TESIS DOCTORAL

UNIQUENESS OF MAXIMAL HYPERSURFACES IN OPEN SPACETIMES AND CALABI-BERNSTEIN TYPE PROBLEMS

Juan Jesús Salamanca Jurado

Universidad de Granada

GRANADA, 2015

Editorial: Universidad de Granada. Tesis Doctorales
Autor: Juan Jesús Salamanca Jurado
ISBN: 978-84-9125-136-1
URI: http://hdl.handle.net/10481/40273

UNIQUENESS OF MAXIMAL HYPERSURFACES IN OPEN

SPACETIMES AND CALABI-BERNSTEIN TYPE PROBLEMS

El doctorando Juan Jesús Salamanca Jurado, y los directores de la tesis Dr. Alfonso Romero Sarabia y Dr. Rafael M. Rubio Ruiz, garantizamos, con nuestra firma, que el trabajo ha sido realizado por el doctorando, bajo la dirección de los directores, y, que, hasta donde nuestro conocimiento alcanza, se han respetado los derechos de otros autores a ser citados cuando se han utilizado sus resultados o publicaciones en este trabajo.

Granada, 20 de marzo de 2015

Fdo.: Juan Jesús Salamanca Jurado

Vº Bº de los Directores

Fdo.: Dr. Alfonso Romero Sarabia Dr. Rafael M. Rubio Ruiz

Este trabajo ha sido realizado gracias a la beca FPI BES-2011-043770 dentro del proyecto de investigación Geometría semi-Riemanniana y Problemas Variacionales en Física Matemática, MTM2010-18099, del Plan Nacional de I+D+I del Ministerio de Ciencia e Innovación y cofinanciado con fondos FEDER de la Unión Europea.

Agradecimientos

Ante todo, quiero dedicar esta memoria a mis padres, por su apoyo y comprensión en esta etapa que ahora se cierra. También quiero agradecer a mi familia, por su cariño mostrado y por estar ahí siempre. En especial lugar guardo a Mª Z., que no pudo acompañarme más.

Por otro lado, deseo expresar mi sincero agradecimiento a Alfonso Romero y Rafael M. Rubio, por su ayuda en esta ardua tarea y continuo apoyo y estímulo.

También, a Magdalena y a Alma por su compañerismo a lo largo de esta rica etapa de mi vida.

Deseo mostrar mi gratitud al Departamento de Geometría y Topología de la Universidad de Granada. Ha sido un placer el haberme encontrado en esta familia. En especial, le agradezco a Miguel Sánchez su confianza depositada en mí.

Por último, al Departamento de Matemáticas de la Universidad de Córdoba, por su apoyo entusiasta.

Contents

Resumen

Las hipersuperficies espaciales son objetos geométricos con un alto grado de interés tanto en Física como en Geometría Lorentziana. Intuitivamente hablando, cada una de ellas representa el espacio físico en un instante de una función tiempo. De manera más precisa, el problema de valores iniciales para cada ecuación fundamental en Relatividad General se formula en términos de una hipersurperficie espacial (véase, por ejemplo, [88] y referencias allí). Incluso más, en Electromagnetismo, una hipersuperficie espacial es un conjunto de datos iniciales que determina unívocamente el futuro tanto para el campo electromagnético que satisface las ecuaciones de Maxwell [102, Thm. 3.11.1], como para las ecuaciones materiales simples [102, Thm. 3.11.2]. En Teoría de la Causalidad, la existencia de cierta hipersuperficie espacial implica que el espaciotiempo tenga un buen comportamiento en relación a dicha teoría. En concreto, un espaciotiempo es globalmente hiperbólico, [83, Def. 14.20], si y sólo si admite una hipersuperficie de Cauchy, [48]. De hecho, cualquier espaciotiempo globalmente hiperbólico admite una hipersuperficie espacial de Cauchy diferenciable S, y es difeomorfo a $\mathbb{R} \times S$, [15].

Si uno desea estudiar una hipersuperficie espacial globalmente, es natural suponer que la métrica que hereda del espaciotiempo ambiente es geodésicamente completa. Desde un punto de vista físico, esta completitud lleva a considerar el espacio físico en toda su extensión.

La geometría extrínseca de una hipersuperficie espacial se codifica en su operador

de Weingarten. De entre todas las funciones definidas a partir de él, la función curvatura media tiene una gran importancia. El caso en el que la curvatura media es constante, y especialmente cuando es idénticamente nula (esto es, para hipersuperficies maximales), es relevante tanto geométricamente como desde el punto de vista de la Relatividad General. Por un lado, cuando una hipersuperficie espacial tiene curvatura media nula, ésta puede conformar un buen conjunto inicial para el problema de Cauchy en Relatividad General [88]. Concretamente, Lichnerowicz probó que el problema de Cauchy con condiciones iniciales sobre una hipersuperficie maximal se reduce a una ecuación diferencial elíptica no lineal de segundo orden y un sistema de ecuaciones diferenciales lineales de primer orden, [71].

Incluso más, las hipersuperficies maximales poseen importancia en el análisis de la dinámica de un campo gravitatorio, o en el problema clásico de los n-cuerpos en el seno de un campo gravitatorio (véase, por ejemplo, [23] y referencias allí).

Por otro lado, cada hipersuperficie maximal puede describir, en algunos casos relevantes, la transición desde una fase expansiva a otra contractiva de un universo relativista. Es más, la existencia de una hipersuperficie de curvatura media constante (y en particular maximal) es necesaria para comprender la estructura de singularidades en el espacio de soluciones de la ecuación de Einstein. Un profundo conocimiento de estas hipersuperficies también es necesario en la prueba de la positividad de la masa gravitatoria. Poseen interés en Relatividad Numérica, donde las hipersuperficies maximales se usan para integrar en el tiempo. Todos estos aspectos físicos pueden ser consultados en [77] y referencias allí.

Desde un punto de vista matemático, una hipersuperficie maximal es (localmente) un punto crítico para un problema variacional natural, esto es, el dado por el funcional área (véase, por ejemplo, [21]). Por otro lado, para entender la estructura de un espaciotiempo es necesario estudiar las hipersuperficies maximales que contiene [13]. Especialmente para algunos espaciotiempos asintóticamente llanos, donde se prueba la existencia de una foliación por hipersuperficies maximales (ver [23] y referencias allí). Los resultados de existencia, y, consecuentemente, de unicidad, aparecen como temas centrales.

En la evolución historia de la investigación sobre hipersuperficies maximales, un hecho sorprendente fue el descubrimiento de nuevos problemas elípticos no lineales. De hecho, la función que define un grafo maximal en el espaciotiempo de Lorentz-Minkowski $(n+1)$-dimensional, $\mathbb{L}^{n+1}$, satisface una EDP elíptica de segundo orden similar a la ecuación de grafos minimales en el espacio Euclídeo $\mathbb{R}^{n+1}$. Sin embargo, se encontró un comportamiento nuevo y sorprendente para sus soluciones enteras (es decir, las definidas en todo $\mathbb{R}^n$): las únicas soluciones enteras a la ecuación de hipersuperficies maximales en $\mathbb{L}^{n+1}$ son las funciones afines que definen hiperplanos espaciales. Este hecho fue probado por Calabi [29] para $n \leq 4$ y después extendido para cualquier n por Cheng y Yau [32] y es conocido por el teorema de Calabi-Bernstein. Recordemos que el teorema de Bernstein para grafos minimales en $\mathbb{R}^{n+1}$ es cierto sólo para $n \leq 7$, [106]. Otro hecho destacable en [32] fue el uso de una nueva herramienta, la que hoy se denomina como principio del máximo generalizado de Omori-Yau [82], [111].

