,,\end{aligned}$$

where we used definition (3.3), $c>1$ and $|p|\leq p^0$ to obtain this. Therefore, we have that

$$\begin{aligned}2\int_{\mathbb{R}^d}|z|^{\omega}\partial_{p^i}(D^{ij}\partial_{p^j}u)\cdot u\,dp\,dx &= \frac{\omega}{mc}\int_{\mathbb{R}^d}\partial_{p^j}(|z|^{\omega-2}p_jp^0)|u|^2\,dz\\ &\quad -2\int_{\mathbb{R}^d}|z|^{\omega}D^{ij}\partial_{p^j}u\cdot\partial_{p^i}u\,dz\\ &\lesssim \omega\int_{\mathbb{R}^d}(1+|z|)|z|^{\omega-2}|u|^2\,dz\,.\end{aligned}$$

Collecting all the above estimates, we find that the following holds for the left hand side of (3.6)

$$(*)\quad \partial_t\int_{\mathbb{R}^d}|z|^{\omega}|u|^2\,dz \lesssim \int_{\mathbb{R}^d}|z|^{\omega}|u|^2\,dz+\omega\int_{\mathbb{R}^d}(1+|z|)|z|^{\omega-2}|u|^2\,dz\,.$$

A direct application of Gronwall's inequality in the above estimate for $\omega = 0$ lead us to

$$\int_{\mathbb{R}^d} |\nabla_z f_c|^2 \, dz \lesssim \int_{\mathbb{R}^d} |\nabla_z f_c^{in}|^2 \, dz \,.$$

In order to prove the general case, we first consider $\omega \geq 2$. It is straightforward to show that

$$\partial_t \int_{\mathbb{R}^d} |z|^\omega |u|^2 \, dz \lesssim \int_{\mathbb{R}^d} (1 + |z|^\omega) |u|^2 \, dz \,,$$

since we can estimate the integral containing the term $(1 + |z|)|z|^{\omega-2}$ in (*) by splitting this expression in two terms, one for $|z| < 1$ and the other for $|z| \geq 1$. Therefore, the bound for the integral of $|(x,p)|^\omega |u|^2$ similarly follows as in the case $\omega = 0$ .

Finally, let $R, k, \alpha > 0$ such that $\alpha + k = \omega$. Under these conditions we are able to find the bound

$$\begin{aligned}
\int_{\mathbb{R}^d} |z|^\alpha |u|^2 \, dz &\leq R^\alpha \int_{\mathbb{R}^d} |u|^2 \, dz + R^{-k} \int_{\mathbb{R}^d} |z|^\omega |u|^2 \, dz \\
&\lesssim \left( \|u\|^2_{L^2(\mathbb{R}^d)} \right)^{\frac{k}{\omega}} \left( \int_{\mathbb{R}^d} |z|^\omega |u|^2 \, dz \right)^{\frac{\alpha}{\omega}} \\
&\lesssim \int_{\mathbb{R}^d} |u|^2 \, dz + \int_{\mathbb{R}^d} |z|^\omega |u|^2 \, dz,
\end{aligned}$$

where we applied Young's inequality with $r = \omega/k$ and $s = \omega/\alpha$, see [65] for instance, and made the choice

$$R = \left( \frac{\int_{\mathbb{R}^d} |z|^\omega |u|^2 \, dz}{\int_{\mathbb{R}^d} |u|^2 \, dz} \right)^{\frac{1}{\omega}} .$$

Thus, the remaining cases are achieved by applying the above interpolation between $\omega = 0, 2$.

2. First, we differentiate equation (3.2) with respect to $p_k$ and obtain that for each $k = 1, \cdots, N$, the following holds

$$\begin{aligned}
\partial_t v_k + mc\frac{p}{p^0} \cdot \nabla_x v_k &= \partial_{p^i} \left[ D^{ij} \partial_{p^j} v_k + \beta p^i v_k \right] \\
&\quad - mc\partial_{p^k} \left( \frac{p}{p^0} \right) \cdot u + \partial_{p^i} \left[ \partial_{p^k} (D^{ij}) v_j \right] + \beta v_k \,,
\end{aligned} \tag{3.7}$$

where $v_k$ is the $k$th component of $v$ and $\beta = \frac{\theta}{m}$. In order to bound the integral of $|(x,p)|^\gamma |v|^2$, we will use a similar argument as in the previous step. Since equation (3.7) contains terms that we have already treated, we can give a first bound as follows

$$\begin{aligned}\partial_t \int_{\mathbb{R}^d} |z|^\gamma |v|^2\, dz \leq\ & [\beta(N+2)+\gamma] \int_{\mathbb{R}^d} |z|^\gamma |v|^2\, dz \\ & + \frac{\gamma}{mc} \int_{\mathbb{R}^d} \partial_{p^j}(|z|^{\gamma-2} p^j p^0)|v|^2\, dz \\ & - 2mc \int_{\mathbb{R}^d} |z|^\gamma \partial_{p^k}\left(\frac{p^i}{p^0}\right) \partial_{x^i} f \partial_{p^k} f\, dz \\ & - 2\gamma \int_{\mathbb{R}^d} |z|^{\gamma-2} p^i \partial_{p^k}(D^{ij}) \partial_{p^j} f \partial_{p^k} f\, dz \\ & - 2 \int_{\mathbb{R}^d} |z|^\gamma \partial_{p^k}(D^{ij}) \partial_{p^j} f \partial_{p^i}(\partial_{p^k} f)\, dz \\ =\ & I_1 + I_2 + I_3 + I_4 + I_5\,. \end{aligned}$$

Notice that $I_1 + I_2$ can be treated as in the previous step. To estimate $I_3$, we take into account that

$$\left| \partial_{p^k}\left(\frac{p^i}{p^0}\right) u_i v_k \right| = \left| \frac{u^k v_k}{p^0} - \frac{p^i u_i p^k v_k}{(p^0)^3} \right| \lesssim \frac{|u||v|}{p^0} \lesssim \frac{|u||v|}{mc}\,,$$

which was obtained from the fact that $mc, |p| < p^0$ and applying Young's inequality with $r = s = 2$. Then, we have

$$I_3 = -2mc \int_{\mathbb{R}^d} |z|^\gamma \partial_{p^k}\left(\frac{p^i}{p^0}\right) \partial_{x^i} f \partial_{p^k} f\, dz \lesssim \int_{\mathbb{R}^d} |z|^\gamma (|v|^2 + |u|^2)\, dz\,.$$

Recall that $\gamma < \omega$ which implies that the term $|z|^\gamma |u|^2$ contained in the last integral is bounded. Now, observe that $I_4$ satisfies

$$-2\gamma \int_{\mathbb{R}^d} |z|^{\gamma-2} p^i \partial_{p^k}(D^{ij}) \partial_{p^j} f \partial_{p^k} f\, dz = -2\gamma \int_{\mathbb{R}^d} \frac{|z|^{\gamma-2}|p|^2}{mcp^0} |v|^2\, dz\,,$$

since

$$\begin{aligned} p^i \partial_{p^k}(D^{ij}) v_j v_k &= \left( \frac{|p|^2 \delta^{jk} + p^j p^k}{mcp^0} - p^i \frac{m^2c^2\delta^{ij} + p^i p^j}{mc(p^0)^2} \cdot \frac{p^k}{p^0} \right) v_j v_k \\ &= \left( \frac{|p|^2 \delta^{jk} + p^j p^k}{mcp^0} - \frac{p^k p^i}{(p^0)^2} D^{ij} \right) v_j v_k \\ &= \left( \frac{|p|^2 \delta^{jk} + p^j p^k}{mcp^0} - \frac{p^j p^k}{mcp^0} \right) v_j v_k = \frac{|p|^2}{mcp^0} |v|^2\,. \end{aligned}$$

Here, definition (3.3) and property $p^i D^{ij} = p^j p^0/mc$ were used. In order to treat $I_5$, we first split $\partial_{p^k}(D^{ij})\partial_{p^j} f \partial_{p^i}\partial_{p^k} f$ as follows

$$\begin{aligned}\partial_{p^k}(D^{ij})\partial_{p^j} f \partial_{p^i}\partial_{p^k} f &= \left[\frac{(p^i\delta^{jk}+p^j\delta^{ik})}{mcp^0} - \frac{p^k D^{ij}}{(p^0)^2}\right]\partial_{p^j} f \partial_{p^i}\partial_{p^k} f \\ &= \frac{p^i}{mcp^0}\partial_{p^k} f \partial_{p^i}\partial_{p^k} f + \frac{p^j}{mcp^0}\partial_{p^j} f \partial_{p^k}\partial_{p^k} f \\ &\quad - \frac{mcp^k}{(p^0)^3}\left[\partial_{p^j} f \partial_{p^j}\partial_{p^k} f + \frac{p^i p^j}{m^2c^2}\partial_{p^j} f \partial_{p^i}\partial_{p^k} f\right].\end{aligned}$$

Next, we see that the contribution to $I_5$ of the last term in the above identity will be

$$\int_{\mathbb{R}^d} |z|^\gamma \frac{p^i p^j p^k}{mc(p^0)^3}\partial_{p^j} f \partial_{p^i}\partial_{p^k} f = -\frac{1}{2}\int_{\mathbb{R}^d} \partial_{p^i}\left[|z|^\gamma \frac{p^i p^j p^k}{mc(p^0)^3}\right] v^j v^k \, dz \,.$$

The previous identity is a result of an integration by parts and the use of the symmetry property from $p^i p^j p^k$. Also, we observe that the second term of $\partial_{p^k}(D^{ij})$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^d} |z|^\gamma \frac{p^j}{mcp^0}\partial_{p^j} f \partial_{p^k}\partial_{p^k} f \, dz = &-\int_{\mathbb{R}^d} \partial_{p^k}\left[\frac{|z|^\gamma p^j}{mcp^0}\right] v^j v^k \, dz \\ &-\int_{\mathbb{R}^d} \frac{|z|^\gamma p^j}{mcp^0}\partial_{p^k}(v^j) v^k \, dz \,.\end{aligned}$$

Notice that the last term in this identity will cancel out with the corresponding first term of $\partial_{p^k}(D^{ij})$. Using the definition of $I_5$, combining all the above identities and applying an integration by parts in the third term of $\partial_{p^k}(D^{ij})$, we obtain

$$\begin{aligned}I_5 &= \frac{1}{mc}\int_{\mathbb{R}^d}\left(2\partial_{p^k}\left[\frac{|z|^\gamma p^j}{p^0}\right] - \partial_{p^i}\left[\frac{|z|^\gamma p^i p^j p^k}{(p^0)^3}\right]\right)\partial_{p^k} f \partial_{p^j} f \, dz \\ &\quad - mc\int_{\mathbb{R}^d} \partial_{p^k}\left[\frac{|z|^\gamma p^k}{(p^0)^3}\right]|v|^2 \, dz \\ &\le \frac{1}{mc}\int_{\mathbb{R}^d}\left(2\partial_{p^k}\left[\frac{|z|^\gamma p^j}{p^0}\right] - \partial_{p^i}\left[\frac{|z|^\gamma p^i p^j p^k}{(p^0)^3}\right]\right)\partial_{p^k} f \partial_{p^j} f \, dz \\ &\quad + \frac{3}{mc}\int_{\mathbb{R}^d} \frac{|z|^\gamma}{p^0}|v|^2 \, dz \,,\end{aligned}$$

where we used

$$\partial_{p^k}\left[\frac{|z|^\gamma p^k}{(p^0)^3}\right] = \frac{N|z|^\gamma + \gamma|p|^2|z|^{\gamma-2}}{(p^0)^3} - \frac{3|p|^2|z|^\gamma}{(p^0)^5}\,,$$

and $p^0 \ge |p|, mc$. For simplicity, we define the matrix

$$\Delta^{jk} = (p^0)^3\left(2\partial_{p^k}\left[\frac{|(x,p)|^\gamma p^j}{p^0}\right] - \partial_{p^i}\left[\frac{|(x,p)|^\gamma p^i p^j p^k}{(p^0)^3}\right]\right).$$

and state that the next inequality holds

$$\Delta^{jk}\partial_{p^j} f \partial_{p^k} f - 2(p^0)^2\gamma|z|^{\gamma-2}|p|^2|v|^2 \leq 2(p^0)^2|(x,p)|^{\gamma}|v|^2\,.$$

As a consequence of this inequality, we see that

$$I_4 + I_5 \leq \frac{5}{m}\int_{\mathbb{R}^d} \frac{|z|^{\gamma}}{p^0}|v|^2\,dz\,,$$

where the fact $c > 1$ was used, and the claim follows as in part 1. In order to prove the remaining statement, we notice that

$$\begin{aligned}
\Delta^{jk}v_j v_k &= (p^0)^3\left(2\partial_{p^k}\left[\frac{|z|^{\gamma}p^j}{p^0}\right] - \partial_{p^i}\left[\frac{|z|^{\gamma}p^i p^j p^k}{(p^0)^3}\right]\right)v_j v_k \\
&= 2(p^0)^2|z|^{\gamma}|v|^2 + p^j p^k v_j v_k\left(2\gamma(p^0)^2|z|^{\gamma-2} - 2|z|^{\gamma}\right) \\
&\quad + p^j p^k v_j v_k\left(3|z|^{\gamma}\frac{|p|^2}{(p^0)^2} - (N+2)|z|^{\gamma} - \gamma|p|^2|z|^{\gamma-2}\right) \\
&\leq 2(p^0)^2|z|^{\gamma}|v|^2 + 2\gamma(p^k v_k)^2(p^0)^2|z|^{\gamma-2} - (p^k v_k)^2(\gamma+N+1)|z|^{\gamma} \\
&\leq 2(p^0)^2|z|^{\gamma}|v|^2 + 2\gamma(p^0)^2|z|^{\gamma-2}|p|^2|v|^2\,,
\end{aligned}$$

where we used $p^j v_j p^k v_k = (p^k v_k)^2 \geq 0$ and $(|p|/p^0)^2 \leq 1$. This last argument completes the proof of the Lemma.

□

It is important to remark that we made sure that all the above estimates do not depend explicitly on any positive power of $c$. As a consequence, $\Gamma_{\omega,\gamma}[f_c(t)]$ will remain bounded in any compact interval of time as $c \to \infty$ since the asymptotic behavior is controlled by a bounded quantity, $\Gamma_{\omega,\gamma}[f_c^{in}]$.

## 3.4 A Nonhomogeneous FP Equation

In this section, we derive a nonhomogeneous Fokker-Planck equation related to our convergence problem. The main reason to consider this equation is due to the fact that a representation formula for solutions can be easily obtained. Although this formula can not be completely given in terms of the initial data, it will turn out that combining the bounds derived in the previous section while estimating the $L^1$ norm of the solution of this non-homogeneous problem is sufficient to prove Theorem 3.2.1. We will achieve this objective by means of the *Duhamel's principle* since the explicit form of the fundamental solution from the homogeneous problem is available. We will also recall some properties of this fundamental solution which are required in the next section. We must proceed in this manner to overcome the fact that the exact form of the fundamental solution of equation (3.2) still remains unknown.

We begin by considering the function $\delta f = f - f_c$, where $f$ and $f_c$ are solutions of (3.1) and (3.2), respectively. It is easy to show that $\delta f$ is a smooth solution of

$$\partial_t \delta f + p \cdot \nabla_x \delta f = \Delta_p \delta f + \frac{\theta}{m} \nabla_p \cdot (p\,\delta f) + g_c\,, \tag{3.8}$$

where

$$\begin{aligned} g_c &= \Delta_p f_c - \partial_{p^i}\left(D^{ij}\partial_{p^j} f_c\right) + \left[\frac{mc}{p^0} - 1\right] p \cdot \nabla_x f_c \\ &= \partial_{p^i}\left([\delta^{ij} - D^{ij}]\partial_{p^j} f_c\right) + \left[\frac{mc}{p^0} - 1\right] p \cdot \nabla_x f_c\,. \end{aligned}$$

Now, recall that the classical Fokker-Planck equation (3.1) has an explicit representation of its fundamental solution. Let $\mathcal{F}(t,x,p,y,w)$ denote this two point Green function. Its exact form is given by

$$\mathcal{F}(t,x,p,y,w) = \left[\frac{\beta \exp\{\beta t\}/4\pi}{\sqrt{a(2\beta,t)t - a^2(\beta,t)}}\right]^d \exp\left\{b(t,x,p,y,w) - \frac{|\beta(x-y)+(p-w)|^2}{4t}\right\},$$

with $\beta = \theta/m$, $a(\beta,t) = \frac{\exp\{\beta t\}-1}{\beta}$ and

$$b(t,x,p,y,w) = -\frac{|a(\beta,t)\left\{\beta(x-y)+(p-w)\right\} + t(w - p\exp\{\beta t\})|^2}{t a(2\beta,t) - a^2(\beta,t)}\,,$$

see [138, Eq. (2.5)], and satisfies the following properties

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathcal{F}(t,z,y,w)\,dp\,dx = 1\,, \\ &|\nabla_w \mathcal{F}|(t-s,z,y,w) \le \frac{C(\alpha,\beta)}{\sqrt{t-s}} \mathcal{F}(t-s,\alpha z,\alpha y,\alpha w)\,, \end{aligned} \tag{3.9}$$

with $0 < \alpha < 1$, see [138, eqs. (2.8), (2.30)]. Then, any solution of (3.1) can be written in terms of $\mathcal{F}$ as follows

$$f(t,x,p) = \int_{\mathbb{R}^d} \mathcal{F}(t,x,p,y,w) f(0,y,w)\,dw\,dy\,.$$

Since (3.8) reduces to (3.1) when $g_c = 0$, Duhamel's principle entails that solutions of (3.8) can be expressed as

$$\begin{aligned} \delta f(t,x,p) = &\int_{\mathbb{R}^d} \mathcal{F}(t,x,p,y,w)\delta f(0,y,w)\,dw\,dy \\ &+ \int_0^t \int_{\mathbb{R}^d} \mathcal{F}(t-s,x,p,y,w) g_c(s,y,w)\,dw\,dy\,ds\,, \end{aligned} \tag{3.10}$$

for $t \geq s$. Notice that the last term in the above identity contains a second order term. In order to avoid this inconvenient, we define $X^i = (\delta^{ij} - D^{ij})\partial_{w^j}$ and perform an integration by parts in the $w$ variable on this term

$$\int_{\mathbb{R}^d} \mathcal{F}\, \partial_{w^i}(X^i f_c)\, dw\, dy = -\int_{\mathbb{R}^d} \partial_{w^i}\mathcal{F}\, X^i f_c\, dw\, dy.$$

Since $g_c = \partial_{p^i}(X^i f_c) + \left[\frac{mc}{p^0} - 1\right] p \cdot \nabla_x f_c$, the above identity allows to express (3.10) as

$$\begin{aligned}\delta f = &\int_{\mathbb{R}^d} \mathcal{F}(t,x,p,y,w)\delta f(0,y,w)\, dw\, dy \\ &- \int_0^t \int_{\mathbb{R}^d} \nabla_w \mathcal{F}(t-s,x,p,y,w) \cdot X f_c(s,y,w)\, dw\, dy\, ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \mathcal{F}(t-s,x,p,y,w) \left[\frac{mc}{w^0} - 1\right] w \cdot \nabla_y f_c(s,y,w)\, dw\, dy\, ds\,. \end{aligned} \tag{3.11}$$

Observe that the latter expression is more suitable to control in the limit for any compact interval of time due to (3.9).

## 3.5 Convergence towards classical solutions

Now, we are in position to prove Theorem 3.2.1 by using all the information developed in the previous sections. In order to prove the result, we must show that the last two terms in (3.11) converge to zero in $L^1$ as $c \to \infty$ since we already know that this happens for the term $\delta f(0)$. The main strategy to achieve the latter consists on giving an explicit rate of convergence for this term. This will be possible due to the explicit dependence on $c$ from the $g_c$ term in the representation formula (3.11) and the finiteness property of $\Gamma_{\omega,\gamma}[f_c(t)]$ given by Lemma 3.3.1, where $\Gamma_{\omega,\gamma}$ is defined by (3.4).

We begin the proof by estimating the following quantities

$$\begin{aligned} &\left|1 - \frac{mc}{w^0}\right| = \left|\frac{\sqrt{m^2c^2 + |w|^2} - mc}{\sqrt{m^2c^2+|w|^2}}\right| = \frac{|w|^2}{w^0(w^0+mc)} \lesssim \frac{|w|}{c}\,, \\ &|X(f_c)| \leq \sup_{i,j} |\delta^{ij} - D^{ij}||\nabla_w f_c| = \sup_{i,j}\left|\left(1 - \frac{mc}{w^0}\right)\delta^{ij} - \frac{w^i w^j}{w^0 mc}\right| |\nabla_w f_c| \\ &\qquad \lesssim c^{-1}|w||\nabla_w f_c|\,. \end{aligned} \tag{3.12}$$

By (3.11) and the above inequalities, we see that the $L^1$ norm of $\delta f$ can be estimated as follows

$$
\begin{aligned}
\|\delta f_c(t)\|_{L^1} \lesssim & \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \mathcal{F}(t,z,y,w)\, dz \right) |\delta f(0,y,w)|\, dw\, dy \\
& + \frac{1}{c} \int_0^t \int_{\mathbb{R}^d} |w|^2 |\nabla_y f_c| \left( \int_{\mathbb{R}^d} \mathcal{F}(t-s,z,y,w) dz \right) dw\, dy\, ds \\
& + \frac{1}{c} \int_0^t \int_{\mathbb{R}^d} |w||\nabla_w f_c| \left( \int_{\mathbb{R}^d} |\nabla_w \mathcal{F}|(t-s,z,y,w) dz \right) dw\, dy\, ds,
\end{aligned}
$$

where we used $z = (x,p)$ for simplicity. Taking into account that the fundamental solution $\mathcal{F}$ satisfies (3.9), the right hand side of the above inequality can be bounded by

$$
\|\delta f(0)\|_{L^1} + \frac{1}{c} \int_0^t \||w|^2 \nabla_y f_c\|_{L^1} + \frac{1}{\sqrt{t-s}} \||w|\nabla_w f_c\|_{L^1} ds\,. \tag{3.13}
$$

In order to conclude the result, we need to bound (3.13) in terms of the $L^2$ moments of the gradient since by Lemma 3.3.1, this moments are finite. First, we interpolate

$$
\begin{aligned}
\int_{\mathbb{R}^d} |w||\nabla_w f_c| dw\, dy \leq & \int_{|(y,w)|\geq 1} |(y,w)||\nabla_w f_c| dw\, dy + \int_{|(y,w)|<1} |\nabla_w f_c| dw\, dy \\
\lesssim & \left( \int_1^\infty r^{d+1-\gamma} dr \right)^{\frac{1}{2}} \||(y,w)|^{\frac{\gamma}{2}} \nabla_w f_c\|_{L^2(\mathbb{R}^d)} \\
& + \left( \int_{\mathbb{R}^d} |\nabla_w f_c|^2 dw\, dy \right)^{\frac{1}{2}},
\end{aligned}
$$

and for $\gamma > d+2$, the integral on the left hand side is finite. By exactly the same argument we see that

$$
\int_{\mathbb{R}^d} |w|^2 |\nabla_y f_c|\, dw\, dy \lesssim \|\nabla_y f_c\|_{L^2(\mathbb{R}^d)} + \left\| |(y,w)|^{\frac{\omega}{2}} \nabla_y f_c \right\|_{L^2(\mathbb{R}^d)},
$$

is also finite for $\omega > d+4$. Using these estimates in (3.13), we find that

$$
\|\delta f(t)\|_{L^1} \lesssim \|\delta f(0)\|_{L^1} + O(1/c)\,.
$$

Therefore, $\delta f(t) \to 0$ in $L^1(\mathbb{R}^d)$ as $c \to \infty$ for every compact interval of time. This concludes the proof of Theorem 3.2.1.

