Universidad de Granada

Facultad de Ciencias

Quantization of categories: weak bialgebras and weak multiplier bialgebras

Thesis submitted by Esperanza López Centella
for the degree of Doctor of Philosophy

Directors of the Thesis: José Gómez Torrecillas and Gabriella Böhm

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Departamento de Álgebra

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Cuantización de categorías: biálgebras débiles y biálgebras multiplicadoras débiles

Tesis presentada por Esperanza López Centella
para obtener el grado de Doctora en Matemáticas

Directores de la Tesis: José Gómez Torrecillas y Gabriella Böhm

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Presentación

La presente memoria de tesis doctoral es presentada por D.ª Esperanza López Centella para optar al título de Doctora en Matemáticas por la Universidad de Granada, en el marco del Programa Oficial de Doctorado en Matemáticas (D24 56 1; RD 1393/2007). Se realiza por tanto de acuerdo con las normas que regulan las ensenãnzas oficiales de Doctorado y del Título de Doctor/a en la Universidad de Granada, aprobadas por el Consejo de Gobierno de la Universidad de Granada en su sesión de 2 de Mayo de 2012, donde en los epígrafes 1 y 5 del Artículo 18° y el epígrafe 2 del Artículo 22° se especifica que

> «La tesis doctoral consistirá en un trabajo original de investigación elaborado por el candidato en cualquier campo del conocimiento que se enmarcará en alguna de las líneas de investigación del Programa de Doctorado en el que está matriculado.»

> «La tesis podrá ser escrita y, en su caso, defendida, en los idiomas habituales para la comunicación científica en su campo de conocimiento. Si la redacción de la tesis se realiza en otro idioma, deberá incluir un amplio resumen en español.»

> «Para garantizar, con anterioridad a su presentación formal, la calidad del trabajo desarrollado se aportará, al menos, una publicación aceptada o publicada en un medio de impacto en el ámbito de conocimiento de la tesis doctoral firmada por el doctorando, que incluya parte de los resultados de la tesis.»

La presente memoria ha sido redactada en base a dos artículos de investigación [14, 15], publicados en los años 2014 y 2015 en revistas de relevancia internacional en el ámbito del Álgebra Cuántica y la Teoría de Categorías y de las estructuras algebraicas asociadas, referenciadas en el Journal of Citations Reports e incluidas en las bases de datos MathScinet (American Matematical Society) y Zentralblatt für Mathematik (European Mathematical Society).

Entre otros motivos, con el fin de optar a la mención internacional en el título de Doctora, la mayor parte de esta memoria está escrita en inglés, lengua que actualmente es de mayoritario uso en la comunicación científica en el área de las matemáticas, respetando así el idioma en que los artículos de investigación en que se basa han sido publicados. No obstante, al ser redactada en una lengua no oficial en España, se incluye un resumen de conclusiones (páginas 213–218) en español.

Los resultados novedosos presentados en esta tesis doctoral han sido obtenidos a lo largo de los anõs 2011–2015 bajo la supervisión del Dr. José Gómez Torrecillas y la Dra. Gabriella Böhm, en el Departamento de Álgebra de la Universidad de Granada. Durante este tiempo, la doctoranda ha sido alumna del Programa Oficial de Doctorado en Matemáticas. Desde Septiembre de 2011 y hasta Septiembre de 2015, ha disfrutado de una Ayuda Predoctoral de Formación de Personal Investigador (Ayuda FPI: BES-2011-044383), otorgada y financiada por el Ministerio de Ciencia e Innovación español y adscrita al Proyecto de Investigación MTM-2010-20940-C02-01, financiado por la Dirección General de Investigación Científica y Técnica, cuyo Investigador Principal es el Dr. José Gómez Torrecillas. La doctoranda ha realizado sus investigaciones siendo miembro del Grupo de Investigación FQM266: *Anillos y módulos*, durante los años 2012–2014, liderado por el Dr. Pascual Jara Martínez, y del Grupo de investigación FQM-379: *Álgebra y Teoría de la información*, durante los anõs 2014 y 2015, liderado por el Dr. Javier Lobillo Borrero; ambos grupos financiados por la Junta de Andalucía. Durante su periodo de estudios de doctorado, la doctoranda ha realizado diversas estancias de investigación en centros extranjeros, a saber:

- Del 1 de Abril al 30 de Junio de 2012, en el Elméleti Fizikai Osztály (Departamento de Física Teórica) del Wigner Research Centre for Physics (Budapest, Hungría), bajo la supervisión de la Dra. Gabriella Böhm; financiada por el Ministerio de Economía y Competitividad español.
- Del 16 de Enero al 22 de Febrero de 2013, en el Elméleti Fizikai Osztály (Departamento de Física Teórica) del Wigner Research Centre for Physics (Budapest, Hungría), bajo la supervisión de la Dra. Gabriella Böhm; financiada por los Nefim Funds of Wigner Research Centre for Physics.
- Del 21 de Septiembre al 21 de Diciembre de 2013, en el Vakgroep Wiskunde (Departamento de Matemáticas) de la Vrije Universiteit Brussel (Bruselas, Bélgica), bajo la supervisión del Dr. Stefaan Caenepeel; financiada por el Ministerio de Economía y Competitividad español.
- Del 15 de Junio al 15 de Agosto de 2015, en el Theory Group del Deutsches Elektronen Synchrotron (Hamburgo, Alemania), bajo la supervisión del Dr. Mikael Rodríguez Chala; financiada por el Ministerio de Economía y Competitividad español.

Signed declaration of the directors of the thesis and the doctoral researcher

The directors of the thesis Dr. José Gómez Torrecillas, University professor in the Department of Algebra at the Universidad de Granada, and Dr. Gabriella Böhm, researcher in the Wigner Research Centre for Physics in Budapest, as well as the doctoral researcher Ms. Esperanza López Centella,

> **CERTIFY**:
>
> By signing this doctoral thesis, *Quantization of categories: weak bialgebras and weak multiplier bialgebras*, that the present work has been undertaken by the doctoral candidate under the supervision of both directors of the thesis and that, to the best of our knowledge, this work respects the rights of other authors to be quoted when their results or publications have been used.

Granada, 2nd September, 2015.

Director of the thesis,	Director of the thesis,	Doctoral researcher,
José Gómez Torrecillas	Gabriella Böhm	Esperanza López Centella

Declaración firmada de los directores de tesis y la doctoranda

Los directores de la tesis Dr. José Gómez Torrecillas, catedrático de Universidad en el Departamento de Álgebra de la Universidad de Granada, y Dra. Gabriella Böhm, investigadora en el Wigner Research Centre for Physics en Budapest, así como la doctoranda D.ª Esperanza López Centella,

> **GARANTIZAMOS**:
>
> al firmar esta tesis doctoral, *Quantization of categories: weak bialgebras and weak multiplier bialgebras* (*Cuantización de categorías: biálgebras débiles y biálgebras multiplicadoras débiles*), que ésta ha sido realizada por la doctoranda bajo la dirección de los directores de la tesis y que, hasta donde nuestro conocimiento alcanza, en la realización de la presente memoria se han respetado los derechos de otros autores a ser citados cuando se han utilizado sus resultados o publicaciones.

Granada, a 2 de septiembre de 2015.

El director de la tesis,	La directora de la tesis,	La doctoranda,
José Gómez Torrecillas	Gabriella Böhm	Esperanza López Centella

Agradecimientos

Escribir esta sección es para mí un verdadero placer y una enorme ilusión. Ineludible y vehementemente anhelada por el inexorable sentimiento de colosal deuda que crea el haber recibido y recibir *tanto y tan bueno*, constituye las primeras páginas de esta tesis, siendo, sin embargo, la última en ser escrita. Y es que, de alguna manera —o más bien, de todas—, no termina de escribirse nunca. Sencillamente, porque hay *cosas* que el lenguaje no sabe —porque no puede— relatar, y porque, francamente, no estimo finito el proceso de agradecimiento por todas ellas. Lo siguiente no deja de ser, por tanto, un cúmulo de *fracasos*. De implacables fracasos al tratar de abarcar con palabras aquello que a estas escapan. Y, por la misma razón, es también un compendio de éxitos. De los grandes éxitos de calidad profesional y humana de todas aquellas personas con quienes he tenido y tengo la gran fortuna de compartir mi trabajo y mis días.

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Es evidente que, en cualquier centro de trabajo, un ambiente agradable favorece enormemente la producción y la confortabilidad de sus trabajadores. Sin duda esa ha sido mi experiencia en el Departamento de Álgebra de la Universidad de Granada desde que comencé a formar parte de él en 2011, cuando aún cursaba mis estudios de máster. Desde ese momento tuve la afectuosa bienvenida de todos sus miembros y, especialmente, el apoyo de quienes —por compartir grupo de investigación o intereses similares— han estado muy cerca: José Luis Bueso Montero, Óscar Cortadellas Izquierdo, Pascual Jara Martínez, Laiachi El Kaoutit, Javier Lobillo Borrero y Gabriel Navarro Garulo. A todos ellos agradezco sus sabios consejos y sus ánimos en esta fascinante carrera de resistencia que es la investigadora. Tampoco mis tareas computacionales habrían sido tan sencillas sin la inestimable *ayuda linuxera* de Francisco Miguel García Olmedo, siempre dispuesto a echarme una mano con el software con el buen humor que lo caracteriza. Me gustaría destacar también la gentileza de Antonio Martínez Cegarra, Pilar Carrasco Carrasco y Antonio Rodríguez Garzón, cuyos buenos días y sonrisas mañaneras han sido una inyección de energía cada jornada en nuestro despacho, así como la cortesía del primero para ponerme en contacto con el nodo de Álgebra en Barcelona a raíz de las jornadas de la Red Española de Topología. Agradezco también la calidez y ejemplar atención en el trato de Antonio López Carmona, y a Antonio Jesús Ureña Alcázar, por las fantásticas rutas de montaña en nuestra preciada sierra *granaína* y todas las *San Alberto* que hemos corrido juntos :). Asimismo, por su gran talante y ayuda en múltiples aspectos, mi agradecimiento a Jerónimo Alaminos Prats. Aprovecho también la ocasión para agradecer a Rafael Payá Albert y Miguel Martín Suárez, quienes a Mario y a mí nos brindaron de manera absolutamente altruista unos maravillosos seminarios a lo largo del cuarto curso de carrera que, como es contrastable, alimentaron favorablemente nuestro entonces incipiente interés por la investigación en Matemáticas. Por ser el promotor precursor de mi apasionado gusto por el Álgebra, agradezco a Manuel Bullejos Lorenzo, de cuyas entusiastas clases pude afortunadamente disfrutar desde mi primer curso en la Licenciatura en Matemáticas.

Durante los dos últimos años de estudios de doctorado, una actividad extraordinariamente gratificante a la par que desafiante (por su particular importancia) ha sido la docente. Personalmente, he disfrutado a lo grande impartiendo clase en el Grado de Ingeniería Informática. Como es natural, en esto han influido muchos factores. En concreto, me gustaría dar las gracias a Jesús García Miranda, quien haciendo alarde de la generosidad que le es propia, nos ha facilitado en todo momento el trabajo a los becarios que nos iniciábamos en la enseñanza, introduciéndonos en el funcionamiento de *la Escuela* y ayudándonos en todo cuanto hemos precisado. A Álvaro Martínez Sevilla y Antonio Jesús Rodríguez Salas, con quienes ha sido un placer coordinar la asignatura de Lógica y Métodos Discretos, así como al resto de compañeros involucrados durante estos dos últimos cursos académicos en la impartición de esta materia: Juan Manuel Urbano Blanco, Evangelina Santos Aláez, Jesús García Miranda y José Carlos Rosales González. Todos ellos me han proporcionado un excelente material teórico y de prácticas y un asesoramiento magnífico en mi iniciación docente. En esta línea, también debo mi gratitud a Luis Miguel Merino González, a quien he tenido el orgullo de tener como mentor en el *Curso de Introducción a la Docencia Universitaria*, y quien me ha facilitado múltiples trámites administrativos, técnicos y de gestión en su calidad de director del Departamento de Álgebra (además de por transmitir alegría con esa gracia especial). Por esto último, también estoy muy agradecida a las secretarias del departamento Encarnación Tello Olmedo, Ángela Gallardo Jiménez y Encarnación Moya Gonzálvez; y al responsable de negociado Juan Antonio Ruiz Rabaneda en el Servicio de Gestión de Investigación del Vicerrectorado de Investigación y Transferencia, por su admirable eficiencia y profesionalidad. Los agradecimientos relativos a mi experiencia docente nunca serían completos sin una mención especial a *mis estudiantes*, verdaderos protagonistas de las clases y quienes sin saberlo me han ensenãdo sobre el método docente mucho más que algunos manuales.

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⟵...—...— gracias —...—...⟶

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A mi Padre, a mi Madre,
a mi Hermano y a mi Hermana.

Porque os quiero con locura y con pasión.

Contents

Chapter 1

Introduction

A quite recent trend in the subject of study of generalizations of the notion of Hopf algebra turns out to be finding the descriptions of these generalizations in a categorical framework. In this spirit, some abstractions of (Hopf) bialgebras —which have been studied intensively on their own right— were shown to be instances of (Hopf) bimonoids in appropriately constructed braided (or even symmetric) monoidal categories. This was done, for example, in [27] for Turaev's group (Hopf) bialgebras [68] and in [28] for Makhlouf and Silvestrov's hom (Hopf) bialgebras [47]. Such a description allows for a unified treatment of all these structures, it conceptually explains the origin of some results obtained earlier by other means and it also makes available the general theory of (Hopf) bimonoids in braided monoidal categories.

Weak (Hopf) bialgebras were introduced by Gabriella Böhm, Florian Nill and Kornél Szlachányi around 1999 in [18] as a generalization of the concept of Hopf algebra. Although in this thesis we are interested in a purely algebraic treatment of them, the first motivations for studying weak bialgebras come from quantum field theory and operator algebras. We refer the reader to [18], [29] and [38] (and the references therein) for more detailed background information about this. A weak bialgebra H, as ordinary bialgebras, has the structures of an algebra and of a coalgebra in which the comultiplication Δ is multiplicative but, in contrast to usual bialgebras, Δ is no longer unital nor the counit ϵ is multiplicative. Instead, the following axioms hold for any elements a, b, c in H:

$$
\begin{aligned}
(\Delta(1)\otimes 1)(1\otimes\Delta(1)) = \quad & \Delta^2(1) \quad = (1\otimes\Delta(1))(\Delta(1)\otimes 1) \\
(\epsilon\otimes\epsilon)((a\otimes 1)\Delta(b)(1\otimes c)) = \quad & \epsilon(abc) \quad = (\epsilon\otimes\epsilon)((a\otimes 1)\Delta^{\mathsf{op}}(b)(1\otimes c))
\end{aligned}
\tag{1.1}
$$

(where Δ^{op} denotes the opposite comultiplication, that is, the resulting map of composing Δ with the canonical flip of vector spaces).

After a solid attempt of locating weak (Hopf) bialgebras [18] in a categorical setting, our first conclusions are summarized as they do not seem to be (Hopf) bimonoids in any braided monoidal category. One of the main aims of this thesis is to describe them rather as (Hopf) bimonoids in so-called *duoidal categories*.

Duoidal categories —term coined by Ross Street in [59]— were introduced by Marcelo Aguiar and Swapneel Mahajan in [5] under the original name '2–monoidal category'. These are categories with two, possibly different, monoidal structures. They are required to be compatible in the sense that the functors and natural transformations defining the first monoidal structure, are comonoidal with respect to the second monoidal structure. Equivalently, the functors and natural transformations defining the second monoidal structure, are monoidal with respect to the first monoidal structure. Whenever both monoidal structures coincide, we re-obtain the notion of braided monoidal category. More details are recalled in Section 2.1.3. A *bimonoid* in a duoidal category is a monoid with respect to the first monoidal structure and a comonoid with respect to the second monoidal structure. The compatibility axioms are formulated in terms of the coherence morphisms between the monoidal structures. In the spirit of [20], a bimonoid is said to be a *Hopf monoid* provided that it induces a right Hopf comonad in the sense of [23], as it is presented in the same Section 2.1.3.

An inspiring example in [5, Example 6.43] says that small categories can be described as bimonoids in an appropriately chosen duoidal category: in the category of spans over a given set (the set of objects). This construction is re-visited in Section 3.1.1. By this motivation we aim to find an appropriate duoidal category whose bimonoids are 'quantum categories'; that is, weak bialgebras. Recall that weak bialgebras are examples of Takeuchi's $\times_R$–bialgebras [65], equivalently, of Lu's bialgebroids [45]; such that the base algebra R carries a separable Frobenius structure [56, 61]. Bialgebroids whose base algebra R is central, were described in [5, Example 6.44] as bimonoids in the duoidal category of R–bimodules. It was also discussed there that arbitrary bialgebroids are beyond this framework because the candidate —Takeuchi's $\times_R$–operation— does not define a monoidal product in general. Nevertheless, under suitable assumptions on R, the situation favorably changes: for any separable Frobenius algebra R, the Takeuchi's $\times_R$–product can be identified with some (twisted) bimodule tensor product over $R \otimes R^{\mathsf{op}}$, as we observe and prove in Section 3.2.1. We use this to equip the category of bimodules over $R \otimes R^{\mathsf{op}}$ for a separable Frobenius algebra R with a duoidal structure. Moreover, we show in Section 3.2.2 that its bimonoids are precisely the weak bialgebras whose base algebra is isomorphic to R.

This interpretation of weak bialgebras as bimonoids allows us to define a category wba of weak bialgebras (by applying a more general construction at the beginning of Chapter 3). Morphisms, from a weak bialgebra H with separable Frobenius base algebra R, to a weak bialgebra H' with separable Frobenius base algebra R', are pairs of coalgebra maps $q : R \to R'$ and $Q : H \to H'$ with additional properties that ensure that they induce a morphism of monoidal comonads —in the sense of [63]— from the monoidal comonad induced by H on the category of $R \otimes R^{\mathsf{op}}$–bimodules to the monoidal comonad induced by H' on the category of $R' \otimes R'^{\mathsf{op}}$–bimodules. These additional properties are proven to be equivalent to $Q : H \to H'$ obeying a weak multiplicativity condition and commuting with the right and left counital maps of H and H', and $q : R \to R'$ commuting with the Nakayama automorphisms of the separable Frobenius algebras R and R'.

As an application of our formulation of the category wba, we generalize the close relation between groups and pointed cosemisimple Hopf algebras (see, for instance, [1]), showing an adjunction between wba and the category cat^0 of small categories with finitely many objects. As it is well-known [9, 53] and recalled in Section 3.3.1, the vector space spanned by any small category with finitely many objects carries a weak bialgebra structure. This turns out to yield the object map of a functor k from the category cat^0 to wba. In Sections 3.3.2 and 3.3.3 we show that it possesses a right adjoint g: For the interval category $\mathbf{2}$ and any weak bialgebra H, we consider the set $\mathsf{g}(H) := \mathsf{wba}(\mathsf{k}(\mathbf{2}), H)$ of morphisms $\mathsf{k}(\mathbf{2}) \to H$ of weak bialgebras. In general, it is isomorphic to a subset of the set of so-called 'group-like elements'; that is, of coalgebra maps from the base field to H (not to be mixed with the weakly group-like elements in [18] and [74]). In favorable situations —for example, if H is cocommutative or H is a weak Hopf algebra— $\mathsf{g}(H)$ is proven to be isomorphic to the set of group-like elements. For any weak bialgebra H, $\mathsf{g}(H)$ is interpreted as the morphism set of a category and it is shown to obey $\mathsf{wba}(\mathsf{k}(\mathsf{C}), H) \cong \mathsf{cat}(\mathsf{C}, \mathsf{g}(H))$, for any small category C with finitely many objects. The unit of this adjunction is a natural isomorphism. The component of the counit at some weak bialgebra H is an isomorphism if and only if H is pointed cosemisimple (as a coalgebra). So we obtain an equivalence between cat^0 and the full subcategory in wba of all pointed cosemisimple weak bialgebras.

Returning to our inspiring example, the Hopf monoids in the duoidal category of spans turn out to be exactly the small groupoids. In the duoidal category of bimodules over $R \otimes R^{\mathsf{op}}$, for a separable Frobenius algebra R, Hopf monoids turn out to be precisely the weak Hopf algebras with base algebra isomorphic to R. In Section 3.3.4 we show that the adjunction $\mathsf{k} \dashv \mathsf{g}$ between cat^0 and wba restricts to an adjunction between the category grp^0 of small groupoids with finitely many objects, and the full subcategory wha in wba of all weak Hopf algebras. Consequently, the equivalence between cat^0 and the full subcategory in wba of all pointed cosemisimple weak bialgebras restricts to an equivalence between grp^0 and the full subcategory in wha of all pointed cosemisimple weak Hopf algebras. As previously announced, this extends the well-known relation between groups and pointed cosemisimple Hopf algebras (see for example [1]), concluding Chapter 3.

In Chapter 4 we introduce a non-unital generalization of weak bialgebras (and multiplier Hopf algebras [69]) with a multiplier-valued comultiplication, meaning that the comultiplication no longer lands in the tensor product of the underlying algebra but in its multiplier algebra [31]. The motivation of this generalization requires the following preliminary analysis.

The most well-known examples of *Hopf algebras* are the linear spans of (arbitrary) *groups* over a field k. Dually, also the vector space of k–valued functions on a *finite* group carries the structure of a Hopf algebra. In the case of *infinite* groups, however, the vector space of k–valued functions —with finite support— possesses no unit. Consequently, it is no longer a Hopf algebra but, more generally, a *multiplier Hopf algebra* [69]. Replacing groups with *finite groupoids*, both their linear spans and the dual vector spaces of k–valued functions carry *weak Hopf algebra* structures [18]. Finally, removing the finiteness constraint in this situation, both the linear spans of arbitrary groupoids, and the vector spaces of k–valued functions with finite support on them are examples of *weak multiplier Hopf algebras* as introduced in the recent paper [72] (see Table 1.1 below).

	C	$k\mathsf{C}$ [1]	$k(\mathsf{C})$ [1]
finite	group	Hopf algebra	Hopf algebra
	groupoid	weak Hopf algebra	weak Hopf algebra
	monoid	bialgebra	bialgebra
	category	weak bialgebra	weak bialgebra
infinite	group	Hopf algebra	multiplier Hopf algebra
	groupoid (finite object set)	weak Hopf algebra	weak multiplier Hopf algebra
	groupoid	weak multiplier Hopf algebra	weak multiplier Hopf algebra
	monoid	?	?
	category (finite object set)	weak bialgebra	?
	category	?	?

Table 1.1: Motivating examples.

Multiplier Hopf algebras [69] were introduced by Alfons Van Daele around 1994 as a non-unital generalization of Hopf algebras with a multiplier-valued comultiplication. Van Daele's approach to multiplier Hopf algebras is based on the principle of using minimal input data. That is, one starts with a non-unital algebra A with an appropriately well-behaving multiplication and a multiplicative map Δ from A to the multiplier algebra of $A \otimes A$. This allows one to define maps T_1 and T_2 from $A \otimes A$ to the multiplier algebra of $A \otimes A$ as

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b),$$

where 1 stands for the unit of the multiplier algebra of A. (If A is a usual, unital bialgebra over a field k, then these maps are the left and right Galois maps for the

[1] $k\mathsf{C}$ and $k(\mathsf{C})$ denote, respectively, the linear span of C and the vector space of finitely supported k–valued functions on C (or of/on its arrow set if C is a category).

A–extension $k \to A$ provided by the unit of A.) The axioms of multiplier Hopf algebra assert first that T_1 and T_2 establish isomorphisms from $A \otimes A$ to $A \otimes A$. Second, T_1 and T_2 are required to obey $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})$ (replacing the coassociativity of Δ in the unital case). These axioms are in turn equivalent to the existence of a counit and an antipode with the expected properties. In particular, if A has a unit, then it is a multiplier Hopf algebra if and only if it is a Hopf algebra.

A similar philosophy is applied in [72, 73] by Alfons Van Daele and Shuanhong Wang to define weak multiplier Hopf algebra. Recall that if A is a weak Hopf algebra over a field k with a unit 1, then its comultiplication Δ is not required to preserve 1 (i.e. $\Delta(1)$ may differ from $1 \otimes 1$). Consequently, the maps T_1 and T_2 are no longer linear automorphisms of $A \otimes A$. Instead, they induce isomorphisms between some canonical vector subspaces determined by the element $\Delta(1)$. In the situation when A is allowed to possess no unit, in [73] the role of $\Delta(1)$ is played by an idempotent element in the multiplier algebra of $A \otimes A$, which is meant to be part of the structure. It is used to single out some canonical vector subspaces of $A \otimes A$. The maps T_1 and T_2 are required to induce isomorphisms between these vector subspaces and the same (coassociativity) axiom $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})$ is imposed. In contrast to the case of multiplier Hopf algebras, however, these axioms do not seem to imply the existence and the expected properties of the counit and the antipode. Therefore, in [73], also the existence of a counit $\epsilon : A \to k$ is assumed (in the sense that $(\epsilon \otimes \mathsf{id})T_1$ and $(\mathsf{id} \otimes \epsilon)T_2$ are equal to the multiplication on A). Adding these counit axioms, the existence of the antipode and *most* of the expected properties of the counit and the antipode do follow. However —at least without requiring that the opposite algebra obeys the same set of axioms, called the *regularity* condition in [73]— some crucial properties seem to be missing (see [73] for several discussions on this issue). Most significantly, in a usual, unital weak Hopf algebra, the counit ϵ is required to obey the two symmetrical conditions (1.1). Interestingly enough, the axioms of weak multiplier Hopf algebra in [73] imply the second equality in (1.1) but apparently not the first one (unless regularity is assumed). In this way, even if a weak multiplier Hopf algebra has a unit, it may not be a usual, unital weak Hopf algebra.

One of our main aims in Chapter 4 is to identify an intermediate class between regular and arbitrary weak multiplier Hopf algebras in [73]. This class should be big enough to contain any usual weak Hopf algebra. On the other hand, its members should have the expected properties like the (separable Frobenius type) structure of the base algebras.

In fact we take a broader perspective in getting to this goal. If considering monoids instead of groups, their linear spans (and vector spaces of base field-valued functions in the finite case) are only *bialgebras*, no longer Hopf algebras. Similarly, the linear spans of small categories with finitely many objects (and the vector spaces of base field-valued functions in the case when also the number of arrows is finite) are only *weak bialgebras* but not weak Hopf algebras. So with the ultimate aim to describe the analogous structures associated to categories without any (or at least with a milder) finiteness assumption, we study *weak multiplier bialgebras*. In this case the existence and the

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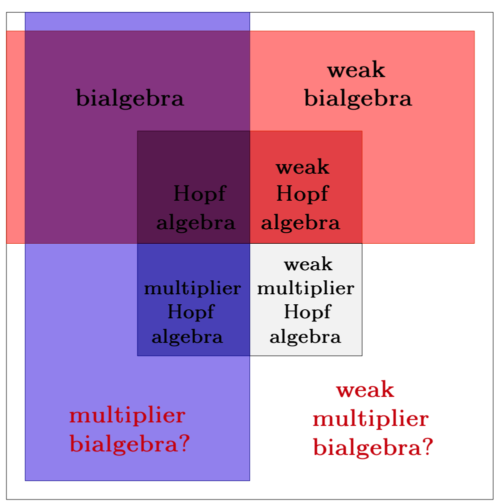


Figure 1.1: Algebraical framework.

In Section 4.2 and Section 4.3 we study some distinguished subalgebras of the multiplier algebra of a weak multiplier bialgebra. They generalize the 'right' and 'left' (also called 'source' and 'target') base algebras of a weak bialgebra. Whenever the comultiplication is 'full' (in the sense of [73]), they are shown to carry firm Frobenius algebra structures arising from a coseparable co-Frobenius coalgebra in the sense of [12]. In Section 4.4 we study an appropriate category of modules (idempotent and non-degenerate non-unital right A–modules) over a regular weak multiplier bialgebra A with a full comultiplication. It is shown to be a monoidal category equipped with a strict monoidal and faithful (in some sense 'forgetful') functor to the category of firm bimodules over the base algebra. In Section 4.5 we introduce the notion of antipode on a regular weak multiplier bialgebra. Whenever the comultiplication is full, the antipode axioms are shown to be equivalent to the projections of the maps T_1 and T_2 to maps between relative tensor products over the base algebras, being isomorphisms. We claim that the one of regular weak multiplier bialgebras possessing an antipode is the desired 'intermediate' class between regular and arbitrary weak multiplier Hopf algebras in which one can answer the questions left open in [73] and which is big enough to contain any unital weak Hopf algebra.

Summarizing, in this thesis we deal with two generalizations —each one in a sense— of the notion of bialgebra: weak bialgebra and weak multiplier bialgebra (this latter being, in turn, a generalization of the first one); and, in some extent, we view both ones as 'quantum' categories: with finitely many objects and with infinite objects respectively. Roughly, the weakening of a weak bialgebra with respect to an ordinary bialgebra is on the compatibility between the algebra and the coalgebra structures; in a weak multiplier bialgebra, instead, the underlying algebra is not supposed to be unital and the comultiplication is multiplier-valued. Our interest in weak bialgebras is essentially giving a categorical approach to them. This allows us to extend to weak bialgebras and small categories (respectively, to weak Hopf algebras and small groupoids) interesting algebraical relations, classical at the level of bialgebras and monoids (respectively, of Hopf algebras and groups). On the other hand, our main goals to define weak multiplier bialgebras are the following. First, filling the conceptual gap of the 'antipodeless' situation of weak multiplier Hopf algebra in [73]: whereas (weak) Hopf algebras are classically defined as (weak) bialgebras admitting the further structure of an antipode, in Van Daele (and Wang)'s approach, (weak) multiplier Hopf algebras are defined directly without considering the antipodeless situation of (weak) multiplier bialgebra. Our definition of weak multiplier bialgebra is supported by the fact that (assuming some further properties like regularity or fullness of the comultiplication), the most characteristic features of weak bialgebras extend to this generalization:

- There is a bijective correspondence between the weak bialgebra structures and the weak multiplier bialgebra structures on any unital algebra.
- The multiplier algebra of a weak multiplier bialgebra contains two canonical commuting anti-isomorphic firm Frobenius algebras; the so-called base algebras. (In the route, multiplier bialgebra is defined as the particular case when the base algebra is trivial; that is, it contains only multiples of the unit element.)
- Appropriately defined modules over a (nice enough) weak multiplier bialgebra constitute a monoidal category via the module tensor product over the base algebra.

Second, our other main aim by defining weak multiplier bialgebras is to introduce a notion that, without doubt and as desirable, generalize both notions: the one of weak bialgebra [18] and the one of multiplier Hopf algebra [69]. Moreover, we provide a concept of antipode for regular weak multiplier bialgebras, in such a way that any regular weak multiplier Hopf algebra in the sense of [73] is a regular weak multiplier bialgebra in the sense of this thesis possessing an antipode; and if a regular weak multiplier bialgebra admits an antipode, then it is also a weak multiplier Hopf algebra —though not necessarily a regular one— in the sense of [73]. We show a wanted intermediate class between regular and arbitrary weak multiplier Hopf algebras, big enough to contain any unital weak Hopf algebra and answering the questions left open in [73] by the aforementioned authors.

Chapter 2

Preliminaries

In this preliminary chapter we recall most of the concepts on which this thesis is based, as well as we fix the notation and establish the conventions adopted throughout the text. Concretely, Section 2.1 presents the main notions needed from category theory: monoidal categories and duoidal categories. In Section 2.2 we collect some definitions on (non-unital) algebras and (non-unital) modules over algebras, and also on their dual counterparts: coalgebras and comodules over them. Section 2.3 is devoted to the study of a particular instance of (co)algebras: the separable Frobenius ones, which will play a crucial role in forthcoming chapters. The goal of Section 2.4 is to succinctly introduce the classical notions of bialgebras and Hopf algebras in order to, further on in Section 2.5, study the weakening by Gabriella Böhm, Florian Nill and Kornél Szlachányi of these concepts that weak bialgebras and weak Hopf algebras [18] are. We aim to expose their principal features, including numerous properties of their so-called base algebras and of the antipode. We refer to [11, 18, 25, 26, 56] for the proofs of most of them. In Section 2.6 we recall the notion of multiplier Hopf algebra [69] due to Alfons Van Daele. Finally, Section 2.7 presents the generalization of this concept that Alfons Van Daele and Shuanhong Wang proposed in [72] under the name of weak multiplier Hopf algebra. We discuss in detail the analogies of the algebraic relations of Hopf algebras with multiplier Hopf algebras and, more deeply, of their respective weak versions. In our walk through all these sections, we systematically analyze our two main sources of motivating examples referred in the introduction: the linear span of various (and more and more general) algebraic structures and the vector spaces of finitely supported field-valued functions on them.

2.1 Categorical notions

We denote categories (always meaning *locally small categories*) in uppercase sans serif math font: C, D, etc. The arrow and object sets of a category C are respectively denoted by C_0 and C_1, and the source and target maps $\mathsf{C}_1 \to \mathsf{C}_0$ by s and t. For any objects A, B in C, by $A \to B$ we mean an arrow with source A and target B. We refer to the set of all these arrows as $\mathsf{C}(A, B)$ and, in most cases, we use Greek letters for them. The identity morphism on A is symbolized by $\mathsf{id}_A, 1_A$ or, shortly, A. Unless otherwise stated, the composition in a category is denoted by juxtaposition. In general, functors are written in uppercase calligraphic font: $\mathcal{F}, \mathcal{G}$, etc. The *singleton category* (that one with a single arrow) is denoted by $\mathbf{1}$, and the *interval category* (with two objects and only one non-identity arrow) by $\mathbf{2}$ and represented as $\circlearrowright S \xrightarrow{\alpha} T \circlearrowleft$. Any label inside a diagram means that the diagram commutes by the argument that the label refers to, holding true in the corresponding context. We denote by cat the category of small categories and functors, and by cat^0 its full subcategory of categories with finitely many objects. For the study of categories and its foundational issues concerning 'size', we refer to the classical references [21, 46].

2.1.1 Monoidal categories

Monoidal categories were first explicitly formulated in [8] by Jean Bénabou in 1963 and, in the same year, by Saunders Mac Lane, under the illustrative names of 'catégories avec multiplication' and 'categories with multiplication'. It was Samuel Eilenberg who introduced the current naming.

In a *monoidal category* $(\mathsf{C}, \circ, I)$, we call $\circ : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ the *monoidal product functor*. For any objects A, B in C, we refer to $A \circ B$ as the *monoidal product* of A and B, and to I as the *unit object* of C. For any objects A, B, C in C, the natural isomorphism $\alpha_{A,B,C} : (A \circ B) \circ C \xrightarrow{\cong} A \circ (B \circ C)$ is called the *associator*, and the natural isomorphisms $\lambda_A : I \circ A \to A$ and $\rho_A : A \circ I \to A$ are known as the *unit constraints* or *unitors*. Often we simultaneously will deal with two monoidal structures on the same category. Their monoidal products will be both of the same type: distinct fibre products or module tensor products. That is why we omit explicitly denoting the associator isomorphisms, identifying the objects $(A \circ B) \circ C$ and $A \circ (B \circ C)$, and using $A \circ B \circ C$ for the identification. However, since their corresponding unitors could take very different forms, we do explicit them, writing λ° and ρ° for the unit constraints associated to the monoidal product $\circ$. For the sake of uniformity, we also adopt both conventions when there is only one monoidal structure in play.

Let $(\mathsf{C}, \circ, I)$ and $(\mathsf{D}, \bullet, J)$ be two monoidal categories. We say that a functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is *comonoidal* if there is a natural transformation

$$\mathcal{F}(A \circ B) \xrightarrow{\mathcal{F}_2^{A,B}} \mathcal{F}(A) \bullet \mathcal{F}(B)$$

from the bifunctor $\mathcal{F}\circ$ to $\bullet(\mathcal{F} \times \mathcal{F})$, and a morphism

$$\mathcal{F}_0 : \mathcal{F}(I) \to J$$

in D, obeying the obvious associativity and unitality conditions. The data $\mathcal{F}_2$ and $\mathcal{F}_0$ are called *coherence* or *structure* maps. More concretely, $\mathcal{F}_2$ is termed the *binary part* and $\mathcal{F}_0$ the *nullary part*.

Dually, a functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is said to be *monoidal* if there is a natural transformation $\mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\mathcal{F}_2^{A,B}} \mathcal{F}(A \circ B)$ and a morphism $\mathcal{F}_0 : J \to \mathcal{F}(I)$ in D, satisfying natural coassociativity and counitality conditions. A monoidal functor is *strict* if its coherence maps are identities. Monoidality of a natural transformation between monoidal functors is defined via the evident compatibility conditions.

Some authors use the names 'colax monoidal functor' (for instance, the authors of [5]) and 'oplax monoidal functor' (used in [66, 76]) for the above concept of comonoidal functor. Also for the dual notion, the names 'monoidal functor' (used in this thesis) and 'lax monoidal functor' (used in [5]) do coexist.

We end this section by recalling a basic result true for any category, which will be repeatedly used further on in Chapter 3.

Proposition 2.1.1. *Let A, B, C be objects in a category C, and let $\varphi : A \to A$ be an idempotent morphism in C. If there exist epimorphisms $\pi : A \twoheadrightarrow B$ and $\pi' : A \twoheadrightarrow C$ which split φ, via monomorphisms $\iota : B \rightarrowtail A$ and $\iota' : C \rightarrowtail A$, then B and C are isomorphic via $\pi'\iota$ and $\pi\iota'$.*

2.1.2 Monoidal comonads

Comonoidal monads were introduced by Ieke Moerdijk in [48] under the name of Hopf monads. In works of Alain Bruguières, Steve Lack and Alexis Virelizier ([23, 24]), this concept has been renamed 'bimonad', being reserved the term 'Hopf monad' for bimonads with an 'antipode', convention also followed in this thesis. In those papers, comonoidal monads on monoidal categories are studied for a generalisation of the classical theory of bialgebras and Hopf algebras over a field. In this text we are especially interested in their dual notion: monoidal comonads.

Let $(\mathcal{H}, \Delta, \epsilon)$ be a comonad on a monoidal category $(\mathsf{C}, \circ, I)$. If the endofunctor $\mathcal{H} : \mathsf{C} \to \mathsf{C}$ is monoidal and Δ and ϵ are monoidal natural transformations, we call the quintuple $\mathbf{H} = (\mathcal{H}, \Delta, \epsilon, \mathcal{H}_2, \mathcal{H}_0)$ a *monoidal comonad*.

The monoidal comonad $\mathbf{H}$ is said to be a *right Hopf comonad* [23] whenever the so-called *right fusion operator* is invertible, that is, for any objects A, B in C,

$$\mathcal{H}A \circ \mathcal{H}B \xrightarrow{\Delta_A \circ \mathcal{H}B} \mathcal{H}^2 A \circ \mathcal{H}B \xrightarrow{\mathcal{H}_2} \mathcal{H}(\mathcal{H}A \circ B)$$

is an isomorphism (natural in A and B).

Let $\mathbf{H'} = (\mathcal{H}', \Delta', \epsilon', \mathcal{H}'_2, \mathcal{H}'_0)$ be a second monoidal comonad on a monoidal category $(\mathsf{D}, \bullet, J)$. A *morphism of monoidal comonads* $\mathbf{H} \to \mathbf{H'}$ [63, Definition 3.1] is a pair $(\mathcal{F} : \mathsf{C} \to \mathsf{C}', \Phi : \mathcal{F}\mathcal{H} \to \mathcal{H}'\mathcal{F})$ where $\mathcal{F}$ is a comonoidal functor and Φ is a comonad morphism (in the sense of [58, §1]) rendering commutative also the diagrams

$$\begin{array}{ccc}
\mathcal{F}(\mathcal{H}A \circ \mathcal{H}B) & \xrightarrow{\mathcal{F}\mathcal{H}_2} & \mathcal{F}\mathcal{H}(A \circ B) \\
\downarrow{\scriptstyle \mathcal{F}_2} & & \downarrow{\scriptstyle \Phi(A \circ B)} \\
\mathcal{F}\mathcal{H}A \bullet \mathcal{F}\mathcal{H}B & & \mathcal{H}'\mathcal{F}(A \circ B) \\
\downarrow{\scriptstyle \Phi A \bullet \Phi B} & & \downarrow{\scriptstyle \mathcal{H}'\mathcal{F}_2} \\
\mathcal{H}'\mathcal{F}A \bullet \mathcal{H}'\mathcal{F}B & \xrightarrow[\mathcal{H}'_2]{} & \mathcal{H}'(\mathcal{F}A \bullet \mathcal{F}B)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}I & \xrightarrow{\mathcal{F}_0} & I' \\
\downarrow{\scriptstyle \mathcal{F}\mathcal{H}_0} & & \\
\mathcal{F}\mathcal{H}I & & \downarrow{\scriptstyle \mathcal{H}'_0} \\
\downarrow{\scriptstyle \Phi I} & & \\
\mathcal{H}'\mathcal{F}I & \xrightarrow[\mathcal{H}'\mathcal{F}_0]{} & \mathcal{H}'I',
\end{array} \tag{2.1}$$

for any objects A, B in C.

2.1.3 Duoidal categories

Duoidal categories were introduced by Marcelo Aguiar and Swapneel Mahajan in [5] under the original name '2–monoidal category'. These are categories with two, possibly different, monoidal structures. They are required to be compatible in the sense that the functors and natural transformations defining the first monoidal structure, are comonoidal with respect to the second monoidal structure. Equivalently, the functors and natural transformations defining the second monoidal structure, are monoidal with respect to the first monoidal structure. Whenever both monoidal structures coincide, we re-obtain the notion of braided monoidal category [5, Section 6.3].

In other words, a *duoidal category* is a quintuple $(\mathsf{C}, \circ, I, \bullet, J)$, where $(\mathsf{C}, \circ, I)$ and $(\mathsf{C}, \bullet, J)$ are monoidal categories, along with a transformation (called the *interchange law*)

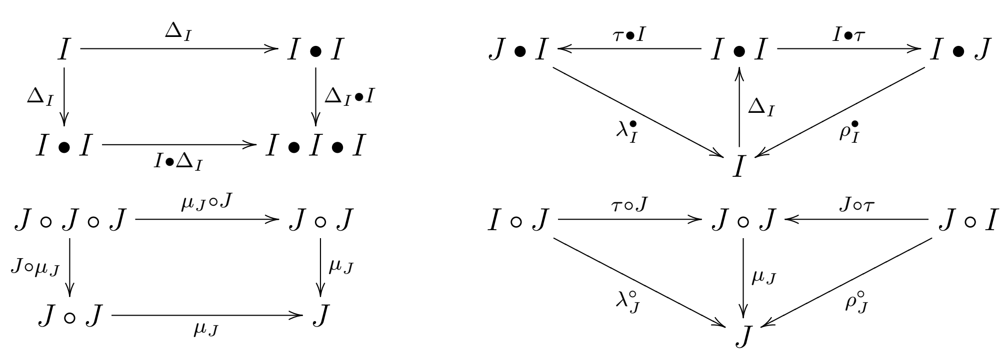
$$\gamma_{A,B,C,D} : (A \bullet B) \circ (C \bullet D) \to (A \circ C) \bullet (B \circ D) \tag{2.2}$$

which is natural in the objects A, B, C, D in C, and three morphisms

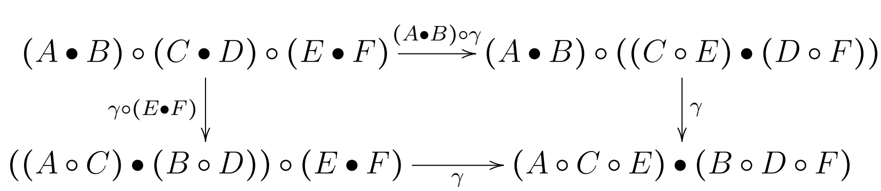
$$\mu_J : J \circ J \to J, \qquad \Delta_I : I \to I \bullet I, \qquad \tau : I \to J \tag{2.3}$$

obeying the axioms below. (If they can be told from the context, we usually omit subscripts from γ referring to the objects on which acts.)

Compatibility of units. The units I and J are compatible in the sense that (J, μ_J, τ) is a monoid in $(\mathsf{C}, \circ, I)$ and (I, Δ_I, τ) is a comonoid in $(\mathsf{C}, \bullet, J)$. Equivalently, the following diagrams commute.



Associativity. The following diagrams commute, for any objects A, B, C, D, E, F in C.



(2.4)

$$\begin{CD}
(A \bullet B \bullet C) \circ (D \bullet E \bullet F) @>{\gamma}>> (A \circ D) \bullet ((B \bullet C) \circ (E \bullet F)) \\
@V{\gamma}VV @VV{(A \circ D) \bullet \gamma}V \\
((A \bullet B) \circ (D \bullet E)) \bullet (C \circ F) @>>{\gamma \bullet (C \circ F)}> (A \circ D) \bullet (B \circ E) \bullet (C \circ F)
\end{CD} \tag{2.5}$$

Unitality. The following diagrams commute, for any objects A, B in C.

$$\begin{CD}
I \circ (A \bullet B) @>{\Delta_I \circ (A \bullet B)}>> (I \bullet I) \circ (A \bullet B) \\
@V{\lambda^{\circ}_{A \bullet B}}VV @VV{\gamma}V \\
A \bullet B @<<{\lambda^{\circ}_A \bullet \lambda^{\circ}_B}< (I \circ A) \bullet (I \circ B)
\end{CD} \tag{2.6}$$

$$\begin{CD}
(A \bullet B) \circ I @>{(A \bullet B) \circ \Delta_I}>> (A \bullet B) \circ (I \bullet I) \\
@V{\rho^{\circ}_{A \bullet B}}VV @VV{\gamma}V \\
A \bullet B @<<{\rho^{\circ}_A \bullet \rho^{\circ}_B}< (A \circ I) \bullet (B \circ I)
\end{CD} \tag{2.7}$$

$$\begin{CD}
J \bullet (A \circ B) @<{\mu_J \bullet (A \circ B)}<< (J \circ J) \bullet (A \circ B) \\
@V{\lambda^{\bullet}_{A \circ B}}VV @AA{\gamma}A \\
A \circ B @<<{\lambda^{\bullet}_A \circ \lambda^{\bullet}_B}< (J \bullet A) \circ (J \bullet B)
\end{CD} \tag{2.8}$$

$$\begin{CD}
(A \circ B) \bullet J @<{(A \circ B) \bullet \mu_J}<< (A \circ B) \bullet (J \circ J) \\
@V{\rho^{\bullet}_{A \circ B}}VV @AA{\gamma}A \\
A \circ B @<<{\rho^{\bullet}_A \circ \rho^{\bullet}_B}< (A \bullet J) \circ (B \bullet J)
\end{CD} \tag{2.9}$$

We denote by duo the category whose objects are duoidal categories and whose morphisms are functors which are comonoidal with respect to both monoidal structures. Note that, in contrast to [5], no compatibility is required between these comonoidal structures. In duo we will use the nomenclatures $\circ$–comonoidality and $\bullet$–comonoidality to refer to the comonoidality of a functor with respect to the monoidal products $\circ$ and $\bullet$, respectively.

Let $(\mathsf{C}, \circ, I, \bullet, J)$ and $(\mathsf{D}, \circ', I', \bullet', J')$ be duoidal categories. A functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is said to be *double comonoidal* [5, Definition 6.55] if it is comonoidal with respect to both monoidal structures, and the following diagrams commute for any objects A, B, C, D in C.

$$\begin{CD}
\mathcal{F}((A \bullet B) \circ (C \bullet D)) @>{\mathcal{F}(\gamma)}>> \mathcal{F}((A \circ C) \bullet (B \circ D)) \\
@V{\mathcal{F}^{\circ}_2}VV @VV{\mathcal{F}^{\bullet}_2}V \\
\mathcal{F}(A \bullet B) \circ' \mathcal{F}(C \bullet D) @. \mathcal{F}(A \circ C) \bullet' \mathcal{F}(B \circ C) \\
@V{\mathcal{F}^{\bullet}_2 \circ' \mathcal{F}^{\bullet}_2}VV @VV{\mathcal{F}^{\circ}_2 \bullet' \mathcal{F}^{\circ}_2}V \\
(\mathcal{F}(A) \bullet' \mathcal{F}(B)) \circ' (\mathcal{F}(C) \bullet' \mathcal{F}(D)) @>>{\gamma'}> (\mathcal{F}(A) \circ' \mathcal{F}(C)) \bullet' (\mathcal{F}(B) \circ' \mathcal{F}(D))
\end{CD} \tag{2.10}$$

$$\begin{array}{ccccc} \mathcal{F}(I\bullet I) & \xleftarrow{\mathcal{F}(\Delta_I)} & \mathcal{F}(I) & \xrightarrow{\mathcal{F}_0^\circ} & I' \\ {\scriptstyle \mathcal{F}_2^\bullet}\downarrow & & & & \downarrow{\scriptstyle \Delta'_I} \\ \mathcal{F}(I)\bullet'\mathcal{F}(I) & & \xrightarrow[\mathcal{F}_0^\circ\bullet'\mathcal{F}_0^\circ]{} & & I'\bullet' I' \end{array} \tag{2.11}$$

$$\begin{array}{ccccc} \mathcal{F}(J\circ J) & \xrightarrow{\mathcal{F}(\mu_J)} & \mathcal{F}(J) & \xrightarrow{\mathcal{F}_0^\bullet} & J' \\ {\scriptstyle \mathcal{F}_2^\circ}\downarrow & & & & \uparrow{\scriptstyle \mu_{J'}} \\ \mathcal{F}(J)\circ'\mathcal{F}(J) & & \xrightarrow[\mathcal{F}_0^\bullet\circ'\mathcal{F}_0^\bullet]{} & & J'\circ' J' \end{array} \tag{2.12}$$

A *bimonoid* in a duoidal category $(\mathsf{C},\circ,I,\bullet,J)$ is a quintuple $(H,\mu,\eta,\Delta,\epsilon)$ where (H,μ,η) is a monoid in $(\mathsf{C},\circ,I)$, (H,Δ,ϵ) is a comonoid in $(\mathsf{C},\bullet,J)$ and both structures are compatible in the sense that the four diagrams below commute.

$$\begin{array}{ccccc} (H\bullet H)\circ(H\bullet H) & & \xrightarrow{\gamma} & & (H\circ H)\bullet(H\circ H) \\ {\scriptstyle \Delta\circ\Delta}\uparrow & & & & \downarrow{\scriptstyle \mu\bullet\mu} \\ H\circ H & \xrightarrow[\mu]{} & H & \xrightarrow[\Delta]{} & H\bullet H \end{array} \tag{2.13}$$

$$\begin{array}{ccc} H\circ H & \xrightarrow{\epsilon\circ\epsilon} & J\circ J \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu_J} \\ H & \xrightarrow[\epsilon]{} & J \end{array} \tag{2.14}$$

$$\begin{array}{ccc} I & \xrightarrow{\eta} & H \\ {\scriptstyle \Delta_I}\downarrow & & \downarrow{\scriptstyle \Delta} \\ I\bullet I & \xrightarrow[\eta\bullet\eta]{} & H\bullet H \end{array} \tag{2.15}$$

$$\begin{array}{ccccc} & & H & & \\ & {\scriptstyle \eta}\nearrow & & \searrow{\scriptstyle \epsilon} & \\ I & & \xrightarrow[\tau]{} & & J \end{array} \tag{2.16}$$

A *morphism of bimonoids* is a morphism of the underlying monoids and comonoids.

Any bimonoid H in $(\mathsf{C},\circ,I,\bullet,J)$ induces monoidal comonads $(-)\bullet H$ and $H\bullet(-)$ on $(M,\circ,I)$, and comonoidal monads $(-)\circ H$ and $H\circ(-)$ on $(M,\bullet,J)$ (see [20]). We say that a bimonoid H is a *Hopf monoid* [20, pages 193-194] if the induced monoidal comonad $(-)\bullet H$ is a right Hopf comonad; that is, if

$$(A\bullet H)\circ(B\bullet H) \xrightarrow{\gamma((A\bullet\Delta)\circ(B\bullet H))} ((A\bullet H)\circ B)\bullet(H\circ H) \xrightarrow{((A\bullet H)\circ B)\bullet\mu} ((A\bullet H)\circ B)\bullet H \tag{2.17}$$

is a natural isomorphism. The above notion of right Hopf comonad is the dual of that of right Hopf monad in [23], where also the left counterpart was defined. The monad-comonad duality and the left-right symmetry allow for analogous versions of definition of Hopf monoid. Replacing the required condition above by that the induced monoidal comonad $H\bullet(-)$ is a left Hopf comonad; or the comonoidal monad $(-)\circ H$ is a right Hopf monad; or $H\circ(-)$ is a left Hopf monad (in the sense of [23]), would result in definitions that equally lead to proving the corresponding results (Proposition 3.3.16 and Proposition 3.3.17) in Chapter 3, as in fact it is shown (Proposition 3.3.18 and Proposition 3.3.19).

2.2 (Non-unital) algebras and coalgebras

The following notations and conventions apply to the entire text of this thesis. Throughout, k will always denote a field and the unadorned symbol $\otimes$ will stand for the usual tensor product of vector spaces over k. The term *linear* will always signify k–linear. Let A and B be k–vector spaces. We call *flip map* the linear map

$$\mathsf{tw} : A \otimes B \to B \otimes A, \quad a \otimes b \mapsto b \otimes a. \tag{2.18}$$

For any subset X of A, we denote by $\langle X \rangle$ the vector subspace linearly spanned by X. The identity map on A is indistinctly denoted by id, id_A and A. For any map $f : A \to B$, we refer to its support as $supp(f)$. For any vector subspace C of A, we denote by $f_{|C}$ the restriction of f to C. By $\mathsf{Lin}(A, B)$ we mean the k–vector space of linear maps $A \to B$ equipped with the componentwise addition and the scalar multiplication. When A and B are the same vector space we simply write $\mathsf{Lin}(A)$. The kernel of any $f \in \mathsf{Lin}(A, B)$ is denoted by $ker(f)$. The single character n will always refer to an (indeterminate) natural number. Frequently, for brevity, those algebraic structures defined by a vector space equipped with further structure maps forming an n–tuple will be denoted by the single vector space, understanding that the rest of structure is given. For any set X, the characteristic function of values in k on a subset S of it is denoted by χ_S. For any $x, y \in X$, $\delta_{x,y}$ means the Kronecker's 'delta symbol'.

In the subsequent equalities of a computation, the label or labels shown above and/or under an equality sign mean that that equality is obtained by properly applying —in the corresponding context— the argument to which the labels refer.

By a *non-unital algebra* (A, μ) over a field k (or *non-unital k–algebra*) we mean a k–vector space equipped with a linear map $\mu : A \otimes A \to A$ (called *multiplication*) satisfying the associativity condition

$$\mu(\mu \otimes \mathsf{id}) = \mu(\mathsf{id} \otimes \mu). \tag{2.19}$$

Working with elements we will normally use juxtaposition to denote the multiplication μ, just writing ab for $\mu(a \otimes b)$. We denote by μ^2 any of the two equal expressions in (2.19). If in addition μ is surjective then A is said to be *idempotent*. If any of the conditions $(ab = 0,\ \forall a \in A)$ and $(ba = 0,\ \forall a \in A)$ implies $b = 0$, then the multiplication

μ is termed *non-degenerate*. We say that A is *firm* if the quotient map $A \otimes_A A \to A$, $a \otimes_A b \mapsto ab$ is bijective. It has *local units*[1] if there is a set E of idempotent elements in A such that for every finite set $\{a_i\}_{i=1}^n \subseteq A$ there is $e \in E$ obeying $ea_i = a_i = a_i e$ for every $i \in \{1, ..., n\}$. For any non-unital algebras (A, μ_A) and (B, μ_B), also the *opposite algebra* $A^{\mathsf{op}} = (A, \mu_A^{\mathsf{op}})$ and $(A \otimes B, \mu_{A\otimes B})$ are non-unital algebras for the *opposite multiplication* $\mu_A^{\mathsf{op}} = \mu \mathsf{tw}$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B)(A \otimes \mathsf{tw} \otimes B)$. Clearly, A^{op} and $A \otimes B$ are idempotent and non-degenerate whenever A and B are so (see e.g. [40, Lemma 1.11]). A *morphism of non-unital algebras* from A to B is a linear map $f : A \to B$ obeying the multiplicativity condition

$$f\mu_A = \mu_B(f \otimes f). \tag{2.20}$$

The non-unital algebra $A \otimes A^{\mathsf{op}}$ is known as the *enveloping algebra* of A and it is denoted by A^e.

Example 2.2.1. Take a small category C. For a field k, let $k\mathsf{C}$ be the k–vector space with basis C_1. For any arrows a and b, let their product be the composite arrow ab if they are composable and zero otherwise. Since the identity arrows of the category give rise to local units, $k\mathsf{C}$ is in this way an idempotent non-unital algebra with a non-degenerate multiplication.

Example 2.2.2. Take again a small category C. For a field k, let $k(\mathsf{C})$ be the vector space of k–valued functions with finite support on C_1. It is a non-unital algebra via the pointwise multiplication $(fg)(c) := f(c)g(c)$ for any $f, g \in k(\mathsf{C})$ and $c \in \mathsf{C}_1$. The characteristic functions χ_F of the finite subsets F of C_1 serve as local units for $k(\mathsf{C})$. Hence $k(\mathsf{C})$ is idempotent with a non-degenerate multiplication.

Let (A, μ) be a non-unital algebra over a field k. A *non-unital right A–module* is a pair (M, ρ) where M is a k–vector space and

$$\rho : M \otimes A \to M, \quad m \otimes a \mapsto ma$$

[1]This definition of local units is the one used in [12, 37], and it can be traced back to [2] and [7]. It is more general than [2, Definition 1.1], since commutativity of the elements of E is not assumed. In fact, the present notion generalizes that of [2] since, when the idempotents of E commute, it is enough to require that for each element $r \in R$ there exists $e \in E$ such that $er = r = re$, see [2, Lemma 1.2]. For more equivalent conditions for the existence of local units, see also [75].

is a linear map (called the right A–*action*) obeying the associativity condition

$$\rho(\mathsf{id}_M \otimes \mu) = \rho(\rho \otimes \mathsf{id}_A). \tag{2.21}$$

A *non-unital left A–module* is a non-unital right module over the opposite algebra A^{op} (equivalently, a k–vector space N endowed with a linear map $A \otimes N \to N$ obeying a symmetric axiom to (2.21)).

If the A–action ρ is a surjective map, then M is said to be *idempotent*[2]. It is called *firm* [55] if the quotient map $M \otimes_A A \to M$, $m \otimes_A a \mapsto ma$ is bijective (where A is regarded as a left A–module with respect to its multiplication). If for any $m \in M$, the condition $ma = 0$ for all $a \in A$ implies $m = 0$, then M is termed *non- degenerate*. For brevity, we will usually omit the summation symbol in the writing of an arbitrary element of a module tensor product $A \otimes_C B$ (for any appropriate A, B and C), writing for example $a_i \otimes_C b_i$ instead of $\sum_i a_i \otimes_C b_i$, with $a_i \in A, b_i \in B$.

The vector spaces of non-unital right A–module maps and of non-unital left A–module maps $A \to A$ are denoted by $\mathsf{End}_A(A)$ and ${}_A\mathsf{End}(A)$ respectively.

A *non-unital A-B–bimodule* is a triple (M, λ, ρ) where (M, λ) is a non-unital left A–module, (M, ρ) is a non-unital right B–module, and their actions obey the compatibility condition

$$\lambda(\mathsf{id}_A \otimes \rho) = \rho(\lambda \otimes \mathsf{id}_B). \tag{2.22}$$

We use the nomenclature A–bimodule to refer to an A-A–bimodule.

By $\mathsf{rmd}(A)$ we will mean the category of idempotent and non-degenerate non-unital right A–modules. The category of firm non-unital A–bimodules (i.e. of bimodules which are firm both as left and right non-unital modules) will be denoted by $\mathsf{bim}^f(A)$. Whenever A is a firm non-unital algebra, $\mathsf{bim}^f(A)$ is a monoidal category via the module tensor product $\otimes_A$ and the unit object A.

By an *algebra over k* we mean a triple (A, μ, η) where (A, μ) is a non-unital k–algebra and $\eta : k \to A$ is a unit for it. Working with elements, we will use the *unit element* $1 := \eta(1)$ instead of η. If (A, μ_A, η_A) and (B, μ_B, η_B) are algebras over a field,

[2] In other many sources —for instance, in [34, 70]—, an A–module M obeying this property is called *unital*. This terminology is motivated by the fact that, by virtue of [34, Proposition 3.3], there is a unique extension of (M, ρ) to a right module $(M, \overline{\rho})$ over $\mathbb{M}(A)$ (the multiplier algebra of A, see page 54), meaning that $\overline{\rho}(m \otimes 1) = m$ for all $m \in M$ (where 1 stands for the unit of $\mathbb{M}(A)$).

a *morphism of algebras* from A to B is a morphism of non-unital algebras $f : A \to B$ respecting their units.

Example 2.2.3. Consider $k\mathsf{C}$ described in Example 2.2.1 for a small category C with finitely many objects. Then the sum of all the identity arrows in C works as a unit for the product, and consequently $k\mathsf{C}$ becomes a k–algebra in this case.

Example 2.2.4. Analogously, for a finite category C, consider $k(\mathsf{C})$ as introduced in Example 2.2.2 (now the assumption of finite support on the k–valued functions becomes redundant). Then the map $\mathsf{C}_1 \to k$ constantly equal to 1 is a unit for the pointwise multiplication, so that $k(\mathsf{C})$ turns out to be a k–algebra.

For an algebra A we define *right* A*–modules*, *left* A*–modules*, A-B*–bimodules* and their corresponding morphisms as their non-unital counterparts obeying the obvious unitality conditions as the case.

Let R be an algebra over a field, and M and N be R^e–bimodules. Then M has the structure of a right R–module via the action $m \otimes r \mapsto m(r \otimes 1)$, and N is a left R–module via $r \otimes n \mapsto n(1 \otimes r)$. Consider the module tensor product $M \otimes_R N$ that these R–modules give rise to, that is, the factor space of $M \otimes N$ by

$$< m(r \otimes 1) \otimes n - m \otimes n(1 \otimes r) > . \tag{2.23}$$

It is an R–bimodule via the actions

$$s(m \otimes_R n)s' := (s \otimes 1)m \otimes_R (1 \otimes s')n \tag{2.24}$$

for any $s, s' \in R$, $m \in M$, $n \in N$. By

$$(s \otimes 1)(m(r \otimes 1)) \otimes (1 \otimes s')n \overset{(2.22)}{=} ((s \otimes 1)m)(r \otimes 1) \otimes (1 \otimes s')n$$

and

$$(s \otimes 1)m \otimes (1 \otimes s')(n(1 \otimes r)) \overset{(2.22)}{=} (s \otimes 1)m \otimes ((1 \otimes s')n)(1 \otimes r),$$

these actions are well defined. The *Takeuchi's product* [65] $M \times_R N$ is defined as the center of this R–bimodule $M \otimes_R N$. It is an R^e–bimodule via the actions:

$$(s \otimes r)(m_t \otimes_R n_t) = (1 \otimes r)m_t \otimes_R (s \otimes 1)n_t \tag{2.25}$$

$$(m_t \otimes_R n_t)(s \otimes r) = m_t(1 \otimes r) \otimes_R n_t(s \otimes 1) \tag{2.26}$$

for any $s \otimes r \in R^e, m_t \otimes_R n_t \in M \times_R N$.

Coalgebras are the dual notion of algebras. A *coalgebra* over a field k is a triple (C, Δ, ε) where C is a k–vector space equipped with linear maps $\Delta : C \to C \otimes C$ (called *comultiplication*) and $\epsilon : C \to k$ (called *counit*) obeying the following *coassociativity* law and *counitary* property.

$$(\Delta \otimes \mathsf{id})\Delta = (\mathsf{id} \otimes \Delta)\Delta \tag{2.27}$$

$$(\epsilon \otimes \mathsf{id})\Delta = \mathsf{id} = (\mathsf{id} \otimes \epsilon)\Delta \tag{2.28}$$

We denote by Δ^2 both equal maps in (2.27). For the comultiplication of a coalgebra, we will use the implicit summation index notation introduced by Robert George Heynemann and Moss Eisenberg Sweedler around 1969. This means that normally we will write $\Delta(c)$ as $c_1 \otimes c_2$ for any $c \in C$, denoting by $c_1 \otimes c_2 \otimes c_3$ both equal sums resulting of applying (2.27) on c. Furthermore, if in a same algebraical expression there is more than one occurence of $\Delta(c)$, in order to distinguish them when passing to Heyneman-Sweedler notation, we will use prime symbols in the subscripts in the following fashion: $c_1 \otimes c_2$, $c_{1'} \otimes c_{2'}$, $c_{1''} \otimes c_{2''}$, and so on. (See [42, Section III.1.6] for further details about this notation.)

The triple $C_{\mathsf{cop}} = (C, \Delta^{\mathsf{op}}, \epsilon)$ with the *opposite comultiplication* $\Delta^{\mathsf{op}} = \mathsf{tw}\Delta$ is a new k–coalgebra called the *coopposite* coalgebra of C.

If $(C, \Delta_C, \epsilon_C)$ and $(D, \Delta_D, \epsilon_D)$ are coalgebras over k, $C \otimes D$ is also a k–coalgebra with structure maps $(\mathsf{id} \otimes \mathsf{tw} \otimes \mathsf{id})(\Delta_C \otimes \Delta_D)$ and $\epsilon_C \otimes \epsilon_D$. A *morphism of coalgebras* from C to D is a linear map $f : C \to D$ satisfying the *comultiplicative* and *counital* conditions

$$\Delta_D f = (f \otimes f)\Delta_C \quad (2.29) \qquad \text{and} \qquad \epsilon_D f = \epsilon_C \quad (2.30).$$

Right and left comodules and bicomodules over a coalgebra [25] are defined dually to right and left modules and bimodules over an algebra.

Let (A, μ, η) and (C, Δ, ϵ) be, respectively, an algebra and a coalgebra over the same field k. For any $f, g \in \mathsf{Lin}(C, A)$, their *convolution product* is defined as

$$f * g := \mu(f \otimes g)\Delta. \tag{2.31}$$

The triple $(\mathsf{Lin}(C, A), *, \eta\epsilon)$ is then a k–algebra, known as the *convolution algebra* of C and A. The dual algebra $C^* = \mathsf{Lin}(C, k)$ of C is a particular instance of a convolution algebra. Every right (left) C–comodule becomes then a left (respectively right) C^*–module. In particular, C becomes a C^*–bimodule. The coalgebra C is called *co-Frobenius* if there exists a monomorphism $C \to C^*$ of left (or right) C^*–modules (see [44]). It is said to be *coseparable* if there is a C–bicomodule retraction (i.e. left inverse) of the comultiplication.

The set of the *group-like* elements of C is defined as

$$G(C) := \{g \in C \ : \ \Delta(g) = g \otimes g, \quad \epsilon(g) = 1\}. \tag{2.32}$$

It is easy to check that the elements of $G(C)$ are linearly independent over k and $kG(C)$ may be regarded as a k–subcoalgebra of C [1, Theorem 2.1.2].

Example 2.2.5. Let k be a field, S a set and kS the k–vector space with basis S. Then, for any $s \in S$, the *diagonal comultiplication* $\Delta : s \mapsto s \otimes s$ and the counit $\epsilon : s \mapsto 1$ endow kS with a k–coalgebra structure.

2.3 Separable Frobenius (co)algebras

Although they would be named Frobenius after, Frobenius algebras began to be studied in the late 1930s and early 1940s by Richard Dagobert Brauer and Cecil James Nesbitt in their works [22] and [51]. At the same time, in [49] and more especially in [50], Tadashi Nakayama discovered the beginnings of a rich duality theory. Essentially, Frobenius algebras are based on the idea of compatibility between an algebra and a coalgebra structure on a same vector space.

On the other hand, the notion of separability of an algebra is a strengthening of that of semisimplicity, and a generalization of that of a separable field extension. Both, separability and Frobenius property, are closely related to each other. In fact, as it is shown in [36], any separable algebra over a field can be endowed with the structure of a particular type (symmetric) Frobenius algebra. Nevertheless, in this text we are interested in the general notion of a Frobenius algebra over a field. The following result provides several characterizations of it.

Proposition 2.3.1. *[4, Theorem 2.1] Let (R, μ, η) be an algebra over a field k. The following assertions are equivalent.*

(i) (R, μ) *and* $(R^*, \cdot)$ *are isomorphic right* R*–modules* [3]*.*

(ii) (R, μ) *and* $(R^*, \cdot)$ *are isomorphic left* R*–modules* [3]*.*

(iii) R *possesses a* k*–coalgebra structure with an* R*–bilinear comultiplication.*

(iv) There exist a linear map $\psi : R \to k$ *and an element* $e_i \otimes f_i \in R \otimes R$ *such that for every* $r \in R$*,*

$$\psi(re_i)f_i = r = e_i\psi(f_ir). \tag{2.33}$$

If an algebra R over a field obeys any of the equivalent assertions of Proposition 2.3.1, it is said to be a *Frobenius algebra*. We call *Frobenius functional* and *Frobenius element* a linear map $\psi : R \to k$ and a distinguished element $e_i \otimes f_i \in R \otimes R$ as in part (iv) of Proposition 2.3.1. We refer to a pair $(\psi, e_i \otimes f_i)$ as a *Frobenius structure* on R, and to each coalgebra structure (δ, ε) existing by part (iii) in the same proposition as a *Frobenius coalgebra structure*. In light of [3, Propositions 2 and 5], if $(\psi, e_i \otimes f_i)$ is a Frobenius structure on R, then all the Frobenius structures on R are of the form $(\psi(-u), e_iu^{-1} \otimes f_i)$ for an invertible element $u \in R$. It is a consequence of its definition (given by the equivalent claims in Proposition 2.3.1) that any Frobenius algebra is finite-dimensional.

Since any Frobenius algebra carries a coalgebra structure, we will also refer to them as Frobenius coalgebras and Frobenius (co)algebras —depending on which aspect is more relevant in the given situation. Clearly, if $(\psi, e_i \otimes f_i)$ and $(\psi', e'_i \otimes f'_i)$ are respective Frobenius structures on algebras R and R', then $(\psi, f_i \otimes e_i)$ and $(\psi \otimes \psi', e_i \otimes e'_j \otimes f_i \otimes f'_j)$ are respectively so on the opposite algebra R^{op} and the tensor product $R \otimes R'$.

---

[3]It is understood that $\varphi \cdot r = \varphi(r-)$ and $r \cdot \varphi = \varphi(-r)$ for any $\varphi \in R^*, r \in R$.

Proposition 2.3.2. *[43, §16E] Let R be an algebra over a field and $(\psi, e_i \otimes f_i)$ a Frobenius structure on it. The following assertions hold true.*

(i) There is a unique algebra automorphism $\theta : R \to R$ such that for any $r, s \in R$

$$\psi(rs) = \psi(\theta(s)r). \tag{2.34}$$

(ii) For any $r \in R$, the following identities hold true.

$$re_i \otimes f_i = e_i \otimes f_i r \tag{2.35}$$
$$e_i r \otimes f_i = e_i \otimes \theta(r) f_i \tag{2.36}$$
$$e_i \otimes \theta(f_i) = \theta^{-1}(e_i) \otimes f_i \tag{2.37}$$
$$\theta(e_i) \otimes f_i = f_i \otimes e_i = e_i \otimes \theta^{-1}(f_i) \tag{2.38}$$

For a Frobenius algebra R, the automorphism θ defined in part (i) of Proposition 2.3.2 is called *Nakayama automorphism*. For any $r \in R$, its explicit expression and that of its inverse are:

$$\theta(r) := \psi(e_i r) f_i \quad (2.39) \qquad \text{and} \qquad \theta^{-1}(r) := e_i \psi(r f_i). \quad (2.40)$$

Proposition 2.3.3. *For any algebra S over a field equipped with an algebra automorphism ζ, there is an automorphism functor $\mathcal{F} : \mathsf{bim}(S^e) \to \mathsf{bim}(S^e)$ defined as follows. For any S^e–bimodule M with actions denoted by juxtaposition, $\mathcal{F}(M) = (M, \cdot)$ and $\mathcal{F}^{-1}(M) = (M, \text{-})$ where*

$$(s \otimes r) \cdot m = (1 \otimes \zeta(r)) m (1 \otimes s), \qquad m \cdot (s \otimes r) = (r \otimes 1) m (s \otimes 1) \tag{2.41}$$
$$(s \otimes r) \text{-} m = (1 \otimes \zeta^{-1}(r)) m (1 \otimes s), \qquad m \text{-} (s \otimes r) = (r \otimes 1) m (s \otimes 1)$$

for any $m \in M$ and $s \otimes r \in S^e$. On morphisms both $\mathcal{F}$ and $\mathcal{F}^{-1}$ act as the identity map.

Proposition 2.3.2 assures that Proposition 2.3.3 holds true, in particular, for Frobenius algebras where ζ can be taken to be the Nakayama automorphism.

Since, by virtue of Abrams's classical theorem ([4, Theorem 3.3]), any R–module over a Frobenius (co)algebra R carries also an R–comodule structure and vice-versa, we will use the nomenclatures R–module, R–comodule or R–(co)module for them — depending on which aspect is more important in the given situation. For a big number of examples and a detailed exposition on many aspects of Frobenius algebras we refer to [43] and [57].

Proposition 2.3.4. *[35, Proposition 1.1] For any algebra (R, μ, η) over a field, the following assertions are equivalent.*

(i) The multiplication $\mu : R \otimes R \to R$ is a split epimorphism of R–bimodules.

(ii) There exists an element $e_i \otimes f_i \in R \otimes R$ such that for any $r \in R$,

$$e_i f_i = 1 \quad (2.42) \qquad \textit{and} \qquad r e_i \otimes f_i = e_i \otimes f_i r. \quad (2.43)$$

An algebra R over a field satisfying any of the equivalent conditions of Proposition 2.3.4 is called a *separable* algebra. In such a case, a distinguished element $e_i \otimes f_i \in R \otimes R$ as in part (ii) of Proposition 2.3.4 is called *separability idempotent*. It is indeed an idempotent in $R \otimes R^{\mathsf{op}}$:

$$e_i e_j \otimes f_j f_i \overset{(2.43)}{=} e_i f_i e_j \otimes f_j \overset{(2.42)}{=} e_j \otimes f_j. \tag{2.44}$$

The separability idempotent in a separable algebra is not unique in general. Clearly, if R and S are separable algebras, then also the opposite algebra R^{op} and the tensor product algebra $R \otimes S$ are so.

We call $(\psi, e_i \otimes f_i)$ a *separable Frobenius structure* on an algebra R if it is a Frobenius structure on R and $e_i \otimes f_i$ is a separability idempotent for R. Then it is natural to call $e_i \otimes f_i$ *separability Frobenius idempotent*.

We denote by sfr the category whose objects are separable Frobenius (co)algebras over a given base field k, and whose morphisms are defined as follows. Given k–algebras R and R' with respective separable Frobenius structures $(\psi, e_i \otimes f_i)$ and $(\psi', e'_i \otimes f'_i)$ and Nakayama automorphisms θ and θ', a morphism from R to R' in sfr is a coalgebra map $q : R \to R'$ such that $\theta' q = q\theta$. Taking into account the expression of the comultiplications of R and R' induced by their separability Frobenius idempotents, the comultiplicativity of q means the identity

$$q(re_i) \otimes q(f_i) = q(r) e'_i \otimes f'_i, \quad \text{for all } r \in R. \tag{2.45}$$

The following results show important features of separable Frobenius algebras which will be exploited in Chapter 3.

Proposition 2.3.5. *Let S be a separable Frobenius algebra with separability Frobenius idempotent $e_i \otimes f_i$, P a right S–module and Q a left S–module. Then the map $P \otimes Q \twoheadrightarrow P \otimes_S Q$ splits via the section $p \otimes_S q \mapsto pe_i \otimes f_i q$.*

Proposition 2.3.6. *Let S be a separable Frobenius algebra with separability Frobenius idempotent $e_i \otimes f_i$ and let P be an S–bimodule. Then the map $P^S \rightarrowtail P$ splits via the section $p \mapsto e_i p f_i$.*

2.4 Bialgebras and Hopf algebras

As the authors of [6] state, determining the origin of Hopf algebras is not a simple task. We learn from that work —aimed precisely to study the beginnings of these objects— that it was Pierre Cartier who, in 1956 and under the name of hyper-algebra, gave the first formal definition of Hopf algebra (although not exactly as we know it nowadays). Nevertheless, it is to Heinz Hopf to whom they owe their name, term coined (originally, in french: *algèbre de Hopf*) by Armand Borel in 1953, in honour to the foundational work of the German mathematician. Leaving aside the interesting historical and mathematical roots of Hopf algebras, the next proposition introduces the definition of a conceptually prior notion: that of a bialgebra.