Otros artículos clásicos que tratan de unicidad de hipersuperficies maximales y espaciales de curvatura media constante completas son [23], [34] y [77]. En [23], Brill y Flaherty reemplazaron el espaciotiempo de Lorentz-Minkowski por un universo espacialmente cerrado, y probaron unicidad global suponiendo que su tensor de Ricci satisface $\mathrm{Ric}(z,z) > 0$ para todo vector tangente temporal z. Esta hipótesis se interpreta como la presencia real de masa en cada punto del espaciotiempo, y es conocida como la Condición de la Energía Ubicua (véase Sección 2 en Capítulo 2). En [77], esta suposición fue relajada por Marsden y Tipler para incluir, por ejemplo, espaciotiempos vacíos no-llanos. Más recientemente, Bartnik probó en [12] teoremas muy generales de existencia y, consecuentemente, apuntó que sería necesario encontrar nuevos resultados de unicidad satisfactorios. Después, en [9], Alías, Romero y Sánchez demostraron nuevos resultados de unicidad para la clase de espaciotiempos generalizados de Robertson-Walker (GRW) espacialmente cerrados (que, claramente, incluyen los espaciotiempos de Robertson-Walker espacialmente cerrados), bajo una condición de energía más débil, la Condición de Convergencia Temporal. En [7], Alías y Montiel probaron que en un espaciotiempo GRW cuya función warping satisface $(\log f)'' \leq 0$, las únicas hipersuperficies espaciales de curvatura media constante son las slices espaciales. Es más, este resultado fue generalizado en [27] por Caballero, Romero y Rubio para una familia más amplia de espaciotiempos. Recientemente, para el caso del espaciotiempo de Einstein-de Sitter, que es un modelo espacialmente abierto, Rubio ha dado nuevos resultados de unicidad y no existencia para hipersuperficies completas maximales y espaciales de curvatura media constante [101].

De entre los objetivos propuestos en el presente trabajo, el primero consiste en determinar qué tipo de espaciotiempos espacialmente abiertos poseen propiedades lo suficientemente adecuadas como para poder obtener resultados de unicidad. La parabolicidad es una buena herramienta que podría ser tenida en cuenta en algún espacio físico. Es más, sería satisfactorio si estos espaciotiempos pudieran ser adecuados para describir algún universo, o al menos ser alguna buena aproximación. Dicha familia consistirá en espaciotiempos GRW convenientes. Recordemos que un espaciotiempo GRW no es sino la variedad producto $I \times F$, de un intervalo I de la recta real $\mathbb{R}$ y una variedad Riemanniana (conexa) $n(\geq 2)$-dimensional (F, g_F), dotada de la métrica Lorentziana

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F)\,,$$

donde π_I y π_F denotan las proyecciones sobre I y F, respectivamente, y f es una función positiva diferenciable sobre I. Representaremos esta variedad Lorentziana por $M = I \times_f F$. El espaciotiempo $(n+1)$-dimensional M es un producto warped, en el sentido de [83, Chap. 7], con base $(I, -dt^2)$, fibra (F, g_F) y función warping f. La familia de espaciotiempos GRW es muy amplia en el sentido de que incluyen espaciotiempos clásicos como el espaciotiempo de Lorentz-Minkowski, el espaciotiempo de Einstein-de Sitter, el espaciotiempo estático de Einstein, y también los espaciotiempos de Robertson-Walker (dimensión cuatro y fibra de curvatura seccional constante).

Cualquier espaciotiempo GRW posee una función tiempo diferenciable global, y, por tanto, es establemente causal [14, p. 64]. Además, si la fibra es completa, entonces es globalmente hiperbólico [14, Thm. 3.66]. Por otro lado, un espaciotiempo GRW no es necesariamente espacialmente homogéneo. La homogeneidad espacial parece una hipótesis deseable para modelar el universo a gran escala. Sin embargo, en otras escalas, esta hipótesis dejaría de ser realista [87]. También, pequeñas deformaciones de la métrica de la fibra de un espaciotiempo de Robertson-Walker dan lugar a nuevos espaciotiempos GRW. Por tanto, los espaciotiempos GRW parecen buenos candidatos para explorar propiedades de estabilidad de un espaciotiempo de Robertson-Walker.

Recientemente, diversos datos experimentales sugieren que habría una dirección privilegiada en el espacio físico. En esta dirección, el universo parece expandirse más rápido que en direcciones ortogonales (véase [65], [66] y [81]). Por tanto, la inhomogeneidad espacial es necesaria de acuerdo con estas evidencias experimentales. Hay también razones teóricas para apoyar el uso de espaciotiempos GRW. Por un lado, hay muchas soluciones exactas a las ecuaciones de Einstein que pertenecen a esta familia. Por otro, la Teoría de la Inflación [72] nos sugiere que es natural pensar que la expansión tuvo que ocurrir en el espacio físico en todo punto y de forma simultánea. Antes de dicha inflación, dicho espacio pudo no ser simétrico. Algunos de los espaciotiempos GRW podrían ser modelos relativistas apropiados para una descripción aproximada a este proceso.

A pesar de la importancia histórica de los espaciotiempos GRW espacialmente cerrados, un número de argumentos teóricos y experimentales sobre el balance total de masa del universo [33] sugieren la conveniencia de considerar modelos cosmológicos espacialmente abiertos. Es más, un espaciotiempo GRW espacialmente cerrado viola el principio holográfico [20, p. 839], mientras que uno con fibra no compacta podría ser un modelo compatible con tal principio [11]. Más precisamente, la entropía contenida en una región acotada de una hipersuperficie espacial no debe exceder la cuarta parte del área de la frontera de dicha región (en unidades de Planck). Esto es, si Ω es una región compacta de una hipersuperficie espacial, y $S(\Omega)$ es la entropía de todos los sistemas materiales en Ω, entonces se debe cumplir

$$S(\Omega) \leq \frac{\text{Area}(\partial\Omega)}{4}.$$

Como muestra el siguiente argumento, puede ocurrir que la desigualdad anterior no se cumpla en espaciotiempos espacialmente cerrados. Supongamos que en un espaciotiempo existe una hipersuperficie espacial compacta tal que tiene un subconjunto abierto propio donde no existe contenido material. Entonces, en dicho subconjunto podemos tomar otro suficientemente pequeño, de tal manera que, aplicando la desigualdad anterior sobre el exterior de este compacto, se obtiene que la entropía se hace arbitrariamente pequeña. Se llega a una contradicción.

De todos modos, sólo la hipótesis de fibra no compacta parece ser muy débil como para considerar de forma completa un espaciotiempo GRW abierto, [68]. Una manera natural de asegurar que el universo es espacialmente inextendible es asumir que la fibra es geodésicamente completa. Por otro lado, sería deseable que aspectos esenciales en el ámbito del análisis geométrico de la fibra de un espaciotiempo GRW espacialmente cerrado se mantuvieran ciertos. Para tales fines, introducimos el siguiente concepto: un espaciotiempo GRW se dice que es espacialmente parabólico si su fibra posee un recubridor universal Riemanniano parabólico (y, por tanto, la fibra también es parabólica). Recordemos que una variedad Riemanniana completa (no compacta) es parabólica si no admite funciones superarmónicas no constantes y no negativas [69]. Por otro lado, si una variedad Riemanniana completa (F, g_F) tiene curvatura de Ricci no negativa (en particular F podría ser $\mathbb{R}^3$), entonces obedece la propiedad fuerte de Liouville [69, Thm. 4.8]; esto es, (F, g_F) no admite funciones positivas armónicas no constantes. Por tanto, la propiedad fuerte de Liouville se satisface en cualquier variedad Riemanniana parabólica sin ninguna hipótesis de curvatura.