## 3.6 Some Remarks

As we have previously seen, the use of momenta to estimate $\nabla_x f_c$ and $\nabla_p f_c$ is essential in order to obtain the result. The different weight condition is a consequence of the degeneracy in the variable $x$ of equation (3.2). Also, the fundamental solution of equation (3.1) helped to complete the argument. Probably if the fundamental solution of equation (3.2) was at our disposal, then some conditions could be weakened or dispensed of.

Now, we conclude the chapter by presenting some modifications in the argument made above to obtain the Newtonian limit that only needs momenta in $p$. To achieve the latter, we also require that the initial data are compactly supported in the $x$ variable. First, we address that Lemma 3.3.1 can be proved for the case where $\Gamma_{\omega,\gamma}$ only depends on momenta in $p$ by following a similar argument as in [3] for $d = 6$. Also, notice that (3.12) can also be estimated as follows

$$\left|1 - \frac{mc}{w^0}\right| \lesssim \frac{|w|^2}{c^2}\,, \quad |X(f_c)| \lesssim \frac{|w|^2}{c^2}|\nabla_w f_c|\,,$$

and now the factor $c^{-2}$ is crucial for the bound of the $L^1$ norm of $\delta f(t)$. By (3.11) and the above estimates we have

$$\begin{aligned}\|\delta f(t)\|_{L^1} \lesssim \|\delta f(0)\|_{L^1} &+ \frac{1}{c^2}\int_0^t \int_{\mathbb{R}^d} |w|^3|\nabla_y f_c|\,dw\,dy\,ds \\ &+ \frac{1}{c^2}\int_0^t \frac{1}{\sqrt{t-s}}\int_{\mathbb{R}^d} |w|^2|\nabla_w f_c|\,dw\,dy\,ds\,.\end{aligned}$$

Before proceeding by an interpolation argument, we observe that by the finite propagation speed property for the relativistic Fokker-Planck equation (3.2), see Proposition 2.3.1, we have that $f_c = 0$ for $|y| \geq R + ct$ when $f_c^{in} = 0$ for $|y| > R$. From here, the dimension becomes relevant. Set $d = 6$. Whence

$$\begin{aligned}\int_{\mathbb{R}^6} |w|^2|\nabla_w f_c|\,dw\,dy &\leq \int_{|y|\lesssim c}\int_{|w|<1} |\nabla_w f_c|\,dw\,dy + \int_{|y|\lesssim c}\int_{|w|\geq 1} |w|^2|\nabla_w f_c|\,dw\,dy \\ &\lesssim c^{3/2}\left(\|\nabla_w f_c\|_{L^2(\mathbb{R}^6)} + \||w|^{\frac{\gamma}{2}}\nabla_w f_c\|_{L^2(\mathbb{R}^6)}\right),\end{aligned}$$

and the integral on the left hand side is $O(c^{3/2})$. Here we used that for $\gamma > 7$

$$\int_1^\infty r^{6-\gamma} dr < \infty.$$

Similarly, the following holds for $\omega > 9$

$$\int_{\mathbb{R}^6} |w|^3|\nabla_y f_c|\,dw\,dy \lesssim c^{3/2}\left(\|\nabla_y f_c\|_{L^2(\mathbb{R}^6)} + \||w|^{\frac{\omega}{2}}\nabla_y f_c\|_{L^2(\mathbb{R}^6)}\right).$$

Then, combining all the above estimates allows to conclude

$$\|\delta f(t)\|_{L^1} \lesssim \|\delta f(0)\|_{L^1} + O(1/\sqrt{c}),$$

and the result follows for this case. Notice that in the latter case ($d = 6$), $7 < \gamma \leq d+2$ and $9 < \omega \leq d+4$ although the rate of convergence is slightly worse. This means that we can dispense of one moment with the compact support property assumption. Unfortunately, the above argument can only be used for $d \leq 6$ since the powers of $c$ in the bounds increase with the dimension. The bright side of this fact is that the physically relevant case is when the dimension is six.

# Chapter 4

# Trend to the equilibrium

In this chapter the long time behavior of solutions for the relativistic Fokker-Planck equation is treated. We will show that these solutions converge exponentially towards the Jüttner distribution function in $L^q$ for $q = 1, 2$. Due to the main difficulties exhibited by the model, the latter problem will be analyzed in the spatially homogeneous case. Moreover, it is shown that the $L^1$ convergence depends on the values of the parameter $\theta$ while in the remaining case, no restrictions are encountered at all.

## 4.1 Background of the problem

In several fields of science and industry the study of evolution systems is highly required. Several models to describe some of these situations have been proposed, analyzed and tested using different tools and perspectives. Also, when a phenomenon in question reaches and remains in certain states of interest, one pays particular attention on how the latter has occurred. For example, if we consider an object falling, then the "ground" state and its position at each instant of time are desired to describe. Under several assumptions, this problem has been deeply studied and in its simplest case, it is known as the free fall motion. Another typical and similar situation illustrating certain mathematical aspects of interest is the movement of a pendulum. In this case, a weighted body is attached to a string, rigid or not, and suspended from a pivot in order this body can swing freely. The simplest case to consider is the sideways motion with a rigid string and gravity as the only acting force. At least three equilibrium states can be identified in the previous situation depending on the mechanism of the pivot and the initial position and velocity of the body: a cyclic state and two ones without motion. The only possible state having mathematical similarities with our current case is the one when the pendulum points towards the same direction as the gravitational field. Both of the previously described profiles are commonly known as stationary or steady states, since their behavior remain invariant with respect to time after their corresponding evolutions reach them. When diffusion, transport and friction are present, Fokker-Planck type models arise in a natural manner as a proposal to describe these phenomena. In particular, these models exhibit desired equilibrium states which are physically relevant as the ones previously mentioned. A quite general Fokker-Planck equation gathering several scenarios of interest for the theory is given by

$$\text{(FP)} \quad \partial_t f + v(p) \cdot \nabla_x f = \nabla_p \cdot (D[\nabla_p f + \nabla_p E f]) , \quad t > 0,\ (x,p) \in \mathbb{R}^{2N},$$

where $f(t,x,p)$ can be though as a density of particles, $D = D(p)$ is the corresponding diffusion matrix, $E(p)$ stands for the energy of each single particle, or also known as *the microscopic energy* of the system, and $v(p)$ is the velocity field of the particles. Notice that equations (3.1) and (3.2) are particular cases of the previous equation. Also, it is worth mentioning that the particular form in which we introduced the previous model is more suitable for the convergence analysis that we will develop. Moreover, when considering the spatially homogeneous case, i.e., $f(t,x,p) = f(t,p)$, and the values of the diffusion matrix and the microscopic energy are $D = \mathrm{Id}$ and $E(p) = |p|^2/2$, equation (FP) becomes into the simplest model in the Fokker-Plank class. In general, one can directly verify that the functions 0 and $e^{-E}$ (up to a multiplicative constant, fixed by the total mass of the system) are equilibrium states of equation (FP). This implies that the non-trivial equilibrium state for equation (3.1), or equivalently for equation (FP) with $D \equiv \mathrm{Id}$ and $E(p) = \theta|p|^2/(2m)$, is the *Maxwellian distribution function*. Then, one might wonder if $f$ reaches either of these states when $t \to \infty$ in some sort of sense. In the case of an affirmative answer, it could be useful to know the rate of convergence. Observe that the equilibrium state $e^{-E}$ is a conserved quantity and solutions of equation (FP) preserve mass. Therefore, the $L^1$ convergence becomes a relevant and meaningful case to analyze even though it is the most difficult and delicate one. There are some criteria in the literature that can be applied to some of the previous models to ensure an exponential rate of convergence. In the spatially homogeneous case [13], Bakry and Emery exploit the geometry behind (FP) by transforming this equation into a diffusive operator in a Riemannian manifold. Then, they define an appropriate positive field operator through the infinitesimal generator of the diffusion process. They found that the problem can be reduced to study the bound from below of an extended version of the Ricci tensor in terms of the metric, which led to a geometric notion. This criterion is known as *the Bakry-Emery bound condition* and guarantees the convergence towards the non-trivial equilibrium state in the $L^1$ sense. In [10], the authors propose different relative entropies to prove the exponential convergence of solutions by deriving Bakry-Emery type conditions from elementary arguments. In addition, they obtain a family of convex Sobolev inequalities associated to these entropies. In [34], Carrillo and Toscani proved that exponential convergence in $L^1$ holds for the simplest Fokker-Planck equation. Their approach allows them to cover different Sobolev spaces. In this case, the fundamental solution of the equation is available and its properties are fully exploited to achieve their results. It is interesting to remark that the previous authors were able to adapt their arguments for the porous medium equation in [33], where they proved that solutions converge to a self-similar profile in $L^1$ with polynomial rate. In the case where $D = \mathrm{Id}$, $E(p) = |p|^2/2$ and $v = p$ in equation (FP), the exponential convergence problem has been solved using two different techniques. In [88], Hérau and Nier used spectral analysis tools for hypoelliptic operators and showed the existence of a spectral gap in the spectrum of the Fokker-Planck operator, which implies the stated convergence. In [139], Villani considered a modified entropy functional for equation (FP). He gave appropriate estimates to bound the derivative of this entropy combined with the validity of a logarithmic Sobolev inequality with respect to the invariant measure of the equation. See also [46]. In the case where solutions of equation (FP) are confined in a torus (i.e., $x \in \mathbb{T}^3$), Calogero was able to show in [22] the exponential trend in the $L^1$ norm to $e^{-E}$ as $t \to \infty$ under suitable conditions on $v$, $E$ and $D$. His arguments were inspired by similar ideas as the ones presented in [139], but those relied on the Riemannian structure of the problem. As a consequence, he also obtained a refined and systematic approach to formulate this exponential convergence problem from simple geometrical conditions.

Now, we describe our strategy to prove the exponential convergence towards the equilibrium in $L^1$ and $L^2$ for spatially homogeneous solutions of the relativistic Fokker-Planck equation. First, we transform the original equation into the equivalent one in a Riemannian manifold and prove all our statements within this formulation. For the $L^1$ case, we will follow an entropy argument combined with the Bakry-Emery bound condition. The main idea is to find an appropriate functional that acts as a Lyapunov function for the relativistic solutions and exponentially converges to the equilibrium as $t \to \infty$. This will be achieved by obtaining an identity for the time derivative of the entropy and by verifying that a logarithmic Sobolev inequality holds. The latter will be ensured by the Bakry-Emery condition for certain values of the parameter $\theta > 0$. The reason to proceed in this manner comes from two essential facts. On the one hand, the exponential rate of the entropy provides an exponential bound in time for the $L^1$ norm between a solution and the equilibrium. On the other hand, second order estimates in time of the entropy lead to the exponential rate for the time derivative and as a consequence, the rate for the entropy and the validity of the logarithmic Sobolev inequality are obtained. In other words, the Bakry-Emery condition is a simpler and sufficient second order condition to verify than performing the analysis described before, since we will only require to calculate and bound elementary expressions. For the $L^2$ case, our approach is based on spectral arguments. We will show that the first non-zero eigenvalue of the elliptic operator associated to the relativistic Fokker-Planck equation is positive. From this fact, we will obtain a Poincaré inequality and proceed similarly as in the $L^1$ case. The main difference is that the $L^2$ norm will now act as a Lyapunov function for relativistic solutions in this space. In general, finding this eigenvalue or equivalently, the *spectral gap* for an elliptic operator can be a hard task. As the problem states, one tries to show that the operator has a finite gap in its spectrum from 0 to the following part of it (possibly continuous). Probably the simplest example for elliptic of operators in dimension greater than one is given by the Laplacian $-\Delta$ under appropriate considerations, see for instance [65, 106]. Finding bounds for the spectral gap is also useful. In [37], Chen and Wang used the variational characterization of the problem for elliptic operators in $\mathbb{R}^d$ and gave estimates of the gap by probabilistic methods. In [141], Wang studies the existence and non-existence of the spectral gap for elliptic operators in a connected, non-compact Riemannian manifold from which he particularly treats the case in $\mathbb{R}^d$. The author found a useful lower bound for the existence of the gap in terms of the radial part of the operator, which reduces the original problem into the one where a basic integrability condition has to be verified. In fact, we are able to apply his result to our present situation.

In the next section, we recall the equivalent formulation of the relativistic Fokker-Planck equation in a Riemannian manifold performed in Chapter 2.

## 4.2 The Spatially Homogeneous case

In this chapter, we focus our analysis on spatially homogeneous solutions of the linear relativistic Fokker-Planck equation from the previous chapter. In this case, these solutions satisfy

$$\partial_t f(t,p) = \partial_{p^i}\left(D^{ij}\partial_{p^j} f(t,p) + \frac{\theta}{m}p^i f(t,p)\right), \tag{4.1}$$

with $(t,p) \in (0,\infty)\times\mathbb{R}^N$ and $f(t,p) \geq 0$. Recall that the diffusion matrix and the microscopic energy are defined by

$$D^{ij} = \frac{mc}{p^0}\left(\delta^{ij} + \frac{p^i p^j}{m^2c^2}\right), \quad p^0 = \sqrt{m^2c^2 + |p|^2}\,.$$

The reason to express the above equation in terms of the parameters relies on the fact that our result in $L^1$ depends on them. Also, the treatment of the convergence problem is simplified by the spatial homogeneity assumption. As a matter of fact, results from [22] should hold when $x \in \mathbb{T}^N$ since this has already been done for $N = 3$. In the previous reference, the author extends the approach that we will present. In particular, one of the required estimates to apply his results will be performed in the next section.

As previously seen in Chapter 2, equation (4.1) can be written in terms of the Jüttner distribution function $\mathscr{J} = e^{-\theta c p^0}$ by using the change of variable $f = \mathscr{J} h$. As we will see in the forthcoming sections, this formulation is more adequate to establish the trend to the equilibrium $\mathscr{J}$. In fact, the explicit form of equation (4.1) in terms of $h$ is

$$\partial_t h(t,p) = \partial_{p^i}\left(D^{ij}\partial_{p^j} h(t,p)\right) - \frac{\theta}{m} p \cdot \nabla_p h(t,p)\,. \tag{4.2}$$

Equivalently, the previous equation can also be interpreted as a Fokker-Plank operator with a Riemannian structure as

$$\partial_t h = \Delta_p^g h + W h, \tag{4.3}$$

where $\Delta_p^g$ is the Laplace-Beltrami operator with respect to the metric $g$ and $Wh = W^i \partial_{p^i} h$ represents a transport operator. The metric $g$ and the vector field $W$ are given by

$$g_{ij} = \frac{1}{mc}\left(p^0 \delta_{ij} - \frac{p_i p_j}{p^0}\right), \quad W^i = -\frac{1 + 2\theta c p^0}{2mcp^0} p^i\,.$$

It is important to remark that the right hand side of (4.3) is also referred in the literature as *the Witten Laplacian*, see [145]. We recall that the matrix $D^{ij}$ is the inverse of the metric $g$ with $\det g = |g| = p^0/mc$, i.e., $D^{kj} g_{jl} = \delta_l^k$. Also, note that

$$Wh = W^i \partial_{p^i} h = D^{ik} W_k \partial_{p^i} h = g(W, \partial_p h)\,,$$

with $W_k = g_{kl} W^l$. In fact, we are able to give a more accurate expression of this field as follows

$$\begin{aligned} g_{kl} W^l &= -\frac{1 + 2\theta c p^0}{2(mc)^2 p^0}\left(p^0 \delta_{kl} - \frac{p_k p_l}{p^0}\right) p^l = -\frac{1 + 2\theta c p^0}{2(p^0)^2} p_k\,, \\ &= -\frac{1}{2}\partial_{p^k} \log p^0 - \theta c \partial_{p^k} p^0 = \partial_{p^k} w\,, \end{aligned}$$

where $w = \log u$ and $u$ denotes the function

$$u = \frac{e^{-\theta c p^0}}{\sqrt{|g|}} = \sqrt{\frac{mc}{p^0}}\, e^{-\theta c p^0}\,. \tag{4.4}$$

Then $Wh = g(\partial_p h, \partial_p \log u) = \partial_p \log u\, h$. This formulation was already used and justified in Chapter 2. In order to recover all the results in terms of $f$, we require to define the probability measure $d\mu_\theta$ as

$$d\mu_\theta = Z^{-1} e^{-\theta c p^0} dp, \quad Z = \int_{\mathbb{R}^N} e^{-\theta c p^0} dp\,,$$

and consider all the involved spaces weighted with this measure. In addition, we assume that a solution of (4.3) is normalized with respect to the probability measure $\mu_\theta$:

$$\|h\|_{L^1(d\mu_\theta)} = \int_{\mathbb{R}^N} h\, d\mu_\theta = 1,$$

since this normalization can always be achieved by rescaling the solution.

## 4.3 The Bakry-Emery Condition

In this section, we show that the exponential trend to the equilibrium in $L^1$ for solutions of (4.1) holds for $\theta > \theta_0 > 0$, where the value of $\theta_0$ will be characterized in our main result. The strategy to achieve this purpose is using an equivalent notion of convergence in terms of equation (4.3) and the weighted space $L^1(d\mu_\theta)$. The latter will be given by an appropriate functional $\mathfrak{D}[h](t)$ that acts as a Lyapunov function for (4.3) and converges exponentially to zero. The key point in the argument comes from the fact that the first variation of $\mathfrak{D}$ will satisfy a Gronwall's inequality for $\theta > \theta_0$ that will be guaranteed by the Bakry-Emery condition.

Now, we proceed by presenting the entropy and the dissipation functionals which are defined by

$$\mathfrak{D}[h] = \int_{\mathbb{R}^N} h \log h\, d\mu_\theta, \quad \mathfrak{I}[h] = \int_{\mathbb{R}^N} g(\partial_p h, \partial_p \log h)\, d\mu_\theta \,.$$

Theses functionals are related by the identity

$$\frac{d}{dt}\mathfrak{D}[h](t) = \int_{\mathbb{R}^N} (\partial_t h + \partial_t h \log h)\, d\mu_\theta = -\mathfrak{I}[h](t)\,, \tag{4.5}$$

which justifies the Lyapunov character of $\mathfrak{D}$ since $\mathfrak{I} \geq 0$ due to $h > 0$ and $g(\partial_p h, \partial_p \log h) = h^{-1} g(\partial_p h, \partial_p h) \geq 0$. In order to verify this identity, we compute

$$\begin{aligned}\int_{\mathbb{R}^N} \partial_t h\, d\mu_\theta &= -Z^{-1}\int_{\mathbb{R}^N} D^{ij}\partial_{p^j} h \partial_{p^i}\left(e^{-\theta c p^0}\right) dp - \int_{\mathbb{R}^N} W h\, d\mu_\theta \\ &= \theta c \int_{\mathbb{R}^N} D^{ij}\frac{p_i}{p^0}\partial_{p^j} h\, d\mu_\theta - \frac{\theta}{m}\int_{\mathbb{R}^N} p^j \partial_{p^j} h\, d\mu_\theta \\ &= 0\,,\end{aligned}$$

where we used equation (4.2)[1] and the following properties

$$\partial_{p^l} e^{-\theta c p^0} = -e^{-\theta c p^0}\frac{\theta c}{p^0} p_l, \quad D^{ij} p_j = \frac{mc}{p^0}\left(\delta^{ij} + \frac{p^i p^j}{m^2 c^2}\right) p_j = \frac{p^0}{mc} p^i \,.$$

[1] Equivalently, we could use (4.3), but recall that the second order operator in (4.2) is not a LB operator. See the proof of Theorem 2.3.2

The identity for $\partial_t h$ was expected in agreement to the fact that solutions of (4.1) preserve mass. Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \partial_t h \log h \, d\mu_\theta &= -Z^{-1} \int_{\mathbb{R}^N} D^{ij} \partial_{p^j} h \partial_{p^i} \left( e^{-\theta c p^0} \log h \right) dp - \int_{\mathbb{R}^N} \log h W h \, d\mu_\theta \\ &= - \int_{\mathbb{R}^N} g(\partial_p h, \partial_p \log h) \, d\mu_\theta \\ &= -\mathfrak{I}[h](t) \, .\end{aligned}$$

In fact, it can be shown that the operator $L = \Delta_p^{(g)} + W$ is symmetric in $L^2(d\mu_\theta)$ from the above identity. For instance, see [22, Lemma 2] for the treatment in the general case of equation (FP).

At this point we are ready to introduce the notion of convergence that we will adopt. We say a solution of (4.3) converges to an equilibrium in the entropic sense if $\mathfrak{D}[h] \to \mathfrak{D}[1] = 0$ as $t \to \infty$. The rate of convergence is exponential if there exists $\lambda > 0$ such that $\mathfrak{D}[h] = O(e^{-\lambda t})$ as $t \to \infty$. A sufficient condition for exponential decay of the entropy is the validity of the following logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^N} h \log h \, d\mu_\theta \leq \alpha \int_{\mathbb{R}^N} g(\partial_p h, \partial_p \log h) \, d\mu_\theta \, , \tag{4.6}$$

for some $\alpha > 0$, and for all sufficiently smooth probability densities $h$ (not necessarily solutions of (4.3)). In order to show the sufficiency of the above inequality, we use (4.6) in (4.5) to obtain

$$\frac{d}{dt}\mathfrak{D}[h] \leq -\frac{1}{\alpha}\mathfrak{D}[h] \Rightarrow \mathfrak{D}[h] \lesssim \exp(-t/\alpha) \, .$$

The main reason to adopt this notion of convergence relies on the fact that the convergence in $L^1(d\mu_\theta)$ is achieved by using the Ciszár-Kullback inequality, see [39]. The previous inequality states that

$$\|h - 1\|_{L^1(d\mu_\theta)} \leq \sqrt{2\mathfrak{D}} \, ,$$

and as an immediate consequence, we obtain the convergence of $h$ to the equilibrium in $L^1(d\mu_\theta)$ with exponential factor $(2\alpha)^{-1}$, or equivalently, a solution of (4.1) satisfies

$$\|f(t) - \mathscr{J}_M\|_{L^1(dp)} \lesssim e^{-t/(2\alpha)}, \tag{4.7}$$

where $\mathscr{J}_M = \frac{M}{Z}\mathscr{J}$ denotes the Jüttner distribution function with mass $M$. Now it is clear that $\mathfrak{D}$ provides a natural convergence notion to an equilibrium for solutions of the relativistic Fokker-Planck equation (4.1) by (4.7). Thus, the problem of the exponential trend to the equilibrium in $L^1$ has been reduced to prove that (4.6) holds. Fortunately, the latter statement can be answered affirmatively, but only for certain values of $\theta$. Before we proceed, it is convenient to give a brief argument that explains why it is possible to prove inequality (4.6). In particular, the latter shows that the exponential convergence of $\mathfrak{D}$ implies (4.6) and also, it will allow to introduce a concept that is required in the proof of this Sobolev inequality.

First, observe that the functional $\mathfrak{D}$ is non-negative and decreasing with respect to $t$. Then, it is expected that this functional converges to its minimum value if $\mathfrak{I}[h] \to 0$ as $t \to \infty$. In order to prove this, we could compute its time derivative. We avoid to do so since the exact form of $\mathfrak{I}'$ will not be used in our forthcoming analysis. We are content with stating that if the next identity holds

$$\frac{d}{dt}\mathfrak{I}[h](t) \leq -\epsilon \mathfrak{I}[h](t)\,, \tag{4.8}$$

then in particular inequality (4.6) will follow. To prove this, we use that $\mathfrak{D}[h] \to 0$ as $t \to \infty$ and see that

$$\mathfrak{D}[h](0) = \int_0^\infty \mathfrak{I}[h](s)\,ds \leq \alpha \mathfrak{I}[h](0)\,,$$

which is precisely the assertion. The crucial part to obtain inequality (4.8) comes from an appropriate bound of the Bakry-Emery-Ricci tensor given by

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} - \nabla_p^2 \log u\,, \tag{4.9}$$

where $u$ is the function defined by (4.4) and Ric and $\nabla_p^2 \log u$ are the Ricci tensor and the Hessian with respect to the metric $g$, respectively. More precisely, if $\widetilde{\mathrm{Ric}}$ satisfies the Bakry- Emery curvature bound condition

$$\widetilde{\mathrm{Ric}} \geq \frac{1}{2\alpha} g\,, \tag{4.10}$$

then (4.8) holds. As a matter of fact, $\mathfrak{I}'$ is constituted by an integral only depending on $\widetilde{\mathrm{Ric}}$ and another non-positive integral term. Now we have that in particular (4.6) follows by our previous argument. See [13, 14] for instance. Condition (4.10) can be thought as a generalized notion of strong convexity with respect to the metric $g$.