Proposition 2.4.1. *[1, Theorem 2.1.1] Let H be a vector space over a field k equipped with linear maps*

$$\mu : H \otimes H \to H, \qquad \eta : k \to H,$$
$$\Delta : H \to H \otimes H, \qquad \epsilon : H \to k,$$

such that (H, μ, η) is a k–algebra and (H, Δ, ϵ) is a k–coalgebra. Then μ and η are k–coalgebra morphisms if and only if Δ and ϵ are k–algebra morphisms.

If a k–vector space H satisfies any of the equivalent conditions of Proposition 2.4.1, then the quintuple $(H, \mu, \eta, \Delta, \epsilon)$ is called a *k–bialgebra*.

Example 2.4.2. Let A be a monoid. Denote by juxtaposition its multiplication and by 1_A its unit. For a field k, let kA be the k–vector space with basis A. The linear extensions of the maps defined by

$$\mu(a \otimes b) = ab, \qquad \eta(1) = 1_A$$
$$\Delta(a) = a \otimes a, \qquad \epsilon(a) = 1$$

for any $a, b \in A$ provide a k–bialgebra structure on kA.

Example 2.4.3. Let k be a field and A a finite monoid, and use the same notation as above for its product and unit. Denoting by $1(-)$ the function $A \to k$ constantly equal to 1, and identifying $k(A) \otimes k(A)$ with $k(A \times A)$, the following maps induce a k–bialgebra structure on the k–vector space $k(A)$ of k–valued functions on A:

$$\begin{array}{rclrcl} \mu(f \otimes g)(a) & = & f(a)g(a), & \eta(1) & = & 1(-) \\ \Delta(f)(a,b) & = & f(ab), & \epsilon(f) & = & f(1_A) \end{array}$$

for any $a, b \in A$.

A *Hopf algebra* $(H, \mu, \eta, \Delta, \epsilon, S)$ over a field k is a k–bialgebra for which the identity map $\mathsf{id} : H \to H$ is invertible in the convolution algebra $\mathsf{Lin}(H)$; in other words, the bialgebra H is equipped with a linear map $S : H \to H$ (called *antipode*) obeying:

$$S * \mathsf{id} = \eta\epsilon = \mathsf{id} * S. \tag{2.46}$$

If H' is another Hopf algebra with antipode S', any morphism $f : H \to H'$ of algebras and coalgebras automatically respects the antipodes of H and H', that is, $S'f = fS$ holds (see [60, Chapter 4]).

Example 2.4.4. If in Example 2.4.2 we consider an arbitrary group G instead of a monoid A, the k–bialgebra kG is in fact a Hopf algebra with the linear extension of the inverse operation of the group as antipode; that is, $S(g) = g^{-1}$ for all $g \in G$. It is called the *group Hopf algebra*.

Example 2.4.5. Analogously, if in Example 2.4.3 we consider a finite group G in place of a finite monoid A, the k–bialgebra $k(G)$ turns out to be a Hopf algebra with the antipode given by $S(f)(g) = f(g^{-1})$ for all $g \in G$.

2.5 Weak bialgebras

Weak bialgebras were introduced by Gabriella Böhm, Florian Nill and Kornél Szlachányi around 1999 in [18]. A weak bialgebra over a field possesses the structure of an algebra and of a coalgebra, but the compatibility between them is weaker than in a bialgebra.

This weakness refers to the multiplicativity of the counit and the comultiplicativity of the unit: they no longer hold. Instead, in [18] they are replaced by the second and third axioms of the definition below.

2.5.1 The weak bialgebra axioms

A *weak bialgebra* [18] over a field k is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ where (H, μ, η) is a k–algebra and (H, Δ, ϵ) is a k–coalgebra satisfying the following axioms.

Multiplicativity of the coproduct.

$$\Delta(ab) = \Delta(a)\Delta(b), \quad \text{for all } a, b \in H. \tag{2.47}$$

Weak multiplicativity of the counit.

$$\epsilon(ab_1)\epsilon(b_2c) = \epsilon(abc) = \epsilon(ab_2)\epsilon(b_1c), \quad \text{for all } a, b, c \in H. \tag{2.48}$$

Weak comultiplicativity of the unit.

$$(\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = \Delta^2(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1) \tag{2.49}$$

In contrast, in [52], $(H, \mu, \eta, \Delta, \epsilon)$ is called a *weak bialgebra* if it is an algebra and a coalgebra in which the comultiplication is multiplicative. In that work, weak bialgebras obeying the weak multiplicativity of the counit axiom (2.48) are termed *monoidal*, and those ones on which the weak comultiplicativity of the unit axiom (2.49) hold are said to be *comonoidal*. These monoidality (respectively, comonoidality) axioms are aimed to make the category of H–modules (respectively, H–comodules) monoidal (see [52, Section 4]).

Example 2.5.1. Example 2.4.2 shows that the linear span of a monoid is a bialgebra. If, instead of a monoid (which may be regarded as a category with a single object), we consider a small category C with finitely many objects, then its linear span $k\mathsf{C}$ still carries an algebra structure as described in Examples 2.2.1 and 2.2.3. However, the diagonal comultiplication given by $\Delta : c \to c \otimes c$ and the counit $\epsilon : c \mapsto 1$ for any $c \in \mathsf{C}_1$ fail to provide a bialgebra structure on it. In fact, ϵ is no longer multiplicative: $\epsilon(cc') = \delta_{s(c),t(c')}$ is not equal to $\epsilon(c)\epsilon(c') = 1$ if c and c' are not composable arrows in C. Moreover, since $\sum_{c \in \mathsf{C}_0} 1_c \otimes 1_c \neq \sum_{c,c' \in \mathsf{C}_0} 1_c \otimes 1_{c'}$, Δ is not unital. So that $k\mathsf{C}$ is not a bialgebra with this natural structure but, more generally, a weak bialgebra over k, as it can be easily checked.

Example 2.5.2. A similar relation to that between Example 2.5.1 and Example 2.4.2 can be observed between the present one and Example 2.4.3. Indeed, replacing in Example 2.4.3 the finite monoid by a finite category C, we can endow the k–vector space $k(\mathsf{C})$ of k–valued functions on C_1 with a weak bialgebra structure via:

$$\mu(f \otimes g)(c) = f(c)g(c), \qquad \eta(1) = 1(-)$$
$$\Delta(f) = \sum_{c,d \in \mathsf{C}_1} f(cd)\chi_{\{c\}} \otimes \chi_{\{d\}}, \qquad \epsilon(f) = \sum_{c \in \mathsf{C}_0} f(c)$$

for all $f \in k(\mathsf{C}), c \in \mathsf{C}_1$. (Recall from Section 2.2 that χ_S denotes the characteristic function of a set S.)

Clearly, for any weak bialgebra, its opposite algebra (with the same coalgebra structure), its coopposite coalgebra (with the same algebra structure) and its opposite-coopposite (equivalently, coopposite-opposite) (co)algebra are also weak bialgebras.

2.5.2 The base algebras of a weak bialgebra

In a weak bialgebra H, define the *counital maps* $H \to H$ by the formulae

$$\sqcap^R(h) = 1_1\epsilon(h1_2) \qquad (2.50) \qquad\qquad \sqcap^L(h) = \epsilon(1_1 h)1_2 \qquad (2.51)$$
$$\overline{\sqcap}^R(h) = 1_1\epsilon(1_2 h) \qquad (2.52) \qquad\qquad \overline{\sqcap}^L(h) = \epsilon(h1_1)1_2. \qquad (2.53)$$

As it will be shown in forthcoming subsections, these counital maps play an important role in weak bialgebra theory in general and, in particular, in the original results of this thesis. In this subsection we focus on presenting a big number of their properties. Most of them are included in [18].

First of all note that, if H is a bialgebra, the four above maps (2.50)-(2.53) are equal to the counit ϵ. So, the novelty of their meaning —which is in fact the interesting part— is actually linked to the weakness of weak bialgebras. In order to guide our intuition on these counital maps, let us recognise their expressions in the weak bialgebra $k\mathsf{C}$ of Example 2.5.1. For any morphism $a \in \mathsf{C}_1$,

$$\sqcap^R_{k\mathsf{C}}(a) = \sum_{c \in \mathsf{C}_0} c\epsilon(ac) = \sum_{c \,\in\, \mathsf{C}_0 \,:\, t(c) = s(a)} c = s(a), \tag{2.54}$$
$$\sqcap^L_{k\mathsf{C}}(a) = \sum_{c \in \mathsf{C}_0} \epsilon(ca)c = \sum_{c \,\in\, \mathsf{C}_0 \,:\, s(c) = t(a)} c = t(a). \tag{2.55}$$

Similarly, we can see that $\overline{\sqcap}^R_{k\mathsf{C}}(a) = t(a)$ and $\overline{\sqcap}^L_{k\mathsf{C}}(a) = s(a)$. Thus, in $k\mathsf{C}$ these maps serve precisely as source and target maps of the category C. Not surprisingly, many authors ([25, 26, 56], etc.) call $\sqcap^R$ and $\sqcap^L$ counital 'source' and 'target' maps.

Putting $b = 1$ in axiom (2.48), the following identities are immediately obtained.

$$\epsilon(ac) = \epsilon(\sqcap^R(a)c) \quad (2.56) \qquad \epsilon(ac) = \epsilon(a \sqcap^L(c)) \quad (2.57)$$
$$\epsilon(ac) = \epsilon(a\overline{\sqcap}^R(c)) \quad (2.58) \qquad \epsilon(ac) = \epsilon(\overline{\sqcap}^L(a)c) \quad (2.59)$$

By them it follows the idempotency of all these counital maps:

$$\sqcap^R \sqcap^R = \sqcap^R, \qquad \overline{\sqcap}^R\overline{\sqcap}^R = \overline{\sqcap}^R, \qquad \sqcap^L\sqcap^L = \sqcap^L, \qquad \overline{\sqcap}^L\overline{\sqcap}^L = \overline{\sqcap}^L; \tag{2.60}$$

as well as the identities

$$\overline{\sqcap}^L\sqcap^R = \overline{\sqcap}^L, \qquad \overline{\sqcap}^R\sqcap^L = \overline{\sqcap}^R, \qquad \sqcap^R\overline{\sqcap}^L = \sqcap^R, \qquad \sqcap^L\overline{\sqcap}^R = \sqcap^L. \tag{2.61}$$

The table below collects, for any weak bialgebra H, the expressions of the counital maps of the weak bialgebras H^{op}, H_{cop} and $H^{\mathsf{op}}_{\mathsf{cop}}$ in terms of the structure maps of H.

$H = (H, \mu, \eta, \Delta, \epsilon)$	$\sqcap^R$	$\sqcap^L$	$\overline{\sqcap}^R$	$\overline{\sqcap}^L$
$H^{\mathsf{op}} = (H, \mu^{\mathsf{op}}, \eta, \Delta, \epsilon)$	$\overline{\sqcap}^R$	$\overline{\sqcap}^L$	$\sqcap^R$	$\sqcap^L$
$H_{\mathsf{cop}} = (H, \mu, \eta, \Delta^{\mathsf{op}}, \epsilon)$	$\overline{\sqcap}^L$	$\overline{\sqcap}^R$	$\sqcap^L$	$\sqcap^R$
$H^{\mathsf{op}}_{\mathsf{cop}} = (H, \mu^{\mathsf{op}}, \eta, \Delta^{\mathsf{op}}, \epsilon)$	$\sqcap^L$	$\sqcap^R$	$\overline{\sqcap}^L$	$\overline{\sqcap}^R$

Table 2.1: Counital maps in the symmetric weak bialgebras.

Any identity in a weak bialgebra has symmetrical versions, obtained by 'rewriting' the identity in question in the opposite, cooposite and opposite-coopposite weak bialgebras. The table above helps on getting them when counital maps are involved in the identity.

Lemma 2.5.3. *[18, Equation (2.4)] In any weak bialgebra H over a field, the following identity holds.*

$$\Delta(1) = 1_1 \otimes \sqcap^L(1_2) = \sqcap^R(1_1) \otimes 1_2 = \sqcap^R(1_2) \otimes \sqcap^L(1_1) \tag{2.62}$$

Consequently,

$$\Delta(1) \in \sqcap^R(H) \otimes \sqcap^L(H). \tag{2.63}$$

Proposition 2.5.4. *[18, Equations (2.7a) and (2.7b)] Let H be a weak bialgebra over a field. Then, for any $h, h' \in H$,*

$$\Delta(\sqcap^R(h)) = 1_1 \otimes 1_2 \sqcap^R(h), \tag{2.64}$$

$$\Delta(\sqcap^L(h)) = \sqcap^L(h) 1_1 \otimes 1_2. \tag{2.65}$$

Therefore, $\sqcap^R(H)$ and $\sqcap^L(H)$ are respectively left and right coideals of the coalgebra H.

As a direct consequence of axiom (2.47) and Proposition 2.5.4, for any elements h, h' of a weak bialgebra H,

$$\Delta(h \sqcap^R(h')) = h_1 \otimes h_2 \sqcap^R(h'), \qquad \Delta(\sqcap^L(h)h') = \sqcap^L(h)h'_1 \otimes h'_2. \tag{2.66}$$

The axiom (2.48) expressing weak multiplicativity of the counit in a weak bialgebra admits the following equivalent reformulation.

Lemma 2.5.5. *[14] Assume that (H, μ, η) and (H, Δ, ϵ) are, respectively, an algebra and a coalgebra over the same field and they obey axioms (2.47) and (2.49). The following assertions are equivalent.*

(i) $\epsilon(ab_1)\epsilon(b_2c) = \epsilon(abc) = \epsilon(ab_2)\epsilon(b_1c)$, *for all $a, b, c \in H$.*

(ii) $\epsilon(a1_1)\epsilon(1_2c) = \epsilon(ac) = \epsilon(a1_2)\epsilon(1_1c)$, *for all $a, c \in H$.*

Proposition 2.5.6. *[18, Lemma 2.5 and Proposition 2.4] Let H be a weak bialgebra over a field. The map $\sqcap^R$ is a right $\sqcap^R(H)$–module map and $\sqcap^L$ is a left $\sqcap^L(H)$–module map. Symmetrically, $\overline{\sqcap}^R$ is a left $\sqcap^R(H)$–module map and $\overline{\sqcap}^L$ is a right $\sqcap^L(H)$–module map. In formulae, the following identities hold true for all $h, h' \in H$.*

$$\begin{aligned} &\sqcap^R(h \sqcap^R(h')) = \sqcap^R(h) \sqcap^R(h'), &&\sqcap^L(\sqcap^L(h)h') = \sqcap^L(h) \sqcap^L(h'), \\ &\overline{\sqcap}^R(\overline{\sqcap}^R(h)\ h') = \overline{\sqcap}^R(h)\ \overline{\sqcap}^R(h'), &&\overline{\sqcap}^L(h\overline{\sqcap}^L(h')) = \overline{\sqcap}^L(h)\overline{\sqcap}^L(h'). \end{aligned} \tag{2.67}$$

Moreover, for any $h, h' \in H$,

$$\sqcap^R(h) \sqcap^L(h') = \sqcap^L(h') \sqcap^R(h). \tag{2.68}$$

As a consequence of (2.56)-(2.59), (2.68) and (2.63), the following identities hold

$$\overline{\sqcap}^R\sqcap^R = \sqcap^R, \qquad \overline{\sqcap}^L\sqcap^L = \sqcap^L, \qquad \sqcap^R\overline{\sqcap}^R = \overline{\sqcap}^R, \qquad \sqcap^L\overline{\sqcap}^L = \overline{\sqcap}^L. \tag{2.69}$$

By (2.67) *and* (2.69), $\sqcap^R(H) = \overline{\sqcap}^R(H)$ *and* $\sqcap^L(H) = \overline{\sqcap}^L(H)$ *are (unital) subalgebras of* H.

The ranges of the counital maps in H are called the *base algebras*. More precisely, by Proposition 2.5.6, the coinciding image of $\sqcap^R$ and $\overline{\sqcap}^R$ in H is called the *right subalgebra*, and the coinciding image of $\sqcap^L$ and $\overline{\sqcap}^L$ is termed the *left subalgebra*.

Proposition 2.5.7. *[18, Equations 2.5a and 2.5b] For any weak bialgebra* H*, the maps* $\sqcap^R$, $\sqcap^L$, $\overline{\sqcap}^R$ *and* $\overline{\sqcap}^L$ *obey the so-called* counital properties

$$h_1 \sqcap^R (h_2) = \sqcap^L(h_1)h_2 = \overline{\sqcap}^R(h_2)h_1 = h_2\overline{\sqcap}^L(h_1) = h, \tag{2.70}$$

and

$$\begin{aligned} \sqcap^R(\sqcap^R(h)h') &= \sqcap^R(hh'), & \sqcap^L(h \sqcap^L (h')) &= \sqcap^L(hh'), \\ \overline{\sqcap}^R(h\overline{\sqcap}^R(h')) &= \overline{\sqcap}^R(hh'), & \overline{\sqcap}^L(\overline{\sqcap}^L(h)\ h') &= \overline{\sqcap}^L(hh') \end{aligned} \tag{2.71}$$

for all $h, h' \in H$.

Proposition 2.5.8. *[18, Lemma 2.3] Let* H *be a weak bialgebra over a field. The following identities hold true for any* $h, h' \in H$.

$$\sqcap^R(h_1) \otimes h_2 = 1_1 \otimes h1_2 \quad (2.72) \qquad h_1 \otimes \sqcap^L(h_2) = 1_1h \otimes 1_2 \quad (2.73)$$

$$h_1 \otimes \sqcap^R(h_2) = h1_1 \otimes \sqcap^R(1_2) \quad (2.74) \qquad \sqcap^L(h_1) \otimes h_2 = \sqcap^L(1_1) \otimes 1_2h \quad (2.75)$$

Moreover,

$$\sqcap^R((hh')_1) \otimes (hh')_2 = \sqcap^R(h'_1) \otimes hh'_2 \tag{2.76}$$

$$(hh')_1 \otimes \sqcap^L((hh')_2) = h_1h' \otimes \sqcap^L(h_2) \tag{2.77}$$

$$\sqcap^R(h_1) \otimes \sqcap^L(h_2) = \sqcap^R(h_2) \otimes \sqcap^L(h_1) \tag{2.78}$$

Theorem 2.5.1. *[18, Proposition 2.11][61] For a weak bialgebra* H*, the pairs*

$$(\epsilon_{|\sqcap^R(H)}, 1_1 \otimes \sqcap^R(1_2)) \qquad \text{and} \qquad (\epsilon_{|\sqcap^L(H)}, \sqcap^L(1_1) \otimes 1_2) \tag{2.79}$$

provide separable Frobenius structures on the right and left subalgebras $\sqcap^R(H)$ *and* $\sqcap^L(H)$*. Their respective Nakayama automorphisms are given by* $\sqcap^R\sqcap^L_{|\sqcap^R(H)}$ *and* $\sqcap^L\sqcap^R_{|\sqcap^L(H)}$.

The Frobenius structures on $\sqcap^R(H)$ and $\sqcap^L(H)$ in Theorem 2.5.1 induce on them the following Frobenius coalgebra structures:

$$(\delta_{\sqcap^R(H)} : \sqcap^R(h) \mapsto 1_1 \otimes \sqcap^R(1_2)\sqcap^R(h), \epsilon_{|\sqcap^R(H)}) \tag{2.80}$$

$$(\delta_{\sqcap^L(H)} : \sqcap^L(h) \mapsto \sqcap^L(h)\sqcap^L(1_1) \otimes 1_2, \epsilon_{|\sqcap^L(H)}). \tag{2.81}$$

Lemma 2.5.9. *Let H be a weak bialgebra over a field. The identities below hold true for any $h, h' \in H$.*

$$\sqcap^R(h\overline{\sqcap}^L(h')) = \sqcap^R(h')\sqcap^R(h) \tag{2.82}$$

$$\sqcap^L(\overline{\sqcap}^R(h)h') = \sqcap^L(h')\sqcap^L(h) \tag{2.83}$$

$$\overline{\sqcap}^R(\sqcap^L(h')h) = \overline{\sqcap}^R(h)\overline{\sqcap}^R(h') \tag{2.84}$$

$$\overline{\sqcap}^L(h'\sqcap^R(h)) = \overline{\sqcap}^L(h)\overline{\sqcap}^L(h') \tag{2.85}$$

Proposition 2.5.10. *[11, Proposition 1.18] For any weak bialgebra H, consider $\sqcap^R(H)$ and $\sqcap^L(H)$ with their coalgebra structures (2.80) and (2.81). Corestriction yields coalgebra maps $\sqcap^R : H \to \sqcap^R(H)$ and $\sqcap^L : H \to \sqcap^L(H)$. Moreover, the maps $\sqcap^R$ and $\overline{\sqcap}^L$ induce mutually inverse anti-algebra and anti-coalgebra isomorphisms between $\sqcap^L(H)$ and $\sqcap^R(H)$.*

Lemma 2.5.11. *Let H be a weak bialgebra and $R := \sqcap^R(H)$. The linear map*

$$\widetilde{\eta} : R^e \to H, \qquad s \otimes r \mapsto s\overline{\sqcap}^L(r) \tag{2.86}$$

is an algebra morphism.

Remark 2.5.12. As a consequence of the results shown along this section, we obtain the equivalence of the following five assertions for a weak bialgebra H:

(i) H is a bialgebra; (ii) $\Delta(1) = 1 \otimes 1$; (iii) $\epsilon(hh') = \epsilon(h)\epsilon(h')$;

(iv) $\sqcap^R(h) = 1\epsilon(h)$; (v) $\sqcap^L(h) = 1\epsilon(h)$.

2.5.3 Weak Hopf algebras

A *weak Hopf algebra* $(H, \mu, \eta, \Delta, \epsilon, S)$ over a field is a weak bialgebra equipped with a linear map $S : H \to H$, called the *antipode*, satisfying the following axioms for all $h \in H$.

$$S(h_1)h_2 = \sqcap^R(h) \tag{2.87}$$

$$h_1 S(h_2) = \sqcap^L(h) \tag{2.88}$$

$$S(h_1)h_2 S(h_3) = S(h) \tag{2.89}$$

In terms of the convolution product (2.31) of $\mathsf{Lin(H)}$, they can be rewritten as

$$S * \mathsf{id} = \sqcap^R, \quad (2.90) \qquad \mathsf{id} * S = \sqcap^L, \quad (2.91) \qquad S * \mathsf{id} * S = S. \quad (2.92)$$

Remark 2.5.13. It is noticeable that the antipode S of a weak Hopf algebra H is no longer a (strict) inverse of the identity map $H \to H$ in the convolution algebra $\mathsf{Lin}(H)$. However, it is a 'weak' inverse in the following sense. By (2.70), the maps $\sqcap^L$ and $\sqcap^R : H \to H$ are idempotent elements in the convolution algebra and they serve as left, respectively, right units for the identity map on H. The antipode is then a linear map $H \to H$ for whom $\sqcap^L$ and $\sqcap^R$ serve as a right, respectively, left unit (cf. (2.92)); and whose convolution products with the identity map in both possible orders yield $\sqcap^L$ and $\sqcap^R$, respectively (cf. (2.90) and (2.91)).

Lemma 2.5.14. *[18, Lemma 2.9] In a weak Hopf algebra H, the following identities hold.*

$$\sqcap^L S = \sqcap^L \sqcap^R = S \sqcap^R, \qquad \sqcap^R S = \sqcap^R \sqcap^L = S \sqcap^L \tag{2.93}$$

$$\overline{\sqcap}^L S = \sqcap^L = S \overline{\sqcap}^R, \qquad \overline{\sqcap}^R S = \sqcap^R = S \overline{\sqcap}^L. \tag{2.94}$$

Proposition 2.5.15. *[18, Theorem 2.10] The antipode S of a weak Hopf algebra H over a field is anti-multiplicative and anti-comultiplicative, that is,*

$$S(hh') = S(h')S(h) \tag{2.95}$$

$$S(h)_1 \otimes S(h)_2 = S(h_2) \otimes S(h_1) \tag{2.96}$$

for all $h, h' \in H$. The unit and the counit are S–invariant:

$$S(1) = 1 \qquad \text{and} \qquad \epsilon S = \epsilon. \tag{2.97}$$

Example 2.5.16. Consider the weak bialgebra described in Example 2.5.1. Replacing the small category C with a small groupoid G with finitely many objects, $k\mathsf{G}$ becomes a weak Hopf algebra. In fact, the map $S : k\mathsf{G} \to k\mathsf{G}$, sending every $a \in \mathsf{G}_1$ to a^{-1} obeys the antipode axioms:

$$\begin{aligned} a_1 S(a_2) &= aS(a) = aa^{-1} = t(a) \overset{(2.55)}{=} \sqcap^L_{k\mathsf{G}}(a), \\ S(a_1)a_2 &= S(a)a = a^{-1}a = s(a) \overset{(2.54)}{=} \sqcap^R_{k\mathsf{G}}(a), \\ S(a_1)a_2 S(a_3) &= S(a)aS(a) = a^{-1}aa^{-1} = a^{-1} = S(a), \end{aligned}$$

for any morphism a in G, see [53, Section 2.5].

Example 2.5.17. Take a finite groupoid G. The vector space $k(\mathsf{G})$ of k–valued functions on G is a weak Hopf algebra with the structure detailed in Example 2.5.2, and the antipode given by $S(f)(a) = f(a^{-1})$ for any $f \in k(\mathsf{G})$ and $a \in \mathsf{G}_1$.

2.6 Multiplier Hopf algebras

Multiplier Hopf algebras [69] were introduced by Alfons Van Daele in 1994. They are a generalization of the concept of Hopf algebra in a different direction of that of a weak Hopf algebra. More precisely, they are a non-unital extension of the notion of Hopf algebra with a multiplier-valued comultiplication, meaning that the comultiplication no longer lands in the tensor product of the underlying algebra but in its multiplier algebra, which is introduced next.

Let A be a non-unital algebra over a field k with a non-degenerate multiplication. A *multiplier* on A [31] is a pair (λ, ρ) of linear maps $A \to A$ such that

$$a\lambda(b) = \rho(a)b \tag{2.98}$$

for all $a, b \in A$. Then it follows that λ is a morphism of right A–modules and ρ is a map of left A–modules. The vector space of multipliers on A —via the componentwise linear structure— is known as the *multiplier algebra* of A and it is denoted by $\mathbb{M}(A)$. It is an algebra via the multiplication $(\lambda', \rho')(\lambda, \rho) = (\lambda'\lambda, \rho\rho')$ (where juxtaposition means composition) and the unit $1 = (\mathsf{id}, \mathsf{id})$. Any element $a \in A$ can be regarded as a multiplier as $(b \mapsto ab, b \mapsto ba)$. This allows us to regard A as a dense two-sided ideal in $\mathbb{M}(A)$, in the sense that the (right and left) annihilators of A in $\mathbb{M}(A)$ are trivial. Indeed, for $(\lambda, \rho) \in \mathbb{M}(A)$ and $a \in A$, $a(\lambda, \rho) = \rho(a)$ and $(\lambda, \rho)a = \lambda(a)$; and —by non-degeneracy of the multiplication— $\rho = 0$ if and only if $\lambda = 0$. So that the inclusion $A \subseteq \mathbb{M}(A)$ always holds true. The opposite one is only true if A possesses a unit. Clearly, $\mathbb{M}(A)^{\mathsf{op}} \cong \mathbb{M}(A^{\mathsf{op}})$. If B denotes a second non-unital algebra with a non-degenerate multiplication, then we have algebra embeddings $A \otimes B \subseteq \mathbb{M}(A) \otimes \mathbb{M}(B) \subseteq \mathbb{M}(A \otimes B)$. None of these inclusions will be explicitly denoted throughout the text. The multiplication in $\mathbb{M}(A)$ will be denoted by $\mu : \mathbb{M}(A) \otimes \mathbb{M}(A) \to \mathbb{M}(A)$.

Example 2.6.1. For a small category C, let $k\mathsf{C}$ be the non-unital k–algebra in Example 2.2.1. As an illustration, let us study its multiplier algebra $\mathbb{M}(k\mathsf{C})$. Since C_1 is a basis for $k\mathsf{C}$, any multiplier (λ,ρ) on $k\mathsf{C}$ can be written as

$$\lambda \;:\; k\mathsf{C} \to k\mathsf{C}, \quad a \mapsto \lambda(a) = \sum_{c\in\mathsf{C}_1} \lambda(a,c)c, \tag{2.99}$$

$$\rho \;:\; k\mathsf{C} \to k\mathsf{C}, \quad a \mapsto \rho(a) = \sum_{c\in\mathsf{C}_1} \rho(a,c)c \tag{2.100}$$

in terms of suitable scalars $\lambda(a,c),\rho(a,c)\in k$ (non-zero only finitely many of them).

Using that λ and ρ are respectively right and left $k\mathsf{C}$–module maps and that $k\mathsf{C}$ has local units, we get for any $a\in\mathsf{C}_1$

$$\lambda(a) = \lambda(t(a)a) = \lambda(t(a))a = \sum_{c\,:\,s(c)=t(a)} \lambda(s(c),c)ca, \tag{2.101}$$

$$\rho(a) = \rho(as(a)) = a\rho(s(a)) = \sum_{c\,:\,t(c)=s(a)} \rho(t(c),c)ac. \tag{2.102}$$

Now, using both identities above, the compatibility condition (2.98) held by (λ,ρ) can be rewritten as

$$\sum_{\substack{c\,:\,s(c)=t(b)\\ t(c)=s(a)}} \lambda(s(c),c)acb = \sum_{\substack{c\,:\,s(c)=t(b)\\ t(c)=s(a)}} \rho(t(c),c)acb$$

for any $a,b\in\mathsf{C}_1$. Taking a and b to be objects in C, it follows by the linear independence of the elements of C_1 that

$$\lambda(s(c),c) = \rho(t(c),c) \tag{2.103}$$

for any $c\in\mathsf{C}_1$. By (2.101), (2.102) and (2.103), we conclude that any multiplier (λ,ρ) on $k\mathsf{C}$ can be given in terms of a single function

$$\mathsf{C}_1 \to k, \quad c \mapsto \lambda(s(c),c) = \rho(t(c),c).$$

Of course, any function $\mathsf{C}_1 \to k$ does not define a multiplier on $k\mathsf{C}$. In fact, the functions which do it are precisely those whose restrictions to the sets

$$L_x := \{c\in\mathsf{C}_1 : s(c)=x\} \quad \text{and} \quad R_x := \{c\in\mathsf{C}_1 : t(c)=x\}$$

have finite support for any object x in C. In this fashion, the vector space $\mathbb{M}(k\mathsf{C})$ can be described as

$$\mathbb{M}(k\mathsf{C}) = \{\varphi : \mathsf{C}_1 \to k \mid supp(\varphi_{|L_x}),\ supp(\varphi_{|R_x}) < \infty \quad \forall x \in \mathsf{C}_0\}.$$

Lastly, consider a morphism d in C and regard it as an element of $\mathbb{M}(k\mathsf{C})$, that is, as the multiplier (λ_d, ρ_d) defined by

$$\lambda_d(a) \coloneqq da \quad \text{and} \quad \rho_d(a) \coloneqq ad \tag{2.104}$$

for any $a \in \mathsf{C}_1$. Comparing it with (2.101) and (2.102), it follows that for any arrow c in C,

$$\lambda_d(s(c), c) = \delta_{c,d} = \rho_d(t(c), c). \tag{2.105}$$

Now that we have introduced the notion of multiplier algebra, on which rests the structure that gives name to this section, let us present its definiton. A *multiplier Hopf algebra* [69, 2.3 Definition] over a field k is a triple (A, μ, Δ) where a (A, μ) is a non-unital k–algebra with a non-degenerate multiplication and $\Delta : A \to \mathbb{M}(A \otimes A)$ is a multiplicative linear map subject to the axioms below.

(i) For any $a, b \in A$, the elements

$$T_1(a \otimes b) \coloneqq \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) \coloneqq (a \otimes 1)\Delta(b) \tag{2.106}$$

of $\mathbb{M}(A \otimes A)$ belong to the two-sided ideal $A \otimes A$. (Notice that, above, 1 stands for the unit of $\mathbb{M}(A)$, introduced on page 54.)

(ii) The comultiplication is coassociative in the sense that

$$(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id}).$$

(iii) The linear maps $T_1, T_2 : A \otimes A \to A \otimes A$ defined in (2.106) are bijective.

Note that axiom (i) makes sense and that it is indeed a requirement. On the one hand, for any $a \in A$, $\Delta(a)$ is an element of $\mathbb{M}(A \otimes A)$ by definition of the comultiplication Δ. On the other hand, for any $b \in A$, $1 \otimes b$ is the multiplier $(c \otimes d \mapsto c \otimes bd, c \otimes d \mapsto c \otimes db)$ in $\mathbb{M}(A) \otimes A \subseteq \mathbb{M}(A) \otimes \mathbb{M}(A) \subseteq \mathbb{M}(A \otimes A)$. Thus, by construction, T_1 lands in $\mathbb{M}(A \otimes A)$.

Axiom (i) is precisely a constraint on its range, requiring it to stay in the ideal $A \otimes A$ of $\mathbb{M}(A \otimes A)$, giving a meaning to the coassociativity axiom as formulated in (ii). An analogous remark can be made on the requirement on T_2.

The notion of multiplier Hopf algebra covers that of Hopf algebra. In fact, if $(H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra, axioms (i) and (ii) obviously hold on H, and the axioms of the antipode assure that the maps $H \otimes H \to H \otimes H$ defined by

$$\begin{aligned} R_1(a \otimes b) &= ((\mathsf{id} \otimes S)\Delta(a))(1 \otimes b), \\ R_2(a \otimes b) &= (a \otimes 1)((S \otimes \mathsf{id})\Delta(b)) \end{aligned}$$

are respective inverses of T_1 and T_2 in (2.106).

Under the assumption of the multiplier Hopf algebra axioms on A, Theorems 3.6 and 4.6 in [69] show the existence of a counit and an antipode with the expected properties. More precisely, they prove the existence of a multiplicative linear map $\epsilon : A \to k$ obeying

$$(\epsilon \otimes \mathsf{id})T_1 = \mu = (\mathsf{id} \otimes \epsilon)T_2, \tag{2.107}$$

and an anti-multiplicative linear map $S : A \to \mathbb{M}(A)$ such that

$$\mu((\mathsf{id} \otimes S)T_2(c \otimes a)(1 \otimes b)) = c\epsilon(a)b, \tag{2.108}$$

$$\mu((c \otimes 1)(S \otimes \mathsf{id})T_1(a \otimes b)) = c\epsilon(a)b \tag{2.109}$$

for all $a, b, c \in A$. The surjectivity of the maps T_1 and T_2 makes clear that the formulas (2.107) determine ϵ. If A possesses a unit, then ϵ is a counit in the usual sense (cf. (2.28)). On the other hand, it can be checked that (2.108) and (2.109) determine S. Again, if A is unital, we obtain the classical formulas (2.46). This leads to prove that a multiplier Hopf algebra with a unit is a Hopf algebra (c.f. [69, Theorem 4.7]). Therefore, in the unital case, both structures (multiplier Hopf algebras and Hopf algebras) turn out to be equivalent.

Example 2.6.2. Let k be a field and G an arbitrary group. Consider the free k–vector space kG spanned by G with the k–algebra structure that G induces on it. Clearly, the linear extension to kG of the map $\Delta : a \to a \otimes a$ $(a \in G)$ is multiplicative and it obeys axioms (i) and (ii) in the definition of multiplier Hopf algebra. Moreover, the maps

$R_1, R_2 : G \otimes G \to G \otimes G$ defined by

$$R_1(a \otimes b) = a \otimes a^{-1}b \quad \text{and} \quad R_2(a \otimes b) := ab^{-1} \otimes b,$$

are left inverses of the maps $T_1, T_2 : G \otimes G \to G \otimes G$ given by

$$T_1(a \otimes b) = a \otimes ab \quad \text{and} \quad T_2(a \otimes b) = ab \otimes b.$$

So that axiom (iii) also holds and hence kG is, with this structure, a multiplier Hopf algebra.

Example 2.6.3. *[69, 2.5 Example]* Let k be a field and G an infinite group. Consider the k–vector space $k(G)$ of finitely supported k–valued functions on G. In contrast to Example 2.4.3 and 2.4.5, now the function $G \to k$ constantly equal to 1 does not work as a unit since it has not finite support. So that $k(G)$ is a non-unital k–algebra with the pointwise multiplication. In this case, $\mathbb{M}(k(G))$ consists of all k–valued functions on G. Moreover, $k(G) \otimes k(G)$ can be naturally identified with finitely supported k–valued functions on $G \times G$, being $\mathbb{M}(k(G) \otimes k(G))$ the space of all k–valued functions on $G \times G$. Let $f, g \in k(G)$ and $(s,t) \in G \times G$. The map $\Delta : k(G) \to k(G \times G)$ defined by

$$\Delta(f)(s,t) := f(st)$$

is clearly multiplicative. For any $f, g \in k(G)$, the maps

$$T_1(f \otimes g)(s,t) := f(st)g(t) \quad \text{and} \quad T_2(f \otimes g)(s,t) := f(s)g(st)$$

have finite support, so they obey axiom (i) in the definition of multiplier Hopf algebra. The coassociativity law (ii) is an immediate consequence of the associativity of the multiplication of G. Finally, the maps defined by

$$R_1(f)(s,t) := f(st^{-1}, t) \quad \text{and} \quad R_2(f)(s,t) := f(s, s^{-1}t)$$

for any $f \in k(G \times G)$, are respective inverses of the above maps T_1 and T_2, hence also axiom (iii) holds.

In [40] a categorical interpretation of multiplier Hopf algebras was proposed.

2.7 Weak multiplier Hopf algebras

Weak multiplier Hopf algebras [73] were introduced by Alfons Van Daele and Shuanhong Wang in 2012. For their definition, as in the case of multiplier Hopf algebras in [69], a philosophy based on minimal input data is applied. However, the weakness of weak multiplier Hopf algebras requires to assume the existence of further structure which in the non-weak multiplier case is derived from the axioms.

As we saw in previous Section 2.6, a Hopf algebra is always a multiplier Hopf algebra. In order to motivate the definition of a weak multiplier Hopf algebra let us briefly analyze the main reason for which a weak Hopf algebra A fails to be a multiplier Hopf algebra. Essentially, it is due to the *weak* comultiplicativity of the unit. Recall that the comultiplication Δ of A is not required to preserve the unit 1 (i.e. $\Delta(1)$ may differ from $1 \otimes 1$). Consequently, the maps T_1 and T_2 are no longer linear automorphisms of $A \otimes A$. Instead, they induce isomorphisms between some canonical vector subspaces determined by the element $\Delta(1)$. Thus, in the situation when the underlying algebra A is allowed to possess no unit, in the definition of weak multiplier Hopf algebra in [73] the role of $\Delta(1)$ is played by an idempotent element E in the multiplier algebra of $A \otimes A$, which is meant to be part of the structure. It is used to single out the canonical vector subspaces $E(A \otimes A)$ and $(A \otimes A)E$ of $A \otimes A$. The maps T_1 and T_2 are required to induce isomorphisms between certain vector subspaces of $A \otimes A$ and the same coassociativity axiom $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})$ is imposed. In contrast to the case of multiplier Hopf algebras, however, these axioms do not seem to imply the existence and the expected properties of the counit and the antipode. Therefore in the definition of a weak multiplier Hopf algebra also the existence of a counit $\epsilon : A \to k$ is assumed (in the same sense of (2.107)). Adding these counit axioms, the existence of the antipode and *most* [4] of the expected properties of the counit and the antipode do follow. The motivating examples are —as the reader at this point can certainly guess— the linear spans of arbitrary groupoids and the vector space of base field-valued functions with finite support on arbitrary groupoids.

Prior to the definition of weak multiplier Hopf algebra, some preliminary results are needed. The following theorem gives sufficient conditions on a multiplicative linear

[4] The emphasis of the word *most* in the sentence will be explained in detail in Chapter 3.

map $A \to \mathbb{M}(B)$ to extend to $\mathbb{M}(A)$.

Theorem 2.7.1. *[72, Proposition A.3] Let A and B be non-unital algebras with non-degenerate multiplications and $\gamma : A \to \mathbb{M}(B)$ be a multiplicative linear map. Assume that there is an idempotent element $e \in \mathbb{M}(B)$ such that*

$$\langle \gamma(a)b \mid a \in A,\ b \in B\rangle = \{eb \mid b \in B\} \quad \textit{and} \quad \langle b\gamma(a) \mid a \in A,\ b \in B\rangle = \{be \mid b \in B\}.$$

Then there is a unique multiplicative linear map $\overline{\gamma} : \mathbb{M}(A) \to \mathbb{M}(B)$ such that $\overline{\gamma}(1) = e$ and $\overline{\gamma}(a) = \gamma(a)$, for all $a \in A$.

It is worth recalling the way in which this extension $\overline{\gamma} : \mathbb{M}(A) \to \mathbb{M}(B)$ is constructed: For any multiplier l on A,

$$\overline{\gamma}(l)x := \sum_i \gamma(la_i)b_i, \quad y\overline{\gamma}(l) := \sum_j c_j\gamma(d_jl) \tag{2.110}$$

for any $x, y \in B$ such that $ex = \sum_i \gamma(a_i)b_i$ and $ye = \sum_j c_j\gamma(d_j)$ for $a_i, d_j \in A$, $b_i, c_j \in B$ (the existence of these elements is assured by the assumptions on γ).

If for some map γ there exists an idempotent element e as in Theorem 2.7.1, then it is clearly unique (cf. [73, Proposition 1.6]).

Let A be an idempotent non-unital algebra over a field k with a non-degenerate multiplication and let $\Delta : A \to \mathbb{M}(A \otimes A)$ be a multiplicative linear map. Assume that there is an idempotent element $E \in \mathbb{M}(A \otimes A)$ such that

$$\langle \Delta(a)(b \otimes b') \mid a, b, b' \in A\rangle = \langle E(b \otimes b') \mid b, b' \in A\rangle \quad \text{and} \tag{2.111}$$

$$\langle (b \otimes b')\Delta(a) \mid a, b, b' \in A\rangle = \langle (b \otimes b')E \mid b, b' \in A\rangle \tag{2.112}$$

as k–vector spaces. Then by Theorem 2.7.1, there exist the extended multiplicative maps $\overline{\Delta} : \mathbb{M}(A) \to \mathbb{M}(A \otimes A)$, $\overline{\Delta \otimes \mathsf{id}} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$ and $\overline{\mathsf{id} \otimes \Delta} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$ such that

$$\overline{\Delta}(1) = E. \tag{2.113}$$

If for any $a \otimes b \in A \otimes A$ the maps $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1)\Delta(b)$ land in $A \otimes A$, and they satisfy $(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})$, then it follows by [72, Proposition A.8] that $(\overline{\mathsf{id} \otimes \Delta})(E) = (\overline{\Delta \otimes \mathsf{id}})(E)$. This allows us to define the idempotent element

$$E^{(3)} := (\overline{\mathsf{id} \otimes \Delta})(E) = (\overline{\Delta \otimes \mathsf{id}})(E) \tag{2.114}$$

in $\mathbb{M}(A \otimes A \otimes A)$. Define also the multiplicative linear map $\Delta^{\mathsf{op}} : A \to \mathbb{M}(A \otimes A)$ via

$$\Delta^{\mathsf{op}}(a)(b \otimes c) := \mathsf{tw}(\Delta(a)(c \otimes b)) \quad \text{and} \quad (b \otimes c)\Delta^{\mathsf{op}}(a) := \mathsf{tw}((c \otimes b)\Delta(a)) \tag{2.115}$$

and the map $\Delta_{13} : A \to \mathbb{M}(A \otimes A \otimes A)$ by

$$\Delta_{13}(a)(b \otimes c \otimes d) := (\mathsf{id} \otimes \mathsf{tw})(\Delta(a)(b \otimes d) \otimes c) \quad \text{and}$$
$$(b \otimes c \otimes d)\Delta_{13}(a) := (\mathsf{id} \otimes \mathsf{tw})((b \otimes d)\Delta(a) \otimes c).$$

By Theorem 2.7.1, taking as idempotent element $\mathsf{tw}E\mathsf{tw}$, also Δ^{op} extends to multiplicative maps $\overline{\Delta^{\mathsf{op}}} : \mathbb{M}(A) \to \mathbb{M}(A \otimes A)$, $\overline{\Delta^{\mathsf{op}} \otimes \mathsf{id}} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$ and $\overline{\mathsf{id} \otimes \Delta^{\mathsf{op}}} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$. The explicit forms of all these maps are easily computed using (2.110). Since we will need them later on in Chapter 4, next we show that one of $\overline{\Delta^{\mathsf{op}} \otimes \mathsf{id}}$ as an illustration. For any $l \in M(A \otimes A), u, v, w, x, y, z \in A$,

$$(\overline{\Delta^{\mathsf{op}} \otimes \mathsf{id}})(l)(u \otimes v \otimes w) := \sum_i [(\Delta^{\mathsf{op}} \otimes \mathsf{id})(l(a_i \otimes b_i))](a'_i \otimes b'_i \otimes c'_i) \tag{2.116}$$
$$(x \otimes y \otimes z)(\overline{\Delta^{\mathsf{op}} \otimes \mathsf{id}})(l) := \sum_i (d'_j \otimes e'_j \otimes f'_j)[(\Delta^{\mathsf{op}} \otimes \mathsf{id})((d_j \otimes e_j)l)] \tag{2.117}$$

where $a_i, b_i, a'_i, b'_i, c'_i, d_j, e_j, d'_j, e'_j, f'_j \in A$ such that

$$(\mathsf{tw}E\mathsf{tw} \otimes 1)(u \otimes v \otimes w) = \sum_i [(\Delta^{\mathsf{op}} \otimes \mathsf{id})(a_i \otimes b_i)](a'_i \otimes b'_i \otimes c'_i),$$
$$(x \otimes y \otimes z)(\mathsf{tw}E\mathsf{tw} \otimes 1) = \sum_j (d'_j \otimes e'_j \otimes f'_j)[(\Delta^{\mathsf{op}} \otimes \mathsf{id})(d_j \otimes e_j)]$$

(the existence of such elements is guaranteed by the previous assumptions (2.111) and (2.112)). Calculating further, we have

$$(\mathsf{tw}E\mathsf{tw} \otimes 1)(u \otimes v \otimes w) = \sum_i (\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})[((\mathsf{id} \otimes \Delta)(b_i \otimes a_i))(c'_i \otimes b'_i \otimes a'_i)]$$

and hence,

$$(1 \otimes E)(w \otimes v \otimes u) = ((\mathsf{id} \otimes \Delta)(b_i \otimes a_i))(c'_i \otimes b'_i \otimes a'_i). \tag{2.118}$$

Proposition 2.7.1. *[73, 1.11 Proposition] Let A be an idempotent non-unital algebra* *over a field with a non-degenerate multiplication and let* $\Delta : A \to \mathbb{M}(A \otimes A)$ *be a multiplicative linear map. Assume that there is an idempotent element* $E \in \mathbb{M}(A \otimes A)$ *such that*

$$\langle \Delta(a)(b \otimes b') \mid a, b, b' \in A \rangle = \langle E(b \otimes b') \mid b, b' \in A \rangle \quad \textit{and}$$
$$\langle (b \otimes b')\Delta(a) \mid a, b, b' \in A \rangle = \langle (b \otimes b')E \mid b, b' \in A \rangle.$$

Then there exist linear maps $G_1, G_2 : A \otimes A \to A \otimes A$ *characterized by the equalities*

$$\begin{aligned}(G_1 \otimes \mathsf{id})[\Delta_{13}(a)(1 \otimes b \otimes c)] &= \Delta_{13}(a)(1 \otimes E)(1 \otimes b \otimes c),\\ (\mathsf{id} \otimes G_2)[(a \otimes b \otimes 1)\Delta_{13}(c)] &= (a \otimes b \otimes 1)(E \otimes 1)\Delta_{13}(c)\end{aligned}$$

for all $a, b, c \in A$.

A *weak multiplier Hopf algebra* (A, μ, Δ) [73, 1.14 Definition] is a non-degenerate idempotent non-unital algebra (A, μ) over a field k equipped with a multiplicative linear map (called *comultiplication*) $\Delta : A \to \mathbb{M}(A \otimes A)$ and a linear map (called *counit*) $\epsilon : A \to k$ subject to the following axioms.

(i) For any $a, b \in A$, the elements

$$T_1(a \otimes b) := \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) := (a \otimes 1)\Delta(b)$$

of $\mathbb{M}(A \otimes A)$ belong to the two-sided ideal $A \otimes A$.

(ii) The comultiplication is coassociative in the sense that

$$(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id}).$$

(iii) The counit obeys

$$(\epsilon \otimes \mathsf{id})T_1 = \mu = (\mathsf{id} \otimes \epsilon)T_2.$$

(iv) There exists an idempotent $E \in \mathbb{M}(A \otimes A)$ giving the ranges of T_1 and T_2:

$$E(A \otimes A) = T_1(A \otimes A) \quad \text{and} \quad (A \otimes A)E = T_2(A \otimes A).$$

(v) The element $E \in \mathbb{M}(A \otimes A)$ satisfies the equality

$$(E \otimes 1)(1 \otimes E) = E^{(3)} = (1 \otimes E)(E \otimes 1)$$

in $\mathbb{M}(A \otimes A \otimes A)$, c.f. (2.114).

(vi) The kernels of the maps T_1 and T_2 are of the form

$$\begin{aligned} ker(T_1) &= (\mathsf{id} - G_1)(A \otimes A) \\ ker(T_2) &= (\mathsf{id} - G_2)(A \otimes A), \end{aligned}$$

where $G_1, G_2 : A \otimes A \to A \otimes A$ are the linear maps in Proposition 2.7.1.

(vii) The smallest k–vector subspaces V and W of A satisfying

$$\Delta(A)(1 \otimes A) \subseteq V \otimes A \quad \text{and} \quad (A \otimes 1)\Delta(A) \subseteq A \otimes W \tag{2.119}$$

are $V = W = A$.

Axiom (vii) is called *fullness* of the coproduct and it is imposed to assure the uniqueness of the counit. If the coproduct is full, it follows that any element of A is a linear combination of elements of the form $(\mathsf{id} \otimes \omega)(\Delta(a)(1 \otimes b))$ with $a, b \in A$ and $\omega \in A^*$, and reciprocally (see [72, 1.11 Lemma]). Roughly speaking, it means that the 'legs' of the comultiplication are all of A.

As in the non-weak case of multiplier Hopf algebras, any weak multiplier Hopf algebra A is proven to possess a map $S : A \to \mathbb{M}(A)$, called the *antipode*, generalizing the properties of the antipode in more restrictive settings (see [73, Proposition 2.4 and Proposition 2.7]).

Following [69, Definition 2.3], a multiplier Hopf algebra (A, μ, Δ) (as introduced in the previous Section 2.6) is called *regular* if $(A, \mu, \Delta^{\mathsf{op}})$ (cf. (2.115)) is also a multiplier Hopf algebra. In the same vein, a weak multiplier Hopf algebra (A, μ, Δ) is said to be *regular* if, for all $a, b \in A$, the elements

$$T_3(a \otimes b) := (1 \otimes b)\Delta(a) \quad \text{and} \quad T_4(a \otimes b) := \Delta(b)(a \otimes 1)$$

of $\mathbb{M}(A \otimes A)$ belong to the two-sided ideal $A \otimes A$, and $(A, \mu, \Delta^{\mathsf{op}})$ (cf. (2.115)) is a weak multiplier Hopf algebra (see Definition 1.1 and Definition 4.1 in [73]). This is equivalent to require $(A, \mu^{\mathsf{op}}, \Delta)$ to be a weak multiplier Hopf algebra. Moreover, in [73, Theorem 4.10], the authors provide a characterization of regular weak multiplier Hopf algebras, proving that a weak multiplier Hopf algebra A is regular if and only if its antipode maps A into A and it is a bijection.

Chapter 3

Categories of bimonoids

A suggestive example in [5, Example 6.43] says that small categories can be described as bimonoids in an appropriately chosen duoidal category: the category of spans over a given set (the set of objects). This construction is revisited in Section 3.1. By this motivation, with the purpose of locating weak bialgebras in a categorical framework, in Section 3.2, we aim to find an appropriate duoidal category whose bimonoids are 'quantum categories'; that is, weak bialgebras. Inspired by the description of bialgebroids whose base algebra R is central, as bimonoids in the duoidal category of R–bimodules [5, Example 6.44], we study the category of bimodules over $R \otimes R^{\mathsf{op}}$ for a separable Frobenius algebra R. Observing that in this case the Takeuchi's $\times_R$–product becomes isomorphic to some (twisted) bimodule tensor product over $R \otimes R^{\mathsf{op}}$, we equip this category with a duoidal structure. Moreover, we show that its bimonoids are precisely the weak bialgebras whose base algebra is isomorphic to R. This interpretation is used to define a category wba of weak bialgebras over a given field. As an application, in Section 3.3, the "free vector space" functor from the category cat^0 of small categories with finitely many objects to wba is shown to possess a right adjoint, given by taking (certain) group-like elements. This adjunction is proven to restrict to the full subcategories of groupoids and of weak Hopf algebras, respectively. As a corollary, we obtain equivalences between cat^0 and the category of pointed cosemisimple weak bialgebras; and between the category of small groupoids with finitely many objects and the category of pointed cosemisimple weak Hopf algebras. This extends the well-known relation between groups and pointed cosemisimple Hopf algebras, see for example [1].

All these results are based on the use of a general categorical construction: to any functor $\mathcal{M}$ from an arbitrary category to the category duo of duoidal categories (recall its definition on page 34), we associate a category $\mathsf{bmd}(\mathcal{M})$ of some bimonoids. On the one hand, when applying it to the functor $\mathsf{span} : \mathsf{set} \to \mathsf{duo}$, we recognize the category $\mathsf{bmd}(\mathsf{span})$ as the category cat. On the other hand, when applying it to the functor $\mathsf{bim}(-^e) : \mathsf{sfr} \to \mathsf{duo}$ constructed in Section 3.2 (recall the definition of sfr on page 44), we show that the objects in $\mathsf{bmd}(\mathsf{bim}(-^e))$ are precisely pairs (R, H) of weak bialgebras H whose right subalgebra is isomorphic to R; and that the morphisms $(R, H) \to (R', H')$ in $\mathsf{bmd}(\mathsf{bim}(-^e))$ can be identified with weakly multiplicative coalgebra maps commuting with $\sqcap^R$ and $\sqcap'^R$, $\overline{\sqcap}^R$ and $\overline{\sqcap}'^R$, and the Nakayama automorphisms of R and R'.

For an arbitrary category S, consider a functor $\mathcal{M} : \mathsf{S} \to \mathsf{duo}$. Let us associate a category (of some bimonoids) to $\mathcal{M}$.

Lemma 3.0.2. *Let X and X' be objects of S and let H and H' be bimonoids in $\mathcal{M}X$ and $\mathcal{M}X'$, respectively. For a morphism $q : X \to X'$ in S and a morphism $Q : (\mathcal{M}q)H \to H'$ in $\mathcal{M}X'$, the following assertions are equivalent.*

(a) The functor $\mathcal{M}q : \mathcal{M}X \to \mathcal{M}X'$ and the natural transformation

$$(\mathcal{M}q)(- \bullet H) \xrightarrow{(\mathcal{M}q)_2^\bullet} (\mathcal{M}q)(-) \bullet' (\mathcal{M}q)H \xrightarrow{(\mathcal{M}q)(-)\bullet' Q} (\mathcal{M}q)(-) \bullet' H'$$

constitute a morphism of monoidal comonads $((\mathcal{M}X, \circ), (-) \bullet H) \to ((\mathcal{M}X', \circ'), (-) \bullet' H')$.

(b) The following diagrams commute, for any objects A, B of $\mathcal{M}X$.

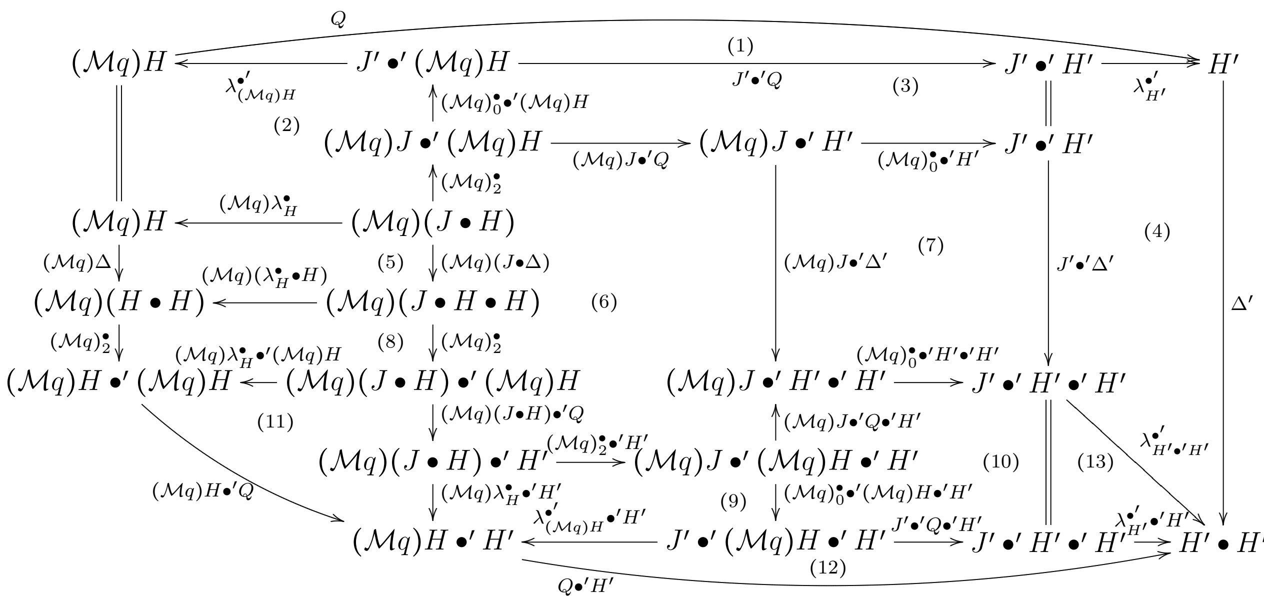
$$\begin{array}{ccccc} (\mathcal{M}q)H & & \xrightarrow{\quad Q \quad} & & H' \\ \downarrow{\scriptstyle (\mathcal{M}q)\Delta} & & & & \downarrow{\scriptstyle \Delta'} \\ (\mathcal{M}q)(H \bullet H) & \xrightarrow[(\mathcal{M}q)_2^\bullet]{} & (\mathcal{M}q)H \bullet' (\mathcal{M}q)H & \xrightarrow[Q\bullet' Q]{} & H' \bullet' H' \end{array} \tag{3.1}$$

$$
\begin{array}{ccccc}
(\mathcal{M}q)H & & \xrightarrow{\quad Q \quad} & & H' \\
{\scriptstyle (\mathcal{M}q)\epsilon}\downarrow & & & & \downarrow{\scriptstyle \epsilon'} \\
(\mathcal{M}q)J & \xrightarrow[(\mathcal{M}q)_0^{\bullet}]{} & J' & = & J'
\end{array}
\tag{3.2}
$$

$$
\begin{array}{ccc}
(\mathcal{M}q)((A \bullet H) \circ (B \bullet H)) & \xrightarrow{(\mathcal{M}q)\gamma} & (\mathcal{M}q)((A \circ B) \bullet (H \circ H)) \\
{\scriptstyle (\mathcal{M}q)_2^{\circ}}\downarrow & & \downarrow{\scriptstyle (\mathcal{M}q)_2^{\bullet}} \\
(\mathcal{M}q)(A \bullet H) \circ' (\mathcal{M}q)(B \bullet H) & & (\mathcal{M}q)(A \circ B) \bullet' (\mathcal{M}q)(H \circ H) \\
{\scriptstyle (\mathcal{M}q)_2^{\bullet} \circ' (\mathcal{M}q)_2^{\bullet}}\downarrow & & \downarrow{\scriptstyle (\mathcal{M}q)_2^{\circ} \bullet' (\mathcal{M}q)(H \circ H)} \\
((\mathcal{M}q)A \bullet' (\mathcal{M}q)H) \circ' ((\mathcal{M}q)B \bullet' (\mathcal{M}q)H) & & ((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' (\mathcal{M}q)(H \circ H) \\
{\scriptstyle \gamma'}\downarrow & & \downarrow{\scriptstyle ((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' (\mathcal{M}q)\mu} \\
((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' ((\mathcal{M}q)H \circ' (\mathcal{M}q)H) & & ((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' (\mathcal{M}q)H \\
{\scriptstyle ((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' (Q \circ' Q)}\downarrow & & \downarrow{\scriptstyle ((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' Q} \\
((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' (H' \circ' H') & \xrightarrow[((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' \mu']{} & ((\mathcal{M}q)A \circ' (\mathcal{M}q)B) \bullet' H'
\end{array}
\tag{3.3}
$$

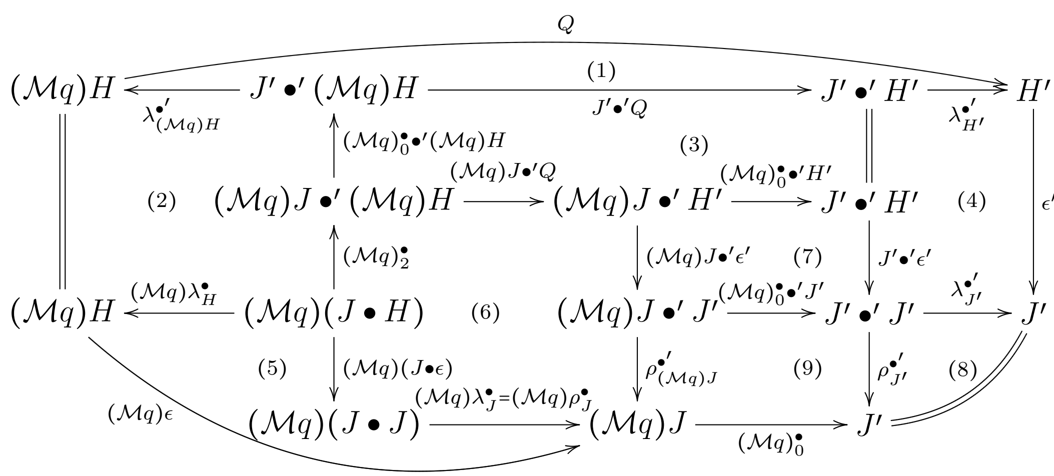
$$
\begin{array}{ccccc}
(\mathcal{M}q)I & & \xrightarrow{\quad (\mathcal{M}q)_0^{\circ} \quad} & & I' \\
{\scriptstyle (\mathcal{M}q)\Delta}\downarrow & & & & \\
(\mathcal{M}q)(I \bullet I) & & & & \downarrow{\scriptstyle \Delta'} \\
{\scriptstyle (\mathcal{M}q)_2^{\bullet}}\downarrow & & & & \\
(\mathcal{M}q)I \bullet' (\mathcal{M}q)I & & & & I' \bullet' I' \\
{\scriptstyle (\mathcal{M}q)I \bullet' (\mathcal{M}q)\eta}\downarrow & & & & \downarrow{\scriptstyle I' \bullet' \eta'} \\
(\mathcal{M}q)I \bullet' (\mathcal{M}q)H & \xrightarrow[(\mathcal{M}q)I \bullet' Q]{} & (\mathcal{M}q)I \bullet' H' & \xrightarrow[(\mathcal{M}q)_0^{\circ} \bullet' H]{} & I' \bullet' H'
\end{array}
\tag{3.4}
$$

Proof. Assume first that the functor $\mathcal{M}q$ and the natural transformation $((\mathcal{M}q)(-) \bullet' Q)(\mathcal{M}q)_2^{\bullet}$ constitute a morphism of monoidal comonads. Diagram (3.1) is the outer path of the diagram:



where (1), (4) and (12) commute by naturality of $\lambda^{\bullet'}$; (2) and (9) do by $\bullet$–comonoidality of $\mathcal{M}q$; (3), (7), (10) and (11) commute by functoriality of the monoidal product $\bullet'$; the commutativity of (5) and (13) follows by the coherence axioms on $\lambda^{\bullet}$ and $\lambda^{\bullet'}$; (6) is commutative by assumption (compatibility with the comultiplications of any morphism of monoidal comonads) and (8) is so by naturality of $(\mathcal{M}q)_2^{\bullet}$. Since $(\mathcal{M}q)\lambda_H^{\bullet}$ is an isomorphism, this proves commutativity of (3.1).

Using again that $(\mathcal{M}q)\lambda_H^{\bullet}$ is an isomorphism, commutativity of (3.2) is derived from



where (1), (4) and (5) commute by naturality of $\lambda^{\bullet'}$ and $\lambda^{\bullet}$; (2) does by $\bullet$–comonoidality of $\mathcal{M}q$; (3) and (7) by functoriality of $\bullet'$; (6) is commutative by assumption; (8) commutes by the identity $\rho^{\bullet'}_{J'} = \lambda^{\bullet'}_{J'}$ (holding true in every monoidal category) and (9) does by naturality of $\rho^{\bullet'}$.

The diagrams (3.3) and (3.4) are literally the diagrams shown in (2.1), rendered commutative by any morphism of monoidal comonads by definition.

For the converse implication, this last argument is valid to justify the commutativity of the diagrams in (2.1), assuming the commutativity of (3.3) and (3.4). As for the compatibility of $(\mathcal{M}q(-) \bullet' Q)(\mathcal{M}q)^{\bullet}_2$ with the comultiplications and the counits concerns, consider respectively the diagrams

$$\begin{CD}
(\mathcal{M}q)(A \bullet H) @>{(\mathcal{M}q)^{\bullet}_2}>> (\mathcal{M}q)A \bullet' (\mathcal{M}q)H @>{(\mathcal{M}q)A\bullet' Q}>> (\mathcal{M}q)A \bullet' H' \\
@V{(\mathcal{M}q)(A\bullet\Delta)}VV @V{(\mathcal{M}q)A\bullet'(\mathcal{M}q)\Delta}VV @V{(\mathcal{M}q)A\bullet'\Delta'}VV \\
(\mathcal{M}q)(A \bullet H \bullet H) @>{(\mathcal{M}q)^{\bullet}_2}>> (\mathcal{M}q)A \bullet' (\mathcal{M}q)(H \bullet H) @. \text{(3.1)} \\
@V{(\mathcal{M}q)^{\bullet}_2}VV @V{(\mathcal{M}q)A\bullet'(\mathcal{M}q)^{\bullet}_2}VV @VVV \\
(\mathcal{M}q)(A \bullet H) \bullet' (\mathcal{M}q)H @>{(\mathcal{M}q)^{\bullet}_2\bullet'(\mathcal{M}q)H}>> (\mathcal{M}q)A \bullet' (\mathcal{M}q)H \bullet' (\mathcal{M}q)H @>{(\mathcal{M}q)A\bullet' Q\bullet' Q}>> (\mathcal{M}q)A \bullet' H' \bullet' H' \\
@V{(\mathcal{M}q)(A\bullet H)\bullet' Q}VV @. @| \\
(\mathcal{M}q)(A \bullet H) \bullet' H' @>>{(\mathcal{M}q)^{\bullet}_2\bullet' H'}> (\mathcal{M}q)A \bullet' (\mathcal{M}q)H \bullet' H' @>>{(\mathcal{M}q)A\bullet' Q\bullet' H'}> (\mathcal{M}q)A \bullet' H' \bullet' H',
\end{CD}$$

where the unlabelled squares commute by naturality and coassociativity of $(\mathcal{M}q)^{\bullet}_2$, and

$$\begin{CD}
(\mathcal{M}q)(A \bullet H) @>{(\mathcal{M}q)^{\bullet}_2}>> (\mathcal{M}q)A \bullet' (\mathcal{M}q)H @>{(\mathcal{M}q)A\bullet' Q}>> (\mathcal{M}q)(A) \bullet' H' \\
@V{(\mathcal{M}q)(A\bullet\epsilon)}VV @V{(\mathcal{M}q)A\bullet'(\mathcal{M}q)\epsilon}VV \text{(3.2)} @V{(\mathcal{M}q)A\bullet'\epsilon'}VV \\
(\mathcal{M}q)(A \bullet J) @>{(\mathcal{M}q)^{\bullet}_2}>> (\mathcal{M}q)A \bullet' (\mathcal{M}q)J @>{(\mathcal{M}q)A\bullet'(\mathcal{M}q)^{\bullet}_0}>> (\mathcal{M}q)A \bullet' J' \\
@V{(\mathcal{M}q)\rho^{\bullet}_A}VV @. @V{\rho^{\bullet'}_{(\mathcal{M}q)A}}VV \\
(\mathcal{M}q)A @= \quad @= (\mathcal{M}q)A,
\end{CD}$$

where the unlabelled squares commute by naturality and counitality of $(\mathcal{M}q)^{\bullet}_2$. □

Remark 3.0.3. If the functor $\mathcal{M}q : \mathcal{M}X \to \mathcal{M}X'$ is double comonoidal in the sense of [5, Definition 6.55] (see this definition on page 34), then the commutativity of the diagrams below assures that of the last two diagrams in part (b) of Lemma 3.0.2.

$$\begin{array}{ccccc} (\mathcal{M}q)(H\circ H) & \xrightarrow{(\mathcal{M}q)^\circ_2} & (\mathcal{M}q)H \circ' (\mathcal{M}q)H & \xrightarrow{Q\circ' Q} & H'\circ' H' \\ {\scriptstyle (\mathcal{M}q)\mu}\downarrow & & & & \downarrow{\scriptstyle \mu'} \\ (\mathcal{M}q)H & & \xrightarrow[Q]{\qquad\qquad} & & H' \end{array} \qquad (3.5)$$

$$\begin{array}{ccc} (\mathcal{M}q)I & \xrightarrow{(\mathcal{M}q)^\circ_0} & I' \\ {\scriptstyle (\mathcal{M}q)\eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ (\mathcal{M}q)H & \xrightarrow[Q]{} & H' \end{array} \qquad (3.6)$$

Indeed, for any objects A, B in $\mathcal{M}X$, the diagrams

$$\begin{array}{ccc} (\mathcal{M}q)(A\bullet H)\circ'(\mathcal{M}q)(B\bullet H) & \xleftarrow[(\mathcal{M}q)^\circ_2]{\qquad} & (\mathcal{M}q)((A\bullet H)\circ(B\bullet H)) \\ {\scriptstyle (\mathcal{M}q)^\bullet_2\circ'(\mathcal{M}q)^\bullet_2}\downarrow & & \downarrow{\scriptstyle (\mathcal{M}q)(\gamma)} \\ ((\mathcal{M}q)A\bullet'(\mathcal{M}q)H)\circ'((\mathcal{M}q)B\bullet'(\mathcal{M}q)H) & (2.10) & (\mathcal{M}q)((A\circ B)\bullet(H\circ H)) \\ {\scriptstyle \gamma'}\downarrow & & \downarrow{\scriptstyle (\mathcal{M}q)^\bullet_2} \\ ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'((\mathcal{M}q)H\circ'(\mathcal{M}q)H) & & (\mathcal{M}q)(A\circ B)\bullet'(\mathcal{M}q)(H\circ H) \\ \Vert & & \downarrow{\scriptstyle (\mathcal{M}q)^\circ_2\bullet'(\mathcal{M}q)(H\circ H)} \\ ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'((\mathcal{M}q)H\circ'(\mathcal{M}q)H) & \xleftarrow[((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet(\mathcal{M}q)^\circ_2]{\qquad} & ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'(\mathcal{M}q)(H\circ H) \\ & & \downarrow{\scriptstyle ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'(\mathcal{M}q)\mu} \\ {\scriptstyle ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'(Q\circ'Q)}\downarrow & (3.5) & ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'(\mathcal{M}q)H \\ & & \downarrow{\scriptstyle ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'Q} \\ ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'(H'\circ'H') & \xrightarrow[((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet\mu']{\qquad} & ((\mathcal{M}q)A\circ'(\mathcal{M}q)B)\bullet'H' \end{array}$$

and

$$\begin{array}{ccccc} (\mathcal{M}q)I & & \xrightarrow{(\mathcal{M}q)^\circ_0} & & I' \\ {\scriptstyle (\mathcal{M}q)\Delta_I}\downarrow & & & & \\ (\mathcal{M}q)(I\bullet I) & & (2.11) & & \downarrow{\scriptstyle \Delta_{I'}} \\ {\scriptstyle (\mathcal{M}q)^\bullet_2}\downarrow & & & & \\ (\mathcal{M}q)I\bullet'(\mathcal{M}q)I & & \xrightarrow{(\mathcal{M}q)^\circ_0\bullet'(\mathcal{M}q)^\circ_0} & & I'\bullet I' \\ {\scriptstyle (\mathcal{M}q)I\bullet'(\mathcal{M}q)\eta}\downarrow & & (3.6) & & \downarrow{\scriptstyle I'\bullet'\eta'} \\ (\mathcal{M}q)I\bullet'(\mathcal{M}q)H & \xrightarrow[(\mathcal{M}q)I\bullet'Q]{} & (\mathcal{M}q)I\bullet'H' & \xrightarrow[(\mathcal{M}q)^\circ_0\bullet'H']{} & I'\bullet'H' \end{array}$$

are commutative. However, in our most important example in Section 3.2, the functors $\mathcal{M}q : \mathcal{M}X \to \mathcal{M}X'$ are not double comonoidal. So we need to cope with the more general situation in Lemma 3.0.2.

Definition 3.0.4. Let $\mathcal{M}$ be a functor from an arbitrary category S to the category duo of duoidal categories. The associated category $\mathsf{bmd}(\mathcal{M})$ is defined to have objects which are pairs (X, H) consisting of an object X of S and a bimonoid H in $\mathcal{M}X$. Morphisms are pairs (q, Q) of a morphism $q : X \to X'$ in S and a morphism $Q : (\mathcal{M}q)H \to H'$ in $\mathcal{M}X'$, obeying the equivalent conditions in Lemma 3.0.2.

Since the composite of any morphisms of monoidal comonads (page 32) is a morphism of monoidal comonads again, the composition of morphisms in $\mathsf{bmd}(\mathcal{M})$ is well defined by their description in part (a) of Lemma 3.0.2.

If S is the singleton category $\mathbf{1}$, then the functors $\mathcal{M} : \mathbf{1} \to \mathsf{duo}$ are in bijection with the objects of duo; that is, with the duoidal categories M. In this case $\mathsf{bmd}(\mathcal{M})$ is the usual category of bimonoids in the duoidal category M: Its objects are the bimonoids and its arrows are the morphisms in M which are both morphisms of monoids (w.r.t. $\circ$) and morphisms of comonoids (w.r.t. $\bullet$). Indeed, if $(*, H)$ and $(*, H')$ are objects in $\mathsf{bmd}(\mathcal{M})$, then a morphism between them in $\mathsf{bmd}(\mathcal{M})$ is given by the pair $(q = \mathsf{id}, Q : (\mathcal{M}\mathsf{id})H = \mathsf{id}_{\mathsf{M}}(H) = H \to H')$, that is, by a morphism $Q : H \to H'$ in M obeying the corresponding conditions in Lemma 3.0.2. It can be easily checked that, with this datum, (3.1), (3.2), (3.3) and (3.4) turn out to be, respectively, the compatibility of Q with the comultiplications, the counits, the multiplications and the units of the bimonoids.

Remark 3.0.5. Note that Definition 3.0.4 is one choice of several symmetric possibilities. With this choice, we obtain the adjunction shown in Subsection 3.3.1 and Subsection 3.3.3. An analogous definition could be based on the monoidal comonad $((MX, \circ), H \bullet (-))$. If applied to the functor $\mathsf{span} : \mathsf{set} \to \mathsf{duo}$ in Subsection 3.1.1, it would lead to the same category of small categories. If applied to the functor $\mathsf{bim}(-^e) : \mathsf{sfr} \to \mathsf{duo}$ in Subsection 3.2, however, it would result in a different notion of morphism between weak bialgebras (related to that in Subsection 3.2.2 by interchanging the roles of the left and right subalgebras). This symmetric variant of the category of weak bialgebras admits a symmetric adjunction with the category of small categories, see also Remark 3.3.11.

As a further symmetry, one can change the notion of morphism between duoidal categories to functors which are monoidal with respect to both monoidal structures. Then two symmetric variants of morphisms between bimonoids can be defined in terms of the induced comonoidal monads $((MX, \bullet), H \circ (-))$ and $((MX, \bullet), (-) \circ H)$. (Note that while weak bialgebra is a self-dual structure [18, page 390], its morphisms in Section 3.2 are not. A category of weak bialgebras whose morphisms are dual to those in Section 3.2 can be obtained by this dual construction. The possibility of finding a contravariant adjunction to the category of small categories has not been investigated in this case.)

3.1 The category cat of small categories

In this section we construct a functor $\mathsf{span} : \mathsf{set} \to \mathsf{duo}$. As for its object part, we need to endow the category $\mathsf{span}(X)$ of spans over a given set X with the structure of a duoidal category. This is recalled from [5, Example 6.17]. As for its morphism part, we construct a functor $\mathsf{span}(q)$ which is comonoidal with respect to both monoidal structures of $\mathsf{span}(X)$ for any map $q : X \to X'$ in set. Next, we prove that the category $\mathsf{bmd}(\mathsf{span})$ associated to it is isomorphic to cat.