La parabolicidad de la fibra de un espaciotiempo GRW puede también ser apoyada en varias razones de índole físco. Por ejemplo, las galaxias se pueden ver como moléculas (véase, por ejemplo, [83, Ch. 12]). Si una sonda se envia al espacio, su movimiento podría ser aproximado por un movimiento Browniano, [51]. De hecho, la distribución de galaxias y su velocidad no son completamente conocidas. Entonces, la parabolicidad favorece que la sonda pueda ser vista en cualquier región, ya que el movimiento Browniano es recurrente en cualquier variedad Riemanniana parabólica [51].

Aunque la familia de espaciotiempos GRW espacialmente parabólicos es muy amplia, existen otros espaciotiempos GRW de interés geométrico que no pertenecen a ella. Por ejemplo, aquellos cuya fibra es el espacio hiperbólico $\mathbb{H}^n$. Diversos principios del máximo pueden servir para caracterizar las hipersuperficies maximales en este contexto. En contraste a la parabolicidad, ahora se necesita imponer algunas hipótesis de curvatura. Los dos principios del máximo que usaremos son: la propiedad fuerte de Liouville, y el principio del máximo generalizado de Omori-Yau. El primero es un principio clásico aplicable a variedades Riemannianas completas con curvatura de Ricci no negativa. El segundo ha mostrado su utilidad para estudiar hipersuperficies espaciales de curvatura media constante y maximales. Notemos que, aunque parecen muy diferentes estas dos formas de atacar los problemas de unicidad, la idea subyacente es común: tener un control sobre el comportamiento de las funciones armónicas, superarmónicas o subarmónicas. De hecho, a lo largo de este trabajo mostraremos ampliamente que algunas funciones distinguidas pueden usarse en ambos casos para obtener resultados de unicidad.

Una vez que se ha establecido el espaciotiempo ambiente, nuestro segundo objetivo en esta memoria es obtener diversos resultados globales de caracterización de hipersuperficies maximales. Cualquier espaciotiempo GRW, $I \times_f F$, posee una familia distinguida de hipersuperficies espaciales, los slices espaciales $\{t_0\} \times F$, $t_0 \in I$. Observemos que un slice espacial es una hipersuperficie de nivel de la función tiempo asociada a la coordenada sobre el intervalo I. En general, un slice espacial $\{t_0\} \times F$ es totalmente umbilical con curvatura media constante, y es maximal (y, por tanto, totalmente geodésico) cuando t_0 sea un punto crítico de la función warping. Nuestra principal finalidad en esta tesis consiste en encontrar condiciones razonables bajo las cuales podamos probar que una hipersuperfice maximal completa sea totalmente geodésica o un slice espacial.

Finalmente, el tercer objetivo que nos proponemos es aplicar los resultados de unicidad paramétricos, que previamente hemos desarrollado, para solucionar nuevos problemas de tipo Calabi-Bernstein. Estos problemas consisten en obtener todas las soluciones de cierta EDP no lineal y elíptica definida sobre la fibra entera (es decir, todas las soluciones enteras). De hecho, trataremos con la ecuación de hipersuperficies maximales sobre una variedad Riemanniana (F, g_F),

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\right) = -\frac{f'(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}\Big(n + \frac{\mid Du \mid^2}{f(u)^2}\Big), \tag{E.1}$$

$$\mid Du \mid < \lambda f(u), \quad 0 < \lambda < 1\,. \tag{E.2}$$

La ecuación (E.1) es la ecuación de Euler-Lagrange para el funcional área. De hecho, significa que la curvatura media del grafo es cero. La ligadura (E.2) no sólo establece que el grafo $\Sigma_u = \{(u(p), p) : p \in F\}$ sea espacial, sino también que su ángulo hiperbólico esté acotado. Desde un punto de vista analítico, (E.2) implica que nuestra ecuación es, de hecho, uniformemente elíptica.

Obtendremos, en esta memoria, condiciones apropiadas bajo las cuales podamos encontrar todas las soluciones enteras a la ecuación (E).

Esta tesis está organizada como sigue. En el Capítulo 2, recordaremos las principales propiedades de los espaciotiempos GRW. Se repasan también algunas condiciones de energía que aparecen de modo natural en Relatividad General, y se mostrará cuándo un espaciotiempo GRW obedece cada una de ellas. Después, estudiaremos las hipersuperficies espaciales, prestando especial atención al caso maximal. Después de analizar el caso 2-dimensional, pasaremos a presentar y examinar en detalle la familia de EDPs relacionadas con grafos maximales en un espaciotiempo GRW de dimensión arbitraria.

El Capítulo 3 está dedicado a repasar el concepto de parabolicidad para el caso $n(\geq 2)$-dimensional. Se revisan diversos resultados bien conocidos que conducen a la parabolicidad de una variedad Riemanniana. El caso de dimensión 2 se muestra de un modo especial, donde brevemente se indica su relación con la curvatura de Gauss. Por otro lado, también se recuerda la definición de cuasi-isometría. Esta herramienta será clave en la consecución de algunas de nuestras técnicas. En la Sección 3.2, presentaremos dos resultados técnicos que permiten asegurar la parabolicidad de una hipersuperficie espacial completa en un espaciotiempo GRW espacialmente parabólico. Primero, obtendremos,

Teorema 3.2.5. *Sea S una hipersuperficie espacial completa en un espaciotiempo GRW espacialmente parabólico. Si el ángulo hiperbólico de S está acotado y la función warping sobre S satisface:*

i) $\sup f(\tau) < \infty$, *y*

ii) $\inf f(\tau) > 0$,

entonces S es parabólica.

La función ángulo hiperbólico de S se define, en cada punto de S, como el ángulo hiperbólico entre el campo vectorial normal unitario N sobre S en el cono temporal de $-\partial_t$, y el campo vectorial coordenado $-\partial_t$ (a lo largo de esta memoria, cualquier espaciotiempo GRW se le dotará de la orientación temporal dada por $-\partial_t$). Notemos que la acotación del ángulo hiperbólico de S implica que la velocidad que el observador instantáneo $-\partial_t(p)$, $p \in S$, mide para $N(p)$ no se aproxima a la velocidad de la luz en el vacío (para más detalles, véase la Sección 2.3). Por otro lado, las hipótesis sobre la función warping también admiten interpretación. Sea $C \subset \{t_0\} \times F$, $t_0 \in I$, un conjunto compacto de un slice espacial. Consideremos el flujo asociado a $-\partial_t$. Entonces, las hipótesis sobre la función warping aseguran que el volumen de C en este flujo no aumenta ni disminuye arbitrariamente (véase también la Sección 3.2). Observemos que $-\partial_t$ es un campo de observadores geodésicos [102].