Now, we are position to prove the main result of this section:

**Theorem 4.3.1.** *There exists a positive value $\theta_0$ such that the logarithmic Sobolev inequality (4.6) is valid for any $\theta > \theta_0$, with constant $\alpha$ given by*

$$\frac{1}{2\alpha} = \begin{cases} \mathscr{P}(mc) = \frac{2\theta mc^2 - 7}{2mc^2}, & \text{if } N = 3,\ \theta_0 < \theta \leq \frac{4}{mc^2}, \\ \mathscr{P}(\gamma(\theta, d))\,, & \text{if } N = 3,\ \theta > \frac{4}{mc^2},\ \text{or } N \neq 3,\ \theta > \theta_0, \end{cases}$$

*where*

$$\gamma(\theta,N)=\frac{mc}{\alpha_N}\left(2\theta mc^2+\sqrt{4\theta^2k^2c^2-3\alpha_N\beta_N}\right),$$

*for some constants* $\alpha_N$, $\beta_N$ *that only depend on the dimension, and* $\mathscr{P}$ *is the rational function*

$$\mathscr{P}(x)=\frac{2\theta cx^3-\alpha_N x^2+2\theta k^2cx-\beta_N k^2}{4mcx^3}.$$

*Proof.* The proof is carried out by using the Bakry-Emery curvature bound condition (4.10). First, we recall that the Ricci tensor of $g$ is given by

$$\mathrm{Ric}_{ij}=\partial_{p^k}\Gamma^k_{ij}-\partial_{p^j}\Gamma^k_{ik}+\Gamma^k_{ij}\Gamma^l_{kl}-\Gamma^l_{ik}\Gamma^k_{jl}\,, \tag{4.11}$$

where $\Gamma^l_{ij}$ are the Christoffel symbols of the second kind associated to the metric $g$. These symbols are obtained from the following relation

$$\Gamma^l_{ij}=\frac{D^{lk}}{2}\left(\partial_{p^j}g_{ki}+\partial_{p^i}g_{kj}-\partial_{p^k}g_{ij}\right).$$

For convenience, we recall that $p^0=\sqrt{m^2c^2+|p|^2}$ and

$$g_{ij}=\frac{1}{mc}\left(p^0\delta_{ij}-\frac{p_ip_j}{p^0}\right).$$

In order to obtain $\Gamma^l_{ij}$, we compute

$$\begin{aligned}\partial_{p^l}g_{rs}=\partial_{p^l}\left[\frac{p^0}{mc}\left(\delta_{rs}-\frac{p_rp_s}{(p^0)^2}\right)\right]&=\frac{p_lg_{rs}}{(p^0)^2}+\frac{2p_lp_rp_s}{mc(p^0)^3}-\frac{\delta_{ls}p_r+\delta_{lr}p_s}{mcp^0}\\ &=a_{lrs}+b_{lrs}+c_{lrs}\,.\end{aligned} \tag{4.12}$$

Observe that the terms $b_{lrs}$ in $\Gamma^l_{ij}$ are equal and the sum of the terms $c_{lrs}$ contribute

$$-\frac{\delta_{jk}p_i+\delta_{ji}p_k}{mcp^0}-\frac{\delta_{ik}p_j+\delta_{ij}p_k}{mcp^0}+\frac{\delta_{ki}p_j+\delta_{kj}p_i}{mcp^0}=-\frac{2\delta_{ij}p_k}{mcp^0}.$$

Then, we see that

$$\begin{aligned}\Gamma^l_{ij}&=\frac{D^{lk}}{2}\left(\frac{p_jg_{ik}+p_ig_{kj}}{(p^0)^2}-\frac{p_kg_{ij}}{(p^0)^2}+\frac{2p_kp_ip_j}{mc(p^0)^3}-\frac{2p_k\delta_{ij}}{mcp^0}\right)\\ &=\frac{1}{2}\left[\frac{p_j\delta^l_i+p_i\delta^l_j}{(p^0)^2}-D^{lk}\frac{p_kg_{ij}}{(p^0)^2}-2D^{lk}\frac{p_k}{mc(p^0)^2}\left(p^0\delta_{ij}-\frac{p_ip_j}{p^0}\right)\right]\\ &=\frac{1}{2}\left(\frac{p_j\delta^l_i+p_i\delta^l_j}{(p^0)^2}-\frac{3p^l}{mcp^0}g_{ij}\right)=\frac{p_j\delta^l_i+p_i\delta^l_j}{2(p^0)^2}-\frac{3p^l}{2m^2c^2}\left(\delta_{ij}-\frac{p_ip_j}{(p^0)^2}\right),\end{aligned}$$

where we used the definition of $g$, the inverse property of the matrix $D$ with respect to $g$ and the identity $D^{lk}p_k = p^l p^0/mc$. Now, we are ready to perform the necessary calculations to obtain the Ricci tensor. The first term in the definition of Ric (4.11) is

$$\begin{aligned} 2\partial_{p^k}\Gamma^k_{ij} &= \partial_{p^k}\left(\frac{p_j\delta^k_i + p_i\delta^k_j}{(p^0)^2} - \frac{3p^k}{mcp^0}g_{ij}\right) \\ &= \frac{2\delta_{ij}}{(p^0)^2} - \frac{2(p_j\delta^k_i + p_i\delta^k_j)p_k}{(p^0)^4} - \partial_{p^k}\left(\frac{3p^k}{mcp^0}g_{ij}\right) \\ &= \frac{2mc}{(p^0)^3}g_{ij} - \frac{2p_ip_j}{(p^0)^4} - \partial_{p^k}\left(\frac{3p^k}{mcp^0}\right)g_{ij} - \frac{3p^k}{mcp^0}\partial_{p^k}\left(g_{ij}\right), \end{aligned}$$

where the definition of the metric $g$ was used again. The last two terms in the above identity can be estimated by using (4.12) as follows

$$\begin{aligned} \partial_{p^k}\left(\frac{3p^k}{mcp^0}g_{ij}\right) &= \frac{3d}{mcp^0}g_{ij} + \frac{3p^k}{mcp^0}\left[\frac{2p_kp_ip_j}{mc(p^0)^3} - \frac{\delta_{ik}p_j + \delta_{jk}p_i}{mcp^0}\right] \\ &= \frac{3d}{mcp^0}g_{ij} - \frac{6p_ip_j}{(p^0)^4}. \end{aligned}$$

Notice that two terms cancel out while estimating the above quantity. The latter implies that

$$2\partial_{p^k}\Gamma^k_{ij} = \frac{2mc}{(p^0)^3}g_{ij} + \frac{4p_ip_j}{(p^0)^4} - \frac{3N}{mcp^0}g_{ij}.$$

Similarly as above, we have that

$$\begin{aligned} 2\partial_{p^j}\Gamma^k_{ik} &= \partial_{p^j}\left[\frac{p_k\delta^k_i + p_i\delta^k_k}{(p^0)^2} - \frac{3p^k}{m^2c^2}\left(\delta_{ik} - \frac{p_ip_k}{(p^0)^2}\right)\right] = (N-2)\partial_{p^j}\left(\frac{p_i}{(p^0)^2}\right) \\ &= (N-2)\left(\frac{\delta_{ij}}{(p^0)^2} - \frac{2p_ip_j}{(p^0)^4}\right) = (N-2)\frac{mc}{(p^0)^3}g_{ij} - (N-2)\frac{p_ip_j}{(p^0)^4}, \end{aligned}$$

and

$$2\partial_{p^k}\Gamma^k_{ij} - 2\partial_{p^j}\Gamma^k_{ik} = (N+2)\frac{p_ip_j}{(p^0)^4} - \frac{3N}{mcp^0}g_{ij} - (N-4)\frac{mc}{(p^0)^3}g_{ij}. \tag{4.13}$$

Now, the remaining terms in (4.11) are obtained as follows

$$\begin{aligned} 4\Gamma^k_{ij}\Gamma^l_{kl} &= \left(\frac{p_j\delta^k_i + p_i\delta^k_j}{(p^0)^2} - \frac{3p^k}{mcp^0}g_{ij}\right)(N-2)\frac{p_k}{(p^0)^2} \\ &= (N-2)\left(\frac{2p_ip_j}{(p^0)^4} - \frac{3|p|^2}{mc(p^0)^3}g_{ij}\right), \end{aligned}$$

and

$$4\Gamma^l_{ik}\Gamma^k_{jl} = \left(\frac{p_k\delta^l_i + p_i\delta^l_k}{(p^0)^2} - \frac{3p^l}{mcp^0}g_{ik}\right)\left(\frac{p_l\delta^k_j + p_j\delta^k_l}{(p^0)^2} - \frac{3p^k}{mcp^0}g_{jl}\right)$$
$$= (N+12)\frac{p_ip_j}{(p^0)^4} - 6\frac{|p|^2\delta^l_i + p_ip^l}{mc(p^0)^3}g_{jl} = (N+6)\frac{p_ip_j}{(p^0)^4} - \frac{6|p|^2}{mc(p^0)^3}g_{ij}\,.$$

The previous quantities lead us to

$$4\Gamma^k_{ij}\Gamma^l_{kl} - 4\Gamma^l_{ik}\Gamma^k_{jl} = (N-10)\frac{p_ip_j}{(p^0)^4} - \frac{3(N-4)|p|^2}{mc(p^0)^3}g_{ij}\,.$$

Therefore, we can combine the identity given above with (4.13) to see that the Ricci tensor (4.11) with respect to $g$ is

$$\mathrm{Ric}_{ij} = 3(N-2)\frac{p_ip_j}{4(p^0)^4} - \frac{2(N-4)m^2c^2 + 6N + 3(N-4)|p|^2}{4mcp^0}g_{ij}$$
$$= 3(N-2)\frac{\delta_{ij}}{4(p^0)^2} - \frac{2(N-1)m^2c^2 + 3(3N-4)(p^0)^2}{4mc(p^0)^3}g_{ij}\,. \tag{4.14}$$

In order to calculate the hessian of $\log u$, we recall that

$$\partial_{p^k} w = \partial_{p^j}\log u = W_j = -p_j\frac{1+2\theta cp^0}{2(p^0)^2}\,,$$

and compute the second order derivatives of the above expression as follows

$$\partial_{p^i}\partial_{p^j}\log u = -\partial_{p^i}\left(p_j\frac{1+2\theta cp^0}{2(p^0)^2}\right)$$
$$= -\delta_{ij}\left(\frac{1+2\theta cp^0}{2(p^0)^2}\right) - \frac{\theta cp_ip_j}{(p^0)^3} + \frac{2p_ip_j}{(p^0)^2}\left(\frac{1+2\theta cp^0}{2(p^0)^2}\right)$$
$$= \frac{p_ip_j}{2(p^0)^4} - \frac{2m^2c^2}{2mcp^0}g_{ij}\left(\frac{1+2\theta cp^0}{2(p^0)^2}\right)\,,$$

where the definitions of the metric $g$ and $p^0$ were used. Similarly as above, we see that the remaining term is obtained as follows

$$\Gamma^k_{ij}\partial_{p^k}\log u = \frac{1}{2}\left(\frac{p_j\delta^k_i + p_i\delta^k_j}{(p^0)^2} - \frac{3p^k}{mcp^0}g_{ij}\right)W_k$$
$$= -\frac{1}{2}\left(\frac{2p_ip_j}{(p^0)^2} - \frac{3|p|^2}{mcp^0}g_{ij}\right)\left(\frac{1+2\theta cp^0}{2(p^0)^2}\right)$$
$$= -\frac{1}{2}\left(2\delta_{ij} - \frac{3(p^0)^2 - m^2c^2}{mcp^0}g_{ij}\right)\left(\frac{1+2\theta cp^0}{2(p^0)^2}\right)\,.$$

Then, the hessian of $\log u$ in local coordinates is

$$(\nabla_p^2 w)_{ij} = \partial_{p^i}\partial_{p^j} \log u - \Gamma_{ij}^k \partial_{p^k} \log u$$
$$= (1+\theta c p^0)\frac{\delta_{ij}}{(p^0)^2} - \frac{mc\, g_{ij}}{2(p^0)^3} - \left(\frac{3(p^0)^2+m^2c^2}{2mcp^0}\right)\left(\frac{1+2\theta cp^0}{2(p^0)^2}\right) g_{ij}$$
$$= (1+\theta c p^0)\frac{\delta_{ij}}{(p^0)^2} - \frac{3m^2c^2+3(p^0)^2+2\theta cp^0(m^2c^2+3(p^0)^2)}{4mc(p^0)^3} g_{ij}\,,$$

where we used the definition of $g$ and the following identity

$$\frac{p_i p_j}{2(p^0)^4} + \delta_{ij}\left(\frac{1+2\theta cp^0}{2(p^0)^2}\right) = \delta_{ij}\left(\frac{1+\theta cp^0}{(p^0)^2}\right) - \frac{\delta_{ij}}{2(p^0)^2} + \frac{p_ip_j}{2(p^0)^4}.$$

Finally, using definition (4.9) and the explicit expression of $(\nabla_p^2 w)_{ij}$ that was obtained before combined with (4.14), we find that

$$\widetilde{\mathrm{Ric}}_{ij} = \mathrm{Ric}_{ij} - (\nabla_p^2 w)_{ij}$$
$$= \frac{6\theta c(p^0)^3 - \alpha(p^0)^2 + 2\theta m^2c^3p^0 - \beta m^2c^2}{4mc(p^0)^3} g_{ij} + \frac{a_N - 4c\theta p^0}{4(p^0)^2}\delta_{ij},$$

where $\alpha = 3(3N-5)$, $\beta = 2N-5$ and $a_N = 3(N-3)-1$ for $N \geq 2$, and $\alpha = -3$, $\beta = -3$, $a_1 = -4$ and $g \equiv mc/p^0$ for $N = 1$, since $\mathrm{Ric} \equiv 0$.

In order to bound $\widetilde{\mathrm{Ric}}_{ij}$ from below in terms of the metric $g$, we require the following estimate

$$\frac{p^0}{mc}|X|^2 \geq g(X,X) = \frac{1}{mcp^0}\left[(mc|X|)^2 + (|p||X|)^2 - (p\cdot X)^2\right] \geq \frac{mc}{p^0}|X|^2,$$

which is valid for all $X \in \mathbb{R}^N$. For $N \leq 3$, we use the right hand side of the above inequality to bound the term containing $a_N$ in $\widetilde{\mathrm{Ric}}$ since $a_N$ is negative. The upper bound of $g$ is used to absorb the corresponding positive term $a_N + 1$ for $N \geq 4$. This is possible by noticing that $\beta = \beta_N + a_N + 1$. The previous information enables to obtain the following lower bound

$$\widetilde{\mathrm{Ric}}(X,X) \geq \left[2\theta c(p^0)^3 - \alpha_N(p^0)^2 + 2\theta m^2c^3p^0 - \beta_N m^2c^2\right]\frac{g(X,X)}{4mc(p^0)^3}$$
$$= \mathscr{P}(p^0) g(X,X)\,,$$

where

$$(\alpha_N, \beta_N) = \begin{cases} (1\,,\,-3) & \text{if } N=1\,,\\ (6N-5\,,\,2N-5) & \text{if } N=2,3\,,\\ (9N-14\,,\,4-N) & \text{if } N\geq 4\,. \end{cases} \tag{4.15}$$

Now, we proceed to prove that $\min\{\mathscr{P}(p^0)\,|\,p^0 \geq mc\}$ is strictly positive if and only if $\theta > \theta_0$ which follows by a standard procedure. First, notice that the positivity of $\mathscr{P}$ holds if the polynomial $Q$ defined by

$$Q(x,\theta) = 2\theta c x^3 - \alpha_N x^2 + 2\theta k^2 c x - \beta_N k^2\,,$$

remains positive for $\theta > \theta_0$ and $x \geq k$, with $k = mc$. We remark that if $\theta \geq \theta_1$, then

$$Q(x,\theta) - Q(x,\theta_1) = 2cx(x^2+k^2)(\theta-\theta_1) \geq 0\,,$$

and as consequence, the value of $\theta_0$ is unique and the minimum possible one in which $Q(x,\theta_0) \geq 0$ holds for all $x \geq k$. In order to show the latter property, we will use the critical points of the derivative of $Q$

$$\frac{d}{dx}Q(x,\theta) = 6\theta c x^2 - 2\alpha_N x + 2\theta k^2 c\,, \quad R_{\mp} = \frac{\alpha_N \mp \sqrt{\alpha_N^2 - 12(\theta c k)^2}}{6\theta c}.$$

Also, the sign of $Q(0,\theta) = -\beta_N k^2$ is required and the fact that there exists at least one real root $x_1$ is relevant (the degree of $Q(x,\theta)$ is three). From here, the analysis is divided in two cases.

Let $N \neq 3$. Notice that $Q(0,\theta) = -\beta_N k^2 \geq 0$ by (4.15), which implies that $x_1 \leq 0$. When $\theta \geq \frac{\alpha_N}{\sqrt{12ck}}$, $x_1$ is the only real root of $Q$ since $R_{\mp}$ are complex or equal. This implies the positivity of $Q$ from the increasing property with respect to $\theta$. Therefore, the value of $\theta_0$ is given by either $k > R_+(\theta_0)$ or $Q(R_+,\theta_0) = 0$. In the latter case, $R_+$ must have multiplicity two from its global minimum property for $x > 0$. For the case $N = 1$, it is easy to show that there exists $\theta_0 > 0$ such that $Q(R_+,\theta_0) = 0$ and $k < R_+(\theta_0)$. This is justified by the fact that $k$ can not be a root of $Q$, since $Q(k,\theta) = k^2(4\theta ck+2) > 0$ for all $\theta > 0$, and if $k \geq R_+$, the value of $Q(R_+,\theta_0)$ will remain strictly positive due to the monotonicity of $Q$ with respect to $\theta$. Therefore, $\theta_0$ is given by $Q(R_+,\theta_0) = 0$ with $\theta_0 < \frac{\alpha_1}{4ck}$. The existence is a consequence of the intermediate value theorem. Now, for $N \neq 1,3$, notice that $k$ can be a root of $Q$ since

$$Q(k,\theta) = k^2(4\theta ck - \alpha_N - \beta_N) = 0 \quad \Leftrightarrow \quad \theta = \frac{\gamma_N}{2ck} = \frac{4N-5}{2ck}.$$

Next, we see that the condition $k \geq R_+(\theta)$ leads to

$$(6\theta ck - \alpha_N)^2 - \alpha_N^2 + 12(\theta ck)^2 = (6\theta ck)^2 + 12(\theta ck)^2 - 12\alpha_N\theta ck \geq 0\,,$$

which implies $\theta \geq \frac{\alpha_N}{4ck}$. We state that the choice $\theta_* = \frac{\gamma_N}{2ck}$ and $k \geq R_+$ can not hold simultaneously unless $N = 4$, i.e., $k$ can not be the biggest root of $Q$, nor one with multiplicity two. This assertion comes from the following

$$\frac{\alpha_N}{4ck} \geq \frac{\alpha_N + \beta_N}{4ck} = \frac{\gamma_N}{2ck},$$

since in this situation $\beta_N \le 0$ by (4.15). Then, we also find in this case that the value of $\theta_0$ is given by $Q(R_+,\theta_0)=0$ with $\frac{\gamma_N}{2ck} < \theta_0 \le \frac{\alpha_N}{4ck}$.

For $N=3$, we have the reversed inequality for the condition $k \ge R_+(\theta)$, since $\frac{\alpha_3}{4ck} \le \frac{\alpha_3+\beta_3}{4ck}$ and $\beta_3 = 1$. Using the fact the value $R_+(\theta)$ corresponds to the global minimum of $Q$ for $x \ge R_+(\theta)$, the latter implies that $Q$ is increasing in the same interval. Now, it is straightforward to show that $Q(k, \frac{\alpha_3}{4ck}) < 0$ and due to the monotonicity of $Q$ with respect to $\theta$, $\theta_0 = \frac{\gamma_3}{2ck}$ must be selected. In fact, $Q(x,\theta_0) = \frac{7}{k}(x-k)(x-x_1)(x-x_2)$, with $x_{1,2} = \frac{k}{7}(3 \pm \sqrt{2}) < k$. We remark that this is the only possible case in which two different intervals of the parameter $\theta$ give two different global minimums of $\mathscr{P}$ for $x \ge k$. In particular, $\mathscr{P}(k)$ is the corresponding minimum for $\theta_0 < \theta \le \frac{4}{ck}$. Therefore, we can conclude that $\mathscr{P}(x) > 0$ and $\min\{\mathscr{P}(x)) \,|\, x > k\} > 0$, for all $\theta > \theta_0$ and $N \ge 1$.

In order to obtain the value of $(2\alpha)^{-1}$, we have to calculate the global minimum of $\mathscr{P}$ in $[mc,\infty)$. From the previous step, we already know that this minimum corresponds to one of the critical points of $\mathscr{P}$. Then, we differentiate $\mathscr{P}$ to see that

$$\begin{aligned}
\frac{d}{dx}\mathscr{P}(x) &= \frac{\left(6\theta c x^2 - 2\alpha_N x + 2\theta k^2 c\right)x - 3Q(x)}{4kx^4} \\
&= \frac{\alpha_N x^2 - 4\theta k^2 c x + 3\beta_N k^2}{4kx^4} \\
&= \frac{\alpha_N (x-\gamma_-)(x-\gamma_+)}{4kx^4}, \qquad \gamma_\mp = \frac{k}{\alpha_N}\left(2\theta k c \mp \sqrt{4\theta^2 k^2 c^2 - 3\alpha_N \beta_N}\right).
\end{aligned}$$

Notice that $\gamma_- \le 0 < \gamma_+$ for $N \ne 3$, and $\gamma_\mp > 0$ for $N=3$. Then, the minimum is achieved at $\mathscr{P}(\gamma_+)$ for all the cases, except when $N=3$ and $\frac{7}{2ck} = \theta_0 < \theta \le \frac{4}{kc}$. In this case, $\mathscr{P}(x)$ attains its minimum at $k$. This follows from the increasing property of the function $g_1(\theta) = k^2 - \gamma_-^2(\theta)$ with respect to $\theta$ and the decreasing one from $g_2(\theta) = k^2 - \gamma_+^2(\theta)$. In fact

$$\begin{aligned}
g_2' &= -2\gamma_+\gamma_+' = -\gamma_-\gamma_+' \frac{6k^2\beta_3}{\alpha_3\gamma_-^2} \\
&= -\frac{6k^4\beta_3}{(\alpha_3)^3\gamma_-^2}\left(2kc + \frac{4\theta k^2 c^2}{\sqrt{4\theta^2 k^2 c^2 - 3\alpha_3\beta_3}}\right)\left(2\theta k c - \sqrt{4\theta^2 k^2 c^2 - 3\alpha_3\beta_3}\right) \\
&= -\frac{ck^5(6\beta_3)^2}{(\alpha_3)^2\gamma_-^2\sqrt{4\theta^2 k^2 c^2 - 3\alpha_3\beta_3}} < 0.
\end{aligned}$$

Now, it is straightforward to show that $g_2(\theta_0) > 0$, since $7+\sqrt{10} < 13$ and $\gamma_+ = k$ for $\theta = \frac{4}{kc}$. A similar computation proves that $g_1' > 0$. Since $g_1(\theta_0) > 0$, $\frac{d}{dx}\mathscr{P}(x)$ only changes sign when $g_2$ does. This justifies that $\mathscr{P}(k)$ is a minimum for $\theta_0 < \theta \le \frac{4}{kc}$. It is important to mention that the other case where an explicit value of the parameter can be obtained is $N = 4$ with $\theta_0 = \frac{11}{2kc}$ since $\beta_4 = 0$ and zero is a root for $\mathscr{P}$. □

We conclude this section with some final remarks. Theorem 4.3.1 ensures an exponential rate of convergence towards the equilibrium for small temperatures of the thermal bath since $\theta \sim T^{-1}$ and $\theta > \theta_0$. Although there are several criteria in the literature for the validity of logarithmic Sobolev inequalities, we were unable to find one that applies for the case independent of $\theta > 0$. In order to prove exponential decay in $L^1$ for all temperatures, one might need to improve the Bakry-Emery curvature bound condition or use a different strategy. For instance in [142, 143], the author considers the possibility of having negative bounds for (4.9). This suggests that a more detailed analysis on $\nabla_p^2 w$ might be fruitful. In [10], the authors study this convergence problem for several circumstances including perturbed and non-symmetric Fokker-Planck operators which might also lead to obtain a logarithmic Sobolev for the original operator. We mentioned all these facts because it is reasonable to believe that the exponential convergence for all possible values of $\theta > 0$ might hold. The results presented in this section are a good starting point.