3.1.1 The functor span

Let us recall the definition of the category of spans over a given set [5, Example 6.17]. For any set X, a *span* over X is a triple (A, s, t) where A is a set and $s, t : A \to X$ are a pair of maps, called the *source* and *target* maps, respectively. A morphism between the spans (A, s, t) and (A', s', t') over X is a map $f : A \to A'$ such that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle s'} & \\ & X & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A' \\ & {\scriptstyle t}\searrow \quad \swarrow{\scriptstyle t'} & \\ & X & \end{array} \tag{3.7}$$

commute. For brevity, we write A instead of (A, s, t), understanding that s and t are given. We denote by $\mathsf{span}(X)$ the category of spans over X. For any spans A and B, define the sets

$$A \circ B \;:=\; \{(a, b) \in A \times B : s(a) = t(b)\}, \tag{3.8}$$

$$A \bullet B \;:=\; \{(a, b) \in A \times B : s(a) = s(b) \text{ and } t(a) = t(b)\}. \tag{3.9}$$

We turn $A \circ B$ and $A \bullet B$ into spans over X by defining, for $(a,b) \in A \circ B$,

$$s(a,b) \coloneqq s(b) \qquad \text{and} \qquad t(a,b) \coloneqq t(a),$$

and for $(a,b) \in A \bullet B$,

$$s(a,b) \coloneqq s(a) = s(b) \qquad \text{and} \qquad t(a,b) \coloneqq t(a) = t(b).$$

Each one of these operations is functorial, that is, they do not act only on spans but also on morphisms between spans in an appropriate manner: preserving identities and composite morphisms. Each one of them endows the category $\mathsf{span}(X)$ with a monoidal structure, with the obvious associators. The unit object I of $(\mathsf{span}(X), \circ)$ is the discrete span $(X, \mathsf{id}, \mathsf{id})$ and the unit object J of $(\mathsf{span}(X), \bullet)$ is the complete span $(X \times X, p_1, p_2)$ with $p_1(x,y) = x$ and $p_2(x,y) = y$. For any span A, the unitors $\lambda^\circ_A, \rho^\circ_A, \lambda^\bullet_A, \rho^\bullet_A$ are given by the projections onto A. Furthermore, $(\mathsf{span}(X), \circ, I, \bullet, J)$ is a duoidal category with the structure below. Let A, B, C, D be spans over X. The interchange law

$$\gamma_{A,B,C,D} : (A \bullet B) \circ (C \bullet D) \to (A \circ C) \bullet (B \circ D)$$

simply sends (a,b,c,d) to (a,c,b,d). The structure map $\Delta_I : I \to I \bullet I$ is the diagonal one and $\mu_J : J \circ J \to J$ and $\tau : I \to J$ are uniquely determined since the object J is terminal in the category $\mathsf{span}(X)$. Explicitly,

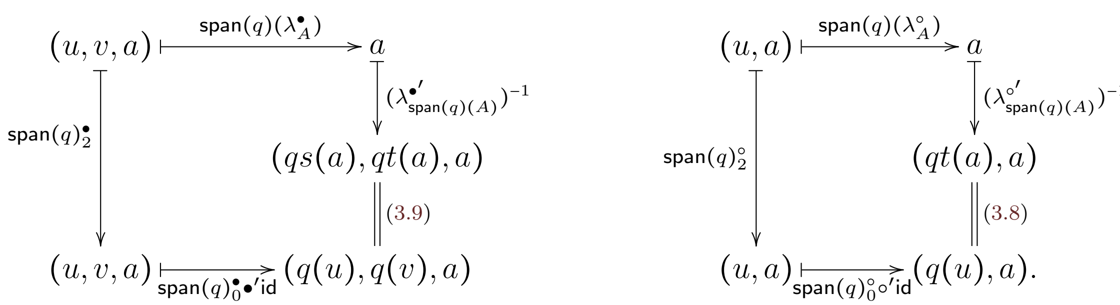
$$\Delta_I : x \mapsto (x,x), \quad \mu_J : ((u,v),(w,u)) \mapsto (w,v) \quad \text{and} \quad \tau : x \mapsto (x,x)$$

for any $x \in I$ and $((u,v),(w,u)) \in J \circ J$. (See [16, Section 5.4], where the category of spans is regarded as a monoidal bicategory Span and any set X is seen to determine, in fact, a naturally Frobenius map-monoidale in the monoidal bicategory resulting by reversing the 2–cells in Span. Then $\mathsf{span}(X)$ is the endo-hom category of X; duoidal via the monoidal products provided by the composition and the convolution). A bimonoid in the duoidal category $\mathsf{span}(X)$ is, equivalently, a small category (see [5, Example 6.43]).

Consider the following functor span from the category set of (small) sets to duo. It sends a set X to the duoidal category $\mathsf{span}(X)$ above. Regarding its action on a map of sets $q : X \to X'$, note that q induces a morphism $\mathsf{span}(q)$ in duo from $\mathsf{span}(X)$ to $\mathsf{span}(X')$: The functor $\mathsf{span}(q)$ takes an object $t : X \leftarrow A \to X : s$ to $qt : X' \leftarrow A \to X' : qs$ and it acts on the morphisms as the identity map. It is easily seen to be a functor which is in addition comonoidal with respect to both monoidal structures $\circ$ and $\bullet$, via the following binary and nullary parts:

$$
\begin{array}{lll}
\mathsf{span}(q)_2^\circ : A \circ B \to A \circ' B, & (a,b) & \mapsto (a,b) \\
\mathsf{span}(q)_0^\circ : X \to X', & x & \mapsto q(x) \\
\mathsf{span}(q)_2^\bullet : A \bullet B \to A \bullet' B, & (a,b) & \mapsto (a,b) \\
\mathsf{span}(q)_0^\bullet : X \times X \to X' \times X', & (x,y) & \mapsto (q(x), q(y)).
\end{array}
$$

In fact, the coassociativity of $\mathsf{span}(q)$ is evident with respect to both monoidal structures. Its left unitality is checked by



Right counitality is proven analogously.

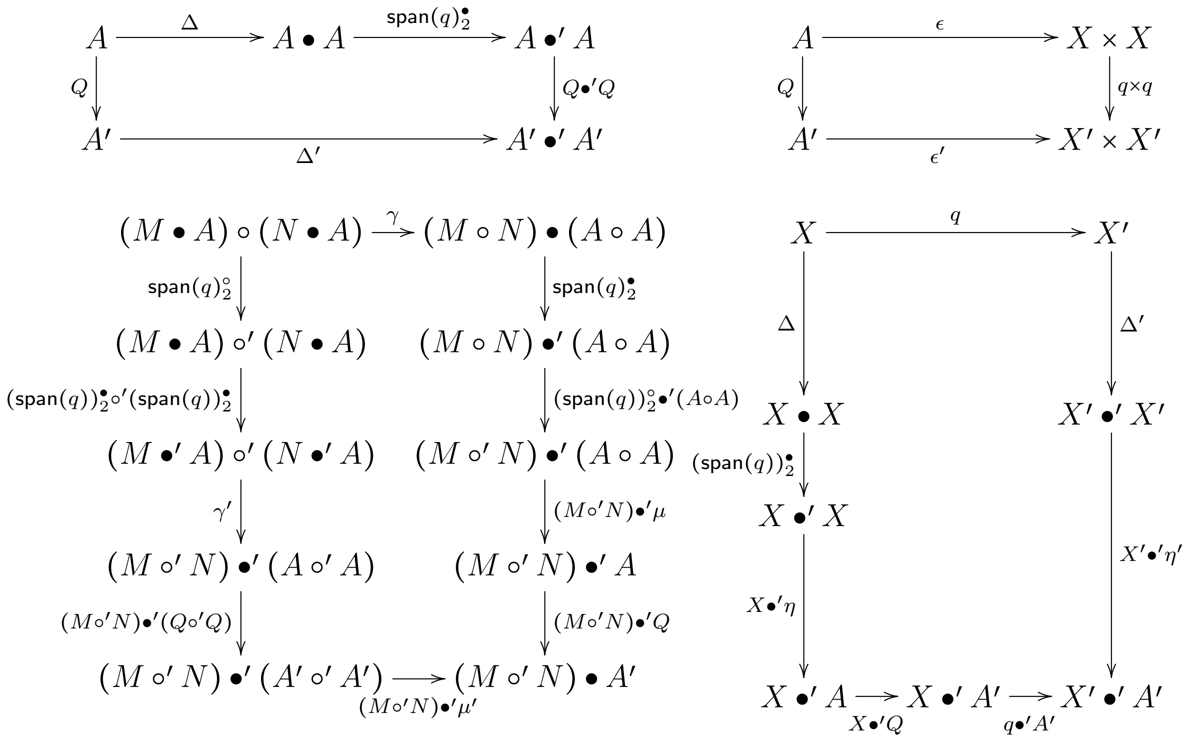
3.1.2 The category bmd(span)

Theorem 3.1.1. *The category* $\mathsf{bmd}(\mathsf{span})$ *is isomorphic to the category of small categories.*

Proof. On the one hand, note that any object (A, s, t) of $(\mathsf{span}(X), \bullet, X \times X)$ has a unique comonoid structure. It is given by the diagonal comultiplication $\Delta : A \to A \bullet A$, $a \mapsto (a, a)$ and the counit $\epsilon : A \to X \times X$, $a \mapsto (s(a), t(a))$. Hence it follows that objects in $\mathsf{bmd}(\mathsf{span})$ are pairs (X, A) of a set X and a monoid A in $\mathsf{span}(X)$ (with the above structure it is easily seen to render commutative diagrams (2.13)-(2.16)) or, equivalently, a small category A with object set X, see [5].

On the other hand, the morphisms in $\mathsf{bmd}(\mathsf{span})$ are pairs $(q : X \to X', Q : A \to A')$

of maps for which $qs = s'Q, qt = t'Q$ and which render commutative the four diagrams in Lemma 3.0.2 (b), which take now the following form.



Evaluating these diagrams on elements of the appropriate set, we see that the first one commutes for any pair of maps $(q : X \to X', Q : A \to A')$; the second one commutes if and only if Q restricts to maps $\{a \in A : ta = y,\ sa = x\} \to \{a' \in A' : t'a' = qy,\ s'a' = qx\}$; the third one commutes if and only if Q preserves composition; and the last one commutes if and only if Q preserves identity arrows. Shortly, these diagrams commute if and only if there is a functor with object map q and morphism map Q. □

Applying the above construction to the restriction of the functor span to the full subcategory of finite sets in set, we obtain the full subcategory cat^0 of small categories with finitely many objects.

3.2 The category wba of weak bialgebras

In this section we construct a functor $\mathsf{bim}(-^e) : \mathsf{sfr} \to \mathsf{duo}$, from the category of separable Frobenius algebras to that of duoidal categories. Concerning its object part, we need to endow the category of R^e–bimodules over a separable Frobenius algebra R with the structure of a duoidal category. This is achieved by considering the monoidal product $\otimes_{R^e}$ and the Takeuchi's product $\times_R$, after showing that in this case —namely, for a separable Frobenius algebra R— $\times_R$ can be identified with some (twisted) bimodule tensor product over R^e (so that it turns out to be a monoidal product). As for its morphism part, for any map $q : R \to R'$ in sfr, we construct a functor $\mathsf{bim}(q^e)$ by using the dual forms of q (obtained via transposition), turning out to be a comonoidal functor with respect to both monoidal structures. Once defined the functor $\mathsf{bim}(-^e)$, we prove that an object (R, H) in the category $\mathsf{bmd}(\mathsf{bim}(-^e))$ (of bimonoids associated to it) is precisely a weak bialgebra H whose right subalgebra is isomorphic to R; and that a morphism in that category is exactly —in the justified language of weak bialgebras— a coalgebra map commuting with $\sqcap^R$ and $\sqcap'^R$, $\overline{\sqcap}^R$ and $\overline{\sqcap}'^R$, the Nakayama automorphisms of R and R' and obeying a so-called weak multiplicativity condition.

3.2.1 The functor $\mathsf{bim}(-^e)$

Let R be an object in sfr, that is, a separable Frobenius algebra. Let $(\psi, e_i \otimes f_i)$ be a separable Frobenius structure on R and $\theta : R \to R$ its Nakayama automorphism. In order to construct the object part of the functor $\mathsf{bim}(-^e) : \mathsf{sfr} \to \mathsf{duo}$, first we present two monoidal structures on $\mathsf{bim}(R^e)$. In what follows, the original R^e–actions of any R^e–bimodule are denoted by juxtaposition.

The first monoidal structure. The category $\mathsf{bim}(R^e)$ of R^e–bimodules is monoidal via the monoidal product $\circ = \otimes_{R^e}$, and the unit $I = R^e$ with the R^e–bimodule structure given by its multiplication as a k–algebra, that is, by the actions

$$(s \otimes r)(x \otimes y) = sx \otimes yr \qquad \text{and} \qquad (x \otimes y)(s \otimes r) = xs \otimes ry. \tag{3.10}$$

Given R^e–bimodules M and N, the unit constraints are

$$\begin{aligned} \lambda^\circ_M : I \circ M \to M, \qquad & (x \otimes y) \circ m \mapsto (x \otimes y)m \\ \rho^\circ_M : M \circ I \to M, \qquad & m \circ (x \otimes y) \mapsto m(x \otimes y). \end{aligned}$$

The product $M \circ N$ is an R^e–bimodule via the actions

$$(s \otimes r)(m \circ n) = (s \otimes r)m \circ n \qquad \text{and} \qquad (m \circ n)(s \otimes r) = m \circ n(s \otimes r). \tag{3.11}$$

Thanks to the separable Frobenius structure of R (and hence of R^e), the canonical R^e–bimodule epimorphism

$$\pi^{\circ}_{M,N} : M \otimes N \to M \circ N, \qquad m \otimes n \mapsto m \circ n$$

is split by

$$\iota^{\circ}_{M,N} : M \circ N \to M \otimes N, \qquad m \circ n \mapsto m(e_i \otimes f_j) \otimes (f_i \otimes e_j)n.$$

Thus, $M \circ N$ is isomorphic to the vector subspace (in fact R^e–subbimodule)

$$\iota^{\circ}_{M,N}(M \circ N) = M(e_i \otimes f_j) \otimes (f_i \otimes e_j)N$$

of $M \otimes N$. Alternatively,

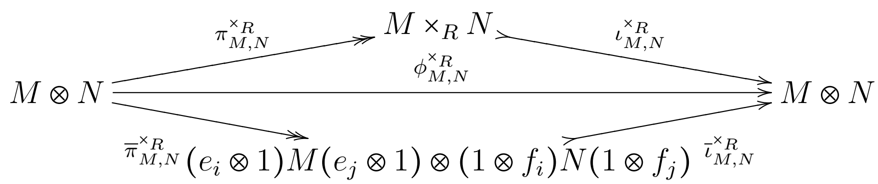
$$M \circ N \cong \{ \ x \in M \otimes N \ : \ \iota^{\circ}_{M,N}\pi^{\circ}_{M,N}x = x \ \}.$$

By [9, Lemma 2.2], the monoids in this monoidal category $(\mathsf{bim}(R^e), \circ, I)$ can be identified with pairs consisting of a k–algebra A and a k–algebra morphism $\eta : R^e \to A$. Let H be a weak bialgebra over a field. By Theorem 2.5.1, its right subalgebra $R := \sqcap^R(H)$ is a separable Frobenius algebra; by Lemma 2.5.11, the map $R^e \to H$, $s \otimes r \mapsto s\overline{\sqcap}^L(r)$ is an algebra morphism. Consequently, H is a monoid in $(\mathsf{bim}(R^e), \circ, I)$.

The second monoidal structure. The Takeuchi's product of R^e–bimodules is defined for *any ring* R. However, at this level of generality it does not define a monoidal product on the category of R^e–bimodules (only a lax monoidal one, see [32]). As we will see below, it is a consequence of the separable Frobenius structure of R that allows us to write the Takeuchi product over it as a (co)module tensor product, what is more, as a split (co)equalizer; serving as a second monoidal product for the category $\mathsf{bim}(R^e)$.

Lemma 3.2.1. *Let R be a separable Frobenius algebra and let $e_i \otimes f_i$ be a separability Frobenius idempotent for R. For any R^e–bimodules M and N, $M \times_R N$ is isomorphic to the R^e–subbimodule $(e_i \otimes 1)M(e_j \otimes 1) \otimes (1 \otimes f_i)N(1 \otimes f_j)$ of $M \otimes N$.*

Proof. Consider the diagram



where the occurring maps are given by:

$$
\begin{aligned}
\pi^{\times_R}_{M,N} &: \; m \otimes n \mapsto (e_i \otimes 1)m \otimes_R (1 \otimes f_i)n, \\
\overline{\pi}^{\times_R}_{M,N} &: \; m \otimes n \mapsto (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j), \\
\iota^{\times_R}_{M,N} &: \; m_t \otimes_R n_t \mapsto m_t(e_j \otimes 1) \otimes n_t(1 \otimes f_j), \\
\overline{\iota}^{\times_R}_{M,N} &: \; (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j) \mapsto (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j), \\
\phi^{\times_R}_{M,N} &: \; m \otimes n \mapsto (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j).
\end{aligned}
$$

The maps $\overline{\pi}^{\times_R}_{M,N}$ and $\overline{\iota}^{\times_R}_{M,N}$ are evidently well defined. As $\iota^{\times_R}_{M,N}$ concerns, it is well defined by

$$
\begin{aligned}
(m(r \otimes 1))(e_j \otimes 1) \otimes n(1 \otimes f_j) &= (m(re_j \otimes 1)) \otimes n(1 \otimes f_j) \\
&\overset{(2.35)}{=} m(e_j \otimes 1) \otimes n(1 \otimes f_j r) \\
&= m(e_j \otimes 1) \otimes (n(1 \otimes r))(1 \otimes f_j);
\end{aligned}
$$

for any $m \otimes n \in M \otimes N$, $r \in R$; and $\pi^{\times_R}_{M,N}$ is also so by Proposition 2.3.6. Moreover, $\pi^{\times_R}_{M,N}$ is surjective: For any element $m_t \otimes_R n_t$ of $M \times_R N$, using in the first equality that $m_t \otimes_R n_t$ belongs to the center of $M \otimes_R N$,

$$
(e_i \otimes 1)m_t \times_R (1 \otimes f_i)n_t = (e_i f_i \otimes 1)m_t \otimes_R n_t \overset{(2.42)}{=} (1 \otimes 1)m_t \otimes_R n_t = m_t \otimes_R n_t.
$$

The identity $\overline{\pi}^{\times_R}_{M,N}\overline{\iota}^{\times_R}_{M,N} = \mathsf{id}$ follows by (2.44); that $\pi^{\times_R}_{M,N}\iota^{\times_R}_{M,N} = \mathsf{id}$, by the following computation. For any $m_t \otimes_R n_t \in M \times_R N$,

$$
\begin{aligned}
\pi^{\times_R}_{M,N}\iota^{\times_R}_{M,N}(m_t \otimes_R n_t) &= (e_j \otimes 1)m_t(e_i \otimes 1) \otimes_R (1 \otimes f_j)n_t(1 \otimes f_i) \\
&= (e_j \otimes 1)m_t \otimes_R (1 \otimes f_j)n_t = m_t \otimes_R n_t.
\end{aligned}
$$

In the second equality we used (2.23) and, in the last one, that $m_t \otimes_R n_t$ is an element of the center of $M \otimes_R N$. By (2.44), $\phi^{\times_R}_{M,N}$ is an idempotent map, $\overline{\iota}^{\times_R}_{M,N} \overline{\pi}^{\times_R}_{M,N} = \phi^{\times_R}_{M,N}$ and $\iota^{\times_R}_{M,N} \pi^{\times_R}_{M,N} = \phi^{\times_R}_{M,N}$. By Proposition 2.1.1, we conclude that $M \times_R N$ is isomorphic to $(e_i \otimes 1)M(e_j \otimes 1) \otimes (1 \otimes f_i)N(1 \otimes f_j)$. □

Any automorphism functor on a monoidal category can be used to twist the monoidal structure. In particular, for any separable Frobenius algebra R, we can use the functor $\mathcal{F}$ given by Proposition 2.3.3 —choosing as ζ the Nakayama automorphism θ of R—, to twist the monoidal category $(\mathsf{bim}(R^e), \circ, I)$ to a new monoidal category $(\mathsf{bim}(R^e), \bullet, J)$. For any R^e–bimodules M and N, define

$$M \bullet N = \mathcal{F}(M) \circ \mathcal{F}(N) \quad \text{and} \quad J = \mathcal{F}^{-1}(I). \tag{3.12}$$

Recall that $\mathcal{F}(M)$ is the same vector space M with the R^e–actions

$$(s \otimes r) \cdot m = (1 \otimes \theta(r))m(1 \otimes s), \qquad m \cdot (s \otimes r) = (r \otimes 1)m(s \otimes 1) \tag{3.13}$$

where the original R^e–actions of M are denoted by juxtaposition. In other words, $M \bullet N$ is the factor space of $M \otimes N$ with respect to the relations

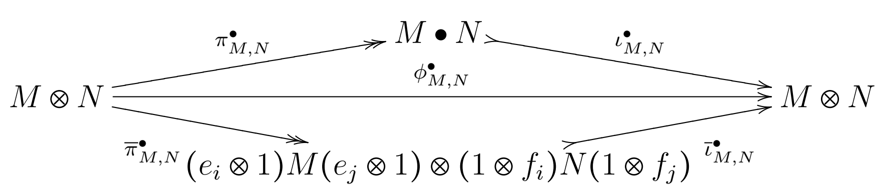
$$\{(r \otimes 1)m(s \otimes 1) \otimes n - m \otimes (1 \otimes \theta(r))n(1 \otimes s)\}. \tag{3.14}$$

On the other hand, $\mathcal{F}^{-1}(M)$ is the same vector space M with the R^e–actions

$$(s \otimes r) \cdot m = (1 \otimes \theta^{-1}(r))m(1 \otimes s), \qquad m \cdot (s \otimes r) = (r \otimes 1)m(s \otimes 1). \tag{3.15}$$

Lemma 3.2.2. *Let R be a separable Frobenius algebra and let $e_i \otimes f_i$ be a separability Frobenius idempotent for R. For any R^e–bimodules M and N, $M \bullet N$ is isomorphic to the R^e–subbimodule $(e_i \otimes 1)M(e_j \otimes 1) \otimes (1 \otimes f_i)N(1 \otimes f_j)$ of $M \otimes N$.*

Proof. Consider the diagram



where the occurring maps are given by:

$$\begin{aligned}
\pi^{\bullet}_{M,N} &: m \otimes n \mapsto m \bullet n, \\
\overline{\pi}^{\bullet}_{M,N} &: m \otimes n \mapsto (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j), \\
\iota^{\bullet}_{M,N} &: m \bullet n \mapsto (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j), \qquad (3.16)\\
\overline{\iota}^{\bullet}_{M,N} &: (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j) \mapsto (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j), \\
\phi^{\bullet}_{M,N} &: m \otimes n \mapsto (e_i \otimes 1)m(e_j \otimes 1) \otimes (1 \otimes f_i)n(1 \otimes f_j).
\end{aligned}$$

The maps $\pi^{\bullet}_{M,N}, \overline{\pi}^{\bullet}_{M,N}$ and $\overline{\iota}^{\bullet}_{M,N}$ are respectively the canonical projections and inclusion. By Proposition 2.3.5, also $\iota^{\bullet}_{M,N}$ is well defined. By (2.44), $\phi^{\bullet}_{M,N}$ is an idempotent map, and we have the identities $\overline{\iota}^{\bullet}_{M,N}\overline{\pi}^{\bullet}_{M,N} = \phi^{\bullet}_{M,N}$, $\iota^{\bullet}_{M,N}\pi^{\bullet}_{M,N} = \phi^{\bullet}_{M,N}$ and $\overline{\pi}^{\bullet}_{M,N}\overline{\iota}^{\bullet}_{M,N} = \mathsf{id}$. The following computation proves that $\pi^{\bullet}_{M,N}\iota^{\bullet}_{M,N} = \mathsf{id}$. For any $m \bullet n \in M \bullet N$,

$$\begin{aligned}
\pi^{\bullet}_{M,N}\iota^{\bullet}_{M,N}(m \bullet n) &= (e_i \otimes 1)m(e_j \otimes 1) \bullet (1 \otimes f_i)n(1 \otimes f_j) \\
&\overset{(3.13)}{=} m \cdot (e_j \otimes e_i) \bullet (f_j \otimes \theta(f_i)) \cdot n \\
&\overset{(3.12)}{=} m \cdot (e_j f_j \otimes \theta^{-1}(f_i)e_i) \bullet n \underset{(2.37)}{\overset{(2.42)}{=}} m \bullet n.
\end{aligned}$$

Proposition 2.1.1 concludes that $M \bullet N$ and $(e_i \otimes 1)M(e_j \otimes 1) \otimes (1 \otimes f_i)N(1 \otimes f_j)$ are isomorphic R^e–bimodules. □

As a consequence of Lemma 3.2.1 and Lemma 3.2.2, there is an isomorphism $\nu_{M,N} : M \times_R N \to M \bullet N$ for any R^e–bimodules M and N for a separable Frobenius algebra R. Let $(\psi, e_i \otimes f_i)$ be a separable Frobenius structure on R. This isomorphism is given by $\nu_{M,N} = \pi^{\bullet}_{M,N}\overline{\iota}^{\bullet}_{M,N}\overline{\pi}^{\times_R}_{M,N}\iota^{\times_R}_{M,N}$ and $\nu^{-1}_{M,N} = \pi^{\times_R}_{M,N}\overline{\iota}^{\times_R}_{M,N}\overline{\pi}^{\bullet}_{M,N}\iota^{\bullet}_{M,N}$, that is,

$$\nu_{M,N} : m_t \otimes_R n_t \mapsto m_t \bullet n_t, \qquad \nu^{-1}_{M,N} : m \bullet n \mapsto (e_i \otimes 1)m \otimes_R (1 \otimes f_i)n.$$

The R^e–bimodule structure of $M \times_R N$ induces thus R^e–actions on $M \bullet N$:

$$(s \otimes r) \cdot (m \bullet n) = \nu((s \otimes r)\nu^{-1}(m \bullet n)) \overset{(2.25)}{=} (1 \otimes r)m \bullet (s \otimes 1)n, \qquad (3.17)$$

$$(m \bullet n) \cdot (s \otimes r) = \nu(\nu^{-1}(m \bullet n)(s \otimes r)) \overset{(2.26)}{=} m(1 \otimes r) \bullet n(s \otimes 1), \qquad (3.18)$$

for any $s \otimes r \in R^e, m \bullet n \in M \bullet N$. Using (3.15), the R^e–bimodule structure of $J =$

$\mathcal{F}^{-1}(I) = R^e$ comes out as

$$(s \otimes r) \cdot (x \otimes y) = x \otimes sy\theta^{-1}(r) \quad \text{and} \quad (x \otimes y) \cdot (s \otimes r) = rxs \otimes y. \tag{3.19}$$

The left and right unit constraints for the monoidal product $\bullet$ are given by

$$\begin{aligned} \lambda^\bullet_M : J \bullet M \to M, \qquad & (x \otimes y) \bullet m \mapsto (x \otimes y) \cdot m = (1 \otimes \theta(y))m(1 \otimes x) \\ \rho^\bullet_M : M \bullet J \to M, \qquad & m \bullet (x \otimes y) \mapsto m \cdot (x \otimes y) = (y \otimes 1)m(x \otimes 1). \end{aligned} \tag{3.20}$$

Moreover,

$$M \bullet N \cong \{\ x \in M \otimes N \ :\ \iota^\bullet_{M,N}\pi^\bullet_{M,N}(x) = x\ \}.$$

Note that, for any $x \otimes y \in R^e$ and $m \in M$,

$$\begin{aligned} &(\psi \otimes \psi \otimes \mathsf{id})\iota^\bullet_{J,M}((x \otimes y) \bullet m) \\ &\overset{(3.16)}{=} (\psi \otimes \psi \otimes \mathsf{id})[(e_i \otimes 1) \cdot (x \otimes y) \cdot (e_j \otimes 1) \otimes (1 \otimes f_i)m(1 \otimes f_j)] \\ &\overset{(3.19)}{=} (\psi \otimes \psi \otimes \mathsf{id})[xe_j \otimes e_i y \otimes (1 \otimes f_i)m(1 \otimes f_j)] \\ &\overset{(2.38)(2.33)}{=} (1 \otimes \theta(y))m(1 \otimes x) \\ &\overset{(3.20)}{=} \lambda^\bullet_M((x \otimes y) \bullet m); \end{aligned} \tag{3.21}$$

and, analogously,

$$(\mathsf{id} \otimes \psi \otimes \psi)\iota^\bullet_{M,J}(m \bullet (x \otimes y)) = \rho^\bullet_M(m \bullet (x \otimes y)). \tag{3.22}$$

Theorem 3.2.1. $(\mathsf{bim}(R^e), \circ, I, \bullet, J)$ *possesses the structure of a duoidal category.*

Proof. Let $(\psi, e_i \otimes f_i)$ be a separable Frobenius structure on R. Given R^e–bimodules A, B, C, D, define the interchange law $\gamma : (A \bullet B) \circ (C \bullet D) \to (A \circ C) \bullet (B \circ D)$ (2.2) by

$$\gamma((a \bullet b) \circ (c \bullet d)) = (a(e_i \otimes 1) \circ c) \bullet (b \circ (1 \otimes f_i)d) \tag{3.23}$$

and the morphisms in (2.3) by

$$\begin{aligned} \tau &: I \to J, & x \otimes y &\mapsto yf_i \otimes xe_i \\ \mu_J &: J \circ J \to J, & (x \otimes y) \circ (p \otimes q) &\mapsto \psi(xq)p \otimes y \\ \Delta_I &: I \to I \bullet I, & x \otimes y &\mapsto (1 \otimes y) \bullet (x \otimes 1). \end{aligned} \tag{3.24}$$

In order to show that γ is well defined, we should check that the map $\widetilde{\gamma} : A\otimes B\otimes C\otimes D \to (A\circ C)\bullet(B\circ D)$ sending $a\otimes b\otimes c\otimes d$ to $(a(e_i\otimes 1)\circ c)\bullet(b\circ(1\otimes f_i)d)$ is R^e–balanced in all of the three occurring tensor products. This is proven by the computations below. In what follows, a, b, c, d are respective elements of A, B, C, D, and r, s, p, q, x, y, of R.

$$\begin{aligned}
\widetilde{\gamma}[a\cdot(1\otimes r)\otimes b\otimes c\otimes d] &= [(r\otimes 1)a(e_i\otimes 1)\circ c]\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(3.13)}{=} [(a(e_i\otimes 1)\circ c)\cdot(1\otimes r)]\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(3.12)}{=} [a(e_i\otimes 1)\circ c]\bullet[(1\otimes r)\cdot(b\circ(1\otimes f_i)d)] \\
&\overset{(3.13)}{=} [a(e_i\otimes 1)\circ c]\bullet[(1\otimes\theta(r))b\circ(1\otimes f_i)d] \\
&\overset{(3.13)}{=} \widetilde{\gamma}[a\otimes(1\otimes r)\cdot b\otimes c\otimes d]
\end{aligned}$$

$$\begin{aligned}
\widetilde{\gamma}[a\cdot(s\otimes 1)\otimes b\otimes c\otimes d] &= [a(se_i\otimes 1)\circ c]\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(2.35)}{=} [(a(e_i\otimes 1)\circ c)]\bullet[b\circ(1\otimes f_i s)d] \\
&= [(a(e_i\otimes 1)\circ c)]\bullet[b(1\otimes s)\circ(1\otimes f_i)d] \\
&\overset{(3.13)}{=} [(a(e_i\otimes 1)\circ c)]\bullet[(s\otimes 1)\cdot b\circ(1\otimes f_i)d] \\
&= \widetilde{\gamma}(a\otimes(s\otimes 1)\cdot b\otimes c\otimes d)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\gamma}[a\otimes b\otimes c\cdot(1\otimes r)\otimes d] &= [a(e_i\otimes 1)\circ c\cdot(1\otimes r)]\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(3.13)}{=} [a(e_i\otimes 1)\circ(r\otimes 1)c]\bullet[b\circ(1\otimes f_i)d] \\
&= [a(e_i r\otimes 1)\circ c]\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(2.37)}{=} [a(e_i\otimes 1)\circ c]\bullet[b\circ(1\otimes\theta(r)f_i)d] \\
&\overset{(3.13)}{=} [a(e_i\otimes 1)\circ c]\bullet[b\circ(1\otimes f_i)(1\otimes r)\cdot d] \\
&= \widetilde{\gamma}(a\otimes b\otimes c\otimes(1\otimes r)\cdot d)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\gamma}[a\otimes b\otimes c\cdot(s\otimes 1)\otimes d] &= [a(e_i\otimes 1)\circ c\cdot(s\otimes 1)]\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(3.13)}{=} [a(e_i\otimes 1)\circ c(s\otimes 1)]\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(3.11)}{=} [a(e_i\otimes 1)\circ c](s\otimes 1)\bullet[b\circ(1\otimes f_i)d] \\
&\overset{(3.12)}{=} [a(e_i\otimes 1)\circ c]\bullet[b\circ(1\otimes f_i)d](1\otimes s) \\
&\overset{(3.11)}{=} [a(e_i\otimes 1)\circ c]\bullet[b\circ(1\otimes f_i)d(1\otimes s)]
\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.13)}{=} [a(e_i \otimes 1) \circ c] \bullet [b \circ (1 \otimes f_i)((s \otimes 1) \cdot d)] \\
&= \widetilde{\gamma}(a \otimes b \otimes c \otimes (s \otimes 1) \cdot d)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\gamma}[(a \otimes b) \cdot (s \otimes 1) \otimes c \otimes d] &\overset{(3.18)}{=} \widetilde{\gamma}[a \otimes b(s \otimes 1) \otimes c \otimes d] \\
&= [a(e_i \otimes 1) \circ c] \bullet [b(s \otimes 1) \circ (1 \otimes f_i)d] \\
&= [a(e_i \otimes 1) \circ c] \bullet [b \circ (1 \otimes f_i)(s \otimes 1)d] \\
&= \widetilde{\gamma}[a \otimes b \otimes c \otimes (s \otimes 1)d] \\
&\overset{(3.17)}{=} \widetilde{\gamma}[a \otimes b \otimes (s \otimes 1) \cdot (c \otimes d)]
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\gamma}[(a \otimes b) \cdot (1 \otimes r) \otimes c \otimes d] &\overset{(3.18)}{=} [a(1 \otimes r) \circ c] \bullet [b \circ (1 \otimes f_i)d] \\
&= [a(1 \otimes r)(e_i \otimes 1) \circ c] \bullet [b \circ (1 \otimes f_i)d] \\
&= [a(e_i \otimes 1) \circ (1 \otimes r)c] \bullet [b \circ (1 \otimes f_i)d] \\
&= \widetilde{\gamma}[a \otimes b \otimes (1 \otimes r)c \otimes d] \\
&\overset{(3.17)}{=} \widetilde{\gamma}[a \otimes b \otimes (1 \otimes r) \cdot (c \otimes d)]
\end{aligned}
$$

By similar steps one can also see that μ_J is well defined:

$$
\begin{aligned}
\mu_J((x \otimes y)(s \otimes r) \otimes (p \otimes q)) &\overset{(3.19)}{=} \mu_J(rxs \otimes y \otimes p \otimes q) \overset{(3.24)}{=} \psi(rxsq)p \otimes y \\
&\overset{(2.34)}{=} \psi(xsq\theta^{-1}(r))p \otimes y \overset{(3.24)}{=} \mu_J(x \otimes y \otimes p \otimes sq\theta^{-1}(r)) \\
&\overset{(3.19)}{=} \mu_J((x \otimes y) \otimes (s \otimes r)(p \otimes q)),
\end{aligned}
$$

and that γ, τ, μ_J and Δ_I are morphisms of R^e–bimodules:

$$
\begin{aligned}
\gamma((a \bullet b) \circ (c \bullet d)) \cdot (s \otimes r) &\overset{(3.23)}{=} [(a(e_i \otimes 1) \circ c) \bullet (b \circ (1 \otimes f_i)d)] \cdot (s \otimes r) \\
&\overset{(3.18)}{=} (a(e_i \otimes 1) \circ c)(1 \otimes r) \bullet (b \circ (1 \otimes f_i)d)(s \otimes 1) \\
&= (a(e_i \otimes 1) \circ c(1 \otimes r)) \bullet (b \circ (s \otimes f_i)d) \\
&= \gamma[(a \bullet b) \circ (c(1 \otimes r) \bullet (s \otimes 1)d)] \\
&\overset{(3.18)}{=} \gamma[(a \bullet b) \circ ((c \bullet d) \cdot (s \otimes r))] \qquad (3.25) \\
&\overset{(3.11)}{=} \gamma[((a \bullet b) \circ (c \bullet d)) \cdot (s \otimes r)],
\end{aligned}
$$

$$
\begin{aligned}
(s\otimes r)\cdot\gamma((a\bullet b)\circ(c\bullet d)) &\overset{(3.23)}{=} (s\otimes r)\cdot[(a(e_i\otimes 1)\circ c)\bullet(b\circ(1\otimes f_i)d)]\\
&\overset{(3.17)}{=} (1\otimes r)[a(e_i\otimes 1)\circ c]\bullet(s\otimes 1)[b\circ(1\otimes f_i)d]\\
&= [(1\otimes r)a(e_i\otimes 1)\circ c]\bullet[(s\otimes 1)b\circ(1\otimes f_i)d]\\
&\overset{(3.23)}{=} \gamma[((1\otimes r)a\bullet(s\otimes 1)b)\circ(c\bullet d)]\\
&\overset{(3.17)}{=} \gamma[((s\otimes r)\cdot(a\bullet b))\circ(c\bullet d)] \qquad (3.26)\\
&\overset{(3.11)}{=} \gamma[(s\otimes r)\cdot((a\bullet b)\circ(c\bullet d))],
\end{aligned}
$$

$$
\begin{aligned}
\tau(x\otimes y)\cdot(s\otimes r) &\overset{(3.24)}{=} (yf_i\otimes xe_i)\cdot(s\otimes r) \overset{(3.19)}{=} ryf_is\otimes xe_i\\
&\overset{(2.35)}{=} ryf_i\otimes xse_i \overset{(3.24)}{=} \tau(xs\otimes ry)\\
&\overset{(3.19)}{=} \tau((x\otimes y)(s\otimes r)),
\end{aligned}
$$

$$
\begin{aligned}
(s\otimes r)\cdot\tau(x\otimes y) &\overset{(3.24)}{=} (s\otimes r)\cdot(yf_i\otimes xe_i) \overset{(3.19)}{=} yf_i\otimes sxe_i\theta^{-1}(r)\\
&\overset{(2.36)}{=} yrf_i\otimes sxe_i \overset{(3.24)}{=} \tau(sx\otimes yr)\\
&= \tau\left((s\otimes r)(x\otimes y)\right),
\end{aligned}
$$

$$
\begin{aligned}
(s\otimes r)\cdot\mu_J((x\otimes y)\circ(p\otimes q)) &\overset{(3.24)}{=} \psi(xq)(s\otimes r)\cdot(p\otimes y) \qquad (3.27)\\
&\overset{(3.19)}{=} \psi(xq)p\otimes sy\theta^{-1}(r)\\
&\overset{(3.24)}{=} \mu_J((x\otimes sy\theta^{-1}(r))\circ(p\otimes q))\\
&\overset{(3.19)(3.11)}{=} \mu_J[(s\otimes r)\cdot((x\otimes y)\circ(p\otimes q))],
\end{aligned}
$$

$$
\begin{aligned}
\mu_J((x\otimes y)\circ(p\otimes q))\cdot(s\otimes r) &\overset{(3.24)}{=} \psi(xq)(p\otimes y)\cdot(s\otimes r) \qquad (3.28)\\
&\overset{(3.19)}{=} \psi(xq)(rps\otimes y)\\
&\overset{(3.24)}{=} \mu_J((x\otimes y)\circ(rps\otimes q))\\
&\overset{(3.19)(3.11)}{=} \mu_J[((x\otimes y)\circ(p\otimes q))\cdot(s\otimes r)],
\end{aligned}
$$

$$
\begin{aligned}
\Delta_I(x\otimes y)\cdot(s\otimes r) &\overset{(3.24)}{=} ((1\otimes y)\bullet(x\otimes 1))\cdot(s\otimes r)\\
&\overset{(3.19)}{=} (1\otimes y)(1\otimes r)\bullet(x\otimes 1)(s\otimes 1)\\
&\overset{(3.24)}{=} \Delta_I(xs\otimes ry) = \Delta_I((x\otimes y)(s\otimes r)),
\end{aligned}
$$

$$
\begin{aligned}
(s\otimes r) \cdot \Delta_I(x\otimes y) &\overset{(3.24)}{=} (s\otimes r) \cdot ((1\otimes y)\bullet(x\otimes 1))\\
&\overset{(3.19)}{=} (1\otimes r)(1\otimes y)\bullet(s\otimes 1)(x\otimes 1)\\
&\overset{(3.24)}{=} \Delta_I(sx\otimes yr) = \Delta_I((s\otimes r)(x\otimes y)).
\end{aligned}
$$

We turn to checking the compatibility between both monoidal structures. This amounts to showing that the just defined maps satisfy the associativity, unitality and compatibility of units conditions from Section 2.1.3. The computations are fairly straightforward. For example, coassociativity of Δ_I and associativity of μ_J are obvious. The counitality of Δ_I and the unitality of μ_J are checked by the following computations.

$$
\begin{aligned}
\lambda_I^\bullet(\tau\bullet \mathsf{id})\Delta_I(x\otimes y) &\overset{(3.24)}{=} \lambda_I^\bullet(\tau\bullet \mathsf{id})((1\otimes y)\bullet(x\otimes 1))\\
&\overset{(3.24)}{=} \lambda_I^\bullet((yf_j\otimes e_j)\bullet(x\otimes 1))\\
&\overset{(3.20)}{=} x\otimes yf_j\theta(e_j) \overset{(2.38)(2.42)}{=} x\otimes y
\end{aligned}
$$

$$
\begin{aligned}
\rho_I^\bullet(I\bullet\tau)\Delta_I(x\otimes y) &\overset{(3.24)}{=} \rho_I^\bullet(I\bullet\tau)((1\otimes y)\bullet(x\otimes 1))\\
&\overset{(3.24)}{=} \rho_I^\bullet((1\otimes y)\bullet(f_i\otimes xe_i))\\
&\overset{(3.20)}{=} (xe_i\otimes 1)(1\otimes y)(f_i\otimes 1) \overset{(2.42)}{=} (x\otimes y)
\end{aligned}
$$

$$
\begin{aligned}
\mu_J(\tau\circ \mathsf{id})((x\otimes y)\circ(p\otimes v)) &\overset{(3.24)}{=} \mu_J((yf_i\otimes xe_i)\circ(p\otimes q))\\
&\overset{(2.36)}{=} \mu_J((f_i\otimes xe_i\theta^{-1}(y))\circ(p\otimes q))\\
&\overset{(3.24)}{=} \psi(f_iq)p\otimes xe_i\theta^{-1}(y) \overset{(2.33)}{=} p\otimes xq\theta^{-1}(y)\\
&\overset{(3.11)}{=} \lambda_J^\circ((x\otimes y)\circ(p\otimes q))
\end{aligned}
$$

$$
\begin{aligned}
\mu_J(\mathsf{id}\circ\tau)((x\otimes y)\circ(p\otimes v)) &\overset{(3.24)}{=} \mu_J((x\otimes y)\circ(qf_i\otimes pe_i))\\
&\overset{(2.36)}{=} \psi(xpe_i)qf_i\otimes y \overset{(2.33)}{=} qxp\otimes y\\
&= (q\otimes 1)(x\otimes y)(p\otimes 1)\\
&\overset{(3.11)}{=} \rho^\circ((x\otimes y)\circ(p\otimes q))
\end{aligned}
$$

Commutativity of diagrams (2.4) and (2.5) is immediate from (3.11), (3.17) and (3.18).

The computations below prove the commutativity of (2.6), (2.7), (2.8) and (2.9).

$$
\begin{aligned}
&\gamma(\Delta_I \circ (A \bullet B))\ (\lambda^\circ_{A\bullet B})^{-1}(a \bullet b) \\
&\overset{(3.11)}{=} \gamma(\Delta_I \circ (A \bullet B))[(1 \otimes 1) \circ (a \bullet b)] \\
&\overset{(3.24)}{=} \gamma[((1 \otimes 1) \bullet (1 \otimes 1)) \circ (a \bullet b)] \\
&\overset{(3.23)}{=} ((1 \otimes 1)(e_j \otimes 1) \circ a) \bullet ((1 \otimes 1) \circ (1 \otimes f_j)b) \\
&= ((e_j \otimes 1) \circ a) \bullet ((1 \otimes f_j) \circ b) \\
&\overset{(3.13)}{=} ((e_j \otimes 1) \circ a) \bullet ((1 \otimes \theta^{-1}(f_j)) \cdot ((1 \otimes 1) \circ b)) \\
&\overset{(3.12)}{=} ((e_j \otimes 1) \circ a) \cdot (1 \otimes \theta^{-1}(f_j)) \bullet ((1 \otimes 1) \circ b)) \\
&\overset{(3.13)}{=} ((\theta^{-1}(f_j) \otimes 1)(e_j \otimes 1) \circ a) \bullet ((1 \otimes 1) \circ b)) \\
&\overset{(2.36)(2.42)}{=} ((1 \otimes 1) \circ a) \bullet ((1 \otimes 1) \circ b) \\
&\overset{(3.11)}{=} ((\lambda^\circ_A)^{-1} \bullet (\lambda^\circ_B)^{-1})(a \bullet b)
\end{aligned}
$$

$$
\begin{aligned}
&(\rho^\circ \bullet \rho^\circ)\gamma\ ((A \bullet B) \circ \Delta_I)\rho^{\circ^{-1}}(a \bullet b) \\
&\overset{(3.11)}{=} (\rho^\circ \bullet \rho^\circ)\gamma((A \bullet B) \circ \Delta_I)[(a \bullet b) \circ (1 \otimes 1)] \\
&\overset{(3.24)}{=} (\rho^\circ \bullet \rho^\circ)\gamma[(a \bullet b) \circ ((1 \otimes 1) \bullet (1 \otimes 1))] \\
&\overset{(3.23)}{=} (\rho^\circ \bullet \rho^\circ)[(a(e_i \otimes 1) \circ (1 \otimes 1)) \bullet (b \circ (1 \otimes f_i))] \\
&\overset{(3.11)}{=} a(e_i \otimes 1) \bullet b(1 \otimes f_i) \overset{(3.12)(2.42)}{=} a \bullet b
\end{aligned}
$$

$$
\begin{aligned}
&(\mu_J \bullet (A \circ B))\gamma((\lambda^\bullet_A)^{-1} \circ (\lambda^\bullet)^{-1})(a \circ b) \\
&\overset{(3.20)}{=} (\mu_J \bullet (A \circ B))\gamma(((1 \otimes 1) \bullet a) \circ ((1 \otimes 1) \circ b)) \\
&\overset{(3.23)}{=} (\mu_J \bullet (A \circ B))(((e_i \otimes 1) \circ (1 \otimes 1)) \bullet (a \circ (1 \otimes f_i)b)) \\
&\overset{(3.24)}{=} \psi(e_i)(1 \otimes 1) \bullet (a \circ (1 \otimes f_i)b) \\
&\overset{(2.33)}{=} (1 \otimes 1) \bullet (a \circ b) \overset{(3.20)}{=} (\lambda^\bullet_{A\circ B})^{-1}(a \circ b)
\end{aligned}
$$

$$
\begin{aligned}
&\rho^\bullet(\mathsf{id} \bullet \mu_J)\ \ \gamma[(a \bullet (x \otimes y)) \circ (b \bullet (p \otimes q))] \\
&\overset{(3.23)}{=} \rho^\bullet(\mathsf{id} \bullet \mu_J)[(a(e_i \otimes 1) \circ b) \bullet ((x \otimes y) \circ (1 \otimes f_i) \cdot (p \otimes q))] \\
&\overset{(3.24)}{=} \rho^\bullet[(a(e_i \otimes 1) \circ b) \bullet (\psi(xq\theta^{-1}(f_i))p \otimes y)] \\
&\overset{(2.34)}{=} \rho^\bullet[(a(e_i \otimes 1) \circ b) \bullet (\psi(f_i xq)p \otimes y)]
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(2.33)}{=} \rho^{\bullet}[(a(xq\otimes 1)\circ b)\bullet(p\otimes y)]\\
&\overset{(3.20)}{=} (y\otimes 1)(a(xq\otimes 1)\circ b)(p\otimes 1)\\
&= (y\otimes 1)a(x\otimes 1)\circ(q\otimes 1)b(p\otimes 1)\\
&\overset{(3.20)}{=} (\rho^{\bullet}\circ\rho^{\bullet})[(a\bullet(x\otimes y))\circ(b\bullet(p\otimes q))]
\end{aligned}
$$

□

Remark 3.2.3. For any commutative algebra R over a field k, a duoidal category $\mathsf{bim}(R)$ of R–bimodules was constructed in [5, Example 6.18] (see also [16, Section 5.2], where the authors study duoidal categories —in particular $\mathsf{bim}(R)$— arising from a special type of pseudo-monoids in a monoidal bicategory). Although the constructions in [5, Example 6.18] and in the current subsection are similar in flavor, they yield inequivalent categories for a commutative separable Frobenius k–algebra R (in which case both can be applied). Indeed, an equivalence $\mathsf{bim}(R) \cong \mathsf{bim}(R^e)$ would imply the Morita equivalence of R^e and $R^e \otimes R^e$; hence $R^e \cong R \cong k$. To say a bit more about the relationship between the categories $\mathsf{bim}(R)$ and $\mathsf{bim}(R^e)$, let R be a commutative separable Frobenius k–algebra. Any R–bimodule M with left and right actions $r \otimes m \mapsto r \triangleright m$ and $m \otimes r \mapsto m \triangleleft r$ can be regarded as an $R^e \cong R \otimes R$–bimodule putting $(s \otimes r)m := r \triangleright m \triangleleft s =: m(s \otimes r)$. This is the object map of a fully faithful embedding (acting on the morphisms as the identity map) from the category $\mathsf{bim}(R)$ in [5, Example 6.18] to the category $\mathsf{bim}(R^e)$ in Theorem 3.2.1 —but it is not an equivalence. It is strict monoidal with respect to the monoidal products $\diamond$ in [5, Example 6.18] and $\circ$ in Theorem 3.2.1 —but not with respect to $\star$ in [5, Example 6.18] and $\bullet$ in Theorem 3.2.1. In fact, it takes the monoidal product $\star$ to $\bullet$ but it does not preserve its monoidal unit. The image of the $\star$–monoidal unit R in [5, Example 6.18] does not serve as a $\bullet$–monoidal unit in our $\mathsf{bim}(R^e)$, while our $\bullet$–monoidal unit R^e does not lie in the image of the above embedding $\mathsf{bim}(R) \to \mathsf{bim}(R^e)$.

Recall (from [5, Appendix C.5.3]) that for a commutative k–algebra R, the duoidal category $\mathsf{bim}(R)$ in [5, Example 6.18] arises via the so-called 'looping principle'. This means the following. If $(\mathsf{V}, \times, \mathbf{1})$ is a monoidal 2–category and C is a V–enriched bicategory, then for any object R of C, the endo-hom object $\mathsf{C}(R, R)$ is a pseudo-monoid in V. By [5, Appendix C.2.4], duoidal categories can be regarded as pseudo-monoids in the monoidal 2–category coMon of monoidal categories, comonoidal functors and comonoidal natural transformations (with monoidal structure provided by the Cartesian product). So via the looping principle, hom objects in a coMon–enriched bicategory are duoidal categories. Below we claim that also the duoidal category $\mathsf{bim}(R^e)$ in Theorem 3.2.1 can be obtained via the looping principle. (See [16] for a more general comment about this.)

Proposition 3.2.4. *There exists a coMon–enriched bicategory C whose objects are separable Frobenius k–algebras and such that, for any object R in C, $\mathsf{C}(R,R) \cong \mathsf{bim}(R^e)$.*

Proof. For any separable Frobenius k–algebras R and S, let $\mathsf{C}(R,S)$ be the category of R^e-S^e–bimodules. As in (3.13), we can regard any R^e-S^e–bimodule M as an $S \otimes R$–bimodule via the actions

$$(s \otimes r) \cdot m \cdot (s' \otimes r') = (r' \otimes \theta(r))m(s' \otimes s),$$

where θ denotes the Nakayama automorphism of R. Hence $\mathsf{C}(R,S)$ is a monoidal category via the $S \otimes R$–module tensor product

$$M \bullet N := M \otimes N / \langle (r \otimes 1)m(s \otimes 1) \otimes n - m \otimes (1 \otimes \theta(r))n(1 \otimes s) \rangle,$$

cf. (3.12). The product $M \bullet N$ is an R^e-S^e–bimodule as in (3.17-3.18). The monoidal unit is $R \otimes S$ with the actions $(r \otimes r')(x \otimes y)(s \otimes s') = rx\theta^{-1}(r') \otimes s'ys$ (which becomes isomorphic to the R^e–bimodule J in (3.19) if $S = R$). For any separable Frobenius k–algebra R, there is a comonoidal functor I_R from the singleton category $\mathbf{1}$ to $\mathsf{C}(R,R)$, sending the single object of $\mathbf{1}$ to the R^e–bimodule I in (3.10). Its comonoidal structure is given (up-to isomorphism) by the R^e–bimodule maps $\tau : I \to J$ and $\Delta_I : I \to I \bullet I$ in (3.24). Coassociativity and counitality of this comonoidal functor follows by coassociativity and counitality of Δ_I. Furthermore, for any separable Frobenius k–algebras S, R and T, there is a comonoidal functor $\circ_{S,R,T} : \mathsf{C}(S,R) \times \mathsf{C}(R,T) \to \mathsf{C}(S,T)$ given by the usual R^e–module tensor product. Denoting by $(\psi, e_i \otimes f_i)$ a separable Frobenius structure on R, its comonoidal structure is given by the maps $(S \otimes R) \circ (R \otimes T) \to S \otimes T$ and $(A \bullet B) \circ (C \bullet D) \to (A \circ C) \bullet (B \circ D)$ defined by

$$(s \otimes r) \circ (r' \otimes t) \quad \mapsto \quad \psi(rr')(s \otimes t) \tag{3.29}$$

$$(a \bullet b) \circ (c \bullet d) \quad \mapsto \quad (a(e_i \otimes 1) \circ c) \bullet (b \circ (1 \otimes f_i)d), \tag{3.30}$$

for any S^e-R^e–bimodules A and B and R^e-T^e–bimodules C and D (compare them with μ_J in (3.24) and γ in (3.23)). By (3.25) and (3.26), the map (3.30) is a bimodule map; by computations similar to (3.27) and (3.28), also the map (3.29) is so.

Naturality of the binary part is immediate. Coassociativity and counitality of the comonoidal functor $\circ_{S,R,T}$ is verified by the same computations used in the proof of Theorem 3.2.1 to check that (2.5), (2.8) and (2.9) hold. The unitors and the associator for the module tensor product $\circ$ give rise to 2–cells

$$\begin{array}{ccccc}
\mathsf{C}(R,S) \times \mathbf{1} & \xleftarrow{\cong} & \mathsf{C}(R,S) & \xrightarrow{\cong} & \mathbf{1} \times \mathsf{C}(R,S) \\
{\scriptstyle \mathsf{C}(R,S)\times I_S}\downarrow & \Rightarrow & \| & \Leftarrow & \downarrow{\scriptstyle I_R \times \mathsf{C}(R,S)} \\
\mathsf{C}(R,S) \times \mathsf{C}(S,S) & \xrightarrow[\circ_{R,S,S}]{} & \mathsf{C}(R,S) & \xleftarrow[\circ_{R,R,S}]{} & \mathsf{C}(R,R) \times \mathsf{C}(R,S)
\end{array}$$

$$\begin{array}{ccc}
(\mathsf{C}(Z,R) \times \mathsf{C}(R,S)) \times \mathsf{C}(S,T) & \xrightarrow{\cong} & \mathsf{C}(Z,R) \times (\mathsf{C}(R,S) \times \mathsf{C}(S,T)) \\
{\scriptstyle \circ_{Z,R,S} \times \mathsf{C}(S,T)}\downarrow & & \downarrow{\scriptstyle \mathsf{C}(Z,R) \times \circ_{R,S,T}} \\
\mathsf{C}(Z,S) \times \mathsf{C}(S,T) & \Rightarrow & \mathsf{C}(Z,R) \times \mathsf{C}(R,T) \\
{\scriptstyle \circ_{Z,S,T}}\downarrow & & \downarrow{\scriptstyle \circ_{Z,R,T}} \\
\mathsf{C}(Z,T) & = & \mathsf{C}(Z,T)
\end{array}$$

in coMon, for any separable Frobenius algebras R, S, T, Z. Indeed, they are shown to be comonoidal natural transformations by computations similar to those in the proof of Theorem 3.2.1 verifying the associativity and the unitality of μ_J and the validity of (2.4), (2.6) and (2.7). They clearly obey the Mac Lane type coherence conditions. This proves that C is a coMon–enriched bicategory. Hence, by [5, Appendix C.2.4] and the looping principle, $\mathsf{C}(R,R) \cong \mathsf{bim}(R^e)$ is a duoidal category. □

This finishes the construction of the object part of the functor $\mathsf{bim}(-^e) : \mathsf{sfr} \to \mathsf{duo}$. Let us now turn to its morphism part.

Let R and R' be separable Frobenius (co)algebras. For any coalgebra morphism $q : R \to R'$, define $q^e : R^e \to R'^e$ by $q^e(s \otimes r) = q(s) \otimes q(r)$. Associated to q, there is a functor $\mathsf{bim}(q^e) : \mathsf{bim}(R^e) \to \mathsf{bim}(R'^e)$. On morphisms it acts as the identity map. On objects, it takes an R^e–bi(co)module P with coactions $\lambda : P \to R^e \otimes P$ and $\rho : P \to P \otimes R^e$ to the R'^e–bi(co)module P with the coactions $(q^e \otimes P)\lambda$ and $(P \otimes q^e)\rho$. The R'^e–actions on P are induced from the R^e–actions by the dual forms of q; that is, by the algebra maps

$$\widetilde{q} : R' \to R,\ r' \mapsto \psi'(r'q(e_i))f_i \qquad \text{and} \qquad \hat{q} : R' \to R,\ r' \mapsto e_i\psi'(q(f_i)r') \tag{3.31}$$

as

$$(r' \otimes s')p(u' \otimes v') = (\widetilde{q}(r') \otimes \hat{q}(s'))p(\hat{q}(u') \otimes \widetilde{q}(v')), \tag{3.32}$$

for $p \in P$, $r', s', u', v' \in R'$. Note that, by (2.33),

$$\hat{q}(e'_i) \otimes f'_i = e_j \otimes q(f_j) \qquad \text{and} \qquad e'_i \otimes \widetilde{q}(f'_i) = q(e_j) \otimes f_j. \tag{3.33}$$

The maps $\widetilde{q}$ and $\hat{q}$ are equal if and only if q commutes with the Nakayama automorphisms of R and R'; that is, $\theta' q = q\theta$. Indeed, if this equality holds, for any $r' \in R'$,

$$\begin{aligned} \widetilde{q}(r') &\overset{(3.31)}{=} \psi'(r'q(e_i))f_i \overset{(2.34)}{=} \psi'(\theta' q(e_i)r')f_i \\ &= \psi'(q\theta(e_i)r')f_i \overset{(2.38)}{=} \psi'(q(f_i)r')e_i \overset{(3.31)}{=} \hat{q}(r'). \end{aligned}$$

Conversely, if $\widetilde{q} = \hat{q}$, for any $r \in R$,

$$\begin{aligned} q\theta(r) &\overset{(2.39)}{=} \psi(e_i r)q(f_i) \overset{(3.33)}{=} f'_i\psi(\widetilde{q}(e'_i)r) \overset{(3.31)}{=} f'_i\psi(\psi'(e'_i q(e_j))f_j r) \\ &= f'_i\psi'(e'_i q(e_j\psi(f_j r))) \overset{(2.33)}{=} f'_i\psi'(e'_i q(r)) \overset{(2.39)}{=} \theta' q(r). \end{aligned}$$

Proposition 3.2.5. *Let R and R' be separable Frobenius (co)algebras and $q : R \to R'$ be a coalgebra morphism which commutes with the Nakayama automorphisms of R and R'. The induced functor* $\mathsf{bim}(q^e) : \mathsf{bim}(R^e) \to \mathsf{bim}(R'^e)$ *is comonoidal with respect to both monoidal structures.*

Proof. The coalgebra morphisms $q : R \to R'$ are in bijective correspondence with the algebra morphisms $\widetilde{q} : R' \to R$ via transposition (or duality)

$$q \mapsto \widetilde{q} = \psi'(-q(e_i))f_i \qquad \widetilde{q} \mapsto q = e'_i\psi(\widetilde{q}(f'_i)-). \tag{3.34}$$

In particular, the Nakayama automorphism and its dual $\widetilde{\theta}$ satisfy

$$\widetilde{\theta}(r) = \psi(r\theta(e_i))f_i \overset{(2.38)}{=} \psi(rf_i)e_i = \theta^{-1}(r).$$

Thus the assumption $\theta' q = q\theta$ can be written equivalently as $\widetilde{q}\theta' = \theta\widetilde{q}$.

The candidate for the binary part of the comonoidal structure with respect to $\circ$ is the (R^e–bimodule) map $\mathsf{bim}(q^e)_2^\circ : \mathsf{bim}(q^e)(M \circ N) \to \mathsf{bim}(q^e)M \circ' \mathsf{bim}(q^e)N$ defined by

$$m \circ n \mapsto m(e_i \otimes f_j) \circ' (f_i \otimes e_j)n.$$

It is evidently coassociative. The nullary part of the $\circ$–comonoidal structure is

$$\mathsf{bim}(q^e)_0^\circ = q^e : R^e \to R'^e, \qquad x \otimes y \mapsto q(x) \otimes q(y).$$

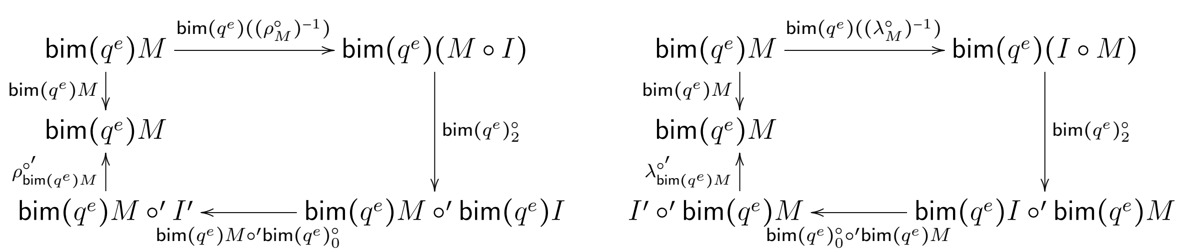
Its R'^e–bimodule map property, that is,

$$sq(x)s' \otimes rq(y)r' = q(\widetilde{q}(s)x\widetilde{q}(s')) \otimes q(\widetilde{q}(r)y\widetilde{q}(r')),$$

is proven by

$$\begin{aligned} q(\widetilde{q}(r')x\widetilde{q}(s')) &\overset{(3.34)}{=} \psi'(r'q(e_i))q(f_i x e_j)\psi'(q(f_j)s') \\ &\overset{(2.45)}{=} \psi'(r'q(e_i))q(f_i x)e'_k\psi'(f'_k s') \\ &\overset{(2.33)}{=} \psi'(r'q(e_i))q(f_i x)s' \overset{(2.45)}{=} \psi'(r'e'_l)f'_l q(x)s' \overset{(2.33)}{=} r'q(x)s', \end{aligned} \tag{3.35}$$

for all $r', s' \in R'$. Right and left counitality; that is, commutativity of the diagrams



—which on elements means the identities

$$m = m(e_i\widetilde{q}q(f_i) \otimes \widetilde{q}q(e_j)f_j) \quad \text{and} \quad m = (\widetilde{q}q(e_i)f_i \otimes e_j\widetilde{q}q(f_j))m$$

— follows from

$$e_j\widetilde{q}q(f_j) \overset{(3.33)}{=} \widetilde{q}(e'_i)\widetilde{q}(f'_i) \overset{(2.20)}{=} \widetilde{q}(e'_if'_i) \overset{(2.42)}{=} \widetilde{q}(1') = 1 \tag{3.36}$$

and

$$\widetilde{q}q(e_j)f_j \overset{(3.33)}{=} \widetilde{q}(e'_i)\widetilde{q}(f'_i) \overset{(2.20)}{=} \widetilde{q}(e'_if'_i) \overset{(2.42)}{=} \widetilde{q}(1') = 1. \tag{3.37}$$

The binary part of the $\bullet$–comonoidal structure is given by the R^e–bimodule map $\mathsf{bim}(q^e)^\bullet_2 : \mathsf{bim}(q^e)(M \bullet N) \to \mathsf{bim}(q^e)M \bullet' \mathsf{bim}(q^e)N$ defined by

$$m \bullet n \mapsto (e_i \otimes 1)m(e_j \otimes 1) \bullet' (1 \otimes f_i)n(1 \otimes f_j).$$

Its coassociativity is obvious. The nullary part is given by $\mathsf{bim}(q^e)^\bullet_0 = q^e : R^e \to R'^e$. Its R'^e–bilinearity, that is,

$$r'q(x)s' \otimes sq(y)\theta'^{-1}(r) = q(\widetilde{q}(r')x\widetilde{q}(s')) \otimes q(\widetilde{q}(s)y\theta^{-1}\widetilde{q}(r)),$$

follows by (3.35) and $\widetilde{q}\theta' = \theta\widetilde{q}$. Right and left counitality; that is, commutativity of the diagrams

$$\begin{array}{ccc}
\mathsf{bim}(q^e)M & \xrightarrow{\mathsf{bim}(q^e)((\rho^\bullet_M)^{-1})} & \mathsf{bim}(q^e)(M \bullet J) \\
{\scriptstyle \mathsf{bim}(q^e)M}\downarrow & & \downarrow{\scriptstyle \mathsf{bim}(q^e)^\bullet_2} \\
\mathsf{bim}(q^e)M & & \\
{\scriptstyle \rho^{\bullet'}_{\mathsf{bim}(q^e)M}}\uparrow & & \\
\mathsf{bim}(q^e)M \bullet' J' & \xleftarrow[\mathsf{bim}(q^e)M \bullet' \mathsf{bim}(q^e)^\bullet_0]{} & \mathsf{bim}(q^e)M \bullet' \mathsf{bim}(q^e)J
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{bim}(q^e)M & \xrightarrow{\mathsf{bim}(q^e)((\lambda^\bullet_M)^{-1})} & \mathsf{bim}(q^e)(J \bullet M) \\
{\scriptstyle \mathsf{bim}(q^e)M}\downarrow & & \downarrow{\scriptstyle \mathsf{bim}(q^e)^\bullet_2} \\
\mathsf{bim}(q^e)M & & \\
{\scriptstyle \lambda^{\bullet'}_{\mathsf{bim}(q^e)M}}\uparrow & & \\
J' \bullet' \mathsf{bim}(q^e)M & \xleftarrow[\mathsf{bim}(q^e)^\bullet_0 \bullet' \mathsf{bim}(q^e)M]{} & \mathsf{bim}(q^e)J \bullet' \mathsf{bim}(q^e)M
\end{array}$$

—which on elements means the identities

$$(\widetilde{q}q(e_i)f_i \otimes 1)m(e_j\widetilde{q}q(f_j) \otimes 1) = m \quad \text{and} \quad (1 \otimes e_i\widetilde{q}q(f_i))m(1 \otimes \widetilde{q}q(e_j)f_j) = m$$

— follows by (3.36) and (3.37). □

3.2.2 The category $\mathsf{bmd}(\mathsf{bim}(-^e))$

By Theorem 3.2.1 and Proposition 3.2.5 there is a functor $\mathsf{bim}(-^e) : \mathsf{sfr} \to \mathsf{duo}$. Our next aim is to describe the corresponding category $\mathsf{bmd}(\mathsf{bim}(-^e))$ as a category of weak bialgebras over k. We begin with identifying in the next two paragraphs the objects of $\mathsf{bmd}(\mathsf{bim}(-^e))$ with weak bialgebras; that is, the bimonoids in $\mathsf{bim}(R^e)$ with weak bialgebras of right subalgebras isomorphic to R.

3.2.2. From weak bialgebras to bimonoids. Let $(H,\mu,\eta,\Delta,\epsilon)$ be a weak bialgebra over a field k and let $R := \sqcap^R(H)$. By Theorem 2.5.1, $(\epsilon_{|R}, 1_1 \otimes \sqcap^R(1_2))$ is a separable Frobenius structure on R. The corresponding Nakayama automorphism and its inverse are the (co)restrictions of $\sqcap^R\sqcap^L$ and $\overline{\sqcap}^R\overline{\sqcap}^L$ to R. In this paragraph we equip H with the structure of a bimonoid in $\mathsf{bim}(R^e)$.

First we construct on H a monoid structure in $\mathsf{bim}(R^e)$. By [9, Lemma 2.2], this amounts to the construction of an algebra homomorphism $R^e \to H$. By Lemma 2.5.11, $\widetilde{\eta}: R^e \to H$ defined as $\widetilde{\eta}(s\otimes r) = s\overline{\sqcap}^L(r)$ is an algebra map. It induces an R^e–bimodule structure on H whose actions we denote by juxtaposition. By virtue of [9, Lemma 2.2], the multiplication μ factorizes through an R^e–bilinear associative multiplication $\widetilde{\mu}: H\circ H \to H$ with unit $\widetilde{\eta}$, so that $(H,\widetilde{\mu},\widetilde{\eta})$ has a structure of monoid in $\mathsf{bim}(R^e)$.

In order to equip H with the structure of a comonoid in $\mathsf{bim}(R^e)$, note that $\Delta: H \to H\otimes H$ factorizes through $H\bullet H$ (via the inclusion $\iota^\bullet_{H,H}: H\bullet H \to H\otimes H$, cf. (3.16)). That is, for any $h\in H$,

$$\begin{aligned}\Delta(h) &= h_1\otimes h_2 \overset{(2.47)}{=} 1_1h_11_{1'}\otimes 1_2h_21_{2'} \\ &\overset{(2.61)}{=} 1_1h_11_{1'}\otimes \overline{\sqcap}^L\sqcap^R(1_2)h_2\overline{\sqcap}^L\sqcap^R(1_{2'}) \\ &\overset{(2.62)}{=} (1_1\otimes 1)h_1(1_{1'}\otimes 1)\otimes(1\otimes\sqcap^R(1_2))h_2(1\otimes\sqcap^R(1_{2'})) \overset{(3.16)}{=} \iota^\bullet_{H,H}(h_1\bullet h_2).\end{aligned} \tag{3.38}$$

As the comultiplication for the bimonoid associated to the weak bialgebra H, consider the corestriction $\widetilde{\Delta}: H \to H\bullet H$ of Δ. It is R^e–bilinear by the R–module map properties of Δ, cf. (2.64); and its coassociativity is obvious. As the counit, consider the map $\widetilde{\epsilon}: H \to R\otimes R$ defined as

$$\widetilde{\epsilon} = (\sqcap^R\otimes\overline{\sqcap}^R)\Delta = (\sqcap^R\otimes\overline{\sqcap}^R)\Delta^{\mathsf{op}}. \tag{3.39}$$

Both defining expressions of $\widetilde{\epsilon}$ are indeed equal, since for any $h\in H$,

$$\sqcap^R(h_1)\otimes\overline{\sqcap}^R(h_2) \overset{(2.61)}{=} \sqcap^R(h_1)\otimes\overline{\sqcap}^R\sqcap^L(h_2) \overset{(2.78)}{=} \sqcap^R(h_2)\otimes\overline{\sqcap}^R\sqcap^L(h_1) \overset{(2.61)}{=} \sqcap^R(h_2)\otimes\overline{\sqcap}^R(h_1).$$

Right R^e–linearity of $\widetilde{\epsilon}$ follows by

$$\begin{aligned}\widetilde{\epsilon}(h(s\otimes r)) &= \widetilde{\epsilon}(hs\overline{\sqcap}^L(r)) \overset{(3.39)(2.68)}{=} 1_1\otimes 1_{1'}\epsilon(1_{2'}hs\overline{\sqcap}^L(r)1_2) = r1_1\otimes 1_{1'}\epsilon(1_{2'}hs1_2)\\ &\overset{(2.68)}{=} r1_1\otimes 1_{1'}\epsilon(1_{2'}h1_2s) \overset{(2.71)}{=} r1_1\otimes 1_{1'}\epsilon(1_{2'}h1_2\sqcap^L(s))\\ &= r1_1s\otimes 1_{1'}\epsilon(1_{2'}h1_2) \overset{(3.19)}{=} \widetilde{\epsilon}(h)\cdot(s\otimes r),\end{aligned}$$

for $h\in H$ and $s\otimes r\in R^e$. The sixth and the third equalities follow, respectively, by the following identities:

$$\begin{aligned}1_1\otimes 1_2\sqcap^L\sqcap^R(h) &\underset{(2.63)}{\overset{(2.61)}{=}} \overline{\sqcap}^R\sqcap^L(1_1)\otimes 1_2\sqcap^L\sqcap^R(h)\\ &\overset{\text{Thm. }2.5.1}{=} \overline{\sqcap}^R(\sqcap^L\sqcap^R(h)\sqcap^L(1_1))\otimes 1_2\\ &\overset{\text{Prop. }2.5.10}{=} \overline{\sqcap}^R\sqcap^L(1_1)\overline{\sqcap}^R\sqcap^L\sqcap^R(h)\otimes 1_2 \underset{(2.61)}{\overset{(2.63)}{=}} 1_1\sqcap^R(h)\otimes 1_2,\end{aligned} \tag{3.40}$$

$$1_1\otimes\overline{\sqcap}^L\sqcap^R(h)1_2 \overset{(2.69)}{=} 1_1\otimes\overline{\sqcap}^L\overline{\sqcap}^R\sqcap^R(h)1_2 = \overline{\sqcap}^R\sqcap^R(h)1_1\otimes 1_2 \overset{(2.69)}{=} \sqcap^R(h)1_1\otimes 1_2 \tag{3.41}$$

where in the second equality of (3.41) we used the symmetric version of (3.40) in A^{op}. Left R^e–linearity of $\widetilde{\epsilon}$ is checked symmetrically. The computation

$$h_1\cdot(\sqcap^R(h_2)\otimes\overline{\sqcap}^R(h_3)) = (\overline{\sqcap}^R(h_3)\otimes 1)h_1(\sqcap^R(h_2)\otimes 1) = \overline{\sqcap}^R(h_3)h_1\sqcap^R(h_2) = h,$$

for any $h\in H$, shows the right counitality of $\widetilde{\Delta}$; left counitality is checked symmetrically. This proves that $(H,\widetilde{\Delta},\widetilde{\epsilon})$ is a comonoid in $\mathsf{bim}(R^e)$.