El Teorema 3.2.5 será la base sobre la que descansarán los resultados principales del Capítulo 4. En este capítulo, bajo ciertas condiciones naturales, se demuestra que una hipersuperficie maximal completa es un slice espacial o es totalmente geodésica. Como ejemplo, tenemos,

Teorema 4.1.1. *Sea S una hipersuperficie maximal completa en un espaciotiempo GRW espacialmente parabólico cuya función warping f es no localmente constante y satisface* $(\log f)'' \leq 0$. *Si el ángulo hiperbólico de S está acotado,* $\sup f(\tau) < \infty$ *e* $\inf f(\tau) > 0$*, entonces S debe ser un slice espacial $t = t_0$, con $f'(t_0) = 0$.*

Notemos que la condición $(\log f)'' \leq 0$ se satisface cuando el espaciotiempo GRW obedece la Condición de Convergencia Temporal. Por otro lado, si combinamos parabolicidad con algunas hipótesis de curvatura, también podemos tratar el caso en que la función warping es constante,

Teorema 4.1.7. *Sea S una hipersuperficie maximal completa en un espaciotiempo GRW espacialmente parabólico y estático, $I \times F$. Si la curvatura de Ricci de la fibra es no negativa y el ángulo hiperbólico de S está acotado, entonces S debe ser totalmente geodésica.*

Como aplicación, nuestros resultados nos conducen a la resolución de nuevos ejemplos de problemas de Calabi-Bernstein para la ecuación de hipersuperficies maximales. Por ejemplo,

Teorema 4.2.1. *Sea $f : I \longrightarrow \mathbb{R}$ una función diferenciable positiva y no localmente constante. Supongamos que f satisface* $(\log f)'' \leq 0$, $\sup f < \infty$ *e* $\inf f > 0$. *Las únicas soluciones enteras a la ecuación* (E) *sobre una variedad Riemanniana parabólica F son las funciones constantes $u = c$, con $f'(c) = 0$.*

Para extender nuestro campo de trabajo, en el Capítulo 5 eliminaremos la hipótesis $\inf f > 0$, y llegaremos a las mismas conclusiones mediante otra aproximación. Desde un punto de vista físico, $\inf f > 0$ parece prohibir la presencia de singularidades inicial y/o final de tipo Big-Bang o Big-Crunch. Intuitivamente, se espera que en la evolución de observadores en caída libre hacia un Big-Crunch el espacio físico disminuya arbitrariamente (una situación análoga ocurre para el Big-Bang). El observador geodésico $\gamma(u) = (-u, p) \in I \times F$, $p \in F$, mide su espacio en reposo como $f(-u)^n \Omega_F(p)$, donde Ω_F es la forma de volumen de F. Si $\inf f > 0$, entonces γ no puede experimentar tal contracción arbitraria.

Esta otra aproximación que se hace en el Capítulo 5 está basada en asegurar la parabolicidad de una cierta métrica conforme a la inducida sobre una hipersuperficie espacial completa.

Teorema 3.2.9. *Sea S una hipersuperficie espacial completa en un espaciotiempo GRW espacialmente parabólico. Si $\sup f(\tau) < \infty$ y el ángulo hiperbólico de S está acotado, entonces S, dotada de la métrica conforme $\hat{g} = \frac{1}{f(\tau)^2}\, g$, es parabólica.*

Para más comentarios que relacionan los Teoremas 3.2.5 y 3.2.9, se puede consultar la Nota 3.2.10. El resultado anterior será fundamental para conseguir los teoremas de unicidad a lo largo del Capítulo 5. Entre ellos,

Teorema 5.1.1. *Sea S una hipersuperficie maximal completa en un espaciotiempo GRW espacialmente parabólico tal que si f es constante, $I \neq \mathbb{R}$. Supongamos que $\sup f(\tau) < \infty$ y que existe una constante positiva σ para la cual $(\log f)''(\tau) \leq (n - 2 + \sigma\, f(\tau))\, (\log f)'(\tau)^2$. Si el ángulo hiperbólico de S está acotado, entonces S debe ser un slice espacial $t = t_0$, con $f'(t_0) = 0$.*

Resaltemos que no sólo estamos considerando una clase mucho más amplia de espaciotiempos GRW, comparados con los del Capítulo 4, sino que también algunas otras hipótesis son ahora más debiles. La razón de este hecho estriba en que en el Capítulo 4 garantizamos la parabolicidad directamente sobre la hipersuperficie maximal, teniendo cierto control sobre su geometría, mientras que con esta otra técnica, la parabolicidad se asegura sobre una métrica conforme.

Se obtienen más resultados de tipo paramétrico que, como en el capítulo previo, resuelven nuevos problemas de tipo Calabi-Bernstein. Uno de ellos es el siguiente,

Teorema 5.2.5. *Sea $f : I \to \mathbb{R}$ una función diferenciable positiva, monótona y que satisface $f \in \mathrm{L}^1(I)$. Las únicas soluciones enteras a la ecuación* (E) *sobre una variedad Riemanniana parabólica F son las constantes $u = c$, con $f'(c) = 0$.*

En el Capítulo 6 nos centramos en el caso de espaciotiempos GRW con fibra de dimensión 2. Estos espaciotiempos (y, en general, otros espaciotiempo de dimensión 3) tienen un mayor interés geométrico que físico. No obstante, permiten investigar propiedades que potencialmente pueden ser extendidas a dimensión superior (por eso en la literatura se les conoce como espaciotiempos de juguete). Prestaremos especial atención a los espaciotiempos GRW cuya fibra tiene curvatura total finita. Físicamente, esta familia de espaciotiempos puede verse como una versión 3-dimensional de los espaciotiempos asintóticamente llanos. Recordemos que una superficie Riemanniana completa (no compacta) M^2 tiene curvatura total finita si la parte negativa de su curvatura de Gauss es integrable (véase, por ejemplo, [69]). Esto es, si K es la curvatura de Gauss de M, entonces M tiene curvatura total finita si

$$\int_M \max\{0, -K\} < \infty\,.$$

En particular, si M tiene curvatura de Gauss no negativa, entonces trivialmente su curvatura total es finita. Extendiendo al conocido teorema de Ahlfors y Blanc-Fiala-Huber, [57], una superficie Riemanniana con curvatura total finita debe ser parabólica [69]. Aquí, en lugar de considerar el caso de superficies maximales, trataremos con superficies espaciales completas S cuya función curvatura media, H, viene controlada por la siguiente desigualdad,

$$H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}\,.$$

Notemos que cualquier superficie maximal satisface claramente la desigualdad. Por tanto, el estudio de esta desigualdad es una extensión natural de nuestro problema original. Por otro lado, podemos dar la siguiente interpretación geométrica de esta desigualdad: el valor absoluto de la curvatura media de S en $p \in S$ no excede la cantidad análoga para el slice espacial $t = t(p)$. En general, una superficie espacial que satisface esta desigualdad no debe tener necesariamente curvatura media constante. No obstante, bajo condiciones razonables sobre el espaciotiempo ambiente, una superficie espacial completa con curvatura media constante contenida entre dos slices espaciales debe satisfacer la desigualdad (véase [24] y [90]).

Observemos que un slice espacial $t = t_0$ obedece la desigualdad para cualquier función warping. Nuestro problema aquí será establecer el recíproco cuando sea posible, es decir, determinar cuándo una superficie espacial completa que satisface dicha desigualdad debe ser un slice espacial.

Daremos respuesta a esta pregunta bajo supociones que generalizan ampliamente a varios trabajos previos [67] y [90], donde la fibra era el plano Euclídeo $\mathbb{R}^2$, y [93], donde la fibra era compacta. Por consiguiente, vamos a considerar un escenario más amplio en el que la fibra tenga curvatura total finita.

La aproximación que haremos para este problema será, en primer lugar, considerar la desigualdad diferencial sobre una superficie Riemanniana completa (no compacta) y obtener condiciones bajo las cuales las funciones constantes son las únicas soluciones. La idea de la prueba es como sigue. Se demostrará que una solución a dicha desigualdad produce un grafo espacial completo con curvatura total finita. Entonces, usaremos la parabolicidad para concluir que la superficie debe de ser un slice espacial mediante un análisis de funciones distinguidas. Así, en el caso no paramétrico, algunos resultados de unicidad se pueden dar. Como ilustración,

Teorema 6.3.3. *Sea (F, g_F) una superficie Riemanniana completa con curvatura total finita y sea $f : I \longrightarrow (0, \infty)$, $I \subset \mathbb{R}$, una función diferenciable no localmente constante y que satisface* $\inf f > 0$ *y* $(\log f)'' \leq 0$. *Entonces, las únicas soluciones enteras a*

$$H(u)^2 \leq \frac{f'(u)^2}{f(u)^2}$$

$$|Du| < \lambda\, f(u)\,, \;\; 0 < \lambda < 1\,,$$

son las constantes.