## 4.4 Exponential convergence in $L^2$

In this section we show that the exponential convergence towards the equilibrium holds without any restrictions in the possible values of the parameter $\theta > 0$ if the $L^1$ framework is abandoned. To achieve the latter, we consider the functional

$$\mathfrak{L}[h] = \int_{\mathbb{R}^N} h^2 \, d\mu_\theta = \|h\|^2_{L^2(d\mu_\theta)},$$

which will act as our new Lyapunov function and the weighted $L^2(d\mu_\theta)$ space is our new framework. We proceed to compute the time derivative of $\mathfrak{L}[h-1]$ in order to verify that solutions of (4.3) are decreasing along this functional. Using (4.3) and integrating by parts, we obtain the following

$$\begin{aligned}\frac{d}{dt}\mathfrak{L}[h-1](t) &= 2\int_{\mathbb{R}^N} (h-1)\partial_t h \, d\mu_\theta = -2\int_{\mathbb{R}^N} g(\partial_p h, \partial_p h)\, d\mu_\theta \\ &\quad - 2\int_{\mathbb{R}^N} \left[D^{ij}\partial_{p^j} h \, \partial_{p^i}\left(e^{-\theta c p^0}\right) e^{\theta c p^0} + Wh\right](h-1) d\mu_\theta \\ &= -2\int_{\mathbb{R}^N} g(\partial_p h, \partial_p h)\, d\mu_\theta \, .\end{aligned}$$

To show the exponential decay rate of $\mathfrak{L}[h-1]$ to the equilibrium, it is sufficient to prove that the following Poincaré inequality

$$\int_{\mathbb{R}^N} (h-1)^2 d\mu_\theta \leq \lambda \int_{\mathbb{R}^N} g(\partial_p h, \partial_p h)\, d\mu_\theta, \quad \text{for some } \lambda > 0\,, \tag{4.16}$$

holds for all sufficiently smooth probability densities $h$. The validity of the Poincaré inequality (4.16) is equivalent to prove the existence of a spectral gap for the operator defined by the right hand side of equation (4.3). More precisely, the spectral gap of this operator is characterized by

$$\lambda_1 = \inf\left\{ \frac{\int_{\mathbb{R}^N} g(\partial_p f, \partial_p f)\, d\mu_\theta}{\|f\|^2_{L^2(d\mu_\theta)} - \|f\|^2_{L^1(d\mu_\theta)}} \;:\; f \in C^1 \cap L^2(d\mu_\theta),\; f \neq \text{constant} \right\},$$

and it is said to exist if $\lambda_1 > 0$. The latter identity is an extension of Raleigh's formula for symmetric elliptic operators in bounded domains with Dirichlet boundary conditions, see [65]. In our present situation, recall that $h > 0$ and $\|h\|_{L^1(d\mu_\theta)} = 1$. These conditions imply

$$\|h\|^2_{L^2(d\mu_\theta)} - \|h\|^2_{L^1(d\mu_\theta)} = \|h-1\|^2_{L^2(d\mu_\theta)},$$

and inequality (4.16) would follow if $\lambda_1 > 0$.

Instead of analyzing the intrinsic variational problem to establish the existence of this spectral gap, we will achieve the latter by applying a criterion due to Wang, see [141]. To do so, we need to consider the operator

$$Lh = a^{ij}\partial_{p^i}\partial_{p^j} h + b^j \partial_{p^j} h\,, \quad p \in \mathbb{R}^N\,, \tag{4.17}$$

and define the following functions

$$\gamma(r) = \sup_{|p|=r} \frac{r[\mathrm{Tr}(a(p)) + p\cdot b(p)]}{a^{ij}p_ip_j} - \frac{1}{r}, \quad \alpha(r) = \inf_{|p|=r} \frac{a^{ij}p_ip_j}{r^2},$$
$$C(r) = \int_1^r \gamma(s)ds, \quad \text{for } r > 0.$$

Then by [141, Th.3.1], the spectral gap for the operator (4.17) is strictly positive provided that there exists a positive function $y \in C([1,\infty))$ such that

$$\sup_{t\geq 1} G_y(t) = \sup_{t\geq 1}\left\{ \frac{1}{y(t)} \int_1^t e^{-C(r)} \int_r^\infty e^{C(s)} \frac{y(s)}{\alpha(s)}\, ds\, dr \right\} < \infty.$$

Before proceeding, it is insightful to briefly recall the idea to prove this criterion. We avoid to give the complete proof of this result because it is quite technical and none of the methods are used in the present work. In [141], the author shows how to bound $\lambda_1$ from below in terms of the smallest eigenvalue of the Neumann problem for $-L$ in $B_R(0)$, the ball of radius $R > 0$ centered at 0, and

$$\lambda^c(r) = \inf\left\{ \int_{\mathbb{R}^N} g(\partial_p f, \partial_p f)\, d\mu_\theta \;:\; \|f\|^2_{L^2(d\mu_\theta)} = 1\, f \in C^1,\; f = 0 \text{ in } B_r(0) \right\}.$$

Then, the author obtains a characterization of the essential spectrum of $L$ in terms of $\lim_{r\to\infty} \lambda^c(r) > 0$, which makes the estimate $\lambda^c(r) > 0$ relevant for the existence of the spectral gap. Also, we can notice from the conditions on $\alpha$, $\gamma$ and $C$ that it is enough to consider radial functions for the estimate on $\lambda_1$. In fact, the function defined by

$$g(|x|) = \int_1^{|x|} e^{-C(r)} \int_r^\infty e^{C(s)} \frac{y(s)}{\alpha(s)}\, ds\, dr,$$

satisfies

$$Lg \le -y(|x|), \quad \text{for } |x| \ge 1, \text{ if } \int_1^\infty e^{C(s)} \frac{y(s)}{\alpha(s)}\, ds < \infty,$$

and this condition will ensure $\lambda^c(1) > 0$.

Finally, we are able to state and prove the following result:

**Theorem 4.4.1.** *The Poincaré inequality* (4.16) *holds for all* $\theta > 0$.

*Proof.* In the particular case of equation (4.3), the corresponding coefficients of the operator (4.17) read as

$$a^{ij} = D^{ij} = \frac{mc}{p^0}\left(\delta^{ij} + \frac{p^i p^j}{m^2c^2}\right), \quad b^j = \frac{Np^j}{mcp^0} - \frac{\theta}{m}p^j,$$

with $p^0(|p|) = \sqrt{m^2c^2 + |p|^2}$. Here, we used the equivalent form of (4.3), namely, equation (4.2). Now, by using the above values we compute the quantities

$$\begin{aligned}\mathrm{Tr}(a(p)) + p\cdot b(p) &= \frac{Np^0}{mc} + \frac{|p|^2}{mcp^0} - \frac{\theta|p|^2}{m}, \\ \frac{a^{ij}p_ip_j}{r^2} &= \frac{mc}{r^2p^0}\left(|p|^2 + \frac{|p|^4}{m^2c^2}\right) = \frac{|p|^2p^0}{r^2mc},\end{aligned}$$

which allow to explicitly obtain for $r > 0$

$$\gamma(r) = \frac{N-1}{r} + \frac{r}{(p^0)^2} - \frac{c\theta r}{p^0}, \quad \alpha(r) = \frac{p^0(r)}{mc}.$$

Next, we observe that

$$\gamma(r) = \frac{d}{dr}\left[(N-1)\log r + \log p^0 - c\theta p^0\right],$$

and as a consequence,

$$C(r) = \int_1^r \gamma(s)ds = (N-1)\log r + \log p^0(r) - c\theta p^0(r) + C,$$

with $C$ representing an integration constant. Then,

$$e^{-C(r)} = \frac{e^{cp^0(r)-C}}{r^{N-1}p^0(r)}, \quad \text{and} \quad \frac{e^{C(s)}}{\alpha(s)} = mc\, s^{N-1}e^{-cp^0(s)+C}.$$

Therefore, the associate function $G_y(t)$ for the operator (4.17) in the current situation is given by

$$G_y(t) = \frac{mc}{y(t)}\int_1^t \frac{e^{c\theta p^0(r)}}{r^{N-1}p^0(r)}\int_r^\infty e^{-c\theta p^0(s)}s^{N-1}y(s)\,ds\,dr.$$

Now, we choose $y(t) = \frac{e^{\beta t}}{t^{N-1}}$ with $\beta < c\theta$. Since $p^0(s) = \sqrt{m^2c^2+s^2} \geq s$, notice that

$$\beta s - c\theta p^0(s) \leq (\beta - c\theta)s,$$

and for $r \geq 1$, the following inequality holds

$$p^0(r) - r = \frac{m^2c^2}{p^0(r)+r} \leq \frac{m^2c^2}{p^0(1)+1} = \sqrt{m^2c^2+1} - 1.$$

The above facts allow to bound $G_y$ as follows

$$\begin{aligned} G_y(t) &\leq \frac{mc}{y(t)}\int_1^t \frac{e^{c\theta p^0(r)}}{r^N}\int_r^\infty e^{(\beta-c\theta)s}\,ds\,dr \\ &= \frac{mc}{(c\theta-\beta)y(t)}\int_1^t e^{c\theta[p^0(r)-r]}\frac{e^{\beta r}}{r^N}\,dr \\ &\leq \frac{mc}{c\theta-\beta}e^{c\theta(\sqrt{m^2c^2+1}-1)}\frac{t^{N-1}}{e^{\beta t}}\int_1^t \frac{e^{\beta r}}{r^N}dr. \end{aligned}$$

Now, using L'Hôpital's rule and the fundamental theorem of calculus, we see that

$$\lim_{t\to\infty}\frac{\int_1^t \frac{e^{\beta r}}{r^d}dr}{\frac{e^{\beta t}}{t^{N-1}}} = \lim_{t\to\infty}\frac{\frac{e^{\beta t}}{t^N}}{\beta\frac{e^{\beta t}}{t^{N-1}} - (N-1)\frac{e^{\beta t}}{t^N}} = \lim_{t\to\infty}\frac{1}{\beta t - N + 1} = 0\,,$$

which guarantees that $\sup_{t\geq 1} G_y(t) < \infty$. Then, we are in position to apply the result by Wang and conclude that the spectral gap for (4.17) is positive.

□

Finally, we remark that the Poincaré inequality (4.16) can also be proven by showing the existence of a spectral gap for elliptic operators, but using a different criterion which can be found in [15]. This result was established by Angst in [8].

# Chapter 5

# The Vlasov-Nordström-Fokker-Planck System

In the present chapter, we consider the existence and uniqueness problem of solutions for the Vlasov-Nordström-Fokker-Planck (VNFP) system. Due to the high technical difficulties exhibited by the system, our results are established in the spatially homogeneous regime. Additionally, we study the asymptotic behavior of the system and prove that solutions possess a non-trivial profile, even in the absence of friction. Finally, we introduce *the ultra-relativistic* Fokker-Planck equation associated to the relativistic model. The reason to do this is justified by the fact that the admissible future attractors for the particle density in the VNFP system might be given by this model. In fact, we derive an explicit representation formula for solutions of this ultra-relativistic equation which enables to identify the candidate for the possible asymptotic profile of the density function for the VNFP system.

## 5.1 Introduction

The Vlasov-Nordström-Fokker-Planck system describes the evolution of the self-gravitating matter experiencing collisions with a fixed background of particles in the framework of a relativistic scalar theory of gravitation. One of the main motivations to consider this system is to obtain a consistent approach to model diffusion dynamics of particle systems when relativistic effects are present. There is already a proposal in the context of General Relativity [20], but due to the well-known complexity of the Einstein field equations, it seems wiser to face a simpler situation as a first step. The VNFP system has the advantage to capture some of the essential features of relativistic gravitational systems undergoing diffusion: the hyperbolic character of the field equation, the invariance under Lorentz transformations and the space-time dependence of the diffusion matrix. These features distinguish the model under study from the Vlasov- Poisson-Fokker-Planck system, which is the non-relativistic analogue of the VNFP system [17, 18, 32, 47, 50, 76, 112]. While the non-relativistic problem has been intensively investigated for a long time, the interest on relativistic diffusion models has only recently started to increase [2, 20, 55, 53, 70, 90, 91, 113]. For these reasons, this chapter has a certain level of importance in what concerns to the development of this field. We recall from Chapter 2 that the VNFP system can be expressed as follows

$$\text{(VFP)}\quad \partial_t f + \nabla_p\left(\sqrt{e^{2\phi}+|p|^2}\right)\cdot\nabla_x f - \nabla_x\left(\sqrt{e^{2\phi}+|p|^2}\right)\cdot\nabla_p f = \sigma e^{2\phi}\partial_{p^i}\left(\frac{e^{2\phi}\delta^{ij}+p^ip^j}{\sqrt{e^{2\phi}+|p|^2}}\partial_{p^j}f\right),$$

$$\text{(N)}\qquad \partial_t^2\phi - \Delta_x\phi = -e^{2\phi}\int_{\mathbb{R}^3}\frac{f}{\sqrt{e^{2\phi}+|p|^2}}\,dp, \qquad t>0,\ x\in\mathbb{R}^3,\ p\in\mathbb{R}^3\,,$$

where $f = f(t, x, p)$ is the density of particles in phase space, $\phi = \phi(t, x)$ is the Nordström gravitational potential generated by the particles, $\sqrt{e^{2\phi}+|p|^2}$ is the microscopic energy of the system and $\sigma > 0$ is a diffusion constant. The remaining physical constants have been set equal to one, i.e., the speed of light $c$, the mass $m$ of the particles and the gravitational constant $G$. The physical interpretation of a solution of (VFP)-(N) is as follows: Space-time is curved by the action of gravitational forces and is given by the manifold $(\mathbb{R}^4, g)$, where $g$ is the conformal Minkowskian metric $g = \exp(2\phi)\eta$ and $\eta$ is the Minkowski metric. It is worth to mention that in the collisionless case ($\sigma = 0$), the VNFP system reduces to the Nordström-Vlasov system [28, 23], a toy model for the full general relativistic Einstein-Vlasov system [6]. In contrast to the collisionless case, particles undergoing diffusion no longer move along the geodesics of space-time. Instead, their trajectories are defined through a system of stochastic differential equations naturally associated to the Fokker-Planck equation (VFP) via Itô formula.

The results in this part of the work concern the global existence and uniqueness of spatially homogeneous solutions ($f = f(t, p), \phi = \phi(t)$) for system (VFP)-(N) and their asymptotic behavior as $t \to \infty$. Remarkably, and in contrast to the non-relativistic case [32], we find that the particle density $f$ does not vanish as $t \to \infty$ in the absence of friction, as one would expect from a diffusion model without this term. In fact, the latest mentioned property of the density resides in the long time behavior of the gravitational potential $\phi$, since it blows-up to $-\infty$ as $t \to \infty$ and it implies that the action of the diffusion operator in the right side of (VFP) without the drift term becomes weaker and weaker as $t \to \infty$. This mechanism can be identified in a simpler context while considering the non-autonomous heat equation

$$\text{(NH)} \qquad \partial_t u = \lambda(t)\Delta_x u, \quad t > 0,\ x \in \mathbb{R}^3 ,$$

where $\lambda(t)$ is a smooth positive function integrable on $(0, \infty)$. Upon introducing the change of variables $\tau(t) = \int_0^t \lambda(s)\, ds$, equation (NH) transforms into the standard, autonomous heat equation. It follows that the solution of (NH) with initial datum $u(0, x) = u_{\text{in}}(x)$ is given in terms of the heat kernel by

$$u(t,x) = \frac{1}{(4\pi\tau(t))^{3/2}} \int_{\mathbb{R}^3} u_0(y)\, e^{-\frac{|x-y|^2}{4\tau(t)}}\, dy.$$

Hence, as $t \to \infty$,

$$u(t,x) \sim \frac{1}{(4\pi\tau_\infty)^{3/2}} \int_{\mathbb{R}^3} u_0 c(y)\, e^{-\frac{|x-y|^2}{4\tau_\infty}}\, dy ,$$

where

$$\tau_\infty = \lim_{t\to\infty} \tau(t) = \int_0^\infty \lambda(s)\, ds < \infty ,$$

i.e., the solution has a non-trivial asymptotic profile. Therefore, we can interpret the diffusion coefficient $e^{2\phi}$ in the right hand side of (VFP) as a new scale in time for the spatially homogeneous case. Then, one might expect as in the previous example that the profile in the absence of a drift term might be related to solutions of

$$\text{(UR)} \qquad \partial_t f = e^{2\phi} \partial_{p^i} \left( \frac{p^i p^j}{|p|} \partial_{p^j} f \right) ,$$

the ultra-relativistic Fokker-Planck equation associated to the relativistic Fokker-Planck equation without drift, since the diffusion matrix coincides in the limit with the relativistic one. It is important to remark that the energy of an ultra-relativistic particle is almost completely determined by its momentum $|p|$. The latter justifies why we refer to (UR) as an ultra-relativistic model since the microscopic energy for the relativistic particles in the case we will consider is $\sqrt{e^{2\phi} + |p|^2} \approx |p|$ as $t \to \infty$.

This chapter proceeds as follows. In the next section, we state and prove a global existence and uniqueness theorem for the VNFP system. Then, we derive the asymptotic behavior of the scalar field, which in particular $\phi \to -\infty$, as $t \to \infty$, linearly in time, and show that the particle density $f$ does not vanish as $t \to \infty$. Since the elliptic part of the relativistic Fokker-Planck equation is not uniformly elliptic and has time dependent coefficients, the standard theory for parabolic equations does not apply in our case and we shall need to rely on stochastic methods to prove existence of solutions. Section 5.3 is devoted to the study of the time asymptotic behavior of the particle density in the ultra-relativistic regime. The main result of this section is Theorem 5.3.2, where we show that solutions of equation (UR) with a drift term satisfy $f \to f_\infty$ in $L^\infty$ as $t \to \infty$, where $f_\infty(p) > 0$ is given by the solution of the linear ultra-relativistic Fokker-Planck equation evaluated at the finite time $T = \|e^{2\phi}\|_{L^1}$. In particular, we are able to compute this limit $f_\infty$ explicitly. The arguments used for the proof of this result are based on those performed for solutions of equation (UR) in [3]. It remains as a very interesting and challenging open problem to prove the analogous result for the long time behavior of solutions in the purely relativistic case.

## 5.2 Global existence and uniqueness

The VNFP system in the spatially homogeneous case becomes

$$\partial_t f = e^{2\phi}\partial_{p^i}\left(\frac{e^{2\phi}\delta^{ij} + p^i p^j}{\sqrt{e^{2\phi}+|p|^2}}\partial_{p^j} f + p^i f\right), \tag{5.1}$$

$$\ddot{\phi} = -e^{2\phi}\int_{\mathbb{R}^3}\frac{f}{\sqrt{e^{2\phi}+|p|^2}}\,dp, \quad t>0,\ p\in\mathbb{R}^3, \tag{5.2}$$

by setting $\sigma = 1$ in (VFP). In this section, we prove the corresponding global existence and uniqueness result of solutions for the previous system in the Banach space $X$ defined as

$$X = \{g:\mathbb{R}^3\to\mathbb{R} : g\in L^1\cap L^2,\ \nabla g\in L^2,\ \text{ and }\ p\to|p|g(p)\in L^1\}.$$

The main strategy to accomplish this purpose is to show that there exists an appropriate sequence of functions that converges to a solution of the VNFP system. In particular, this sequence is given by an iterative scheme procedure. In order to define the iterative scheme, we require to study the Cauchy problems for the nonlinear equation (5.2), when $f$ is known, and for the associated linear equation of (5.1). Although the last problem is not particularly difficult in this case, the argument relies on the use of stochastic methods to obtain the result. We remark that this is probably one of the main difficulties to overcome for the general model. Also, we will derive essential bounds to establish the asymptotic behavior of solutions as well as conditions to ensure uniqueness. Now, the main result of this section reads as follows:

**Theorem 5.2.1.** *Given* $(f_{\rm in}, \phi_{\rm in}, \psi_{\rm in}) \in X\times\mathbb{R}^2$, *with* $f_{\rm in}\geq 0$ *a.e., there exists a solution of* (5.1)–(5.2) *such that*

$$(f(0,p), \phi(0), \dot{\phi}(0)) = (f_{\rm in}, \phi_{\rm in}, \psi_{\rm in})$$

*and*

$$(f,\phi) \in L^\infty((0,\infty);X) \times C^1((0,\infty)) \cap W^{2,\infty}_{\mathrm{loc}}([0,\infty)).$$

*Moreover, there exist constants* $\alpha, \beta, \varepsilon, C > 0$ *such that*

$$-C - \alpha t \le \phi(t) \le C - \beta t, \quad |\dot\phi(t)| < C, \quad -Ce^{-\alpha t} \le \ddot\phi(t) \le 0\,, \tag{5.3}$$

$$\mu(\{p : |f(t,p)| > \varepsilon)\}) > C, \tag{5.4}$$

*where* $\mu$ *denotes the Lebesgue measure,* $f \ge 0$ *a.e. and the total mass is conserved, i.e.,*

$$\|f(t)\|_{L^1(\mathbb{R}^3)} = \|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^3)}\,.$$

*Finally, if the initial datum* $f_{\mathrm{in}}$ *satisfies*

$$\int_{\mathbb{R}^3} (1+|p|^2)^{\delta+1} \left[ |\nabla_p f_{\mathrm{in}}|^2 + |\nabla_p^2 f_{\mathrm{in}}|^2 \right] dp < \infty\,, \tag{5.5}$$

*for some* $\delta > 1/2$*, then the estimate*

$$\int_{\mathbb{R}^3} (1+|p|^2)^{\delta+1} |\nabla_p f|^2\, dp + (1+t)^{-1} \int_{\mathbb{R}^3} (1+|p|^2)^{\delta+1} |\nabla_p^2 f|^2\, dp < C\,, \tag{5.6}$$

*holds for all* $t > 0$*, and the solution is unique.*

Notice that estimate (5.4) shows that $f$ does not vanish, not even asymptotically so. The crucial ingredient to prove (5.4) is the uniform estimate $\int_{\mathbb{R}^3} |p| f\, dp \le C$, which is a direct consequence of the field decay, see (5.15). We also remark this estimate remains valid even in the absence of friction with no difference in the argument between cases. A very intuitive, simple and formal computation explains this fact. We can see $e^{2\phi}$ in equation (5.1) as time rescaling factor and as a consequence, equation (5.1) becomes into one to be solved in finite time due to the $L^1$ integrability property of this factor inherited from estimates (5.3). This is the main reason to consider the associated ultra-relativistic model (UR) with or without a drift term.