Our next aim is to show that the compatibility conditions —expressed by diagrams (2.13), (2.14), (2.15) and (2.16)— hold between the above monoid and comonoid structures of H. For any $h,h'\in H$,

$$\begin{aligned}(\widetilde{\mu}\bullet\widetilde{\mu})\gamma(\widetilde{\Delta}\circ\widetilde{\Delta})(h\circ h') &= (\widetilde{\mu}\bullet\widetilde{\mu})\gamma((h_1\bullet h_2)\circ(h'_1\bullet h'_2))\\ &\overset{(3.23)}{=} (\widetilde{\mu}\bullet\widetilde{\mu})((h_1(1_1\otimes 1)\circ h'_1)\bullet(h_2\circ(1\otimes\sqcap^R(1_2))h'_2))\\ &\overset{(2.61)(2.62)}{=} h_11_1h'_1\bullet h_21_2h'_2 \overset{(2.47)}{=} (hh')_1\bullet(hh')_2\\ &= \widetilde{\Delta}\widetilde{\mu}(h\circ h'),\end{aligned}$$

$$\mu_J(\widetilde{\epsilon}\circ\widetilde{\epsilon})(h\circ h') \overset{(3.39)}{=} \mu_J[(\sqcap^R(h_1)\otimes\overline{\sqcap}^R(h_2))\circ(\sqcap^R(h'_1)\otimes\overline{\sqcap}^R(h'_2))]$$

$$
\begin{aligned}
&\overset{(3.24)}{=} \epsilon(\sqcap^R(h_1)\overline{\sqcap}^R(h'_2))\sqcap^R(h'_1)\otimes\overline{\sqcap}^R(h_2)\\
&\overset{(2.58)}{=} \epsilon(\sqcap^R(h_1)h'_2)\sqcap^R(h'_1)\otimes\overline{\sqcap}^R(h_2)\\
&\overset{(2.50)(2.64)}{=} \sqcap^R(\sqcap^R(h_1)h')\otimes\overline{\sqcap}^R(h_2)\overset{(2.71)}{=}\sqcap^R(h_1h')\otimes\overline{\sqcap}^R(h_2)\\
&\overset{(2.47)}{=} \sqcap^R(h_1 1_1 h')\otimes\overline{\sqcap}^R(h_2 1_2)\overset{(2.73)}{=}\sqcap^R(h_1h'_1)\otimes\overline{\sqcap}^R(h_2\sqcap^L(h'_2))\\
&\overset{(*)}{=} \sqcap^R(h_1h'_1)\otimes\overline{\sqcap}^R(h_2h'_2)\overset{(2.47)}{=}\sqcap^R((hh')_1)\otimes\overline{\sqcap}^R((hh')_2)\\
&\overset{(3.24)(3.39)}{=} \widetilde{\epsilon}\widetilde{\mu}(h\circ h'),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Delta}\widetilde{\eta}(s\otimes r) &= \widetilde{\Delta}(s\overline{\sqcap}^L(r))\overset{(2.64)(2.65)}{=}1_1\overline{\sqcap}^L(r)\bullet s1_2\\
&\overset{(2.61)(2.62)}{=}(1_1\otimes r)1\bullet(s\otimes\sqcap^R(1_2))1\overset{(3.12)}{=}(\overline{\sqcap}^R(1_2)1_1\otimes r)1\bullet(s\otimes 1)1\\
&\overset{(2.70)}{=}(1\otimes r)1\bullet(s\otimes 1)1=(\widetilde{\eta}\bullet\widetilde{\eta})[(1\otimes r)\bullet(s\otimes 1)]\\
&\overset{(3.24)}{=}(\widetilde{\eta}\bullet\widetilde{\eta})\Delta_I(s\otimes r),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\epsilon}\widetilde{\eta}(s\otimes r) &= \widetilde{\epsilon}(s\overline{\sqcap}^L(r))\overset{(3.39)}{=}\sqcap^R(\overline{\sqcap}^L(r)1_1)\otimes\overline{\sqcap}^R(s1_2)\\
&\overset{(*)}{=}\sqcap^R(r1_1)\otimes\overline{\sqcap}^R(s1_2)\overset{(2.67)}{=}r1_1\otimes s\overline{\sqcap}^R(1_2)\\
&\overset{(2.62)(2.61)}{=}r\sqcap^R(1_2)\otimes s1_1\overset{(3.24)}{=}\tau(s\otimes r).
\end{aligned}
$$

In the equalities marked with $(*)$ we used that for all $h,h'\in H$,

$$
\overline{\sqcap}^R(hh')\overset{(2.71)}{=}\overline{\sqcap}^R(h\overline{\sqcap}^R(h'))\overset{(2.61)}{=}\overline{\sqcap}^R(h\overline{\sqcap}^R\sqcap^L(h'))\overset{(2.71)}{=}\overline{\sqcap}^R(h\sqcap^L(h'))
$$

and symmetrically, $\sqcap^R(hh')=\sqcap^R(\overline{\sqcap}^L(h)h')$. Therefore, we conclude that $(H,\widetilde{\mu},\widetilde{\eta},\widetilde{\Delta},\widetilde{\epsilon})$ is a bimonoid in $\mathsf{bim}(R^e)$.

3.2.3. From bimonoids to weak bialgebras. Take now a bimonoid $(H,\widetilde{\mu},\widetilde{\Delta},\widetilde{\eta},\widetilde{\epsilon})$ in $\mathsf{bim}(R^e)$, for some separable Frobenius (co)algebra R over the field k. In this paragraph we equip H with the structure of a weak bialgebra over k, whose right subalgebra is isomorphic to R.

Let $(\psi,e_i\otimes f_i)$ be a separable Frobenius structure on R. First we construct an associative and unital k–algebra structure on H, via the multiplication and the unit defined by

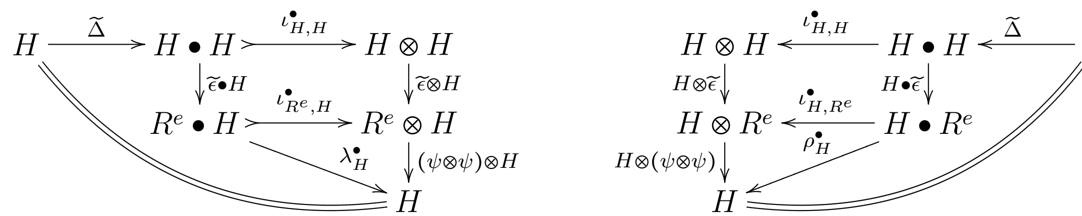
$$
\mu: H\otimes H\xrightarrow{\pi^\circ_{H,H}}H\circ H\xrightarrow{\widetilde{\mu}}H \quad\text{and}\quad \eta: k\xrightarrow{\eta_{R^e}}R^e\xrightarrow{\widetilde{\eta}}H, \tag{3.42}
$$

where η_{R^e} denotes the unit of the k–algebra R^e.

Next, we can make H to be a k–coalgebra via the comultiplication and the counit

$$\Delta : H \xrightarrow{\widetilde{\Delta}} H \bullet H \rightarrowtail^{\iota^\bullet_{H,H}} H \otimes H \quad \text{and} \quad \epsilon : H \xrightarrow{\widetilde{\epsilon}} R^e \xrightarrow{\psi \otimes \psi} k. \tag{3.43}$$

Indeed, Δ is evidently coassociative and it is counital by commutativity of



where the triangles at the bottom commute by (3.22).

Our next aim is to show that the above algebra and coalgebra structures of H combine into a weak bialgebra. In doing so, we use both Heynemann-Sweedler notations $\widetilde{\Delta}(h) = h_{\tilde{1}} \bullet h_{\tilde{2}}$ and $\Delta(h) = h_1 \otimes h_2$, for any $h \in H$.

We begin with checking the multiplicativity of the comultiplication Δ; that is, axiom (2.47). For any $h \in H$,

$$\Delta(h) \overset{(3.43)}{=} \iota^\bullet_{H,H}\widetilde{\Delta}(h) \overset{(3.16)}{=} (e_j \otimes 1)h_{\tilde{1}}(e_i \otimes 1) \otimes (1 \otimes f_j)h_{\tilde{2}}(1 \otimes f_i), \tag{3.44}$$

hence

$$\begin{aligned} \Delta(h)\Delta(h') &\overset{(3.44)}{=} (e_j \otimes 1)h_{\tilde{1}}(e_i \otimes 1)h'_{\tilde{1}}(e_k \otimes 1) \otimes (1 \otimes f_j)h_{\tilde{2}}(1 \otimes f_i)h'_{\tilde{2}}(1 \otimes f_k) \\ &= \iota^\bullet_{H,H}(\widetilde{\mu} \bullet \widetilde{\mu})\gamma(\widetilde{\Delta} \circ \widetilde{\Delta})\pi^\circ_{H,H}(h \otimes h') \overset{(2.13)}{=} \iota^\bullet_{H,H}\widetilde{\Delta}\widetilde{\mu}\pi^\circ_{H,H}(h \otimes h') \\ &\overset{(3.43)(3.42)}{=} \Delta\mu(h \otimes h') = \Delta(hh'), \end{aligned}$$

for all $h, h' \in H$. Next we check axiom (2.49), expressing weak comultiplicativity of the unit. From (2.15) on the bimonoid H it follows that, for any $r \otimes s \in R^e$,

$$\Delta\widetilde{\eta}(r \otimes s) \overset{(3.43)}{=} \iota^\bullet_{H,H}\widetilde{\Delta}\widetilde{\eta}(r \otimes s) \overset{(2.15)}{=} \iota^\bullet_{H,H}(\widetilde{\eta} \bullet \widetilde{\eta})\Delta_I(r \otimes s) \overset{(3.24)}{=} \widetilde{\eta}(e_i \otimes s) \otimes \widetilde{\eta}(r \otimes f_i). \tag{3.45}$$

With this identity at hand, the weak comultiplicativity of the unit is checked by

$$\begin{aligned}(H\otimes\Delta)\Delta(1) &\overset{(3.45)}{=} \widetilde{\eta}(e_i\otimes 1)\otimes\Delta\widetilde{\eta}(1\otimes f_i) \overset{(2.15)}{=} \widetilde{\eta}(e_i\otimes 1)\otimes\widetilde{\eta}(e_j\otimes f_i)\otimes\widetilde{\eta}(1\otimes f_j)\\ &= \widetilde{\eta}(e_i\otimes 1)\otimes\widetilde{\eta}(1\otimes f_i)\widetilde{\eta}(e_j\otimes 1)\otimes\widetilde{\eta}(1\otimes f_j)\\ &\overset{(3.45)}{=} (\Delta(1)\otimes 1)(1\otimes\Delta(1)).\end{aligned}$$

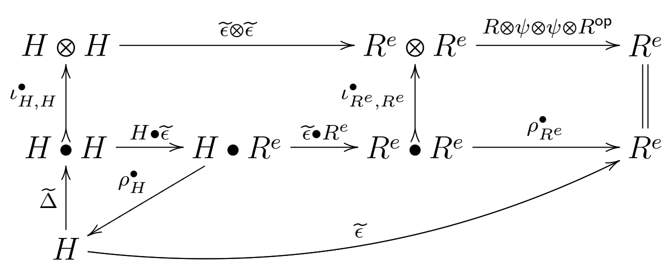
Since $\widetilde{\eta}(1\otimes r)\widetilde{\eta}(s\otimes 1) = \widetilde{\eta}(s\otimes r) = \widetilde{\eta}(s\otimes 1)\widetilde{\eta}(1\otimes r)$, for all $r, s \in R$, also $(1\otimes\Delta(1))(\Delta(1)\otimes 1) = (H\otimes\Delta)\Delta(1)$. Finally, we check that the axiom on weak multiplicativity of the counit —in its equivalent form given by part (ii) in Lemma 2.5.5— holds. This starts with proving the equality

$$\widetilde{\epsilon} = (\sqcap\otimes\overline{\sqcap})\iota^{\bullet}_{H,H}\widetilde{\Delta} = (\sqcap\otimes\overline{\sqcap})\Delta \tag{3.46}$$

in terms of the maps

$$\sqcap := (H \xrightarrow{\widetilde{\epsilon}} R\otimes R^{\mathsf{op}} \xrightarrow{R\otimes\psi} R) \qquad \text{and} \qquad \overline{\sqcap} := (H \xrightarrow{\widetilde{\epsilon}} R\otimes R^{\mathsf{op}} \xrightarrow{\psi\otimes R^{\mathsf{op}}} R^{\mathsf{op}}).$$

Equality (3.46) is proven by commutativity of the following diagram, noting that $\rho^{\bullet}_H$ is an isomorphism.



The bottom-right region and the top-left region commute by the R^e–bimodule map property of $\widetilde{\epsilon}$. The bottom-left region commutes by counitality of $\widetilde{\Delta}$. Commutativity of the top-right region follows by

$$\begin{aligned}&(R\otimes\psi\otimes\psi\otimes R^{\mathsf{op}})\iota^{\bullet}_{R^e,R^e}((x\otimes y)\bullet(u\otimes v))\\ &\quad= (R\otimes\psi\otimes\psi\otimes R^{\mathsf{op}})((e_i\otimes 1)\cdot(x\otimes y)\cdot(e_j\otimes 1)\otimes(1\otimes f_i)\cdot(u\otimes v)\cdot(1\otimes f_j))\\ &\quad\overset{(3.19)}{=} (R\otimes\psi\otimes\psi\otimes R^{\mathsf{op}})(xe_j\otimes e_iy\otimes f_ju\otimes v\theta^{-1}(f_i))\\ &\quad\overset{(2.33)(2.36)}{=} xu\otimes vy \overset{(3.19)}{=} (v\otimes 1)\cdot(x\otimes y)\cdot(u\otimes 1) \overset{(3.20)}{=} \rho^{\bullet}_{R^e,R^e}((x\otimes y)\bullet(u\otimes v))\end{aligned}$$

for any $(x\otimes y)\bullet(u\otimes v)\in R^e\bullet R^e$. For all $h, h'\in H$,

$$\sqcap((hh')_1)\otimes\overline{\sqcap}((hh')_2) \overset{(3.46)}{=} \widetilde{\epsilon}\widetilde{\mu}(h\circ h') \overset{(2.14)}{=} \mu_J(\widetilde{\epsilon}\circ\widetilde{\epsilon})(h\circ h') \overset{(3.46)(3.24)}{=} \psi(\sqcap(h_1)\overline{\sqcap}(h'_2))\sqcap(h'_1)\otimes\overline{\sqcap}(h_2). \tag{3.47}$$

Using the R^e–bilinearity of $\widetilde{\epsilon}$ together with (3.46) in the second equality,

$$\begin{aligned}\epsilon(h1_1)\epsilon(1_2h') &\overset{(3.43)}{=} (\psi\otimes\psi)\widetilde{\epsilon}(h(e_i\otimes 1))(\psi\otimes\psi)\widetilde{\epsilon}((1\otimes f_i)h')\\ &\overset{(3.19)}{=} \psi(\sqcap(h_1)e_i)\psi\overline{\sqcap}(h_2)\psi\sqcap(h'_1)\psi(\overline{\sqcap}(h'_2)\theta^{-1}(f_i))\\ &\overset{(2.33)(2.34)}{=} \psi(\sqcap(h_1)\overline{\sqcap}(h'_2))\psi\sqcap(h'_1)\psi\overline{\sqcap}(h_2)\\ &\overset{(3.47)}{=} \psi\sqcap((hh')_1)\psi\overline{\sqcap}((hh')_2)\\ &\overset{(3.46)}{=} (\psi\otimes\psi)\widetilde{\epsilon}(hh') = \epsilon(hh'),\end{aligned}$$

where in the first equality, in addition, we used (3.45) and that the multiplication μ of the k–algebra H is R^e–balanced and R^e–bilinear. Symmetrically,

$$\begin{aligned}\epsilon(h1_2)\epsilon(1_1h') &\overset{(3.43)}{=} (\psi\otimes\psi)\widetilde{\epsilon}(h(1\otimes f_i))(\psi\otimes\psi)\widetilde{\epsilon}((e_i\otimes 1)h')\\ &\overset{(3.19)}{=} \psi(f_i\sqcap(h_1))\psi\overline{\sqcap}(h_2)\psi\sqcap(h'_1)\psi(e_i\overline{\sqcap}(h'_2))\\ &\overset{(2.33)}{=} \psi(\sqcap(h_1)\overline{\sqcap}(h'_2))\psi\sqcap(h'_1)\psi\overline{\sqcap}(h_2)\\ &\overset{(3.47)}{=} \psi\sqcap((hh')_1)\psi\overline{\sqcap}((hh')_2)\\ &\overset{(3.46)}{=} (\psi\otimes\psi)\widetilde{\epsilon}(hh') = \epsilon(hh').\end{aligned}$$

We have so far constructed a weak bialgebra structure on H. It remains to check that $\sqcap^R(H)$ is isomorphic to the given separable Frobenius (co)algebra R. With this purpose, consider the map

$$\sigma : R\to H, \qquad r\mapsto\widetilde{\eta}(r\otimes 1). \tag{3.48}$$

For any $s, r\in R$,

$$\epsilon\widetilde{\eta}(r\otimes s) \overset{(3.43)}{=} (\psi\otimes\psi)\widetilde{\epsilon}\widetilde{\eta}(r\otimes s) \overset{(2.16)}{=} (\psi\otimes\psi)\tau(r\otimes s) \overset{(3.24)}{=} \psi(sf_i)\psi(re_i) \overset{(2.33)}{=} \psi(sr) \tag{3.49}$$

and

$$\sqcap^R\sigma(r) \overset{(2.50)(3.45)}{\underset{(2.20)}{=}} \widetilde{\eta}(e_i\otimes 1)\epsilon\widetilde{\eta}(r\otimes f_i) \overset{(3.49)}{=} \widetilde{\eta}(e_i\otimes 1)\psi(f_ir) \overset{(2.33)}{=} \widetilde{\eta}(r\otimes 1) \overset{(3.48)}{=} \sigma(r). \tag{3.50}$$

This proves that σ corestricts to a map $R \to \sqcap^R(H)$, to be denoted also by σ. This restricted map $\sigma : R \to \sqcap^R(H)$ is our candidate to establish the desired isomorphism of separable Frobenius (co)algebras. Since $\widetilde{\eta}$ is a k–algebra morphism, so is σ. Comultiplicativity of σ is proven by

$$\sigma(r)\widetilde{\eta}(e_i \otimes 1) \otimes \sqcap^R \widetilde{\eta}(1 \otimes f_i) \overset{(3.50)}{=} \widetilde{\eta}(re_i \otimes 1) \otimes \widetilde{\eta}(f_i \otimes 1) \overset{(3.48)}{=} \sigma(re_i) \otimes \sigma(f_i).$$

Finally, it is checked that σ is also counital by applying (3.49) for $s = 1$. By [54, Proposition A.3], this proves that σ is an isomorphism of separable Frobenius (co)algebras.

Theorem 3.2.4. *Let R be a separable Frobenius algebra over a field k. A bimonoid in the duoidal category* $\mathsf{bim}(R^e)$ *in Theorem 3.2.1 is, equivalently, a weak bialgebra over k whose right subalgebra is isomorphic to R (as a separable Frobenius algebra).*

Proof. In light of Paragraphs 3.2.2 and 3.2.3, we only have to prove the bijectivity of the correspondence described in them. Starting with a weak bialgebra $(H, \mu, \eta, \Delta, \epsilon)$, and applying to it the above constructions, the resulting weak bialgebra has the same structure as H, as the following shows. The resulting multiplication is the unique map which yields $\mu\pi^{\circ}_{H,H}$ if composed with $\pi^{\circ}_{H,H}$. Hence it is equal to μ. The resulting unit map multiplies an element of k by $1\overline{\sqcap}^L(1) = 1$ hence it is equal to η. The resulting comultiplication is equal to Δ by (3.38). The resulting counit sends $h \in H$ to

$$(\epsilon_{|R} \sqcap^R \otimes \epsilon_{|R}\overline{\sqcap}^R)\Delta(h) \overset{(2.56)(2.58)}{=} (\epsilon \otimes \epsilon)\Delta(h) \overset{(2.28)}{=} \epsilon(h).$$

Conversely, consider a bimonoid $(H, \widetilde{\mu}, \widetilde{\eta}, \widetilde{\Delta}, \widetilde{\epsilon})$ in $\mathsf{bim}(R^e)$ and the bimonoid obtained by applying to it the constructions in Paragraphs 3.2.3 and 3.2.2. By construction, they have identical multiplications and comultiplications. Concerning the unit and the counit, note that in the weak bialgebra in Paragraph 3.2.2,

$$\begin{aligned} \sqcap^R(h) &= \widetilde{\eta}(e_i \otimes 1)(\psi \otimes \psi)\widetilde{\epsilon}(h\widetilde{\eta}(1 \otimes f_i)) \\ &= \widetilde{\eta}(e_i \otimes 1)\psi(f_i \sqcap (h_1))\psi\overline{\sqcap}(h_2) = \widetilde{\eta}(\sqcap(h_1) \otimes 1)\epsilon(h_2) = \widetilde{\eta}(\sqcap(h) \otimes 1), \end{aligned}$$

for all $h \in H$. In the first equality we wrote the definition (2.50) of $\sqcap^R$; in the second one we used the right R^e–linearity of $\widetilde{\epsilon}$, (3.46) and (3.19); in the third equality we used the Frobenius property (2.33) and $\psi\overline{\sqcap} = \epsilon$, and in the last one we used the counit property of ϵ. By similar computations also the idempotent maps $\overline{\sqcap}^L$ and $\overline{\sqcap}^R$ —in the weak bialgebra associated in Paragraph 3.2.2 to the bimonoid $(H, \widetilde{\mu}, \widetilde{\eta}, \widetilde{\Delta}, \widetilde{\epsilon})$— can be expressed as

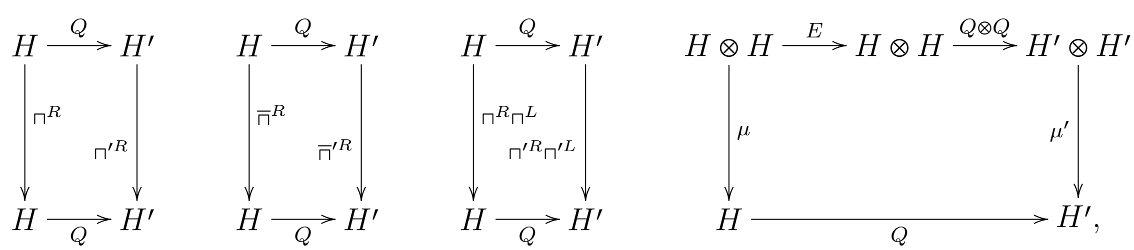
$$\overline{\sqcap}^L = \widetilde{\eta}(1 \otimes \sqcap(-)) \qquad \text{and} \qquad \overline{\sqcap}^R = \widetilde{\eta}(\overline{\sqcap}(-) \otimes 1).$$

So the counits differ by the isomorphism $\sigma \otimes \sigma$ by (3.46). Finally, in the bimonoid obtained by applying both constructions, the unit map takes $s \otimes r \in R \otimes R^{\mathsf{op}}$ to $\sigma(s)\overline{\sqcap}^L\sigma(r) \overset{(3.48)}{=} \widetilde{\eta}(s \otimes 1)\widetilde{\eta}(1 \otimes r) \overset{(2.20)}{=} \widetilde{\eta}(s \otimes r)$. □

By Theorem 3.2.4, an object of $\mathsf{bmd}(\mathsf{bim}(-^e))$ is given by a weak bialgebra. We make no notational distinction between a weak bialgebra H and the corresponding bimonoid in the bi(co)module category $\mathsf{bim}(R^e)$, where R is the right subalgebra $\sqcap^R(H)$.

By [56, 61], a weak bialgebra with right subalgebra R can be regarded as a right R–bialgebroid (or '$\times_R$–bialgebra' in [65]) supplemented by a separable Frobenius structure on R. However, since for arbitrary algebras R we cannot equip the category of R^e–bimodules with a duoidal structure, we cannot extend Theorem 3.2.4 to interpret arbitrary bialgebroids as bimonoids in an appropriate duoidal category.

Theorem 3.2.5. *Let H and H' be weak bialgebras with respective right subalgebras R and R'. A morphism in $\mathsf{bmd}(\mathsf{bim}(-^e))$ from (R,H) to (R',H') is, equivalently, a coalgebra map $Q: H \to H'$, rendering commutative the diagrams*[1]



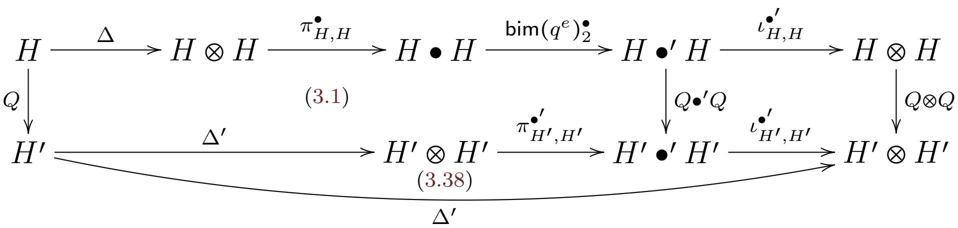
where $E(h \otimes h') := h1_1 \otimes \sqcap^R(1_2)h'$.

Proof. Let us take first a morphism in $\mathsf{bmd}(\mathsf{bim}(-^e))$, and see that it obeys the properties in the claim. A morphism in $\mathsf{bmd}(\mathsf{bim}(-^e))$ is given by a morphism $q: R \to R'$

[1] We will refer to the commutativity of these diagrams, from left to right, as *source* (sc), *target* (tc), *Nakayama* (Nkc) and *weak multiplicativity* (wmc) conditions.

in sfr and a morphism $Q : \mathsf{bim}(q^e)H \to H'$ in $\mathsf{bim}(R'^e)$, rendering commutative the four diagrams in part (b) of Lemma 3.0.2.

Let us check first that Q is a coalgebra map. In order to prove that it is comultiplicative, we need to see that the top row of



is equal to the comultiplication Δ of H. Computing its value on $h \in H$, we get

$$(\widetilde{q}(e'_k)e_i \otimes 1)h_1(e_j\widetilde{q}(e'_l) \otimes 1) \otimes (1 \otimes f_i\widetilde{q}(f'_k))h_2(1 \otimes \widetilde{q}(f'_l)f_j).$$

It is equal to

$$(e_i \otimes 1)h_1(e_j \otimes 1) \otimes (1 \otimes f_i)h_2(1 \otimes f_j) \overset{(2.61)(2.62)}{=} 1_1h_11_{1'} \otimes 1_2h_21_{2'} \overset{(2.47)}{=} \Delta(h)$$

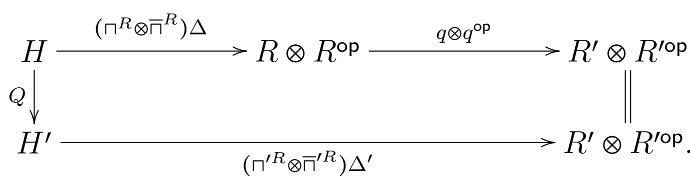
by

$$\begin{aligned} \widetilde{q}(e'_k)e_i \otimes f_i\widetilde{q}(f'_k) &\overset{(2.35)}{=} \widetilde{q}(e'_k)\widetilde{q}(f'_k)e_i \otimes f_i \overset{(2.20)}{=} \widetilde{q}(e'_kf'_k)e_i \otimes f_i \\ &\overset{(2.42)}{=} \widetilde{q}(1')e_i \otimes f_i = e_i \otimes f_i \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} e_j\widetilde{q}(e'_l) \otimes \widetilde{q}(f'_l)f_j &\overset{(2.36)}{=} e_j \otimes \widetilde{q}(f'_l)\theta\widetilde{q}(e'_l)f_j = e_j \otimes \widetilde{q}(f'_l)\widetilde{q}\theta'(e'_l)f_j \\ &\overset{(2.38)}{=} e_j \otimes \widetilde{q}(e'_l)\widetilde{q}(f'_l)f_j \overset{(2.20)(2.42)}{=} e_j \otimes f_j, \end{aligned} \tag{3.52}$$

where, in the second equality of the last computation, we used the commutativity of $q : R \to R'$ in sfr with the respective Nakayama automorphisms θ and θ' of R and R' in its equivalent form $\widetilde{q}\theta' = \theta\widetilde{q}$. This proves the comultiplicativity of Q. In order to see that Q is counital as well, observe that condition (3.2) takes now the form



Composing both paths around it with $R'\otimes\epsilon'_{|R'}$ and with $\epsilon'_{|R'}\otimes R'$, respectively, we obtain

$$\sqcap'^{R} Q(h) = q\sqcap^{R}(h) \qquad \text{and} \qquad \overline{\sqcap}'^{R} Q(h) = q\overline{\sqcap}^{R}(h); \tag{3.53}$$

and composing either one of these equalities with $\epsilon'_{|R'}$ we have the counitality of Q proven.

Let us check now that Q satisfies the required weak multiplicativity condition; that is, it renders commutative the last diagram in the claim. Since (q,Q) is a morphism in $\mathsf{bmd}(\mathsf{bim}(-^e))$ by assumption, it renders commutative diagram (3.3) for any R^e–bimodules A and B. Let us evaluate both paths around it on an arbitrary element $(a\bullet h)\circ(b\bullet h')\in(A\bullet H)\circ(B\bullet H)$. On the one hand, we have

$$
\begin{gathered}
(a\bullet h)\circ(b\bullet h')\\
\downarrow{\scriptstyle \mathsf{bim}(q^e)_2^\circ}\\
(a\bullet h)\cdot(e_i\otimes f_j)\circ'(f_i\otimes e_j)\cdot(b\bullet h')\\
=(a(1\otimes f_j)\bullet h(e_i\otimes 1))\circ'((1\otimes e_j)b\bullet(f_i\otimes 1)h')\\
\downarrow{\scriptstyle \mathsf{bim}(q^e)_2^\bullet\circ'\mathsf{bim}(q^e)_2^\bullet}\\
[(e_p\otimes 1)a(e_q\otimes f_j)\bullet'(1\otimes f_p)h(e_i\otimes f_q)]\circ'\\
[(e_k\otimes e_j)b(e_l\otimes 1)\bullet'(f_i\otimes f_k)h'(1\otimes f_l)]\\
\downarrow{\scriptstyle \gamma'}\\
[(e_p\otimes 1)a(e_q\widetilde{q}(e'_m)\otimes f_j)\circ'(e_k\otimes e_j)b(e_l\otimes 1)]\bullet'\\
[(1\otimes f_p)h(e_i\otimes f_q)\circ'(f_i\otimes f_k\widetilde{q}(f'_m))h'(1\otimes f_l)]\\
\downarrow{\scriptstyle (\mathsf{bim}(q^e)_2^\bullet A\circ'\mathsf{bim}(q^e)_2^\bullet B)\bullet'(Q\circ' Q)}\\
[(e_p\otimes 1)a(e_q\widetilde{q}(e'_m)\otimes f_j)\circ'(e_k\otimes e_j)b(e_l\otimes 1)]\bullet'\\
Q[(1\otimes f_p)h(e_i\otimes f_q)]\circ' Q[(f_i\otimes f_k\widetilde{q}(f'_m))h'(1\otimes f_l)]\\
\\
\overset{(2.35)(2.42)}{\underset{(2.20)}{=}}[(e_p\otimes 1)a(e_q\otimes f_j)\circ'(e_k\otimes e_j)b(e_l\otimes 1)]\bullet'\\
Q[(1\otimes f_p)h(e_i\otimes f_q)]Q[(f_i\otimes f_k)h'(1\otimes f_l)].
\end{gathered}
$$

On the other hand,

$$(a\bullet h)\circ(b\bullet h')$$
$$\downarrow\gamma'$$
$$(a(e_i\otimes 1)\circ b)\bullet(h\circ(1\otimes f_i)h'))$$
$$\downarrow \mathsf{bim}(q^e)_2^{\bullet}$$
$$=[(e_p\otimes 1)a(e_i\otimes 1)\circ b(e_q\otimes 1)]\bullet'[(1\otimes f_p)h\circ(1\otimes f_i)h'(1\otimes f_q)]$$
$$\downarrow \mathsf{bim}(q^e)_2^{\circ}\bullet'\mathsf{bim}(q^e)(H\circ H)$$
$$[(e_p\otimes 1)a(e_i\otimes 1)(e_k\otimes f_l)\circ'(f_k\otimes e_l)b(e_q\otimes 1)]\bullet'[(1\otimes f_p)h\circ(1\otimes f_i)h'(1\otimes f_q)]$$
$$\downarrow(\mathsf{bim}(q^e)_2^{\circ}H\circ'\mathsf{bim}(q^e)_2^{\circ}H)\bullet'\mathsf{bim}(q^e)\mu$$
$$[(e_p\otimes 1)a(e_ie_k\otimes f_l)\circ'(f_k\otimes e_l)b(e_q\otimes 1)]\bullet'[(1\otimes f_p)h(1\otimes f_i)h'(1\otimes f_q)]$$
$$\downarrow(\mathsf{bim}(q^e)_2^{\circ}H\circ'\mathsf{bim}(q^e)_2^{\circ}H)\bullet'Q$$
$$[(e_p\otimes 1)(a(e_ie_k\otimes f_l)\circ'(f_k\otimes e_l)b(e_q\otimes 1)]\bullet'Q[(1\otimes f_p)h(1\otimes f_i)h'(1\otimes f_q)]$$

So, by commuativity of (3.3), the identity

$$\begin{gathered}((e_p\otimes 1)a(e_ie_k\otimes f_l)\circ'(f_k\otimes e_l)b(e_q\otimes 1))\bullet'Q((1\otimes f_p)h(1\otimes f_i)h'(1\otimes f_q))= \\ ((e_p\otimes 1)a(e_q\otimes f_j)\circ'(e_k\otimes e_j)b(e_l\otimes 1))\bullet'Q((1\otimes f_p)h(e_i\otimes f_q))Q((f_i\otimes f_k)h'(1\otimes f_l))\end{gathered}\tag{3.54}$$

holds for any $(a\bullet h)\circ(b\bullet h')\in(A\bullet H)\circ(B\bullet H)$. Take $A=B=R^e\otimes R^e$ with the R^e–actions

$$(r\otimes s)((x\otimes y)\otimes(v\otimes w))(r'\otimes s'):=(rx\otimes ys)\otimes(vr'\otimes s'w).$$

Putting $a=b=1\otimes 1\otimes 1\otimes 1$, and applying $(\iota^{\circ'}\otimes H')\iota^{\bullet'}$ to the resulting equality, it follows by the R'^e–bilinearity of Q and (3.51) that

$$\begin{aligned}&e_p\otimes 1\otimes e_ie_k\otimes f_l\otimes f_k\otimes e_l\otimes e_q\otimes 1\otimes Q((1\otimes f_p)h(1\otimes f_i)h'(1\otimes f_q))\\ =\;&e_p\otimes 1\otimes e_q\otimes f_j\otimes e_k\otimes e_j\otimes e_l\otimes 1\otimes Q((1\otimes f_p)h(e_i\otimes f_q))Q((f_i\otimes f_k)h'(1\otimes f_l)).\end{aligned}$$

Applying ψ to the first, third, fifth and seventh tensorands in the last equality, we get

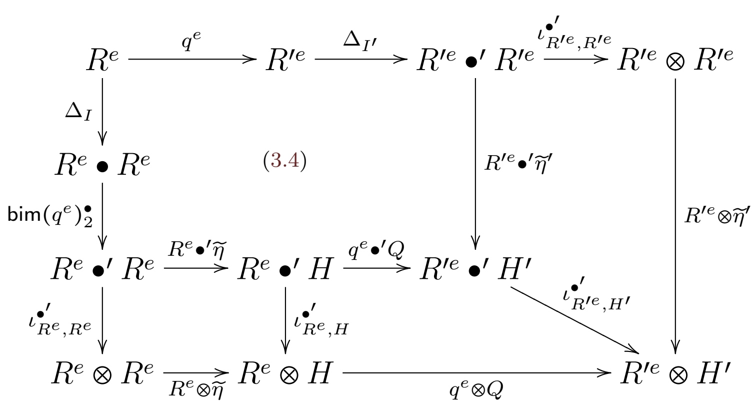
$$1\otimes f_l\otimes e_l\otimes 1\otimes Q(hh')=1\otimes f_j\otimes e_j\otimes 1\otimes Q(h(e_i\otimes 1))Q((f_i\otimes 1)h').$$

This is equivalent to

$$Q(hh') = Q(h1_1)Q(\sqcap^R(1_2)h'),$$

that is, commutativity of the last diagram in the claim.

Next we check that q can be uniquely reconstructed from Q —namely, it is the (co)restriction to $R \to R'$ of $Q : H \to H'$. Evaluating the equal paths around



at $1_1 \otimes \sqcap^R(1_2)r \in R^e$, we obtain

$$q(1_{1'}) \otimes q(\sqcap^R(1_2)r) \otimes Q(1_1 1_{2'}) = 1'_1 \otimes q(\sqcap^R(1_2)r) \otimes q(1_1)1'_2.$$

Applying $\epsilon'_{|R'} \otimes \epsilon'_{|R'} \otimes H'$ to both sides of the identity above and using counitality of q, counitality of the comultiplication (2.80) of R and (2.33), we conclude $Q(r) = q(r)$.

Comparing this identity $Q(r) = q(r)$ with (3.53), the source and target conditions (that is, compatibility of Q with $\sqcap^R$ and $\overline{\sqcap}^R$, meaning the commutativity of the first two diagrams in the claim) follow. By Theorem 2.5.1, the Nakayama condition (that is, compatibility of Q with $\sqcap^R \sqcap^L$, commutativity of the third diagram in the claim) is equivalent to the assumed commutativity of q with the Nakayama automorphisms.

Conversely, assume that $Q : H \to H'$ is a coalgebra map rendering commutative the four diagrams in the statement. We construct its mate $q : R \to R'$ together with whom they constitute a morphism in $\mathsf{bmd}(\mathsf{bim}(-^e))$.

By commutativity of any of the first two diagrams in the claim, Q restricts to a map $q : R \to R'$. Indeed, for $r \in R$, $\sqcap'^R Q(r) \overset{\text{(sc)}}{=} Q \sqcap^R (r) \overset{(2.60)}{=} Q(r)$, so $Q(r) \in R'$, proving the existence of $q : R \to R'$, $r \mapsto Q(r)$. Let us see that the restriction $q : R \to R'$ of Q is a morphism in sfr. First of all, that it is a coalgebra map. Take $y \in R$. Since Q respects the counits, $\epsilon'_{|R'}q(y) = \epsilon' Q(y) = \epsilon_{|R}(y)$. Moreover, q is comultiplicative by

$$\begin{aligned} Q(y)1'_1 \otimes \sqcap'^R(1'_2) &\overset{(2.72)}{=} Q(y)_1 \otimes \sqcap'^R(Q(y)_2) \overset{(2.29)}{=} Q(y_1) \otimes \sqcap'^R Q(y_2) \\ &\overset{(sc)}{=} Q(y_1) \otimes Q\sqcap^R(y_2) \overset{(2.72)}{=} Q(y1_1) \otimes Q\sqcap^R(1_2). \end{aligned}$$

By commutativity of the third diagram in the claim, q commutes with the Nakayama automorphisms (see Theorem 2.5.1). Hence it is a morphism in sfr, as needed.

In order for Q to be a morphism in $\mathsf{bim}(R'^e)$, it has to be an R'^e–bimodule map. Below we check that Q is a right R'^e–module map; left R'^e–module map property is similarly proven.

$$\begin{aligned} Q(h\widetilde{q}(r')) &\overset{(3.34)}{=} Q(h1_1)\epsilon'(q\sqcap^R(1_2)r') \overset{(2.72)}{=} Q(h_1)\epsilon'(q\sqcap^R(h_2)r') \\ &\overset{(sc)}{=} Q(h_1)\epsilon'(\sqcap'^R Q(h_2)r') \overset{(2.71)}{=} Q(h_1)\epsilon'(Q(h_2)r') \\ &\overset{(2.29)}{=} Q(h)_1\epsilon'(Q(h)_2 r') \overset{(2.28)(2.72)}{=} Q(h)r' \end{aligned}$$

$$\begin{aligned} Q(h\overline{\sqcap}^L\widetilde{q}(r')) &\underset{(2.69)}{\overset{(2.28)(2.65)}{=}} \epsilon(h_1\overline{\sqcap}^L\widetilde{q}(r'))Q(h_2) \overset{(3.34)}{=} \epsilon(h_1\overline{\sqcap}^L\sqcap^R(1_2))\epsilon'(r'q(1_1))Q(h_2) \\ &\underset{(2.63)(2.60)}{\overset{(2.69)(2.59)}{=}} \epsilon(h_1 1_2)\epsilon'(\overline{\sqcap}'^L(r')q(1_1))Q(h_2) \\ &\overset{(2.68)}{=} \epsilon(h_1 1_2)\epsilon'(q(1_1)\overline{\sqcap}'^L(r'))Q(h_2) \\ &\overset{(2.50)}{=} \epsilon'(q\sqcap^R(h_1)\overline{\sqcap}'^L(r'))Q(h_2) \overset{(sc)}{=} \epsilon'(\sqcap'^R Q(h_1)\overline{\sqcap}'^L(r'))Q(h_2) \\ &\overset{(2.29)}{=} \epsilon'(Q(h)_1\overline{\sqcap}'^L(r'))Q(h)_2 \underset{(2.69)}{\overset{(2.28)(2.65)}{=}} Q(h)\overline{\sqcap}'^L(r') \end{aligned}$$

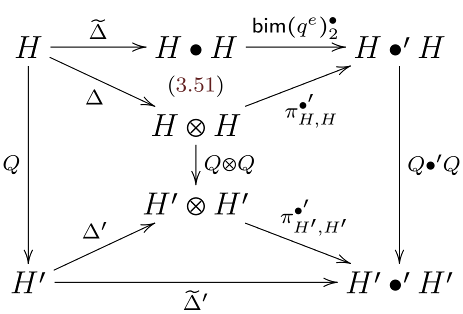
It remains to show that the morphisms $q : R \to R'$ in sfr and $Q : H \to H'$ in $\mathsf{bim}(R'^e)$ obey the conditions in part (b) of Lemma 3.0.2. Commutativity of diagram (3.3) was seen to be equivalent to the identity (3.54). It holds by the following computation, for all $h, h' \in H$, $a \in A$ and $b \in B$ for any R^e–bimodules A and B.

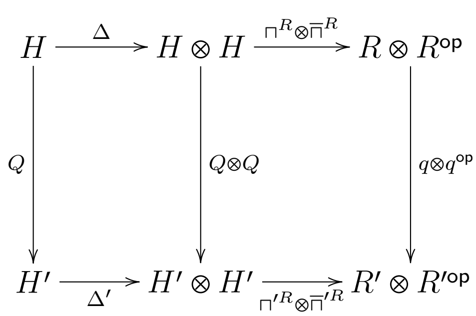
$$\begin{aligned} &((e_p \otimes 1)a(e_q \otimes f_j) \circ' (e_k \otimes e_j)b(e_l \otimes 1)) \bullet' Q((1 \otimes f_p)h(e_i \otimes f_q))Q((f_i \otimes f_k)h'(1 \otimes f_l)) \\ \overset{(wm)}{=} &((e_p \otimes 1)a(e_q \otimes f_j) \circ' (e_k \otimes e_j)b(e_l \otimes 1)) \bullet' Q((1 \otimes f_p)h(1 \otimes f_q)(1 \otimes f_k)h'(1 \otimes f_l)) \\ \underset{(2.36)}{\overset{(2.38)}{=}} &((e_p \otimes 1)a(e_q e_k \otimes f_j) \circ' (f_k \otimes e_j)b(e_l \otimes 1)) \bullet' Q((1 \otimes f_p)h(1 \otimes f_q)h'(1 \otimes f_l)) \end{aligned}$$

Commutativity of diagram (3.4) is checked by

$$\begin{aligned}
(q(1_1)\otimes q(y))\bullet' Q\widetilde{\eta}(x\otimes\sqcap^R(1_2)) &= (1\otimes q(y))\bullet' Q\widetilde{\eta}(x\otimes\sqcap^R(1_2))(1\otimes q(1_1))\\
&= (1\otimes q(y))\bullet' Q(\widetilde{\eta}(x\otimes\sqcap^R(1_2))(1\otimes\widetilde{q}q(1_1)))\\
&= (1\otimes q(y))\bullet' Q\widetilde{\eta}(x\otimes\widetilde{q}q(1_1)\sqcap^R(1_2))\\
&= (1\otimes q(y))\bullet' Q\widetilde{\eta}(x\otimes\widetilde{q}(1'_1)\widetilde{q}\sqcap'^R(1'_2))\\
&= (1\otimes q(y))\bullet' Q\widetilde{\eta}(x\otimes 1)\\
&= (1\otimes q(y))\bullet' \widetilde{\eta}'(q(x)\otimes 1),
\end{aligned}$$

for any $x,y\in R$. In the first equality we used the definition of $\bullet'$ (cf. (3.12)). In the second and third equalities we used the right R'^e–linearity of Q and the right R^e–linearity of $\widetilde{\eta}$, respectively. In the fourth equality we used (3.33); in the penultimate equality we used that $\widetilde{q}$ is an algebra map together with (2.70); and in the last equality we used that Q restricts to q on R. The following commutative diagrams show that (3.1) and (3.2) hold.





□

We conclude by Theorem 3.2.4 and Theorem 3.2.5 that the category $\mathsf{bmd}(\mathsf{bim}(-^e))$ has weak bialgebras as its objects and morphisms as in Theorem 3.2.5. Thus we can regard it as the category of weak bialgebras and introduce the notation wba for it.

As it was already pointed out, applying results from [63], we know from Lemma 3.0.2 that the morphisms in wba are closed under the composition. It is also easy to see this directly. Indeed, if both morphisms $Q: H\to H'$ and $Q': H'\to H''$ render commutative the first three diagrams in Theorem 3.2.5, then so does their composite evidently. If Q and Q' make commutative the last diagram in Theorem 3.2.5, then so does their composite: For any $h, h' \in H$,

$$\begin{aligned} Q'Q(hh') &\overset{\text{(wmc)}}{=} Q'[Q(h1_1)Q(\sqcap^R(1_2)h')] \overset{\text{(wmc)}}{=} Q'[Q(h1_1)1'_1]Q'[\sqcap'^R(1'_2)Q(\sqcap^R(1_2)h')] \\ &\overset{(3.32)}{=} Q'Q(h1_1\widetilde{q}(1'_1))Q'Q(\widetilde{q}\sqcap'^R(1'_2)\sqcap^R(1_2)h') \\ &\overset{(3.51)}{=} Q'Q(h1_1)Q'Q(\sqcap^R(1_2)h'). \end{aligned}$$

While the notion of weak bialgebra is self-dual, the morphisms in Theorem 3.2.5 are not. (They are coalgebra morphisms but not algebra morphisms.) The dual counterpart of wba; that is, a category of weak bialgebras with the dual notion of morphisms, would be obtained from a construction based on a symmetric form of Definition 3.0.4 (see the discussion in Remark 3.0.5).

The morphisms in Theorem 3.2.5 look different from all other kinds of morphisms between weak bialgebras discussed previously in [62, Section 1.4]. However, if we restrict to morphisms $Q : H \to H'$ whose (co)restriction $q : \sqcap^R(H) \to \sqcap'^R(H')$ is the identity map, they are in particular unit preserving $\sqcap^R(H) = \sqcap'^R(H')$–bimodule maps; hence also morphisms of algebras (see also Remark 3.0.3). That is to say, they are *'strict morphisms'* of weak bialgebras in the sense of [62, Section 1.4]. For usual (non-weak) bialgebras H and H' over the field k, any morphism $H \to H'$ in Theorem 3.2.5 (co)restricts to the identity map $\sqcap^R(H) \cong k \to \sqcap'^R(H') \cong k$. Hence wba contains the usual category of k–bialgebras —in which morphisms are algebra and coalgebra morphisms— as a full subcategory.

3.3 Application: Adjunction between cat^0 and wba

In this section we use the just defined category wba of weak bialgebras to show the existence of an adjoint pair between it and the category cat^0 of small categories with finitely many objects.

3.3.1 The "free vector space functor"

Let k be a field. For any small category C with finite object set, the free k–vector space $k\mathsf{C}$ spanned by its set of morphisms has a structure of weak bialgebra, as described in Example 2.5.1. This assignment gives the object map of a functor $\mathsf{k} : \mathsf{cat}^0 \to \mathsf{wba}$ as Proposition 3.3.1 shows.

Proposition 3.3.1. *Let* C *and* C' *be small categories with finite object sets. For any functor* $\mathcal{F}:\mathsf{C}\to\mathsf{C}'$*, the linear extension* $k\mathcal{F}:k\mathsf{C}\to k\mathsf{C}'$ *is a morphism in* wba.

Proof. First, note that $k\mathcal{F}$ is a morphism of k–coalgebras because it sends group-like elements to group-like elements; and group-like elements provide a basis in $k\mathsf{C}$. We need to show that the four diagrams in Theorem 3.2.5 commute for $Q=k\mathcal{F}$. As for the first two concerns, for any basis element $c\in\mathsf{C}_1$,

$$\begin{aligned}(k\mathcal{F})\sqcap^R_{k\mathsf{C}}(c) &= (k\mathcal{F})s(c)=\mathcal{F}s(c)=s'\mathcal{F}(c)=\sqcap^R_{k\mathsf{C}'}\mathcal{F}(c)=\sqcap^R_{k\mathsf{C}'}(k\mathcal{F})(c),\\ (k\mathcal{F})\,\overline{\sqcap}^R_{k\mathsf{C}}(c) &= (k\mathcal{F})t(c)=\mathcal{F}t(c)=t'\mathcal{F}(c)=\overline{\sqcap}^R_{k\mathsf{C}'}\mathcal{F}(c)=\overline{\sqcap}^R_{k\mathsf{C}'}(k\mathcal{F})(c).\end{aligned}$$

The commutativity of the third diagram in Theorem 3.2.5 becomes redundant by $\sqcap^L_{k\mathsf{C}}=\overline{\sqcap}^R_{k\mathsf{C}}$. In order to check that the fourth diagram commutes, let us first note that any element in the range of the map $E=(-)1_1\otimes\sqcap^R(1_2)(-):k\mathsf{C}\otimes k\mathsf{C}\to k\mathsf{C}\otimes k\mathsf{C}$ is of the form

$$\sum_{x\in\mathsf{C}_0}(\sum_{c\in\mathsf{C}}\lambda_c c)x\otimes x(\sum_{c'\in\mathsf{C}}\lambda_{c'}c')=\sum_{x\in\mathsf{C}_0}(\sum_{c\,:\,s(c)=x}\lambda_c c)\otimes(\sum_{c'\,:\,t(c')=x}\lambda_{c'}c')=\sum_{c,c'\,:\,s(c)=t(c')}\lambda_c\lambda_{c'}c\otimes c';$$

and if $s(c)=t(c')$, then

$$(k\mathcal{F})\mu(c\otimes c')=\mathcal{F}(c.c')=\mathcal{F}(c).\mathcal{F}(c')=\mu'(k\mathcal{F}\otimes k\mathcal{F})(c\otimes c'),$$

where we denoted by . the composition in C (in order to distinguish from the multiplication in $k\mathsf{C}$, denoted by juxtaposition). □

3.3.2 Group-like elements in a weak bialgebra

In forthcoming Subsection 3.3.3 we will construct the right adjoint g of the "free vector space" functor k in Subsection 3.3.1. Recall that for any small category C, the set of morphisms is in a bijective correspondence with the set of functors from the interval category $\mathbf{2}=\circlearrowright S\xrightarrow{\alpha}T\circlearrowleft$ to C. So if the right adjoint g of k exists, then for any weak bialgebra H over the field k, the set of morphisms in $\mathsf{g}(H)$ is isomorphic to $\mathsf{cat}^0(\mathbf{2},\mathsf{g}(H))\cong\mathsf{wba}(k\mathbf{2},H)$. This motivates the study of the set $\mathsf{wba}(k\mathbf{2},H)$ for any weak bialgebra H, with the aim of finding the way to look at it as the set of morphisms in an appropriate category.

Definition 3.3.2. *For any weak bialgebra H, define the subset*

$$\mathsf{g}(H) = \left\{ g \in H : \begin{array}{llll} \Delta(g) &= g \otimes g, & \Delta \sqcap^R(g) &= \sqcap^R(g) \otimes \sqcap^R(g) \\ \epsilon(g) &= 1, & \Delta \overline{\sqcap}^R(g) &= \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g) \end{array} \right\}$$

of the set of group-like elements in H.

Remark 3.3.3. Let us stress that for a general weak bialgebra H, the set $\mathsf{g}(H)$ is strictly smaller than the set $\{g \in H \ : \ \Delta(g) = g \otimes g,\ \epsilon(g) = 1\}$ of group-like elements.

For example, let us consider the free k–vector space on the basis provided by the morphisms of the interval category **2**. It is a weak bialgebra via the dual of the weak bialgebra structure in Example 2.5.1. In terms of Kronecker's delta, it has the unique multiplication such that $pq = \delta_{p,q}p$, for all $p, q \in \{S, T, \alpha\}$, the unit $S + T + \alpha$, the unique comultiplication for which

$$\Delta(S) = S \otimes S, \qquad \Delta(T) = T \otimes T, \qquad \Delta(\alpha) = T \otimes \alpha + \alpha \otimes S$$

and the unique counit for which $\epsilon(S) = \epsilon(T) = 1$ and $\epsilon(\alpha) = 0$. In this weak bialgebra

$$\sqcap^R_{k\mathbf{2}}(S) = \overline{\sqcap}^R_{k\mathbf{2}}(S) = S + \alpha, \qquad \sqcap^R_{k\mathbf{2}}(T) = \overline{\sqcap}^R_{k\mathbf{2}}(T) = T, \qquad \sqcap^R_{k\mathbf{2}}(\alpha) = \overline{\sqcap}^R_{k\mathbf{2}}(\alpha) = 0.$$

Thus there are two group-like elements S and T but only T belongs to $\mathsf{g}(k\mathbf{2})$.

As we shall see below, there are some distinguished classes of weak bialgebras H, however, in which $\mathsf{g}(H)$ coincides with the set of group-like elements in H.

In contrast to usual bialgebras, where the unit element is always group-like, there are weak bialgebras H in which the set of group-like elements (and therefore the subset $\mathsf{g}(H)$) is empty. Consider, for example, the groupoid with two objects S and T and only one non-identity isomorphism $\alpha : S \to T$. The free k–vector space on the basis provided by its morphisms, is a weak bialgebra via the dual of the weak bialgebra structure in Example 2.5.1. It has the unique multiplication such that $pq = \delta_{p,q}p$, for all $p, q \in \{S, T, \alpha, \alpha^{-1}\}$, the unit $S + T + \alpha + \alpha^{-1}$, the unique comultiplication for which

$$\begin{array}{llll} \Delta(S) = & S \otimes S + \alpha^{-1} \otimes \alpha, & \Delta(T) & = T \otimes T + \alpha \otimes \alpha^{-1} \\ \Delta(\alpha) = & T \otimes \alpha + \alpha \otimes S, & \Delta(\alpha^{-1}) & = S \otimes \alpha^{-1} + \alpha^{-1} \otimes T, \end{array}$$

and the unique counit for which $\epsilon(S) = \epsilon(T) = 1$ and $\epsilon(\alpha) = \epsilon(\alpha^{-1}) = 0$. In this weak bialgebra there is no group-like element.

Lemma 3.3.4. *For a weak bialgebra H, any element $g \in H$ such that $\Delta(g) = g \otimes g$ obeys the following identities.*

$$\text{(i)} \quad g \sqcap^R (g) = g = \overline{\sqcap}^R(g) g \quad \text{and} \quad \sqcap^L (g) g = g = g \overline{\sqcap}^L(g). \tag{3.55}$$

(ii) All elements $\sqcap^R(g)$, $\overline{\sqcap}^R(g)$, $\sqcap^L(g)$, $\overline{\sqcap}^L(g)$ are idempotent.

(iii) If in addition $g \in \mathsf{g}(H)$, then

$$\sqcap^R \sqcap^L(g) = \overline{\sqcap}^R(g) \quad \text{and} \quad \sqcap^L \sqcap^R(g) = \overline{\sqcap}^L(g). \tag{3.56}$$

Proof. The equalities in part (i) follow from $\Delta(g) = g \otimes g$ and (2.70). The statements in part (ii) are obtained by applying $\sqcap^R$, $\overline{\sqcap}^R$, $\sqcap^L$ and $\overline{\sqcap}^L$, respectively, to the equalities in part (i), and taking into account the module map properties (2.67). For $g \in \mathsf{g}(H)$,

$$\overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g) = \Delta \overline{\sqcap}^R(g) = 1_1 \otimes 1_2 \overline{\sqcap}^R(g). \tag{3.57}$$

Applying to both sides $\mathsf{id} \otimes \sqcap^R$ and multiplying on the right the result by $g \otimes 1$, by the application of part (i) we get

$$g \otimes \overline{\sqcap}^R(g) = 1_1 g \otimes \sqcap^R(1_2) \overline{\sqcap}^R(g).$$

Application of $\epsilon \otimes \mathsf{id}$ to both sides of this equality yields

$$\overline{\sqcap}^R(g) = \sqcap^R \sqcap^L (g) \overline{\sqcap}^R(g). \tag{3.58}$$

On the other hand, applying to both sides of (3.57) $\sqcap^L \otimes \sqcap^R \sqcap^L$ and multiplying on the right the result by $g \otimes 1$, we obtain

$$g \otimes \sqcap^R \sqcap^L (g) = 1_2 g \otimes \sqcap^R \sqcap^L (g) 1_1,$$

where we used (2.61), part (i), (2.62), (2.67), anti-multiplicativity of $\sqcap^R : \sqcap^L(H) \to$

$\sqcap^R(H)$ (cf. Proposition 2.5.10), and (2.62). Thus by applying $\epsilon \otimes \mathsf{id}$, we get

$$\sqcap^R \sqcap^L(g) = \sqcap^R \sqcap^L (g)\overline{\sqcap}^R(g). \tag{3.59}$$

Comparing (3.58) and (3.59), we conclude on the first equality in part (iii). The other equality in part (iii) follows by applying in $H^{\mathsf{op}}_{\mathsf{cop}}$ the just proven identity. □

Proposition 3.3.5. *For a cocommutative weak bialgebra H, the set of group-like elements and the set $\mathsf{g}(H)$ are equal; that is, $\mathsf{g}(H) = \{g \in H : \Delta(g) = g \otimes g, \epsilon(g) = 1\}$.*

Proof. It follows immediately from the cocommutativity of H that $\sqcap^L = \overline{\sqcap}^R$ and $\sqcap^R = \overline{\sqcap}^L$, so that $\sqcap^R(H)$ and $\sqcap^L(H)$ are coinciding commutative separable Frobenius subalgebras in H, with separability Frobenius idempotent $\Delta(1) = 1_1 \otimes 1_2$ (Theorem 2.5.1). Hence if $\Delta(g) = g \otimes g$, then

$$\begin{aligned} \Delta \sqcap^R (g) &= 1_1 \otimes \sqcap^R(g)1_2 = 1_1 \otimes \sqcap^R(g) \sqcap^R (g)1_2 \\ &= \sqcap^R(g)1_1 \otimes \sqcap^R(g)1_2 = \sqcap^R(g1_1) \otimes \sqcap^R(g1_2) = \sqcap^R(g) \otimes \sqcap^R(g). \end{aligned}$$

In the first equality we used (2.64) and, in the second one, part (ii) of Lemma 3.3.4. In the third equality we used the commutativity of the algebra $\sqcap^R(H)$ and the Frobenius property (2.36) for the separability idempotent $\Delta(1)$. In the fourth equality we used $\Delta(1) \in \sqcap^R(H) \otimes \sqcap^R(H)$ and (2.67). In the last equality we used the multiplicativity of the comultiplication (cf. (2.47)) and that $\Delta(g) = g \otimes g$.

The identity $\Delta\overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g)$ follows symmetrically (by applying $\Delta \sqcap^R (g) = \sqcap^R(g) \otimes \sqcap^R(g)$ in H^{op}, see Table 2.1). □

Example 3.3.6. Consider the cocommutative weak bialgebra $k\mathsf{C}$ presented in Example 2.5.1. Proposition 3.3.5 says that $G(k\mathsf{C}) = \mathsf{g}(k\mathsf{C})$. It is also easy to see this directly: Indeed, an element in $k\mathsf{C}$ is evidently a group-like if and only if it is a basis element, so that $G(k\mathsf{C}) = \mathsf{C}_1$; taking into account (2.54) and (2.55), it immediately follows that also $\mathsf{g}(k\mathsf{C}) = \mathsf{C}_1$.

Lemma 3.3.7. *Let H be a weak Hopf algebra and $g \in H$ such that $\Delta(g) = g \otimes g$. Then the following assertions hold.*

(i) $\Delta \sqcap^L (g) = \sqcap^L(g) \otimes \sqcap^L(g)$ *and* $\Delta \sqcap^R (g) = \sqcap^R(g) \otimes \sqcap^R(g)$.

(ii) $S^2 \sqcap^R (g) = \sqcap^R(g)$ *and* $S^2 \sqcap^L (g) = \sqcap^L(g)$.

(iii) $\overline{\sqcap}^L(g) = \sqcap^R(g)$ *and* $\sqcap^L(g) = \overline{\sqcap}^R(g)$.

(iv) $S^2(g) = g$.

(v) $\sqcap^R S(g) = \overline{\sqcap}^R(g)$ *and* $\overline{\sqcap}^R S(g) = \sqcap^R(g)$*;* $\sqcap^L S(g) = \overline{\sqcap}^L(g)$ *and* $\overline{\sqcap}^L S(g) = \sqcap^L(g)$.

(vi) $\Delta\overline{\sqcap}^L(g) = \overline{\sqcap}^L(g) \otimes \overline{\sqcap}^L(g)$ *and* $\Delta\overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) \otimes \overline{\sqcap}^R(g)$.

Proof. (i). Since $\Delta(g) = g \otimes g$, it follows that

$$\begin{aligned} \Delta \sqcap^L (g) &\overset{(2.88)}{=} \Delta(g_1 S(g_2)) = \Delta(gS(g)) \overset{(2.47)(2.96)}{=} g_1 S(g_{2'}) \otimes g_2 S(g_{1'}) \\ &= gS(g) \otimes gS(g) \overset{(2.5.2)}{=} \sqcap^L(g) \otimes \sqcap^L(g), \end{aligned}$$

and symmetrically for $\sqcap^R(g)$ (by applying the above identity in $H^{\mathsf{op}}_{\mathsf{cop}}$).

(ii). By the weak Hopf algebra axioms and part (i),

$$\sqcap^L \sqcap^R(g) \overset{(2.88)}{=} \sqcap^R(g)_1 S(\sqcap^R(g)_2) \overset{\text{part (i)}}{=} \sqcap^R(g) S \sqcap^R (g) \overset{(2.93)}{=} \sqcap^R(g) \sqcap^L \sqcap^R(g). \tag{3.60}$$

Symmetrically,

$$\begin{aligned} \sqcap^R (g) &\overset{(2.60)}{=} \sqcap^R \sqcap^R (g) \overset{(2.87)}{=} S(\sqcap^R(g)_1) \sqcap^R (g)_2 \\ &\overset{\text{part (i)}}{=} S \sqcap^R (g) \sqcap^R (g) \overset{(2.93)}{=} \sqcap^L \sqcap^R (g) \sqcap^R (g). \end{aligned} \tag{3.61}$$

The right hand sides of (3.60) and (3.61) are equal by (2.68), proving

$$\sqcap^L \sqcap^R(g) = \sqcap^R(g). \tag{3.62}$$

Applying $\sqcap^R$ to both sides of (3.62) and using (2.93), we conclude on $S^2 \sqcap^R (g) = \sqcap^R(g)$. The other equality is proven symmetrically.

(iii). As a matter of fact,

$$\sqcap^R(g) \overset{(3.62)}{=} \sqcap^L \sqcap^R (g) \overset{(2.61)}{=} \overline{\sqcap}^L \sqcap^L \sqcap^R(g) \overset{(3.62)}{=} \overline{\sqcap}^L \sqcap^R (g) \overset{(2.61)}{=} \overline{\sqcap}^L(g).$$

The other equality is proven symmetrically (by applying the above identity in $H^{\mathsf{op}}_{\mathsf{cop}}$).

(iv). If $\Delta(g) = g \otimes g$, then

$$gS(g)g = g_1 S(g_2) g_3 \overset{(2.87)}{=} g_1 \sqcap^R (g_2) \overset{(3.55)}{=} g. \tag{3.63}$$

Hence

$$\begin{aligned} g &\overset{(3.63)}{=} gS(g)g \overset{(3.63)}{=} gS(gS(g)g)g \overset{(2.95)}{=} gS(g)S^2(g)S(g)g \\ &= g_1 S(g_2) S^2(g) S(g_{1'}) g_{2'} \overset{(2.88)(2.87)}{=} \sqcap^L(g) S^2(g) \sqcap^R (g) \\ &\overset{\text{part (ii)}}{=} S^2 \sqcap^L (g) S^2(g) S^2 \sqcap^R (g) \overset{(2.95)}{=} S^2(\sqcap^L(g) g \sqcap^R (g)) \overset{(3.55)}{=} S^2(g). \end{aligned}$$

(v). The first claim follows by $\overline{\sqcap}^R(g) = \overline{\sqcap}^R S^2(g) = \sqcap^R S(g)$, cf. part (iv) and (2.93). The second claim is immediate by (2.93). The remaining two claims follow symmetrically (by applying these previous ones in $H^{\mathsf{op}}_{\mathsf{cop}}$).

(vi). This is immediate by parts (i) and (iii). □

From parts (i) and (vi) of Lemma 3.3.7 we obtain the following.

Corollary 3.3.8. *In a weak Hopf algebra H, $\mathsf{g}(H) = \{g \in H \ : \ \Delta(g) = g \otimes g, \epsilon(g) = 1\}$.*

Our motivation of the study of the set $\mathsf{g}(H)$ in a weak bialgebra H comes from the following.

Proposition 3.3.9. *For any weak bialgebra H over a field k, there is a bijection between the sets $\mathsf{wba}(k\mathbf{2}, H)$ and $\mathsf{g}(H)$.*

Proof. Let $\gamma \in \mathsf{wba}(k\mathbf{2}, H)$ and consider $g_\gamma := \gamma(\alpha)$ (where α stands for the only non-identity morphism in $\mathbf{2}$). Let us see that $g_\gamma \in \mathsf{g}(H)$:

$$\begin{aligned} \Delta(g_\gamma) &= \Delta\gamma(\alpha) = (\gamma \otimes \gamma)\Delta_{k\mathbf{2}}(\alpha) \overset{(2.29)}{=} \gamma(\alpha) \otimes \gamma(\alpha) = g_\gamma \otimes g_\gamma, \\ \epsilon(g_\gamma) &= \epsilon\gamma(\alpha) \overset{(2.30)}{=} \epsilon_{k\mathbf{2}}(\alpha) = 1, \\ \Delta \sqcap^R (g_\gamma) &= \Delta \sqcap^R \gamma(\alpha) \overset{(\text{sc})(2.29)}{=} (\gamma \otimes \gamma)\Delta_{k\mathbf{2}} \sqcap^R_{k\mathbf{2}} (\alpha) = \gamma \sqcap^R_{k\mathbf{2}} (\alpha) \otimes \gamma \sqcap^R_{k\mathbf{2}} (\alpha) \\ &\overset{(\text{sc})}{=} \sqcap^R\gamma(\alpha) \otimes \sqcap^R\gamma(\alpha) = \sqcap^R(g_\gamma) \otimes \sqcap^R(g_\gamma), \\ \Delta\overline{\sqcap}^R(g_\gamma) &= \Delta\overline{\sqcap}^R\gamma(\alpha) \overset{(\text{tc})(2.29)}{=} (\gamma \otimes \gamma)\Delta_{k\mathbf{2}}\overline{\sqcap}^R_{k\mathbf{2}}(\alpha) = \gamma\overline{\sqcap}^R_{k\mathbf{2}}(\alpha) \otimes \gamma\overline{\sqcap}^R_{k\mathbf{2}}(\alpha) \\ &\overset{(\text{tc})}{=} \overline{\sqcap}^R\gamma(\alpha) \otimes \overline{\sqcap}^R\gamma(\alpha) = \overline{\sqcap}^R(g_\gamma) \otimes \overline{\sqcap}^R(g_\gamma). \end{aligned}$$

Conversely, let $g \in \mathsf{g}(H)$ and consider the linear map $\gamma_g : k\mathbf{2} \to H$, given by

$$\gamma_g(S) = \sqcap^R(g), \qquad \gamma_g(T) = \overline{\sqcap}^R(g), \qquad \gamma_g(\alpha) = g, \tag{3.64}$$

(where S and T are the objects of the category $\mathbf{2}$ and the same symbols stand for their unit morphisms). By Theorem 3.2.5, to check that γ_g is a morphism in $\mathsf{wba}(k\mathbf{2}, H)$ it should be proven first that γ_g is a coalgebra map. This follows by noting that —since $\epsilon\sqcap^R = \epsilon$ and $\epsilon\overline{\sqcap}^R = \epsilon$ (cf. (2.56) and (2.58))— for any morphism c in $\mathbf{2}$,

$$\Delta\gamma_g(c) = \gamma_g(c) \otimes \gamma_g(c) = (\gamma_g \otimes \gamma_g)\Delta_{k\mathbf{2}}(c) \qquad \text{and} \qquad \epsilon\gamma_g(c) = \epsilon(g) = 1 = \epsilon_{k\mathbf{2}}(c).$$

Next, γ_g can be seen to obey the source condition:

$$\begin{aligned}
\sqcap^R\gamma_g(S) &\overset{(3.64)}{=} \sqcap^R \sqcap^R (g) \overset{(2.60)}{=} \sqcap^R(g) \overset{(3.64)}{=} \gamma_g(S) \overset{(2.54)}{=} \gamma_g \sqcap^R_{k\mathbf{2}} (S),\\
\sqcap^R\gamma_g(T) &\overset{(3.64)}{=} \sqcap^R\overline{\sqcap}^R(g) \overset{(2.61)}{=} \overline{\sqcap}^R(g) \overset{(3.64)}{=} \gamma_g(T) \overset{(2.54)}{=} \gamma_g \sqcap^R_{k\mathbf{2}} (T),\\
\sqcap^R\gamma_g(\alpha) &\overset{(3.64)}{=} \sqcap^R(g) \overset{(3.64)}{=} \gamma_g(S) = \gamma_g s(\alpha) \overset{(2.54)}{=} \gamma_g \sqcap^R_{k\mathbf{2}} (\alpha).
\end{aligned}$$

The target condition is checked analogously. The Nakayama condition on γ_g is proven by

$$\begin{aligned}
\sqcap^R \sqcap^L \gamma_g(S) &\overset{(3.64)}{=} \sqcap^R \sqcap^L \sqcap^R(g) \overset{(3.56)}{=} \sqcap^R\overline{\sqcap}^L(g)\\
&\overset{(2.61)}{=} \sqcap^R(g) \overset{(3.64)}{=} \gamma_g(S) \overset{(2.54)}{=} \gamma_g \sqcap^R_{k\mathbf{2}} \sqcap^L_{k\mathbf{2}}(S),
\end{aligned}$$

$$\begin{aligned}
\sqcap^R \sqcap^L \gamma_g(T) &\overset{(3.64)}{=} \sqcap^R \sqcap^L \overline{\sqcap}^R(g) \overset{(2.61)}{=} \sqcap^R \sqcap^L (g)\\
&\overset{(3.56)}{=} \overline{\sqcap}^R(g) \overset{(3.64)}{=} \gamma_g(T) \overset{(2.54)}{=} \gamma_g \sqcap^R_{k\mathbf{2}} \sqcap^L_{k\mathbf{2}}(T),
\end{aligned}$$

$$\sqcap^R \sqcap^L \gamma_g(\alpha) \overset{(3.64)}{=} \sqcap^R \sqcap^L (g) \overset{(3.56)}{=} \overline{\sqcap}^R(g) \overset{(3.64)}{=} \gamma_g(T) \overset{(2.54)}{=} \gamma_g \sqcap^R_{k\mathbf{2}} \sqcap^L_{k\mathbf{2}}(g).$$

Finally, the weak multiplicativity condition in Theorem 3.2.5 translates to four equalities in parts (i) and (ii) of Lemma 3.3.4, see

$$\begin{aligned}
\gamma_g(S)\gamma_g(S) &= \sqcap^R(g) \sqcap^R (g) = \sqcap^R(g) = \gamma_g(S), &\qquad \gamma_g(\alpha)\gamma_g(S) &= g \sqcap^R (g) = g = \gamma_g(\alpha),\\
\gamma_g(T)\gamma_g(T) &= \overline{\sqcap}^R(g)\overline{\sqcap}^R(g) = \overline{\sqcap}^R(g) = \gamma_g(T), &\qquad \gamma_g(T)\gamma_g(\alpha) &= \overline{\sqcap}^R(g)\, g = g = \gamma_g(\alpha).
\end{aligned}$$

These constructions clearly yield mutually inverse maps between the sets $\mathsf{g}(H)$ and $\mathsf{wba}(k\mathbf{2},H)$. Indeed, for any $g \in \mathsf{g}(H)$, $g_{\gamma_g} = \gamma_g(\alpha) = g$; and for any $\gamma \in \mathsf{wba}(k\mathbf{2},H)$, γ_{g_γ} is the map that sends S to $\sqcap^R(g_\gamma) = \sqcap^R(\gamma(\alpha)) = \gamma(S)$, α to $g_\gamma = \gamma(\alpha)$ and T to $\overline{\sqcap}^R(g_\gamma) = \overline{\sqcap}^R(\gamma(\alpha)) = \gamma(T)$; shortly, $\gamma_{g_\gamma} = \gamma$. □

Proposition 3.3.10. *For any weak bialgebra H, there is a category with morphism set $\mathsf{g}(H)$ in Definition 3.3.2. The object set is $\{r \in \sqcap^R(H) = \overline{\sqcap}^R(H) : \Delta(r) = r\otimes r,\ \epsilon(r) = 1\}$ and the identity morphisms are given by the evident inclusion into $\mathsf{g}(H)$. The source map is given by the restriction of $\sqcap^R$ and the target map is given by the restriction of $\overline{\sqcap}^R$. The composition is given by the restriction of the multiplication in H.*

Proof. First we check that $\mathsf{g}(H)$ is closed under the composition. Let $g, g' \in \mathsf{g}(H)$ such that $\sqcap^R(g) = \overline{\sqcap}^R(g')$. Then

$$\begin{aligned} \Delta(gg') &\overset{(2.47)}{=} \Delta(g)\Delta(g') = (g\otimes g)(g'\otimes g') = gg'\otimes gg' \qquad \text{and} \\ \epsilon(gg') &\overset{(3.55)}{=} \epsilon(g\overline{\sqcap}^R(g')) = \epsilon(g \sqcap^R(g)) \overset{(3.55)}{=} \epsilon(g) = 1. \end{aligned}$$

Since

$$\sqcap^R(gg') \overset{(2.71)}{=} \sqcap^R(\sqcap^R(g)g') = \sqcap^R(\overline{\sqcap}^R(g')g') \overset{(3.55)}{=} \sqcap^R(g') \tag{3.65}$$

and

$$\overline{\sqcap}^R(gg') \overset{(2.71)}{=} \overline{\sqcap}^R(g\overline{\sqcap}^R(g')) = \overline{\sqcap}^R(g\sqcap^R(g)) \overset{(3.55)}{=} \overline{\sqcap}^R(g), \tag{3.66}$$

also the following identities hold:

$$\begin{aligned} \Delta\sqcap^R(gg') &\overset{(3.65)}{=} \Delta\sqcap^R(g') = \sqcap^R(g')\otimes\sqcap^R(g') = \sqcap^R(gg')\otimes\sqcap^R(gg'), \\ \Delta\overline{\sqcap}^R(gg') &\overset{(3.66)}{=} \Delta\overline{\sqcap}^R(g) = \overline{\sqcap}^R(g)\otimes\overline{\sqcap}^R(g) = \overline{\sqcap}^R(gg')\otimes\overline{\sqcap}^R(gg'), \end{aligned}$$

and we conclude that $gg' \in \mathsf{g}(H)$. Associativity of the composition is evident because of associativity of the multiplication. The object set is clearly a subset of the morphism set (taking into account the idempotency of $\sqcap^R$ and that $\overline{\sqcap}^R\sqcap^R = \sqcap^R$, cf. (2.60) and (2.61)); and for any $g \in \mathsf{g}(H)$, both $\sqcap^R(g)$ and $\overline{\sqcap}^R(g)$ belong to the object set. The restrictions of $\sqcap^R$ and $\overline{\sqcap}^R$ give the source and target maps, respectively, by part (i) of Lemma 3.3.4. It follows by (3.65) and (3.66) that the composition is compatible with the source and target maps. □

The category in Proposition 3.3.10 is also denoted by $\mathsf{g}(H)$.

Remark 3.3.11. For an arbitrary weak bialgebra H, the construction of the category $\mathsf{g}(H)$ in Proposition 3.3.10 is not symmetric under the simultaneous replacements $\sqcap^R \leftrightarrow \overline{\sqcap}^L$, $\overline{\sqcap}^R \leftrightarrow \sqcap^L$. This is a consequence of the choice we made in the definition of morphisms between bimonoids (so in particular in the definition of morphisms in wba), see Remark 3.0.5. In light of part (iii) of Lemma 3.3.7, the symmetry of the category $\mathsf{g}(H)$ under the simultaneous replacements $\sqcap^R \leftrightarrow \overline{\sqcap}^L$, $\overline{\sqcap}^R \leftrightarrow \sqcap^L$ is restored whenever H is a weak Hopf algebra.