En el caso no paramétrico la topología del grafo se controla por la topología de la fibra; sin embargo, en el caso paramétrico este hecho deja de ocurrir. Entonces, necesitaremos imponer hipótesis extra para obtener el control topológico necesario. Más precisamente, exigiremos que la superficie recubra con un número finito de hojas a la fibra.

Teorema 6.4.1. *Sea $M = I \times_f F$ un espaciotiempo GRW cuya función warping es no localmente constante, y cuya fibra 2-dimensional tiene curvatura total finita. Sea S una superficie espacial completa en M, tal que recubra a la fibra con un número finito de hojas, la función warping esté acotada sobre S y $(\log f)''(\tau) \leq 0$. Supongamos que la desigualdad*

$$H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}$$

ocurre sobre S. Entonces S es un slice espacial.

El Capítulo 6 se cierra con la Sección 6, donde se discuten algunas interpretaciones físicas. De hecho, se analizarán algunas estimaciones para la energía total que una superficie espacial puede tener en nuestros espaciotiempos ambientes. Estas estimaciones serán posibles gracias al hecho de que la superficie espacial tiene curvatura total finita. Observemos que la energía total de una superficie espacial está acotada superior e inferiormente por invariantes intrínsecos. Es más, si la superficie es un slice espacial, entonces su energía total está acotada superiormente por un múltiplo de la característica de Euler-Poincaré de F.

Finalmente, el Capítulo 7 está dedicado a estudiar hipersuperficies maximales en ciertos espaciotiempos GRW, esta vez mediante el uso de principios del máximo apropiados. La idea común con los capítulos anteriores es obtener cierto control de las funciones superarmónicas, subarmónicas y/o armónicas. Primero, usando la propiedad fuerte de Liouville, probamos

Teorema 7.1.2. *Sea S una hipersuperficie maximal completa en un espaciotiempo*

GRW estático cuya fibra tiene curvatura seccional no negativa. Si S está acotada superior o inferiormente, entonces S debe ser un slice espacial.

A continuación, consideraremos el principio del máximo generalizado de Omori-Yau. Para tal fin, encontraremos condiciones satisfactorias bajo las cuales se pueda asegurar la aplicabilidad de este principio sobre ciertas hipersuperficies. Entonces, distintos análisis de funciones distinguidas conducirán a diversos resultados de unicidad. Por ejemplo,

Teorema 7.1.9. *Sea S una hipersuperficie maximal completa en un espaciotiempo GRW cuya función warping es no localmente constante, y cuya fibra tiene curvatura seccional acotada inferiormente. Supongamos que* $(\log f)''(\tau) \leq 0$ *y S está contenida entre dos slices espaciales. Si S tiene ángulo hiperbólico acotado, entonces S debe ser un slice espacial.*

Notemos que la naturaleza de las hipótesis son análogas a las requeridas en los capítulos previos, mientras que la parabolicidad de la fibra se reemplaza por hipótesis de curvatura sobre ella.

Por otro lado, el principio del máximo generalizado puede ser usado para obtener más información geométrica para nuestros propósitos. En la Sección 7.2 se presentan algunos resultados de no existencia. Aquí, la hipótesis decisiva es la suposición de ausencia de puntos críticos de la función warping.

Finalmente, se presentan una breve discusión de las conclusiones de esta memoria, así como diversas líneas de investigación futuras de interés.

Chapter 1

Introduction

Spacelike hypersurfaces are geometrical objects which have high interest for Physics and Lorentzian Geometry. Roughly speaking, each of them represents the physical space in an instant of a time function. More precisely, the initial value problem for each fundamental equation within General Relativity is formulated in terms of a spacelike hypersurface (see, for instance, [88] and references therein). Even more, in Electromagnetism, a spacelike hypersurface is an initial data set which univocally determines the future of the electromagnetic field which satisfies the Maxwell equations [102, Thm. 3.11.1] and for the simple matter equations [102, Thm. 3.11.2]. In Causality Theory, the mere existence of a certain spacelike hypersurface implies that the spacetime obeys a certain causal property. For instance, a spacetime is globally hyperbolic [83, Def. 14.20] if and only if it admits a Cauchy hypersurface, [48]. In fact, any globally hyperbolic spacetime admits a smooth spacelike Cauchy hypersurface S and then, it is diffeomorphic to $\mathbb{R} \times S$, [15]. Hence, spacelike hypersurfaces are remarkable due to their physical interest. Let us remark that the completeness of a spacelike hypersurface is required whenever we study its global properties, and also, from a physical viewpoint, completeness implies that the whole physical space is take into consideration.

The extrinsic geometry of a spacelike hypersurface is codified by its shape operator. Among the functions defined by it, the mean curvature function has a great importance. The case when we have constant mean curvature is relevant both in General Relativity and Lorentzian Geometry, specially when it vanishes (i.e., the maximal case). On the one hand, they can constitute an initial set for the Cauchy problem [88]. Specifically, Lichnerowicz proved that a Cauchy problem with initial conditions on a maximal hypersurface is reduced to a second-order non-linear elliptic differential equation and a first-order linear differential system [71].

Moreover, these hypersurfaces are important in order to analyze the dynamics of a gravitational field or the classical n-body problem in a gravitational field (see, for instance, [23] and references therein).

On the other hand, each maximal hypersurface can describe, in some relevant cases, the transition between the expanding and contracting phases of a relativistic universe. Moreover, the existence of constant mean curvature (and in particular maximal) hypersurfaces is necessary for the study of the structure of singularities in the space of solutions to the Einstein equations. Also, the deep understanding of this kind of hypersurfaces is essential to proof the positivity of the gravitational mass. They are also interesting for Numerical Relativity, where maximal hypersurfaces are used to integrate forward in time. All these physical aspects can be found in [77] and references therein.

From a mathematical point of view, it is necessary to study the maximal hypersurfaces of a spacetime in order to understand its structure [13]. Especially, for some asymptotically flat spacetimes, the existence of a foliation by maximal hypersurfaces is established (see, for instance, [23] and references therein). The existence results and, consequently, uniqueness appear as kernel topics.

A maximal hypersurface is (locally) a critical point for a natural variational problem, namely of the area functional (see, for instance, [21]).

Throughout the history of the research on maximal hypersurfaces, the discovery of new non-linear elliptic problems was a striking fact. In fact, the function defining a maximal graph in the $(n+1)$-dimensional Lorentz-Minkowski spacetime, $\mathbb{L}^{n+1}$, satisfies a elliptic second order PDE similar to the equation of minimal graphs in the Euclidean space $\mathbb{R}^{n+1}$. However, a new and surprising behavior in its entire solutions was found: the affine functions defining spacelike hyperplanes are the only entire solutions to the maximal hypersurface equation in $\mathbb{L}^{n+1}$. This result was previously proved by Calabi [29] for $n \leq 4$ and later extended for any n in the seminal paper by Cheng and Yau [32]. That is why it is usually called the Calabi-Bernstein theorem. Let us recall that the Bernstein theorem for entire minimal graphs in $\mathbb{R}^{n+1}$ holds true only for $n \leq 7$, [106]. Another important achievement in [32] was the introduction of a new procedure, the so-called Omori-Yau generalized maximum principle [82], [111].