Since the proof of Theorem 5.2.1 is considerably long, we divide it into several subsections for a more systematic and comprehensive reading.

### 5.2.1 The Nordström equation

In this section, we devote our attention to study the Cauchy problem for the Nordström field equation

$$\ddot\phi(t) = -H_f(t,\phi), \quad t > 0\,, \tag{5.7a}$$

$$\phi(0) = \phi_{\mathrm{in}}, \ \dot\phi(0) = \psi_{\mathrm{in}}\,, \tag{5.7b}$$

with

$$H_f(t,\phi) = e^{2\phi}\int_{\mathbb{R}^3}\frac{f}{p^0}\,dp\,, \quad p^0 = p^0(\phi,p) = \sqrt{e^{2\phi}+|p|^2}\,, \tag{5.7c}$$

where we assume that $0 \le f \in C((0,\infty); L^1(\mathbb{R}^3))$ is given. Let us begin with some observations concerning the above system. Since the function $x \to e^{2x}/p^0(x,p)$ is convex and monotonically increasing, with derivative $(e^{2x}+2|p|^2)/(p^0)^3$, we obtain the following estimate

$$\begin{aligned}|H_f(t,\phi_2) - H_f(t,\phi_1)| &\le \partial_\phi H_f(t,\phi_*)|\phi_2-\phi_1| \\ &= e^{2\phi_*}|\phi_2-\phi_1|\int_{\mathbb{R}^3} f(t,p)\frac{e^{2\phi_*}+2|p|^2}{(e^{2\phi_*}+|p|^2)^{3/2}}\,dp \\ &\le 2\|f(t)\|_{L^1(\mathbb{R}^3)}e^{\phi_*}|\phi_2-\phi_1|\,,\end{aligned} \tag{5.8}$$

where $\phi_* = \max\{\phi_1,\phi_2\}$. Next, we transform equation (5.7a) into a system of the form $\dot{y} = F(t,y)$ by using the change of variables

$$\psi = \dot{\phi}, \quad y = (\phi,\psi), \quad F(t,y) = (y_2, -H_f(t,y_1)).$$

From the regularity assumption on $f$ and estimate (5.8), the function $F$ is uniformly continuous for $t>0$ and locally Lipschitz in $y$. Then, it follows by Picard's theorem that the Cauchy problem (5.7) has a unique local classical solution. Moreover, it is straightforward to obtain the following estimates

$$-\mathcal{K}_f(t)e^{\phi(t)} \le -H_f(t,\phi) = \ddot{\phi}(t) \le 0\,, \tag{5.9a}$$

$$\psi_{\text{in}} - \mathcal{K}_f(t)\int_0^t e^{\phi(s)}\,ds \le \dot{\phi}(t) \le \psi_{\text{in}}\,, \tag{5.9b}$$

$$\psi_{\text{in}}t + \phi_{\text{in}} - \mathcal{K}_f(t)\int_0^t\int_0^s e^{\phi(\tau)}\,d\tau\,ds \le \phi(t) \le \psi_{\text{in}}t + \phi_{\text{in}}\,, \tag{5.9c}$$

where

$$\mathcal{K}_f(t) = \sup_{s\in(0,t)}\|f(s)\|_{L^1(\mathbb{R}^3)}\,. \tag{5.10}$$

These estimates imply that $\phi \in W^{2,\infty}((0,T))$, with the following bound

$$\|\phi\|_{W^{2,\infty}((0,T))} \le C_T\mathcal{K}_f(T)\,, \tag{5.11}$$

for all $T>0$. Hence we have proven the following result:

**Proposition 5.2.1.** *The Cauchy problem* (5.7) *has a unique global solution* $\phi \in C^2((0,\infty))$. *Moreover, this solution satisfies the bounds* (5.9)–(5.11), *for all* $t\in[0,T]$ *and* $T>0$.

In order to end this section, we remark some facts concerning the asymptotic behavior of $\phi$. Under additional conditions on $f$, we can ensure that $\phi \to -\infty$ as $t\to\infty$. We do not do so since this circumstance arises naturally when considering the global existence in time for the VNFP system. The argument used for the VNFP system can be easily adapted in this situation.

### 5.2.2 The linear Fokker-Planck equation

In this section, we assume $\phi \in C^2((0,\infty)) \cap W^{1,\infty}_{\mathrm{loc}}([0,\infty))$ is given and denote by $D[\phi]$ the diffusion matrix with entries

$$D^{ij}[\phi] = \frac{e^{2\phi}\delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}} = (e^{2\phi}\delta^{ij} + p^i p^j)(p^0)^{-1}\,,$$

where $p^0 = (e^{2\phi} + |p|^2)^{1/2}$. Now, we consider the Cauchy problem for the linear Fokker-Planck equation

$$\partial_t f = e^{2\phi}\partial_{p^i}(D^{ij}[\phi]\partial_{p^j} f + p^i f), \quad t > 0,\ p \in \mathbb{R}^3\,, \tag{5.12a}$$

$$f(0,p) = f_{\mathrm{in}}(p), \tag{5.12b}$$

The purpose of this subsection is to prove the following result:

**Proposition 5.2.2.** *Given $0 \le f_{\mathrm{in}} \in C^2_c(\mathbb{R}^3)$, there exists a unique classical solution of the Cauchy problem (5.12), with $f > 0$. Moreover, $f$ satisfies*

$$\|f(t)\|_{L^1(\mathbb{R}^3)} = \|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^3)}\,, \qquad \|f(t)\|_{L^q(\mathbb{R}^3)} \le Ce^{\alpha(t)}\|f_{\mathrm{in}}\|_{L^q(\mathbb{R}^3)}\,, \tag{5.13}$$

*for all $q > 1$, where*

$$\alpha(t) = C\int_0^t \mathcal{Q}_\phi(s)\,ds\,, \quad \mathcal{Q}_\phi(t) = e^{2\phi(t)} + e^{\phi(t)} + (\dot{\phi}(t))_+, \tag{5.14}$$

*and $(z)_+ = \min(0,z)$. Finally, for all $\gamma \ge 0$, there exists a constant $C > 0$, which depends only on $\gamma$, such that*

$$\int_{\mathbb{R}^3} (p^0)^{2\gamma}\left(f + |\nabla_p f|^2\right) dp \le Ce^{\alpha(t)} \int_{\mathbb{R}^3} (p^0)^{2\gamma}\left(f_{\mathrm{in}} + |\nabla_p f_{\mathrm{in}}|^2\right) dp\,, \tag{5.15}$$

$$\|(p^0)^\gamma \nabla_p^2 f\|^2_{L^2} \le Ce^{\alpha(t)}\left(\|(p^0)^\gamma \nabla_p^2 f_{\mathrm{in}}\|^2_{L^2} + \int_0^t \|(p^0)^\gamma \nabla_p f\|^2_{L^2}\,ds\right)\,. \tag{5.16}$$

*Proof.* We divide the proof in several steps. In order to prove existence, we employ methods from the theory of stochastic differential equations and diffusion processes developed in [11]. Our objective is to show that the system

$$dP = b(s,P)\,ds + G(s,P)\cdot dW\,, \quad P(t;x,t) = p\,. \tag{5.17}$$

admits a unique solution $P(s;x,t)$ for any $t \in [-T,0]$ and $t \le s \le 0$. Here, $dW$ denotes the standard Wiener process, $G(s,p)$ is the positive definite matrix with entries

$$G^{ij} = \frac{\sqrt{2}e^{\bar{\phi}(s)}}{(p^0)^{1/2}}\left(e^{\bar{\phi}(s)}\delta^{ij} + \frac{p^i p^j}{e^{\bar{\phi}(s)} + p^0}\right)\,, \quad p^0 = \sqrt{e^{2\bar{\phi}(s)} + |p|^2}\,,$$

and the vector field $b$ is given by

$$b = e^{2\bar{\phi}}\partial_{p^j}(D^{ij}[\bar{\phi}]) + e^{2\bar{\phi}}p^j.$$

According to the theory of stochastic differential equations, system (5.17) has associated the following *backward* Kolmogorov equation

$$\partial_t \bar{f} + b^i \partial_{p^i}\bar{f} + \frac{1}{2}d^{ij}\partial_{p^i}\partial_{p^j}\bar{f} = 0, \quad t<0,\ p\in\mathbb{R}^3, \tag{5.18a}$$

$$\bar{f}(0,p) = f_{\text{in}}(p), \tag{5.18b}$$

with $d^{ij} = G^{ik}G^{kj}$. Also, the Feynman-Kac formula ensures that

$$\bar{f}(t,p) = \mathbb{E}[f_{\text{in}}(P(0,p;t))] \tag{5.19}$$

is a candidate to be a classical positive solution of problem (5.18) in $[-T,0]$. Then, we require to adhere our formulation in terms of the previous setting as follows. For $t<0$ and $p\in\mathbb{R}^3$, we define the functions

$$\bar{f}(t,p) = e^{-3\tau(-t)}f(-t,p), \quad \bar{\phi}(t) = \phi(-t), \quad \tau(t) = \int_0^t e^{2\phi(s)}\,ds,$$

and multiply (5.12a) by the integral factor $e^{-3\tau(t)}$ to transform the Cauchy problem (5.12) into (5.18), where

$$b^i(t,p) = \frac{3e^{2\bar{\phi}}}{p^0}p^i + e^{2\bar{\phi}}p^i, \quad d^{ij}(t,p) = 2e^{2\bar{\phi}}D^{ij}[\bar{\phi}]\,.$$

Notice that

$$\begin{aligned} e^{-2\bar{\phi}}p^0|p|^4G^{il}G^{lj} &= 2e^{2\bar{\phi}}|p|^4\delta^{ij} + 2p^ip^j|p|^2(e^{\bar{\phi}}-p^0)^2 + 4e^{\bar{\phi}}p^ip^j|p|^2(p^0-e^{\bar{\phi}}) \\ &= 2(e^{2\bar{\phi}}|p|^4\delta^{ij} + p^ip^j|p|^4) = 2D^{ij}[\bar{\phi}]|p|^4p^0\,, \end{aligned}$$

and as a consequence, $G$ is the unique square root of $d$. Next, we need to find growth estimates on $b$, $G$ and their first and second derivatives. To achieve this, we will heavily use the fact that $e^{2\bar{\phi}}$, $|p|^2 \le (p^0)^2 = e^{2\bar{\phi}} + |p|^2$. Then, we see that

$$|b(t,p)| + |G(t,p)| \le Ce^{2\bar{\phi}} + e^{\frac{\bar{\phi}}{2}}\left(Ce^{\bar{\phi}} + |p|\right)\,.$$

Now, we compute

$$\begin{aligned} \partial_{p^j}b^i(t,p) &= \frac{3e^{2\bar{\phi}}}{(p^0)^3}\left(\delta^{ij}(p^0)^2 - p^ip^j\right) + e^{2\bar{\phi}}\delta^{ij}\,, \\ \partial_{p^k}\partial_{p^j}b^i(t,p) &= -\frac{3e^{2\bar{\phi}}}{(p^0)^5}\left((\delta^{ij}p^k + \delta^{ik}p^j - \delta^{jk}p^i)(p^0)^2 + 3p^kp^ip^j\right)\,, \end{aligned}$$

and whence we see that for every $i, j, k$, the following bounds hold

$$|\partial_{p^j} b^i(t,p)| \le C\frac{e^{2\bar{\phi}}}{(p^0)^3}((p^0)^2 + |p|^2) + Ce^{2\bar{\phi}} \le C(e^{\bar{\phi}} + e^{2\bar{\phi}}),$$
$$|\partial_{p^k}\partial_{p^j} b^i(t,p)| \le C\frac{|p|}{(p^0)^3}((p^0)^2 + |p|^2) \le C.$$

Similarly, we obtain

$$\partial_{p_k} G^{ij}(t,p) = \frac{e^{\bar{\phi}}(p^0)^{-5/2}}{\sqrt{2}(e^{\bar{\phi}} + p^0)^2}\Big[2(p^0)^2(e^{\bar{\phi}} + p^0)(\delta^i_k p^j + \delta^j_k p^i)$$
$$-p_k p^i p^j e^{\bar{\phi}} - 3p_k p^i p^j p^0 - e^{\bar{\phi}}(e^{\bar{\phi}} + p^0)^2 p_k \delta^{ij}\Big],$$

where we used

$$\partial_{p^k}(\delta^{ij} e^{\bar{\phi}}(e^{\bar{\phi}} + p^0) + p^i p^j) = \delta^{ik} p^j + \delta^{jk} p^i + \delta^{ij} p^k \frac{e^{\bar{\phi}}}{p^0},$$
$$\partial_{p^k}((p^0)^{1/2}(e^{\bar{\phi}} + p^0)) = p^k \frac{(3p^0 + e^{\bar{\phi}})}{2(p^0)^{3/2}}.$$

Therefore, we have that for every $i, j, k$,

$$|\partial_{p_k} G^{ij}(t,p)| \le C\frac{e^{\bar{\phi}/2}(p^0)^{-2}}{(e^{\bar{\phi}} + p^0)^2}\left[|p|\,(p^0)^2\left(e^{\bar{\phi}} + p^0\right) + e^{\bar{\phi}}|p|\left(e^{\bar{\phi}} + p^0\right)^2\right]$$
$$\le Ce^{\bar{\phi}/2}.$$

As before, second derivatives can be bounded using similar estimates.

Finally, let $T > 0$ be fixed. Then, $|b(t,p)| + |G(t,p)| \le C_T(1 + |p|)$, and the first and second derivatives of $b$ and $G$ with respect to $p$ are uniformly bounded for $t \in [-T, 0]$. These estimates are exactly those ones required to apply [11, Th. 9.4.4] and conclude that (5.19) is a classical solution of (5.18). From here, the existence and uniqueness property for solutions of (5.12) follows by transforming back into the original variables and by applying the estimate that will be performed on $\|f(t)\|_{L^2(\mathbb{R}^3)}$ to the difference of two solutions, for instance.

Next, we show that classical solutions satisfy the estimates (5.13). Let $\xi \in C_c^\infty([0,\infty))$ be a non-increasing function such that

$$\xi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2, \end{cases}$$

and define the function $\xi_n(p) = \xi(\frac{|p|}{n})$, for $p \in \mathbb{R}^3$ and $n \in \mathbb{N}$, with $n \geq 1$. Then, $\xi_n \in C_c^\infty(\mathbb{R}^3)$ is a cut-off function satisfying the following properties

$$\xi_n(p) = \begin{cases} 1 & \text{if } |p| \leq n\,, \\ 0 & \text{if } |p| \geq 2n\,, \end{cases}$$

which clearly implies that $0 \leq \xi_n \leq 1$ . Also, we have that $|\nabla_p \xi_n| \leq C/n$ and $|\Delta_p \xi_n| \leq C/n^2$. By a direct computation, we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3} \xi_n f^q\, dp = &- q(q-1)e^{2\phi}\int_{\mathbb{R}^3} \xi_n f^{q-2} D^{ij}[\phi]\partial_{p^i} f \partial_{p^j} f\, dp \\ &+ e^{2\phi}\int_{\mathbb{R}^3} f^q \left[\partial_{p^j}(D^{ij}[\phi]\partial_{p^i}\xi_n) - p^i\partial_{p^i}\xi_n\right] dp \\ &+ 3(q-1)e^{2\phi}\int_{\mathbb{R}^3} f^q \xi_n dp\,, \end{aligned} \tag{5.20}$$

for all $q \geq 1$. By the positivity of $D$, the first term in the right side of (5.20) is non-positive. From the properties of the cutoff function, the term in square brackets in the last integral satisfies

$$[\dots] \leq \frac{C_T}{n}, \quad \text{for all } t \in [0,T] \text{ and all } T > 0\,.$$

Hence, using again the properties of the cut-off function and Gronwall's inequality, identity (5.20) is bounded as follows

$$\|f(t)\|_{L^q(\mathbb{R}^3)} \leq C_T\,.$$

Substituting again in (5.20), we obtain the following inequalities

$$\begin{aligned} \|f_{\text{in}}\|_{L^1(\mathbb{R}^3)} - \frac{C_T}{n} \leq \|f(t)\|_{L^1(\mathbb{R}^3)} \leq \|f_{\text{in}}\|_{L^1(\mathbb{R}^3)} + \frac{C_T}{n}\,, \\ \|f(t)\|_{L^q(\mathbb{R}^3)} \leq e^{3\frac{(q-1)}{q}\tau(t)}\|f_{\text{in}}\|_{L^q(\mathbb{R}^3)} + \frac{C_T}{n}\,, \end{aligned}$$

which allow to conclude (5.13) by taking the limit $n \to \infty$. In order to prove estimates (5.15)–(5.16), we present a formal proof; all the computations can be rigorously made by introducing the cut-off function $\xi$ as above.

Taking the time derivative of $(e^{2\phi} + |p|^2)^\gamma f$, integrating this quantity over $\mathbb{R}^3$ and using equation (5.12a), we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma f\, dp = &\, 2\gamma e^{2\phi}\dot{\phi}\int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma-1} f\, dp \\ &+ e^{2\phi}\int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma\, \partial_{p^i}(D^{ij}[\phi]\partial_{p^j} f + p^i f)\, dp\,. \end{aligned}$$

In order to bound the right hand side of the previous identity, we perform an integration by parts and use the following identity

$$p^i \partial_{p^i}[(e^{2\phi} + |p|^2)^\gamma] = 2\gamma |p|^2 (e^{2\phi} + |p|^2)^{\gamma-1} > 0,$$

to see that

$$\begin{aligned}\frac{d}{dt}\|(p^0)^{2\gamma} f\|_{L^1} &\le C(\dot\phi)_+ \int_{\mathbb{R}^3} (p^0)^{2\gamma} f\, dp - e^{2\phi} \int_{\mathbb{R}^3} p^i \partial_{p^i}[(p^0)^{2\gamma}]\, f\, dp \\ &\quad + e^{2\phi} \int_{\mathbb{R}^3} \partial_{p^j}\left\{D^{ij}[\phi]\partial_{p^i}[(p^0)^{2\gamma}]\right\} f\, dp \\ &\le C(\dot\phi)_+ \|(p^0)^{2\gamma} f\|_{L^1} + e^{2\phi} \int_{\mathbb{R}^3} \partial_{p^j}\left\{D^{ij}[\phi]\partial_{p^i}[(p^0)^{2\gamma}]\right\} f\, dp\,.\end{aligned}$$

Here, $(\cdot)_+$ denotes the positive part. The bracketed portion $\partial_{p^j}\{...\}$ of the second term in the above inequality requires to recall that $\partial_{p^j} D^{ij}[\phi] = 3p^i/p^0$ and the next computation

$$\partial_{p^j}\partial_{p^i}[(e^{2\phi} + |p|^2)^\gamma] = 4\gamma(\gamma-1)p^i p^j (e^{2\phi} + |p|^2)^{\gamma-2} + 2\gamma\delta^{ij}(e^{2\phi} + |p|^2)^{\gamma-1}\,,$$

so we can obtain the bound

$$\begin{aligned}\partial_{p^j}\{\dots\} &= 6\gamma(e^{2\phi} + |p|^2)^{\gamma-1/2} + 4\gamma(\gamma - 1/2)(e^{2\phi} + |p|^2)^{\gamma-3/2}|p|^2 \\ &\le Ce^{-\phi}(e^{2\phi} + |p|^2)^\gamma\,.\end{aligned}$$

From the previous inequality, we find

$$\frac{d}{dt}\int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma\, f\, dp \le C(e^\phi + (\dot\phi)_+) \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma\, f\, dp\,,$$

which again by Gronwall's inequality the following estimate holds

$$\int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma\, f\, dp \le C \exp\left( C \int_0^t \mathcal{Q}_\phi(s)\, ds \right).$$

As to the estimate on $\nabla_p f$, we recall that $(p^0)^2 = e^{2\phi} + |p|^2$ and compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} (p^0)^{2\gamma} |\nabla_p f|^2\, dp &= 2\gamma e^{2\phi}\dot\phi \int_{\mathbb{R}^3} (p^0)^{2(\gamma-1)} |\nabla_p f|^2\, dp \\ &\quad + 5e^{2\phi} \int_{\mathbb{R}^3} (p^0)^{2\gamma} |\nabla_p f|^2\, dp \\ &\quad - e^{2\phi} \int_{\mathbb{R}^3} p^i \partial_{p^i}[(p^0)^{2\gamma}] |\nabla_p f|^2\, dp \\ &\quad + \underbrace{2e^{2\phi} \int_{\mathbb{R}^3} (p^0)^{2\gamma} \nabla_p f \cdot \nabla_p(\partial_{p^i}(D^{ij}[\phi]\partial_{p^j} f))\, dp}_{(*)}\,. \qquad (5.21)\end{aligned}$$

Similarly as in the previous case, we will be able to show that

$$\frac{d}{dt}\int_{\mathbb{R}^3}(p^0)^{2\gamma}\,|\nabla_p f|^2\,dp \le C(e^{2\phi}+e^{\phi}+(\dot{\phi})_+)\int_{\mathbb{R}^3}(p^0)^{2\gamma}\,|\nabla_p f|^2\,dp,$$

since we already know how to handle the first three terms in (5.21) and after bounding the remaining one, an application of Gronwall's inequality will lead us to complete the proof of (5.15). Now, we estimate $(*)$ in (5.21). To do so, we first integrate by parts in the variable $p^i$ and then, after straightforward calculations, we obtain

$$(*) = I_1 + I_2 + I_3 + I_4\,,$$

where

$$\begin{aligned}
I_1 &= -2e^{2\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma}D^{ij}[\phi]\partial_{p^i}\nabla_p f\cdot\partial_{p^j}\nabla_p f\,dp\,,\\
I_2 &= -4\gamma e^{2\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma}A^j_k[\phi]\partial_{p^j}f\,\partial_{p^k}f\,dp\,,\\
I_3 &= 2\gamma e^{2\phi}\int_{\mathbb{R}^3}\nabla_p\cdot(p\,(e^{2\phi}+|p|^2)^{\gamma-1/2})|\nabla_p f|^2\,dp\,,\\
I_4 &= -2e^{2\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma}\partial_{p^i}\partial_{p^k}f(\partial_{p_k}D^{ij}[\phi])\partial_{p^j}f\,dp\,,
\end{aligned}$$

and

$$A^j_k[\phi] = \frac{p_i\partial_{p^k}D^{ij}}{(p^0)^2} = \frac{p_i(p^i\delta^j_k+p^j\delta^i_k)}{(e^{2\phi}+|p|^2)^{3/2}} - \frac{p_ip_kD^{ij}}{(e^{2\phi}+|p|^2)^2} = \delta^j_k\frac{|p|^2}{(p^0)^3}\,. \tag{5.22}$$

By the positivity of $D$ and $A$, we have $I_1 + I_2 \le 0$. In $I_3$, we compute

$$\nabla_p\cdot(p\;(p^0)^{2\gamma-1}) = 3(p^0)^{2\gamma-1} + (2\gamma-1)(p^0)^{2\gamma-3}|p|^2 \le Ce^{-\phi}(p^0)^{2\gamma},$$

and thus the integral $I_3$ is bounded by

$$I_3 \le Ce^{\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma}\,|\nabla_p f|^2\,dp\,.$$

The integral $I_4$ requires some further work. Integrating by parts with respect to the $p^k$ derivative in the $\partial_{p^i}\partial_{p^k}f$ term, we obtain

$$\begin{aligned}
I_4 = 2e^{2\phi}&\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma}(\Delta_p D^{ij}[\phi])\partial_{p^i}f\partial_{p^j}f\\
&+4\gamma e^{2\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma}B^{ij}\partial_{p^i}f\,\partial_{p^j}f\\
&+2e^{2\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma}\partial_{p^i}f(\partial_{p_k}D^{ij})\partial_{p^j}\partial_{p^k}f\,dp\,,
\end{aligned}$$

where

$$B^{ij} = \frac{p\cdot\nabla_p D^{ij}[\phi]}{(e^{2\phi}+|p|^2)} = \frac{2p^ip^j}{(e^{2\phi}+|p|^2)^{3/2}} - \frac{|p|^2}{(e^{2\phi}+|p|^2)^2}D^{ij}[\phi]\,, \tag{5.23}$$
$$\Delta_p D^{ij}[\phi] = \frac{1}{\sqrt{e^{2\phi}+|p|^2}}\left(2\delta^{ij} - \frac{4p^ip^j}{e^{2\phi}+|p|^2}\right) - \frac{3e^{2\phi}}{(e^{2\phi}+|p|^2)^2}D^{ij}[\phi]\,.$$

By the symmetry of $D$, the last integral is equal to $-I_4$ and thus we have obtained

$$I_4 = I_{4A} + I_{4B},$$

where

$$I_{4A} = e^{2\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^\gamma(\Delta_p D^{ij}[\phi])\partial_{p^i}f\partial_{p^j}f\,dp\,,$$
$$I_{4B} = 2\gamma e^{2\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^\gamma B^{ij}\partial_{p^i}f\,\partial_{p^j}f\,dp\,.$$

Due to the positivity of $D^{ij}$ and the Cauchy-Schwarz inequality, it is straightforward to obtain the bounds

$$B^{ij}x_ix_j \le Ce^{-\phi}|x|^2, \qquad \Delta_p D^{ij}[\phi]x_ix_j \le Ce^{-\phi}|x|^2\,, \tag{5.24}$$

for all $x\in\mathbb{R}^3$, and as a consequence, we have

$$I_4 \le Ce^{\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^\gamma\,|\nabla_p f|^2\,dp\,.$$

Collecting all the above estimates, we find that the term $(*)$ in (5.21) satisfies

$$(*) \le Ce^{\phi}\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^\gamma\,|\nabla_p f|^2\,dp\,, \tag{5.25}$$

which is the remaining bound to complete the proof of (5.15).