Proposition 3.3.12. *Any morphism $H \to H'$ in wba restricts to a functor $\mathsf{g}(H) \to \mathsf{g}(H')$.*

Proof. Let $Q : H \to H'$ be a morphism in wba. First we need to see that it restricts to a map $\mathsf{g}(Q) = Q_{|\mathsf{g}(H)} : \mathsf{g}(H) \to \mathsf{g}(H')$. Since Q is in particular a coalgebra map, it follows for all $g \in \mathsf{g}(H)$ that

$$\Delta' Q(g) \overset{(2.29)}{=} (Q \otimes Q)\Delta(g) = Q(g) \otimes Q(g) \quad \text{and} \quad \epsilon' Q(g) \overset{(2.28)}{=} \epsilon(g) = 1.$$

Since Q is comultiplicative and it satisfies the source and target conditions,

$$\begin{aligned} \Delta' \sqcap'^R Q(g) &\overset{(sc)(2.29)}{=} (Q \otimes Q)\Delta \sqcap^R (g) = Q \sqcap^R (g) \otimes Q \sqcap^R (g) \overset{(sc)}{=} \sqcap'^R Q(g) \otimes \sqcap'^R Q(g), \\ \Delta' \overline{\sqcap}'^R Q(g) &\overset{(tc)(2.29)}{=} (Q \otimes Q)\Delta \overline{\sqcap}^R (g) = Q \overline{\sqcap}^R (g) \otimes Q \overline{\sqcap}^R (g) \overset{(tc)}{=} \overline{\sqcap}'^R Q(g) \otimes \overline{\sqcap}'^R Q(g). \end{aligned}$$

This proves $Q(g) \in \mathsf{g}(H')$. Also from the compatibility of Q with $\sqcap^R$ and $\overline{\sqcap}^R$, it follows that $\mathsf{g}(Q)$ respects the source and target maps as well as the unit morphisms. By the weak multiplicativity condition, $\mathsf{g}(Q)$ preserves the composition: For all $g, g' \in \mathsf{g}(H)$ such that $\sqcap^R(g) = \overline{\sqcap}^R(g')$,

$$\begin{aligned} Q(gg') &\overset{(wmc)}{=} Q(g1_1)Q(\sqcap^R(1_2)g') \overset{(2.72)}{=} Q(g_1)Q(\sqcap^R(g_2)g') \\ &= Q(g)Q(\sqcap^R(g)g') = Q(g)Q(\overline{\sqcap}^R(g')g') \overset{(3.55)}{=} Q(g)Q(g'). \end{aligned}$$

□

The group-like elements in any coalgebra over a field are linearly independent (see [1, Theorem 2.1.2]). Hence the elements of $\mathsf{g}(H)$ in a weak bialgebra H are linearly independent. As the right subalgebra $\sqcap^R(H)$ of H is finite dimensional (since it is separable Frobenius, cf. Theorem 2.5.1), this proves that the cardinality of the object set of $\mathsf{g}(H)$ —that is, of the set $\mathsf{g}(H)\cap\sqcap^R(H)$— is finite. So we conclude by Proposition 3.3.10 and Proposition 3.3.12 that there is a functor g from wba to the category cat^0.

3.3.3 The right adjoint of the "free vector space" functor

The aim of this subsection is to show that the functor g in Subsection 3.3.2 is right adjoint of the "free vector space" functor k in Subsection 3.3.1. That is, to prove the following.

Theorem 3.3.1. *For any small category C with finitely many objects, and for any weak bialgebra H over a given field k, there is a bijection $\mathsf{wba}(\mathsf{k}(\mathsf{C}),H)\cong\mathsf{cat}^0(\mathsf{C},\mathsf{g}(H))$ which is natural in C and H. Moreover, the image of $1_{\mathsf{k}(-)}$ under this bijection (that is, the unit of the adjunction $\mathsf{k}\dashv\mathsf{g}$) is a natural isomorphism.*

Proof. First we show that the to-be-unit of the adjunction $\mathsf{k}\dashv\mathsf{g}$ is a natural isomorphism. That is, for any category C with finitely many objects the functor $\mathsf{C}\to\mathsf{gk}(\mathsf{C})$, $c\mapsto c$ is an isomorphism. This amounts to checking its bijectivity on the sets of morphisms. Injectivity is obvious. In order to see its surjectivity, let us take some $p\in\mathsf{gk}(\mathsf{C})$. Let us write $p=\sum_{c\in\mathsf{C}_1}\lambda_c c$, with $\lambda_c\in k$ non-zero at most for finitely many $c\in\mathsf{C}_1$. Then from the requirement that p is group-like,

$$\Delta(p)=p\otimes p=\sum_{c,d\in\mathsf{C}_1}\lambda_c\lambda_d c\otimes d.$$

By linearity of Δ,

$$\Delta(p)=\sum_{c\in\mathsf{C}_1}\lambda_c\Delta(c)=\sum_{c\in\mathsf{C}_1}\lambda_c c\otimes c.$$

Since $\{c\otimes d\}_{c,d\in\mathsf{C}_1}$ is a linearly independent subset in $k\mathsf{C}\otimes k\mathsf{C}$, we conclude that λ_c is non-zero at most for one element $c\in\mathsf{C}_1$. On the other hand, as

$$1=\epsilon(p)=\lambda_c\epsilon(c)=\lambda_c,$$

we have $p=c\in\mathsf{C}_1$.

We claim next that the desired bijection $\phi_{\mathsf{C},H}:\mathsf{wba}(\mathsf{k}(\mathsf{C}),H)\to\mathsf{cat}^0(\mathsf{C},\mathsf{g}(H))$ takes any morphism $Q:k\mathsf{C}\to H$ to $Q_{|\mathsf{C}_1}$, its restriction to $\mathsf{C}_1\cong\mathsf{gk}(\mathsf{C})$. By Proposition 3.3.12,

Q restricts to a functor $\mathsf{C}_1 \cong \mathsf{gk}(\mathsf{C}) \to \mathsf{g}(H)$; so that $\phi_{\mathsf{C},H}$ is well defined. Naturality of $\phi_{\mathsf{C},H}$ is evident. Since C_1 is a basis of the vector space $k\mathsf{C}$, the map $\phi_{\mathsf{C},H}$ is injective. In order to show surjectivity of $\phi_{\mathsf{C},H}$, consider some functor $\mathcal{F} : \mathsf{C} \to \mathsf{g}(H)$. Since C_1 is a basis of the vector space $k\mathsf{C}$, it can be extended to a unique linear map $\widetilde{\mathcal{F}} : k\mathsf{C} \to H$. Let us see that $\widetilde{\mathcal{F}}$ is a morphism of weak bialgebras and hence $\mathcal{F} = \phi_{\mathsf{C},H}(\widetilde{\mathcal{F}})$. For any $c \in \mathsf{C}_1$, $\mathcal{F}(c) \in \mathsf{g}(H)$ so $\Delta\mathcal{F}(c) = \mathcal{F}(c) \otimes \mathcal{F}(c)$ and $\epsilon\mathcal{F}(c) = 1$. Thus $\mathcal{F}$ extends to a coalgebra map $\widetilde{\mathcal{F}}$. The weak multiplicativity of $\widetilde{\mathcal{F}}$ follows from the fact that $\mathcal{F}$ preserves the composition. Indeed, denoting by . the composition in C, for $c, d \in \mathsf{C}_1$,

$$\widetilde{\mathcal{F}}(c(1_{k\mathsf{C}})_1)\widetilde{\mathcal{F}}(\sqcap^R((1_{k\mathsf{C}})_2)d) = \delta_{s(c),t(d)}\mathcal{F}(c)\mathcal{F}(d) = \delta_{s(c),t(d)}\mathcal{F}(c.d) = \widetilde{\mathcal{F}}(cd).$$

Since $\mathcal{F}$ preserves the source and target maps, $\widetilde{\mathcal{F}}$ commutes with $\sqcap^R$ and $\overline{\sqcap}^R$. Finally, by part (iii) of Lemma 3.3.4,

$$\sqcap^R \sqcap^L \mathcal{F}(c) = \overline{\sqcap}^R \mathcal{F}(c) = \mathcal{F}t(c) = \mathcal{F} \sqcap^R_{k\mathsf{C}} \sqcap^L_{k\mathsf{C}}(c)$$

for all $c \in \mathsf{C}_1$, hence $\sqcap^R \sqcap^L \widetilde{\mathcal{F}} = \widetilde{\mathcal{F}} \sqcap^R_{k\mathsf{C}} \sqcap^L_{k\mathsf{C}}$ follows by linearity. □

The counit of the above adjunction $\mathsf{k} \dashv \mathsf{g}$ is not an isomorphism in general (as it is not so for usual, non-weak bialgebras; see for instance [1]). Consider for example the weak bialgebra on the vector space $k\mathbf{2}$ from Remark 3.3.3. This weak bialgebra $k\mathbf{2}$ is three dimensional, while applying to it the functor kg we get a one dimensional weak bialgebra. So they cannot be isomorphic. Another counterexample is the following: For any (non-zero) weak bialgebra H for which there are no group-like elements in $\sqcap^R(H)$, $\mathsf{kg}(H)$ is the zero dimensional weak bialgebra.

Proposition 3.3.13. *The component $\phi^{-1}_{\mathsf{g}(H),H}(\mathsf{g}(H)) : \mathsf{kg}(H) \to H$ of the counit of the adjunction $\mathsf{k} \dashv \mathsf{g} : \mathsf{wba} \to \mathsf{cat}^0$ is an isomorphism if and only if H is a pointed cosemisimple weak bialgebra.*

Proof. Assume that H is a pointed cosemisimple weak bialgebra. By cosemisimplicity, $H = \oplus_{i\in I} H_i$ where each H_i is a cosimple subcoalgebra of H. Since H is pointed, each H_i is 1–dimensional, i.e. $H = \oplus_{g\in G(H)} kg$ (where $G(H)$ is the set of group-like elements of H, cf. (2.32)). So that H is cocommutative. By Proposition 3.3.5, $G(H) = \mathsf{g}(H)$. As a consequence, $H \cong \mathsf{kg}(H)$. The converse is clear since $\mathsf{kg}(H)$ is obviously a pointed cosemisimple coalgebra. □

Corollary 3.3.14. *The functors* k *and* g *induce an equivalence between the category of all small categories with finitely many objects, and the full subcategory of* wba *of all pointed cosemisimple weak bialgebras over a given field* k.

Since over an algebraically closed field every cocommutative coalgebra is pointed (see for instance [1, Theorem 2.3.3]), we get the following alternative form of Corollary 3.3.14.

Corollary 3.3.15. *If* k *is an algebraically closed field, then the functors* k *and* g *induce an equivalence between the category of all small categories with finitely many objects, and the full subcategory of* wba *of all cocommutative cosemisimple weak bialgebras.*

3.3.4 Restriction to Hopf monoids

The aim of this subsection is to study and compare the full subcategories of Hopf monoids in the category of spans over a given set in Subsection 3.1.2 and in the one of R^e–bimodules for a separable Frobenius algebra R in Subsection 3.2.2.

Proposition 3.3.16. *For any set* X*, a Hopf monoid in* $\mathsf{span}(X)$ *is precisely a groupoid with object set* X.

Proof. In light of Theorem 3.1.1, a bimonoid H in $\mathsf{span}(X)$ is a small category. Let H be a Hopf monoid in $\mathsf{span}(X)$ and consider the induced monoidal comonad $(-)\bullet H$. By assumption, the map

$$\beta_{A,B} : (A\bullet H)\circ(B\bullet H) \xrightarrow{(((A\bullet H)\circ B)\bullet\mu)\gamma((A\bullet\Delta)\circ(B\bullet H))} ((A\bullet H)\circ B)\bullet H$$

is an isomorphism for any objects A, B in $\mathsf{span}(X)$. So in particular, for $A = B = J = X\times X$, it is an isomorphism from

$$((X\times X)\bullet H)\circ((X\times X)\bullet H) \cong H\circ H$$

to

$$(((X\times X)\bullet H)\circ(X\times X))\bullet H \cong \{(h,h')\in H\times H : t(h) = t(h')\} =: H\times_t H.$$

Call it υ. It sends (h,h') to (h,hh'). We can write its inverse υ^{-1} in the form $(h,h') \mapsto (l(h,h'), r(h,h'))$, in terms of some maps l and r from $H \times H$ to H satisfying the conditions

$$sl(h,h') = tr(h,h') \qquad l(h,h') = h \quad (3.67) \qquad l(h,h')r(h,h') = h' \quad (3.68)$$

for all $h,h' \in H$ such that $t(h) = t(h')$, and

$$r(h,hh') = h' \tag{3.69}$$

for all $h,h' \in H$ such that $s(h) = t(h')$. Using (3.67) to simplify (3.68) and substituting $h' = t(h)$ in it, we obtain

$$hr(h,t(h)) = t(h) \tag{3.70}$$

so that $r(h,t(h))$ is a right inverse of h. As the following computation proves, it is also its left inverse.

$$r(h,t(h))h \overset{(3.69)}{=} r(h,hr(h,t(h))h) \overset{(3.70)}{=} r(h,h) \overset{(3.69)}{=} s(h)$$

Since this construction is valid for every $h \in H$, we showed that H is a groupoid.

Conversely, if H is a groupoid with object set X, then $\beta_{A,B} : (a,h,b,h') \mapsto (a,h,b,hh')$ is an isomorphism with the inverse $\beta^{-1}_{A,B} : (a,h,b,h') \mapsto (a,h,b,h^{-1}h')$, for any $h,h' \in H$ and any $a \in A, b \in B$ of objects A,B in $\mathsf{span}(X)$. Therefore, H is a Hopf monoid. □

Proposition 3.3.17. *For any separable Frobenius algebra R, a Hopf monoid in $\mathsf{bim}(R^e)$ is precisely a weak Hopf algebra with right subalgebra isomorphic to R.*

Proof. By Theorem 3.2.4, a bimonoid in $\mathsf{bim}(R^e)$ is precisely a weak bialgebra H whose right subalgebra is isomorphic to R. Assume that H is a weak Hopf algebra with the antipode $S : H \to H$. Then the map (2.17) —which takes now the explicit form

$$\beta_{A,B}((a \bullet h) \circ (b \bullet h')) = ((a \bullet h_1) \circ b) \bullet h_2 h'$$

— is an isomorphism with the inverse

$$\beta^{-1}_{A,B}(((a \bullet h) \circ b) \bullet h') = (a \bullet h_1) \circ (b \bullet S(h_2)h').$$

In order to show that $\beta^{-1}_{A,B}$ is well defined, we should check that the map $\widetilde{\beta}^{-1}_{A,B} : A \otimes H \otimes$

$B \otimes H' \to (A \bullet H) \circ (B \bullet H')$ sending $a \otimes h \otimes b \otimes h'$ to $(a \bullet h_1) \circ (b \bullet S(h_2)h')$ is R^e–balanced in all of the three occurring tensor products. This follows by the computations below for any $h, h' \in H, a \in A, b \in B$ and $s, r \in R$.

$$\begin{aligned}
\widetilde{\beta}^{-1}_{A,B}(a \cdot (1 \otimes r) \otimes h \otimes b \otimes h') &\overset{(3.13)}{=} \widetilde{\beta}^{-1}_{A,B}((r \otimes 1)a \otimes h \otimes b \otimes h') \\
&= ((r \otimes 1)a \bullet h_1) \circ (b \bullet S(h_2)h') \\
&\overset{(3.14)}{=} (a \bullet (1 \otimes \theta(r))h_1) \circ (b \bullet S(h_2)h') \\
&= (a \bullet \sqcap^L(r)h_1) \circ (b \bullet S(h_2)h') \\
&\overset{(2.66)}{=} (a \bullet (\sqcap^L(r)h)_1) \circ (b \bullet S((\sqcap^L(r)h)_2)h') \\
&= \widetilde{\beta}^{-1}_{A,B}[a \otimes \sqcap^L(r)h \otimes b \otimes h'] \\
&\overset{(3.13)}{=} \widetilde{\beta}^{-1}_{A,B}[a \otimes (1 \otimes r) \cdot h \otimes b \otimes h']
\end{aligned}$$

$$\begin{aligned}
\widetilde{\beta}^{-1}_{A,B}(a \cdot (s \otimes 1) \otimes h \otimes b \otimes h') &\overset{(3.13)}{=} \widetilde{\beta}^{-1}_{A,B}(a(s \otimes 1) \otimes h \otimes b \otimes h') \\
&= (a(s \otimes 1) \bullet h_1) \circ (b \bullet S(h_2)h') \\
&\overset{(3.14)}{=} (a \bullet h_1(1 \otimes s)) \circ (b \bullet S(h_2)h') \\
&= (a \bullet h_1 \overline{\sqcap}^L(s)) \circ (b \bullet S(h_2)h') \\
&\overset{(2.61)(2.66)}{=} (a \bullet (h\overline{\sqcap}^L(s))_1) \circ (b \bullet S((h\overline{\sqcap}^L(s))_2)h') \\
&= (a \bullet (h(1 \otimes s))_1) \circ (b \bullet S((h(1 \otimes s))_2)h') \\
&\overset{(3.13)}{=} (a \bullet ((s \otimes 1) \cdot h)_1) \circ (b \bullet S(((s \otimes 1) \cdot h)_2)h') \\
&= \widetilde{\beta}^{-1}_{A,B}(a \otimes (s \otimes 1) \cdot h \otimes b \otimes h')
\end{aligned}$$

$$\begin{aligned}
\widetilde{\beta}^{-1}_{A,B}(a \otimes h(s \otimes 1) \otimes b \otimes h') &= \widetilde{\beta}^{-1}_{A,B}(a \otimes hs \otimes b \otimes h') \\
&= (a \bullet h_1) \circ (b \bullet S(h_2 s)h') \\
&\overset{(2.95)}{=} (a \bullet h_1) \circ (b \bullet S(s)S(h_2)h') \\
&\overset{(2.60)(2.93)}{=} (a \bullet h_1) \circ (b \bullet \sqcap^L(s)S(h_2)h') \\
&\overset{(3.13)}{=} (a \bullet h_1) \circ (b \bullet (1 \otimes s) \cdot S(h_2)h') \\
&\overset{(3.12)}{=} (a \bullet h_1) \circ (b \cdot (1 \otimes s) \bullet S(h_2)h') \\
&\overset{(3.13)}{=} (a \bullet h_1) \circ ((s \otimes 1)b \bullet S(h_2)h') \\
&= \widetilde{\beta}^{-1}_{A,B}(a \otimes h \otimes (s \otimes 1)b \otimes h')
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\beta}^{-1}_{A,B}(a(1\otimes r)\otimes h\otimes b\otimes h') &= (a(1\otimes r)\bullet h_1)\circ(b\bullet S(h_2)h') \\
&\overset{(3.18)}{=} (a\bullet h_1)\mathbin{\cdot\!\cdot}(1\otimes r)\circ(b\bullet S(h_2)h') \\
&= (a\bullet h_1)\circ(1\otimes r)\mathbin{\cdot\!\cdot}(b\bullet S(h_2)h') \\
&\overset{(3.17)}{=} (a\bullet h_1)\circ((1\otimes r)b\bullet S(h_2)h') \\
&= \widetilde{\beta}^{-1}_{A,B}[a\otimes h\otimes(1\otimes r)b\otimes h']
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\beta}^{-1}[a\otimes h\otimes b(s\otimes 1)\otimes h'] &= (a\bullet h_1)\circ(b(s\otimes 1)\bullet S(h_2)h') \\
&\overset{(3.13)}{=} (a\bullet h_1)\circ(b\cdot(s\otimes 1)\bullet S(h_2)h') \\
&\overset{(3.12)}{=} (a\bullet h_1)\circ(b\bullet(s\otimes 1)\cdot S(h_2)h') \\
&\overset{(3.13)}{=} (a\bullet h_1)\circ(b\bullet S(h_2)h'(1\otimes s)) \\
&= (a\bullet h_1)\circ(b\bullet S(h_2)h'\overline{\sqcap}^L(s)) \\
&= \widetilde{\beta}^{-1}_{A,B}[a\otimes h\otimes b\otimes h'\overline{\sqcap}^L(s))] \\
&= \widetilde{\beta}^{-1}_{A,B}[a\otimes h\otimes b\otimes(s\otimes 1)\cdot h']
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\beta}^{-1}_{A,B}[a\otimes(r\otimes 1)h\otimes b\otimes h'] &= (a\bullet h_1)\circ(b\bullet S(rh_2)h') \\
&\overset{(2.95)}{=} (a\bullet h_1)\circ(b\bullet S(h_2)S(r)h') \\
&\overset{(2.60)(2.93)}{=} (a\bullet h_1)\circ(b\bullet S(h_2)\sqcap^L(r)h') \\
&= \widetilde{\beta}^{-1}_{A,B}[a\otimes h\otimes b\otimes\sqcap^L(r)h'] \\
&= \widetilde{\beta}^{-1}_{A,B}[a\otimes h\otimes b\otimes(1\otimes\theta(r))h'] \\
&\overset{(3.13)}{=} \widetilde{\beta}^{-1}_{A,B}[a\otimes h\otimes b\otimes(1\otimes r)\cdot h']
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\beta^{-1}_{A,B}\beta_{A,B}((a\bullet h)\circ(b\bullet h')) &= (a\bullet h_1)\circ(b\bullet S(h_2)h_3h') \\
&\overset{(2.87)}{=} (a\bullet h_1)\circ(b\bullet\sqcap^R(h_2)h') \\
&\overset{(3.17)}{=} (a\bullet h_1)\circ(\sqcap^R(h_2)\otimes 1)\mathbin{\cdot\!\cdot}(b\bullet h') \\
&= (a\bullet h_1)\mathbin{\cdot\!\cdot}(\sqcap^R(h_2)\otimes 1)\circ(b\bullet h') \\
&\overset{(3.18)}{=} (a\bullet h_1\sqcap^R(h_2))\circ(b\bullet h') \\
&\overset{(2.70)}{=} (a\bullet h)\circ(b\bullet h'),
\end{aligned}
$$

$$
\begin{aligned}
\beta_{A,B}\beta^{-1}_{A,B}(((a\bullet h)\circ b)\bullet h') &= ((a\bullet h_1)\circ b)\bullet h_2S(h_3)h' \\
&\overset{(2.88)}{=} ((a\bullet h_1)\circ b)\bullet \sqcap^L(h_2)h' \\
&\overset{(2.61)}{=} ((a\bullet h_1)\circ b)\bullet \overline{\sqcap}^L\sqcap^R\sqcap^L\overline{\sqcap}^R(h_2)h' \\
&\overset{(3.13)}{=} ((a\bullet h_1)\circ b)\bullet (1\otimes \overline{\sqcap}^R(h_2))\cdot h' \\
&\overset{(3.12)}{=} ((a\bullet h_1)\circ b)\cdot (1\otimes \overline{\sqcap}^R(h_2))\bullet h' \\
&\overset{(3.13)}{=} (\overline{\sqcap}^R(h_2)\otimes 1)((a\bullet h_1)\circ b)\bullet h' \\
&\overset{(3.11)}{=} ((\overline{\sqcap}^R(h_2)\otimes 1) \cdot\cdot (a\bullet h_1)\circ b)\bullet h' \\
&\overset{(3.17)}{=} ((a\bullet (\overline{\sqcap}^R(h_2)\otimes 1)h_1)\circ b)\bullet h' \\
&= ((a\bullet \overline{\sqcap}^R(h_2)h_1))\circ b\bullet h' \\
&\overset{(2.70)}{=} ((a\bullet h)\circ b)\bullet h'.
\end{aligned}
$$

Conversely, assume that $\beta_{A,B}$ is an isomorphism for any objects A, B in $\mathsf{bim}(R^e)$. Then it is an isomorphism, in particular, for $A = B = R^e\otimes R^e$ with the R^e–actions

$$(r\otimes s)((x\otimes y)\otimes(v\otimes w))(r'\otimes s') := (rx\otimes ys)\otimes(vr'\otimes s'w).$$

The maps

$$
\begin{array}{lcl}
R\otimes R\otimes R\otimes H1_1\otimes\sqcap^R(1_2)H & \overset{\xi_1}{\to} & ((R^e\otimes R^e)\bullet H)\circ((R^e\otimes R^e)\bullet H) \\
x\otimes y\otimes z\otimes h1_1\otimes\sqcap^R(1_2)h' & \mapsto & (((1\otimes x)\otimes(1\otimes 1))\bullet h)\circ(((1\otimes y)\otimes(1\otimes z))\bullet h')
\end{array}
$$

$$
\begin{array}{lcl}
R\otimes R\otimes R\otimes 1_1H\otimes 1_2H & \overset{\xi_2}{\to} & (((R^e\otimes R^e)\bullet H)\circ(R^e\otimes R^e))\bullet H \\
x\otimes y\otimes z\otimes 1_1h\otimes 1_2h' & \mapsto & ((((1\otimes x)\otimes(1\otimes 1))\bullet h)\circ((1\otimes y)\otimes(1\otimes z)))\bullet h',
\end{array}
$$

are evidently isomorphisms with inverses given by

$$
\begin{array}{lcl}
((R^e\otimes R^e)\bullet H)\circ((R^e\otimes R^e)\bullet H) & \overset{\xi_1^{-1}}{\to} & R\otimes R\otimes R\otimes H1_1\otimes\sqcap^R(1_2)H \\
(((w\otimes x)\otimes(y\otimes z))\bullet h)\circ(((s\otimes t)\otimes(u\otimes v))\bullet h') & \mapsto & x\otimes tz\otimes v\otimes((y\otimes w)\cdot h)1_1 \\
& & \otimes\sqcap^R(1_2)((u\otimes s)\cdot h')
\end{array}
$$

$$
\begin{array}{lcl}
(((R^e\otimes R^e)\bullet H)\circ(R^e\otimes R^e))\bullet H & \overset{\xi_2^{-1}}{\to} & R\otimes R\otimes R\otimes 1_1H\otimes 1_2H \\
((((w\otimes x)\otimes(y\otimes z))\bullet h)\circ((s\otimes t)\otimes(u\otimes v)))\bullet h' & \mapsto & x\otimes tz\otimes v\otimes 1_1((y\otimes w)\cdot h\cdot(s\otimes 1)) \\
& & \otimes 1_2((u\otimes 1)\cdot h')
\end{array}
$$

(where $\cdot$ refers to the R^e–actions in (3.13)). So that the composite $\xi_2^{-1}\beta_{R^e\otimes R^e, R^e\otimes R^e}\xi_1$, given by

$$\begin{array}{rcl} R\otimes R\otimes R\otimes H1_1\otimes \sqcap^R(1_2)H & \to & R\otimes R\otimes R\otimes 1_1H\otimes 1_2H \\ x\otimes y\otimes z\otimes h1_1\otimes \sqcap^R(1_2)h' & \mapsto & x\otimes y\otimes z\otimes h_1\otimes h_2h', \end{array}$$

also is so. Then the Galois map $H1_1\otimes\sqcap^R(1_2)H \to 1_1H\otimes 1_2H$, $h1_1\otimes\sqcap^R(1_2)h' \mapsto h_1\otimes h_2h'$ is an isomorphism. This means, equivalently, that H is a weak Hopf algebra (see [56, Corollary 6.2] for the details of this equivalent characterization of weak Hopf algebras among weak bialgebras). □

In the preliminary Chapter 2, we pointed out that the definition of Hopf monoid in a duoidal category given on page 35 is one choice of several symmetric possibilities; and that those others —based on the notions of left Hopf comonad and left and right Hopf monad [23]— would also lead to prove our results. Next we detail these notions and we provide the symmetric versions of Proposition 3.3.16 and Proposition 3.3.17 for each one of them.

Let H be a bimonoid in a duoidal category $(\mathsf{C},\circ,I,\bullet,J)$. The induced monoidal comonad $H\bullet(-)$ is said to be a *left Hopf comonad* [23] if

$$(H\bullet A)\circ(H\bullet B) \xrightarrow{\gamma((H\bullet A)\circ(\Delta\bullet B))} (H\circ H)\bullet(A\circ(H\bullet B)) \xrightarrow{\mu\bullet(A\circ(H\bullet B))} H\bullet(A\circ(H\bullet B)) \quad (3.71)$$

is a natural isomorphism. Dually, the comonoidal monad $H\circ(-)$ is a *left Hopf monad* [23] if

$$H\circ(A\bullet(H\circ B)) \xrightarrow{\gamma(\Delta\circ(A\bullet(H\circ B)))} (H\circ A)\bullet((H\circ H)\circ B) \xrightarrow{(H\circ A)\bullet(\mu\circ B)} (H\circ A)\bullet(H\circ B) \quad (3.72)$$

is a natural isomorphism. Symmetrically, we call the comonoidal monad $(-)\circ H$ a *right Hopf monad* [23] if

$$((A\circ H)\bullet B)\circ H \xrightarrow{\gamma(((A\circ H)\bullet B)\circ\Delta)} (A\circ(H\circ H))\bullet(B\circ H) \xrightarrow{(A\circ\mu)\bullet(B\circ H)} (A\circ H)\bullet(B\circ H) \quad (3.73)$$

is a natural isomorphism.

Proposition 3.3.18. *For any set X and any bimonoid H in $\mathsf{span}(X)$, the following assertions hold.*

(i) The induced monoidal comonad $H \bullet (-)$ is a left Hopf comonad if and only if H is a groupoid with object set X.

(ii) The induced comonoidal monad $(-) \circ H$ is a right Hopf monad if and only if H is a groupoid with object set X.

(iii) The induced comonoidal monad $H \circ (-)$ is a left Hopf monad if and only if H is a groupoid with object set X.

Proof. Throughout the proof, X and $X \times X$ refer to the units $I = (X, \mathsf{id}, \mathsf{id})$ and $J = (X \times X, p_1, p_2)$ of $\mathsf{span}(X)$ (page 73). In light of Theorem 3.1.1, a bimonoid H in $\mathsf{span}(X)$ is a small category.

(i). Define the sets $H * H := \{(h,h') \in H \times H : t(h) = s(h')\}$ and $H \times_s H := \{(h,h') \in H \times H : s(h) = s(h')\}$. First, observe that applying the same reasoning as in the proof of Proposition 3.3.16 on H^{op} (meaning H with its opposite monoid structure), it is concluded that H^{op} is a groupoid (and hence also that H is so), by using that

$$\nu : H * H = H^{\mathsf{op}} \circ H^{\mathsf{op}} \longrightarrow H^{\mathsf{op}} \times_t H^{\mathsf{op}} = H \times_s H,$$

taking (h,h') to $(h, h'h)$, is an isomorphism.

By assumption, the map

$$\varsigma := [\mu \bullet ((X \times X) \circ (H \bullet (X \times X)))]\gamma[(H \bullet (X \times X)) \circ (\Delta \bullet (X \times X))]$$

from

$$(H \bullet (X \times X)) \circ (H \bullet (X \times X)) \cong H \circ H$$

to

$$H \bullet ((X \times X) \circ (H \bullet (X \times X))) \cong H \times_s H$$

is an isomorphism, sending (h,h') to (hh', h'). Then, denoting by t the usual flip map $(h,h') \mapsto (h',h)$ of sets, also the composite morphism

$$H * H \xrightarrow{\mathsf{t}_{|H*H}} H \circ H \xrightarrow{\varsigma} H \times_s H \xrightarrow{\mathsf{t}_{|H \times_s H}} H \times_s H$$

sending (h,h') to $(h, h'h)$ is so. Since it is equal to ν, we conclude by the observation at the beginning of the proof that H is a groupoid.

Conversely, if H is a groupoid with object set X, then the fusion morphism (h,a,h',b) $\mapsto (hh',a,h',b)$ in (3.71) is an isomorphism with the inverse $(h,a,h',b) \mapsto (hh'^{-1},a,h',b)$ for any $h,h' \in H$ and any $a \in A, b \in B$ of any objects A,B in $\mathsf{span}(X)$. Therefore, H is a Hopf monoid.

(ii). The hypothesis of the claim assures that the map $[(X\circ\mu)\bullet((X\times X)\circ H)]\gamma[((X\circ H)\bullet(X\times X))\circ\Delta]$ from

$$((X\circ H)\bullet(X\times X))\circ H \cong H\circ H$$

to

$$(X\circ H)\bullet((X\times X)\circ H) \cong H\times_s H,$$

taking (h,h') to (hh',h'), is an isomorphism. This is precisely the map ς in the proof of part (i), so that by the same reasoning we conclude that H is a groupoid.

Reciprocally, if H is a groupoid with object set X, then the fusion morphism $(a,h,b,h') \mapsto (a,hh',b,h')$ in (3.73) is an isomorphism with the inverse $(a,h,b,h') \mapsto (a,hh'^{-1},b,h')$ for any $h,h' \in H$ and any $a \in A, b \in B$ of any objects A,B in $\mathsf{span}(X)$. Therefore, H is a Hopf monoid.

(iii). By assumption, the map $[(H\circ(X\times X))\bullet(\mu\circ X)]\gamma[\Delta\circ((X\times X)\bullet(H\circ X))]$ from

$$H\circ((X\times X)\bullet(H\circ X)) \cong H\circ H$$

to

$$(H\circ(X\times X))\bullet(H\circ X) \cong H\times_t H$$

is an isomorphism, sending (h,h') to (h,hh'). It is equal to the map υ in the proof of Proposition 3.3.16. Thus, by the same reasoning that in Proposition 3.3.16, it is proven that H is a groupoid.

Conversely, if H is a groupoid with object set X, then the fusion morphism (h,a,h',b) $\mapsto (h,a,hh',b)$ in (3.72) is an isomorphism with the inverse $(h,a,h',b) \mapsto (h,a,h^{-1}h',b)$ for any $h,h' \in H$ and any $a \in A, b \in B$ of any objects A,B in $\mathsf{span}(X)$. This concludes that H is a Hopf monoid. □

Proposition 3.3.19. *For any separable Frobenius algebra R and any bimonoid H in*

$\mathsf{bim}(R^e)$ *the following assertions hold.*

(i) The induced monoidal comonad $H \bullet (-)$ *is a left Hopf comonad if and only if* H *is a weak Hopf algebra with right subalgebra isomorphic to* R.

(ii) The induced comonoidal monad $(-) \circ H$ *is a right Hopf monad if and only if* H *is a weak Hopf algebra with right subalgebra isomorphic to* R.

(iii) The induced comonoidal monad $H \circ (-)$ *is a left Hopf monad if and only if* H *is a weak Hopf algebra with right subalgebra isomorphic to* R.

Proof. By Theorem 3.2.4, a bimonoid in $\mathsf{bim}(R^e)$ is precisely a weak bialgebra H whose right subalgebra is isomorphic to R. As usual, the original R^e–actions of H are denoted by juxtaposition, while $\cdot$ refers to the R^e–actions in (3.13).

(i). If H is a weak Hopf algebra with antipode S, the fusion morphism (3.71) and its inverse are given by

$$\begin{aligned} \kappa_{A,B}((h \bullet a) \circ (h' \bullet b)) &= hh'_1 \bullet (a \circ (h'_2 \bullet b)) \\ \kappa^{-1}_{A,B}(h \bullet (a \circ (h' \bullet b))) &= (hS(h'_1) \bullet a) \circ (h'_2 \bullet b) \end{aligned}$$

for any $h, h' \in H$ and any elements a, b of any R^e–bimodules A, B.

Conversely, if $\kappa_{A,B}$ is an isomorphism for any objects A, B in $\mathsf{bim}(R^e)$, then also is so the composite $\zeta_2^{-1}\kappa_{R^e\otimes R^e, R^e\otimes R^e}\zeta_1$, given by

$$\begin{aligned} R \otimes R \otimes R \otimes H1_1 \otimes \sqcap^R(1_2)H &\to R \otimes R \otimes R \otimes H1_1 \otimes H1_2 \\ x \otimes y \otimes z \otimes h1_1 \otimes \sqcap^R(1_2)h' &\mapsto x \otimes y \otimes z \otimes hh'_1 \otimes h'_2, \end{aligned}$$

where

$$\begin{aligned} R \otimes R \otimes R \otimes H1_1 \otimes \sqcap^R(1_2)H &\overset{\zeta_1}{\to} ((R^e \otimes R^e) \bullet H) \circ ((R^e \otimes R^e) \bullet H) \\ x \otimes y \otimes z \otimes h1_1 \otimes \sqcap^R(1_2)h' &\mapsto (h \bullet ((x \otimes 1) \otimes (1 \otimes 1))) \circ (h' \bullet ((y \otimes 1) \otimes (z \otimes 1))) \end{aligned}$$

$$\begin{aligned} R \otimes R \otimes R \otimes H1_1 \otimes H1_2 &\overset{\zeta_2}{\to} H \bullet ((R^e \otimes R^e) \circ (H \bullet (R^e \otimes R^e))) \\ x \otimes y \otimes z \otimes h1_1 \otimes h'1_2 &\mapsto h \bullet (((x \otimes 1) \otimes (1 \otimes 1)) \circ (h' \bullet ((y \otimes 1) \otimes (z \otimes 1)))), \end{aligned}$$

and

$$\begin{array}{lcl}
(H\bullet(R^e\otimes R^e))\circ(H\bullet(R^e\otimes R^e)) & \overset{\zeta_1^{-1}}{\to} & R\otimes R\otimes R\otimes H1_1\otimes\sqcap^R(1_2)H \\
(h\bullet((w\otimes x)\otimes(y\otimes z)))\circ(h'\bullet((s\otimes t)\otimes(u\otimes v))) & \mapsto & w\otimes ys\otimes u\otimes(h\cdot(z\otimes\theta^{-1}(x)))1_1 \\
 & & \otimes\sqcap^R(1_2)(h'\cdot(v\otimes\theta^{-1}(t)))
\end{array}$$

$$\begin{array}{lcl}
H\bullet((R^e\otimes R^e)\circ(H\bullet(R^e\otimes R^e))) & \overset{\zeta_2^{-1}}{\to} & R\otimes R\otimes R\otimes H1_1\otimes H1_2 \\
h\bullet(((w\otimes x)\otimes(y\otimes z))\circ(h'\bullet(((s\otimes t)\otimes(u\otimes v)))) & \mapsto & w\otimes ys\otimes u\otimes(h\cdot(z\otimes\theta^{-1}(x)))1_1 \\
 & & \otimes(h'\cdot(v\otimes\theta^{-1}(t)))1_2,
\end{array}$$

and $R^e\otimes R^e$ is regarded as an R^e–bimodule via the actions

$$(r\otimes s)((x\otimes y)\otimes(v\otimes w))(r'\otimes s') := (rx\otimes ys)\otimes(vr'\otimes s'w).$$

Then the Galois map $H1_1\otimes\sqcap^R(1_2)H\to H1_1\otimes H1_2$, $h1_1\otimes\sqcap^R(1_2)h'\mapsto hh'_1\otimes h'_2$ is an isomorphism. In light of [56, Corollary 6.2] this means, equivalently, that H is a weak Hopf algebra.

(ii). If H is a weak Hopf algebra with antipode S, the fusion morphism in (3.73) and its inverse are given by

$$\begin{array}{lcl}
\sigma_{A,B}(((a\circ h)\bullet b)\circ h') & = & (a\circ hh'_1)\bullet(b\circ h'_2) \\
\sigma^{-1}_{A,B}((a\circ h)\bullet(b\circ h')) & = & ((a\circ hS(h'_1))\bullet b)\circ h'_2
\end{array}$$

for any $h,h'\in H$ and any elements a,b of any R^e–bimodules A,B.

Conversely, if $\sigma_{A,B}$ is an isomorphism for any objects A,B in $\mathsf{bim}(R^e)$, then also is so the composite $\varrho_2^{-1}\sigma_{R^e\otimes R^e,R^e\otimes R^e}\varrho_1$, given by

$$\begin{array}{lcl}
R\otimes R\otimes R\otimes H1_1\otimes\sqcap^R(1_2)H & \to & R\otimes R\otimes R\otimes H1_1\otimes H1_2 \\
x\otimes y\otimes z\otimes h1_1\otimes\sqcap^R(1_2)h' & \mapsto & x\otimes y\otimes z\otimes hh'_1\otimes h'_2,
\end{array}$$

where

$$\begin{array}{lcl}
R\otimes R\otimes R\otimes H1_1\otimes\sqcap^R(1_2)H & \overset{\varrho_1}{\to} & (((R^e\otimes R^e)\circ H)\bullet(R^e\otimes R^e))\circ H \\
x\otimes y\otimes z\otimes h1_1\otimes\sqcap^R(1_2)h' & \mapsto & ((((y\otimes x)\otimes(1\otimes 1))\circ h)\bullet((z\otimes 1)\otimes(1\otimes 1)))\circ h'
\end{array}$$

$$\begin{array}{lcl}
R\otimes R\otimes R\otimes H1_1\otimes H1_2 & \overset{\varrho_2}{\to} & ((R^e\otimes R^e)\circ H)\bullet((R^e\otimes R^e)\circ H) \\
x\otimes y\otimes z\otimes h1_1\otimes h'1_2 & \mapsto & (((y\otimes x)\otimes(1\otimes 1))\circ h)\bullet(((z\otimes 1)\otimes(1\otimes 1))\circ h'),
\end{array}$$

and

$$\begin{array}{lcl} (((R^e \otimes R^e) \circ H) \bullet (R^e \otimes R^e)) \circ H & \overset{\varrho_1^{-1}}{\to} & R \otimes R \otimes R \otimes H1_1 \otimes \sqcap^R(1_2)H \\ (((w \otimes x) \otimes (y \otimes z)) \circ h) \bullet ((s \otimes t) \otimes (u \otimes v))) \circ h' & \mapsto & x \otimes \theta^{-1}(t) w \otimes s \otimes ((y \otimes z) h (v \otimes 1)) 1_1 \\ & & \otimes \sqcap^R (1_2)((u \otimes 1) h') \end{array}$$

$$\begin{array}{lcl} ((R^e \otimes R^e) \circ H) \bullet ((R^e \otimes R^e) \circ H) & \overset{\varrho_2^{-1}}{\to} & R \otimes R \otimes R \otimes H1_1 \otimes H1_2 \\ (((w \otimes x) \otimes (y \otimes z)) \circ h) \bullet (((s \otimes t) \otimes (u \otimes v)) \circ h') & \mapsto & x \otimes \theta^{-1}(t) w \otimes s \otimes ((y \otimes z) h (v \otimes 1)) 1_1 \\ & & \otimes ((u \otimes 1) h') 1_2, \end{array}$$

and $R^e \otimes R^e$ is seen as an R^e–module via the actions

$$(r \otimes s)((x \otimes y) \otimes (v \otimes w))(r' \otimes s') := (rx \otimes ys) \otimes (vr' \otimes s'w).$$

Then the Galois map $H1_1 \otimes \sqcap^R(1_2)H \to H1_1 \otimes H1_2$, $h1_1 \otimes \sqcap^R(1_2)h' \mapsto hh'_1 \otimes h'_2$ is an isomorphism. By [56, Corollary 6.2] this means, equivalently, that H is a weak Hopf algebra.

(iii). If H is a weak Hopf algebra with antipode S, the fusion morphism (3.72) and its inverse are given by

$$\begin{array}{rcl} \omega_{A,B}(h \circ (a \bullet (h' \circ b))) & = & (h_1 \circ a) \bullet (h_2 h' \circ b) \\ \omega_{A,B}^{-1}(h \circ a) \bullet (h' \circ b) & = & h_1 \circ (a \bullet (S(h_2) h' \circ b)) \end{array}$$

for any $h, h' \in H$ and any elements a, b of any R^e–bimodules A, B.

Conversely, if $\omega_{A,B}$ is an isomorphism for any objects A, B in $\mathsf{bim}(R^e)$, then also is so the composite $\varsigma_2^{-1} \omega_{R^e \otimes R^e, R^e \otimes R^e} \varsigma_1$, given by

$$\begin{array}{lcl} R \otimes R \otimes R \otimes H1_1 \otimes \sqcap^R(1_2)H & \to & R \otimes R \otimes R \otimes 1_1 H \otimes 1_2 H \\ x \otimes y \otimes z \otimes h1_1 \otimes \sqcap^R(1_2)h' & \mapsto & x \otimes y \otimes z \otimes h_1 \otimes h_2 h', \end{array}$$

where

$$\begin{array}{lcl} R \otimes R \otimes R \otimes H1_1 \otimes \sqcap^R(1_2)H & \overset{\varsigma_1}{\to} & H \circ ((R^e \otimes R^e) \bullet (H \circ (R^e \otimes R^e))) \\ x \otimes y \otimes z \otimes h1_1 \otimes \sqcap^R(1_2)h' & \mapsto & h \circ (((1 \otimes 1) \otimes (y \otimes z)) \bullet (h' \circ ((1 \otimes 1) \otimes (x \otimes 1)))) \end{array}$$

$$\begin{array}{lcl} R \otimes R \otimes R \otimes 1_1 H \otimes 1_2 H & \overset{\varsigma_2}{\to} & (H \circ (R^e \otimes R^e)) \bullet (H \circ (R^e \otimes R^e)) \\ x \otimes y \otimes z \otimes 1_1 h \otimes 1_2 h' & \mapsto & (h \circ ((1 \otimes 1) \otimes (y \otimes z))) \bullet (h' \circ ((1 \otimes 1) \otimes (x \otimes 1))), \end{array}$$

and

$$\begin{array}{lcl} H\circ((R^e\otimes R^e)\bullet(H\circ(R^e\otimes R^e))) & \overset{\varsigma_1^{-1}}{\to} & R\otimes R\otimes R\otimes H1_1\otimes\sqcap^R(1_2)H \\ h\circ(((w\otimes x)\otimes(y\otimes z))\bullet(h'\circ((s\otimes t)\otimes(u\otimes v)))) & \mapsto & u\otimes yv\otimes z\otimes(h(1\otimes x))1_1 \\ & & \otimes\sqcap^R(1_2)((1\otimes\theta(w))h'(s\otimes t)) \end{array}$$

$$\begin{array}{lcl} (H\circ(R^e\otimes R^e))\bullet(H\circ(R^e\otimes R^e)) & \overset{\varsigma_2^{-1}}{\to} & R\otimes R\otimes R\otimes 1_1H\otimes 1_2H \\ (h\circ((w\otimes x)\otimes(y\otimes z)))\bullet(h'\circ(((s\otimes t)\otimes(u\otimes v)))) & \mapsto & u\otimes yv\otimes z\otimes 1_1(h(w\otimes x)) \\ & & \otimes 1_2(h'(s\otimes t)), \end{array}$$

and $R^e\otimes R^e$ is considered an R^e–module via the actions

$$(r\otimes s)((x\otimes y)\otimes(v\otimes w))(r'\otimes s') := (rx\otimes ys)\otimes(vr'\otimes s'w).$$

Then the Galois map $H1_1\otimes\sqcap^R(1_2)H\to 1_1H\otimes 1_2H$, $h1_1\otimes\sqcap^R(1_2)h'\mapsto h_1\otimes h_2h'$ is an isomorphism. In view of [56, Corollary 6.2] this means, equivalently, that H is a weak Hopf algebra. □

Remark 3.3.20. In the recent paper [16], many equivalent characterizations of a Hopf algebra over a field are generalized for any naturally Frobenius map-monoidale M in a monoidal bicategory M, and any monoidal comonad on M. Regarding any monoidal comonad on M as a bimonoid in the duoidal hom-category $\mathsf{M}(M,M)$ (see [59]), and extending to that setting various conditions distinguishing classical Hopf algebras among bialgebras, in particular, a notion of antipode —which was missing to date— is defined in that context. Under suitable assumptions —the existence of certain conservative functors and the splitting of idempotent 2–cells in M— all those Hopf-like conditions are shown to be equivalent ([16, Theorem 7.16]). Applying these conditions to a small category H (regarded as a monoidal comonad on a suitable naturally Frobenius map-monoidale [16, 5.4]), all of them are equivalent to H being a groupoid [16, 8.2]. Applying these conditions to a weak bialgebra H (regarded as a monoidal comonad on a suitable naturally Frobenius map-monoidale [16, 5.3]), all of them are equivalent to H being a weak Hopf algebra [16, 8.4]. In other words, Proposition 3.3.16, Proposition 3.3.17, Proposition 3.3.18 and Proposition 3.3.19 can be seen as particular instances of [16, Theorem 7.16].

Let us take the full subcategory grp^0 of groupoids in the category of small categories with finitely many objects. The morphisms in grp^0 are functors (so that they are compatible with the inverse operation on the morphisms). Similarly, let us consider the full subcategory wha of weak Hopf algebras in wba. Its morphisms are the coalgebra maps $H \to H'$ rendering commutative the diagrams in Theorem 3.2.5. Note that there is no reason to expect that all of them will be compatible with the antipodes (that is, the equality $S'Q = QS$ will hold). In fact, compatibility with the antipodes is equivalent to $\sqcap'^L Q = Q\sqcap^L$ holding true. Indeed, if $S'Q = QS$, the identity $\sqcap'^L Q = Q\sqcap^L$ immediately follows by using (2.94) and the source condition. Conversely, if $\sqcap'^L Q = Q\sqcap^L$ is assumed, then for any $h \in H$,

$$\begin{aligned}
S'Q(h) &\overset{(2.89)}{=} S'(Q(h)_1)Q(h)_2S'(Q(h)_3) \overset{(2.88)}{=} S'(Q(h)_1)\sqcap'^L(Q(h)_2)\\
&\overset{(2.29)}{=} S'Q(h_1)\sqcap'^L(Q(h_2)) = S'Q(h_1)Q\sqcap^L(h_2)\\
&\overset{(2.88)}{=} S'Q(h_1)Q(h_2S(h_3)) \overset{(\text{wmc})}{=} S'Q(h_1)Q(h_21_1)Q(\sqcap^R(1_2)S(h_3))\\
&\overset{(2.63)(2.66)}{=} S'(Q(h_11_1)_1)Q((h_11_1)_2)Q(\sqcap^R(1_2)S(h_2))\\
&\overset{(2.87)}{=} \sqcap'^R Q(h_11_1)Q(\sqcap^R(1_2)S(h_2)) \overset{(\text{sc})}{=} Q(\sqcap^R(h_11_1))Q(\sqcap^R(1_2)S(h_2))\\
&\overset{(2.63)(2.67)}{=} Q(\sqcap^R(h_1)1_1)Q(\sqcap^R(1_2)S(h_2)) \overset{(\text{wmc})}{=} Q(\sqcap^R(h_1)S(h_2)) \overset{(2.89)}{=} QS(h).
\end{aligned}$$

Theorem 3.3.2. *The adjunction in Subsection 3.3.3 restricts to an iso unit adjunction between* grp^0 *and* wha.

Proof. First we check that $\mathsf{k} : \mathsf{cat}^0 \to \mathsf{wba}$ restricts to a functor $\mathsf{grp}^0 \to \mathsf{wha}$. If G is a groupoid, then $k\mathsf{G}$ has a weak Hopf algebra structure via the structure in Example 2.5.16. On the other hand, also $\mathsf{g} : \mathsf{wba} \to \mathsf{cat}^0$ restricts to a functor $\mathsf{wha} \to \mathsf{grp}^0$. That is, if H is a weak Hopf algebra, then $\mathsf{g}(H)$ is a groupoid (with many finitely objects) with the inverse operation $\mathsf{g}(H) \to \mathsf{g}(H)$, $g \mapsto S(g)$. In order to see that $S(g)$ is indeed an element of $\mathsf{g}(H)$, note that $\Delta S(g) = (S \otimes S)\Delta^{\mathsf{op}}(g) = S(g) \otimes S(g)$ and $\epsilon S(g) = \epsilon(g) = 1$ follow from the fact that S is an anti-coalgebra map (Proposition 2.5.15). By part (v) of Lemma 3.3.7 also the other two conditions on elements of $\mathsf{g}(H)$ hold true and the to-be-inverse operation $g \mapsto S(g)$ is compatible with the source and target maps. Moreover, for any $g \in \mathsf{g}(H)$, it works as an inverse by

$$gS(g) = g_1S(g_2) \overset{(2.88)}{=} \sqcap^L_{|\mathsf{g}(H)}(g) \qquad \text{and} \qquad S(g)g = S(g_1)g_2 \overset{(2.87)}{=} \sqcap^R_{|\mathsf{g}(H)}(g).$$

□

The following corollaries are immediate consequences of Corollary 3.3.14 and Corollary 3.3.15, respectively.

Corollary 3.3.21. *The functors* k *and* g *induce an equivalence between the category of all small groupoids with finitely many objects, and the full subcategory of* wha *of all pointed cosemisimple weak Hopf algebras over a given field k.*

Corollary 3.3.22. *If k is an algebraically closed field, then the functors* k *and* g *induce an equivalence between the category of all small groupoids with finitely many objects, and the full subcategory of* wha *of all cocommutative cosemisimple weak Hopf algebras.*

Example 3.3.23. Assume k to be a field of characteristic 0, and let N be a positive integer. The 'algebraic quantum torus'; that is, the algebra

$$H = k\langle u, v, v^{-1} \mid u^N = 1, vu = quv\rangle$$

with $q \in k$ such that $q^N = 1$, is a double crossed product weak Hopf algebra of the group Hopf algebra $k\langle v, v^{-1}\rangle$ and the N–dimensional weak Hopf algebra $B \coloneqq k\langle u \mid u^N = 1\rangle$ with the comultiplication

$$\Delta(u^n) = \frac{1}{N}\sum_{j=1}^{N}(u^{j+n} \otimes u^{-j}),$$

the counit defined by $\epsilon(1) = N$, $\epsilon(u^n) = 0$ if $u^n \neq 1$ and the antipode $S = \mathsf{id}$ (see [13, Example 9]).

For any Nth root of unity $\omega \in k$ (possibly, different from q), there is a group-like element $g_\omega = \frac{1}{N}\sum_{j=1}^{N}\omega^j u^j$ in B. Thus, if k contains a primitive Nth root of unity (so that the set $T \coloneqq \{\omega \in k : \omega^N = 1\}$ has N elements) then, as coalgebras,

$$B = \bigoplus_{\omega\in T} kg_\omega \qquad \text{and} \qquad H = \bigoplus_{\omega\in T,\ m\in\mathbb{Z}} kg_\omega v^m.$$

We deduce from Corollary 3.3.21 that in this case H is isomorphic to the groupoid weak Hopf algebra $k\mathsf{G}$, where $\mathsf{G} = \{g_\omega v^m \,|\, \omega \in T, m \in \mathbb{Z}\}$. This groupoid has N objects $\{g_\omega \,|\, \omega \in T\}$, but it is not finite. Since $g_\omega g_{\omega'} = 0$ if $\omega \neq \omega'$, and $g_\omega^2 = g_\omega$, we get that two morphisms $g_\omega v^m, g_\nu v^n$ of G are composable if and only if $\omega = \nu q^m$, and, in such a case, $g_\omega v^m g_\nu v^n = g_\omega v^{m+n}$.

Chapter 4

Weak multiplier bialgebras

In the previous Chapter 3 we provided, among other things, a categorical approach to weak bialgebras. In this chapter we introduce a non-unital generalization of weak bialgebras with a multiplier-valued comultiplication. This means that, in contrast to weak bialgebras, the underlying algebra A of the structure is not supposed to have a unit and the 'comultiplication' no longer lands in $A \otimes A$ but in its multiplier algebra. As we pointed out in the introductory Chapter 1, this generalization is well motivated both in practice and in theory. On the one hand, its motivation comes from the wish to have an algebraical structure for which, given a category with non-finite object set, the linear span of its arrow set and the vector space of finitely supported functions on its arrow set are particular instances. On the other hand, we would like to fill the conceptual gap of the *antipodeless* situation of (weak) multiplier Hopf algebras [69, 72]; as well as identifying a class of the new objects, intermediate between regular and arbitrary weak multiplier Hopf algebras, big enough to contain any unital weak Hopf algebra and whose members should have the expected properties like the structure of the base algebras. In Section 4.1 we provide several equivalent formulations of the definition of a *weak multiplier bialgebra over a field*, as well as various sources of examples. In a remarkable analogy with the unital case (that is, weak bialgebra setting), in Section 4.2 we study certain canonical subalgebras of the multiplier algebra of a weak multiplier bialgebra. Also in this more general context these non-unital algebras are proven to carry a rich structure: under appropriate assumptions on the weak multiplier bialgebra, they turn out to be coseparable co-Frobenius coalgebras. In the spirit of extending further features of weak bialgebras to this generalization, in Section 4.4 we study appropriately defined modules over a weak multiplier bialgebra. They are shown to constitute a monoidal category via the (co)module tensor product over the base (co)algebra. Finally, in Section 4.5, we provide a notion for weak multiplier bialgebras that, as it will be justified, deserves to be called *antipode*. We end the present chapter discussing the relation of weak multiplier bialgebras to Van Daele and Wang's (regular and arbitrary) weak multiplier Hopf algebras.

4.1 The weak multiplier bialgebra axioms

In this section we introduce the central notion of this chapter: weak multiplier bialgebra. Several equivalent forms of the axioms are presented and their first consequences are drawn. At the end of the section, we collect some illustrative examples.

Definition 4.1.1. A *weak multiplier bialgebra* A over a field k is given by

- an idempotent non-unital k–algebra with a non-degenerate multiplication $\mu : A \otimes A \to A$,
- an idempotent element E in $\mathbb{M}(A \otimes A)$,
- a multiplicative linear map $\Delta : A \to \mathbb{M}(A \otimes A)$ (called the *comultiplication*),
- and a linear map $\epsilon : A \to k$ (called the *counit*),

which are subject to the axioms below.

(i) For any elements $a, b \in A$, the elements

$$T_1(a \otimes b) := \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) := (a \otimes 1)\Delta(b) \tag{4.1}$$

of $\mathbb{M}(A \otimes A)$ belong to the two-sided ideal $A \otimes A$.

(ii) The comultiplication is coassociative in the sense that

$$(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) = (\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id}).$$

(iii) The counit obeys

$$(\epsilon \otimes \mathsf{id})T_1 = \mu = (\mathsf{id} \otimes \epsilon)T_2.$$

(iv) In terms of the idempotent element E,

$$\langle \Delta(a)(b \otimes c) \mid a, b, c \in A \rangle = \langle E(b \otimes c) \mid b, c \in A \rangle \quad \text{and}$$
$$\langle (b \otimes c)\Delta(a) \mid a, b, c \in A \rangle = \langle (b \otimes c)E \mid b, c \in A \rangle.$$

(v) The idempotent element E satisfies the equality

$$(E \otimes 1)(1 \otimes E) = E^{(3)} = (1 \otimes E)(E \otimes 1)$$

in $\mathbb{M}(A \otimes A \otimes A)$, cf. (2.114).

(vi) For any $a, b, c \in A$,

$$(\epsilon \otimes \mathsf{id})((1 \otimes a)E(b \otimes c)) = (\epsilon \otimes \mathsf{id})(\Delta(a)(b \otimes c)) \quad \text{and}$$
$$(\epsilon \otimes \mathsf{id})((a \otimes b)E(1 \otimes c)) = (\epsilon \otimes \mathsf{id})((a \otimes b)\Delta(c)).$$

From these axioms it follows immediately that

$$E\Delta(a) = \Delta(a) = \Delta(a)E, \tag{4.2}$$

for all $a \in A$. Indeed, for any $a, b, c \in A$,

$$\Delta(a)(b \otimes c) = E(x_i \otimes y_i) = E^2(x_i \otimes y_i) = E\Delta(a)(b \otimes c),$$

where the existence of $x_i \otimes y_i \in A \otimes A$ in the first equality is assured by axiom (iv); and the second and third equalities follow, respectively, by the idempotency of E and the first equality of the own chain of equalities.

Remark 4.1.2. In a weak multiplier bialgebra, the idempotent element E and the counit ϵ are uniquely determined in fact by the multiplication μ and the comultiplication Δ. The uniqueness of E follows by the uniqueness of the idempotent element in Theorem 2.7.1. We will come back to the uniqueness of ϵ later in this section (cf. Theorem 4.1.1).

Definition 4.1.3. A weak multiplier bialgebra A is said to be *regular* if also the elements

$$T_3(a \otimes b) := (1 \otimes b)\Delta(a) \quad \text{and} \quad T_4(a \otimes b) := \Delta(b)(a \otimes 1) \tag{4.3}$$

of $\mathbb{M}(A \otimes A)$ belong to the two-sided ideal $A \otimes A$, for all $a, b \in A$.

For an equivalent formulation of the *regularity* condition for weak multiplier bialgebras see [10, Definition 1.1 and Theorem 1.2].

Remark 4.1.4. Note that the same term *regular* has a different meaning preceding weak multiplier bialgebra or preceding weak multiplier Hopf algebra (compare them on page 63). In fact, a weak multiplier bialgebra is regular if T_3 and T_4 exist (as maps to $A \otimes A$), and a weak multiplier Hopf algebra is regular if T_3 and T_4 are weakly invertible; equivalently, the antipode (coming from the weakly invertible maps T_1 and T_2) is invertible [73, Theorem 4.10].

The following is easily derived using the non-degeneracy of the multiplication.

Lemma 4.1.5. *For a regular weak multiplier bialgebra A over a field, the following identities are equivalent.*

(i) $T_1 = \mathsf{tw} T_4 \mathsf{tw}$.

(ii) $T_2 = \mathsf{tw} T_3 \mathsf{tw}$.

(iii) $\Delta = \Delta^{\mathsf{op}}$ *(cf. (2.115)).*

Definition 4.1.6. A regular weak multiplier bialgebra A over a field is said to be *cocommutative* if it satisfies the equivalent identities in Lemma 4.1.5.

Below we shall provide some equivalent forms of axiom (vi) in Definition 4.1.1. In particular, this will allow us to prove the uniqueness of the counit.

Proposition 4.1.7. *For any weak multiplier bialgebra A over a field, and for any $a \in A$, the linear maps $A \to A$,*

$$b \mapsto (\epsilon \otimes \mathsf{id})T_2(a \otimes b) \quad \textit{and} \quad b \mapsto (\epsilon \otimes \mathsf{id})((a \otimes b)E) \tag{4.4}$$

define a multiplier $\overline{\sqcap}^L(a)$ on A, giving rise to a linear map $\overline{\sqcap}^L : A \to \mathbb{M}(A)$.

Proof. For any $a, b, c \in A$,

$$\begin{aligned} c((\epsilon\otimes\mathsf{id})T_2(a\otimes b)) &\overset{(4.1)}{=} (\epsilon\otimes\mathsf{id})((a\otimes c)\Delta(b)) \\ &\overset{(vi)}{=} (\epsilon\otimes\mathsf{id})((a\otimes c)E(1\otimes b)) = ((\epsilon\otimes\mathsf{id})((a\otimes c)E))b. \end{aligned}$$

□

So that, any element a of a weak multiplier bialgebra A, defines a multiplier $\overline{\sqcap}^L(a) \in \mathbb{M}(A)$ given by

$$\overline{\sqcap}^L(a)b := (\epsilon\otimes\mathsf{id})T_2(a\otimes b) \quad (4.5) \quad \text{and} \quad b\overline{\sqcap}^L(a) := (\epsilon\otimes\mathsf{id})((a\otimes b)E) \quad (4.6)$$

for any $b \in A$.

Proposition 4.1.8. *Let A be a weak multiplier bialgebra over a field. For any $a, b \in A$, the following assertions hold.*

(1) $(\mathsf{id}\otimes\overline{\sqcap}^L)T_2(a\otimes b) = (ab\otimes 1)E$ *as elements of* $\mathbb{M}(A\otimes A)$.

(2) $(a\otimes 1)E$ *belongs to the non-unital subalgebra* $A\otimes\mathbb{M}(A)$ *of* $\mathbb{M}(A\otimes A)$.

(3) $(a\otimes 1)E(1\otimes b)$ *belongs to the non-unital subalgebra* $A\otimes A$ *of* $\mathbb{M}(A\otimes A)$.

Proof. (1). For any $a, b, c, d \in A$,

$$\begin{aligned} (c\otimes d)((\mathsf{id}\otimes\overline{\sqcap}^L)T_2(a\otimes b)) &= (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})[(T_2(ca\otimes b)\otimes d)(1\otimes E)] \\ &\overset{(iv)}{=} (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})[(T_2(ca\otimes b)\otimes d)(E\otimes 1)(1\otimes E)] \\ &\overset{(v)}{=} (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})[(T_2\otimes\mathsf{id})(ca\otimes b\otimes d)E^{(3)}] \\ &= (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})(T_2\otimes\mathsf{id})[(ca\otimes b\otimes d)(1\otimes E)] \\ &\overset{(iii)}{=} (c\otimes d)(ab\otimes 1)E. \end{aligned}$$

In the first equality we used the definition of $\overline{\sqcap}^L$ in (4.5) and the left A–module map property of T_2. The fourth equality follows by

$$\begin{aligned} ((T_2\otimes\mathsf{id})(a\otimes b\otimes c))E^{(3)} &\overset{(2.114)}{=} (a\otimes 1\otimes 1)(\Delta\otimes\mathsf{id})(b\otimes c)(\overline{\Delta\otimes\mathsf{id}})(E) \\ &= (a\otimes 1\otimes 1)((\Delta\otimes\mathsf{id})((b\otimes c)E)) \\ &\overset{(4.1)}{=} (T_2\otimes\mathsf{id})((a\otimes b\otimes c)(1\otimes E)). \end{aligned}$$

(2). In the equality in (1), the left hand side belongs to $A \otimes \mathbb{M}(A)$ hence so does the right hand side. Since A is idempotent by assumption, this proves (2).

(3) follows immediately from (2), since A is an ideal of $\mathbb{M}(A)$. □

Proposition 4.1.9. *Let A be an idempotent non-unital algebra over a field k with a non-degenerate multiplication, $\Delta : A \to \mathbb{M}(A \otimes A)$ be a multiplicative linear map, $\epsilon : A \to k$ be a linear map and E be an idempotent element in $\mathbb{M}(A \otimes A)$. Assume that the axioms (i)-(v) —but not necessarily (vi)— in Definition 4.1.1 hold. If*

$$(\mathsf{id} \otimes \epsilon)((a \otimes 1)E(b \otimes c)) = (\mathsf{id} \otimes \epsilon)(\Delta(a)(b \otimes c)) \tag{4.7}$$

for all $a, b, c \in A$, then the following assertions hold.

(1) The linear maps

$$b \mapsto (\mathsf{id} \otimes \epsilon)(E(b \otimes a)) \quad \textit{and} \quad b \mapsto (\mathsf{id} \otimes \epsilon)T_1(b \otimes a) \tag{4.8}$$

define a multiplier $\overline{\sqcap}^R(a)$ on A, giving rise to a linear map $\overline{\sqcap}^R : A \to \mathbb{M}(A)$.

(2) $(\overline{\sqcap}^R \otimes \mathsf{id})T_1(a \otimes b) = E(1 \otimes ab)$ as elements of $\mathbb{M}(A \otimes A)$.

(3) $E(1 \otimes a)$ belongs to the non-unital subalgebra $\mathbb{M}(A) \otimes A$ of $\mathbb{M}(A \otimes A)$.

Proof. (1). For any $a, b, c \in A$,

$$\begin{aligned} a((\mathsf{id} \otimes \epsilon)E(b \otimes c)) &= (\mathsf{id} \otimes \epsilon)((a \otimes 1)E(b \otimes c)) \\ &\overset{(4.7)}{=} (\mathsf{id} \otimes \epsilon)(\Delta(a)(b \otimes c)) \overset{(4.1)}{=} (\mathsf{id} \otimes \epsilon)(T_1(a \otimes c))b. \end{aligned}$$

(2). For any $a, b, c, d \in A$,

$$\begin{aligned} ((\overline{\sqcap}^R \otimes \mathsf{id})T_1(a \otimes b))(c \otimes d) &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})((E \otimes 1)(c \otimes T_1(a \otimes bd)) \\ &\overset{(4.2)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})((E \otimes 1)(1 \otimes E)(c \otimes T_1(a \otimes bd)) \\ &\overset{(v)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(E^{(3)}(\mathsf{id} \otimes T_1)(c \otimes a \otimes bd)) \\ &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})((\mathsf{id} \otimes T_1)((E \otimes 1)(c \otimes a \otimes bd)) \\ &\overset{(iii)}{=} E(1 \otimes ab)(c \otimes d). \end{aligned}$$

In the first equality we used the definition of $\overline{\sqcap}^R$ in (4.8) and the right A–module map property of T_1. The fourth equality follows by

$$\begin{aligned} E^{(3)}((\mathsf{id}\otimes T_1)(c\otimes a\otimes b)) &\overset{(2.114)}{=} (\overline{\mathsf{id}\otimes\Delta})(E)(\mathsf{id}\otimes\Delta)(c\otimes a)(1\otimes 1\otimes b) \\ &= ((\mathsf{id}\otimes\Delta)(E(c\otimes a))(1\otimes 1\otimes b) \\ &\overset{(4.1)}{=} (\mathsf{id}\otimes T_1)((E\otimes 1)(c\otimes a\otimes b)), \end{aligned}$$

for any $a,b,c\in A$.
(3). In the equality in (2), the left hand side belongs to $\mathbb{M}(A)\otimes A$ hence so does the right hand side. Since A is idempotent by assumption, this proves (3). □

By virtue of Proposition 4.1.9, any element a in a weak multiplier bialgebra A defines a multiplier $\overline{\sqcap}^R(a)\in\mathbb{M}(A)$ given by

$$\overline{\sqcap}^R(a)b := (\mathsf{id}\otimes\epsilon)(E(b\otimes a)) \quad (4.9) \quad \text{and} \quad b\overline{\sqcap}^R(a) := (\mathsf{id}\otimes\epsilon)T_1(b\otimes a) \quad (4.10)$$

for any $b\in A$.

Proposition 4.1.10. *Let A be an idempotent non-unital algebra over a field k with a non-degenerate multiplication, $\Delta : A\to\mathbb{M}(A\otimes A)$ be a multiplicative linear map, $\epsilon : A\to k$ be a linear map and E be an idempotent element in $\mathbb{M}(A\otimes A)$. Assume that the axioms (i)-(v) —but not necessarily (vi)— in Definition 4.1.1 hold. The following assertions are equivalent* [1].

(1) $(\epsilon\otimes\mathsf{id})((a\otimes b)E(1\otimes c)) = (\epsilon\otimes\mathsf{id})((a\otimes b)\Delta(c))$ *for all* $a,b,c\in A$.

(2) $(\mathsf{id}\otimes\epsilon)((a\otimes 1)E(b\otimes c)) = (\mathsf{id}\otimes\epsilon)(\Delta(a)(b\otimes c))$ *for all* $a,b,c\in A$.

(3) $(a\otimes 1)E(1\otimes c)\in A\otimes A$ *and* $(\epsilon\otimes\epsilon)((a\otimes 1)E(1\otimes c)) = \epsilon(ac)$ *for all* $a,c\in A$.

(4) $(\epsilon\otimes\epsilon)((a\otimes 1)\Delta(b)(1\otimes c)) = \epsilon(abc)$ *for all* $a,b,c\in A$.

Proof. (1)⇒(3). Note that (1) is in fact the second one of the axioms in Definition 4.1.1 (vi). Hence the same reasoning used to prove Proposition 4.1.8 (3) shows that for all $a,b\in A$, $(a\otimes 1)E(1\otimes b)\in A\otimes A$; so that (1) is equivalent to $(\epsilon\otimes\mathsf{id})((a\otimes 1)E(1\otimes c)) =$

[1] The proof of (4) ⇒ (1) was kindly communicated to us by Alfons Van Daele.

$(\epsilon \otimes \mathsf{id})T_2(a \otimes c)$ for all $a, c \in A$. Applying ϵ to both sides of this equality and using the counitality axiom (iii), we obtain the equality in (3).

(2)⇒(3). By part (3) in Proposition 4.1.9, $E(1 \otimes c) \in \mathbb{M}(A) \otimes A$ for any $c \in A$ hence $(a \otimes 1)E(1 \otimes c) \in A \otimes A$ for any $a, c \in A$. Then (2) is equivalent to

$$(\mathsf{id} \otimes \epsilon)((a \otimes 1)E(1 \otimes c)) = (\mathsf{id} \otimes \epsilon)T_1(a \otimes c) \tag{4.11}$$

for all $a, c \in A$. Applying ϵ to both sides of this equality and using the counitality axiom (iii), we obtain the equality in (3).

(3)⇒(1). For any $a, c, d \in A$,

$$\begin{aligned}
(\epsilon \otimes \mathsf{id}) \quad & ((a \otimes 1)E(1 \otimes c))d \overset{\text{(iii)}}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes T_1)((a \otimes 1)E(1 \otimes c) \otimes d)) \\
\overset{(4.1)}{=} \quad & (\epsilon \otimes \epsilon \otimes \mathsf{id})((\mathsf{id} \otimes \Delta)((a \otimes 1)E(1 \otimes c))(1 \otimes 1 \otimes d)) \\
= \quad & (\epsilon \otimes \epsilon \otimes \mathsf{id})((\overline{\mathsf{id} \otimes \Delta})(a \otimes 1)(\overline{\mathsf{id} \otimes \Delta})(E)(\mathsf{id} \otimes \Delta)(1 \otimes c)(1 \otimes 1 \otimes d)) \\
\overset{\text{(v)(2.113)}}{=} \quad & (\epsilon \otimes \epsilon \otimes \mathsf{id})((a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes E)(1 \otimes T_1(c \otimes d))) \\
\overset{(4.2)}{=} \quad & (\epsilon \otimes \epsilon \otimes \mathsf{id})((a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes T_1(c \otimes d))) \\
\overset{(3)}{=} \quad & (\epsilon \otimes \mathsf{id})((a \otimes 1)T_1(c \otimes d)) = ((\epsilon \otimes \mathsf{id})T_2(a \otimes c))d,
\end{aligned}$$

where in the third equality we used that $\mathsf{id} \otimes \Delta : A \otimes A \to \mathbb{M}(A \otimes A \otimes A)$ extends to $\overline{\mathsf{id} \otimes \Delta} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A \otimes A \otimes A)$ (see page 60) and the multiplicativity of $\overline{\mathsf{id} \otimes \Delta}$, and in the fourth one we used, in addition to (v) and (2.113), the idempotency of $E \in \mathbb{M}(A \otimes A)$.

(3)⇒(2). For any $a, c, d \in A$,

$$\begin{aligned}
d(\mathsf{id} \otimes \epsilon) \quad & ((a \otimes 1)E(1 \otimes c)) \overset{\text{(iii)}}{=} (\mathsf{id} \otimes \epsilon \otimes \epsilon)(T_2 \otimes \mathsf{id})(d \otimes (a \otimes 1)E(1 \otimes c)) \\
\overset{(4.1)}{=} \quad & (\mathsf{id} \otimes \epsilon \otimes \epsilon)((d \otimes 1 \otimes 1)(\Delta \otimes \mathsf{id})((a \otimes 1)E(1 \otimes c))) \\
= \quad & (\mathsf{id} \otimes \epsilon \otimes \epsilon)((d \otimes 1 \otimes 1)(\Delta \otimes \mathsf{id})(a \otimes 1)(\overline{\Delta \otimes \mathsf{id}})(E)(\overline{\Delta \otimes \mathsf{id}})(1 \otimes c)) \\
\overset{\text{(v)(2.113)}}{=} \quad & (\mathsf{id} \otimes \epsilon \otimes \epsilon)(T_2(d \otimes a) \otimes 1)(E \otimes 1)(1 \otimes E)(1 \otimes 1 \otimes c)) \\
\overset{(4.2)}{=} \quad & (\mathsf{id} \otimes \epsilon \otimes \epsilon)((T_2(d \otimes a) \otimes 1)(1 \otimes E)(1 \otimes 1 \otimes c)) \\
\overset{(3)}{=} \quad & (\mathsf{id} \otimes \epsilon)(T_2(d \otimes a)(1 \otimes c)) = d(\mathsf{id} \otimes \epsilon)T_1(a \otimes c),
\end{aligned}$$

where in the third equality we used that $\overline{\mathsf{id} \otimes \Delta}$ is a multiplicative extension of $\mathsf{id} \otimes \Delta$ (see page 60), and in the fourth one we used, in addition to (v) and (2.113), the idempotency of $E \in \mathbb{M}(A \otimes A)$.