Other classical papers dealing with uniqueness of complete maximal and constant mean curvature spacelike hypersurfaces are [23], [34] and [77]. In [23], Brill and Flaherty replaced Lorentz-Minkowski spacetime by a spatially closed universe, and proved uniqueness in the large by assuming that its Ricci tensor satisfies $\mathrm{Ric}(z,z) > 0$ for all the timelike tangent vectors z. This assumption may be interpreted as the fact that there is real present matter at every point of the spacetime. It is known as the Ubiquitous Energy Condition (see Section 2 in Chapter 2). In [77], this energy condition was relaxed by Marsden and Tipler to include, for instance, non-flat vacuum spacetimes. More recently, Bartnik proved very general existence theorems in [12], and consequently, he claimed that it would be necessary to find new satisfactory uniqueness results. Later, in [9] Alías, Romero and Sánchez proved new uniqueness results for the class of spatially closed generalized Robertson-Walker (GRW) spacetimes (which clearly includes spatially closed Robertson-Walker spacetimes), under a weaker energy condition, the so-called Timelike Convergence Condition. In [7], Alías and Montiel proved that in a GRW spacetime whose warping function satisfies $(\log f)'' \leq 0$, the spacelike slices are the only compact constant mean curvature spacelike hypersurfaces. Furthermore, this result was generalized in [27] by Caballero, Romero and Rubio for a larger class of spacetimes. In the case of the Einstein-de

Sitter spacetime, which is a spatially open model, Rubio gave new uniqueness and non-existence results for complete maximal and constant mean hypersurfaces [101].

Firstly, this thesis is aimed at inquiring about what kind of open spacetimes have rich properties in order to provide uniqueness results. Parabolicity is a good feature that should be taken into account within a physical space. Moreover, it would be satisfactory if these spacetimes could describe the universe in some environment. This family would be GRW spacetimes suitable for our purposes. Let us recall that by a GRW spacetime we mean a product manifold $I \times F$, of an open interval I of the real line $\mathbb{R}$ and an $n(\geq 2)$-dimensional (connected) Riemannian manifold $(F, g_{_F})$, endowed with the Lorentzian metric

$$\overline{g} = -\pi_{_I}^*(dt^2) + f(\pi_{_I})^2 \, \pi_{_F}^*(g_{_F}) \, ,$$

where $\pi_{_I}$ and $\pi_{_F}$ denote the projections onto I and F, respectively, and f is a positive smooth function on I. We will represent this Lorentzian manifold by $M = I \times_f F$. The $(n+1)$-dimensional spacetime M is a warped product, in the sense of [83, Chap. 7], with base $(I, -dt^2)$, fiber $(F, g_{_F})$ and warping function f. The family of GRW spacetimes is very large since it does not only includes classical spacetimes as the Lorentz-Minkowski spacetime, the Einstein-de Sitter spacetime, the static Einstein spacetime, but also the Robertson-Walker spacetimes (dimension four and fiber of constant sectional curvature).

Any GRW spacetime has a smooth global time function, and therefore it is stably causal [14, p. 64]. In addition, if the fiber is complete, then a GRW spacetime is globally hyperbolic [14, Thm. 3.66]. On the other hand, a GRW spacetime is not necessarily spatially homogeneous. Roughly speaking, spatial homogeneity seems to be a desirable assumption so as to shape the universe in the large. However, in a more precise scale this condition might be unrealistic [87]. Furthermore, small deformations in the metric of a Robertson-Walker spacetime's fiber also fit into the class of GRW spacetimes. Therefore, GRW spacetimes appear to be nice candidates to explore stability properties of Robertson-Walker spacetimes.

Recently, several experimental data have suggested that there is a preferable direction in the physical space. In this direction the universe seems to be expanding faster than it does in orthogonal directions, (see [65], [66] and [81]). Hence, spatial inhomogeneity is required according to these experimental data. There are also reasons of theoretical nature to support the use of GRW spacetimes. On the one hand, there are many exact solutions to Einstein equations which lie on the family of GRW spacetimes. On the other hand, the theory of inflation is nowadays commonly accepted [72]. In this environment, it is natural to think that the expansion must have occurred anywhere in the physical space and simultaneously. Prior to the inflation, the physical space may not be symmetric in a large scale. Therefore, GRW spacetimes may be suitable relativistic models to approach this process.

In spite of the historical importance of spatially closed GRW spacetimes, a number of observational and theoretical arguments about the total mass balance of the universe [33] suggest the convenience of taking into consideration open cosmological models. Even more, a spatially closed GRW spacetime violates the holographic principle [20, p. 839] whereas a GRW spacetime with non-compact fiber could be a suitable model that follows that principle [11]. More precisely, the entropy contained in any spatial region cannot exceed a quarter of the area of the region's boundary (in Planck units). That is, if Ω is a compact region of a spacelike hypersurface, and $S(\Omega)$ denotes the entropy of all matter systems in Ω, then

$$S(\Omega) \leq \frac{\text{Area}(\partial\Omega)}{4} \, .$$

The following argument shows that the previous inequality cannot be held in some spatially closed GRW spacetimes. Let us consider that a spacetime has a compact spacelike hypersurface such that it contains a matter system that does not occupy the whole of it. That is, the hypersurface has a proper compact subset with no matter system. In that subset we may consider another small enough, in such a way that, applying the previous inequality on the exterior of this compact subset, we have that the entropy becomes arbitrarily small. We found a contradiction.

Nevertheless, the assumed non-compact fiber seems to be too weak to consider such a GRW spacetime as a suitable model for a whole open universe, [68]. A natural way to assert that the universe is spatially inextensible is to suppose that this fiber is geodesically complete. On the other hand, it would be desirable that the essential aspects of the rich geometric analysis of the fiber of a spatially closed GRW spacetime remain true. In order to do that, we shall introduce the following notion. A GRW spacetime is said to be spatially parabolic if its fiber has a parabolic universal Riemannian covering (therefore, the fiber is so). Let us recall that a (non-compact) complete Riemannian manifold is parabolic provided that it does not admit non-constant non-negative superharmonic function, [69]. Notice that whenever a complete Riemannian manifold (F, g_F) has non-negative Ricci curvature (in particular F may be $\mathbb{R}^3$), the strong Liouville property remains on it [69, Thm. 4.8], i.e., (F, g_F) admits no non-constant positive harmonic function. Note that the strong Liouville property holds true on any parabolic Riemannian manifold without any curvature assumption.

The parabolicity of a GRW spacetime's fiber could also be supported by some physical reasons. For instance, galaxies can be understood as molecules (see, for instance, [83, Ch. 12]). If a sonde is sent to the space, its motion may be approached by a Brownian motion, [51]. In fact, the distribution of galaxies and their velocities are not completely known. Parabolicity may favor that the sonde could be observed in any region, since the Brownian motion is recurrent in any parabolic Riemannian manifold [51].

The family of spatially parabolic GRW spacetimes is very large, although some other interesting GRW spacetimes do not belong to this family. For instance, those GRW spacetimes whose fiber is the hyperbolic space $\mathbb{H}^n$ are excluded. Maximum principles can help to deal with this environment. In contrast to parabolicity, some curvature assumptions should be imposed here. The two maximum principles that we will be using are: the strong Liouville property and the Omori-Yau generalized maximum principle. The first one is a classical principle that works on complete Riemannian manifolds with non-negative Ricci curvature. The second one has proved its utility to study maximal and constant mean curvature spacelike hypersurfaces.