To prove (5.16), we use that $g_k = \partial_{p^k}f$ satisfies, for all $k=1,2,3$,

$$\begin{aligned}\partial_t g_k &= e^{2\phi}\partial_{p^i}(D^{ij}\partial_{p^j}g_k + p^ig_k) + e^{2\phi}g_k + e^{2\phi}\partial_{p^i}[(\partial_{p^k}D^{ij})g_j]\\ &= e^{2\phi}\mathrm{FP}[g_k] + e^{2\phi}g_k + e^{2\phi}\partial_{p^i}[(\partial_{p^k}D^{ij})g_j]\,,\end{aligned}$$

and thus

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}(p^0)^{2\gamma}\nabla_p g^k\cdot\nabla_p g_k &= 2\gamma e^{2\phi}\dot\phi\int_{\mathbb{R}^3}(e^{2\phi}+|p|^2)^{\gamma-1}\nabla_p g^k\cdot\nabla_p g_k\,dp\\ &\quad + 2e^{2\phi}\int_{\mathbb{R}^3}(p^0)^{2\gamma}\nabla_p g^k\cdot\nabla_p\left(\mathrm{FP}[g_k]+g_k\right)dp\\ &\quad + 2e^{2\phi}\int_{\mathbb{R}^3}(p^0)^{2\gamma}\nabla_p g^k\cdot\nabla_p\{\partial_{p^i}[(\partial_{p^k}D^{ij})g_j]\}\,dp\\ &= II + III + IV\,.\end{aligned} \tag{5.26}$$

The term $II + III$ can be treated in a similar manner as $(*)$ in (5.21), with $f$ replaced by $g_k$, and thus by (5.25), it satisfies the bound

$$II + III \le C((\dot{\phi})_+ + e^{\phi} + e^{2\phi}) \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma}\, \nabla_p g^k \cdot \nabla_p g_k \, dp$$

$$\le C((\dot{\phi})_+ + e^{\phi} + e^{2\phi}) \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma}\, |\nabla_p^2 f|^2 \, dp\,. \tag{5.27}$$

Expanding the term $IV$ in (5.26) we obtain

$$\begin{aligned} IV = {} & 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma}\, \nabla_p g^k \cdot \nabla_p(\partial_{p^i}\partial_{p^k} D^{ij}) g_j \, dp \\ & + 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma} (\partial_{p^i}\partial_{p^k} D^{ij}) \nabla_p g^k \cdot \nabla_p g_j \, dp \\ & + 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma}\, \nabla_p g^k \cdot \nabla_p(\partial_{p^k} D^{ij}) \partial_{p^i} g_j \, dp \\ & + 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma} (\partial_{p^k} D^{ij}) \nabla_p g^k \cdot \nabla_p \partial_{p^i} g_j \, dp \\ = {} & IV_1 + IV_2 + IV_3 + IV_4\,. \end{aligned} \tag{5.28}$$

In $IV_4$, we integrate by parts in the $p^i$ derivative acting on $g_j$ and obtain

$$\begin{aligned} IV_4 = {} & -4\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma-1} (p_i \partial_{p^k} D^{ij}) \nabla_p g^k \cdot \nabla_p g_j \, dp \\ & - 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma} (\partial_{p^i}\partial_{p^k} D^{ij}) \nabla_p g^k \cdot \nabla_p g_j \, dp \\ & - 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma} (\partial_{p^k} D^{ij}) \nabla_p \partial_{p^i} g^k \cdot \nabla_p g_j \, dp \end{aligned} \tag{5.29}$$

$$= IV_{4A} + IV_{4B} + IV_{4C}. \tag{5.30}$$

Note that $IV_2 + IV_{4B} = 0$. In $IV_{4C}$ we integrate by parts in the $p^k$ derivative within $g_k = \partial_{p^k} f$, so that

$$\begin{aligned} IV_{4C} = {} & 4\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma-1} (p_k \partial_{p^k} D^{ij}) \nabla_p g_i \cdot \nabla_p g_j \, dp \\ & + 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma} (\Delta_p D^{ij}) \nabla_p g_i \cdot \nabla_p g_j \, dp \\ & + 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma} (\partial_{p^k} D^{ij}) \nabla_p g_i \cdot \nabla_p \partial_{p^j} g^k \, dp. \end{aligned}$$

By the symmetry of $D$, the third term in the right hand side of the latter equation equals $-IV_{4C}$, hence

$$\begin{aligned} IV_{4C} = {} & e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma} (\Delta_p D^{ij}) \nabla_p g_i \cdot \nabla_p g_j \\ & + 2\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^{\gamma-1} (p^k \partial_{p^k} D^{ij}) \nabla_p g_i \cdot \nabla_p g_j \, dp. \end{aligned} \tag{5.31}$$

Substituting (5.31) into (5.30) and then, returning to (5.28) we obtain

$$
\begin{aligned}
IV = {} & 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma \nabla_p(\partial_{p^i}\partial_{p^k} D^{ij}) \cdot (\nabla_p g^k) g_j \, dp \\
& + 2e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma \nabla_p(\partial_{p^k} D^{ij}) \cdot \nabla_p g^k \partial_{p^i} g_j \, dp \\
& + e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma (\Delta_p D^{ij}) \nabla_p g_i \cdot \nabla_p g_j \, dp \\
& - 4\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma A^j_k \nabla_p g^k \cdot \nabla_p g_j \, dp \\
& + 2\gamma e^{2\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma B^{ij} \nabla_p g_i \cdot \nabla_p g_j \, dp,
\end{aligned}
\tag{5.32}
$$

where ${A^j}_k$ and $B^{ij}$ are given by (5.22), (5.23). Recall that $A$ is positive definite and that the estimates (5.24) hold. Furthermore, using the following estimates

$$
|\partial_{p^k}\partial_{p^l} D^{ij}| \le Ce^{-\phi}, \quad |\nabla_p(\partial_{p^i}\partial_{p^k} D^{ij})| \le Ce^{-2\phi}\,, \tag{5.33}
$$

we see that equation (5.32) entails

$$
\begin{aligned}
IV \le {} & C\int_{\mathbb{R}^3} (p^0)^{2\gamma} |\nabla_p g_k|\,|g_j| \, dp + Ce^{\phi} \int_{\mathbb{R}^3} (p^0)^{2\gamma} (|\nabla_p g_k| + |\nabla_p g_i|) |\nabla_p g_j| \, dp \\
\le {} & C \left( \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla^2_p f|^2 \, dp \right)^{1/2} \left( \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla_p f|^2 \, dp \right)^{1/2} \\
& + e^{\phi} \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla^2_p f|^2 \, dp,
\end{aligned}
\tag{5.34}
$$

where we used Hölder's inequality in the last step. Now, we substitute the bounds (5.27) and (5.34) into (5.26) in order to obtain

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^3} (p^0)^{2\gamma} |\nabla^2_p f|^2 \, dp \le {} & C(e^{2\phi} + e^{\phi} + (\dot\phi)_+) \int_{\mathbb{R}^3} (p^0)^{2\gamma} |\nabla^2_p f|^2 \, dp \\
& + C \left( \int_{\mathbb{R}^3} (p^0)^{2\gamma} |\nabla^2_p f|^2 \, dp \right)^{\frac12} \left( \int_{\mathbb{R}^3} (p^0)^{2\gamma} |\nabla_p f|^2 \, dp \right)^{\frac12},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\frac{d}{dt} \left( \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla^2_p f|^2 \, dp \right)^{\frac12} \le {} & CQ_\phi(s) \left( \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla^2_p f|^2 \, dp \right)^{\frac12} \\
& + C \left( \int_{\mathbb{R}^3} (e^{2\phi} + |p|^2)^\gamma |\nabla_p f|^2 \, dp \right)^{\frac12},
\end{aligned}
$$

which by Gronwall's inequality gives (5.16). To conclude the proof, we show the validity of estimates (5.33). For our convenience, we recall that the diffusion matrix is given by

$$D^{ij}[\phi] = \frac{e^{2\phi}\delta^{ij} + p^i p^j}{\sqrt{e^{2\phi} + |p|^2}}\,,$$

which clearly is bounded by $p^0 = \sqrt{e^{2\phi} + |p|^2}$. The first derivatives of $D$ are given by

$$\begin{aligned}\partial_{p^k} D^{ij} &= \frac{\delta^i{}_k p^j + \delta^j{}_k p^i}{\sqrt{e^{2\phi} + |p|^2}} - \frac{(p_k/p^0)(\delta^i{}_k p^j + \delta^j{}_k p^i)}{e^{2\phi} + |p|^2} \\ &= \frac{\delta^i{}_k p^j + \delta^j{}_k p^i}{\sqrt{e^{2\phi} + |p|^2}} - \frac{p_k D^{ij}}{e^{2\phi} + |p|^2}\,, \end{aligned} \tag{5.35}$$

and therefore,

$$\partial_{p^k} D^{ij} \le C\frac{|p|}{p^0} + \frac{|p|^2}{(p^0)^2} \le C\,,$$

where we used the boundedness property of $D$. Moreover

$$\begin{aligned}\partial_{p^k}\partial_{p^l} D^{ij} =& \frac{\delta^i{}_k\delta^j{}_l + \delta^j{}_k\delta^i{}_l}{\sqrt{e^{2\phi} + |p|^2}} - \frac{(\delta^i{}_k p^j + \delta^j{}_k p^i)p_l}{(e^{2\phi} + |p|^2)^{3/2}} \\ &- \frac{\partial_{p^l} D^{ij} p_k}{e^{2\phi} + |p|^2} - \frac{D^{ij}\delta_{kl}}{e^{2\phi} + |p|^2} + \frac{2D^{ij}p_k p_l}{(e^{2\phi} + |p|^2)^2}\,,\end{aligned}$$

and each term in the right hand side is bounded in modulus by $Ce^{-\phi}$, which proves the first estimate in (5.33). Furthermore

$$\partial_{p^l}(\partial_{p^k}\partial_{p^i} D^{ij}) = -3\frac{\delta^j{}_k p_l + \delta^j{}_l p_k + \delta_{kl}p^j}{(e^{2\phi} + |p|^2)^{3/2}} - 9\frac{p^j p_k p_l}{(e^{2\phi} + |p|^3)^{5/2}},$$

and each term in the right hand side is bounded in modulus by $Ce^{-2\phi}$, which proves the second estimate in (5.33). □

We would like to finish this section by making some comments:

- The above proof can be used to show the existence and uniqueness of solutions of (5.12a) without friction with some slight modifications.
- It would be interesting if some improvements could be made while estimating (5.21). For instance, the use of a lower derivative bound instead of (5.33) would allow us to obtain a better regularity while dealing with the long time behavior of solutions of the VNFP system.

- As we mentioned in the introduction, there a no analytical results that cover our former situation and it does not seem likely that there is an easy way to develop one without the use of stochastic methods. That is why the importance of the previous result. In fact, in order to cover different growth conditions on the coefficients or considering coefficients with singularities in unbounded domains, it appears more reasonable to make use of stochastic tools rather than the analytical ones. A good example of this situation can be found in [125].

### 5.2.3 Existence

In this section, we prove the existence of solutions for the VNFP system (5.1)–(5.2). The procedure is quite standard. First, we define an iterative scheme using the linear Fokker-Planck equation (5.12) and the Nordström equation (5.7) with regular initial data, since we can use a density argument to obtain the general result in our context. Next, we employ properties (5.11) and (5.13) in order to obtain weak convergence of the sequence defined in the previous step and then, a compactness argument will be applied to show that this sequence will converge to a solution of VNFP system in $L^2(\mathbb{R}^3)\times C^1(0,T)$. Finally, we will also prove that the mass of $f$ is conserved from the weak convergence in $L^1$ of the sequence.

Let $0 \leq f_{\text{in}} \in C_c^2(\mathbb{R}^3)$ and $T > 0$ be given. We consider the sequence $(f_n, \phi_n)$ which will be defined iteratively as follows:

- For $n = 0$, we set $(f_0, \phi_0) = (f_{\text{in}}, \phi_{\text{in}})$.
- Assuming that the pair $(f_n, \phi_n)$ is given, we define $(f_{n+1}, \phi_{n+1})$ as the unique solution of the system

$$\partial_t f_{n+1} = e^{2\phi_n}\partial_{p^i}(D^{ij}[\phi_n]\partial_{p^j} f_{n+1} + p^i f_{n+1}), \quad f_{n+1}(0,p) = f_{\text{in}}(p),$$
$$\ddot{\phi}_{n+1} = -H(t,\phi_{n+1}, f_{n+1}), \quad (\phi_{n+1}(0), \dot{\phi}_{n+1}(0)) = (\phi_{\text{in}}, \psi_{\text{in}}),$$

where

$$H(t,\phi_{n+1}, f_{n+1}) = e^{2\phi_{n+1}} \int_{\mathbb{R}^3} \frac{f_{n+1}}{\sqrt{e^{2\phi_{n+1}} + |p|^2}}\, dp.$$

It follows by an induction argument and propositions 5.2.1–5.2.2 that the sequence $(f_n, \phi_n)$ consists of smooth functions. Moreover, by (5.13),

$$\|f_n(t)\|_{L^1(\mathbb{R}^3)} = \|f_{\text{in}}\|_{L^1(\mathbb{R}^3)}, \quad \|f_n(t)\|_{L^2(\mathbb{R}^3)} \leq Ce^{\alpha(T)}\|f_{\text{in}}\|_{L^2(\mathbb{R}^3)},$$

and therefore the function $\mathcal{K}_{f_n}(t)$ given by (5.10) is equibounded along the sequence $f_n$. Thus, we have by (5.11) that

$$\|\phi_n\|_{W^{2,\infty}((0,T))} \leq C_T.$$

We infer that the function $\mathcal{Q}_{\phi_n}(t)$ given by (5.14) is equibounded along the sequence $\phi_n$. Hence, using (5.15) we see that

$$\|\nabla_p f_n(t)\|_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} |p| f_n(t)\, dp \leq C_T, \quad \text{for all } t \in [0,T].$$

It follows that there exist

$$f \in L^\infty((0,T); H^1(\mathbb{R}^3)), \qquad \phi \in W^{2,\infty}((0,T)),$$

and a subsequence, still denoted by $(f_n, \phi_n)$, such that

$$f_n \rightharpoonup f \text{ in } L^2((0,T)\times\mathbb{R}^3), \qquad \phi_n \overset{*}{\rightharpoonup} \phi \text{ in } W^{2,\infty}(0,T), \quad \text{as } n \to \infty.$$

By a standard diagonal sequence argument, we can choose $(f_n, \phi_n)$ to be independent of $T > 0$. Moreover, we also have

$$f_n(t,\cdot) \rightharpoonup f(t,\cdot) \text{ in } H^1(\mathbb{R}^3) \quad \text{for all } t \in [0,T].$$

By compactness, we may extract a subsequence $f_n(t,\cdot)$ and $(\phi_n, \dot{\phi}_n)$ such that converge strongly in $L^2(\mathbb{R}^3)$ and uniformly on $[0,T]$, respectively. This convergence is strong enough to pass to the limit in the equations and conclude that $(f,\phi)$ is a solution of the spatially homogeneous VNFP system (5.1)–(5.2). Also, we deduce that $f \in L^\infty\left((0,T]; L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)\right)$ and $\phi \in C^1$. Next, we show that $f_n(t,\cdot) \rightharpoonup f(t,\cdot)$ in $L^1(\mathbb{R}^3)$ (up to subsequences) which in particular implies the mass conservation of $f$. First, it is immediate that $f_n$ is equibounded, due to mass conservation. Also, if $\delta > 0$ and $\Omega \subset \mathbb{R}^3$ is any measurable set, with $|\Omega| < \delta$, we have

$$\int_\Omega f_n\, dp \leq \|f_n\|_{L^2(\mathbb{R}^3)} (\delta)^{1/2},$$

and as a consequence, the sequence does not concentrate due to boundedness of $f_n$ in $L^2(\mathbb{R}^3)$. Moreover, the sequence is tight, because $|p|\, f_n$ is bounded in $L^1(\mathbb{R}^3)$ and for every ball centered at the origin of radius $r > 0$, we have

$$\int_{\mathbb{R}^3/B_r(0)} f_n\, dp \leq \frac{1}{r} \int_{\mathbb{R}^3/B_r(0)} |p| f_n\, dp.$$

Then, the Dunford-Pettis theorem ensures the weak convergence in $L^1(\mathbb{R}^3)$ of the sequence $f_n(t,\cdot)$ for $t \in (0,T]$.

### 5.2.4 Uniform estimates and asymptotic behavior of the field

In this section, we extend our previous results to the case where $t \in (0,\infty)$ uniformly, which will imply that

$$|p| f \in L^\infty\left((0,\infty); L^1(\mathbb{R}^3)\right) \quad f, \nabla_p f \in L^\infty\left((0,\infty); L^2(\mathbb{R}^2)\right).$$

Moreover, we establish estimate (5.3) and prove that (5.6) holds when the initial data satisfy (5.5). The latter is an immediate consequence of showing that $\phi(t) \to -\infty$ as $t \to \infty$, since the latter is necessary in order to obtain $\alpha(t) < \infty$ in (5.14) for all $t > 0$ ($\alpha(t)$ is increasing). In fact, we have that $\dot{\phi}$ is decreasing which implies that the limit

$$\dot{\phi}_\infty = \lim_{t\to\infty} \dot{\phi}(t)$$

exists. The following lemma describes the asymptotic behavior of $\dot{\phi}(t)$, which in particular ensures that $\lim_{t\to\infty} \dot{\phi}(t) = -\infty$ can not occur:

**Lemma 5.2.1.** *Let* $(f, \phi)$ *be a solution of* (5.1)–(5.2). *Then, under the same assumptions as in Theorem 5.2.1, we have* $\dot{\phi}_\infty < 0$.

*Proof.* Let

$$M = \|f(t)\|_{L^1(\mathbb{R}^3)} = \|f_{\rm in}\|_{L^1(\mathbb{R}^3)}, \quad \mathcal{E}(t) = \int f\sqrt{e^{2\phi} + |p|^2}\, dp + \frac{1}{2}\dot{\phi}^2 ,$$

be the mass and the energy of the solution constructed in Section 5.2.3. Recall that $\mathcal{E}(t) < \infty$ due to (5.15). By Hölder's inequality,

$$\begin{aligned} M^2 &\leq \left(\int_{\mathbb{R}^3} \frac{f}{\sqrt{e^{2\phi} + |p|^2}}\, dp\right) \left(\int_{\mathbb{R}^3} f\sqrt{e^{2\phi} + |p|^2}\, dp\right) \\ &\leq \left(\int_{\mathbb{R}^3} \frac{f}{\sqrt{e^{2\phi} + |p|^2}}\, dp\right) \mathcal{E}(t) . \end{aligned} \tag{5.36}$$

Now, by a direct formal computation we have

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \int_{\mathbb{R}^3} \partial_t f\sqrt{e^{2\phi} + |p|^2}\, dp + \dot{\phi}\ddot{\phi} + \int_{\mathbb{R}^3} \frac{f e^{2\phi}\dot{\phi}}{\sqrt{e^{2\phi} + |p|^2}}\, dp \\ &= 3e^{2\phi}\int_{\mathbb{R}^3} f\, dp - \int_{\mathbb{R}^3} \frac{e^{2\phi}|p|^2 f}{\sqrt{e^{2\phi} + |p|^2}}\, dp \\ &\leq 3e^{2\phi}\int_{\mathbb{R}^3} f\, dp , \end{aligned}$$

whence

$$\mathcal{E}(t) \leq \mathcal{E}(0) + 3M\int_0^t e^{2\phi(s)}\, ds. \tag{5.37}$$

The previous inequality holds for the solution constructed in the previous section, as it follows by applying the above formal calculation to the sequence $(f_n, \phi_n)$ and then passing to the (strong) limit as $n \to \infty$. Using (5.37) in (5.36), we arrive at

$$\int_{\mathbb{R}^3} \frac{f}{\sqrt{e^{2\phi} + |p|^2}}\, dp \geq \frac{M^2}{\mathcal{E}(0) + 3M\int_0^t e^{2\phi(s)} ds}.$$

Using the previous inequality yields

$$\ddot{\phi} = -e^{2\phi}\int \frac{f}{\sqrt{e^{2\phi}+|p|^2}}\,dp \le -\frac{M^2 e^{2\phi}}{\mathcal{E}(0)+3M\int_0^t e^{2\phi(s)}ds}$$
$$= -\frac{M}{3}\frac{d}{dt}\log\left[\mathcal{E}(0)+3M\int_0^t e^{2\phi(s)}ds\right],$$

which by an integration in time, the following bound for $\dot{\phi}$ is obtained

$$\dot{\phi}(t) \le \dot{\phi}(0) - \frac{M}{3}\log\left[\mathcal{E}(0)+3M\int_0^t e^{2\phi(s)}ds\right] + \frac{M}{3}\log\mathcal{E}(0). \tag{5.38}$$

If $\dot{\phi}_\infty \ge 0$, then $\dot{\phi}$ is positive for all $t\in[0,\infty)$, since $\dot{\phi}$ is decreasing by the fact $\ddot{\phi}<0$. Hence the right side of (5.38) tends to $-\infty$ as $t\to\infty$ since the integral inside the log function diverges to infinity, which contradicts our initial assumption. Thus $\dot{\phi}_\infty < 0$ must hold. □

The previous lemma easily yields the desired estimates. In fact, since $\dot{\phi}_\infty < 0$ and $\dot{\phi}$ is decreasing, there exists $t_0\ge 0$ such that $\dot{\phi}(t)<\dot{\phi}(t_0)<0$, for all $t\ge t_0$. Hence

$$\phi(t) = \phi(t_0) + \int_{t_0}^t \dot{\phi}(s)\,ds \le \Big(\phi(t_0)+|\dot{\phi}(t_0)|t_0\Big) - |\dot{\phi}(t_0)|t,$$

and therefore, $\phi(t)\le C-\beta t$ holds for some $\beta, C>0$. Using this fact within (5.9), we obtain (5.3). Finally, since $\mathcal{Q}_\phi(t) = e^{2\phi(t)}+e^{\phi(t)}+(\dot{\phi}(t))_+ = e^{2\phi(t)}+e^{\phi(t)}$, for $t\ge t_0$, we have

$$\int_0^\infty \mathcal{Q}_{\phi(t)}\,dt \le C\left(1+\int_{t_0}^\infty e^{-\beta t}\,dt\right) < C,$$

and then, estimates $|p|f, |\nabla_p f|^2 \in L^\infty\big((0,\infty);L^1(\mathbb{R}^3)\big)$ and (5.6) follows from (5.15)–(5.16).