(1) (and (3)) ⇒(4). For any $a, c, d \in A$,

$$(\epsilon \otimes \epsilon)((a \otimes 1)\Delta(b)(1 \otimes c)) \overset{(1)}{=} (\epsilon \otimes \epsilon)((a \otimes 1)E(1 \otimes bc)) \overset{(3)}{=} \epsilon(abc).$$

(4)⇒(1). For the idea of the reasoning below, we are grateful to Alfons Van Daele. In view of axiom (iv) in Definition 4.1.1, (1) is equivalent to

$$(\epsilon \otimes \mathsf{id})((a \otimes b)\Delta(c)(1 \otimes d)) = (\epsilon \otimes \mathsf{id})((a \otimes b)\Delta(cd)) \;\forall a, b, c, d \in A,$$

hence by the non-degeneracy of the multiplication, also to

$$(\epsilon \otimes \mathsf{id})T_2(a \otimes c)d = (\epsilon \otimes \mathsf{id})T_2(a \otimes cd) \;\forall a, c, d \in A.$$

So we will prove it in this last form. For any $c, d \in A$, denote $c_i \otimes d_i := T_1(c \otimes d)$. Then for any $b \in A$,

$$T_1(bc \otimes d) = \Delta(b)(c_i \otimes d_i) = T_1(b \otimes d_i)(c_i \otimes 1). \tag{4.12}$$

With this information in mind, for any $a, b, c, d \in A$,

$$\begin{aligned}
((\epsilon \otimes \mathsf{id})T_2(a \otimes b))cd &= (\epsilon \otimes \mathsf{id})[T_2(a \otimes b)(1 \otimes cd)] \\
&\overset{\text{(iii)}}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes T_1)[(T_2(a \otimes b)(1 \otimes c)) \otimes d] \\
&\overset{(4.12)}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})(a \otimes b \otimes d_i))(1 \otimes c_i \otimes 1)] \\
&\overset{\text{(ii)}}{=} (\epsilon \otimes \epsilon \otimes \mathsf{id})[((T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(a \otimes b \otimes d_i))(1 \otimes c_i \otimes 1)] \\
&\overset{(4)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes 1)T_1(b \otimes d_i)(c_i \otimes 1)] \\
&\overset{(4.12)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes 1)T_1(bc \otimes d)] = ((\epsilon \otimes \mathsf{id})T_2(a \otimes bc))d,
\end{aligned}$$

so we conclude by the non-degeneracy of the multiplication. □

Lemma 4.1.11. *Let A be an idempotent non-unital algebra over a field k with a non-degenerate multiplication, $\Delta : A \to \mathbb{M}(A \otimes A)$ be a multiplicative linear map, $\epsilon : A \to k$ be a linear map and E be an idempotent element in $\mathbb{M}(A \otimes A)$. Assume that also the ranges of the maps T_3 and T_4 in Definition 4.1.3 are in the ideal $A \otimes A$, and that the axioms (i)-(v) —but not necessarily (vi)— in Definition 4.1.1 hold. Then, $A^{\mathsf{op}} := (A, \mu^{\mathsf{op}}, \Delta, \epsilon, E)$ obeys the same assumptions made on A.*

Proof. Clearly, (A, μ^{op}) is an idempotent non-unital algebra and μ^{op} is non-degenerate. As $\mathbb{M}(A^{\mathsf{op}} \otimes A^{\mathsf{op}}) = \mathbb{M}((A \otimes A)^{\mathsf{op}}) \cong \mathbb{M}(A \otimes A)^{\mathsf{op}}$, the multiplicativity of Δ on A^{op} follows immediately from its multiplicativity on A. Since the maps T_1 and T_2 in A^{op} are exactly the maps T_3 and T_4 (cf. (4.3)), axiom (i) on A^{op} follows by the assumption that the ranges of the maps T_3 and T_4 in are in $A \otimes A$. In order to check the coassoaciativity axiom (ii), introduce the notations $T_1(b \otimes d) =: b_i \otimes d_i$ and $T_2(e \otimes b) =: e_j \otimes b'_j$ for any $b, d, e \in A$. For any $a, b, c, d, e \in A$,

$$
\begin{aligned}
&(e \otimes 1 \otimes 1)[(T_4 \otimes \mathsf{id})(\mathsf{id} \otimes T_3)(a \otimes b \otimes c)](1 \otimes 1 \otimes d) \\
&\overset{(4.3)(4.1)}{=} (e \otimes 1 \otimes c)[(T_4 \otimes \mathsf{id})(a \otimes b_i \otimes d_i)] \\
&\overset{(4.3)(4.1)}{=} (1 \otimes 1 \otimes c)[(T_2 \otimes \mathsf{id})(e \otimes b_i \otimes d_i)](a \otimes 1 \otimes 1) \\
&\overset{(ii)}{=} (1 \otimes 1 \otimes c)[(\mathsf{id} \otimes T_1)(e_j a \otimes b'_j \otimes d)] \\
&\overset{(4.3)(4.1)}{=} [(\mathsf{id} \otimes T_3)(e_j a \otimes b'_j \otimes c)](1 \otimes 1 \otimes d) \\
&\overset{(4.3)(4.1)}{=} [(\mathsf{id} \otimes T_3)((e \otimes 1)T_4(a \otimes b) \otimes c)](1 \otimes 1 \otimes d) \\
&= (e \otimes 1 \otimes 1)[(\mathsf{id} \otimes T_3)(T_4 \otimes \mathsf{id})(a \otimes b \otimes c)](1 \otimes 1 \otimes d),
\end{aligned}
$$

from which we conclude

$$
(T_4 \otimes \mathsf{id})(\mathsf{id} \otimes T_3) = (\mathsf{id} \otimes T_3)(T_4 \otimes \mathsf{id}). \tag{4.13}
$$

As for the counitality axiom (iii), for any $a, b, c \in A$,

$$
\begin{aligned}
(\epsilon \otimes \mathsf{id})T_3(a \otimes b)c &\overset{(4.3)(4.1)}{=} b(\epsilon \otimes \mathsf{id})T_1(a \otimes c) \overset{(iii)}{=} bac, \\
c(\mathsf{id} \otimes \epsilon)T_4(a \otimes b) &\overset{(4.3)(4.1)}{=} (\mathsf{id} \otimes \epsilon)T_2(c \otimes b)a \overset{(iii)}{=} cba,
\end{aligned}
$$

and hence $(\epsilon \otimes \mathsf{id})T_3 = \mu^{\mathsf{op}} = (\mathsf{id} \otimes \epsilon)T_4$. Writing out on A^{op} both assertions in axiom (iv) and the three expressions requested to be equal in axiom (v), one literally re-obtains those ones in axioms (iv) and (v) on A, what concludes the proof of the claim. □

The following analogous version of Proposition 4.1.10 is immediate.

Proposition 4.1.12. *Let A be an idempotent non-unital algebra over a field k with a non-degenerate multiplication, $\Delta : A \to \mathbb{M}(A \otimes A)$ be a multiplicative linear map, $\epsilon : A \to k$ be a linear map and E be an idempotent element in $\mathbb{M}(A \otimes A)$. Assume* *that also the ranges of the maps T_3 and T_4 in Definition 4.1.3 are in the ideal $A \otimes A$, and that the axioms (i)-(v) —but not necessarily (vi)— in Definition 4.1.1 hold. The following assertions are equivalent.*

(1) $(\epsilon \otimes \mathsf{id})((1 \otimes a)E(b \otimes c)) = (\epsilon \otimes \mathsf{id})(\Delta(a)(b \otimes c)$*, for any* $a, b, c \in A$.

(2) $(\mathsf{id} \otimes \epsilon)((a \otimes b)E(c \otimes 1)) = (\mathsf{id} \otimes \epsilon)((a \otimes b)\Delta(c))$*, for any* $a, b, c \in A$.

(3) $(1 \otimes a)E(c \otimes 1) \in A \otimes A$ *and* $(\epsilon \otimes \epsilon)((1 \otimes a)E(c \otimes 1)) = \epsilon(ac)$*, for any* $a, c \in A$.

(4) $(\epsilon \otimes \epsilon)((1 \otimes a)\Delta(b)(c \otimes 1)) = \epsilon(abc)$*, for any* $a, b, c \in A$.

Proof. By Lemma 4.1.11, $A^{\mathsf{op}} := (A, \mu^{\mathsf{op}}, \Delta, \epsilon, E)$ obeys the hypotheses of Proposition 4.1.10. Parts (1), (2), (3) and (4) follow by applying Proposition 4.1.10 on it. □

Lemma 4.1.13. *Let $(A, \mu, \Delta, \epsilon, E)$ be a regular weak multiplier bialgebra over a field. Then, $A^{\mathsf{op}} := (A, \mu^{\mathsf{op}}, \Delta, \epsilon, E)$, $A_{\mathsf{cop}} := (A, \mu, \Delta^{\mathsf{op}}, \epsilon, \mathsf{tw}E\mathsf{tw})$ and $A^{\mathsf{op}}_{\mathsf{cop}} := (A, \mu^{\mathsf{op}}, \Delta^{\mathsf{op}}, \epsilon,$ $\mathsf{tw}E\mathsf{tw})$ are also regular weak multiplier bialgebras over the same field (Δ^{op} as defined in (2.115)). They will be referred to as the opposite, coopposite and opposite-coopposite structures of A, respectively.*

Proof. In view of Lemma 4.1.11, in order to conclude that A is a weak multiplier bialgebra, it is enough to observe that axiom (vi) on A^{op} says the same as on A. Regularity of A^{op} is evident from the fact that the maps T_3 and T_4 on A^{op} are the maps T_1 and T_2 on A, and axiom (i) on A.

Next, we prove that A_{cop} also obeys the axioms of a weak multiplier bialgebra. Since A is regular, the elements $\mathsf{tw}T_3\mathsf{tw}(a \otimes b), \mathsf{tw}T_4\mathsf{tw}(a \otimes b)$ belong to $A \otimes A$ for any $a, b \in A$, proving axiom (i). Using the notations $T_3(b \otimes a) =: b_i \otimes a_i$ and $T_4(c \otimes b) =: c_j \otimes b'_j$ for any $a, b, c \in A$, the coassociativity axiom (ii) is proven by

$$\begin{aligned}
(\mathsf{id} \otimes \mathsf{tw}T_4\mathsf{tw})(\mathsf{tw}T_3\mathsf{tw} \otimes \mathsf{id})(a \otimes b \otimes c) &= (\mathsf{id} \otimes \mathsf{tw})(a_i \otimes T_4(c \otimes b_i)) \\
&= (\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})(T_4(c \otimes b_i) \otimes a_i) \\
&\overset{(4.13)}{=} (\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})(c_j \otimes T_3(b'_j \otimes a)) \\
&= (\mathsf{tw} \otimes \mathsf{id})(T_3(b'_j \otimes a) \otimes c_j) \\
&= (\mathsf{tw}T_3\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw}T_4\mathsf{tw})(a \otimes b \otimes c)
\end{aligned}$$

for all $a, b, c \in A$. Using part (i) of the current proposition in the second equalities, axiom (iii) follows by

$$\begin{aligned}(\epsilon \otimes \mathsf{id})\mathsf{tw}T_4\mathsf{tw}(a \otimes b) &= (\mathsf{id} \otimes \epsilon)T_4(b \otimes a) = \mu^{\mathsf{op}}(b \otimes a) = \mu(a \otimes b)\\ (\mathsf{id} \otimes \epsilon)\mathsf{tw}T_3\mathsf{tw}(a \otimes b) &= (\epsilon \otimes \mathsf{id})T_3(b \otimes a) = \mu^{\mathsf{op}}(b \otimes a) = \mu(a \otimes b)\end{aligned}$$

for any $a, b \in A$. Using the definition (2.115) of Δ^{op}, the first and second assertions in axiom (iv) on A_{cop} follow, respectively, by the first and second assertions in the same axiom on A. In order to check axiom (v), introduce the notations $E(c \otimes b) =: c_i \otimes b_i$ and $E(b \otimes a) =: b'_j \otimes a_j$ for any $a, b, c \in A$. For any $a, b, c \in A$,

$$\begin{aligned}(\mathsf{tw}E\mathsf{tw} \otimes 1)(1 \otimes \mathsf{tw}E\mathsf{tw})(a \otimes b \otimes c) &= (\mathsf{tw}E\mathsf{tw} \otimes 1)(a \otimes b_i \otimes c_i) \qquad (4.14)\\ &= (\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})(c_i \otimes E(b_i \otimes a))\\ &\overset{(v)}{=} (\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})(E(c \otimes b'_j) \otimes a_j)\\ &= (1 \otimes \mathsf{tw}E\mathsf{tw})(a_j \otimes b'_j \otimes c)\\ &= (1 \otimes \mathsf{tw}E\mathsf{tw})(\mathsf{tw}E\mathsf{tw} \otimes 1)(a \otimes b \otimes c)\end{aligned}$$

Let us check now that the elements $(1 \otimes \mathsf{tw}E\mathsf{tw})(\mathsf{tw}E\mathsf{tw} \otimes 1) = (\mathsf{tw}E\mathsf{tw} \otimes 1)(1 \otimes \mathsf{tw}E\mathsf{tw})$ of $\mathbb{M}(A \otimes A \otimes A)$ are also equal to $(\overline{\Delta^{\mathsf{op}} \otimes \mathsf{id}})(\mathsf{tw}E\mathsf{tw})$. For any $u, v, w \in A$,

$$\begin{aligned}&(\overline{\Delta^{\mathsf{op}} \otimes \mathsf{id}})(\mathsf{tw}E\mathsf{tw})(u \otimes v \otimes w)\\ =\ &\sum_i [(\Delta^{\mathsf{op}} \otimes \mathsf{id})\mathsf{tw}E\mathsf{tw}(a_i \otimes b_i)](a'_i \otimes b'_i \otimes c'_i)\\ =\ &(\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})[((\mathsf{id} \otimes \Delta)E(b_i \otimes a_i))(c'_i \otimes b'_i \otimes a'_i)]\\ \overset{(v)}{=}\ &(\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})[(E \otimes 1)(1 \otimes E)((\mathsf{id} \otimes \Delta)(b_i \otimes a_i))(c'_i \otimes b'_i \otimes a'_i)]\\ \overset{(2.118)}{=}\ &(\mathsf{id} \otimes \mathsf{tw})(\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{tw})[(E \otimes 1)(1 \otimes E)(w \otimes v \otimes u)]\\ =\ &(1 \otimes \mathsf{tw}E\mathsf{tw})(\mathsf{tw}E\mathsf{tw} \otimes 1)(u \otimes v \otimes w),\end{aligned}$$

where $a_i, b_i, a'_i, b'_i, c'_i$ are elements of A such that $(\mathsf{tw}E\mathsf{tw} \otimes 1)(u \otimes v \otimes w) = \sum_i [(\Delta^{\mathsf{op}} \otimes \mathsf{id})(a_i \otimes b_i)](a'_i \otimes b'_i \otimes c'_i)$ (their existence is assured by axiom (iv)). In the third equality we used, in addition, the multiplicativity of $\overline{\Delta \otimes \mathsf{id}}$, and in the last equality we used the identity of the third and last lines in (4.14). The equality of $\overline{\mathsf{id} \otimes \Delta^{\mathsf{op}}}(\mathsf{tw}E\mathsf{tw})$ with $(\mathsf{tw}E\mathsf{tw} \otimes 1)(1 \otimes \mathsf{tw}E\mathsf{tw})$ is similarly proven. The first assertion in axiom (vi) on A_{cop}

requires $(\mathsf{id} \otimes \epsilon)((a \otimes 1)E(b \otimes c))$ to be equal to $(\mathsf{id} \otimes \epsilon)(\Delta(a)(b \otimes c))$ for any $a, b, c \in A$, what indeed holds by the equivalent form of axiom (vi) on A in part (2) in Proposition 4.1.10. The second assertion in the same axiom (vi) on A_{cop} means the equality of $(\mathsf{id} \otimes \epsilon)((a \otimes b)E(c \otimes 1))$ and $(\mathsf{id} \otimes \epsilon)((a \otimes b)\Delta(c))$, what is assured by the equivalent form of axiom (vi) on A in part (2) in Proposition 4.1.12. Since the maps T_3 and T_4 on A_{cop} are the maps $\mathsf{tw}T_2\mathsf{tw}$ and $\mathsf{tw}T_1\mathsf{tw}$ on A, regularity of A_{cop} follows from axiom (i) on A.

As a consequence of the considerations above, the opposite structure of A_{cop} and the coopposite structure of A^{op} are both (identical) regular weak multiplier bialgebras, to be denoted by $A^{\mathsf{op}}_{\mathsf{cop}}$. □

In view of Lemma 4.1.13, a weak multiplier bialgebra over a field is regular if and only if the opposite of its underlying algebra is a weak multiplier bialgebra too, via the same comultiplication, counit and idempotent element. For a regular weak multiplier bialgebra A, some of the axioms in Definition 4.1.1 can be re-written in the following equivalent forms.

(ii) $\Leftrightarrow (T_4 \otimes \mathsf{id})(\mathsf{id} \otimes T_3) = (\mathsf{id} \otimes T_3)(T_4 \otimes \mathsf{id})$,

(iii) $\Leftrightarrow (\epsilon \otimes \mathsf{id})T_3 = \mu^{\mathsf{op}} = (\mathsf{id} \otimes \epsilon)T_4$.

Note also that, keeping the regularity assumption on A, by evaluating both sides of (ii) on any $a \otimes b \otimes c \in A \otimes A \otimes A$, multiplying on the left by $1 \otimes 1 \otimes d$ and simplifying on the right by $1 \otimes 1 \otimes c$, the coassociativity axiom (ii) admits the alternative form

$$(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_3) = (\mathsf{id} \otimes T_3)(T_2 \otimes \mathsf{id}). \tag{4.15}$$

In order to reduce the computations and to shorten the proofs, Lemma 4.1.13 will be exploited in many of the results in this chapter in the following sense. After showing some claims of the corresponding statements, we may like to stress the fact that the other ones may *symmetrically* follow. More precisely, if we say, for instance, that an assertion q follows by some proven assertion p and the symmetry $A - A^{\mathsf{op}}$, this means that p formulated on A^{op} yields exactly the assertion q what, indeed, proves q.

The following table collects the expressions of the T_i maps in the regular weak multiplier bialgebras A^{op}, A_{cop} and $A^{\mathsf{op}}_{\mathsf{cop}}$ for any regular weak multiplier bialgebra A.

A $(A,\mu,\Delta,\epsilon,E)$	A^{op} $(A,\mu^{\mathsf{op}},\Delta,\epsilon,E)$	A_{cop} $(A,\mu,\Delta^{\mathsf{op}},\epsilon,\mathsf{tw}E\mathsf{tw})$	$A^{\mathsf{op}}_{\mathsf{cop}}$ $(A,\mu^{\mathsf{op}},\Delta^{\mathsf{op}},\epsilon,\mathsf{tw}E\mathsf{tw})$
T_1	T_3	$\mathsf{tw}T_4\mathsf{tw}$	$\mathsf{tw}T_2\mathsf{tw}$
T_2	T_4	$\mathsf{tw}T_3\mathsf{tw}$	$\mathsf{tw}T_1\mathsf{tw}$
T_3	T_1	$\mathsf{tw}T_2\mathsf{tw}$	$\mathsf{tw}T_4\mathsf{tw}$
T_4	T_2	$\mathsf{tw}T_1\mathsf{tw}$	$\mathsf{tw}T_3\mathsf{tw}$

Table 4.1: T_i maps in the symmetric weak multiplier bialgebras.

Theorem 4.1.1. *The counit of a weak multiplier bialgebra A over a field k is uniquely determined by the multiplication and the comultiplication.*

Proof. We have seen in Remark 4.1.2 that the idempotent element E is uniquely fixed. Let $\epsilon,\epsilon' : A \to k$ be counits for A. Then for all $a,b,c \in A$,

$$\begin{aligned}
(\epsilon\otimes\epsilon')[(a\otimes 1)\Delta(b)(1\otimes c)] &= (\epsilon\otimes\epsilon')[(a\otimes 1)T_1(b\otimes c)]\\
&= (\epsilon\otimes\epsilon\otimes\epsilon')[(a\otimes 1\otimes 1)(E\otimes 1)(1\otimes T_1(b\otimes c))]\\
&\overset{\text{(iv)}}{=} (\epsilon\otimes\epsilon\otimes\epsilon')[(a\otimes 1\otimes 1)(E\otimes 1)(1\otimes E)(1\otimes T_1(b\otimes c))]\\
&\overset{\text{(v)}}{=} (\epsilon\otimes\epsilon\otimes\epsilon')[(a\otimes 1\otimes 1)(\overline{\mathsf{id}\otimes\Delta})(E(1\otimes b))(1\otimes 1\otimes c)]\\
&= (\epsilon\otimes\epsilon\otimes\epsilon')[(a\otimes 1\otimes 1)(\mathsf{id}\otimes T_1)(E(1\otimes b)\otimes c)]\\
&\overset{\text{(iii)}}{=} (\epsilon\otimes\epsilon')[(a\otimes 1)E(1\otimes bc)] = (\epsilon\otimes\epsilon')T_1(a\otimes bc) \overset{\text{(iii)}}{=} \epsilon'(abc).
\end{aligned}$$

In the second and the penultimate equalities we used Proposition 4.1.10 (3) and (2) (in its alternative form (4.11)) for ϵ and for ϵ', respectively. In the fifth equality we used Propostion 4.1.9 (3). Symmetrically, using Proposition 4.1.10 (3) for ϵ' in the second equality, Proposition 4.1.8 (2) in the fifth equality and Proposition 4.1.10 (1) for ϵ in the penultimate equality,

$$\begin{aligned}
(\epsilon\otimes\epsilon')[(a\otimes 1)\,\Delta(b)\,(1\otimes c)] &= (\epsilon\otimes\epsilon')[T_2(a\otimes b)(1\otimes c)]\\
&= (\epsilon\otimes\epsilon'\otimes\epsilon')[(T_2(a\otimes b)\otimes 1)(1\otimes E)(1\otimes 1\otimes c)]\\
&\overset{\text{(iv)}}{=} (\epsilon\otimes\epsilon'\otimes\epsilon')[(T_2(a\otimes b)\otimes 1)(E\otimes 1)(1\otimes E)(1\otimes 1\otimes c)]\\
&\overset{\text{(v)}}{=} (\epsilon\otimes\epsilon'\otimes\epsilon')[(a\otimes 1\otimes 1)(\overline{\Delta\otimes\mathsf{id}})((b\otimes 1)E)(1\otimes 1\otimes c)]\\
&= (\epsilon\otimes\epsilon'\otimes\epsilon')[(T_2\otimes\mathsf{id})(a\otimes(b\otimes 1)E)(1\otimes 1\otimes c)]\\
&\overset{(iii)'}{=} (\epsilon\otimes\epsilon')[(ab\otimes 1)E(1\otimes c)] = (\epsilon\otimes\epsilon')T_2(ab\otimes c) \overset{(iii)'}{=} \epsilon(abc),
\end{aligned}$$

where the label (iii)' refers to the application of axiom (iii) to ϵ'. So we conclude by the idempotency of A that $\epsilon = \epsilon'$. □

Two main sources of examples of weak multiplier bialgebras are regular weak multiplier Hopf algebras in [73] and weak bialgebras [18, 52] (possessing units), as we shall see in the next two theorems.

Theorem 4.1.2. *If an idempotent non-unital algebra A over a field with a non-degenerate multiplication possesses a regular weak multiplier Hopf algebra structure in the sense of [73] (recalled in Section 2.7), then A is also a (regular) weak multiplier bialgebra via the same structure maps.*

Proof. Axioms (i), (ii), (iii) and (v) in Definition 4.1.1 are parts of the definition of regular weak multiplier Hopf algebra in [73]. Since A is an idempotent non-unital algebra by assumption, the axioms $E(A \otimes A) = T_1(A \otimes A)$ and $(A \otimes A)E = T_2(A \otimes A)$ in [73] imply our axiom (iv). It remains to prove that axiom (vi) holds true. By [73, Proposition 2.3], for any weak multiplier Hopf algebra A over a field, there exists a linear map $R_1 : A \otimes A \to A \otimes A$ such that $T_1 R_1(a \otimes b) = E(a \otimes b)$ for all $a, b \in A$. Then applying $\epsilon \otimes \mathsf{id}$ to both sides and using the counitality axiom (iii) in Definition 4.1.1, it follows that

$$\mu R_1(a \otimes b) = (\epsilon \otimes \mathsf{id})[E(a \otimes b)] \qquad \forall a, b \in A. \tag{4.16}$$

For any $a, b, c \in A$,

$$T_1[(a \otimes 1)R_1(b \otimes c)] = \Delta(a)(T_1 R_1(b \otimes c)) = \Delta(a)E(b \otimes c) \overset{\text{(iv)}}{=} \Delta(a)(b \otimes c).$$

Applying $\epsilon \otimes \mathsf{id}$ to both sides and using the counitality axiom (iii) and (4.16),

$$(\epsilon \otimes \mathsf{id})((1 \otimes a)E(b \otimes c)) = (\epsilon \otimes \mathsf{id})(\Delta(a)(b \otimes c)).$$

So the first axiom in (vi) holds true. The assumption about regularity —which has not yet been used so far— allows for a symmetric verification of the second axiom in (vi). □

For *arbitrary* weak multiplier Hopf algebras in [73], however, the second axiom in Definition 4.1.1 (vi) does not seem to hold. Consequences of this will be discussed further in Section 4.5. The following result shows, in particular, that the notion of weak multiplier bialgebra extends that of weak bialgebra (recalled in 2.5.3).

Theorem 4.1.3. *For an algebra A over a field, there is a bijective correspondence between*

- *weak bialgebra structures on A,*
- *and weak multiplier bialgebra structures on A.*

Proof. An algebra A is clearly idempotent with a non-degenerate multiplication, and its multiplier algebra $\mathbb{M}(A)$ coincides with A. So in this case the axioms in Definition 4.1.1 (i) become trivial identities and any weak multiplier bialgebra structure on A is regular. By axioms (ii) and (iii), a weak multiplier bialgebra structure on A is given by a coassociative counital comultiplication $A \to A \otimes A$ which is a multiplicative map, and a compatible idempotent element of $A \otimes A$. By the uniqueness of the idempotent element E (see Remark 4.1.2) obeying axiom (iv), it follows that $E = \Delta(1)$. Then axiom (v) is the usual weak bialgebra axiom (2.49) expressing the weak comultiplicativity of the unit. By Proposition 4.1.10 and Proposition 4.1.12, axiom (vi) is equivalent to the usual weak bialgebra axiom (2.48) expressing the weak multiplicativity of the counit (cf. parts (4) of the quoted propositions). □

Among weak bialgebras A over a field, bialgebras are distinguished by the equivalent properties that $\Delta(1) = 1 \otimes 1$, or $\epsilon(ab) = \epsilon(a)\epsilon(b)$ for all $a, b \in A$, or $\overline{\sqcap}^L(a) = \epsilon(a)1$ for all $a \in A$, or $\overline{\sqcap}^R(a) = \epsilon(a)1$ for all $a \in A$ (see Remark 2.5.12). As shown in the next theorem, these properties (in appropriate forms) remain equivalent also for a weak multiplier bialgebra A.

Theorem 4.1.4. *Let A be a weak multiplier bialgebra over a field. The following assertions are equivalent.*

(1) $E = 1$ *as elements of* $\mathbb{M}(A \otimes A)$.

(2) $\epsilon(ab) = \epsilon(a)\epsilon(b)$ *for all* $a, b \in A$.

(3) $\overline{\sqcap}^L(a) = \epsilon(a)1$ *as elements of* $\mathbb{M}(A)$, *for all* $a \in A$.

(4) $\overline{\sqcap}^R(a) = \epsilon(a)1$ *as elements of* $\mathbb{M}(A)$, *for all* $a \in A$.

Proof. (1)⇒(2). For any $a, b \in A$, $\epsilon(a)\epsilon(b) \overset{(1)}{=} (\epsilon \otimes \epsilon)[(a \otimes 1)E(1 \otimes b)] = \epsilon(ab)$, where the last equality follows by Proposition 4.1.10 (3).

(2)⇒(1). Using Proposition 4.1.8 (1) in the first equality, it follows for any $a, b, c, d \in A$ that

$$\begin{aligned} (ab \otimes 1)E(1 \otimes cd) &= ((\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b))(1 \otimes cd) \\ &\overset{(4.4)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes T_2)(T_2 \otimes \mathsf{id})(a \otimes b \otimes c))(1 \otimes 1 \otimes d)] \\ &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})[(T_2(a \otimes b) \otimes 1)(1 \otimes T_1(c \otimes d))] \\ &\overset{(2)}{=} (\mathsf{id} \otimes \epsilon)T_2(a \otimes b) \otimes (\epsilon \otimes \mathsf{id})T_1(c \otimes d) \overset{\text{(iii)}}{=} ab \otimes cd, \end{aligned}$$

from which we conclude by the density of $A \otimes A$ in $\mathbb{M}(A \otimes A)$.

(1)⇒(3). For any $a, b \in A$, $b\overline{\sqcap}^L(a) \overset{(4.4)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes b)E] \overset{(1)}{=} b\epsilon(a)$, from which we conclude by the density of A in $\mathbb{M}(A)$.

(1)⇒(4). For any $a, b \in A$, $\overline{\sqcap}^R(a)b \overset{(4.6)}{=} (\mathsf{id} \otimes \epsilon)(E(b \otimes a)) \overset{(1)}{=} \epsilon(a)b$, from which we conclude by the density of A in $\mathbb{M}(A)$.

(3)⇒(1). Using Proposition 4.1.8 (1) in the first equality, it follows for any $a, b \in A$ that

$$(ab \otimes 1)E = (\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b) \overset{(3)}{=} (\mathsf{id} \otimes \epsilon)T_2(a \otimes b) \otimes 1 \overset{\text{(iii)}}{=} ab \otimes 1,$$

from which we conclude by the density of $A \otimes A$ in $\mathbb{M}(A \otimes A)$.

(4)⇒(1). Using Proposition 4.1.9 (2) in the first equality, it follows for any $a, b \in A$ that

$$E(1 \otimes ab) = (\overline{\sqcap}^R \otimes \mathsf{id})T_1(a \otimes b) \overset{(4)}{=} 1 \otimes (\epsilon \otimes \mathsf{id})T_1(a \otimes b) \overset{\text{(iii)}}{=} 1 \otimes ab,$$

from which we conclude by the density of $A \otimes A$ in $\mathbb{M}(A \otimes A)$. □

Whenever the equivalent conditions in Theorem 4.1.4 hold for a weak multiplier bialgebra over a field, it would be most natural to term it a *multiplier bialgebra*. Note, however, that this notion is different from both notions in [40] and [67] which were given the same name. The present one is covered by the definition of *regular multiplier bialgebra* in a braided monoidal category given in [17] applied to the category of vector spaces.

The next four families of examples do not belong to any of the previously discussed classes.

Example 4.1.14. Take a small category C possibly with infinitely many objects and arrows. For a fixed field, consider the idempotent non-unital algebra $k\mathsf{C}$ with a non-degenerate multiplication from Example 2.2.1. It can be equipped with the structure of a regular weak multiplier bialgebra. The comultiplication takes an arrow a to $a \otimes a$ regarded as an element of the multiplier algebra $\mathbb{M}(k\mathsf{C} \otimes k\mathsf{C})$. The counit takes each arrow to 1. For any arrows a, b in C, the idempotent element E in $\mathbb{M}(k\mathsf{C} \otimes k\mathsf{C})$ is given by $E(a \otimes b) = a \otimes b$ if $t(a) = t(b)$ and $E(a \otimes b) = 0$ otherwise; and $(a \otimes b)E = a \otimes b$ if $s(a) = s(b)$ and $(a \otimes b)E = 0$ otherwise. All these maps are then linearly extended.

Example 4.1.15. Take again a small category C possibly with infinitely many objects and arrows. For any arrows a and b of common source, assume that there are only finitely many arrows c such that $ca = b$. Symmetrically, for any arrows a and b of common target, assume that there are only finitely many arrows c such that $ac = b$. (These assumptions evidently hold for a groupoid.) For a fixed field k, consider $k(\mathsf{C})$, the non-unital k–algebra of k–valued functions of finite support on the arrow set C_1, see Example 2.2.2. It carries the structure of a regular weak multiplier bialgebra. In terms of the characteristic functions $\chi_{\{c\}}$ of the one element subsets $\{c\}$ of C_1, the comultiplication Δ takes $f \in k(\mathsf{C})$ to the multiplier $\Delta(f)$ described by

$$\Delta(f)(g \otimes h) = \sum_{c,d \in \mathsf{C}_1} g(c)h(d)f(cd)\chi_{\{c\}} \otimes \chi_{\{d\}} = (g \otimes h)\Delta(f)$$

for any $g, h \in k(\mathsf{C})$. Note that in this sum there are only finitely many non-zero terms since g, h and f have finite supports. The maps T_j (for $j \in \{1, 2, 3, 4\}$) land in $k(\mathsf{C}) \otimes k(\mathsf{C})$ by the assumption that we made about the set of arrows. The counit takes f to the sum of the values $f(1_x)$ for the *identity* arrows 1_x (which contains finitely many non-zero terms by assumption). The idempotent element E in $\mathbb{M}(k(\mathsf{C}) \otimes k(\mathsf{C}))$ is given by

$$E(g \otimes h) = \sum_{\substack{c, d \in C_1 \\ s(c) = t(d)}} g(c)h(d)\chi_{\{c\}} \otimes \chi_{\{d\}} = (g \otimes h)E \quad \forall g, h \in k(\mathsf{C}).$$

It was shown in [73] that whenever the categories in the above examples are groupoids, then both constructions yield regular weak multiplier Hopf algebras in the sense of [73].

Example 4.1.16. In this example we show that any direct sum of weak multiplier bialgebras over a field —so in particular any infinite direct sum of weak bialgebras over a field —is a weak multiplier bialgebra.

For any index set I, consider a family of idempotent non-unital algebras $\{A_j\}_{j\in I}$ over a field k with non-degenerate multiplications μ_j. Let $A := \bigoplus_{j\in I} A_j$ denote the direct sum vector space with the inclusions $i_j : A_j \to A$ and the projections $p_j : A \to A_j$. The elements of A are the I–tuples $\underline{a} = \{a_j \in A_j\}_{j\in I}$ such that $a_j := p_j(\underline{a})$ is non-zero only for finitely many indices $j \in I$. Clearly, A can be equipped with the structure of an idempotent non-unital algebra with a non-degenerate multiplication $\mu : \underline{a} \otimes \underline{b} \mapsto \underline{a}\,\underline{b}$, uniquely characterized by $p_j(\underline{a}\,\underline{b}) = a_j b_j$, for any $\underline{a}, \underline{b} \in A$ and $j \in I$ (so that i_j becomes multiplicative as well).

The multiplier algebra of A is isomorphic to the direct (in fact, Cartesian) product $\prod_{j\in I} \mathbb{M}(A_j)$, regarded as an algebra via the factorwise multiplication. (Its elements are I–tuples $\{\omega_j \in \mathbb{M}(A_j)\}_{j\in I}$ without any restriction on the number of the non-zero elements.) Indeed, $i_j(A_j)$ is an ideal in A for any $j \in I$. Hence for any $\omega \in \mathbb{M}(A)$, any $j \in I$, and any $a, b \in A_j$,

$$\omega i_j(ab) = \omega(i_j(a)i_j(b)) = (\omega i_j(a))i_j(b)$$

is an element of $i_j(A_j)$. So by the idempotency of A_j, $\omega i_j(a) \in i_j(A_j)$, and symmetrically, $i_j(a)\omega \in i_j(A_j)$, for any $j \in I$ and $a \in A_j$. This proves the existence of multipliers $\omega_j \in \mathbb{M}(A_j)$ such that

$$i_j(\omega_j a) := \omega i_j(a) \qquad \text{and} \qquad i_j(a\omega_j) := i_j(a)\omega, \qquad \forall a \in A_j.$$

Hence there is a map

$$\varphi : \mathbb{M}(A) \to \prod_{j\in I} \mathbb{M}(A_j), \qquad \omega \mapsto \{\omega_j\}_{j\in I}. \tag{4.17}$$

It has the inverse $\{\omega_j\}_{j\in I} \mapsto \omega$ such that $p_j(\omega\underline{a}) = \omega_j a_j$ and $p_j(\underline{a}\omega) = a_j\omega_j$ for all $\underline{a} \in A$ and $j \in I$.

Let us take now two families $\{A_j\}_{j\in I}$ and $\{B_j\}_{j\in I}$ of idempotent non-unital algebras with non-degenerate multiplications, together with a family of multiplicative linear maps $\{\gamma_j : A_j \to \mathbb{M}(B_j)\}_{j\in I}$ and idempotent elements $\{e_j \in \mathbb{M}(B_j)\}_{j\in I}$ such that for all

$j \in I$, $\gamma_j(A_j)B_j = e_jB_j$ and $B_j\gamma_j(A_j) = B_je_j$. Then it follows by Theorem 2.7.1 that there exist unique multiplicative linear maps $\{\overline{\gamma}_j : \mathbb{M}(A_j) \to \mathbb{M}(B_j)\}_{j\in I}$ extending γ_j such that $\overline{\gamma}_j(1_j) = e_j$. Put $A := \oplus_{j\in I}A_j$ and $B := \oplus_{j\in I}B_j$ as before and in terms of the map (4.17) define

$$e := \varphi^{-1}(\{e_j\}_{j\in I}) \in \mathbb{M}(B) \quad \text{and} \quad \gamma : A \to \mathbb{M}(B), \qquad \underline{a} \mapsto \varphi^{-1}(\{\gamma_j(a_j)\}_{j\in I}).$$

Then for all $j \in I$, $p_j(\gamma(A)B) = \gamma_j(A_j)B_j = e_jB_j = p_j(eB)$, so that $\gamma(A)B = eB$. Symmetrically, $B\gamma(A) = Be$. Thus by Theorem 2.7.1, γ extends to a unique multiplicative linear map $\overline{\gamma} : \mathbb{M}(A) \to \mathbb{M}(B)$ such that $\overline{\gamma}(1) = e$. Explicitly,

$$p_j(\overline{\gamma}(\omega)\underline{b}) = \overline{\gamma}_j(\omega_j)b_j \qquad \text{and} \qquad p_j(\underline{b}\overline{\gamma}(\omega)) = b_j\overline{\gamma}_j(\omega_j), \tag{4.18}$$

for all $j \in I$, $\underline{b} \in B$ and $\omega \in \mathbb{M}(A)$.

Assume next that for all $j \in I$, A_j carries a weak multiplier bialgebra structure with comultiplication $\Delta_j : A_j \to \mathbb{M}(A_j \otimes A_j)$, counit $\epsilon_j : A_j \to k$ and idempotent element $E_j \in \mathbb{M}(A_j \otimes A_j)$. Since $A \otimes A \cong \oplus_{j,l\in I}A_j \otimes A_l$, its multiplier algebra $\mathbb{M}(A \otimes A)$ is isomorphic to $\prod_{j,l\in I}\mathbb{M}(A_j \otimes A_l)$. Hence $\mathbb{M}(A \otimes A)$ has a non-unital subalgebra $\prod_{j\in I}\mathbb{M}(A_j \otimes A_j)$. In terms of the map (4.17), we define

$$\begin{array}{rll} \Delta : & A \to \prod_{j\in I}\mathbb{M}(A_j \otimes A_j) \subset \mathbb{M}(A \otimes A), & \underline{a} \mapsto \varphi^{-1}(\{\Delta_j(a_j)\}_{j\in I}) \\ \epsilon : & A \to k, & \underline{a} \mapsto \sum_{j\in I}\epsilon_j(a_j) \\ E \in & \prod_{j\in I}\mathbb{M}(A_j \otimes A_j) \subset \mathbb{M}(A \otimes A), & E := \varphi^{-1}(\{E_j\}_{j\in I}). \end{array}$$

Note that the counit ϵ is well defined because the sum has only finitely many non-zero terms. This equips A with the structure of a weak multiplier bialgebra. Moreover, if A_j is a regular weak multiplier bialgebra for all $j \in I$, then so is the direct sum $A = \oplus_{j\in I}A_j$.

Example 4.1.17. Recently, in [33], Kenny De Commer and Thomas Timmermann showed that the total algebra associated to a partial bialgebra has the structure of a regular weak multiplier bialgebra. Next, we briefly present these notions and refer to the reader to sections 1.1, 1.2 and 1.3 in [33] for details about them.

Let I be a set and consider $I^2 = I \times I$ as the pair groupoid with $\cdot$ denoting composition. An element $X = (s,t) \in I^2$ has source s and target t; we denote them by X_l and X_r respectively. If $X = (s,t), Y = (t,u) \in I^2$ we write $X \cdot Y = (s,u)$. An I*–partial algebra* (or *partial algebra over I*) $\underline{A} = (\underline{A}, \mu)$ (over $\mathbb{C}$) is a set I together with

- for each $X = (s,t) \in I^2$ a vector space $A(X) = A\Big(s,t\Big) = {}_sA_t$ (possibly the zero vector space),

- for each X, Y with $X_r = Y_l$ a multiplication map

$$\mu(X,Y) : A(X) \otimes A(Y) \to A(X \cdot Y), \qquad a \otimes b \mapsto ab$$

- and elements $\mathbf{1}(s) = \mathbf{1}_s \in A\Big(s,s\Big)$ (the units),

such that the obvious associativity and unit conditions are satisfied. The vector space $A(X \cdot Y)$ is defined to be $\{0\}$ when $X_r \neq Y_l$; then $\mu(X,Y)$ is let to be the zero map.

The *total algebra* A of an I–partial algebra $\underline{A}$ is the vector space

$$A = \bigoplus_{X \in I^2} A(X)$$

endowed with the unique multiplication whose restriction to $A(X) \otimes A(Y)$ coincides with $\mu(X,Y)$.

Regard now the elements of I^2 as column vectors and denote by $*$ the (vertical) composition. Now we say that $X = \binom{s}{t} \in I^2$ has source s and target t and we denote them by X_u and X_d respectively. If $X = \binom{s}{t}, Y = \binom{t}{u} \in I^2$ then $X * Y = \binom{s}{u}$. An *I–partial coalgebra* (or *partial coalgebra over I*) $\underline{A} = (\underline{A}, \Delta)$ (over $\mathbb{C}$) consists of a set I together with

- for each $X = \binom{s}{t} \in I^2$ a vector space $A(X) = A\binom{s}{t} = A^s_t$,

- for each X, Y with $X_d = Y_u$ a comultiplication map

$$\Delta\binom{X}{Y} : A(X * Y) \to A(X) \otimes A(Y), \qquad a \mapsto a_{(1)X} \otimes a_{(2)Y},$$

- and counit maps $\epsilon_s : A\binom{s}{s} \to \mathbb{C}$,

obeying the obvious coassociativity and counitality conditions. Analogously to the above, we make the convention that $A(X * Y) = \{0\}$ and $\Delta\binom{X}{Y}$ is the zero map when

$X_d \neq Y_u$. Similarly, a counit map is seen as the zero functional on $A(X)$ when $X = \binom{s}{t}$ with $s \neq t$. For brevity, if $X = \binom{s}{t}$ and $Y = \binom{t}{u}$, we write Δ_t instead of $\Delta\binom{X}{Y}$, as the other indices are determined by the element to which Δ_t is applied. For similar reason, we drop the index from ϵ_s and simply write ϵ.

Let $M_2(I)$ be the set of 4–tuples of elements of I arranged as 2×2–matrices. It can be endowed with two compositions, namely $\cdot$ (viewing $M_2(I)$ as a row vector of column vectors) and $*$ (viewing $M_2(I)$ as a column vector of row vectors). For any $X \in M_2(I)$, we write $X = \left(X_l, X_r\right) = \binom{X_u}{X_d} = \begin{pmatrix} X_{lu} & X_{ru} \\ X_{ld} & X_{rd} \end{pmatrix}$.

Note that, in what follows, the index of Δ will now be a 1×2 vector in I^2 as we will deal with partial coalgebras over I^2. A *partial bialgebra* $\underline{A} = (\underline{A}, \mu, \Delta)$ consists of a set I and a collection of vector spaces $A(X)$ for $X \in M_2(I)$ such that

- $A\left(X_l, X_r\right)$ constitute an I^2–partial algebra,
- $A\binom{X_u}{X_d}$ constitute an I^2–partial coalgebra,

and on which the following compatibility conditions hold.

(1) For all $s, t, t', u \in I$, one has

$$\Delta_{t,t'}(\mathbf{1}\binom{s}{u}) = \delta_{t,t'}\mathbf{1}\binom{s}{t} \otimes \mathbf{1}\binom{t}{u}.$$

(2) For all $X, Y \in M_2(I)$ with $X_r = Y_l$ and all $a \in A(X)$ and $b \in A(Y)$,

$$\epsilon(ab) = \epsilon(a)\epsilon(b).$$

(3) For all $s \in I$, $\epsilon(\mathbf{1}\binom{s}{s}) = 1$.

(4) For each $X \in M_2(I)$ and each $a \in A(X)$, the assignment $(p,q) \to \Delta_{(p,q)}(a)$ has finite support in either one of the variables when the other variable has been fixed.

(5) For all $a \in A(X)$ and $b \in A(Y)$ with $X_r = Y_l$,

$$\Delta_{(p,q)}(ab) = \sum_m \Delta_{(p,m)}(a)\Delta_{(m,q)}(b).$$

As a consequence of all the above information (see [33, Lemma 1.18]), it follows that for each element a in the total algebra A of an I–partial bialgebra $\underline{A}$, there exists a unique multiplier $\Delta(a) \in \mathbb{M}(A \otimes A)$ such that

$$\Delta_{pq}(a) = (1 \otimes \lambda_p)\Delta(a)(1 \otimes \lambda_q) = (\rho_p \otimes 1)\Delta(a)(\rho_q \otimes 1) \tag{4.19}$$

for all $p, q \in I$, all $X \in M_2(I)$ and all $a \in A(X)$, where $\lambda_k = \sum_l \mathbf{1}\binom{k}{l}$, $\rho_l = \sum_k \mathbf{1}\binom{k}{l} \in \mathbb{M}(A)$. The resulting map $\Delta : A \to M(A \otimes A)$, $a \mapsto \Delta(a)$ is a homomorphism. Moreover, the element $E = \sum_{s,t,u} \mathbf{1}\binom{s}{t} \otimes \mathbf{1}\binom{t}{u} = \sum_t \rho_t \otimes \lambda_t$ is a well-defined idempotent in $\mathbb{M}(A \otimes A)$, and it satisfies $\Delta(A)(A \otimes A) = E(A \otimes A)$ and $(A \otimes A)\Delta(A) = (A \otimes A)E$ ([33, Lemma 1.19]). Proposition 1.20 and Remark 1.21 in [33] prove that the total algebra of $\underline{A}$ is, with this structure, a regular weak multiplier bialgebra.

4.2 The base algebras

Let A be a weak multiplier bialgebra over a field. The aim of this section is to study the properties of the maps

$$\overline{\sqcap}^L : A \to \mathbb{M}(A), \qquad a \mapsto (\epsilon \otimes \mathsf{id})((a \otimes 1)E) \tag{4.20}$$

in (4.4) and

$$\overline{\sqcap}^R : A \to \mathbb{M}(A), \qquad a \mapsto (\mathsf{id} \otimes \epsilon)(E(1 \otimes a)) \tag{4.21}$$

in (4.8) in a remarkable analogy with the unital case. Their images in $\mathbb{M}(A)$ are termed as the *base algebras* (they are indeed subalgebras of $\mathbb{M}(A)$ by Lemma 4.2.6 below), and they will be investigated further in the next Section 4.3.

The following two examples explicitly show the base algebras of some families of weak multiplier bialgebras.

Example 4.2.1. For a small category C, consider the weak multiplier bialgebra $k\mathsf{C}$ presented in Example 4.1.14. The base algebra $\sqcap^R(k\mathsf{C})$ is easily seen to be equal to $k\mathsf{C}_0$. Indeed, for any $a, b \in \mathsf{C}_1$ such that $s(a) = s(b)$,

$$b \sqcap^R (a) \overset{(4.23)}{=} (\mathsf{id} \otimes \epsilon)((b \otimes a)E) = b\epsilon(a) = b;$$

and for any $a, b \in \mathsf{C}_1$ such that $s(a) \neq s(b)$, $b \sqcap^R (a) = 0$. That is, $\sqcap^R(a) = s(a)$ for any

$a \in \mathsf{C}_1$, from which we conclude $\sqcap^R(k\mathsf{C}) = k\mathsf{C}_0$. A symmetrical computation checks that also $\sqcap^L(k\mathsf{C}) = k\mathsf{C}_0$.

Example 4.2.2. Let C be a small category such that for any arrows a and b of common source, there are only finitely many arrows c such that $ca = b$; and that, for any arrows a and b of common target, there are only finitely many arrows c such that $ac = b$. Let $k(\mathsf{C})$ be the weak multiplier bialgebra shown in Example 4.1.15. The base algebra $\sqcap^R(k(\mathsf{C}))$ is described as follows. For any $g, h \in k(\mathsf{C})$,

$$\begin{aligned} g \sqcap^R(h) &\overset{(4.23)}{=} (\mathsf{id} \otimes \epsilon)((g \otimes h)E) = \sum_{\substack{c,d \in \mathsf{C}_1 \\ s(c)=t(d)}} g(c)h(d)\epsilon(\chi_{\{d\}})\chi_{\{c\}} \\ &= \sum_{\substack{c,d \in \mathsf{C}_1 \\ s(c)=t(d)}} g(c)h(d)\Big(\sum_{x \in \mathsf{C}_0} \chi_{\{d\}}(1_x)\Big)\chi_{\{c\}} = \sum_{c \in \mathsf{C}_1} g(c)h(s(c))\chi_{\{c\}} \\ &= g(-)h(s(-)). \end{aligned}$$

(Notice that above, in the last expression, juxtaposition means pointwise multiplication.) Hence, $\sqcap^R(k(\mathsf{C}))$ consists of the functions $f : \mathsf{C}_1 \to k$ such that $f(c) = f(d)$ if $s(c) = s(d)$, for any $c, d \in \mathsf{C}_1$. In other words, $\sqcap^R(k(\mathsf{C})) \cong k(\mathsf{C}_0)$. The map $\sqcap^R(k(\mathsf{C})) \to k(\mathsf{C}_0)$ is given by restriction, sending $f \in \sqcap^R(k(\mathsf{C}))$ to $f_{|\mathsf{C}_0}$; and its inverse $k(\mathsf{C}_0) \to \sqcap^R(k(\mathsf{C}))$, by precomposition with the source map, taking $f \in k(\mathsf{C}_0)$ to $f(s(-))$. Analogously, $\sqcap^L(k(\mathsf{C}))$ consists of the functions $f : \mathsf{C}_1 \to k$ such that $f(c) = f(d)$ if $t(c) = t(d)$, for any $c, d \in \mathsf{C}_1$; that is, $\sqcap^L(k(\mathsf{C})) \cong k(\mathsf{C}_0)$, via the isomorphism given by $\sqcap^L(k(\mathsf{C})) \to k(\mathsf{C}_0), f \mapsto f_{|\mathsf{C}_0}$ and $k(\mathsf{C}_0) \to \sqcap^L(k(\mathsf{C})), f \mapsto f(t(-))$.

Lemma 4.2.3. *For any weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\epsilon(\overline{\sqcap}^L(a)b) = \epsilon(ab) \qquad \textit{and} \qquad \epsilon(a\overline{\sqcap}^R(b)) = \epsilon(ab).$$

Proof. For any $a, b \in A$,

$$\begin{aligned} \epsilon(\overline{\sqcap}^L(a)b) &\overset{(4.5)}{=} (\epsilon \otimes \epsilon)T_2(a \otimes b) \overset{(iii)}{=} \epsilon(ab) \\ \epsilon(a\overline{\sqcap}^R(b)) &\overset{(4.10)}{=} (\epsilon \otimes \epsilon)T_1(a \otimes b) \overset{(iii)}{=} \epsilon(ab). \end{aligned}$$

□

Lemma 4.2.4. *For any weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\overline{\sqcap}^L(\overline{\sqcap}^L(a)b) = \overline{\sqcap}^L(ab) \qquad \textit{and} \qquad \overline{\sqcap}^R(a\overline{\sqcap}^R(b)) = \overline{\sqcap}^R(ab).$$

Proof. For any $a, b \in A$, using Lemma 4.2.3 together with part (2) in Proposition 4.1.8 and part (3) in Proposition 4.1.9 in the second equalities,

$$\begin{aligned}
\overline{\sqcap}^L(\overline{\sqcap}^L(a)b) &\overset{(4.20)}{=} (\epsilon \otimes \mathsf{id})[(\overline{\sqcap}^L(a)b \otimes 1)E] = (\epsilon \otimes \mathsf{id})[(ab \otimes 1)E] \overset{(4.20)}{=} \overline{\sqcap}^L(ab),\\
\overline{\sqcap}^R(a\overline{\sqcap}^R(b)) &\overset{(4.21)}{=} (\mathsf{id} \otimes \epsilon)(E(1 \otimes a\overline{\sqcap}^R(b))) = (\mathsf{id} \otimes \epsilon)(E(1 \otimes ab)) \overset{(4.21)}{=} \overline{\sqcap}^R(ab).
\end{aligned}$$

□

Lemma 4.2.5. *For any weak multiplier bialgebra A over a field, and any $a \in A$,*

$$\begin{aligned}
\overline{\Delta}\overline{\sqcap}^L(a) &= (\overline{\sqcap}^L(a) \otimes 1)E &&= E(\overline{\sqcap}^L(a) \otimes 1) \qquad \textit{and}\\
\overline{\Delta}\overline{\sqcap}^R(a) &= (1 \otimes \overline{\sqcap}^R(a))E &&= E(1 \otimes \overline{\sqcap}^R(a)).
\end{aligned}$$

Proof. For any $a \in A$, $(a \otimes 1)E \in A \otimes \mathbb{M}(A)$ by Proposition 4.1.8 (2). Hence, using additionally the idempotency of E in the fifth equalities,

$$\begin{aligned}
\overline{\Delta}\overline{\sqcap}^L(a) &\overset{(4.20)}{=} \overline{\Delta}(\epsilon \otimes \mathsf{id})[(a \otimes 1)E] = (\epsilon \otimes \mathsf{id})(\mathsf{id} \otimes \overline{\Delta})[(a \otimes 1)E]\\
&\overset{(2.20)}{=} (\epsilon \otimes \mathsf{id})[(\mathsf{id} \otimes \overline{\Delta})(a \otimes 1)(\overline{\mathsf{id} \otimes \Delta})(E)]\\
&\overset{(2.113)(v)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes 1 \otimes 1)(1 \otimes E)(1 \otimes E)(E \otimes 1)]\\
&\overset{(v)}{=} (\epsilon \otimes \mathsf{id})[(a \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes E)]\\
&= ((\epsilon \otimes \mathsf{id})[(a \otimes 1)E] \otimes 1)E \overset{(4.20)}{=} (\overline{\sqcap}^L(a) \otimes 1)E,
\end{aligned}$$

$$\begin{aligned}
\overline{\Delta}\overline{\sqcap}^R(a) &\overset{(4.21)}{=} \overline{\Delta}(\mathsf{id} \otimes \epsilon)[E(1 \otimes a)] = (\mathsf{id} \otimes \epsilon)(\overline{\Delta \otimes \mathsf{id}})[E(1 \otimes a)]\\
&\overset{(2.20)}{=} (\mathsf{id} \otimes \epsilon)[(\overline{\Delta \otimes \mathsf{id}})(E)(\overline{\Delta} \otimes \mathsf{id})(1 \otimes a)]\\
&\overset{(2.113)(v)}{=} (\mathsf{id} \otimes \epsilon)[(1 \otimes E)(E \otimes 1)(E \otimes 1)(1 \otimes 1 \otimes a)]\\
&\overset{(v)}{=} (\mathsf{id} \otimes \epsilon)[(E \otimes 1)(1 \otimes E)(1 \otimes 1 \otimes a)]\\
&= E(1 \otimes (\mathsf{id} \otimes \epsilon)[E(1 \otimes a)]) = E(1 \otimes \overline{\sqcap}^R(a)).
\end{aligned}$$

For any $a, c, d \in A$, using the notations $(c \otimes d)E =: c_i \otimes d_i$, $E(c \otimes d) =: c'_j \otimes d'_j$,

$$\begin{aligned}
(c \otimes d)(\overline{\sqcap}^L(a) \otimes 1)E &\overset{(4.6)}{=} [(\epsilon \otimes \mathsf{id})((a \otimes c)E) \otimes d]E \\
&= (\epsilon \otimes \mathsf{id} \otimes \mathsf{id})[(a \otimes c \otimes d)(E \otimes 1)(1 \otimes E)] \\
&\overset{(v)}{=} (\epsilon \otimes \mathsf{id} \otimes \mathsf{id})[(a \otimes c \otimes d)(1 \otimes E)(E \otimes 1)] \\
&= (\epsilon \otimes \mathsf{id})((a \otimes c_i)E) \otimes d_i \\
&\overset{(4.6)}{=} c_i \overline{\sqcap}^L(a) \otimes d_i = (c \otimes d)E(\overline{\sqcap}^L(a) \otimes 1),
\end{aligned}$$

$$\begin{aligned}
E(1 \otimes \overline{\sqcap}^R(a))(c \otimes d) &\overset{(4.9)}{=} E[c \otimes (\mathsf{id} \otimes \epsilon)(E(d \otimes a))] \\
&= (\mathsf{id} \otimes \mathsf{id} \otimes \epsilon)[(E \otimes 1)(1 \otimes E)(c \otimes d \otimes a)] \\
&\overset{(v)}{=} (\mathsf{id} \otimes \mathsf{id} \otimes \epsilon)[(1 \otimes E)(E \otimes 1)(c \otimes d \otimes a)] \\
&= (\mathsf{id} \otimes \mathsf{id} \otimes \epsilon)(c'_j \otimes E(d'_j \otimes a)) \\
&\overset{(4.9)}{=} c'_j \otimes \overline{\sqcap}^R(a) d'_j = (1 \otimes \overline{\sqcap}^R(a))E(c \otimes d).
\end{aligned}$$

□

Lemma 4.2.6. *For any weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\overline{\sqcap}^L(a\overline{\sqcap}^L(b)) = \overline{\sqcap}^L(a)\overline{\sqcap}^L(b) \qquad \textit{and} \qquad \overline{\sqcap}^R(\overline{\sqcap}^R(a)b) = \overline{\sqcap}^R(a)\overline{\sqcap}^R(b).$$

Proof. For any $a, b, c \in A$,

$$\begin{aligned}
\overline{\sqcap}^L(a\overline{\sqcap}^L(b))c &\overset{(4.5)}{=} (\epsilon \otimes \mathsf{id})T_2(a\overline{\sqcap}^L(b) \otimes c) \overset{(4.1)(4.2)}{=} (\epsilon \otimes \mathsf{id})[(a\overline{\sqcap}^L(b) \otimes 1)E\Delta(c)] \\
&= (\epsilon \otimes \mathsf{id})[(a \otimes 1)\Delta(\overline{\sqcap}^L(b)c)] \overset{(4.1)}{=} (\epsilon \otimes \mathsf{id})T_2(a \otimes \overline{\sqcap}^L(b)c) \\
&\overset{(4.5)}{=} \overline{\sqcap}^L(a)\overline{\sqcap}^L(b)c,
\end{aligned}$$

$$\begin{aligned}
c\overline{\sqcap}^R(\overline{\sqcap}^R(a)b) &\overset{(4.10)}{=} (\mathsf{id} \otimes \epsilon)T_1(c \otimes \overline{\sqcap}^R(a)b) \overset{(4.1)(4.2)}{=} (\mathsf{id} \otimes \epsilon)(\Delta(c)E(1 \otimes \overline{\sqcap}^R(a)b)) \\
&= (\mathsf{id} \otimes \epsilon)(\Delta(c\overline{\sqcap}^R(a))(1 \otimes b)) \overset{(4.1)}{=} (\mathsf{id} \otimes \epsilon)T_1(c\overline{\sqcap}^R(a) \otimes b) \overset{(4.10)}{=} \\
&= c\overline{\sqcap}^R(a)\overline{\sqcap}^R(b),
\end{aligned}$$

where in the third equalities we used Lemma 4.2.5 and the multiplicativity of $\overline{\Delta}$. By the density of A in $\mathbb{M}(A)$, we conclude the claim. □

As an immediate consequence of the previous lemma, for any weak multiplier bialgebra A, the ranges of $\overline{\sqcap}^L$ and of $\overline{\sqcap}^R$ are non-unital subalgebras of the multiplier algebra $\mathbb{M}(A)$.

Lemma 4.2.7. *For any weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\overline{\sqcap}^R(a)\overline{\sqcap}^L(b) = \overline{\sqcap}^L(b)\overline{\sqcap}^R(a).$$

Proof. For any $a, b \in A$,

$$\begin{aligned}
\overline{\sqcap}^R(a)\overline{\sqcap}^L(b) &\overset{(4.21)(4.20)}{=} (\mathsf{id} \otimes \epsilon)[E(1 \otimes a)](\epsilon \otimes \mathsf{id})[(b \otimes 1)E] \\
&= (\epsilon \otimes \mathsf{id} \otimes \epsilon)[(1 \otimes E)(b \otimes 1 \otimes a)(E \otimes 1)] \\
&= (\epsilon \otimes \mathsf{id} \otimes \epsilon)[(b \otimes 1 \otimes 1)(1 \otimes E)(E \otimes 1)(1 \otimes 1 \otimes a)] \\
&\overset{(v)}{=} (\epsilon \otimes \mathsf{id} \otimes \epsilon)[(b \otimes 1 \otimes 1)(E \otimes 1)(1 \otimes E)(1 \otimes 1 \otimes a)] \\
&= (\epsilon \otimes \mathsf{id})[(b \otimes 1)E](\mathsf{id} \otimes \epsilon)[E(1 \otimes a)] \\
&\overset{(4.20)(4.21)}{=} \overline{\sqcap}^L(b)\overline{\sqcap}^R(a).
\end{aligned}$$

□

Lemma 4.2.8. *For any weak multiplier bialgebra A over a field, and for any $a, b, c, d \in A$,*

$$(ab \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_1(c \otimes d)) = ((\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b))(1 \otimes cd).$$

Proof. Both expressions in the claim are equal to $(ab \otimes 1)E(1 \otimes cd)$, see Proposition 4.1.8 (1) and Proposition 4.1.9 (2). □

Whenever A is a *regular* weak multiplier bialgebra over a field, the above considerations can be repeated in the opposite weak multiplier bialgebra algebra A^{op}. That is, for any $a \in A$ we can define multipliers $\sqcap^R(a), \sqcap^L(a) \in \mathbb{M}(A)$ by

$$\sqcap^R(a)b := (\mathsf{id} \otimes \epsilon)T_3(b \otimes a) \qquad (4.22) \qquad \text{and} \qquad b\sqcap^R(a) := (\mathsf{id} \otimes \epsilon)((b \otimes a)E) \qquad (4.23)$$

$$\sqcap^L(a)b := (\epsilon \otimes \mathsf{id})(E(a \otimes b)) \qquad (4.24) \qquad \text{and} \qquad b\sqcap^L(a) := (\epsilon \otimes \mathsf{id})T_4(a \otimes b) \qquad (4.25)$$

for any $b \in A$. They obey the following properties, for all $a, b, c, d \in A$.

$$(1 \otimes ab)E = (\sqcap^R \otimes \mathsf{id})T_3(b \otimes a) \text{ and } E(ab \otimes 1) = (\mathsf{id} \otimes \sqcap^L)T_4(b \otimes a). \quad (4.26)$$

$$\epsilon(a \sqcap^L (b)) = \epsilon(ab) \text{ and } \epsilon(\sqcap^R(a)b) = \epsilon(ab). \quad (4.27)$$

$$\sqcap^L(a \sqcap^L (b)) = \sqcap^L(ab) \text{ and } \sqcap^R (\sqcap^R(a)b) = \sqcap^R(ab). \quad (4.28)$$

$$\overline{\Delta} \sqcap^L (a) = (\sqcap^L(a) \otimes 1)E = E(\sqcap^L(a) \otimes 1) \text{ and} \quad (4.29)$$

$$\overline{\Delta} \sqcap^R (a) = (1 \otimes \sqcap^R(a))E = E(1 \otimes \sqcap^R(a)).$$

$$\sqcap^L(\sqcap^L(a)b) = \sqcap^L(a) \sqcap^L (b) \text{ and } \sqcap^R (a \sqcap^R (b)) = \sqcap^R(a) \sqcap^R (b). \quad (4.30)$$

$$\sqcap^L(a) \sqcap^R (b) = \sqcap^R(b) \sqcap^L (a). \quad (4.31)$$

$$((\sqcap^R \otimes \mathsf{id})T_3(a \otimes b))(cd \otimes 1) = (1 \otimes ba)((\mathsf{id} \otimes \sqcap^L)T_4(d \otimes c)). \quad (4.32)$$

The following table collects the expressions of the generalized counital maps in each one of the regular weak multiplier bialgebras $A^{\mathsf{op}}, A_{\mathsf{cop}}$ and $A^{\mathsf{op}}_{\mathsf{cop}}$ in terms of a regular weak multiplier bialgebra A.

A $(A, \mu, \Delta, \epsilon, E)$	A^{op} $(A, \mu^{\mathsf{op}}, \Delta, \epsilon, E)$	A_{cop} $(A, \mu, \Delta^{\mathsf{op}}, \epsilon, \mathsf{tw}E\mathsf{tw})$	$A^{\mathsf{op}}_{\mathsf{cop}}$ $(A, \mu^{\mathsf{op}}, \Delta^{\mathsf{op}}, \epsilon, \mathsf{tw}E\mathsf{tw})$
$\overline{\sqcap}^R$	$\sqcap^R$	$\sqcap^L$	$\overline{\sqcap}^L$
$\overline{\sqcap}^L$	$\sqcap^L$	$\sqcap^R$	$\overline{\sqcap}^R$
$\sqcap^R$	$\overline{\sqcap}^R$	$\overline{\sqcap}^L$	$\sqcap^L$
$\sqcap^L$	$\overline{\sqcap}^L$	$\overline{\sqcap}^R$	$\sqcap^R$

Table 4.2: Generalized counital maps in the symmetric weak multiplier bialgebras.

In a weak bialgebra —weak multiplier bialgebra possessing a unit, by Theorem 4.1.3—, the above maps (which turn out to be precisely the counital maps (2.50)-(2.53) defined in Subsection 2.5.2) behave as generalized counits: $\mu(\sqcap^L \otimes \mathsf{id})\Delta = \mathsf{id} = \mu(\mathsf{id} \otimes \sqcap^R)\Delta$ and $\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})\Delta = \mathsf{id} = \mu^{\mathsf{op}}(\mathsf{id} \otimes \overline{\sqcap}^R)\Delta$ (see Proposition 2.5.7). In the following lemma these identities are generalized to weak multiplier bialgebras.

Lemma 4.2.9. *For a regular weak multiplier bialgebra A over a field, the following equalities hold.*

(1) $\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})T_3 = \mu^{\mathsf{op}}$.

(2) $\mu^{\mathsf{op}}(\mathsf{id} \otimes \overline{\sqcap}^R)T_4 = \mu^{\mathsf{op}}$.

(3) $\mu(\sqcap^L \otimes \mathsf{id})T_1 = \mu$.

(4) $\mu(\mathsf{id} \otimes \sqcap^R)T_2 = \mu$.

Proof. We spell out the proof only for (1), all other assertions follow by the symmetries shown in Table 4.2. Let $a, b, c \in A$ and put $a_i \otimes b_i := T_3(a \otimes b) = (1 \otimes b)\Delta(a)$. Then,

$$\begin{aligned}(\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})T_3(a \otimes b))c &= \mu^{\mathsf{op}}[((\overline{\sqcap}^L \otimes \mathsf{id})T_3(a \otimes b))(c \otimes 1)] \\ &= \mu^{\mathsf{op}}(\overline{\sqcap}^L(a_i)c \otimes b_i) \overset{(4.5)}{=} b_i(\epsilon \otimes \mathsf{id})[(a_i \otimes 1)\Delta(c)] \\ &= (\epsilon \otimes \mathsf{id})[(a_i \otimes b_i)\Delta(c)] = (\epsilon \otimes \mathsf{id})[(1 \otimes b)\Delta(a)\Delta(c)] \\ &\underset{(4.3)}{\overset{(2.20)}{=}} (\epsilon \otimes \mathsf{id})T_3(ac \otimes b) \overset{(\mathrm{iii})}{=} bac,\end{aligned}$$

so we conclude by non-degeneracy of the multiplication. □

Remark 4.2.10. (See [10, Lemma 1.5].) For any regular weak multiplier bialgebra A, it follows by the idempotency of A and Lemma 4.2.9, that the vector space A is spanned by elements of the form

(1) $a\overline{\sqcap}^L(b)$ for $a, b \in A$; (2) $\overline{\sqcap}^R(b)a$ for $a, b \in A$;

(3) $\sqcap^L(b)a$ for $a, b \in A$; (4) $a \sqcap^R(b)$ for $a, b \in A$.

Using this together with Lemma 4.2.6 and (4.30), we conclude that $\overline{\sqcap}^L(A), \overline{\sqcap}^R(A), \sqcap^L(A)$ and $\sqcap^R(A)$ are idempotent non-unital algebras.

Lemma 4.2.11. *For a regular weak multiplier bialgebra A over a field, and any $a, b \in A$,*

$$\epsilon(\sqcap^L(a)b) = \epsilon(\overline{\sqcap}^R(b)a) \quad \textit{and} \quad \epsilon(a \sqcap^R(b)) = \epsilon(b\overline{\sqcap}^L(a)).$$

Proof. In light of (4.24) and (4.9), respectively, left and right sides of the first equality are equal to $(\epsilon \otimes \epsilon)[E(a \otimes b)]$ hence also to each other. The second equality follows by the first one and the symmetry $A - A^{\mathsf{op}}$ (or $A - A^{\mathsf{op}}_{\mathsf{cop}}$ on equal footing). □

Lemma 4.2.12. *For a regular weak multiplier bialgebra A over a field, and any $a \in A$, the following equalities hold.*

$$(\sqcap^R(a) \otimes 1)E = (1 \otimes \overline{\sqcap}^L(a))E \qquad \textit{and} \qquad E(\overline{\sqcap}^R(a) \otimes 1) = E(1 \otimes \sqcap^L(a)).$$

Proof. By axiom (iv) in Definition 4.1.1, the first equality in the claim is equivalent to

$$(\sqcap^R(a)\otimes 1)T_1(b\otimes c) = (1\otimes \overline{\sqcap}^L(a))T_1(b\otimes c), \quad \forall a,b,c\in A. \tag{4.33}$$

Using the identities

$$\begin{aligned} c\sqcap^R(a)b &\overset{(4.22)}{=} (\mathsf{id}\otimes\epsilon)[(c\otimes a)\Delta(b)] \\ &\overset{(4.1)}{=} (\mathsf{id}\otimes\epsilon)[(1\otimes a)T_2(c\otimes b)] \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} (\epsilon\otimes\mathsf{id})[(a\otimes 1)T_1(b\otimes c)] &\overset{(4.1)}{=} (\epsilon\otimes\mathsf{id})[T_2(a\otimes b)(1\otimes c)] \\ &\overset{(4.4)}{=} \overline{\sqcap}^L(a)bc \end{aligned} \tag{4.35}$$

(for any $a,b,c\in A$) in the first and the fourth equalities, respectively, one computes

$$\begin{aligned} (d\sqcap^R(a)\otimes 1)T_1(b\otimes c) &\overset{(4.34)}{=} (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})[(1\otimes a\otimes 1)((T_2\otimes\mathsf{id})(\mathsf{id}\otimes T_1)(d\otimes b\otimes c))] \\ &\overset{(ii)}{=} (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})[(1\otimes a\otimes 1)((\mathsf{id}\otimes T_1)(T_2\otimes\mathsf{id})(d\otimes b\otimes c))] \\ &\overset{(4.1)}{=} (\mathsf{id}\otimes\epsilon\otimes\mathsf{id})[(1\otimes a\otimes 1)((\mathsf{id}\otimes T_1)((d\otimes 1)\Delta(b)\otimes c))] \\ &\overset{(4.35)}{=} (1\otimes\overline{\sqcap}^L(a))(d\otimes 1)\Delta(b)(1\otimes c) \\ &\overset{(4.1)}{=} (d\otimes\overline{\sqcap}^L(a))T_1(b\otimes c), \end{aligned}$$

for all $a,b,c,d\in A$. So we conclude (4.33) by the non-degeneracy of the multiplication, and hence $(\sqcap^R(a)\otimes 1)E = (1\otimes\overline{\sqcap}^L(a))E$ holds for all $a\in A$. The other equality follows by the proven identity and the symmetry $A - A^{\mathsf{op}}$. □

Lemma 4.2.13. *For any regular weak multiplier bialgebra A over a field, and for any $a,b,c,d\in A$,*

$$\begin{aligned} &(1\otimes ab)((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(c\otimes d)) = ((\sqcap^R\otimes\mathsf{id})T_2^{\mathsf{op}}(a\otimes b))(dc\otimes 1) \quad \textit{and} \\ &((\overline{\sqcap}^L\otimes\mathsf{id})T_3(a\otimes b))(cd\otimes 1) = (1\otimes ba)((\mathsf{id}\otimes\sqcap^L)T_1^{\mathsf{op}}(c\otimes d)), \end{aligned}$$

where $T_1^{\mathsf{op}} = \mathsf{tw}\,T_1$ and $T_2^{\mathsf{op}} = \mathsf{tw}\,T_2$.

Proof. For any $a, b, c, d \in A$,

$$\begin{aligned}(1 \otimes 1 \otimes a)((\mathsf{id} \otimes T_1^{\mathsf{op}}\,\mathsf{tw})\ (T_4 \otimes \mathsf{id})(c \otimes d \otimes b)) &\\ \overset{(4.3)}{=} &\ (1 \otimes 1 \otimes a)(\mathsf{id} \otimes T_1^{\mathsf{op}}\mathsf{tw})(\Delta(d)(c \otimes 1) \otimes b)\\ \overset{(4.1)}{=} &\ (1 \otimes 1 \otimes a)(1 \otimes \Delta^{\mathsf{op}}(b))(\Delta(d)(c \otimes 1) \otimes 1)\\ \overset{(4.1)}{=} &\ (1 \otimes T_2^{\mathsf{op}}(a \otimes b))(\Delta(d) \otimes 1)(c \otimes 1 \otimes 1)\\ \overset{(4.3)}{=} &\ ((T_3 \otimes \mathsf{id})(\mathsf{id} \otimes T_2^{\mathsf{op}})(d \otimes a \otimes b))(c \otimes 1 \otimes 1).\end{aligned}$$

Applying $\mathsf{id} \otimes \epsilon \otimes \mathsf{id}$ to both sides, and using the identities (4.8) and (4.22), we obtain the first equality in the claim. The second equality follows by the proven identity and the symmetry $A - A^{\mathsf{op}}_{\mathsf{cop}}$. □

Lemma 4.2.14. *Let A be a regular weak multiplier bialgebra over a field. For any $a, b \in A$,*

$$\sqcap^R(a\overline{\sqcap}^R(b)) = \sqcap^R(a)\overline{\sqcap}^R(b) = \overline{\sqcap}^R(\sqcap^R(a)b) \quad \textit{and}$$
$$\sqcap^L(\overline{\sqcap}^L(a)b) = \overline{\sqcap}^L(a) \sqcap^L (b) = \overline{\sqcap}^L(a \sqcap^L (b)).$$

Proof. The second equality follows by the first one and the symmetry $A - A^{\mathsf{op}}$; the third and fourth equalities follow by the first and second ones and the symmetry $A - A^{\mathsf{op}}_{\mathsf{cop}}$. So let us prove the first identity in the claim. Applying the multiplicativity of $\overline{\Delta} : \mathbb{M}(A) \to \mathbb{M}(A \otimes A)$, Lemma 4.2.5 and (4.2) in the second equality,

$$T_3(\overline{\sqcap}^R(a)b \otimes c) \overset{(4.3)}{=} (1 \otimes c)\Delta(\overline{\sqcap}^R(a)b) = (1 \otimes c\overline{\sqcap}^R(a))\Delta(b) \overset{(4.3)}{=} T_3(b \otimes c\overline{\sqcap}^R(a)),$$

for any $a, b, c \in A$. Using the above identity in the second equality,

$$\sqcap^R(a\overline{\sqcap}^R(b))c \overset{(4.22)}{=} (\mathsf{id} \otimes \epsilon)T_3(c \otimes a\overline{\sqcap}^R(b)) = (\mathsf{id} \otimes \epsilon)T_3(\overline{\sqcap}^R(b)c \otimes a) \overset{(4.22)}{=} \sqcap^R(a)\overline{\sqcap}^R(b)c.$$

□

Lemma 4.2.15. *Let A be a regular weak multiplier bialgebra over a field. For any $a, b \in A$, the following hold.*

$$\sqcap^R(a\overline{\sqcap}^L(b)) = \sqcap^R(b) \sqcap^R (a) \qquad \sqcap^L(\overline{\sqcap}^R(a)b) = \sqcap^L(b) \sqcap^L (a)$$
$$\overline{\sqcap}^R(\sqcap^L(b)a) = \overline{\sqcap}^R(a)\overline{\sqcap}^R(b) \qquad \overline{\sqcap}^L(b \sqcap^R (a)) = \overline{\sqcap}^L(a)\overline{\sqcap}^L(b).$$

Proof. We only prove the first assertion explicitly. The second one follows by it and the symmetry $A - A^{\mathsf{op}}_{\mathsf{cop}}$; the third and fourth equalities in the claim follow by the first and second ones and the symmetry $A - A^{\mathsf{op}}$. For any $a, b, c \in A$,

$$\begin{aligned}
\sqcap^R(a\overline{\sqcap}^L(b))c &\overset{(4.22)}{=} (\mathsf{id}\otimes\epsilon)T_3(c\otimes a\overline{\sqcap}^L(b)) \overset{(4.3)}{=} (\mathsf{id}\otimes\epsilon)[(1\otimes a\overline{\sqcap}^L(b))\Delta(c)] \\
&= (\mathsf{id}\otimes\epsilon)[(\sqcap^R(b)\otimes a)\Delta(c)] \overset{(4.3)}{=} \sqcap^R(b)(\mathsf{id}\otimes\epsilon)T_3(c\otimes a) \\
&\overset{(4.22)}{=} \sqcap^R(b)\sqcap^R(a)c,
\end{aligned}$$

where the third equality follows by (4.2) and Lemma 4.2.12. So we conclude by the density of A in $\mathbb{M}(A)$. □

The following theorem introduces the important notions of *right* and *left full* comultiplication. Roughly speaking, they mean that the respective 'legs' of the comultiplication of a regular weak multiplier bialgebra A are all of A.