Although parabolicity and maximum principles seem to be too very different factors to be considered together, the underlying idea is common: they have a certain control over the behavior of superharmonic, subharmonic or harmonic functions. In fact, throughout this thesis we will show that some distinguished functions can be used to deal with uniqueness results in both cases.

Once ambient spacetimes have been established, our second objective is to provide several global characterization results for maximal hypersurfaces. Any GRW spacetime $I \times_f F$ possesses a family of distinguished spacelike hypersurfaces, the so-called (embedded) spacelike slices $\{t_0\} \times F$, $t_0 \in I$. Notice that a spacelike slice is a level hypersurface of the time function associated to the coordinate on the interval I. In general, a spacelike slice $\{t_0\} \times F$ is totally umbilical and it has constant mean curvature. Besides, it is maximal (and hence totally geodesic) whenever t_0 is a critical point in the warping function. Throughout this thesis we will say that a spacelike hypersurface $x : S \to I \times_f F$ is an (immersed) spacelike slice provided that $\pi_I \circ x$ is a constant t_0, i.e., if $x(S)$ is contained in $t = t_0$. Our main aim consists on finding reasonable conditions under which a complete maximal hypersurface has to be a spacelike slice or totally geodesic.

Finally, our third goal is to apply our previously developed parametric uniqueness results to solve new Calabi-Bernstein type problems. That is, to obtain all the solutions to certain non-linear elliptic PDE defined on the whole fiber (i.e., all the entire solutions). In fact, we will deal with the maximal hypersurface equation on a Riemannian manifold (F, g_F),

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\right) = -\frac{f'(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}\Big(n + \frac{\mid Du \mid^2}{f(u)^2}\Big), \qquad \text{(E.1)}$$

$$\mid Du \mid < \lambda f(u), \quad 0 < \lambda < 1\,. \qquad \text{(E.2)}$$

Equation (E.1) is the Euler-Lagrange equation for the area functional. In fact, it means that the mean curvature of the graph vanishes. The constrain (E.2) establishes that the graph $\Sigma_u = \{(u(p), p), p \in F\}$ is spacelike and its hyperbolic angle is bounded. From an analytical point of view, (E.2) assures that our equation is uniformly elliptic.

In this thesis, we will actually obtain suitable conditions under which all the entire solutions to equation (E) can be found.

This report is organized as follows. In Chapter 2, we recall the main properties of GRW spacetimes. Some energy conditions arising in a natural way in General Relativity are also reviewed, and it will be showed when a GRW spacetime obeys each of them. Then, spacelike hypersurfaces will be examined, paying special attention to the maximal case. After analyzing the 2-dimensional case, we will continue by presenting and examining the family of PDEs equations related to a maximal graph in a GRW spacetime.

Chapter 3 is devoted to revise the notion of parabolicity in the case $n(\geq 2)$-dimensional. We review several well-known results that lead to parabolicity of a Riemannian manifold. The 2-dimensional case arises in a special way, where it is briefly indicated the relation between parabolicity and curvature. On the other hand, the definition of quasi-isometry is also recalled. This feature will be a central key in the consecution of our techniques. Next, in Section 3.2 we present two technical results that allow us to assure parabolicity on a complete spacelike hypersurface in a spatially parabolic GRW spacetime. Firstly, we obtain,

Theorem 3.2.5. *Let S be a complete spacelike hypersurface in a spatially parabolic GRW spacetime. If the hyperbolic angle of S is bounded and the warping function on S satisfies:*

i) $\sup f(\tau) < \infty$, *and*

ii) $\inf f(\tau) > 0$,

then, S is parabolic.

The hyperbolic angle function of S is defined as the hyperbolic angle between the unit normal vector field N on S in the time cone of $-\partial_t$, and the coordinate vector field $-\partial_t$ (throughout this memory, any GRW spacetime is assumed to have the time orientation defined by $-\partial_t$). Note that the boundedness of the hyperbolic angle of S implies that the speed which the instantaneous observer $-\partial_t(p)$, $p \in S$, measures from $N(p)$ does not approach to the speed of light in vacuum (for more details, see Section 2.3). On the other hand, the assumptions on the warping function also admit a nice interpretation. Let $C \subset \{t_0\} \times F$, $t_0 \in I$, be a compact set of a spacelike slice. Let us consider the flow associated to $-\partial_t$. Then, the hypothesis on the warping function assures that the volume of C in this flow neither increase nor decrease arbitrarily (see also Section 3.2). Notice that $-\partial_t$ is a geodesic reference frame [102].

Theorem 3.2.5 will be the basis on which the main results of Chapter 4 are built. In this chapter, under certain natural assumptions, a complete maximal hypersurface is proved to be a spacelike slice or totally geodesic. As an example, we can provide,

Theorem 4.1.1. *Let S be a complete maximal hypersurface of a spatially parabolic GRW spacetime whose warping function f is non-locally constant and satisfies $(\log f)'' \leq 0$. If the hyperbolic angle of S is bounded,* $\sup f(\tau) < \infty$ *and* $\inf f(\tau) > 0$*, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Note that the assumption $(\log f)'' \leq 0$ is satisfied when the GRW spacetime obeys the Timelike Convergent Condition. On the other hand, if we combine parabolicity with some curvature hypothesis, the case in which the warping function is constant can be handled,

Theorem 4.1.7. *Let S be a complete maximal hypersurface in a static spatially parabolic GRW spacetime $I \times F$. If the Ricci curvature of the fiber is non-negative and the hyperbolic angle of S is bounded, then S must be totally geodesic.*

When applied, our results lead to new examples of Calabi-Bernstein problems for the maximal hypersurface equation. For instance,

Theorem 4.2.1. *Let $f : I \longrightarrow \mathbb{R}$ be a non-locally constant positive smooth function. Assume f satisfies $(\log f)'' \leq 0$, $\sup f < \infty$ and $\inf f > 0$. The only entire solutions to the equation* (E) *on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

In order to go deeper, in Chapter 5 we will drop the assumption $\inf f > 0$ and we will reach to the same conclusions using a different approach. From a physical point of view, the assumption $\inf f > 0$ seems to forbid the presence of initial and/or final singularities type Big-Bang or Big-Crunch. On the other hand, it is intuitively expected that in the evolution of free falling observers into a Big-Crunch, or from a Big-Bang, the physical space would decrease arbitrarily. The geodesic observer $\gamma(u) = (-u, p) \in I \times F$, $p \in F$, measures its restspace as $f(-u)^n \Omega_F(p)$, where Ω_F is the volume form of F. Therefore, if $\inf f > 0$, γ cannot experiment such arbitrarily contraction in either its future or its past. This other one is based in assuring parabolicity on a complete spacelike hypersurface whenever it is endowed with a certain conformal metric.

Theorem 3.2.9. *Let S be a complete spacelike hypersurface in a spatially parabolic GRW spacetime. If $\sup f(\tau) < \infty$ and the hyperbolic angle of S is bounded, then S, endowed with the conformal metric $\hat{g} = \frac{1}{f(\tau)^2}\, g$, is parabolic.*

For more comments relating Theorem 3.2.5 and 3.2.9 see Remark 3.2.10. This result will be essential to attain the uniqueness theorems in Chapter 5. Among them, we get,

Theorem 5.1.1. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime which is not a complete static one. Suppose that $\sup f(\tau) < \infty$ and there exists a positive constant σ such that $(\log f)''(\tau) \leq (n-2+\sigma\, f(\tau))\,(\log f)'(\tau)^2$. If the hyperbolic angle of S is bounded, then S must be a spacelike slice $t = t_0$, with*

$f'(t_0) = 0$.