### 5.2.5 Non-vanishing property

Now, we proceed to show that the asymptotic behavior in time of the density function $f$ in the VNFP system is non-trivial. Since $f$ is uniformly bounded in $L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$, estimate (5.4) will follow if we are able to prove the following bound

$$\|f(t)\|_{L^q(\mathbb{R}^3)} \ge C, \quad \text{for some } q\in(1,2). \tag{5.39}$$

The previous lower bound condition is known as the $p, q, r$-Theorem, see [106] for details. To establish (5.39), we first note that for all $R > 0$

$$
\begin{aligned}
0 < M = \int_{\mathbb{R}^3} f\, dp &\leq \int_{|p|\leq R} f\, dp + \frac{1}{R}\int_{|p|\geq R} |p| f\, dp \\
&\leq (4\pi)^{1-\frac{1}{q}} \|f(t)\|_{L^q(\mathbb{R}^3)} R^{3-\frac{3}{q}} + \frac{1}{R}\int_{\mathbb{R}^3} |p| f\, dp.
\end{aligned}
$$

To optimize the previous inequality, we choose

$$
R = \left[ \frac{(4\pi)^{\frac{1}{q}-1} \int_{\mathbb{R}^3} |p| f\, dp}{(3-\frac{3}{q}) \|f(t)\|_{L^q(\mathbb{R}^3)}} \right]^{\frac{q}{4q-3}},
$$

which allows to obtain the estimate

$$
M \leq C \|f(t)\|_{L^q(\mathbb{R}^3)}^{\frac{q}{4q-3}} \left( \int_{\mathbb{R}^3} |p| f\, dp \right)^{\frac{3(q-1)}{4q-3}}.
$$

Since $\int_{\mathbb{R}^3} |p| f\, dp \leq C$, identity (5.39) follows.

### 5.2.6 Uniqueness

The purpose of this section is to prove the uniqueness of solutions for the VNFP system stated in Theorem 5.2.1. Due to the nonlinear character of the system, conditions (5.6) and $\delta > 1/2$ are sufficient to ensure that at most one solution exist. It is unclear if the latter conditions are also necessary, but we hope that when the proof is presented, the difficulty to use weaker conditions can become evident. We will proceed by deriving a homogenous Gronwall's type inequality on the difference of two solutions with the same initial data. For brevity, we limit ourselves to a formal derivation assuming all the regularity of solutions for the forthcoming computations. However, after regularizing with a mollifying test function $\xi \in C_c^\infty((0,T) \times \mathbb{R}^3)$ of the form $\xi(t,p) = \theta(t)\mu(p)$, for an appropriate choice of $\theta$ and $\mu$, one may work with only the proven regularity of solutions and make the proof completely rigorous (an example of an application of this argument can be found for instance in [18]).

Let $\delta > 1/2$ be given and $(f_1, \phi_1)$, $(f_2, \phi_2)$, be two regular solutions of the VNFP system (5.1)–(5.2) with the same initial data. Define $h = f_1 - f_2$ and $\psi = \phi_1 - \phi_2$, for simplicity on our calculations. First, we compute the time evolution of $h$ as follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(1+|p|^2)^\delta h^2 dp &= e^{2\phi_1}\int_{\mathbb{R}^3}(1+|p|^2)^\delta h\partial_{p^i}(D^{ij}[\phi_1]\partial_{p^j}h+p^ih)\,dp\\ &\quad+\int_{\mathbb{R}^3}\partial_{p^i}\left[(e^{2\phi_1}D^{ij}[\phi_1]-e^{2\phi_2}D^{ij}[\phi_2])\partial_{p^j}f_2\right]\\ &\qquad\times(1+|p|^2)^\delta h\,dp\\ &\quad+(e^{2\phi_1}-e^{2\phi_2})\int_{\mathbb{R}^3}(1+|p|^2)^\delta h\partial_{p^i}(p^if_2)\,dp\\ &=I_1+I_2+I_3\,.\end{aligned}$$

Integrating by parts and using the positivity of $D$, the first integral in the above identity satisfies

$$\frac{I_1}{e^{2\phi_1}}\le\frac{3}{2}\int_{\mathbb{R}^3}(1+|p|^2)^\delta h^2\,dp+\delta\int_{\mathbb{R}^3}\partial_{p^j}\left[(1+|p|^2)^{\delta-1}p_iD^{ij}[\phi_1]\right]h^2\,dp\,.$$

Using the properties of $D^{ij}[\phi_1]$, we notice that

$$\begin{aligned}\partial_{p^j}\left[(1+|p|^2)^{\delta-1}p_iD^{ij}[\phi_1]\right]&=\partial_{p^j}\left[(1+|p|^2)^{\delta-1}p_j\sqrt{e^{2\phi_1}+|p|^2}\right]\\ &\le\frac{(\delta-1)|p|^2(1+|p|^2)^{\delta-2}(e^{2\phi_1}+|p|^2)}{\sqrt{e^{2\phi_1}+|p|^2}}\\ &\quad+\frac{4(1+|p|^2)^{\delta-1}(e^{2\phi_1}+|p|^2)}{\sqrt{e^{2\phi_1}+|p|^2}}\\ &\le C_T\frac{(1+|p|^2)^\delta}{e^{\phi_1}}\,,\end{aligned}$$

and as a consequence, we are able to find the following bound

$$I_1\le C_T\|(1+|p|^2)^{\frac{\delta}{2}}(f_1-f_2)(t)\|^2_{L^2(\mathbb{R}^3)}.$$

In order to bound the second integral $I_2$, we require the bounds

$$|e^{2\phi_1}D^{ij}[\phi_1]-e^{2\phi_2}D^{ij}[\phi_2]|\le C_T\sqrt{1+|p|^2}\,|\psi|\,,\tag{5.40}$$

$$\left|\partial_{p^i}\left(e^{2\phi_1}D^{ij}[\phi_1]-e^{2\phi_2}D^{ij}[\phi_2]\right)\right|\le C_T\,|\psi|\,,\tag{5.41}$$

and Hölder's inequality. Then, we see that

$$\begin{aligned}I_2&\le C_T|\psi|\int_{\mathbb{R}^3}(1+|p|^2)^\delta|h|\left(|\nabla_pf_2|+\sqrt{1+|p|^2}|\nabla_p^2f_2|\right)dp\\ &\le C_T|\psi|\left(\int_{\mathbb{R}^3}(1+|p|^2)^\delta h^2\,dp\right)^{1/2}\\ &\quad\times\left[\left(\int_{\mathbb{R}^3}(1+|p|^2)^\delta|\nabla_pf_2|^2\,dp\right)^{1/2}+\left(\int_{\mathbb{R}^3}(1+|p|^2)^{\delta+1}|\nabla_p^2f_2|^2\,dp\right)^{1/2}\right]\\ &\le C_T|\psi|\cdot\|(1+|p|^2)^{\frac{\delta}{2}}h(t)\|_{L^2(\mathbb{R}^3)}\le C_T\left(|\psi|^2+\|(1+|p|^2)^{\frac{\delta}{2}}h(t)\|^2_{L^2(\mathbb{R}^3)}\right)\,,\end{aligned}$$

where we used (5.6) for the term in square brackets. To prove inequalities (5.40) and (5.41), we use (5.35), straightforward estimates and the Mean Value Theorem as follows. First, we calculate and bound

$$\begin{aligned}\partial_{p^i}\left(e^{2\phi_1}D^{ij}[\phi_1]-e^{2\phi_2}D^{ij}[\phi_2]\right) &= (e^{2\phi_1}-e^{2\phi_2})\partial_{p^i}\left(D^{ij}[\phi_1]\right)\\ &\quad + e^{2\phi_2}\partial_{p^i}\left(D^{ij}[\phi_1]-D^{ij}[\phi_2]\right)\\ &\le |e^{2\phi_1}-e^{2\phi_2}|\frac{3|p|}{\sqrt{e^{2\phi_1}+|p|^2}}\\ &\quad + e^{2\phi_2}\left|\frac{3p^j}{\sqrt{e^{2\phi_1}+|p|^2}}-\frac{3p^j}{\sqrt{e^{2\phi_2}+|p|^2}}\right|\\ &\le C|e^{2\phi_1}-e^{2\phi_2}|\,,\end{aligned}$$

where we used

$$\begin{aligned}\left|\frac{1}{\sqrt{e^{2\phi_1}+|p|^2}}-\frac{1}{\sqrt{e^{2\phi_2}+|p|^2}}\right| &\le |e^{2\phi_1}-e^{2\phi_2}|\frac{(\sqrt{e^{2\phi_2}+|p|^2}\sqrt{e^{2\phi_1}+|p|^2})^{-1}}{\sqrt{e^{2\phi_1}+|p|^2}+\sqrt{e^{2\phi_2}+|p|^2}}\\ &\le |e^{2\phi_1}-e^{2\phi_2}|\frac{(|p|\sqrt{e^{2\phi_2}+|p|^2})^{-1}}{\sqrt{e^{2\phi_2}+|p|^2}}\\ &\le |e^{2\phi_1}-e^{2\phi_2}|(|p|e^{2\phi_2})^{-1}\,,\end{aligned}$$

since $\sqrt{e^{2\phi_1}+|p|^2}+\sqrt{e^{2\phi_2}+|p|^2}\ge\sqrt{e^{2\phi_k}+|p|^2}\ge|p|,e^{\phi_k}$, for $k=1,2$, and (5.41) is proved. Next, we consider the function $L(x)=e^{2x}D^{ij}[x]$ and compute

$$L'(x)=2(e^{2x}\delta^{ij}+p^ip^j)\frac{e^{2x}}{p^0}+\left[2\delta^{ij}(p^0)^2-(e^{2x}\delta^{ij}+p^ip^j)\right]\frac{e^{4x}}{(p^0)^3}\,.$$

Thus, for all bounded $x$

$$|L'(x)| \quad\le\quad C(p^0)^2\frac{e^{2x}}{p^0}+C(p^0)^2\frac{e^{4x}}{(p^0)^3}\le Cp^0e^{2x}\le C(1+|p|)\,.$$

For the last integral,

$$\begin{aligned}I_3 &\le |e^{2\phi_1}-e^{2\phi_2}|\int_{\mathbb{R}^3}(1+|p|^2)^\delta(3|h||f_2|+|h||p||\nabla f_2|)\,dp\\ &\le C_T|\psi|\|(1+|p|^2)^{\frac{\delta}{2}}h(t)\|_{L^2(\mathbb{R}^3)}\\ &\quad\times\left(\|(1+|p|^2)^{\frac{\delta}{2}}f_2(t)\|_{L^2(\mathbb{R}^3)}+\|(1+|p|^2)^{\frac{\delta+1}{2}}\nabla f_2(t)\|_{L^2(\mathbb{R}^3)}\right)\\ &\le C_T\left(|\psi|^2+\|(1+|p|^2)^{\frac{\delta}{2}}h(t)\|^2_{L^2(\mathbb{R}^3)}\right).\end{aligned}$$

Combining $I_1$, $I_2$ and $I_3$, we have the bound

$$\|(1+|p|^2)^{\frac{\delta}{2}}h(t)\|^2_{L^2(\mathbb{R}^3)}\le C_T\left(\int_0^t\|(1+|p|^2)^{\frac{\delta}{2}}h(s)\|^2_{L^2(\mathbb{R}^3)}\,ds+\|\psi\|^2_{L^\infty(0,t)}\right).$$

Recalling the definitions of $h$, $\psi$, and using Gronwall's inequality in the previous inequality, we obtain

$$\|(1+|p|^2)^{\frac{\delta}{2}}(f_1-f_2)(t)\|_{L^2(\mathbb{R}^3)} \le C_T\|\phi_1-\phi_2\|_{L^\infty((0,t))}. \tag{5.42}$$

Finally, we compute the time evolution of the difference $\phi_1-\phi_2$ and integrate it twice to obtain

$$\begin{aligned}
\phi_1-\phi_2 &= -\int_0^t\int_0^s\int_{\mathbb{R}^3}\left(\frac{f_1e^{2\phi_1}}{\sqrt{e^{2\phi_1}+|p|^2}}-\frac{f_2e^{2\phi_2}}{\sqrt{e^{2\phi_2}+|p|^2}}\right)dp\,d\tau\,ds\\
&= -\int_0^t\int_0^s\int_{\mathbb{R}^3} f_1\left(\frac{e^{2\phi_1}}{\sqrt{e^{2\phi_1}+|p|^2}}-\frac{e^{2\phi_2}}{\sqrt{e^{2\phi_2}+|p|^2}}\right)dp\,d\tau\,ds\\
&\quad -\int_0^t\int_0^s\int_{\mathbb{R}^3}\frac{e^{2\phi_2}}{\sqrt{e^{2\phi_2}+|p|^2}}(f_1-f_2)\,dp\,d\tau\,ds\\
&= I_4+I_5.
\end{aligned}$$

In the first integral, we simply use the Mean Value Theorem so that

$$\left|\frac{e^{2\phi_1}}{\sqrt{e^{2\phi_1}+|p|^2}}-\frac{e^{2\phi_2}}{\sqrt{e^{2\phi_2}+|p|^2}}\right| \le C_T|\phi_1-\phi_2|.$$

Next, we use the fact $f\in L^\infty((0,T);L^1(\mathbb{R}^3))$ and see that we are able to estimate $I_4$ as follows

$$I_4 \le C_T\int_0^t \|\phi_1-\phi_2\|_{L^\infty((0,s))}\,ds.$$

For $I_5$, we use Hölder's inequality, so that

$$\begin{aligned}
\int_{\mathbb{R}^3}\frac{h}{\sqrt{e^{2\phi_2}+|p|^2}}\,dp &= \int_{\mathbb{R}^3}(e^{2\phi_2}+|p|^2)^{-\frac{1+\delta}{2}}\left(h(e^{2\phi_2}+|p|^2)^{\frac{\delta}{2}}\right)dp\\
&\le \left(\int_{\mathbb{R}^3}(e^{2\phi_2}+|p|^2)^{-(1+\delta)}\,dp\right)^{\frac{1}{2}}\|(e^{2\phi_2}+|p|^2)^{\frac{\delta}{2}}h(t)\|_{L^2}\\
&\le C_T\|(1+|p|^2)^{\frac{\delta}{2}}h(t)\|_{L^2(\mathbb{R}^3)},
\end{aligned}$$

where we used $\delta>1/2$ in order to have a finite term in the second line. Hence, recalling the definition of $\psi=\phi_1-\phi_2$ and collecting the estimates obtained for $I_4$ and $I_5$, we find the following bound

$$|\psi(t)| \le C_T\int_0^t\left(\|\psi\|_{L^\infty((0,s))}+\sup_{\tau\in(0,s)}\|(1+|p|^2)^{\frac{\delta}{2}}h(\tau)\|_{L^2(\mathbb{R}^3)}\right)ds. \tag{5.43}$$

Finally, using (5.42) within (5.43) we have that

$$\|\phi_1-\phi_2\|_{L^\infty((0,t))} \le C_T\int_0^t\|\phi_1-\phi_2\|_{L^\infty((0,s))}\,ds\,,$$

holds for all $t\in[0,T)$ and conclude that $\phi_1=\phi_2$ and $f_1=f_2$ a.e. on $[0,T]\times\mathbb{R}^3$, for all $T>0$ again by Gronwall's inequality.

## 5.3 The ultra-relativistic case

In this section, we devote our attention to the long time behavior of solutions of the ultra-relativistic Fokker-Planck equation associated to the relativistic model considered in the previous section. The main properties of the gravitational potential $\phi$ that are required in the forthcoming analysis are smoothness of $\phi$ and

$$\lim_{t\to+\infty}\phi(t)=-\infty\,,\qquad \int_0^\infty e^{2\phi(t)}\,dt<\infty\,,$$

which also holds for solutions of the VNFP system. In order to achieve this goal, we first consider a reduced equation which arises by setting $\phi\equiv-\infty$, or $e^{2\phi}\equiv 0$ within the diffusion matrix $D[\phi]$. This is motivated by the previous result in which we found that $e^{2\phi(t)}\to 0$ as $t\to\infty$ for solutions of VNFP, and hence one expects the asymptotic behavior of the density $f$ to mimic that of a solution to the reduced equation. Unfortunately, we are not able to prove this conjecture, but we think the contents in this section might lead to obtain the result in the future.

We begin this section by investigating the existence problem of solutions for the ultra-relativistic Fokker-Planck equation (URFP)

$$\partial_t g=\partial_{p^i}\left(D^{ij}_\infty\partial_{p^j}g\right),\quad t>0,\ p\in\mathbb{R}^3, \tag{5.44}$$

where

$$D^{ij}_\infty=\lim_{\phi\to-\infty}D^{ij}[\phi]=\lim_{\phi\to-\infty}\frac{e^{2\phi}\delta^{ij}+p^ip^j}{\sqrt{e^{2\phi}+|p|^2}}=\frac{p^ip^j}{|p|}.$$

We will see that for any given initial datum in an adequate functional space, equation (5.44) launches a unique solution. Before we accurately state and prove this result, we prefer to roughly justify why this equation is an ultra-relativistic model for the relativistic Fokker-Planck equation in which the diffusion matrix is given by $D^{ij}[\phi]$ in (5.44) instead of $D^{ij}_\infty$, i.e.,

$$\partial_t g=\partial_{p^i}\left(D^{ij}[\phi]\partial_{p^j}g\right),\quad t>0,\ p\in\mathbb{R}^3. \tag{5.45}$$

Given a particle with mass $m$, we denote by $c$ the speed of light and its relativistic energy by $E=\sqrt{(pc)^2+(mc^2)^2}$, where $pc$ is the momentum of the particle. We say that the particle is ultra-relativistic when its relativistic energy can be approximated by its momentum, i.e., $E\sim pc$. The latter can occur when either the mass is very small in comparison with its momentum or $pc\gg mc^2$. In our present situation, the relativistic microscopic energy $p^0=\sqrt{e^{2\phi}+|p|^2}$ approaches $|p|$ as time goes to infinity, where we have set $m=c=1$. From a formal perspective, one expects that solutions of this equation are similar to the ones of relativistic Fokker-Planck equation (5.45) for large times due to the limiting behavior of the matrix $D$. Therefore, giving an explicit representation of solutions for equation (5.44) could be useful to compare both models. In fact, we will see that this representation can be written in terms of the $\alpha$th modified Bessel function of the first kind $\mathcal{I}_\alpha[x]$, see [1, Eq. 9.6.19, pag. 376], defined by

$$\mathcal{I}_\alpha[x] = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos(\alpha\theta)\,d\theta, \quad \alpha\in\mathbb{N}. \tag{5.46}$$

Now, we are ready to state and prove our first result of this section which ensures the existence of a unique solution for equation (5.44) under suitable conditions:

**Proposition 5.3.1.** *Let $g_{\text{in}}\in L^1(\mathbb{R}^3)$ be given and expressed in spherical coordinates as $g_{\text{in}} = \mathfrak{g}_{\text{in}}(r,\omega)$, where $r>0$ and $\omega = p/r\in S^2$. Then, there exists a unique global solution $g$ of (5.44) which satisfies $g(0,p) = g_{\text{in}}(p)$ and $g\in L^\infty((0,\infty);L^1(\mathbb{R}^3))$, where*

$$\mathfrak{g}(t,r,\omega) = \frac{e^{-\frac{r}{t}}}{tr}\int_0^\infty \mathfrak{g}_{\text{in}}(z,\omega) z e^{-\frac{z}{t}}\,\mathcal{I}_2\left[2\frac{\sqrt{rz}}{t}\right]dz, \tag{5.47}$$

*and $\mathcal{I}_2[x]$ is given by (5.46). Moreover, if $g_{\text{in}}\geq 0$ and $g_{\text{in}}\in L^\gamma(\mathbb{R}^3)$, then $g(t,p) = \mathfrak{g}(t,r,\omega)\geq 0$ and $g(t,\cdot)\in L^\gamma(\mathbb{R}^3)$. Finally, the following estimate holds*

$$\|g(t)\|_{L^\gamma(\mathbb{R}^3)} \leq \|g_{\text{in}}\|_{L^\gamma(\mathbb{R}^3)}, \tag{5.48}$$

*for any $\gamma\in[1,\infty]$, with equality for $\gamma = 1$.*

*Proof.* First, let us consider smooth initial data, since we are able to use a standard approximation procedure to obtain the general result. Next, we derive some properties of (5.44). Notice that the operator

$$\mathcal{L}u = \partial_{p^i}\left(\frac{p^ip^j}{|p|}\partial_{p^j}u\right) \tag{5.49}$$

is purely radial, i.e., it is invariant under rotations. In order to prove this, we consider an orthogonal matrix $Q = [q^{ij}]$ and define the functions $v$ and $w$ which are related by the identity $v(p) = w(z)$, where $z = Qp$. By the chain rule, we can compute the following derivatives

$$\partial_{p^i}v(t,p) = q^k{}_i\partial_{z^k}w(t,Qp),$$

and combining this fact with $|z| = |Qp| = |p|$, we find that

$$\begin{aligned}\mathcal{L}v &= 3\frac{p^i}{|p|}\partial_{p^i}v + \frac{p^ip^j}{|p|}\partial_{p^i}\partial_{p^j}v = 3\frac{q^k{}_ip^i}{|p|}\partial_{z^k}w + \frac{q^k{}_ip^iq^l{}_jp^j}{|p|}\partial_{z^k}\partial_{z^l}w\\ &= 3\frac{z^k}{|z|}\partial_{z^k}w + \frac{z^kz^l}{|z|}\partial_{z^k}\partial_{z^l}w = \mathcal{L}w.\end{aligned}$$

The previous argument implies that the angular variables can be treated as constant parameters, i.e., fixing $\omega \in S^2$ and defining $v^\omega_{\text{in}}(r) = \mathfrak{g}_{\text{in}}(r,\omega)$, it is enough to find a solution of equation (5.49) in terms of $v(p) = v^\omega(t,r)$. To achieve this purpose, we take into account the following quantities

$$\frac{\partial r}{\partial p^j} = \frac{p_j}{r}, \quad \frac{\partial v}{\partial p^j} = \partial_r v^\omega \frac{p_j}{r}, \quad \frac{\partial^2 v}{\partial p^i \partial p^j} = \partial_r^2 v^\omega \frac{p_i p_j}{r^2} + \partial_r v^\omega \left( \frac{\delta_{ij}}{r} - \frac{p_i p_j}{r^3} \right),$$

and substitute in (5.49) to see that

$$\frac{p^i p^j}{|p|} \partial_{p^i} \partial_{p^j} v + 3 \frac{p^i}{|p|} \partial_{p^i} v = \frac{p^i p^j}{r} \left[ \partial_r^2 v^\omega \frac{p_i p_j}{r^2} + \partial_r v^\omega \left( \frac{\delta_{ij}}{r} - \frac{p^i p^j}{r^3} \right) \right] + 3 \frac{p^i p_i}{r^2} \partial_r v^\omega,$$

which implies that $v^\omega(t,r)$ is solution of

$$\partial_t v = r \partial_r^2 v + 3 \partial_r v, \quad v^\omega(0,r) = v^\omega_{\text{in}}(r). \tag{5.50}$$

Therefore, a solution $g(t,p) = \mathfrak{g}(t,r,\omega)$ of our original problem is given by $\mathfrak{g}(t,r,\omega) = v^\omega(t,r)$.