Theorem 4.2.1. *For a regular weak multiplier bialgebra A over a field k, the following assertions are equivalent to each other.*

(1) The comultiplication is right full *in the sense that*

$$\langle(\mathsf{id}\otimes\omega)T_1(a\otimes b)\mid a,b\in A,\omega\in\mathsf{Lin}(A,k)\rangle = A.$$

(2) The comultiplication is right full *in the sense that*

$$\langle(\mathsf{id}\otimes\omega)T_3(a\otimes b)\mid a,b\in A,\omega\in\mathsf{Lin}(A,k)\rangle = A.$$

(3) $\langle(\mathsf{id}\otimes\epsilon)T_1(a\otimes b)\mid a,b\in A\rangle = A.$

(4) $\langle(\mathsf{id}\otimes\epsilon)T_3(a\otimes b)\mid a,b\in A\rangle = A.$

(5) $\{\sqcap^R(a)\mid a\in A\} = \{\overline{\sqcap}^R(a)\mid a\in A\}.$

The following assertions are equivalent to each other, too.

(1)' The comultiplication is left full *in the sense that*

$$\langle(\omega\otimes\mathsf{id})T_2(a\otimes b)\mid a,b\in A,\omega\in\mathsf{Lin}(A,k)\rangle = A.$$

(2)' The comultiplication is left full *in the sense that*

$$\langle (\omega \otimes \mathsf{id}) T_4(a \otimes b) \mid a, b \in A, \omega \in \mathsf{Lin}(A,k) \rangle = A.$$

(3)' $\langle (\epsilon \otimes \mathsf{id}) T_2(a \otimes b) \mid a, b \in A \rangle = A.$

(4)' $\langle (\epsilon \otimes \mathsf{id}) T_4(a \otimes b) \mid a, b \in A \rangle = A.$

(5)' $\{\sqcap^L(a) \mid a \in A\} = \{\overline{\sqcap}^L(a) \mid a \in A\}.$

Proof. We only prove the equivalence of the first five assertions. The equivalence of the second quintuple follows symmetrically by applying the first one in $A^{\mathsf{op}}_{\mathsf{cop}}$.

(1)⇔(2) is proven in [72, Lemma 1.11]; we also prove it below. Reasoning by *reductio ad absurdum*, suppose that A is not spanned by the elements of the left hand side of the equality in (2). Then there exists $0 \neq \varphi \in \mathsf{Lin}(A,k)$ vanishing on all such elements; equivalently, $\omega(\varphi \otimes \mathsf{id}) T_3(a \otimes b) = 0$ for all $\omega \in \mathsf{Lin}(A,k)$ and all $a, b \in A$. Thus, $(\varphi \otimes \mathsf{id}) T_3(a \otimes b) = 0$ for all $a, b \in A$. Consequently, $(1 \otimes b) T_1(a \otimes c) = T_3(a \otimes b)(1 \otimes c) \in ker(\varphi) \otimes A$ for any $a, b, c \in A$. Hence, $T_1(a \otimes c) \in ker(\varphi) \otimes A$ for all $a, c \in A$, what contradicts (1). The converse is analogously proven.

(3)⇒(1) and (4)⇒(2) are trivial.

(1) and (2)⇒(5). For any $a, b, c \in A$,

$$\begin{aligned}(1 \otimes ab)E(1 \otimes c) &\overset{(4.26)}{=} ((\sqcap^R \otimes \mathsf{id}) T_3(b \otimes a))(1 \otimes c) \overset{(4.3)}{=} (\sqcap^R \otimes \mathsf{id})((1 \otimes a)\Delta(b)(1 \otimes c)) \\ &\overset{(4.1)}{=} (1 \otimes a)((\sqcap^R \otimes \mathsf{id}) T_1(b \otimes c)),\end{aligned}$$

$$\begin{aligned}(1 \otimes a)E(1 \otimes cd) &= (1 \otimes a)((\overline{\sqcap}^R \otimes \mathsf{id}) T_1(c \otimes d)) \overset{(4.1)}{=} (\overline{\sqcap}^R \otimes \mathsf{id})((1 \otimes a)\Delta(c)(1 \otimes d)) \\ &\overset{(4.3)}{=} ((\overline{\sqcap}^R \otimes \mathsf{id}) T_3(c \otimes a))(1 \otimes d),\end{aligned}$$

where in the first equality of the second chain of equalities we used Proposition 4.1.9 (2). By the non-degeneracy of the multiplication and the idempotency of A, it follows that

$$(\sqcap^R \otimes \mathsf{id}) T_1(a \otimes b) = (1 \otimes a)E(1 \otimes b) = (\overline{\sqcap}^R \otimes \mathsf{id}) T_3(b \otimes a)$$

for any $a, b \in A$. Then (1) implies

$$\{\sqcap^R(a) \mid a \in A\} = \langle (\mathsf{id} \otimes \omega)[(1 \otimes a)E(1 \otimes b)] \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle$$

and (2) implies

$$\{\overline{\sqcap}^R(a) \mid a \in A\} = \langle (\mathsf{id} \otimes \omega)[(1 \otimes a)E(1 \otimes b)] \mid a, b \in A, \omega \in \mathsf{Lin}(A, k)\rangle,$$

proving the claim.

(5)⇒(3). By Remark 4.2.10, $\langle a \sqcap^R (b) \mid a, b \in A\rangle = A$. Hence by (5),

$$A = \langle a\overline{\sqcap}^R(b) \mid a, b \in A\rangle \overset{(4.8)}{=} \langle (\mathsf{id} \otimes \epsilon)T_1(a \otimes b) \mid a, b \in A\rangle.$$

(5)⇒(4) follows by (5)⇒(3) and the symmetry $A - A^{\mathsf{op}}$. □

Note that, if a weak multiplier bialgebra A has a unit, then the comultiplication is automatically right and left full. This will also hold if A has no unit but the counit is multiplicative (that is, if A is a multiplier bialgebra, cf. Theorem 4.1.4). However, in a general (regular) weak multiplier bialgebra, we do not assume the existence of a unit neither the multiplicativity of the counit.

4.3 Firm separable Frobenius structure of the base algebras

In a weak bialgebra, the coinciding range of the maps $\overline{\sqcap}^L$ and $\sqcap^L$, and the coinciding range of $\overline{\sqcap}^R$ and $\sqcap^R$ in the previous section, carry the structures of anti-isomorphic *separable Frobenius algebras* (with units; see Theorem 2.5.1). The aim of this section is to see that in a regular weak multiplier bialgebra with a left and right full comultiplication, the base algebras still carry the structures of anti-isomorphic coseparable and co-Frobenius coalgebras. Consequently, they are firm Frobenius algebras in the sense of [12] (recalled in the preliminary Section 2.2).

It follows immediately from Lemma 4.2.6 that for any weak multiplier bialgebra A, the ranges of $\overline{\sqcap}^L$ and of $\overline{\sqcap}^R$ are non-unital subalgebras of the multiplier algebra $\mathbb{M}(A)$. We turn to proving that —whenever A is regular with a left and right full comultiplication— they carry coalgebra structures as well. First we look for the candidate counit.

Proposition 4.3.1. *Let A be a regular weak multiplier bialgebra over a field k with a right full comultiplication. Then the counit ϵ determines a linear map*

$$\varepsilon : \{\sqcap^R(a) \mid a \in A\} \to k, \qquad \sqcap^R(a) \mapsto \epsilon(a). \tag{4.36}$$

Proof. In order to see that ε is a well defined linear map, we need to show that $\sqcap^R(p) = 0$ implies $\epsilon(p) = 0$, for any $p \in A$. Since A is idempotent by assumption, we can write any element p of A as a linear combination $\sum_i a^i b^i$ in terms of finitely many elements $a^i, b^i \in A$. So omitting throughout the summation symbol for brevity, it is enough to prove that $\sqcap^R(a^i b^i) = 0$ implies $\epsilon(a^i b^i) = 0$, for any finite set of elements $a^i, b^i \in A$. If $\sqcap^R(a^i b^i) = 0$, then

$$\begin{aligned} 0 &\overset{(4.27)}{=} (\mathsf{id} \otimes \epsilon)[(1 \otimes a^i b^i)((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(h \otimes d))(1 \otimes c)] \\ &= (\epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes \sqcap^R)T_2(a^i \otimes b^i))(c \otimes dh)] \\ &= (\epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes \sqcap^R)T_2(a^i \otimes b^i))(\overline{\sqcap}^R(c) \otimes 1)]dh, \end{aligned}$$

for all $c, d, h \in A$. In the second equality above, we used the first statement in Lemma 4.2.13 and the third equality follows by Lemma 4.2.3. By Theorem 4.2.1, the vector subspaces $\sqcap^R(A) := \{\sqcap^R(a) \mid a \in A\}$ and $\overline{\sqcap}^R(A) := \{\overline{\sqcap}^R(a) \mid a \in A\}$ of $\mathbb{M}(A)$ coincide. So by the density of A in $\mathbb{M}(A)$, the map

$$\sqcap^R(A) \to \sqcap^R(A), \qquad \overline{\sqcap}^R(c) \mapsto (\epsilon \otimes \mathsf{id})[((\mathsf{id} \otimes \sqcap^R)T_2(a^i \otimes b^i))(\overline{\sqcap}^R(c) \otimes 1)]$$

is the zero map. The map $\mathsf{Lin}(\sqcap^R(A), k) \otimes \sqcap^R(A) \to \mathsf{Lin}(\sqcap^R(A), \sqcap^R(A))$, $\Phi \otimes r \mapsto \Phi(-)r$ is injective. Indeed, denote by I, J two index sets and by $\{\Phi_i\}_{i\in I}$,$\{r_j\}_{j\in J}$ respective bases of the k–vector spaces $\mathsf{Lin}(\sqcap^R(A), k)$, $\sqcap^R(A)$. If $\sum_{i\in I, j\in J} \lambda_{ij} \Phi(-)_i r_j = 0$ ($\lambda_{i,j} \in k$ for every $i \in I, j \in J$, non-zero for at most finitely many of them), then $\sum_{i\in I, j\in J} \lambda_i \Phi(s)_i r_j = 0$ for all $s \in \sqcap^R(A)$ and, hence, $\sum_{i\in I} \lambda_{i,j} \Phi_i(s) = 0$ for all $s \in \sqcap^R(A)$. This implies that $\sum_{i\in I} \lambda_{ij} \Phi_i(-) = 0$, from what follows $\lambda_{i,j} = 0$ for all $i \in I, j \in J$. Using this and the notation $T_2(a^i \otimes b^i) =: a'_j \otimes b'_j$ we conclude that

$$\epsilon(a'_j -) \otimes \sqcap^R(b'_j) \in \mathsf{Lin}(\sqcap^R(A), k) \otimes \sqcap^R(A)$$

is equal to zero. Applying to it the evaluation map $\mathsf{Lin}(\sqcap^R(A),k)\otimes\sqcap^R(A)\to k$, $\Phi\otimes x\mapsto\Phi(x)$, and using Lemma 4.2.9 (4) in the second equality, we prove that

$$\epsilon(a'_j\sqcap^R(b'_j)) = \epsilon\mu(\mathsf{id}\otimes\sqcap^R)T_2(a^i\otimes b^i) = \epsilon(a^ib^i)$$

is equal to zero as needed. □

Proposition 4.3.2. *Let A be a regular weak multiplier bialgebra over a field k with a left full comultiplication. Then the counit ϵ determines a linear map*

$$\varepsilon:\{\sqcap^L(a)\mid a\in A\}\to k,\qquad\qquad \sqcap^L(a)\mapsto\epsilon(a). \tag{4.37}$$

Proof. It follows by applying Proposition 4.3.1 to the regular weak multiplier bialgebra $A^{\mathsf{op}}_{\mathsf{cop}}$ with a right full comultiplication. □

For the construction of the comultiplication on $\sqcap^R(A)$, the following technical lemma is needed.

Lemma 4.3.3. *Let A be a regular weak multiplier bialgebra over a field. For any $a,b\in A$, the following assertions hold.*

(1) The element $(\mathsf{id}\otimes\overline{\sqcap}^R)T_4(a\otimes b)$ of $A\otimes\mathbb{M}(A)$ depends on a and b only through the product ba.

(2) The element $(\mathsf{id}\otimes\sqcap^R)T_2(a\otimes b)$ of $A\otimes\mathbb{M}(A)$ depends on a and b only through the product ab.

Proof. We only explicitly prove part (1), part (2) follows from it and the symmetry $A-A^{\mathsf{op}}$. Applying twice the first identity in Lemma 4.2.13, for all $a,b,c,d,f,g\in A$

$$\begin{aligned}(1\otimes cd)((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(a\otimes b))(f\otimes g) &= ((\sqcap^R\otimes\mathsf{id})T_2^{\mathsf{op}}(c\otimes d))(baf\otimes g)\\ &= (1\otimes cd)((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(f\otimes ba))(1\otimes g).\end{aligned}$$

If $ba=b'a'$ for some $a,b,a',b'\in A$, using the above identity in the first and third equalities,

$$\begin{aligned}((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(a\otimes b))(f\otimes g) &= ((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(f\otimes ba))(1\otimes g)\\ &= ((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(f\otimes b'a'))(1\otimes g)\\ &= ((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(a'\otimes b'))(f\otimes g),\end{aligned}$$

for all $f,g\in A$, proving $(\mathsf{id}\otimes\overline{\sqcap}^R)T_4(a\otimes b)=(\mathsf{id}\otimes\overline{\sqcap}^R)T_4(a'\otimes b')$. □

Proposition 4.3.4. *For a regular weak multiplier bialgebra A over a field, the following assertions hold.*

(1) The maps $A\otimes A\to A\otimes A$,

$$\begin{aligned}a\otimes bc &\mapsto ((\overline{\sqcap}^R\otimes\mathsf{id})T_4^{\mathsf{op}}(c\otimes b))(a\otimes 1)\quad\textit{and}\\ ab\otimes c &\mapsto (1\otimes c)((\mathsf{id}\otimes\sqcap^R)T_2(a\otimes b))\end{aligned}$$

(where $T_4^{\mathsf{op}}=\mathsf{tw}T_4$) determine an element of $\mathbb{M}(A\otimes A)$, to be denoted by F.

(2) For any element $a\in A$, and $F\in\mathbb{M}(A\otimes A)$ as in (1), $(\sqcap^R(a)\otimes 1)F$ and $F(1\otimes\sqcap^R(a))$ are equal elements of $\sqcap^R(A)\otimes\sqcap^R(A)$, to be denoted by $\delta_{\sqcap^R(A)}\sqcap^R(a)$.

(3) The map

$$\delta_{\sqcap^R(A)}:\sqcap^R(A)\to\sqcap^R(A)\otimes\sqcap^R(A),\qquad \sqcap^R(a)\mapsto(\sqcap^R(a)\otimes 1)F=F(1\otimes\sqcap^R(a))$$

provides a $\sqcap^R(A)$–bimodule section of the multiplication in $\sqcap^R(A)$.

(4) The map $\delta_{\sqcap^R(A)}:\sqcap^R(A)\to\sqcap^R(A)\otimes\sqcap^R(A)$ in part (3) is a coassociative comultiplication.

Proof. (1). Both maps in the claim are well defined by Lemma 4.3.3. They define a multiplier by the first statement in Lemma 4.2.13, as the following computation proves. For any $a,b,c,d,e,f,g\in A$,

$$\begin{aligned}&(dg\otimes ef)\,[((\overline{\sqcap}^R\otimes\mathsf{id})T_4^{\mathsf{op}}(c\otimes b))(a\otimes 1)]\\ &\quad= \mathsf{tw}[(ef\otimes dg)((\mathsf{id}\otimes\overline{\sqcap}^R)T_4(c\otimes b))(1\otimes a)]\\ &\quad= \mathsf{tw}[(ef\otimes 1)[((\sqcap^R\otimes\mathsf{id})T_2^{\mathsf{op}}(d\otimes g))(bc\otimes 1)(1\otimes a)]]\\ &\quad= (1\otimes ef)((\mathsf{id}\otimes\sqcap^R)T_2(d\otimes g))(a\otimes bc).\end{aligned}$$

(2). Centrality of F in the $\sqcap^R(A)$–bimodule $\mathbb{M}(A \otimes A)$ follows by the following computation, for all $a, b, c, d \in A$.

$$\begin{aligned}(bc \otimes d)(\sqcap^R(a) \otimes 1)F &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)T_2(b \otimes c \sqcap^R(a))) \\ &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)(T_2(b \otimes c)\overline{\Delta} \sqcap^R(a))) \\ &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)(T_2(b \otimes c)(1 \otimes \sqcap^R(a)))) \\ &= (1 \otimes d)((\mathsf{id} \otimes \sqcap^R)T_2(b \otimes c))(1 \otimes \sqcap^R(a)) \\ &= (bc \otimes d)F(1 \otimes \sqcap^R(a))\end{aligned}$$

The second equality follows by the explicit form of T_2 and the multiplicativity of $\overline{\Delta}$. In the third equality we used that by (4.29), $\overline{\Delta} \sqcap^R(a) = E(1 \otimes \sqcap^R(a))$, and by (4.2), $T_2(b \otimes c)E = T_2(b \otimes c)$. The fourth equality follows by (4.30). The stated elements belong to $\sqcap^R(A) \otimes \sqcap^R(A)$ by the following reasoning. Let $a, b, c, d, f \in A$, and denote $a_i \otimes b_i := T_2(a \otimes b)$ and $f_j \otimes d_j := T_4(f \otimes d)$. Then,

$$\begin{aligned}(\sqcap^R(ab) \otimes 1)F(c \otimes df) &= \sqcap^R(ab)\overline{\sqcap}^R(d_j)c \otimes f_j = \sqcap^R(ab\overline{\sqcap}^R(d_j))c \otimes f_j \\ &= \sqcap^R(a_i)c \otimes \sqcap^R(b_i)df = (\sqcap^R \otimes \sqcap^R)T_2(a \otimes b)(c \otimes df).\end{aligned}$$

The second equality follows by Lemma 4.2.14 and the third one follows by the first assertion in Lemma 4.2.13. This proves

$$\delta_{\sqcap^R(A)} \sqcap^R(ab) = (\sqcap^R \otimes \sqcap^R)T_2(a \otimes b), \quad \forall a, b \in A. \tag{4.38}$$

So by the idempotency of A, $\delta_{\sqcap^R(A)} \sqcap^R(a) \in \sqcap^R(A) \otimes \sqcap^R(A)$, for all $a \in A$.

(3). Using Lemma 4.2.9 (4) in the last equality, for any $a, b \in A$,

$$\mu\delta_{\sqcap^R(A)} \sqcap^R(ab) \overset{(4.38)}{=} \mu(\sqcap^R \otimes \sqcap^R)T_2(a \otimes b) \overset{(4.30)}{=} \sqcap^R\mu(\mathsf{id} \otimes \sqcap^R)T_2(a \otimes b) = \sqcap^R(ab).$$

(4). Let us use Heyneman-Sweedler type index notation $\delta_{\sqcap^R(A)}(r) =: r_1 \otimes r_2$ for any $r \in \sqcap^R(A)$, where implicit summation is understood. It follows by part (3) that $\sqcap^R(A)$ is an idempotent non-unital algebra. So the coassociativity of $\delta_{\sqcap^R(A)}$ follows by

$$\begin{aligned}(\delta_{\sqcap^R(A)} \otimes \mathsf{id})\delta_{\sqcap^R(A)}(sr) &= \delta_{\sqcap^R(A)}(sr_1) \otimes r_2 = s_1 \otimes s_2 r_1 \otimes r_2 \\ &= s_1 \otimes \delta_{\sqcap^R(A)}(s_2 r) = (\mathsf{id} \otimes \delta_{\sqcap^R(A)})\delta_{\sqcap^R(A)}(sr),\end{aligned}$$

for all $s, r \in \sqcap^R(A)$. In the first and the penultimate equalities we used that $\delta_{\sqcap^R(A)}$ is a morphism of left $\sqcap^R(A)$–modules (part (3)) and in the second and the last equalities we used that it is a morphism of right $\sqcap^R(A)$-modules (part (3)). □

The following proposition, collecting symmetric results to the previous ones, is immediate.

Proposition 4.3.5. *For a regular weak multiplier bialgebra A over a field, the following assertions hold.*

(a) The maps $A \otimes A \to A \otimes A$,

$$a \otimes bc \mapsto ((\sqcap^L \otimes \mathsf{id})T_1(b \otimes c))(a \otimes 1) \quad \textit{and}$$
$$ab \otimes c \mapsto (1 \otimes c)((\mathsf{id} \otimes \overline{\sqcap}^L)T_3^{\mathsf{op}}(b \otimes a))$$

(where $T_3^{\mathsf{op}} = \mathsf{tw}T_3$) determine an element of $\mathbb{M}(A \otimes A)$, to be denoted by F_2. For any element $a \in A$, $(\overline{\sqcap}^L(a) \otimes 1)F_2$ and $F_2(1 \otimes \overline{\sqcap}^L(a))$ are equal elements of $\overline{\sqcap}^L(A) \otimes \overline{\sqcap}^L(A)$, to be denoted by $\delta_{\overline{\sqcap}^L(A)}\overline{\sqcap}^L(a)$. The map

$$\delta_{\overline{\sqcap}^L(A)} : \overline{\sqcap}^L(A) \to \overline{\sqcap}^L(A) \otimes \overline{\sqcap}^L(A), \qquad \overline{\sqcap}^L(a) \mapsto (\overline{\sqcap}^L(a) \otimes 1)F_2 = F_2(1 \otimes \overline{\sqcap}^L(a))$$

provides a $\overline{\sqcap}^L(A)$–bimodule section of the multiplication in $\overline{\sqcap}^L(A)$ and a coassociative comultiplication.

(b) The maps $A \otimes A \to A \otimes A$,

$$ba \otimes c \mapsto ((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(a \otimes b))(1 \otimes c) \quad \textit{and}$$
$$a \otimes bc \mapsto (a \otimes 1)((\sqcap^R \otimes \mathsf{id})T_2^{\mathsf{op}}(b \otimes c))$$

(where $T_2^{\mathsf{op}} = \mathsf{tw}T_2$) determine an element of $\mathbb{M}(A \otimes A)$, to be denoted by F_3. For any element $a \in A$, $F_3(\overline{\sqcap}^R(a) \otimes 1)$ and $(1 \otimes \overline{\sqcap}^R(a))F_3$ are equal elements of $\overline{\sqcap}^R(A) \otimes \overline{\sqcap}^R(A)$, to be denoted by $\delta_{\overline{\sqcap}^R(A)}\overline{\sqcap}^R(a)$. The map

$$\delta_{\overline{\sqcap}^R(A)} : \overline{\sqcap}^R(A) \to \overline{\sqcap}^R(A) \otimes \overline{\sqcap}^R(A), \qquad \overline{\sqcap}^R(a) \mapsto F_3(\overline{\sqcap}^R(a) \otimes 1) = (1 \otimes \overline{\sqcap}^R(a))F_3$$

provides a coassociative comultiplication in $\overline{\sqcap}^R(A)$.

(c) *The maps* $A \otimes A \to A \otimes A$,

$$ba \otimes c \mapsto ((\mathsf{id} \otimes \sqcap^L)T_1^{\mathsf{op}}(b \otimes a))(1 \otimes c) \quad \textit{and}$$
$$a \otimes bc \mapsto (a \otimes 1)((\overline{\sqcap}^L \otimes \mathsf{id})T_3(c \otimes b))$$

(where $T_1^{\mathsf{op}} = \mathsf{tw}T_1$*) determine an element of* $\mathbb{M}(A \otimes A)$*, to be denoted by* F_4*. For any element* $a \in A$*,* $F_4(\sqcap^L(a) \otimes 1)$ *and* $(1 \otimes \sqcap^L(a))F_4$ *are equal elements of* $\sqcap^L(A) \otimes \sqcap^L(A)$*, to be denoted by* $\delta_{\sqcap^L(A)} \sqcap^L(a)$*. The map*

$$\delta_{\sqcap^L(A)} : \sqcap^L(A) \to \sqcap^L(A) \otimes \sqcap^L(A), \qquad \sqcap^L(a) \mapsto F_4(\sqcap^L(a) \otimes 1) = (1 \otimes \sqcap^L(a))F_4$$

provides a coassociative comultiplication in $\sqcap^L(A)$*.*

Proof. Part (a) follows by applying parts (1), (2), (3) and (4) in Proposition 4.3.4 to the regular weak multiplier bialgebra A_{cop}. Parts (b) and (c) follow by applying parts (1), (2) and (4) in the same proposition to A^{op} and $A^{\mathsf{op}}_{\mathsf{cop}}$. □

The elements F, F_2, F_3 and F_4 of $\mathbb{M}(A \otimes A)$ from Proposition 4.3.4 and 4.3.5 obey the identities:

$$F_3 = \mathsf{tw}F\mathsf{tw} \quad \text{and} \quad F_4 = \mathsf{tw}F_2\mathsf{tw}. \tag{4.39}$$

More details on these maps are shown further on in Table 4.5.

Theorem 4.3.1. *Let* A *be a regular weak multiplier bialgebra over a field* k *with a right full comultiplication. Then* $\sqcap^R(A)$ *is a coalgebra via the counit* $\varepsilon : \sqcap^R(A) \to k$ *in Proposition 4.3.1 and the comultiplication* $\delta_{\sqcap^R(A)} : \sqcap^R(A) \to \sqcap^R(A) \otimes \sqcap^R(A)$ *in Proposition 4.3.4 (3).*

Proof. The map $\delta_{\sqcap^R(A)}$ is a coassociative comultiplication by Proposition 4.3.4 (4). It remains to prove the counitality of $\delta_{\sqcap^R(A)}$. For any $a, b \in A$,

$$(\mathsf{id} \otimes \varepsilon)\delta_{\sqcap^R(A)} \sqcap^R(ab) \overset{(4.38)}{=} (\mathsf{id} \otimes \varepsilon)(\sqcap^R \otimes \sqcap^R)T_2(a \otimes b) = \sqcap^R(\mathsf{id} \otimes \epsilon)T_2(a \otimes b) \overset{\text{(iii)}}{=} \sqcap^R(ab).$$

In order to prove counitality on the other side, we need an alternative expression of δ. To this end, note that for any $a, b, c \in A$,

$$(\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_2(b \otimes c)] = (\mathsf{id} \otimes \epsilon)[(b \otimes 1)T_3(c \otimes a)] \overset{(4.22)}{=} b \sqcap^R(a)c. \tag{4.40}$$

On the other hand, for any $a, b, c \in A$,

$$\begin{aligned}(\mathsf{id}\otimes\epsilon)[T_3(b\otimes a)(1\otimes c)] &= (\mathsf{id}\otimes\epsilon)[(1\otimes a)T_1(b\otimes c)] \\ &= (\mathsf{id}\otimes\epsilon)[(1\otimes\overline{\sqcap}^L(a))T_1(b\otimes c)] \\ &= (\mathsf{id}\otimes\epsilon)[(\sqcap^R(a)\otimes 1)T_1(b\otimes c)],\end{aligned} \tag{4.41}$$

where the second equality follows by Lemma 4.2.3 and the third one follows by (4.2) and Lemma 4.2.12. Therefore,

$$\begin{aligned}(\sqcap^R\otimes\epsilon)[T_3(b\otimes a)(1\otimes c)] &\overset{(4.41)}{=} (\sqcap^R\otimes\epsilon)[(\sqcap^R(a)\otimes 1)T_1(b\otimes c)] \\ &\overset{(4.28)}{=} (\sqcap^R\otimes\epsilon)[(a\otimes 1)T_1(b\otimes c)] \\ &\overset{(4.1)}{=} (\sqcap^R\otimes\epsilon)[T_2(a\otimes b)(1\otimes c)].\end{aligned} \tag{4.42}$$

With these identities at hand, for any $a, b, c, d, f, g \in A$,

$$\begin{aligned}&(f\otimes g)\ \ ((\sqcap^R\otimes\sqcap^R)T_3(b\otimes a))(c\otimes d) \\ &\overset{(4.40)}{=} (f\otimes 1)(\sqcap^R\otimes\epsilon\otimes\mathsf{id})[(T_3(b\otimes a)\otimes 1)(1\otimes T_2^{\mathsf{op}}(g\otimes d))](c\otimes 1) \\ &\overset{(4.42)}{=} (f\otimes 1)(\sqcap^R\otimes\epsilon\otimes\mathsf{id})[(T_2(a\otimes b)\otimes 1)(1\otimes T_2^{\mathsf{op}}(g\otimes d))](c\otimes 1) \\ &\overset{(4.40)}{=} (f\otimes g)((\sqcap^R\otimes\sqcap^R)T_2(a\otimes b))(c\otimes d),\end{aligned} \tag{4.43}$$

so that by (4.38),

$$\delta_{\sqcap^R(A)}\sqcap^R(ab) = (\sqcap^R\otimes\sqcap^R)T_3(b\otimes a), \qquad \forall a, b \in A. \tag{4.44}$$

Using this expression of δ,

$$(\varepsilon\otimes\mathsf{id})\delta_{\sqcap^R(A)}\sqcap^R(ab) \overset{(4.44)}{=} (\varepsilon\otimes\mathsf{id})(\sqcap^R\otimes\sqcap^R)T_3(b\otimes a) \overset{(4.36)}{=} \sqcap^R(\epsilon\otimes\mathsf{id})T_3(b\otimes a) \overset{(\mathrm{iii})}{=} \sqcap^R(ab).$$

□

Theorem 4.3.2. *Let A be a regular weak multiplier bialgebra over a field k with a left full comultiplication. Then $\sqcap^L(A)$ is a coalgebra via the counit $\varepsilon : \sqcap^L(A) \to k$ in Proposition 4.3.2 and the comultiplication $\delta_{\sqcap^L(A)} : \sqcap^L(A) \to \sqcap^L(A)\otimes\sqcap^L(A)$ in Proposition 4.3.5 (c).*

Proof. It follows by applying Theorem 4.3.1 to the regular weak multiplier bialgebra $A^{\mathsf{op}}_{\mathsf{cop}}$ with a right full comultiplication. □

Lemma 4.3.6. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then the comultiplication $\delta_{\sqcap^R(A)}$ and the counit ε in Theorem 4.3.1 satisfy the following identities, for all $a, b \in A$.*

(1) $\delta_{\sqcap^R(A)}\overline{\sqcap}^R(ab) = (\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_4^{\mathsf{op}}(b \otimes a) = (\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_1^{\mathsf{op}}(a \otimes b).$

(2) $\varepsilon\overline{\sqcap}^R(a) = \epsilon(a).$

Proof. (1). Symmetrically to the derivation of (4.38), for any $a, b, c, d, f \in A$ denote $T_4(b \otimes a) =: b_i \otimes a_i$ and $T_2(c \otimes d) =: c_j \otimes d_j$. Then

$$\begin{aligned}(cd \otimes f)\delta_{\sqcap^R(A)}\overline{\sqcap}^R(ab) &= (cd \otimes f)F(1 \otimes \overline{\sqcap}^R(ab)) = c_j \otimes f \sqcap^R (d_j)\overline{\sqcap}^R(ab) \\ &= c_j \otimes f\overline{\sqcap}^R(\sqcap^R(d_j)ab) = cd\overline{\sqcap}^R(a_i) \otimes f\overline{\sqcap}^R(b_i) \\ &= (cd \otimes f)((\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_4^{\mathsf{op}}(b \otimes a)).\end{aligned}$$

The third equality follows by Lemma 4.2.14 and the fourth equality follows by the first assertion in Lemma 4.2.13. Now, applying the equality (4.43) in A^{op} (see Table 2.1), we obtain for all $a, b \in A$

$$(\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_4(b \otimes a) = (\overline{\sqcap}^R \otimes \overline{\sqcap}^R)T_1(a \otimes b).$$

(2). Applying Proposition 4.3.1 to A^{op}, there is a linear map $\overline{\sqcap}^R(A) \to k$, $\overline{\sqcap}^R(a) \mapsto \epsilon(a)$. Using part (1) and axiom (iii) in Definition 4.1.1 for A^{op}, it can be seen to be the counit for $\delta_{\sqcap^R(A)}$ proving that it is equal to ε. □

Remark 4.3.7. Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Equivalently, $\sqcap^R(A) = \overline{\sqcap}^R(A)$ as sets (see Theorem 4.2.1). Then $\delta_{\sqcap^R(A)}$ from part (1) in Proposition 4.3.4 and $\delta_{\overline{\sqcap}^R(A)}$ from part (2) in Proposition 4.3.5 obey the following relation: For any $a \in A$,

$$\delta_{\overline{\sqcap}^R(A)}\overline{\sqcap}^R(a) = F_3(\overline{\sqcap}^R(a) \otimes 1) \overset{(4.39)}{=} \mathsf{tw}F\mathsf{tw}(\overline{\sqcap}^R(a) \otimes 1) = \mathsf{tw}\delta_{\sqcap^R(A)}\overline{\sqcap}^R(a).$$

Analogously, for a regular weak multiplier bialgebra with a left full comultiplication, it holds $\delta_{\sqcap^L(A)} \sqcap^L (a) = \mathsf{tw}\delta_{\overline{\sqcap}^L(A)} \sqcap^L (a)$. These formulas stress the fact that $\delta_{\overline{\sqcap}^R(A)}$ and

$\delta_{\sqcap^L(A)}$ do not induce, in general, bimodule sections of the multiplication in $\overline{\sqcap}^R(A)$ and $\sqcap^L(A)$ respectively (in contrast to $\delta_{\sqcap^R(A)}$ and $\delta_{\overline{\sqcap}^L(A)}$). However, taking into account Lemma 4.3.6, they give a sufficient condition for this to happen: whenever the regular weak multiplier bialgebra A is cocommutative in the sense of Definition 4.1.6 (in such a case, it follows that $\delta_{\overline{\sqcap}^R(A)} = \delta_{\sqcap^R(A)}$ and $\delta_{\sqcap^L(A)} = \delta_{\overline{\sqcap}^L(A)}$).

The following theorem describes the rich algebraic structure carried by the base algebras. Such a result was obtained for *regular* weak multiplier Hopf algebras in [71].

Theorem 4.3.3. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then the following assertions hold.*

(1) Via the coalgebra structure in Theorem 4.3.1 and the restriction of the multiplication in $\mathbb{M}(A)$, $\sqcap^R(A)$ is a coseparable coalgebra, hence a firm Frobenius non-unital algebra.

(2) The multiplication in the firm Frobenius non-unital algebra in part (1) is non-degenerate. Moreover, it has local units.

(3) The coalgebra $\sqcap^R(A)$ in part (1) is a co-Frobenius coalgebra. Hence there exists a unique isomorphism of non-unital algebras $\vartheta : \sqcap^R(A) \to \sqcap^R(A)$ —known as the Nakayama automorphism— *such that $\varepsilon(sr) = \varepsilon(\vartheta(r)s)$, for all $s, r \in \sqcap^R(A)$.*

Proof. (1). By Theorem 4.3.1, $\sqcap^R(A)$ is a coalgebra. By Proposition 4.3.4 (3), the multiplication in $\sqcap^R(A)$ is a bicomodule retraction (i.e. left inverse) of the comultiplication. This precisely means a coseparable coalgebra structure. Then $\sqcap^R(A)$ is a firm Frobenius non-unital algebra by the considerations in [12, Section 6.4].

(2). For some $a \in A$, assume that $\sqcap^R(a)\overline{\sqcap}^R(b) = \sqcap^R(a\overline{\sqcap}^R(b)) = 0$, for all $b \in A$ (where the first equality follows by Lemma 4.2.14). Then also

$$0 = \varepsilon \sqcap^R (a\overline{\sqcap}^R(b)) = \epsilon(a\overline{\sqcap}^R(b)) = \epsilon(ab), \quad \forall b \in A,$$

where the last equality follows by Lemma 4.2.3. This implies that

$$0 = (\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_2(b \otimes c)] = (\mathsf{id} \otimes \epsilon)[(b \otimes 1)T_3(c \otimes a)] \overset{(4.22)}{=} b \sqcap^R (a)c, \quad \forall b, c \in A,$$

proving $\sqcap^R(a) = 0$. Since $\sqcap^R(A) = \overline{\sqcap}^R(A)$ by Theorem 4.2.1, this proves the non-degeneracy of the multiplication on the right. Non-degeneracy on the left is proven symmetrically (applying the reasoning above in A^{op}). The existence of local units follows by [12, Proposition 7].

(3). In light of part (2), it follows by [12, Proposition 7] that the coalgebra $\sqcap^R(A)$ in Theorem 4.3.1 is left and right co-Frobenius. So the existence of the Nakayama automorphism follows by [30, Section 6]. □

The following symmetric version is immediate.

Theorem 4.3.4. *Let A be a regular weak multiplier bialgebra over a field k with a left full comultiplication. Then the following assertions hold.*

(1) Via the coalebra structure in Theorem (4.3.2) and the restriction of the multiplication in $\mathbb{M}(A)$, $\sqcap^L(A)$ is a coseparable coalgebra, hence a firm Frobenius non-unital algebra.

(2) The multiplication in the firm Frobenius non-unital algebra in part (1) is non-degenerate. Moreover, it has local units.

(3) The coalgebra $\sqcap^L(A)$ in part (1) is a co-Frobenius coalgebra (hence its counit has a Nakayama automorphism).

Proof. It follows by applying Theorem 4.3.3 on the regular weak multiplier bialgebra $A^{\mathsf{op}}_{\mathsf{cop}}$ with a right full comultiplication. □

Our next aim is to find a more explicit expression of the Nakayama automorphisms in Theorem 4.3.3 (3) and Theorem 4.3.4 (3).

Lemma 4.3.8. *For a regular weak multiplier bialgebra A over a field, the following assertions hold.*

(1) If the comultiplication is left full, then there is a linear anti-multiplicative map

$$\sigma : \sqcap^L(A) = \overline{\sqcap}^L(A) \to \sqcap^R(A), \qquad \overline{\sqcap}^L(a) \mapsto \sqcap^R(a).$$

(2) If the comultiplication is left full, then there is a linear anti-multiplicative map

$$\overline{\sigma} : \sqcap^L(A) = \overline{\sqcap}^L(A) \to \overline{\sqcap}^R(A), \qquad \sqcap^L(a) \mapsto \overline{\sqcap}^R(a).$$

(3) If the comultiplication is right full, then there is a linear anti-multiplicative map

$$\tau : \sqcap^R(A) = \overline{\sqcap}^R(A) \to \overline{\sqcap}^L(A), \qquad \sqcap^R(a) \mapsto \overline{\sqcap}^L(a).$$

(4) If the comultiplication is right full, then there is a linear anti-multiplicative map

$$\overline{\tau} : \sqcap^R(A) = \overline{\sqcap}^R(A) \to \sqcap^L(A), \qquad \overline{\sqcap}^R(a) \mapsto \sqcap^L(a).$$

If the comultiplication is both left and right full, then $\tau = \sigma^{-1}$ and $\overline{\tau} = \overline{\sigma}^{-1}$.

Proof. We prove part (1) explicitly; parts (2), (3) and (4) follow, respectively, by part (1) and the symmetries $A - A^{\mathsf{op}}, A - A_{\mathsf{cop}}, A - A^{\mathsf{op}}_{\mathsf{cop}}$. Let ε the counit of the coalgebra $\sqcap^L(A) = \overline{\sqcap}^L(A)$ (cf. Theorem 4.3.2). If $\overline{\sqcap}^L(a) = 0$, then for all $b, c \in A$,

$$\begin{aligned} 0 &= (\mathsf{id} \otimes \varepsilon(\overline{\sqcap}^L(a)\overline{\sqcap}^L(-)))T_2(b \otimes c) = (\mathsf{id} \otimes \varepsilon\overline{\sqcap}^L(a\overline{\sqcap}^L(-)))T_2(b \otimes c) \\ &= (\mathsf{id} \otimes \epsilon(a\overline{\sqcap}^L(-)))T_2(b \otimes c) = (\mathsf{id} \otimes \epsilon(- \sqcap^R(a)))T_2(b \otimes c) \\ &= (\mathsf{id} \otimes \epsilon)[(b \otimes 1)\Delta(c)(1 \otimes \sqcap^R(a))] = (\mathsf{id} \otimes \epsilon)T_2(b \otimes c \sqcap^R(a)) \overset{\text{(iii)}}{=} bc \sqcap^R(a), \end{aligned}$$

proving that $\sqcap^R(a) = 0$. The second equality follows by Lemma 4.2.6 and the fourth one follows by Lemma 4.2.11. In the penultimate equality we applied axiom (iv) in Definition 4.1.1, (4.29) and the multiplicativity of $\overline{\Delta}$. This proves the existence of the stated linear map σ. Using Lemma 4.2.6 in the first equality and Lemma 4.2.15 in the penultimate equality,

$$\sigma(\overline{\sqcap}^L(a)\overline{\sqcap}^L(b)) = \sigma\overline{\sqcap}^L(a\overline{\sqcap}^L(b)) = \sqcap^R(a\overline{\sqcap}^L(b)) = \sqcap^R(b)\sqcap^R(a) = (\sigma\overline{\sqcap}^L(b))(\sigma\overline{\sqcap}^L(a)),$$

for any $a, b \in A$; that is, σ is anti-multiplicative. □

Proposition 4.3.9. *Let A be a regular weak multiplier bialgebra over a field with a left and right full comultiplication. Then the maps σ and $\overline{\sigma}$ in Lemma 4.3.8 are anti-*

coalgebra isomorphisms. Moreover, the Nakayama automorphism of $\sqcap^R(A)$ *is equal to* $\sigma\overline{\sigma}^{-1}$ *and the Nakayama automorphism of* $\sqcap^L(A)$ *is equal to* $\overline{\sigma}^{-1}\sigma$.

Proof. By the $A - A^{\mathsf{op}}_{\mathsf{cop}}$ symmetric counterpart of Lemma 4.3.6 (1), for any $a, b \in A$, $\delta_{\overline{\sqcap}^L(A)}\overline{\sqcap}^L(ab) = (\overline{\sqcap}^L \otimes \overline{\sqcap}^L)T_3^{\mathsf{op}}(b \otimes a)$. Therefore,

$$(\sigma \otimes \sigma)\delta^{\mathsf{op}}_{\overline{\sqcap}^L(A)}\overline{\sqcap}^L(ab) = (\sigma\overline{\sqcap}^L \otimes \sigma\overline{\sqcap}^L)T_3(b \otimes a) = (\sqcap^R \otimes \sqcap^R)T_3(b \otimes a) \overset{(4.44)}{=} \delta \sqcap^R (ab),$$

so that σ is anti-comultiplicative. By the left-right symmetric counterpart of Lemma 4.3.6 (3), $\varepsilon\sigma\overline{\sqcap}^L(a) = \varepsilon \sqcap^R (a) = \epsilon(a) = \varepsilon\overline{\sqcap}^L(a)$ for any $a \in A$, proving that σ is an anti-coalgebra map. That also $\overline{\sigma}$ is an anti-coalgebra homomorphism is proven by the above and the symmetry $A - A^{\mathsf{op}}$.

Applying Lemma 4.2.3 in the second equality, it follows for all $a, b \in A$ that

$$\epsilon(a\overline{\sigma} \sqcap^L (b)) = \epsilon(a\overline{\sqcap}^R(b)) = \epsilon(ab) \overset{(4.27)}{=} \epsilon(a \sqcap^L (b)).$$

Since $\sqcap^L(A) = \overline{\sqcap}^L(A)$ by Theorem 4.2.1, this implies that $\epsilon(a\overline{\sigma}\overline{\sqcap}^L(b)) = \epsilon(a\overline{\sqcap}^L(b))$, for all $a, b \in A$. Using this identity in the fourth equality and Lemma 4.2.11 in the fifth one,

$$\begin{aligned}\varepsilon(\sqcap^R(a)\overline{\sigma}\,\sigma^{-1} \sqcap^R (b)) = \varepsilon(\sqcap^R(a)\overline{\sigma}\overline{\sqcap}^L(b)) &\overset{(4.30)}{=} \varepsilon \sqcap^R (a\overline{\sigma}\overline{\sqcap}^L(b)) = \epsilon(a\overline{\sigma}\overline{\sqcap}^L(b)) \\ = \quad \epsilon(a\overline{\sqcap}^L(b)) = \epsilon(b \sqcap^R (a)) &= \varepsilon \sqcap^R (b \sqcap^R (a)) \overset{(4.30)}{=} \varepsilon(\sqcap^R(b) \sqcap^R (a)),\end{aligned}$$

for all $a, b \in A$. This proves that $\sigma\overline{\sigma}^{-1}$ is the Nakayama automorphism of $\sqcap^R(A)$ and, by the symmetry $A - A^{\mathsf{op}}_{\mathsf{cop}}$, also that $\overline{\sigma}^{-1}\sigma$ is the Nakayama automorphism of $\sqcap^L(A)$. □

Finally, we focus on finding a relation between the multipliers E and F.

Lemma 4.3.10. *Let* A *be a regular weak multiplier bialgebra over a field. Then for all* $a \in A$,

$$(1 \otimes \overline{\sqcap}^L(a))E \in \sqcap^R(A) \otimes \overline{\sqcap}^L(A) \quad \textit{and} \quad E(1 \otimes \sqcap^L(a)) \in \overline{\sqcap}^R(A) \otimes \sqcap^L(A).$$

In particular, if the comultiplication is right and left full, then we can regard E *as an element of* $\mathbb{M}(\sqcap^R(A) \otimes \sqcap^L(A)^{\mathsf{op}})$.

Proof. For any $c,d,f \in A$, $(f\otimes 1)T_4(d\otimes c) = (f\otimes 1)\Delta(c)(d\otimes 1) = T_2(f\otimes c)(d\otimes 1)$. Hence multiplying on the left both sides of (4.32) by $f\otimes 1$ and simplifying on the right the resulting equality by $d\otimes 1$, we obtain the identity

$$(f\otimes 1)((\sqcap^R\otimes \mathsf{id})T_3(a\otimes b))(c\otimes 1) = (1\otimes ba)((\mathsf{id}\otimes \sqcap^L)T_2(f\otimes c)),$$

for all $a,b,c,f\in A$. Using this identity in the fourth equality and Lemma 4.2.14 in the third one,

$$\begin{aligned}
(a\otimes 1)(1\otimes \overline{\sqcap}^L(bc))\ E\ (d\otimes f) &= (1\otimes \overline{\sqcap}^L(bc))(a\otimes 1)E(d\otimes f)\\
&\overset{(4.26)}{=} (1\otimes \overline{\sqcap}^L(bc))((\mathsf{id}\otimes \sqcap^L)T_2(a\otimes d))(1\otimes f)\\
&= (1\otimes \overline{\sqcap}^L)[(1\otimes bc)((\mathsf{id}\otimes \sqcap^L)T_2(a\otimes d))](1\otimes f)\\
&= (1\otimes \overline{\sqcap}^L)[(a\otimes 1)((\sqcap^R\otimes \mathsf{id})T_3(c\otimes b))(d\otimes 1)](1\otimes f)\\
&= (a\otimes 1)((\sqcap^R\otimes \overline{\sqcap}^L)T_3(c\otimes b))(d\otimes f),
\end{aligned}$$

for all $a,b,c,d,f\in A$. This proves

$$(1\otimes \overline{\sqcap}^L(bc))E = (\sqcap^R\otimes \overline{\sqcap}^L)T_3(c\otimes b) \qquad \forall b,c\in A, \tag{4.45}$$

hence by the idempotency of A also the first claim. The second claim follows from the first one and the symmetry $A - A^{\mathsf{op}}$. ☐

Let A be a regular weak multiplier bialgebra over a field with a right and left full comultiplication. The non-unital algebra $\sqcap^R(A)\otimes\sqcap^L(A)$ is idempotent by Proposition 4.3.4 (3) and its symmetric counterpart. Hence the multiplicative and bijective map $\mathsf{id}\otimes\sigma : \sqcap^R(A)\otimes\sqcap^L(A)^{\mathsf{op}} \to \sqcap^R(A)\otimes\sqcap^R(A)$ in Lemma 4.3.8 (1) is non-degenerate and thus extends to an algebra homomorphism $\overline{\mathsf{id}\otimes\sigma} : \mathbb{M}(\sqcap^R(A)\otimes\sqcap^L(A)^{\mathsf{op}}) \to \mathbb{M}(\sqcap^R(A)\otimes\sqcap^R(A))$.

Proposition 4.3.11. *Let A be a regular weak multiplier bialgebra over a field with a right and left full comultiplication. Then $(\overline{\mathsf{id}\otimes\sigma})(E) = F$ as elements of $\mathbb{M}(\sqcap^R(A)\otimes\sqcap^R(A))$.*

Proof. For any $a, b, c \in A$,

$$
\begin{aligned}
((\overline{\mathsf{id} \otimes \sigma})(E))(\sqcap^R(a) \otimes \sqcap^R(bc)) &= (\overline{\mathsf{id} \otimes \sigma})[(1 \otimes \overline{\sqcap}^L(bc))E(\sqcap^R(a) \otimes 1)] \\
&\overset{(4.45)}{=} (\overline{\mathsf{id} \otimes \sigma})[((\sqcap^R \otimes \overline{\sqcap}^L)T_3(c \otimes b))(\sqcap^R(a) \otimes 1)] \\
&= ((\sqcap^R \otimes \sqcap^R)T_3(c \otimes b))(\sqcap^R(a) \otimes 1) \\
&\overset{(4.44)}{=} F(\sqcap^R(a) \otimes \sqcap^R(bc)),
\end{aligned}
$$

where in the first and the third equalities we used part (1) of Lemma 4.3.8 and the multiplicativity of $\overline{\mathsf{id} \otimes \sigma}$. □

4.4 Monoidal category of modules

Bialgebras over a field can be characterized by the property that the category of their (left or right) modules is monoidal such that the forgetful functor to the category of vector spaces is strict monoidal. More generally, the category of (left or right) modules over a weak bialgebra is monoidal such that the forgetful functor to the category of bimodules over the (separable Frobenius) base algebra is strict monoidal (see e.g. [64]). The aim of this section is to prove a similar property of regular weak multiplier bialgebras with a (left or right) full comultiplication. The key point in doing so is to find the appropriate notion of module in the absence of an algebraic unit.

Recall from the preliminary Section 2.2 that whenever A is a firm non-unital algebra —that is, the quotient map $A \otimes_A A \to A$, $a \otimes_A b \mapsto ab$ is bijective— the category $\mathsf{bim}^f(A)$ of firm non-unital A–bimodules is monoidal via the module tensor product $\otimes_A$ and the neutral object A.

Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. By Theorem 4.3.3 (1), $R := \sqcap^R(A)$ is a firm non-unital algebra so there is a monoidal category $\mathsf{bim}^f(R)$. Recall from the aforementioned preliminary section that $\mathsf{rmd}(A)$ denotes the category of idempotent and non-degenerate non-unital right A–modules (page 38).

Proposition 4.4.1. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Any object of $\mathsf{rmd}(A)$ can be regarded as a firm R–bimodule. This gives rise to a functor $U : \mathsf{rmd}(A) \to \mathsf{bim}^f(R)$, acting on the morphisms as the identity map.*

Proof. By [34, Proposition 3.3] we know that the A–actions on any object M of $\mathsf{rmd}(A)$ can be extended to $\sqcap^R(A)$ (in fact, to $\mathbb{M}(A)$). Using that M is an idempotent non-unital A–module, define the R–actions on M with the help of the map τ in Lemma 4.3.8 (3) by

$$(ma)\cdot\sqcap^R(b) := m(a\sqcap^R(b)) \quad \text{and} \quad \sqcap^R(b)\cdot(ma) := m(a(\tau\sqcap^R(b))) = m(a\overline{\sqcap}^L(b)).$$

In order to see that these actions are well defined, assume that $\sum_i m_i a_i = 0$. Then, omitting throughout the summation symbol for brevity, for all $b, c \in A$,

$$\begin{aligned} 0 &= (m_i a_i)(\sqcap^R(b)c) = m_i(a_i(\sqcap^R(b)c)) = m_i((a_i\sqcap^R(b))c) = (m_i(a_i\sqcap^R(b)))c, \\ 0 &= (m_i a_i)(\overline{\sqcap}^L(b)c) = m_i(a_i(\overline{\sqcap}^L(b)c)) = m_i((a_i\overline{\sqcap}^L(b))c) = (m_i(a_i\overline{\sqcap}^L(b)))c. \end{aligned}$$

So by the non-degeneracy of M, $0 = m_i(a_i\sqcap^R(b)) = m_i(a_i\overline{\sqcap}^L(b))$ proving that the right and the left R–actions on M are well defined. Associativity of both actions is proven by the associativity of the multiplication in $\mathbb{M}(A)$ and the anti-multiplicativity of τ (cf. Lemma 4.3.8 (3)): For any $a, b, c \in A$,

$$\begin{aligned} (ma)\cdot(\sqcap^R(c)\sqcap^R(b)) &= m(a\sqcap^R(c)\sqcap^R(b)) = (m(a\sqcap^R(c)))\cdot\sqcap^R(b) \\ &= ((ma)\cdot\sqcap^R(c))\cdot\sqcap^R(b), \\ (\sqcap^R(c)\sqcap^R(b))\cdot ma &= m(a(\tau(\sqcap^R(c)\sqcap^R(b))) = m(a(\tau\sqcap^R(b)\tau\sqcap^R(c)) \\ &= \sqcap^R(c)\cdot(m(a(\tau\sqcap^R(b)))) = \sqcap^R(c)\cdot(\sqcap^R(b)\cdot(ma)). \end{aligned}$$

The following computation checks that the left and right R–actions commute (cf. (2.22)): For any $a, b, c \in A$ and $m \in M$,

$$\begin{aligned} \sqcap^R(c)\cdot((ma)\cdot\sqcap^R(b)) &= \sqcap^R(c)\cdot(m(a\sqcap^R(b))) = m(a\sqcap^R(b)\overline{\sqcap}^L(c)) \\ &= m(a\overline{\sqcap}^L(c)\sqcap^R(b)) = (m(a\overline{\sqcap}^L(c)))\cdot\sqcap^R(b) \\ &= (\sqcap^R(c)\cdot(ma))\cdot\sqcap^R(b). \end{aligned}$$

The third equality follows by Lemma 4.2.7 (since by the right fullness of the comultiplication $\sqcap^R(A) = \overline{\sqcap}^R(A)$, see Theorem 4.2.1). Finally, R has local units by Theorem 4.3.3 (2). So in order to see that M is a firm non-unital R–bimodule, it is enough to see that it is idempotent as a left and as a right non-unital R–module. Since both the non-unital algebra A and the non-unital module M are idempotent, any element of M can be written as a linear combination of elements of the form $m(ab) = m(a_i \sqcap^R(b_i)) = (ma_i)\cdot \sqcap^R(b_i)$, in terms of $m \in M$ and $a,b \in A$, where $a_i \otimes b_i := T_2(a\otimes b)$ and the first equality follows by Lemma 4.2.9 (4). Symmetrically, any element of M can be written as a linear combination of elements of the form $n(cd) = n(c_j\overline{\sqcap}^L(d_j)) = \sqcap^R(d_j)\cdot(nc_j)$, in terms of $n \in M$ and $c,d \in A$, where $d_j \otimes c_j := T_3(d\otimes c)$ and the first equality follows by Lemma 4.2.9 (1).

With respect to the stated R–actions, any map of non-unital right A–modules is evidently a morphism of non-unital R–bimodules. This proves the existence of the stated functor U. □

Proposition 4.4.2. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then $R := \sqcap^R(A)$ carries the structure of an idempotent and non-degenerate non-unital right A–module. The functor U in Proposition 4.4.1 takes this object R of $\mathsf{rmd}(A)$ to the non-unital R–bimodule R with the actions provided by the multiplication.*

Proof. For any $a,b \in A$, put

$$\sqcap^R(a) \triangleleft b := \sqcap^R(\sqcap^R(a)b) \overset{(4.28)}{=} \sqcap^R(ab).$$

It is clearly a well defined action; by the associativity of the product in A, it is associative. Let us see that it is idempotent. For any $a,b \in A$, denote $b_i \otimes a_i := T_4(b\otimes a)$. By the right fullness of the comultiplication, $\sqcap^R(A) = \overline{\sqcap}^R(A)$, cf. Theorem 4.2.1. So by Lemma 4.2.9 (2),

$$\overline{\sqcap}^R(a_i) \triangleleft b_i = \sqcap^R(\overline{\sqcap}^R(a_i)b_i) = \sqcap^R(ab).$$

By the idempotency of A, this proves the surjectivity of the A–action on R. In order to see its non-degeneracy, assume that for some $a \in A$, $\sqcap^R(ab) = 0$ for all $b \in A$. Then for all $b,c,d \in A$,

$$\begin{aligned} 0 &= (\mu(\mathsf{id}\otimes\sqcap^R)[(1\otimes a)T_2(b\otimes c)])d \\ &= (\mu(\mathsf{id}\otimes\sqcap^R)[(b\otimes 1)T_3(c\otimes a)])d \\ &\overset{(4.22)}{=} b(\mu(\mathsf{id}\otimes\mathsf{id}\otimes\epsilon)(\mathsf{id}\otimes T_3\mathsf{tw})(T_3\otimes\mathsf{id})(c\otimes a\otimes d)) \end{aligned}$$

$$\begin{aligned}
&= b((\mathsf{id}\otimes\epsilon)(\mu\otimes\mathsf{id})(\mathsf{id}\otimes T_3\mathsf{tw})(T_3\otimes\mathsf{id})(c\otimes a\otimes d))\\
&= b((\mathsf{id}\otimes\epsilon)[(1\otimes a)\Delta(c)\Delta(d)])\\
&\overset{(2.20)}{=} b((\mathsf{id}\otimes\epsilon)[(1\otimes a)\Delta(cd)])\\
&\overset{(4.3)}{=} b((\mathsf{id}\otimes\epsilon)T_3(cd\otimes a)) \overset{(4.22)}{=} b\sqcap^R(a)cd.
\end{aligned}$$

By the density of A in $\mathbb{M}(A)$, this proves $\sqcap^R(a)=0$ hence the non-degeneracy of the action. Applying the functor $U:\mathsf{rmd}(A)\to\mathsf{bim}^f(R)$ in Proposition 4.4.1 to the object R of $\mathsf{rmd}(A)$ above, the right action in the resulting non-unital R–bimodule comes out as the right multiplication. Indeed,

$$\sqcap^R(ab)\cdot\sqcap^R(c)=\sqcap^R(a)\triangleleft(b\sqcap^R(c))=\sqcap^R(ab\sqcap^R(c))\overset{(4.30)}{=}\sqcap^R(ab)\sqcap^R(c),$$

for all $a,b,c\in A$. The left R–action is also given by the multiplication since

$$\sqcap^R(c)\cdot\sqcap^R(ab)=\sqcap^R(a)\triangleleft(b\overline{\sqcap}^L(c))=\sqcap^R(ab\overline{\sqcap}^L(c))=\sqcap^R(c)\sqcap^R(ab),$$

where the last equality follows by Lemma 4.2.15. □

Lemma 4.4.3. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Regard any objects M and N of $\mathsf{rmd}(A)$ as firm non-unital $R:=\sqcap^R(A)$–bimodules as in Proposition 4.4.1. Then the non-unital R–module tensor product $M\otimes_R N$ is isomorphic to*

$$\langle (m\otimes n)((a\otimes b)E)\mid m\in M,\ n\in N,\ a,b\in A\rangle.$$

Proof. By Theorem 4.3.3 (1), R is a coseparable coalgebra. Then by [19, Proposition 2.17], $M\otimes_R N$ is isomorphic to the image of the idempotent map $\theta:M\otimes N\to M\otimes N$,

$$m\cdot\sqcap^R(a)\otimes n\mapsto (m\cdot(-)\otimes(-)\cdot n)\delta_R\sqcap^R(a),$$

where $\delta_R:R\to R\otimes R$ is the (R–bilinear) comultiplication in Proposition 4.3.4 (3). Let us obtain a more explicit expression of this map θ. For this, note first that for all $a,b,c,d\in A$,

$$\begin{aligned}
(\sqcap^R\otimes\mathsf{id})[(ab\otimes cd)E] &\overset{(4.26)}{=} (\sqcap^R\otimes\mathsf{id})[(ab\otimes 1)((\sqcap^R\otimes\mathsf{id})T_3(d\otimes c))]\\
&\overset{(4.30)}{=} (\sqcap^R(ab)\otimes 1)((\sqcap^R\otimes\mathsf{id})T_3(d\otimes c)) \overset{(4.26)}{=} (\sqcap^R(ab)\otimes cd)E,
\end{aligned}\tag{4.46}$$

hence $(\sqcap^R \otimes \mathsf{id})[(ab \otimes 1)E] = (\sqcap^R(ab) \otimes 1)E$. By the idempotency of A and Lemma 4.2.9 (4), any element of A can be written as a linear combination of elements of the form $a\sqcap^R(bc)$ —so any element of M can be written as a linear combination of elements of the form $ma\sqcap^R(bc)$— in terms of $m \in M$, $a,b,c \in A$. Now

$$\begin{aligned} \theta(ma \sqcap^R (bc) \otimes nd) &\overset{(4.38)}{=} (ma \otimes nd)((\sqcap^R \otimes \overline{\sqcap}^L)T_2(b \otimes c)) \\ &= (ma \otimes nd)((\sqcap^R \otimes \mathsf{id})[(bc \otimes 1)E]) \\ &\overset{(4.46)}{=} (m \otimes n)((a \sqcap^R (bc) \otimes d)E), \end{aligned} \tag{4.47}$$

where the second equality follows by Proposition 4.1.8 (1). This proves that the image of θ is spanned by the stated elements $(m \otimes n)((a \otimes b)E)$. □

Proposition 4.4.4. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Regard any objects M and N of $\mathsf{rmd}(A)$ as firm non-unital $R := \sqcap^R(A)$–bimodules as in Proposition 4.4.1. Then the non-unital R–module tensor product $M \otimes_R N$ carries the structure of an idempotent and non-degenerate non-unital A–module too.*

Proof. Observe that, for each $c \in A$, there is a well defined linear map $\varphi_c : M \otimes N \to M \otimes N$ given by

$$\varphi_c(ma \otimes nb) := (ma \otimes n)T_3(c \otimes b) = (m \otimes nb)T_2(a \otimes c) = (m \otimes n)((a \otimes b)\Delta(c))$$

for any $a, b \in A$, $m \in M$ and $n \in N$. This map is R–balanced:

$$\begin{aligned} \varphi_c((ma) \cdot \sqcap^R(d) \otimes nb) &= (m \otimes n)((a \sqcap^R (d) \otimes b)\Delta(c)) \\ &\overset{(4.2)}{=} (m \otimes n)(a \otimes b)(\sqcap^R(d) \otimes 1)E\Delta(c) \\ &= (m \otimes n)(a \otimes b)(1 \otimes \overline{\sqcap}^L(d))E\Delta(c) \\ &\overset{(4.2)}{=} (m \otimes n)(a \otimes b)(1 \otimes \overline{\sqcap}^L(d))\Delta(c) \\ &= (ma \otimes n(b\overline{\sqcap}^L(d)))\Delta(c) = \varphi_c(ma \otimes \sqcap^R(d) \cdot (nb)), \end{aligned}$$

where in the third equaility we used Lemma 4.2.12. By this R–balancement, φ_c induces an action of A on $M \otimes_R N$ defined by

$$(ma \otimes_R nb)c = \pi((m \otimes n)((a \otimes b)\Delta(c))),$$

where $\pi : M \otimes N \to M \otimes_R N$ denotes the canonical epimorphism. This action is associative by the multiplicativity of Δ. In order to see that it is idempotent and non-degenerate, let us apply the isomorphism in Lemma 4.4.3. It takes the above A–action on $M \otimes_R N$ to

$$(m \otimes n)((a \otimes b)E)c = (m \otimes n)((a \otimes b)\Delta(c)). \tag{4.48}$$

It is an idempotent action by axiom (iv) in Definition 4.1.1. In order to see that it is non-degenerate, assume that $(m \otimes n)((a \otimes b)\Delta(c)) = 0$ for all $c \in A$. Then

$$0 = (m \otimes n)((a \otimes b)\Delta(c))(d \otimes f) = (m \otimes n)(a \otimes b)(\Delta(c)(d \otimes f)) \quad \forall c, d, f \in A$$

implies, by axiom (iv) in Definition 4.1.1, that

$$0 = (m \otimes n)(a \otimes b)(E(c \otimes d)) = (m \otimes n)((a \otimes b)E)(c \otimes d) \quad \forall c, d \in A.$$

By [40, Lemma 1.11], $M \otimes N$ is a non-degenerate non-unital $A \otimes A$–module. Hence $0 = (m \otimes n)((a \otimes b)E)$, proving the non-degeneracy of the non-unital A–module $M \otimes_R N$.

Applying the functor $U : \mathsf{rmd}(A) \to \mathsf{bim}^f(R)$ in Proposition 4.4.1 to the object $M \otimes_R N$ of $\mathsf{rmd}(A)$ above, it follows by Lemma 4.2.5 and (4.29) that the resulting non-unital R–bimodule has the actions

$$\sqcap^R(a) \cdot (m \otimes_R n) \cdot \sqcap^R(b) = (\sqcap^R(a) \cdot m) \otimes_R (n \cdot \sqcap^R(b)).$$

□

Theorem 4.4.1. *Let A be a regular weak multiplier bialgebra over a field with a right full comultiplication. Then $\mathsf{rmd}(A)$ is a monoidal category and the functor $U : \mathsf{rmd}(A) \to \mathsf{bim}^f(R)$ in Proposition 4.4.1 is strict monoidal.*

Proof. In view of Proposition 4.4.2 and Proposition 4.4.4, we only need to show that the associativity and unit constraints of $\mathsf{bim}^f(R)$ —if evaluated on objects of $\mathsf{rmd}(A)$— are morphisms of non-unital A–modules. Take any objects M, N, P in $\mathsf{rmd}(A)$. In view of Lemma 4.4.3, $(M \otimes_R N) \otimes_R P$ is isomorphic to the vector subspace of $M \otimes N \otimes P$ spanned by the elements of the form

$$\begin{aligned}((m \otimes n)((a \otimes b)E) \otimes p)((c \otimes d)E) &\overset{(4.48)}{=} (m \otimes n \otimes p)((a \otimes b \otimes 1)(\Delta \otimes \mathsf{id})((c \otimes d)E)) \\ &\overset{(v),(iv)}{=} (m \otimes n \otimes p)((a \otimes b \otimes 1)(\Delta(c) \otimes d)(1 \otimes E)),\end{aligned}$$

(for $a,b,c,d \in A$, $m \in M$, $n \in N$ and $p \in P$) hence in light of axiom (iv) in Definition 4.1.1, by elements of the form

$$(m \otimes n \otimes p)((a \otimes b \otimes d)(E \otimes 1)(1 \otimes E)) \overset{(v)}{=} (m \otimes n \otimes p)((a \otimes b \otimes d)(1 \otimes E)(E \otimes 1))$$

(for $a,b,d \in A$, $m \in M$, $n \in N$ and $p \in P$). A symmetric computation shows the isomorphism of the same vector subspace of $M \otimes N \otimes P$ to $M \otimes_R (N \otimes_R P)$, and the associator isomorphism $(M \otimes_R N) \otimes_R P \to M \otimes_R (N \otimes_R P)$ is given by the composite of these isomorphisms. Its A–module map property is thus equivalent to the equality of both induced actions

$$\begin{aligned} &((m \otimes n \otimes p)((a \otimes b \otimes c)(E \otimes 1)(1 \otimes E)))d \\ &\quad = ((m \otimes n)((a \otimes b)E) \otimes p)T_3(d \otimes c) \\ &\quad = (m \otimes nb \otimes p)((T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_3)(a \otimes d \otimes c)) \end{aligned} \tag{4.49}$$

and

$$\begin{aligned} &((m \otimes n \otimes p)((a \otimes b \otimes c)(1 \otimes E)(E \otimes 1)))d \\ &\quad = (m \otimes (n \otimes p)((b \otimes c)E))T_2(a \otimes d) \\ &\quad = (m \otimes nb \otimes p)((\mathsf{id} \otimes T_3)(T_2 \otimes \mathsf{id})(a \otimes d \otimes c)), \end{aligned} \tag{4.50}$$

where we used the equivalent forms $(m \otimes n)((a \otimes b)E)c = (ma \otimes n)T_3(c \otimes b) = (m \otimes nb)T_2(a \otimes c)$ of the action in (4.48). The actions (4.49) and (4.50) are equal by (4.15).

In order to see that the left unit constraint $R \otimes_R M \to M$, $\sqcap^R(a) \otimes_R m \mapsto m\overline{\sqcap}^L(a)$ is a morphism of non-unital right A–modules, take any $\sqcap^R(a) \otimes_R mb \in R \otimes_R M$. Applying to it the left unit constraint and next the action by any $c \in A$ results in $mb\overline{\sqcap}^L(a)c$. On the other hand, acting first by $c \in A$ on $\sqcap^R(a) \otimes_R mb$ yields $\pi((\sqcap^R(a) \otimes m)T_3(c \otimes b))$, where we used the notation $\pi : R \otimes M \to R \otimes_R M$ for the canonical epimorphism. Applying now the left unit constraint, we obtain $m(\mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})[(a \otimes 1)T_3(c \otimes b)])$. Using Lemma 4.2.4, Lemma 4.2.5 and Lemma 4.2.9 (1) in the first, second and last equalities, respectively, we see that for any $a,b,c \in A$

$$\begin{aligned} \mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})[(a \otimes 1)T_3(c \otimes b)] &= \mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})[(\overline{\sqcap}^L(a) \otimes 1)T_3(c \otimes b)] \\ &= \mu^{\mathsf{op}}(\overline{\sqcap}^L \otimes \mathsf{id})T_3(\overline{\sqcap}^L(a)c \otimes b) = b\overline{\sqcap}^L(a)c. \end{aligned}$$

This proves that the left unit constraint in $\mathsf{bim}^f(R)$ evaluated on an object M of $\mathsf{rmd}(A)$ is a morphism of non-unital A–modules. A symmetric reasoning applies for the right unit constraint. □

4.5 The antipode

The *antipode* of a Hopf algebra A is defined as the convolution inverse of the identity map $A \to A$ (cf. (2.31) and (2.46)). In a *weak* Hopf algebra, the antipode is no longer a strict inverse of the identity map in the convolution algebra of the linear maps $A \to A$. However, it is a 'weak' inverse in some sense (see Remark 2.5.13). In what follows, we equip a weak multiplier bialgebra with an antipode in the same spirit: as a generalized convolution inverse. For this, our first step is to get a generalization of the convolution product of a (weak) bialgebra.

Let A be a regular weak multiplier bialgebra over a field. By Proposition 4.3.4 (1), for all $a, b \in A$ $(ab \otimes 1)F = (\mathsf{id} \otimes \sqcap^R)T_2(a \otimes b)$ is an element of $A \otimes \mathbb{M}(A)$. So by the idempotency of A, $(a \otimes 1)F \in A \otimes \mathbb{M}(A)$ for all $a \in A$, allowing for the definition of a linear map

$$G_1 : A \otimes A \to A \otimes A, \qquad a \otimes b \mapsto (a \otimes 1)F(1 \otimes b). \tag{4.51}$$

The notation G_1 is motivated by the fact that it is the same map appearing under the same name in [73, Proposition 1.11]:

Proposition 4.5.1. *Let A be a regular weak multiplier bialgebra. The map (4.51) satisfies the equality*

$$(G_1 \otimes \mathsf{id})[\Delta_{13}(a)(1 \otimes b \otimes c)] = \Delta_{13}(a)(1 \otimes E)(1 \otimes b \otimes c), \qquad \forall a, b, c \in A.$$

Hence it is the same map denoted by G_1 in [73, Proposition 1.11].

Proof. For any $a, b, c, d \in A$,

$$\begin{aligned}(G_1 \otimes \mathsf{id})[\Delta_{13}(a)(1 \otimes bd \otimes c)] &= \Delta_{13}(a)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes b) \otimes c)\\ &= \Delta_{13}(a)(1 \otimes (\mathsf{id} \otimes \sqcap^L)T_4(d \otimes b))(1 \otimes 1 \otimes c)\\ &= \Delta_{13}(a)(1 \otimes E)(1 \otimes bd \otimes c).\end{aligned}$$

The first equality follows by Proposition 4.3.4 (1), the second one follows by Lemma

4.2.12 and the last equality follows by (4.26). So we conclude by the idempotency of A. ☐

Symmetrically to (4.51), the elements F_2, F_3 and F_4 in Proposition 4.3.5 induce maps $A \otimes A \to A \otimes A$ defined as:

$$\begin{aligned} G_2 :\ & a \otimes b \mapsto (a \otimes 1)F_2(1 \otimes b), \\ G_3 :\ & a \otimes b \mapsto (1 \otimes b)F_3(a \otimes 1), \\ G_4 :\ & a \otimes b \mapsto (1 \otimes b)F_4(a \otimes 1). \end{aligned}$$

More explicit forms of all these maps are given by

$$\begin{aligned} G_1(a \otimes bc) &= (a \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(c \otimes b)), \quad (4.52) \\ G_2(a \otimes bc) &= (a \otimes 1)((\sqcap^L \otimes \mathsf{id})T_1(b \otimes c)), \\ G_3(a \otimes bc) &= ((\sqcap^R \otimes \mathsf{id})T_2^{\mathsf{op}}(b \otimes c))(a \otimes 1), \\ G_4(a \otimes bc) &= ((\overline{\sqcap}^L \otimes \mathsf{id})T_3(c \otimes b))(a \otimes 1), \end{aligned}$$

and, equivalently, by

$$\begin{aligned} G_1(ab \otimes c) &= ((\mathsf{id} \otimes \sqcap^R)T_2(a \otimes b))(1 \otimes c), \quad (4.53) \\ G_2(ab \otimes c) &= ((\mathsf{id} \otimes \overline{\sqcap}^L)T_3^{\mathsf{op}}(b \otimes a))(1 \otimes c), \quad (4.54) \\ G_3(ab \otimes c) &= (1 \otimes c)((\mathsf{id} \otimes \overline{\sqcap}^R)T_4(b \otimes a)), \\ G_4(ab \otimes c) &= (1 \otimes c)((\mathsf{id} \otimes \sqcap^L)T_1^{\mathsf{op}}(a \otimes b)), \end{aligned}$$

for any $a, b, c \in A$. They obey the following relations:

$$G_3 = \mathsf{tw} G_1 \mathsf{tw} \quad \text{and} \quad G_4 = \mathsf{tw} G_2 \mathsf{tw}.$$

The table below collects the form of all these maps (and of $E_1 := E(- \otimes -) : A \otimes A \to A \otimes A$ and $E_2 := (- \otimes -)E : A \otimes A \to A \otimes A$) in a regular weak multiplier bialgebra and in its opposite, copposite and opposite-coopposite structures.

In [73], the form (4.51) of G_1 was proven for *regular* weak multiplier Hopf algebras, but it was left open if it has the above form for *arbitrary* weak multiplier Hopf algebras.

A $(A,\mu,\Delta,\epsilon,E)$	A^{op} $(A,\mu^{\mathsf{op}},\Delta,\epsilon,E)$	A_{cop} $(A,\mu,\Delta^{\mathsf{op}},\epsilon,\mathsf{tw}E\mathsf{tw})$	$A^{\mathsf{op}}_{\mathsf{cop}}$ $(A,\mu^{\mathsf{op}},\Delta^{\mathsf{op}},\epsilon,\mathsf{tw}E\mathsf{tw})$
E_1	E_2	$\mathsf{tw}E_1\mathsf{tw}$	$\mathsf{tw}E_2\mathsf{tw}$
F	F_3	F_2	F_4
G_1	G_3	G_2	G_4
G_2	G_4	G_1	G_3
G_3	G_1	G_4	G_2
G_4	G_2	G_3	G_1

Table 4.3: E_i, F_i and G_i in the symmetric weak multiplier bialgebras.

Proposition 4.5.2. *Let A be a regular weak multiplier bialgebra over a field. If the comultiplication is left and right full, then the following hold.*

(1) The image of the map $E_1 : A\otimes A \to A\otimes A$, $a\otimes b \mapsto E(a\otimes b)$ is isomorphic to the non-unital $\sqcap^L(A)$–module tensor square of A with respect to the actions

$$\sqcap^L(a)\cdot b := \sqcap^L(a)b \qquad \textit{and} \qquad b\cdot \sqcap^L(a) := \overline{\sqcap}^R(a)b, \quad \textit{for } a,b\in A.$$

(2) The image of the map $G_1 : A\otimes A \to A\otimes A$ in (4.51) is isomorphic to the non-unital $\sqcap^R(A)$–module tensor square of A with respect to the actions

$$\sqcap^R(a)\cdot b := \sqcap^R(a)b \qquad \textit{and} \qquad b\cdot \sqcap^R(a) := b\sqcap^R(a), \quad \textit{for } a,b\in A.$$

Proof. (1). By the $A - A^{\mathsf{op}}_{\mathsf{cop}}$ symmetric version of (4.47), E_1 is equal to the map

$$a\otimes \sqcap^L(bc)d \mapsto ((\overline{\sqcap}^R\otimes\sqcap^L)T_1(b\otimes c))(a\otimes d)$$

whose image is equal to the stated module tensor product.