Next, we derive a representation formula for solutions of the spherically symmetric heat equation in six dimensions which will automatically imply the existence and uniqueness of solutions for equation (5.44). Let $u(t,x)$ be the solution of the Cauchy problem

$$\begin{aligned} \partial_t u &= \Delta u, \quad t > 0, \ x \in \mathbb{R}^6, \\ u(0,x) &= u_{\text{in}}(x), \quad x \in \mathbb{R}^6. \end{aligned}$$

Solutions of the previous system are explicitly given by

$$u(t,x) = \frac{1}{(4\pi t)^3} \int_{\mathbb{R}^6} e^{-\frac{|x-y|^2}{4t}} u_{\text{in}}(y)\, dy. \tag{5.51}$$

Recall that spherically symmetric solutions of the heat equation in six dimensions, i.e., $u(t,x) = \mathfrak{u}(t,w)$ with $w = |x|$, satisfy

$$\partial_t \mathfrak{u} = \partial_w^2 \mathfrak{u} + \frac{5}{w} \partial_w \mathfrak{u}. \tag{5.52}$$

Actually, the reason to consider this solution is simple: Observe that if $\mathfrak{u}$ solves (5.52), then $v^\omega(t,r) = \mathfrak{u}(t, 2\sqrt{r})$ solves (5.50). In fact, since $\partial_t \mathfrak{g} = \partial_t \mathfrak{u}$, $\partial_r \mathfrak{g} = 2w^{-1} \partial_w \mathfrak{u}$ and $\partial_r^2 \mathfrak{g} = 4w^{-2} \partial_w^2 \mathfrak{u} - 4w^{-3} \partial_w \mathfrak{u}$, we see that

$$\begin{aligned} \partial_t \mathfrak{u} = r \partial_r^2 \mathfrak{g} + 3 \partial_r \mathfrak{g} &= \frac{w^2}{4} \left( \frac{4}{w^2} \partial_w^2 \mathfrak{g} - \frac{4}{w^3} \partial_w \mathfrak{u} \right) + \frac{6}{w} \partial_w \mathfrak{u} \\ &= \partial_w^2 \mathfrak{u} + \frac{5}{w} \partial_w \mathfrak{u} = \partial_t \mathfrak{u}\,. \end{aligned}$$

Returning to the representation formula problem, let $u(0,x) = \mathfrak{u}_{\text{in}}(w)$ be spherically symmetric, with $|x| = w$ and passing to hyperspherical coordinates in the integral on the right side of (5.51), we obtain

$$\begin{aligned} \mathfrak{u}(t,w) &= \frac{e^{-\frac{w^2}{4t}}}{(4\pi t)^3} \int_{\mathbb{R}^6} e^{-\frac{(|y|^2 - 2y\cdot x)}{4t}} \mathfrak{u}_{\text{in}}(|y|)\, dy \\ &= \frac{8\pi^2}{3} \frac{e^{-\frac{w^2}{4t}}}{(4\pi t)^3} \int_0^\infty \mathfrak{u}_{\text{in}}(s)\, e^{-\frac{s^2}{4t}} s^5 \int_0^\pi \exp\left(\frac{ws\cos\theta}{2t}\right) \sin^4\theta \, d\theta\, ds. \end{aligned} \tag{5.53}$$

Evaluating the angular integral gives

$$\begin{aligned} \int_0^\pi \exp\left(\frac{ws\cos\theta}{2t}\right)\sin^4\theta\, d\theta &= \frac{3t}{ws}\int_0^\pi \exp\left(\frac{ws\cos\theta}{2t}\right)\sin\theta\sin(2\theta)\, d\theta \\ &= 3\left(\frac{2t}{ws}\right)^2 \int_0^\pi \exp\left(\frac{ws\cos\theta}{2t}\right)\cos(2\theta)\, d\theta \\ &= 12\pi\left(\frac{t}{ws}\right)^2 \mathcal{I}_2\left[\frac{ws}{2t}\right], \end{aligned}$$

where $\mathcal{I}_2[x]$ is given by (5.46) for $\alpha = 2$. Substituting this expression into (5.53), we obtain

$$\mathfrak{u}(t,w) = \frac{1}{2}\frac{e^{-\frac{w^2}{4t}}}{w^2 t}\int_0^\infty \mathfrak{u}_{\text{in}}(s) e^{-\frac{s^2}{4t}} s^3\, \mathcal{I}_2\left[\frac{ws}{2t}\right] ds\,. \tag{5.54}$$

Now, we can proceed to prove formula (5.47). By making the substitution $w = 2\sqrt{r}$ and the change of variables $s = 2\sqrt{z}$ in (5.54), the solution of the 3-dimensional ultra-relativistic Fokker-Planck equation (5.44) with initial datum $g_{\text{in}}(p) = v^{\omega}_{\text{in}}(z)$ is given by (5.47) as claimed.

Finally, we prove assertion (5.48). Observe that $g(t,p) = \mathfrak{g}(t,r,\omega)$ given by (5.47) belongs to $L^\infty((0,\infty); L^1(\mathbb{R}^3))$ for initial data in $L^1(\mathbb{R}^3)$. The latter fact is equivalent to prove that $g$ preserves mass, i.e., we have equality in (5.48). this property is obtained from the following

$$\begin{aligned} \int_{\mathbb{R}^3} g(t,p)\, dp &= \int_{S^2}\int_0^\infty r^2 \mathfrak{g}(t,r,\omega)\, dr\, d\omega \\ &= \frac{1}{t}\int_{S^2}\int_0^\infty z\, \mathfrak{g}_{\text{in}}(z,\omega)\left(\int_0^\infty r e^{-\frac{r}{t}-\frac{z}{t}} \mathcal{I}_2\left[2\frac{\sqrt{rz}}{t}\right] dr\right) dz\, d\omega\,. \end{aligned}$$

The integral within round brackets equals $e^{z/t} tz$ as a consequence of its Gaussian character. Thus

$$\int_{\mathbb{R}^3} g(t,p)\, dp = \int_{S^2}\int_0^\infty z^2 \mathfrak{g}_{\text{in}}(z,\omega)\, dz\, d\omega,$$

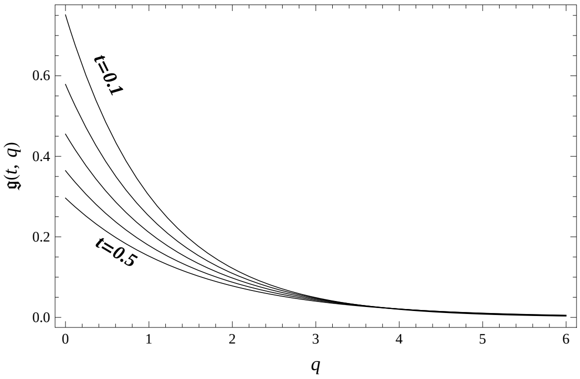


**Figure 5.1:** Numerical depiction of a spherically symmetric solution $\mathfrak{g}(t,q)$ of the ultra-relativistic Fokker-Planck equation (5.44) for constant values of $t$, from $t = 0.1$ (top curve) until $t = 0.5$ (bottom curve), and initial datum $\mathfrak{g}_{\text{in}}(q) = e^{-q}$.

and $\|g(t)\|_{L^1(\mathbb{R}^3)} = \|g_{\text{in}}\|_{L^1(\mathbb{R}^3)}$ follows. Similarly, it is easy to prove that $g(t,p)$ given by (5.47) satisfies estimate (5.48) by using again the Gaussian character of the kernel

$$K(r,z) = e^{-\frac{r}{t}-\frac{z}{t}}\mathcal{I}_2\left[2\frac{\sqrt{rz}}{t}\right],$$

and Hölder's inequality as in the case of equation (5.51) for any dimension. □

As a direct consequence of identity (5.47), solutions of equation (5.44) are spherically symmetric when the initial data also possess the previous property. Figure 5.1 shows a numerical simulation for certain values of $t$ for a spherically symmetric solution with initial datum $\mathfrak{g}_{\text{in}}(q) = e^{-q}$. In particular, we will see that Proposition 5.3.2 is an extension for solutions within this class of a modified version of equation (5.47) (with or without a drift term). In fact, the preceding result already allows to answer the analogous question for the ultra-relativistic system with a scalar field and friction term using a simple change of variables.

**Corollary 5.3.1.** *Assume $h_{\text{in}}(p) \in L^1(\mathbb{R}^3)$, with $h_{\text{in}}(p) \geq 0$. Solutions of*

$$\text{(5.55)} \qquad \begin{aligned} &\partial_t h = e^{2\phi}\partial_{p^i}\left(\frac{p^ip^j}{|p|}\partial_{p^j}h + p^ih\right), \quad t>0,\ p\in\mathbb{R}^3,\\ &h(0,p) = h_{\text{in}}(p), \end{aligned}$$

*are given by*

$$\text{(5.56)} \qquad h(t,p) = e^{3\tau(t)}g(\sigma(t), q(t,p)),$$

*where*

$$\tau(t) = \int_0^t e^{2\phi(s)}\, ds, \quad \sigma(t) = e^{\tau(t)} - 1, \quad q(t,p) = e^{\tau(t)} p,$$

*and $g(\sigma, q)$ is the corresponding solution of equation (5.44) with the same initial datum. Moreover, we have the representation formula*

$$h(t,p) = \frac{e^{3\tau(t) - \frac{r(t)}{\sigma(t)}}}{\sigma(t) r(t)} \int_0^\infty \mathfrak{h}_{\text{in}}(z,\omega) z e^{-\frac{z}{\sigma(t)}}\, \mathcal{I}_2 \left[ 2\frac{\sqrt{r(t)z}}{\sigma(t)} \right] dz\,, \tag{5.57}$$

*where $r(t) = e^{\tau(t)}|p|$.*

*Proof.* The result follows by using an appropriate integral factor and by rescaling time and momentum to account the friction term and the gravitational potential, respectively. Let

$$\tau(t) = \int_0^t e^{2\phi(s)}\, ds, \quad \sigma(t) = e^{\tau(t)} - 1 = \int_0^t e^{\tau(s)} e^{2\phi(s)}\, ds, \quad q(t,p) = e^{\tau(t)} p\,.$$

and make the change of variables $h(t,p) = e^{3\tau(t)} \tilde{h}(t, q(t,p))$. Then, both sides of (5.55) transforms into

$$\partial_t h = 3e^{3\tau} e^{2\phi} \tilde{h} + e^{3\tau} \left( \partial_t \tilde{h} + e^{2\phi} q \cdot \nabla_q \tilde{h} \right),$$

and

$$\begin{aligned} e^{2\phi} \partial_{p^i} \left( p^i h + \frac{p^i p^j}{|p|} \partial_{p^j} h \right) &= 3e^{3\tau} e^{2\phi} \tilde{h} + e^{3\tau} e^{2\phi} q \cdot \nabla_q \tilde{h} \\ &\quad + e^{3\tau} e^{2\phi} \partial_{p^i} \left( \frac{q^i q^j}{|q|} \partial_{q^j} \tilde{h} \right) \\ &= \partial_t h - e^{3\tau} \partial_t \tilde{h} + e^{3\tau} e^{2\phi} e^{\tau} \partial_{q^i} \left( \frac{q^i q^j}{|q|} \partial_{q^j} \tilde{h} \right), \end{aligned}$$

where we used the fact $\nabla_p h = e^\tau \nabla_q \tilde{h}$. Next, we use definition (5.49) and $d\sigma/dt = e^{2\phi} e^\tau$ to see that the resulting equation for $\tilde{h}$ is

$$\partial_t \tilde{h} = (\sigma)' \partial_{q^i} \mathcal{L} \tilde{h}\,.$$

Once again, we apply another change of variables in time $\sigma = e^\tau - 1$ and $\tilde{h}(t,q) = g(\sigma, q)$, since $\partial_t \tilde{h} \partial_\sigma t = \partial_\sigma g$ and hence, the unknown function $g(\sigma, q)$ satisfies the parabolic equation

$$\partial_\sigma g = \partial_{q^i} \left( \frac{q^i q^j}{|q|} \partial_{q^j} g \right),$$

and any solution $h$ must be of the form (5.56). Finally, the representation of solutions follows in view of Proposition 5.3.1. □

It is clear that following similar lines as in the previous proof can be used to obtain a representation formula for the corresponding URFP equation without friction. In fact, we have that

$$h(t,p) = g(\tau(t),p),$$

is the solution of equation

$$\partial_t h = e^{2\phi}\partial_{p^i}\left(\frac{p^i p^j}{|p|}\partial_{p^j} h\right), \tag{5.58}$$

with non-negative initial datum $h(0,p) = h_{\text{in}}(p) \in L^1(\mathbb{R}^3)$. Surprisingly, this model exhibits a nontrivial asymptotic profile as $t \to \infty$. Basically, the factor $e^{2\phi}$ prevents solutions to vanish in the absence of friction as a result of its integrability in $(0,\infty)$, since this quantity is part of the time rescaling $\tau$ in the previous proposition. Then, the resulting self-similar solution can be interpreted as an intermediate stage. More precisely, the $L^1$ norm of $e^{2\phi}$ is finite, but it could be large enough so that solutions of (5.58) are close to zero in the limit. In the case of equation (5.55), see Proposition (5.3.2) below, the profile could be close to the ultra-relativistic Jüttner distribution function $e^{-|p|}$[1] depending on the size of $\left\|e^{2\phi}\right\|_{L^1}$.

Before stating and proving our next result, it is convenient to introduce some notation. Given $\phi = \phi(t)$ such that $e^{2\phi} \in L^1((0,\infty))$, and a function $u = u(p)$ with the representation $u(p) = \mathfrak{u}(r,\omega)$ in spherical coordinates, we define the operator

$$\mathcal{T}_\phi[u] = \frac{e^{3T-\frac{R}{S}}}{SR}\int_0^\infty \mathfrak{u}(z,\omega) z e^{-\frac{z}{S}}\,\mathcal{I}_2\left[2\frac{\sqrt{Rz}}{S}\right] dz\,, \tag{5.59a}$$

where

$$T = \|e^{2\phi}\|_{L^1(\mathbb{R}^3)}, \quad S = e^T - 1 \quad R = e^T r. \tag{5.59b}$$

Also, we recall some bounds satisfied by the modified Bessel function (5.46), see eqns. (9.6.7), (9.7.1), (9.6.26), pags. 375, 377, in [1], [102]: for $\alpha \in \mathbb{N}$, there exists $C > 0$ such that

$$\mathcal{I}_\alpha[x] \le C x^\alpha, \qquad \text{for } x \le 1, \tag{5.60}$$

$$\mathcal{I}_\alpha[x] \le C x^{-1/2} e^x \le C e^x, \qquad \text{for } x \ge 1, \tag{5.61}$$

$$\mathcal{I}'_\alpha[x] = \mathcal{I}_{\alpha-1}[x] - \frac{\alpha}{x}\mathcal{I}_\alpha[x]. \tag{5.62}$$

Given a solution $h$ of (5.55), we want to show that the long time asymptotic profile of $h$ is exactly (5.59). To avoid the need of technical estimates on Bessel functions in Lebesgue spaces, we choose to study the limit in the $L^\infty$ norm. Now, we are in position to prove the main result of this section.

[1] This function is also known in the literature as the *Laplace* distribution function.

**Proposition 5.3.2.** *Let $\phi = \phi(t)$ be such that $e^{2\phi} \in L^1((0,\infty))$ and $h(t,p)$ be the positive solution of (5.55) with initial datum $h(0,p) = \mathfrak{h}_{\rm in}(r,\omega)$, cf. Corollary 5.3.1. We assume that the spherically symmetric function $\bar{\mathfrak{h}}_{\rm in}(r) = \sup_{\omega\in S^2} \mathfrak{h}_{\rm in}(r,\omega)$ satisfies*

$$\int_0^\infty r(1+r)^2 \bar{\mathfrak{h}}_{\rm in}(r)\, dr < \infty \,. \tag{5.63}$$

*Then, for all $t > 1$,*

$$\|h(t) - \mathcal{T}_\phi[h_{\rm in}]\|_{L^\infty(\mathbb{R}^3)} \le C \int_t^\infty e^{2\phi(s)}\, ds, \tag{5.64}$$

*where $C$ depends on $\tau(1)$ and $T$.*

*Proof.* By (5.57) and the definition of $\mathcal{T}$ (5.59), we have

$$\begin{aligned} h - \mathcal{T}_\phi[h_{\rm in}] &= \frac{e^{2\tau(t)}}{\sigma(t) r} \int_0^\infty z\, \mathfrak{h}_{\rm in}(z,\omega)\, \mathcal{I}_2\left[2\frac{\sqrt{r(t)z}}{\sigma(t)}\right] e^{-\frac{r(t)+z}{\sigma(t)}}\, dz \\ &\quad - \frac{e^{2T}}{Sr} \int_0^\infty z\, \mathfrak{h}_{\rm in}(z,\omega)\, \mathcal{I}_2\left[2\frac{\sqrt{Rz}}{S}\right] e^{-\frac{R+z}{S}}\, dz \\ &= \frac{1}{r}\int_0^\infty z\mathfrak{h}_{\rm in}(z,\omega)\, \left[H(\tau(t),r,z) - H(T,r,z)\right] dz, \end{aligned}$$

where

$$H(\tau,r,z) := (\bar{\sigma}(\tau))^{-1} e^{2\tau - \frac{e^\tau r + z}{\bar{\sigma}(\tau)}} \mathcal{I}_2\left[\frac{2\sqrt{e^\tau}}{\bar{\sigma}(\tau)}\sqrt{rz}\right], \quad \bar{\sigma}(\tau) = e^\tau - 1\,.$$

Since $t > 1$, and $\tau(t)$ is increasing, we have $\tau_1 := \tau(1) < \tau(t) < T$. By the Mean Value Theorem, we estimate

$$|h - \mathcal{T}_\phi[h_{\rm in}]| \le (T - \tau(t))\, \frac{1}{r}\int_0^\infty \mathfrak{h}_{\rm in}(z,\omega) \sup_{\tau\in(\tau_1,T)} |\partial_\tau H(\tau,q,z)| z\, dz,$$

with

$$T - \tau(t) = \int_t^\infty e^{2\phi(s)} ds.$$

We prove below that

$$\frac{1}{r}\sup_{\tau\in(\tau_1,T)} |\partial_\tau H(\tau,r,z)| \le C(1+z)^2. \tag{5.65}$$

Hence, assumption (5.63) allows to obtain

$$|h - \mathcal{T}_\phi[h_{\rm in}]| \le (T - \tau(t)) \int_0^\infty \bar{\mathfrak{h}}_{\rm in}(z) z(1+z)^2\, dz \le C\int_t^\infty e^{2\phi(s)} ds,$$

which proves (5.64). It remains to establish (5.65). First, we consider the following quantities

$$\gamma(\tau) = \frac{d}{d\tau}\frac{e^{\tau/2}}{\bar{\sigma}(\tau)} = \frac{e^{\tau/2}}{2\bar{\sigma}(\tau)} - \frac{e^{\tau/2}e^{\tau}}{\bar{\sigma}^2(\tau)} = -\frac{e^{\tau/2}}{2\bar{\sigma}^2(\tau)}\left(e^{\tau}+1\right)$$

$$\beta(\tau) = \frac{\bar{\sigma}(\tau)}{e^{2\tau-\frac{e^{\tau}r+z}{\bar{\sigma}(\tau)}}}\frac{d}{d\tau}\frac{e^{2\tau-\frac{e^{\tau}r+z}{\bar{\sigma}(\tau)}}}{\bar{\sigma}(\tau)} = 2 - \frac{e^{\tau}}{\bar{\sigma}(\tau)}\left[r - \frac{e^{\tau}r+z}{\bar{\sigma}(\tau)}+1\right]$$
$$= \frac{2\bar{\sigma}^2(\tau) - e^{\tau}\bar{\sigma}(\tau) + e^{\tau}(r+z)}{\bar{\sigma}^2(\tau)}.$$

Using the recurrence relation (5.62) and the previous calculations, we find

$$\frac{\bar{\sigma}(\tau)\partial_\tau H}{e^{2\tau-\frac{e^{\tau}r+z}{\bar{\sigma}(\tau)}}} = \beta(\tau)\mathcal{I}_2\left[\frac{2\sqrt{e^{\tau}}}{\bar{\sigma}(\tau)}\sqrt{rz}\right] + 2\sqrt{rz}\gamma(\tau)\mathcal{I}_2'\left[\frac{2\sqrt{e^{\tau}}}{\bar{\sigma}(\tau)}\sqrt{rz}\right]$$
$$= \left(\beta(\tau) + \frac{e^{\tau}+1}{\bar{\sigma}(\tau)}\right)\mathcal{I}_2\left[\frac{2\sqrt{e^{\tau}}}{\bar{\sigma}(\tau)}\sqrt{rz}\right] + 2\sqrt{rz}\gamma(\tau)\mathcal{I}_1\left[\frac{2\sqrt{e^{\tau}}}{\bar{\sigma}(\tau)}\sqrt{rz}\right].$$

Notice that for $\tau \in (\tau_1, T)$ and $2\sqrt{e^{\tau}rz} \le \bar{\sigma}(\tau)$, the bound (5.60) satisfied by the modified Bessel function combined with the following identities

$$\gamma(\tau)\frac{2\sqrt{e^{\tau}}}{\bar{\sigma}(\tau)} = -(e^{\tau}+1)\frac{e^{\tau}}{\bar{\sigma}^3(\tau)}, \qquad \beta(\tau) + \frac{e^{\tau}+1}{\bar{\sigma}(\tau)} \le \frac{e^{\tau}}{\bar{\sigma}(\tau)}\left(2 + \frac{r+z}{\bar{\sigma}(\tau)}\right),$$

allows to deduce the estimate

$$\frac{1}{r}|\partial_\tau H| \le C\frac{e^{4\tau}}{\bar{\sigma}^4}z\left(1 + \frac{r+z}{\bar{\sigma}(\tau)}\right)\exp\left\{-\frac{e^{\tau}r+z}{\bar{\sigma}(\tau)}\right\} \le Cz(1+r+z).$$

For $2\sqrt{e^{\tau}rz} \ge \bar{\sigma}(\tau)$, the bound (5.61) implies that

$$\frac{1}{r}|\partial_\tau H| \le C\frac{e^{2\tau}}{\bar{\sigma}^2}\left(\frac{1}{r} + \frac{1+z/r}{\bar{\sigma}(\tau)} + \frac{\sqrt{z}/\sqrt{r}}{\bar{\sigma}(\tau)}\right)\exp\left\{-\frac{(\sqrt{e^{\tau}r}-\sqrt{z})^2}{\sigma(\tau)}\right\}$$
$$\le C(1+z)^2,$$

for $\tau \in (\tau_1, T)$. Therefore, we have

$$\sup_{\tau\in(\tau_1,T)}\frac{1}{r}|\partial_\tau H(\tau,r,z)| \le C(1+z)^2,$$

as desired. This completes the proof of the proposition. □

We remark that condition (5.63) is slightly stronger than requiring a bounded first moment of $h_{\text{in}}(p)$ in $L^1(\mathbb{R}^3)$. In particular, if $h_{\text{in}}$ is spherically symmetric, then (5.63) is automatically implied by any initial data $h_{\text{in}} \in X$ satisfying (5.5). Also, the analogue result holds for solutions of equation (5.58) by following similar lines as in the previous result, see [4].

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