(2). Since $\sqcap^R(A)$ is a coseparable coalgebra by Theorem 4.3.3 (1), it follows by [19, Proposition 2.17] that the stated module tensor product is isomorphic to the image of the map

$$a\sqcap^R(b)\otimes c \mapsto (a\otimes 1)(\delta_{\sqcap^R(A)}\sqcap^R(b))(1\otimes c) = (a\sqcap^R(b)\otimes 1)F(1\otimes c) = G_1(a\sqcap^R(b)\otimes c),$$

where $F\in\mathbb{M}(A\otimes A)$ appeared in Proposition 4.3.4 (1) and $\delta_{\sqcap^R(A)} : \sqcap^R(A)\to\sqcap^R(A)\otimes$

$\sqcap^R(A)$ is the comultiplication in Proposition 4.3.4 (3). Since by Lemma 4.2.9 (4) and by the idempotency of A any element of A is a linear combination of elements of the form $a \sqcap^R (b)$, for $a, b \in A$, we have the claim proven. □

For any weak multiplier bialgebra A over a field, consider the vector space

$$\mathcal{L} := \left\{ L : A \otimes A \to A \otimes A \mid \begin{array}{l} L(a \otimes bc) = L(a \otimes b)(1 \otimes c) \ \forall a, b, c \in A, \\ (T_2 \otimes \mathsf{id})(\mathsf{id} \otimes L) = (\mathsf{id} \otimes L)(T_2 \otimes \mathsf{id}) \end{array} \right\}.$$

There is a linear map

$$\mathcal{L} \to \mathsf{Lin}(A, \mathsf{End}_A(A)), \qquad L \mapsto [\lambda_L : a \mapsto (\epsilon \otimes \mathsf{id})L(a \otimes -)]. \tag{4.55}$$

With its help, for any $a, b, c \in A$ and $L \in \mathcal{L}$,

$$\begin{aligned}((\mathsf{id} \otimes \lambda_L) T_2 (a \otimes b))(1 \otimes c) &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes L)(T_2 \otimes \mathsf{id})(a \otimes b \otimes c) \\ &= (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes L)(a \otimes b \otimes c) \overset{\text{(iii)}}{=} (a \otimes 1)L(b \otimes c).\end{aligned} \tag{4.56}$$

Applying this together with the non-degeneracy of the multiplication in $A \otimes A$, we conclude that the map (4.55) is injective. Clearly, $\mathcal{L}$ is an algebra via the composition of maps. For $L, L' \in \mathcal{L}$ and $a \in A$,

$$\lambda_{L'L}(a) = (\epsilon \otimes \mathsf{id})L'L(a \otimes -) = \mu(\lambda_{L'} \otimes \mathsf{id})L(a \otimes -) \tag{4.57}$$

(where $\mu : \mathsf{End}_A(A) \otimes A \to A$ denotes the evaluation map $\Phi \otimes a \mapsto \Phi a \equiv \Phi(a)$), generalizing the convolution product $(\lambda_{L'} * \lambda_L)(a) = \mu(\lambda_{L'} \otimes \lambda_L)\Delta(a)$ (cf. (2.31)) of endomorphisms $\lambda_{L'}$ and λ_L on a (weak) bialgebra.

Proposition 4.5.3. *Let A be a regular weak multiplier bialgebra over a field.*

(1) The maps T_1, $E_1 := E(- \otimes -)$ and G_1 in (4.51) from $A \otimes A$ to $A \otimes A$ are elements of $\mathcal{L}$.

(2) The map (4.55) takes the elements of $\mathcal{L}$ in part (1) to $[a \mapsto a(-)]$, $[a \mapsto \sqcap^L(a)(-)]$ and $[a \mapsto \sqcap^R(a)(-)]$, respectively.

(3) $E_1^2 = E_1$, $G_1^2 = G_1$ and $E_1 T_1 = T_1 = T_1 G_1$.

Proof. (1). Evidently, all of T_1, E_1 and G_1 are right non-unital A–module maps. The compatibility of T_1 with T_2 (in the definition of $\mathcal{L}$) is axiom (ii) in Definition 4.1.1. The compatibility of E_1 with T_2 follows in the same way as in [73, Proposition 2.2]: For all $a, b, c \in A$,

$$\begin{aligned}(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(a \otimes b \otimes c) &= (a \otimes 1 \otimes 1)(\Delta \otimes \mathsf{id})(E(b \otimes c)) \\ &\overset{(v)}{=} (a \otimes 1 \otimes 1)(1 \otimes E)(E \otimes 1)(\Delta(b) \otimes c) \\ &\overset{(iv)}{=} (1 \otimes E)(a \otimes 1 \otimes 1)(\Delta(b) \otimes c) \\ &= (\mathsf{id} \otimes E_1)(T_2 \otimes \mathsf{id})(a \otimes b \otimes c).\end{aligned}$$

It remains to prove the compatibility of G_1 with T_2. Denoting $T_4(d \otimes c) =: d_i \otimes c_i$ and using the multiplicativity of $\overline{\Delta}$ in the second equality, and Lemma 4.2.5 together with (4.2) in the third one, it follows for any $a, b, c, d \in A$ that

$$\begin{aligned}(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)(a \otimes b \otimes cd) &\overset{(4.52)}{=} (a \otimes 1 \otimes 1)(\Delta \otimes \mathsf{id})[(b \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c))] \\ &= (a \otimes 1 \otimes 1)(\Delta(b)\overline{\Delta}\overline{\sqcap}^R(c_i) \otimes d_i) \\ &= (a \otimes 1 \otimes 1)(\Delta(b)(1 \otimes \overline{\sqcap}^R(c_i)) \otimes d_i) \\ &= (a \otimes 1 \otimes 1)(\Delta(b) \otimes 1)(1 \otimes (\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c)) \\ &\overset{(4.52)}{=} (\mathsf{id} \otimes G_1)(T_2 \otimes \mathsf{id})(a \otimes b \otimes cd),\end{aligned}$$

from which we conclude by the idempotency of A.

(2). Take any $a, b \in A$. By axiom (iii), $(\epsilon \otimes \mathsf{id})T_1(a \otimes b) = ab$. By (4.24), $(\epsilon \otimes \mathsf{id})E_1(a \otimes b) = \sqcap^L(a)b$. Finally, using Lemma 4.2.3 in the second equality, it follows for all $a, b, c \in A$ that

$$\begin{aligned}(\epsilon \otimes \mathsf{id})G_1(a \otimes bc) &\overset{(4.52)}{=} (\mathsf{id} \otimes \epsilon)[(1 \otimes a)(\mathsf{id} \otimes \overline{\sqcap}^R)T_4(c \otimes b)] \\ &= (\mathsf{id} \otimes \epsilon)[(1 \otimes a)T_4(c \otimes b)] \\ &\overset{(4.3)}{=} (\mathsf{id} \otimes \epsilon)[T_3(b \otimes a)(c \otimes 1)] \overset{(4.22)}{=} \sqcap^R(a)bc.\end{aligned}$$

(3). $E_1^2 = E_1$ is evident by the fact that E is an idempotent element of $\mathbb{M}(A \otimes A)$. By Lemma 4.2.5, for any $a, b, c \in A$,

$$(1 \otimes \overline{\sqcap}^R(a))T_4(b \otimes c) = T_4(b \otimes \overline{\sqcap}^R(a)c). \tag{4.58}$$

For any $b, c \in A$, denote $T_4(c \otimes b) =: c_i \otimes b_i$. Then by Lemma 4.2.9 (2),

$$\overline{\sqcap}^R(b_i)c_i = \mu^{\mathsf{op}}(\mathsf{id} \otimes \overline{\sqcap}^R)T_4(c \otimes b) = bc. \tag{4.59}$$

With these identities at hand, and applying Lemma 4.2.6 in the second equality, it follows for $a, b, c, d \in A$ that

$$\begin{aligned} G_1^2(a \otimes bcd) &\overset{(4.52)}{=} (a\overline{\sqcap}^R(b_i) \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c_i)) \\ &= (a \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})[(\overline{\sqcap}^R(b_i) \otimes 1)T_4^{\mathsf{op}}(d \otimes c_i)]) \\ &\overset{(4.58)}{=} (a \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes \overline{\sqcap}^R(b_i)c_i) \\ &\overset{(4.59)}{=} (a \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes bc)) \overset{(4.52)}{=} G_1(a \otimes bcd), \end{aligned}$$

proving $G_1^2 = G_1$. The equality $E_1T_1 = T_1$ is immediate by (4.2). Finally, using again the notation $T_4(c \otimes b) =: c_i \otimes b_i$, for all $a, b, c \in A$

$$T_1G_1(a\otimes bc) \overset{(4.52)}{=} \Delta(a\overline{\sqcap}^R(b_i))(1\otimes c_i) = \Delta(a)(1\otimes\overline{\sqcap}^R(b_i)c_i) \overset{(4.59)}{=} \Delta(a)(1\otimes bc) = T_1(a\otimes bc).$$

The second equality follows by the multiplicativity of $\overline{\Delta}$, Lemma 4.2.5 and axiom (iv). □

Applying the same reasoning as in [73, Proposition 2.3], the following is shown.

Proposition 4.5.4. *Let A be a regular weak multiplier bialgebra over a field. If there is a linear map $R_1 : A \otimes A \to A \otimes A$ such that $R_1T_1 = G_1$, $T_1R_1 = E_1$ and $R_1T_1R_1 = R_1$, then $R_1 \in \mathcal{L}$.*

Proof. For any $a, b, c \in A$,

$$\begin{aligned} R_1(a \otimes bc) &= R_1T_1R_1(a \otimes bc) = R_1(T_1R_1(a \otimes b)(1 \otimes c)) \\ &= R_1(T_1(R_1(a \otimes b)(1 \otimes c))) = R_1T_1R_1(a \otimes b)(1 \otimes c) = R_1(a \otimes b)(1 \otimes c). \end{aligned}$$

In the first equality we used the assumption $R_1T_1R_1 = R_1$; in the second one we used the assumption $T_1R_1 = E_1$ and that $E_1 \in \mathcal{L}$ by Proposition 4.5.3 (1); in the third equality we used that, by the same proposition, also $T_1 \in \mathcal{L}$; and in the last equality we used the assumption that $R_1T_1 = G_1$ and that $G_1 \in \mathcal{L}$ by the mentioned proposition. As for the other condition concerns,

$$\begin{aligned}(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1) &= (T_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1T_1R_1) = (\mathsf{id} \otimes R_1T_1)(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1) \\ &= (\mathsf{id} \otimes R_1)(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(\mathsf{id} \otimes R_1) = (\mathsf{id} \otimes R_1T_1R_1)(T_2 \otimes \mathsf{id}) \\ &= (\mathsf{id} \otimes R_1)(T_2 \otimes \mathsf{id}),\end{aligned}$$

where in the first, second and third equalities the same arguments as in the above first, second and third equalities were used respectively. □

Symmetrically to the above considerations, we can define

$$\mathcal{R} := \left\{ K : A \otimes A \to A \otimes A \mid \quad \begin{array}{l} K(ab \otimes c) = (a \otimes 1)K(b \otimes c) \ \forall a, b, c \in A, \\ (\mathsf{id} \otimes T_1)(K \otimes \mathsf{id}) = (K \otimes \mathsf{id})(\mathsf{id} \otimes T_1) \end{array} \right\}.$$

There is an injective linear map

$$\mathcal{R} \to \mathsf{Lin}(A, {}_A\mathsf{End}(A)), \qquad K \mapsto [\rho_K : a \mapsto (\mathsf{id} \otimes \epsilon)K(- \otimes a)] \tag{4.60}$$

such that

$$(a \otimes 1)((\rho_K \otimes \mathsf{id})T_1(b \otimes c)) = K(a \otimes b)(1 \otimes c), \qquad \forall a, b, c \in A. \tag{4.61}$$

For $K, K' \in \mathcal{R}$,

$$\rho_{K'K}(a) = \mu(\mathsf{id} \otimes \rho_{K'})K(- \otimes a), \quad \forall a \in A, \tag{4.62}$$

(where $\mu : A \otimes {}_A\mathsf{End}(A) \to A$ denotes the evaluation map $a \otimes \Phi \mapsto a\Phi \equiv \Phi(a)$.) The linear map (4.60) takes the elements T_2, $E_2 := (- \otimes -)E$ and G_2 (cf. (4.54)) of $\mathcal{R}$ to $[a \mapsto (-)a]$, $[a \mapsto (-)\sqcap^R(a)]$ and $[a \mapsto (-)\sqcap^L(a)]$, respectively. The equalities $E_2^2 = E_2$, $G_2^2 = G_2$ and $E_2T_2 = T_2 = T_2G_2$ hold. If there is a linear map $R_2 : A \otimes A \to A \otimes A$ such that $R_2T_2 = G_2$, $T_2R_2 = E_2$ and $R_2T_2R_2 = R_2$, then $R_2 \in \mathcal{R}$.

Proposition 4.5.5. *Let A be a regular weak multiplier bialgebra over a field and for $i \in \{1, 2\}$, let $E_i, G_i : A \otimes A \to A \otimes A$ be the same maps as before. Then the following hold.*

(1) $(\mathsf{id} \otimes E_1)(E_2 \otimes \mathsf{id}) = (E_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)$.

(2) $(\mathsf{id} \otimes G_1)(G_2 \otimes \mathsf{id}) = (G_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)$.

(3) $(\mathsf{id} \otimes G_1)(E_2 \otimes \mathsf{id}) = (E_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)$.

(4) $(\mathsf{id} \otimes E_1)(G_2 \otimes \mathsf{id}) = (G_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)$.

Proof. Assertion (1) is evident and (2) is checked by the following computation for any $a, b, c, d, e \in A$, where we use the notations $T_3^{\mathsf{op}}(b \otimes a) =: b_i \otimes a_i$ and $T_4^{\mathsf{op}}(e \otimes d) =: e_j \otimes d_j$.

$$\begin{aligned}
(\mathsf{id} \otimes G_1)(G_2 \otimes \mathsf{id})(ab \otimes c \otimes de) &\overset{(4.54)}{=} (\mathsf{id} \otimes G_1)([(\mathsf{id} \otimes \overline{\sqcap}^L)T_3^{\mathsf{op}}(b \otimes a)](1 \otimes c) \otimes de) \\
&= (\mathsf{id} \otimes G_1)(b_i \otimes \overline{\sqcap}^L(a_i)c \otimes de) \\
&\overset{(4.52)}{=} x_i \otimes (\overline{\sqcap}^L(y_i)c \otimes 1)[(\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(e \otimes d)] \\
&= b_i \otimes \overline{\sqcap}^L(a_i)c\overline{\sqcap}^R(e_j) \otimes d_j \\
&\overset{(4.54)}{=} (G_2 \otimes \mathsf{id})(ab \otimes c\overline{\sqcap}^R(e_j) \otimes d_j) \\
&= (G_2 \otimes \mathsf{id})(ab \otimes (c \otimes 1)[(\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(e \otimes d)]) \\
&\overset{(4.52)}{=} (G_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)(ab \otimes c \otimes de)
\end{aligned}$$

Concerning (3), take any $a, b, c, d \in A$ and denote $T_4(d \otimes c) =: d_j \otimes c_j$ and $E_2(a \otimes b) =: a_i \otimes b_i$. Then using Lemma 4.2.5 in the fourth equality,

$$\begin{aligned}
(\mathsf{id} \otimes G_1)(E_2 \otimes \mathsf{id})(a \otimes b \otimes cd) &\overset{(4.52)}{=} a_i \otimes ((b_i \otimes 1)[(\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c)]) \\
&= (a_i \otimes b_i)(1 \otimes \overline{\sqcap}^R(c_j)) \otimes d_j = (a \otimes b)E(1 \otimes \overline{\sqcap}^R(c_j)) \otimes d_j \\
&= (a \otimes b\overline{\sqcap}^R(c_j))E \otimes d_j = (a \otimes [(b \otimes 1)((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c))])(E \otimes 1) \\
&\overset{(4.52)}{=} (E_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)(a \otimes b \otimes cd).
\end{aligned}$$

Part (4) is proven symmetrically. □

Consider the vector subspace

$$\mathcal{M} := \{(L, K) \in \mathcal{L} \times \mathcal{R} \mid a((\epsilon \otimes \mathsf{id})L(b \otimes c)) = ((\mathsf{id} \otimes \epsilon)K(a \otimes b))c, \ \forall a, b, c \in A\}$$

of $\mathcal{L} \times \mathcal{R}$. The maps (4.55) and (4.60) induce a linear map

$$\mathcal{M} \to \mathsf{Lin}(A, \mathbb{M}(A)), \qquad (L, K) \mapsto [a \mapsto (\lambda_L(a), \rho_K(a))].$$

For $(L,K) \in \mathcal{M}$, assume that $L = 0$. Then for any $a \in A$, in $(\lambda_L(a), \rho_K(a)) \in \mathbb{M}(A)$ the component $\lambda_L(a)$ is zero. Hence also $\rho_K(a) = 0$ for any $a \in A$ so $\rho_K = 0$. Thus by the injectivity of (4.60), also $K = 0$. Symmetrically, $K = 0$ implies $L = 0$.

By part (2) of Proposition 4.5.3 and its $A - A^{\mathsf{op}}_{\mathsf{cop}}$ symmetric counterpart, (T_1, T_2), (E_1, G_2) and (G_1, E_2) are elements of $\mathcal{M}$. For $i \in \{1,2\}$, assume that there exist linear maps $R_i : A \otimes A \to A \otimes A$ such that $R_i T_i = G_i$, $T_i R_i = E_i$ and $R_i T_i R_i = R_i$. Then $(R_1, R_2) \in \mathcal{L} \times \mathcal{R}$, and our next aim is to show that in fact $(R_1, R_2) \in \mathcal{M}$.

Proposition 4.5.6. *Let A be a regular weak multiplier bialgebra over a field and for $i \in \{1,2\}$, let $E_i, G_i, R_i : A \otimes A \to A \otimes A$ be the same maps as before. Then the following hold.*

(1) $(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id}) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)$.

(2) $(\mathsf{id} \otimes R_1)(E_2 \otimes \mathsf{id}) = (E_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)$.

(3) $(\mathsf{id} \otimes G_1)(R_2 \otimes \mathsf{id}) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)$.

(4) $(\mathsf{id} \otimes R_1)(G_2 \otimes \mathsf{id}) = (G_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)$.

(5) $(\mathsf{id} \otimes R_1)(R_2 \otimes \mathsf{id}) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)$.

Proof. (1). Applying part (4) of Proposition 4.5.5 in the second equality and its part (1) in the penultimate equality,

$$\begin{aligned}(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id}) &= (\mathsf{id} \otimes E_1)(G_2 R_2 \otimes \mathsf{id}) = (G_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id}) \\ &= (R_2 T_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(R_2 \otimes \mathsf{id}) \overset{E_1 \in \mathcal{L}}{=} (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(T_2 R_2 \otimes \mathsf{id}) \\ &= (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1)(E_2 \otimes \mathsf{id}) = (R_2 E_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1) = (R_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1).\end{aligned}$$

(2).

$$\begin{aligned}(\mathsf{id} \otimes R_1)(E_2 \otimes \mathsf{id}) &= (\mathsf{id} \otimes R_1 E_1)(E_2 \otimes \mathsf{id}) = (\mathsf{id} \otimes R_1)(E_2 \otimes \mathsf{id})(\mathsf{id} \otimes E_1) \\ &= (\mathsf{id} \otimes R_1)(E_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1 R_1) = (\mathsf{id} \otimes R_1)(\mathsf{id} \otimes T_1)(E_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1) \\ &= (\mathsf{id} \otimes G_1)(E_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1) = (E_2 \otimes \mathsf{id})(\mathsf{id} \otimes G_1)(\mathsf{id} \otimes R_1) \\ &= (E_2 \otimes \mathsf{id})(\mathsf{id} \otimes R_1)\end{aligned}$$

(3) and (4). Part (3) is analogously proven to (4), which is checked by

$$\begin{aligned}(\mathsf{id}\otimes R_1)(G_2\otimes \mathsf{id})(\mathsf{id}\otimes R_1E_1)(G_2\otimes \mathsf{id}) &= (\mathsf{id}\otimes R_1)(G_2\otimes \mathsf{id})(\mathsf{id}\otimes E_1)\\ &= (\mathsf{id}\otimes R_1)(G_2\otimes \mathsf{id})(\mathsf{id}\otimes T_1R_1) = (\mathsf{id}\otimes R_1)(\mathsf{id}\otimes T_1)(G_2\otimes \mathsf{id})(\mathsf{id}\otimes R_1)\\ &= (\mathsf{id}\otimes G_1)(G_2\otimes \mathsf{id})(\mathsf{id}\otimes R_1) = (G_2\otimes \mathsf{id})(\mathsf{id}\otimes G_1)(\mathsf{id}\otimes R_1)\\ &= (G_2\otimes \mathsf{id})(\mathsf{id}\otimes R_1).\end{aligned}$$

(5). Using part (1) of the current proposition in the second equality and its part (3) in the penultimate equality,

$$\begin{aligned}(\mathsf{id}\otimes R_1)(R_2\otimes \mathsf{id}) &= (\mathsf{id}\otimes R_1E_1)(R_2\otimes \mathsf{id}) = (\mathsf{id}\otimes R_1)(R_2\otimes \mathsf{id})(\mathsf{id}\otimes E_1)\\ &= (\mathsf{id}\otimes R_1)(R_2\otimes \mathsf{id})(\mathsf{id}\otimes T_1R_1) \overset{R_2\in\mathcal{R}}{=} (\mathsf{id}\otimes R_1T_1)(R_2\otimes \mathsf{id})(\mathsf{id}\otimes R_1)\\ &= (\mathsf{id}\otimes G_1)(R_2\otimes \mathsf{id})(\mathsf{id}\otimes R_1) = (R_2\otimes \mathsf{id})(\mathsf{id}\otimes G_1R_1) = (R_2\otimes \mathsf{id})(\mathsf{id}\otimes R_1).\end{aligned}$$

□

Corollary 4.5.7. *Let A be a regular weak multiplier bialgebra over a field and for $i\in\{1,2\}$, let $T_i, E_i, G_i : A\otimes A\to A\otimes A$ be the same maps as before. Assume that there exist linear maps $R_i : A\otimes A\to A\otimes A$ such that $R_iT_i = G_i$, $T_iR_i = E_i$ and $R_iT_iR_i = R_i$. Then $(R_1, R_2)\in\mathcal{M}$, hence there is a corresponding linear map $S := (\lambda_{R_1}, \rho_{R_2}) : A\to\mathbb{M}(A)$.*

Proof. Using in the second equality that $(G_1, E_2)\in\mathcal{M}$, it follows for any $a, b, c\in A$ that

$$\begin{aligned}a(\epsilon\otimes\mathsf{id})R_1(b\otimes c) &= a(\epsilon\otimes\mathsf{id})G_1R_1(b\otimes c)\\ &= \mu(\mathsf{id}\otimes\epsilon\otimes\mathsf{id})(E_2\otimes\mathsf{id})(\mathsf{id}\otimes R_1)(a\otimes b\otimes c)\\ &= \mu(\mathsf{id}\otimes\epsilon\otimes\mathsf{id})(T_2R_2\otimes\mathsf{id})(\mathsf{id}\otimes R_1)(a\otimes b\otimes c)\\ &\overset{\text{(iii)}}{=} \mu(\mu\otimes\mathsf{id})(R_2\otimes\mathsf{id})(\mathsf{id}\otimes R_1)(a\otimes b\otimes c).\end{aligned}$$

Symmetrically, using in the second equality that $(E_1, G_2)\in\mathcal{M}$,

$$\begin{aligned}(\mathsf{id}\otimes\epsilon)R_2(a\otimes b)c &= (\mathsf{id}\otimes\epsilon)G_2R_2(a\otimes b)c\\ &= \mu(\mathsf{id}\otimes\epsilon\otimes\mathsf{id})(\mathsf{id}\otimes E_1)(R_2\otimes\mathsf{id})(a\otimes b\otimes c)\\ &= \mu(\mathsf{id}\otimes\epsilon\otimes\mathsf{id})(\mathsf{id}\otimes T_1R_1)(R_2\otimes\mathsf{id})(a\otimes b\otimes c)\\ &\overset{\text{(iii)}}{=} \mu(\mathsf{id}\otimes\mu)(\mathsf{id}\otimes R_1)(R_2\otimes\mathsf{id})(a\otimes b\otimes c).\end{aligned}$$

We conclude that the expressions above are equal by the associativity of μ and Proposition 4.5.6 (5). ☐

The map $S : A \to \mathbb{M}(A)$ in Corollary 4.5.7 —whenever it exists— will be termed the *antipode* for the following reason.

Theorem 4.5.1. *For any regular weak multiplier bialgebra A over a field, there is a bijective correspondence between the following data.*

(1) For $i \in \{1,2\}$, a linear map $R_i : A \otimes A \to A \otimes A$ such that $R_iT_i = G_i$, $T_iR_i = E_i$ and $R_iT_iR_i = R_i$.

(2) A linear map $S : A \to \mathbb{M}(A)$ satisfying for all $a, b, c \in A$

(vii) $T_1[((\mathsf{id} \otimes S)T_2(a \otimes b))(1 \otimes c)] = \Delta(a)(b \otimes c)$,

(viii) $T_2[(a \otimes 1)((S \otimes \mathsf{id})T_1(b \otimes c))] = (a \otimes b)\Delta(c)$,

(ix) $\mu(S \otimes \mathsf{id})[E(a \otimes 1)] = S(a)$ *(equivalently,* $\mu(\mathsf{id} \otimes S)[(1 \otimes a)E] = S(a)$*).*

Proof. (1)↦(2). By Corollary 4.5.7, there is a linear map $(\lambda_{R_1}, \rho_{R_2}) =: S : A \to \mathbb{M}(A)$. Using in the third equality that $T_1R_1 = E_1$,

$$\begin{aligned} T_1[((\mathsf{id} \otimes S)T_2(a \otimes b))(1 \otimes c)] &\overset{(4.56)}{=} T_1[(a \otimes 1)R_1(b \otimes c)] \underset{(4.1)}{\overset{(2.20)}{=}} \Delta(a)T_1R_1(b \otimes c) \\ &= \Delta(a)E(b \otimes c) \overset{(4.2)}{=} \Delta(a)(b \otimes c) \end{aligned}$$

for all $a, b, c \in A$, so that (vii) holds. Symmetrically, (viii) follows by $T_2R_2 = E_2$. Using in the second equality $R_1E_1 = R_1$,

$$\mu(S \otimes \mathsf{id})[E(a \otimes b)] \overset{(4.57)}{=} \lambda_{R_1E_1}(a)b = \lambda_{R_1}(a)b = S(a)b,$$

for all $a, b \in A$, proving the first form of (ix). The second form follows symmetrically by $R_2E_2 = R_2$.

(2)↦(1). By axiom (iv) in Definition 4.1.1,

$$\mathsf{Im}(T_1) \subseteq \langle E(a \otimes b) \mid a, b \in A\rangle = \langle \Delta(a)(b \otimes c) \mid a, b, c \in A\rangle.$$

Conversely, by (vii), $\langle \Delta(a)(b \otimes c) \mid a, b, c \in A\rangle \subseteq \mathsf{Im}(T_1)$, so that $\mathsf{Im}(T_1) = \mathsf{Im}(E_1)$. By Proposition 4.5.3 (3), $T_1G_1 = T_1$ so that $\mathsf{Ker}(G_1) \subseteq \mathsf{Ker}(T_1)$. In order to see the converse, note that applying $\mathsf{id} \otimes \epsilon$ to both sides of (viii) and making use of the counitality axiom (iii) and (4.22), we conclude, since A is non-degenerate, that

$$\mu(S \otimes \mathsf{id})T_1 = \mu(\sqcap^R \otimes \mathsf{id}). \tag{4.63}$$

Assume that for some $b, c \in A$, $T_1(b \otimes c) = 0$. Then for all $a \in A$,

$$\begin{aligned} 0 &= (\mathsf{id} \otimes \mu)(\mathsf{id} \otimes S \otimes \mathsf{id})(T_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(a \otimes b \otimes c) \\ &\overset{\text{(ii)}}{=} (\mathsf{id} \otimes \mu)(\mathsf{id} \otimes S \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(T_2 \otimes \mathsf{id})(a \otimes b \otimes c) \\ &\overset{(4.63)}{=} (\mathsf{id} \otimes \mu)(\mathsf{id} \otimes \sqcap^R \otimes \mathsf{id})(T_2 \otimes \mathsf{id})(a \otimes b \otimes c) \overset{(4.53)}{=} G_1(ab \otimes c) = (a \otimes 1)G_1(b \otimes c). \end{aligned}$$

In the last equality we used that G_1 in (4.51) is a morphism of left non-unital A–modules. By the non-degeneracy of the multiplication in $A \otimes A$, this proves $G_1(b \otimes c) = 0$ hence $\mathsf{Ker}(G_1) = \mathsf{Ker}(T_1)$. By the same reasoning applied in [73, Proposition 2.3], the above information about the image and the kernel of T_1 implies that there is a linear map $R_1 : A \otimes A \to A \otimes A$ with the desired properties. A bit more explicitly, for any $a, b \in A$,

$$R_1 : T_1(a \otimes b) \mapsto G_1(a \otimes b), \tag{4.64}$$

gives R_1 on $\mathsf{Im}(E_1) = \mathsf{Im}(T_1)$, while R_1 is defined as zero on $\mathsf{Ker}(E_1)$. The map R_2 is constructed symmetrically. Note that we did not make use of property (ix) so far.

It remains to see the bijectivity of the above correspondence. From the expression (4.64) of R_1, it is clear that it does not depend on the actual choice of the map S in part (2) (only on its existence). Hence starting with the data (R_1, R_2) as in part (1), we get from the relation $R_1 T_1 = G_1$ that R_1 must be defined by (4.64) on $\mathsf{Im}(T_1)$; and because $R_1 = R_1 E_1$, R_1 must be equal to zero on $\mathsf{Ker}(E_1)$. Similarly for R_2. Conversely, starting with a map S as in part (2) and iterating the above constructions $S \mapsto (R_1, R_2) \mapsto (\lambda_{R_1}, \rho_{R_2})$, we obtain the map $\lambda_{R_1} : A \to \mathsf{End}_A(A)$ taking $a \in A$ to

$$\begin{aligned} b &\mapsto (\epsilon \otimes \mathsf{id})R_1(a \otimes b) = (\epsilon \otimes \mathsf{id})G_1 R_1(a \otimes b) = \mu(\sqcap^R \otimes \mathsf{id})R_1(a \otimes b) \\ &\overset{(4.63)}{=} \mu(S \otimes \mathsf{id})T_1 R_1(a \otimes b) = \mu(S \otimes \mathsf{id})[E(a \otimes b)]. \end{aligned}$$

In the second equality we used Proposition 4.5.3 (2). This element $\lambda_{R_1}(a)b$ is equal to

$S(a)b$ for all $a, b \in A$ if and only if the first form of (ix) holds. Symmetrically, $a\rho_{R_2}(b)$ is equal to $aS(b)$ for all $a, b \in A$ if and only if the second form of (ix) holds, what proves in particular the equivalence of both stated forms of (ix). □

Theorem 4.5.1 implies in particular that if the antipode exists then it is unique.

Remark 4.5.8. Let us stress that the antipode axioms in part (2) of Theorem 4.5.1 imply the identities

$$\mu(S \otimes \mathsf{id})T_1 = \mu(\sqcap^R \otimes \mathsf{id}), \quad \mu(\mathsf{id} \otimes S)T_2 = \mu(\mathsf{id} \otimes \sqcap^L), \quad \mu(S \otimes \mathsf{id})E_1 = \mu(S \otimes \mathsf{id}) \tag{4.65}$$

and, equivalently,

$$\mu(S \otimes \mathsf{id})T_1 = \mu(\sqcap^R \otimes \mathsf{id}), \quad \mu(\mathsf{id} \otimes S)T_2 = \mu(\mathsf{id} \otimes \sqcap^L), \quad \mu(\mathsf{id} \otimes S)E_2 = \mu(\mathsf{id} \otimes S), \tag{4.66}$$

expressing the requirement that S is the (widely generalized) convolution inverse of the map $A \to \mathbb{M}(A)$, $a \mapsto (a(-), (-)a)$. Indeed, the third identity in (4.65) is literally item (ix) in part (2) of Theorem 4.5.1. As for the first and second ones concerns, for any $a, b, c \in A$,

$$\begin{aligned} c\mu(\sqcap^R \otimes \mathsf{id})(a \otimes b) &\overset{(4.22)}{=} c(\mathsf{id} \otimes \epsilon)T_3(b \otimes a) \overset{(4.3)}{=} (\mathsf{id} \otimes \epsilon)((c \otimes a)\Delta(b)) \\ &\overset{(viii)}{=} (\mathsf{id} \otimes \epsilon)T_2[(c \otimes 1)((S \otimes \mathsf{id})T_1(a \otimes b))] \\ &\overset{(iii)}{=} c\mu(S \otimes \mathsf{id})T_1(a \otimes b), \end{aligned}$$

$$\begin{aligned} \mu(\mathsf{id} \otimes \sqcap^L)(a \otimes b)c &\overset{(4.25)}{=} (\epsilon \otimes \mathsf{id})T_4(b \otimes a)c \overset{(4.3)}{=} (\epsilon \otimes \mathsf{id})(\Delta(a)(b \otimes c)) \\ &\overset{(vii)}{=} (\epsilon \otimes \mathsf{id})T_1[((\mathsf{id} \otimes S)T_2(a \otimes b))(1 \otimes c)] \\ &\overset{(iii)}{=} \mu[(\mathsf{id} \otimes S)T_2(a \otimes b)(1 \otimes c)]. \end{aligned}$$

However, the identities in (4.65) (or, equivalently, in (4.66)) do not seem to be equivalent to the axioms (vii)-(ix).

Let us compare now the structure of a weak multiplier bialgebra with an antipode in the just presented sense with a (regular or not) weak multiplier Hopf algebra in [73]. Combining Theorem 4.1.2 and Theorem 4.5.1, we conclude that any regular weak multiplier Hopf algebra in the sense of [73] is a regular weak multiplier bialgebra in the sense of this thesis possessing an antipode. On the other hand, if a regular weak multiplier bialgebra in the sense of this thesis admits an antipode, then it is also a weak multiplier Hopf algebra —though not necessarily a regular one— in the sense of [73]. That is to say, regular weak multiplier bialgebras possessing an antipode here are *between regular and arbitrary* weak multiplier Hopf algebras in [73].

In view of Theorem 4.1.3, an algebra possesses a weak Hopf algebra structure as in [18] if and only if via the same structure maps, it is a regular weak multiplier bialgebra with an antipode.

From Theorem 4.5.1 and Example 4.1.16, we obtain the following example.

Example 4.5.9. For a family $\{A_j\}_{j\in I}$ of regular weak multiplier bialgebras over a field, labelled by any index set I, the direct sum regular weak multiplier bialgebra $\oplus_{j\in I}A_j$ in Example 4.1.16 possesses an antipode if and only if A_j does, for all $j \in I$. In this case, for any $\underline{a} \in A$, $S(\underline{a}) = \varphi^{-1}(\{S_j(a_j)\}_{j\in I})$ in terms of the map (4.17) and the antipode S_j of A_j.

Our next task is to investigate the properties of the antipode.

Lemma 4.5.10. *Let A be a regular weak multiplier bialgebra over a field. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. For $i \in \{1,2\}$, denote by $R_i : A \otimes A \to A \otimes A$ the corresponding maps in Theorem 4.5.1 (1). Then the following hold.*

$$\mu R_1 = \mu(\sqcap^L \otimes \mathsf{id}) \qquad \textit{and} \qquad \mu R_2 = \mu(\mathsf{id} \otimes \sqcap^R).$$

Proof. For any $a, b, c \in A$,

$$a(\mu R_1(b \otimes c)) \overset{(4.56)}{=} (\mu(\mathsf{id} \otimes S)T_2(a \otimes b))c \overset{(4.65)}{=} a \sqcap^L (b)c.$$

This proves the first assertion and the second one is proven symmetrically. □

Lemma 4.5.11. *Let A be a regular weak multiplier bialgebra over a field. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. For $i \in \{1,2\}$, denote by $R_i : A \otimes A \to A \otimes A$ the corresponding maps in Theorem 4.5.1 (1). Then the following hold.*

$$\mu(\sqcap^R \otimes \mathsf{id})R_1 = \mu(S \otimes \mathsf{id}) \qquad \textit{and} \qquad \mu(\mathsf{id} \otimes \sqcap^L)R_2 = \mu(\mathsf{id} \otimes S).$$

Proof. Using part (2) of Proposition 4.5.3 in the first equality, $G_1R_1 = R_1$ in the third one, and the relation between S and λ_{R_1} in Corollary 4.5.7 in the last one, it follows for all $a, b \in A$ that

$$\mu(\sqcap^R \otimes \mathsf{id})R_1(a \otimes b) = \mu(\lambda_{G_1} \otimes \mathsf{id})R_1(a \otimes b) \overset{(4.57)}{=} \lambda_{G_1R_1}(a)b = \lambda_{R_1}(a)b = S(a)b.$$

This proves the first assertion and the second one is proven symmetrically. □

Although the following theorem is contained in [73, Proposition 3.5], we prefer to give an alternative proof not referring to Heyneman-Sweedler type indices.

Theorem 4.5.2. *Let A be a regular weak multiplier bialgebra over a field. If A possesses an antipode $S : A \to \mathbb{M}(A)$, then it is anti-multiplicative.*

Proof. For $i \in \{1, 2\}$, denote by $R_i : A \otimes A \to A \otimes A$ the maps in Theorem 4.5.1 (1). Consider the composite map

$$W := \mu^2(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes \mu \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{id} \otimes R_1)(\mathsf{id} \otimes \mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes \mathsf{id} \otimes R_1)(\mathsf{id} \otimes \mathsf{tw} \otimes \mathsf{id})$$

from $A \otimes A \otimes A \otimes A$ to A. We shall evaluate it on an arbitrary element $a \otimes b \otimes c \otimes d$ in two different ways. In one case, we will get $aS(bc)d$ and in the other case it will yield $aS(c)S(b)d$. To begin with, compute

$$\begin{aligned}(\mu(\mathsf{id} \otimes S) \otimes \mathsf{id})(\mathsf{id} \otimes T_1) &\overset{(4.60)}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(R_2 \otimes \mathsf{id})(\mathsf{id} \otimes T_1) \\ &\overset{R_2 \in \mathcal{R}}{=} (\mathsf{id} \otimes \epsilon \otimes \mathsf{id})(\mathsf{id} \otimes T_1)(R_2 \otimes \mathsf{id}) \overset{\text{(iii)}}{=} (\mathsf{id} \otimes \mu)(R_2 \otimes \mathsf{id}).\end{aligned}$$

With its help,

$$W(a \otimes b \otimes c \otimes d) = a(\mu(S \otimes \mathsf{id})T_1(\mu \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(\mathsf{tw} \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(c \otimes b \otimes d))).$$

Next, for all $b, c, d \in A$,

$$T_1(\mu \otimes \mathsf{id})(\mathsf{id} \otimes R_1)(b \otimes c \otimes d) \quad = \quad \Delta(b)(T_1R_1(c \otimes d)) \overset{\text{(iv)}}{=} T_1(b \otimes d)(c \otimes 1).$$

Using this computation,

$$
\begin{aligned}
W(a\otimes b\otimes c\otimes d) &= a(\mu(S\otimes \mathsf{id})[(T_1R_1(b\otimes d))(c\otimes 1)]) \\
&= a(\mu(S\otimes \mathsf{id})E_1(bc\otimes d)) \overset{(4.65)}{=} aS(bc)d
\end{aligned}
$$

for all $a,b,c,d \in A$. On the other hand, using Lemma 4.5.10 in the first and last equalities,

$$
\mu R_2(\mathsf{id}\otimes \mu) = \mu(\mathsf{id}\otimes \sqcap^R\mu) \overset{(4.28)}{=} \mu(\mathsf{id}\otimes \sqcap^R)(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})) = \mu R_2(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})),
$$

hence

$$
\begin{aligned}
W = \mu^2(R_2\otimes \mathsf{id})&(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})\otimes \mathsf{id}) \\
&(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(\mathsf{id}\otimes \mathsf{tw}\otimes \mathsf{id})(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(\mathsf{id}\otimes \mathsf{tw}\otimes \mathsf{id}).
\end{aligned}
$$

Moreover, for any $a,b,c,d \in A$,

$$
\begin{aligned}
R_2(a\otimes \sqcap^R(b)c)(1\otimes d) &\overset{(4.61)}{=} (a\otimes 1)((S\otimes \mathsf{id})T_1(\sqcap^R(b)c\otimes d)) \\
&\overset{(4.29)(4.2)}{=} (a\otimes 1)((S\otimes \mathsf{id})((1\otimes \sqcap^R(b))T_1(c\otimes d))) \\
&= (a\otimes \sqcap^R(b))((S\otimes \mathsf{id})T_1(c\otimes d)).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mu^2 \quad &(R_2\otimes \mathsf{id})(\mathsf{id}\otimes \mu(\sqcap^R\otimes \mathsf{id})\otimes \mathsf{id})(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(a\otimes b\otimes c\otimes d) \\
&= \mu[(a\otimes \sqcap^R(b))((S\otimes \mathsf{id})T_1R_1(c\otimes d))] \\
&= \mu[(a\otimes \sqcap^R(b))((S\otimes \mathsf{id})(E(c\otimes d)))] \\
&= \mu[(a\otimes 1)((S\otimes \mathsf{id})((1\otimes \sqcap^R(b))E(c\otimes d)))] \\
&\overset{(4.29)}{=} a(\mu(S\otimes \mathsf{id})E_1(c\otimes \sqcap^R(b)d)) \overset{(4.65)}{=} aS(c)\sqcap^R(b)d.
\end{aligned}
$$

Substituting this identity in the latest expression of W and applying Lemma 4.5.11, we obtain

$$
W(a\otimes b\otimes c\otimes d) = \mu^3(\mathsf{id}\otimes S\otimes \sqcap^R\otimes \mathsf{id})(\mathsf{id}\otimes \mathsf{id}\otimes R_1)(a\otimes c\otimes b\otimes d) = aS(c)S(b)d
$$

for any $a,b,c,d \in A$. By the density of A in $\mathbb{M}(A)$, this proves $S(bc) = S(c)S(b)$, for all $b,c \in A$. □

The following proposition is contained in [73, Proposition 3.6]. However, in our setting a much shorter proof can be given.

Proposition 4.5.12. *Let A be a regular weak multiplier bialgebra over a field which possesses an antipode $S : A \to \mathbb{M}(A)$. Whenever the comultiplication is left and right full, S is a non-degenerate map.*

Proof. Using the idempotency of the non-unital algebra A, Lemma 4.2.9 (1), the fact that $\sqcap^L(A) = \overline{\sqcap}^L(A)$ (cf. Theorem 4.2.1) and (4.65),

$$A = A^2 \subseteq A\overline{\sqcap}^L(A) = A\sqcap^L(A) \subseteq AS(A) \subseteq A$$

so that $A = AS(A)$. A symmetrical reasoning shows that also $A = S(A)A$. □

We conclude by Theorem 2.7.1 that in the situation in Proposition 4.5.12 the antipode extends to algebra homomorphisms $\overline{S} : \mathbb{M}(A)^{\mathsf{op}} \to \mathbb{M}(A)$, $\overline{\mathsf{id} \otimes S} : \mathbb{M}(A \otimes A^{\mathsf{op}}) \cong \mathbb{M}(A^{\mathsf{op}} \otimes A)^{\mathsf{op}} \to \mathbb{M}(A \otimes A)$, $\overline{S \otimes \mathsf{id}} : \mathbb{M}(A \otimes A) \to \mathbb{M}(A^{\mathsf{op}} \otimes A) \cong \mathbb{M}(A \otimes A^{\mathsf{op}})^{\mathsf{op}}$ and $\overline{S \otimes S} : \mathbb{M}(A \otimes A)^{\mathsf{op}} \to \mathbb{M}(A \otimes A)$.

Lemma 4.5.13. *Let A be a regular weak multiplier bialgebra over a field. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. Then for any $a, b \in A$, the following hold.*

$$S(a\overline{\sqcap}^L(b)) = \sqcap^R(b)S(a) \qquad S(\overline{\sqcap}^L(b)a) = S(a)\sqcap^R(b)$$
$$S(a\overline{\sqcap}^R(b)) = \sqcap^L(b)S(a) \qquad S(\overline{\sqcap}^R(b)a) = S(a)\sqcap^L(b).$$

Proof. Using Lemma 4.2.12 in the second equality, it follows for any $a, b, c \in A$ that

$$\begin{aligned} aS(b\overline{\sqcap}^L(c)) &\overset{(4.66)}{=} \mu(\mathsf{id} \otimes S)[(a \otimes b\,\overline{\sqcap}^L(c))E] \\ &= \mu(\mathsf{id} \otimes S)[(a\sqcap^R(c) \otimes b)E] \overset{(4.66)}{=} a\sqcap^R(c)S(b). \end{aligned}$$

By the density of A in $\mathbb{M}(A)$, this proves the first claim. It also implies that

$$S(a)\sqcap^R(c)S(b)d = S(a)S(b\overline{\sqcap}^L(c))d = S(\overline{\sqcap}^L(c)a)S(b)d$$

for all $a, b, c, d \in A$, where in the second equality we used the anti-multiplicativity of S (cf. Theorem 4.5.2). Using the non-degeneracy of S and the density of A in $\mathbb{M}(A)$, we have the second claim proven. The remaining assertions follow symmetrically. □

In view of Proposition 4.1.8 and (4.26), in any regular weak multiplier bialgebra A over a field, we may regard E as an element of $\mathbb{M}(A \otimes A^{\mathsf{op}})$. The following proposition —and thus its Corollary 4.5.15— was proven in [73] only for regular weak multiplier Hopf algebras.

Proposition 4.5.14. *Let A be a regular weak multiplier bialgebra over a field with a left and right full comultiplication. Assume that A possesses an antipode $S : A \to \mathbb{M}(A)$. Then the elements $E \in \mathbb{M}(A \otimes A^{\mathsf{op}})$ and $F \in \mathbb{M}(A \otimes A)$ in Proposition 4.3.4 (1) are related via the extensions of S as*

$$\overline{(\mathsf{id} \otimes S)}(E) = F \qquad \text{and} \qquad \overline{(S \otimes \mathsf{id})}(F) = E^{\mathsf{op}},$$

where $(a \otimes 1)E^{\mathsf{op}}(1 \otimes b) := \mathsf{tw}[(1 \otimes a)E(b \otimes 1)]$ and $(1 \otimes b)E^{\mathsf{op}}(a \otimes 1) := \mathsf{tw}[(b \otimes 1)E(1 \otimes a)]$ define $E^{\mathsf{op}} \in \mathbb{M}(A^{\mathsf{op}} \otimes A)$.

Proof. Since A is idempotent and S is non-degenerate (by Proposition 4.5.12), any element of $A \otimes A$ can be written as a linear combination of elements of the form $ab \otimes cS(d)$, in terms of $a, b, c, d \in A$. Moreover, using the anti-multiplicativity of S in the first equality, applying Proposition 4.1.8 (1) in the second equality, Lemma 4.5.13 in the third one and Proposition 4.3.4 (1) in the last one, it follows for any $a, b, c, d \in A$ that

$$\begin{aligned}(ab \otimes cS(d))\overline{(\mathsf{id} \otimes S)}(E) &= (1 \otimes c)((\mathsf{id} \otimes S)[(ab \otimes 1)E(1 \otimes d)]) \\ &= (1 \otimes c)((\mathsf{id} \otimes S)[((\mathsf{id} \otimes \overline{\sqcap}^L)T_2(a \otimes b))(1 \otimes d)]) \\ &= (1 \otimes cS(d))((\mathsf{id} \otimes \sqcap^R)T_2(a \otimes b)) = (ab \otimes cS(d))F.\end{aligned}$$

This proves the first assertion. Symmetrically, in order to prove the second one, write any element of $A \otimes A$ as a linear combination of elements of the form $aS(b) \otimes cd$. Using again the anti-multiplicativity of S in the first equality, Proposition 4.3.4 (1) in the second equality, Lemma 4.5.13 in the third one and (4.26) in the fourth one, it follows for any $a, b, c, d \in A$ that

$$\begin{aligned}(aS(b) \otimes 1)\overline{(S \otimes \mathsf{id})}(F)(1 \otimes cd) &= (a \otimes 1)((S \otimes \mathsf{id})[F(b \otimes cd)]) \\ &= (a \otimes 1)((S \otimes \mathsf{id})[((\overline{\sqcap}^R \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c))(b \otimes 1)]) \\ &= (aS(b) \otimes 1))((\sqcap^L \otimes \mathsf{id})T_4^{\mathsf{op}}(d \otimes c))\end{aligned}$$

$$
\begin{aligned}
&= (aS(b) \otimes 1)\mathsf{tw}(E(cd \otimes 1)) \\
&= \mathsf{tw}((1 \otimes aS(b))E(cd \otimes 1)) \\
&= (aS(b) \otimes 1)E^{\mathsf{op}}(1 \otimes cd).
\end{aligned}
$$

□

Corollary 4.5.15. *Let A be a regular weak multiplier bialgebra over a field with a left and right full comultiplication. If A possesses an antipode $S : A \to \mathbb{M}(A)$, then S is anti-comultiplicative in the sense of the commutative diagram*

$$
\begin{array}{ccc}
A^{\mathsf{op}} & \xrightarrow{\quad S \quad} & \mathbb{M}(A) \\
{\scriptstyle \Delta^{\mathsf{op}}}\downarrow & & \downarrow{\scriptstyle \overline{\Delta}} \\
\mathbb{M}(A \otimes A)^{\mathsf{op}} & \xrightarrow[\overline{S \otimes S}]{} & \mathbb{M}(A \otimes A).
\end{array}
$$

Proof. By [73, Proposition 3.7], $\overline{\Delta}S(a) = ((\overline{S \otimes S})\Delta^{\mathsf{op}}(a))E$, for all $a \in A$. By Proposition 4.5.14, $E = (\overline{S \otimes S})(E^{\mathsf{op}})$ so that by the anti-multiplicativity of $\overline{S \otimes S}$,

$$
\begin{aligned}
\overline{\Delta}S(a) &= ((\overline{S \otimes S})\Delta^{\mathsf{op}}(a))E = ((\overline{S \otimes S})\Delta^{\mathsf{op}}(a))(\overline{S \otimes S})(E^{\mathsf{op}}) \\
&= (\overline{S \otimes S})(E^{\mathsf{op}}\Delta^{\mathsf{op}}(a)) \overset{(4.2)}{=} (\overline{S \otimes S})\Delta^{\mathsf{op}}(a).
\end{aligned}
$$

□

Chapter 5

Conclusions and further research proposals

The main achievements of this thesis, as well as various new questions related to them, are collected below.

First, for a separable Frobenius algebra R, we describe weak bialgebras with right subalgebra isomorphic to R as bimonoids in the duoidal category of $R \otimes R^{\mathsf{op}}$–bimodules (Theorem 3.2.4). In this, two points are specially worth to be highlighted. On the one hand, that the duoidal structure of the category of bimodules over $R \otimes R^{\mathsf{op}}$ is given by the monoidal products $\otimes_{R^e}$ and the Takeuchi's $\times_R$–product [65] (Theorem 3.2.1). On the other hand, that this is possible because the Takeuchi's $\times_R$–product, when considered over a separable Frobenius algebra R, is proven to be a bimodule tensor product (Lemma 3.2.1 and Lemma 3.2.2) turning out to be, consequently, a monoidal product.

Secondly, this interpretation of weak bialgebras allows us to define a category wba of weak bialgebras over a given field. We use this to extend the well-known relation between groups and cosemisimple pointed Hopf algebras in the following sense. We prove that the 'free vector space' functor $\mathsf{k} : \mathsf{cat}^0 \to \mathsf{wba}$ (from the category of small categories with finitely many objects, Section 3.3.1) possesses a right adjoint (Proposition 3.3.10, Proposition 3.3.12 and Theorem 3.3.1) given by taking a suitable subset of the set of group-like elements (Definition 3.3.2; namely, those group-like elements g whose both right projections —$\sqcap^R(g)$ and $\overline{\sqcap}^R(g)$— are group-like elements too; Theorem 3.3.1). For a general weak bialgebra, this subset is shown to be proper, that is, it is indeed strictly smaller than the set of group-like elements (Remark 3.3.3). Nevertheless, if the weak bialgebra in question is cocommutative or if it has an antipode, then it is exactly the set of group-like elements (Proposition 3.3.5 and Lemma 3.3.7). We prove that this adjunction restricts to the full subcategories of weak Hopf algebras of wba and the category of small groupoids of cat^0 (Theorem 3.3.2); and that it becomes an equivalence by respectively restricting us to the categories of pointed cosemisimple weak bialgebras (Corollaries 3.3.14 and 3.3.21), and pointed cosemisimple weak Hopf algebras (Corollaries 3.3.21 and 3.3.22).

We propose weak multiplier bialgebras (Definition 4.1.1) as a non-unital generalization of weak bialgebras with a multiplier-valued comultiplication. On the one hand, weak multiplier bialgebras fill the conceptual gap of the *antipodless* situation of weak multiplier Hopf algebras [73]. On the other hand, our definition is supported by the fact that (assuming some further properties like *regularity* (Definition 4.1.3) or *fullness* (Theorem 4.2.1) of the comultiplication), the most characteristic features of usual, unital, weak bialgebras extend to this generalization:

(1) There is a bijective correspondence between the weak bialgebra structures and the weak multiplier bialgebra structures on any unital algebra (Theorem 4.1.3).

(2) The multiplier algebra of a weak multiplier bialgebra contains two canonical commuting anti-isomorphic firm Frobenius non-unital algebras; the so-called base algebras (Theorem 4.3.1, Theorem 4.3.3, Proposition 4.3.4, Proposition 4.3.5). By generalizing to the multiplier setting several equivalent properties that distinguish bialgebras among weak bialgebras, we also propose a notion of *multiplier bialgebra* based on this: it is defined as the particular case of weak multiplier bialgebra when the base algebra is trivial; that is, it contains only multiples of the unit element (Theorem 4.1.4).

(3) Appropriately defined modules (i.e. idempotent and non-degenerate non-unital right A–modules) over a regular weak multiplier bialgebra A with a full comultiplication constitute a monoidal category via the module tensor product over the base algebra (Theorem 4.4.1).

Moreover, we introduce the notion of antipode on a regular weak multiplier bialgebra

(Theorem 4.5.1). Whenever the comultiplication is full, the antipode axioms are shown to be equivalent to the projections of the maps T_1 and T_2, to maps between relative tensor products over the base algebras, being isomorphisms (Remark 4.5.8). We claim that the one of regular weak multiplier bialgebras possessing an antipode is the desired 'intermediate' class between regular and arbitrary weak multiplier Hopf algebras in

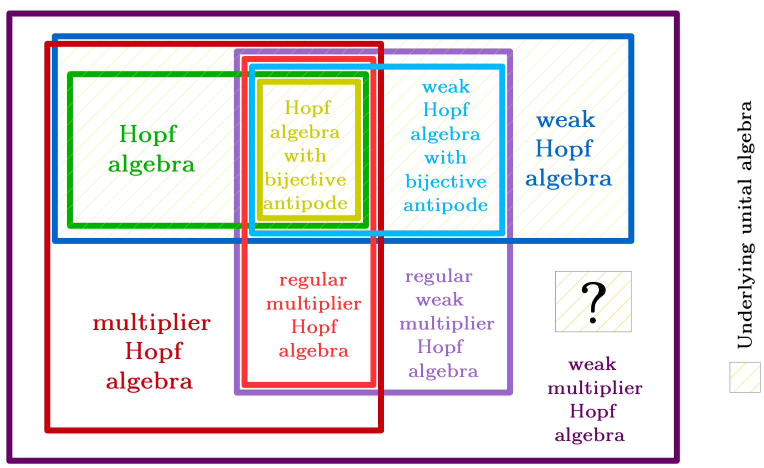


Figure 5.1: Sets relations I.

In the Figure 5.1 above, the question mark inside the small striped rectangle in the dark violet rectangle of weak multiplier Hopf algebras stress the fact that, to the date, it is not known whether any weak multiplier Hopf algebra with underlying algebra is a

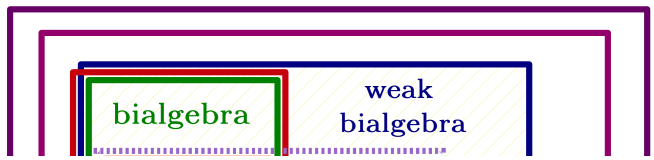


Figure 5.3: Sets relations III.

Many interesting questions and problems arise from the work collected in this thesis. Some of them are being addressed at present, others are future projects that will lead to success with almost no doubts, and there are also some more challenging research proposals whose approach certainly requires the development of new and innovative tools. Let us briefly outline some of these ideas.

In Chapter 3 we proved that the category cat^0 of small categories with finitely many objects and the category wba of weak bialgebras are related by an adjunction via the 'free vector space functor' k and certain functor g (Theorem 3.3.1). As it has been previously pointed out, this adjunction extends the well-known relation between the category of finite monoids and bialgebras over a field. Example 4.1.14 shows that the linear span of a small category (without any constraint on the finiteness of its object set) has the structure of a weak multiplier bialgebra. In order to extend to the category cat of (arbitrary) small categories the adjunction $\mathsf{k} \dashv \mathsf{g}$ between cat^0 and wba, a category wmb of weak multiplier bialgebras has to be first defined —what in fact is interesting on its own. For this purpose, we need to find out the right notion of morphism of weak multiplier bialgebras; in such a way that the category wmb (which it gives rise to), contains wba as a subcategory, and that a functor between small categories induces a morphism in wmb.

On the other hand, since cat has a natural structure of bicategory considering natural transformations as 2–morphisms (2–cells), it would be of interest to study the adjunction $\mathsf{k} \dashv \mathsf{g}$ between cat^0 and wba (or, more generally, between cat and wmb after achieving the first exposed goal) at the level of bicategories, having previously added an appropriate notion of 2–morphisms to the categories wba and wmb. This seems to be a not hard question with at most technical difficulties.

Once the category wmb of weak multiplier bialgebras is defined, a big challenge would be to obtain a 'characterization' of it. Recall that bialgebras over a field can be characterized by the property that the category of their (left or right) modules is monoidal such that the forgetful functor to the category of vector spaces is strict monoidal. More generally, the category of (left or right) modules over a weak bialgebra is monoidal such that the forgetful functor to the category of bimodules over the (separable Frobenius) base algebra is strict monoidal (see e.g. [64]). In Chapter 4 we proved that for any regular weak multiplier bialgebra A over a field, the category of idempotent and non-degenerate non-unital right A–modules is monoidal, and that the functor from it to the category of firm non-unital $\sqcap^R(A)$–bimodules is strict monoidal (Theorem

4.4.1). It would be really interesting to further investigate this relation, in order to get a definite statement, that is, a characterization of weak multiplier bialgebras by these classical means. However, in this situation, the 'forgetful functor' from the category of (co)modules is not (co)monadic; hence the well-developed theory of lifting is not applicable. Consequently, some really fresh ideas and innovative tools are needed to face this problem —which potentially entails some risk and also a high gain. In this regard, noteworthy is the work [40] by Kris Janssen and Joost Vercruysse, where it is shown that, over a commutative ring k, a (non-unital, idempotent, non-degenerate, k–projective) k–algebra is a multiplier bialgebra (in a different sense that in the present thesis, cf. Theorem 4.1.4) if and only if the category of its algebra extensions and both the categories of its left and right modules are monoidal and fit, together with the category of k–modules, into a diagram of strict monoidal forgetful functors.

Another big interest in the theory of weak multiplier bialgebras is a more deep study of the non-regular case as well as the find of examples of non-regular weak multiplier bialgebras. Examples 4.1.14, 4.1.15, 4.1.16 and 4.1.17 present weak multiplier bialgebras obeying the regularity condition (Definition 4.1.3). Non-regular examples would stress even more the relevance and necessity of the well justified weak multiplier bialgebras theory, taking advantage of its potential.

Concerning applications, in spite of their recent birth, weak multiplier bialgebras are already having a notable impact in related areas, allowing progress in further research projects. This is evidenced by the work [33] by Kenny De Commer and Thomas Timmermann, where they generalize Hayashi's definition of a compact quantum group of face type [39] to the case where the commutative base algebra is no longer finite-dimensional, by relying on the notion of weak multiplier bialgebra. In addition, in [41], Byung-Jay Kahng and Alfons Van Daele, show that a weak multiplier bialgebra with a regular and right and left full comultiplication is a regular weak multiplier Hopf algebra if there is a faithful set of integrals. As pointed out by the authors, the relevance of this result lies in the aid that it represents for the development of the theory of locally compact quantum groupoids in the operator algebra setting, being precisely the prospect of such a theory the motivation of their study.

Conclusiones y propuestas de investigación futura

Esta sección trata de ser un compendio de los principales logros de esta tesis, así como de algunas cuestiones relacionadas sugiriendo nuevas direcciones en esta investigación.

En primer lugar, para un álgebra Frobenius separable R, las biálgebras débiles con subálgebra derecha isomorfa a R son descritas como bimonoides en la categoría duoidal de $R \otimes R^{\mathsf{op}}$–bimódulos (Teorema 3.2.4). Entre los detalles de esta descripción merece ser especialmente destacado que, sobre un álgebra Frobenius separable R, el producto $\times_R$ de Takeuchi [65] resulta ser un producto tensor de bimódulos (y, por tanto, monoidal; Lema 3.2.1 y Lema 3.2.2) sirviendo, junto con el producto tensor $\times_{R^e}$ de módulos sobre R^e, como producto monoidal para la estructura duoidal de $\mathsf{bim}(R^e)$ (Teorema 3.2.1).

En segundo lugar, esta interpretación de biálgebras débiles permite definir una categoría wba de biálgebras débiles sobre un cuerpo dado. Esta categoría es usada para extender la bien conocida relación entre grupos y álgebras de Hopf cosemisimples y punteadas, en el siguiente sentido. Probamos que el funtor $\mathsf{k} : \mathsf{cat}^0 \to \mathsf{wba}$ (de la categoría de categorías pequeñas con un número finito de objetos) posee un funtor adjunto por la derecha (Proposición 3.3.10, Proposición 3.3.12 y Teorema 3.3.1), dado por un subconjunto del conjunto de elementos group-like (Definición 3.3.2; a saber, aquellos elementos group-like g cuyas proyecciones derecha —$\sqcap^R(g)$ and $\overline{\sqcap}^R(g)$— son también elementos group-like). Para una biálgebra débil general, este subconjunto es propio, esto es, es estrictamente más pequeño que el conjunto de elementos group-like (Observación 3.3.3). No obstante, si la biálgebra débil en cuestión es coconmutativa o tiene antípoda, entonces se trata exactamente del conjunto de elementos group-like (Proposición 3.3.5 y Lema 3.3.7). Además, probamos que esta adjunción restringe a las categorías plenas de álgebras de Hopf débiles de wba y de grupoides pequeños de cat^0 (Teorema 3.3.2); y que resulta ser una equivalencia al restringirnos respectivamente a las categorías de biálgebras débiles cosemisimples y punteadas (Corolarios 3.3.14 y 3.3.21), y de álgebras de Hopf débiles cosemisimples y punteadas (Corolarios 3.3.21 y 3.3.22).

Por otro lado, proponemos las biálgebras multiplicadoras débiles (Definición 4.1.1) como una generalización no unital de las biálgebras débiles y con comultiplicación valuada en el álgebra de multiplicadores. Por una parte, nuestra definición es avalada por el hecho de que (asumiendo algunas otras propiedades como *regularidad* (Definición 4.1.3) o *plenitud* (Teorema 4.2.1) de la comultiplicación), la mayoría de las características de las biálgebras débiles (unitales, usuales) extienden a esta generalización:

(1) Existe una correspondencia biyectiva entre las estructuras de biálgebra débil y las de biálgebra multiplicadora débil sobre cualquier álgebra unital (Teorema 4.1.3).

(2) El álgebra de multiplicadores de una biálgebra multiplicadora débil contiene dos álgebras no unitales Frobenius firmes canónicas, que conmutan entre sí; las llamadas álgebras base (Teorema 4.3.1, Teorema 4.3.3, Proposición 4.3.4, Proposición 4.3.5). Basándonos en esto y generalizando al 'ambiente de multiplicadores' varias propiedades equivalentes que distinguen a las biálgebras entre las biálgebras débiles, también proponemos una noción de *biálgebra multiplicadora*. Ésta es definida como el caso particular de biálgebra multiplicadora débil en que el álgebra base es trivial; esto es, contiene sólo múltiplos del elemento unidad (Teorema 4.1.4).

(3) Definidos apropiadamente (i.e. A–módulos por la derecha no unitales, idempotentes y no degenerados), los módulos sobre una biálgebra multiplicadora débil A regular constituyen una categoría monoidal vía el producto tensor sobre el álgebra base (Teorema 4.4.1).

Por otra parte, las biálgebras multiplicadoras débiles resuelven la brecha conceptual de la situación *sin antípoda* de las álgebras multiplicadoras de Hopf débiles [73]. Además, introducimos la noción de antípoda sobre una biálgebra multiplicadora débil regular (Teorema 4.5.1), probando que la de biálgebras multiplicadoras débiles regulares con antípoda es la clase 'intermedia' deseada entre las álgebras multiplicadoras de

Hopf débiles regulares y las álgebras multiplicadoras de Hopf débiles arbitrarias; que

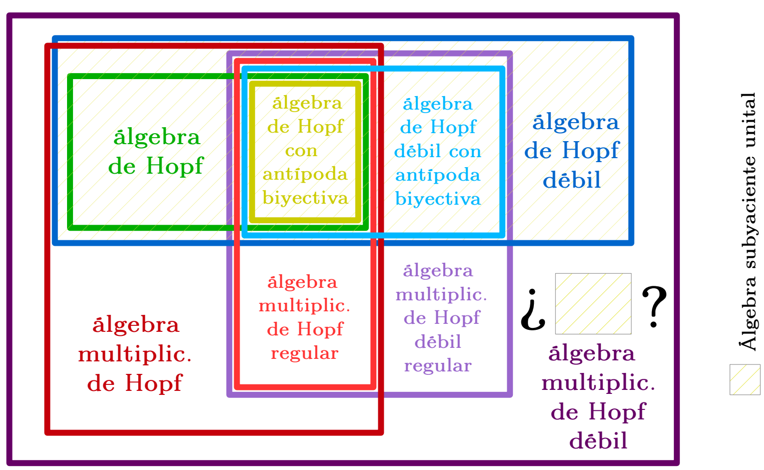


Figure 5.4: Relaciones de conjuntos I.

En la Figura 5.4 de arriba, los signos de interrogación a los lados del rectángulo pequeño y rayado dentro del rectángulo de álgebras multiplicadoras débiles, enfatizan el hecho de que, hasta la fecha, no se conoce si toda álgebra multiplicadora de Hopf débil con álgebra subyaciente unital es un álgebra de Hopf débil, como fue señalado en el introductorio Capítulo 1, y en [73, página 29] por Alfons Van Daele y Shuanhong Wang. Por el momento no se conoce ningún ejemplo de álgebra multiplicadora de Hopf débil unital

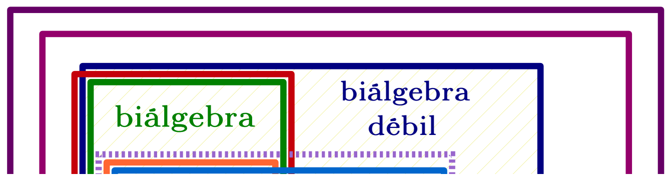


Figure 5.6: Relaciones de conjuntos III.

De cara a investigaciones futuras, muchas son las cuestiones y problemas interesantes que se derivan de esta tesis. Algunas de ellas ya están siendo abordadas; otras formarán parte de próximos proyectos. Además, también damos cuenta de objetivos más desafiantes que ciertamente requieren un gran desarrollo de nuevas e innovadoras herramientas. En lo que sigue, esbozamos brevemente algunas de estas propuestas.

En el Capítulo 3 probamos que la categoría cat^0 de categorías pequeñas con un número finito de objetos y la categoría wba de biálgebras debiles son adjuntas vía el funtor k 'espacio vectorial libre' y cierto funtor g de tipo 'group-like' (Teorema 3.3.1). Como ya ha sido señalado previamente, esta adjunción generaliza la bien conocida relación entre la categoría de monoides finitos y la de biálgebras sobre un cuerpo. El Ejemplo 4.1.14 muestra que el espacio vectorial generado por el conjunto de morfismos de una categoría pequeña (sin ninguna restricción sobre la finitud de su conjunto de objetos) tiene estructura de biálgebra multiplicadora débil. Con el fin de extender a la categoría cat de categoría pequeñas (arbitrarias) la adjunción $\mathsf{k} \dashv \mathsf{g}$ entre cat^0 y wba, primero se ha de definir apropiadamente una categoría wmb de biálgebras multiplicadoras débiles —lo cual tiene interés por sí mismo. Para ello, es preciso dar una definición de morfismo entre biálgebras multiplicadoras débiles de tal forma que la categoría wmb (a la que dé lugar), contenga wba como subcategoría, y que un funtor entre categorías pequeñas induzca un morfismo en wmb.

Por otra parte, dado que cat tiene una estructura natural de bicategoría con las transformaciones naturales como 2–morfismos (2–celdas), sería de interés estudiar la adjunción $\mathsf{k} \dashv \mathsf{g}$ entre cat^0 y wba (o, más generalmente, entre cat y wmb una vez conseguido el primer objetivo expuesto) al nivel de bicategorías, dando previamente unas nociones adecuadas de 2–morfismos en las categorías wba y wmb. Esto parece ser un estudio factible con, a lo sumo, ciertas dificultades de tipo técnico.

Después de formalizar la categoría de biálgebras multiplicadoras débiles, un gran reto sería obtener una caracterización de ella. Recordemos que las biálgebras sobre un cuerpo pueden ser caracterizadas por la propiedad de que la categoría de sus módulos (por la derecha o por la izquierda) es monoidal tal que el funtor olvido a la categoría de espacios vectoriales es monoidal estricto. Más generalmente, la categoría de módulos (por la izquierda o por la derecha) de una biálgebra débil es monoidal tal que el funtor olvido a la categoría de bimódulos sobre el álgebra base (separable Frobenius) es monoidal estricto (e.g. [64]). En el Capítulo 4 probamos que para cualquier biálgebra multiplicadora débil A regular sobre un cuerpo, la categoría de A–módulos por la derecha no unitales idempotentes y no degenerados es monoidal, y que el funtor desde ella a la categoría de $\sqcap^R(A)$–bimódulos no unitales firmes es monoidal estricta (Teorema 4.4.1). Resulta muy interesante investigar más allá esta relación con el objetivo de conseguir una caracterización de las biálgebras multiplicadoras débiles en los términos expuestos. Sin embargo, en esta situación, el funtor olvido de la categoría de (co)módulos no es (co)monádico; de ahí, la bien desarrollada teoría de 'elevación' no es aplicable. En consecuencia, este estudio requiere de ideas realmente innovadoras —lo que potencialmente conlleva un cierto riesgo y también un gran beneficio. A este respecto, cabe destacar el trabajo [40] de Kris Janssen y Joost Vercruysse, donde los autores prueban que, sobre un anillo conmutativo k, una k–álgebra (posiblemente no unital, idempotente, no degenerada, k–proyectiva) es una biálgebra multiplicadora (en un sentido diferente que en la presente tesis, cf. Teorema 4.1.4) si y sólo si la categoría de sus extensiones de álgebra y las categorías de sus módulos por la derecha y por la izquierda son monoidales y hacen conmutar, junto con la categoría de k–módulos, un diagrama de funtores olvido monoidales.

Otro gran interés en la teoría de biálgebras multiplicadoras débiles es el desarrollo de un estudio más profundo sobre el caso no regular, así como el hallazgo de ejemplos en este caso. Los Ejemplos 4.1.14, 4.1.15, 4.1.16 y 4.1.17 presentan biálgebras multiplicadoras débiles que satisfacen la condición de regularidad (Definición 4.1.3). Ejemplos no regulares destacarían aún más la importancia y necesidad de la ya bien justificada teoría de biálgebras multiplicadoras débiles, exprimiendo su potencialidad.

En cuanto a aplicaciones, a pesar de su reciente nacimiento, las biálgebras multiplicadoras ya están repercutiendo notablemente en áreas de investigación relacionadas, favoreciendo el progreso de otros proyectos y líneas con temática afín. Prueba de ello es el trabajo [33] de Kenny De Commer y Thomas Timmermann, donde la definición de grupo cuántico compacto de tipo *face* debida a Hayashi [39] es generalizada al caso en que el álgebra base conmutativa no es finito dimensional, basándose en la noción de biálgebra multiplicadora débil. Adicionalmente, Byung-Jay Kahng y Alfons Van Daele demuestran en [41] que una biálgebra multiplicadora débil con comultiplicación regular y plena por la derecha y por la izquierda es un álgebra multiplicadora de Hopf débil regular si existe cierto conjunto de integrales. Este resultado, tal y como señalan sus autores, contribuye de manera relevante en el desarrollo de la teoría de grupoides cuánticos localmente compactos en el ámbito de álgebras de operadores, siendo precisamente esta teoría la motivación de su estudio.

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Index

Symbol index

Categories

$\mathbf{1}$	—	Singleton category
$\mathbf{2}$	—	Interval category
cat	—	Small categories
cat^0	—	Small categories with finitely many objects
sfr	—	Separable Frobenius algebras
duo	—	Duoidal categories
$\mathsf{rmd}(A)$	—	Idempotent and non-degenerate non-unital right A–modules
$\mathsf{bim}^f(A)$	—	Firm non-unital A–bimodules
$\mathsf{bim}(A)$	—	A–bimodules
wba	—	Weak bialgebras
wha	—	Weak Hopf algebras
gpd	—	Small groupoids
set	—	Sets
$\mathsf{span(X)}$	—	Spans over a set X
$\mathsf{bmd}(\mathcal{M})$	—	Bimonoids associated to a functor $\mathcal{M}$ from an arbitrary category to duo
coMon	—	2–category of monoidal categories, comonoidal functors and comonoidal natural transformations

Morphisms

s	—	Source of a category

$A^{\mathsf{op}}_{\mathsf{cop}}$	—	Opposite-coopposite ((co)algebra) of A
$\mathbb{M}(A)$	—	Multiplier algebra of an algebra A
$ker(f)$	—	Kernel of a linear function f
$supp(f)$	—	Support of a function f
$G(C)$	—	Group-like elements of a coalgebra C
$k\mathsf{C}$	—	Free vector space spanned by C
$k(\mathsf{C})$	—	Vector space of finitely supported k–valued functions on $k\mathsf{C}$
$\mathsf{Lin}(A,B)$	—	Vector space of linear maps $A \to B$
$\mathsf{End}_A(A)$	—	Vector space of non-unital right A–module maps
${}_A\mathsf{End}(A)$	—	Vector space of non-unital left A–module maps $A \to A$

Sets

$\mathbb{N}$	—	Set of natural numbers
$\mathbb{C}$	—	Set of complex numbers

Arrows

$\Rightarrow$	—	Implies/2–cells (depending on the context)
$\Leftrightarrow$	—	If and only if, equivalent
$\dashv$	—	Adjunction
$\twoheadrightarrow$	—	Surjection
$\rightarrowtail$	—	Injection

Products

$\otimes$	—	Tensor product of vector spaces over a field
$\times$	—	Cartesian product
$\otimes_R$	—	Module tensor product over an algebra R
$\times_R$	—	Takeuchi's product over an algebra R
$\circ$	—	Monoidal product
$\bullet$	—	Monoidal product

Relations

$=$	—	Equal
$\subseteq$	—	Inclusion of sets
$\cong$	—	Isomorphism, isomorphic
$\geq$	—	Greater or equal than
$<$	—	Strictly lesser than

Operators

$\cap$	—	Intersection of sets
$\sum$	—	Sum
$\mid$	—	Restriction (e.g. $f_{\mid X}$)/Corestriction (e.g. $f^{\mid X}$)

Miscellaneous

∞	—	Infinity
$\forall$	—	For all
$\square$	—	*Quod erat demonstrandum*, end of the proof
n	—	Indeterminate natural number

Acronymous

w.r.t	—	With respect to
i.e.	—	*Isto es* (that is)
e.g.	—	*Exempli gratia* (for example)
cf.	—	*Confer* (compare)

Gracias, :).