Notice that not only are we considering a wider class of GRW spacetimes in comparison with Chapter 4, but also some hypothesis are now lessened. The reason for this is that in Chapter 4 we take parabolicity for granted directly on the maximal hypersurface, having certain control over its geometry, whilst with this other technique, parabolicity is assured on a suitable conformal metric on the maximal hypersurface.

As in the previous chapter, new Calabi-Bernstein type problems are solved. One of them is the following,

Theorem 5.2.5. *Let $f : I \to \mathbb{R}^+$ be a positive monotone smooth function which satisfies $f \in \mathrm{L}^1(I)$. The only entire solutions to equation* (E) *on a parabolic Riemannian manifold F are the constants $u = c$, with $f'(c) = 0$.*

In Chapter 6 we focus on the case of GRW spacetimes with a two-dimensional fiber. This kind of Lorentzian manifolds are usually called toy spacetimes since they are easier to deal with, and allow us to investigate properties which potentially could be extendible to a higher dimension. We will pay attention to GRW spacetimes whose fiber has finite total curvature. Physically, this family of GRW spacetimes may be regarded as a 3-dimensional version of asymptotically flat spacetimes. Let us recall that a complete (non-compact) Riemannian surface M^2 has finite total curvature providing that the negative part of its Gauss curvature is integrable (see, for instance, [69]). That is, if K is the Gaussian curvature of M, then M has finite total curvature if

$$\int_M \max\{0, -K\} < \infty \,.$$

In particular, if M has non-negative Gaussian curvature, then it must trivially have finite total curvature. Generalizing the theorem of Ahlfors and Blanc-Fiala-Huber, [57], a Riemannian surface with finite total curvature must be parabolic [69]. Here, instead of considering the case of maximal surfaces, we deal with complete spacelike surfaces S whose mean curvature function H is controlled by the following inequality,

$$H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}\,.$$

Note that any maximal surface trivially satisfies the inequality. Hence, the study of this inequality is a natural extension of our original problem. Moreover, it also admits a geometrical interpretation: the absolute value of the mean curvature of S at $p \in S$ does not exceed the analogous quantity for the spacelike slice $t = t(p)$. Note that any spacelike surface which satisfies the inequality does not necessarily to have a constant mean curvature. However, under reasonable assumptions on the ambient spacetime, a complete spacelike surface with constant mean curvature which lies between two spacelike slices must satisfy this inequality (see [24] and [90]).

Notice that a spacelike slice $t = t_0$ satisfies the inequality for any warping function. Hence, our problem resides in establishing the converse, i.e., when a complete spacelike surface which satisfies the inequality must be a spacelike slice.

We will provide some answers to this question under assumptions which widely generalize some previous works [67] and [90], where the fiber was the Euclidean plane $\mathbb{R}^2$, and [93] where the fiber was compact. Consequently, we will consider a wider framework where the fiber has finite total curvature.

The approach to this problem is, as first instance, to consider the differential inequality on a complete (non-compact) Riemannian surface and provide conditions under which the constant functions are the only solutions. The idea of the proof is as follows. We will see that a solution to the inequality provides a complete spacelike graph in this class of GRW spacetimes such that it has finite total curvature. Then, parabolicity appears to hint that the surface must be a spacelike slice by analyzing distinguished functions. Hence, for the non-parametric case, some uniqueness results are supplied. As an illustration,

Theorem 6.3.3. *Let (F, g_F) be a complete Riemannian surface with finite total curvature and let $f : I \longrightarrow (0,\infty)$, $I \subset \mathbb{R}$ be a smooth function such that f is* *non-locally constant,* $\inf f > 0$ *and* $(\log f)'' \leq 0$*. Then, the only entire solutions to*

$$H(u)^2 \leq \frac{f'(u)^2}{f(u)^2}$$

$$|Du| < \lambda\, f(u)\,,\ \ 0 < \lambda < 1\,,$$

are the constants.

The topology of the graph is controlled by the topology of the fiber in the non-parametric case. However, this fact does not occur in the parametric case. Therefore, we need to impose some extra hypothesis in order to get the required topological control. Basically, this control is achieved by requiring that the surface covers the fiber with a finite number of sheets.

Theorem 6.4.1. *Let* $M = I \times_f F$ *be a GRW spacetime whose warping function is non-locally constant, and whose* 2*-dimensional fiber has finite total curvature. Let* S *be a complete spacelike surface in* M*, such that it covers the fiber with a finite number of sheets, the warping function is bounded on* S *and* $(\log f)''(\tau) \leq 0$*. Suppose that the inequality*

$$H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}$$

holds on S*. Then* S *is a spacelike slice.*

Section 6 ends with some physical interpretations. In fact, we provide some estimates of the total energy which a spacelike surface can have in our ambient spacetimes. These estimates are possible due to the fact that the spacelike surface has finite total curvature. We observe that the total energy of a spacelike surface is bounded by intrinsic invariants from above and from below. Moreover, if the surface is a spacelike slice, its total energy is bounded by the Euler-Poincaré characteristic of F from above.

Finally, Chapter 7 is devoted to study maximal hypersurfaces in now different GRW spacetimes. This time using suitable maximum principles. The common idea shared with previous chapters is to obtain certain control over the superharmonic, subharmonic and/or harmonic functions. Firstly, using the strong Liouville property we prove

Theorem 7.1.2. *Let S be a complete maximal hypersurface in a static GRW spacetime whose fiber has non-negative sectional curvature. If S is bounded from below or from above, then S must be a spacelike slice.*

After that, we consider the generalized Omori-Yau maximum principle. In order to make use of it, we shall find satisfactory conditions under which this principle remains on such a hypersurface. Then, an analysis of distinguished functions leads to several uniqueness results, for instance,

Theorem 7.1.9. *Let S be a complete maximal hypersurface in a GRW spacetime whose warping function is non-locally constant and whose fiber has sectional curvature bounded from below. Assume that $(\log f)''(\tau) \leq 0$ and S lies between two spacelike slices. If S has bounded hyperbolic angle, then S must be a spacelike slice.*

Notice that the nature of the assumptions on the maximal hypersurface are analogous to those required on previous chapters, while parabolicity on the fiber is replaced by a curvature assumption on it.

In another environment, the generalized maximum principle can be used in order to obtain more geometrical information for our purposes. In Section 7.2 some nonexistence results are presented. Here, the determining assumption is the absence of critical points in the warping function.

Finally, we present a brief discussion about our conclusions and we provide several interesing guidelines that may be very useful for future research.

Chapter 2

Preliminaries

This chapter is devoted to present the geometrical background that will come in handy throughout this memory. Firstly, we focus on the ambient spacetimes, the generalized Robertson-Walker spacetimes. Some of their mathematical and physical properties are analyzed. Secondly, we shall consider several energy assumptions which will be accepted. These energy assumptions are borrowed from General Relativity. Indeed, if a spacetime satisfies the Einstein equations with a physically reasonable stress-energy tensor, then it must obey some of these energy assumptions. From a geometric point of view, they are established considering the curvature of the spacetime. Later, we devote a section to recall the basics of spacelike hypersurfaces. The geometry of a GRW spacetime favours to consider several natural functions, to the extent that, when restricted to a spacelike hypersurface, their Laplacian is computable. Finally, we consider the maximal hypersurface equation (associated to a GRW spacetime). We prove its deduction from a variational point of view and some of its most interesting features are explained. Among them, it is highlighted that an entire solution does not define, in general, a complete maximal graph.

2.1 GRW spacetimes

Let (F, g_F) be an $n(\geq 2)$-dimensional (connected) Riemannian manifold. Let us consider a positive smooth function f defined on an open interval $I \subseteq \mathbb{R}$. The product space $I \times F$ can be endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F)\,, \tag{2.1}$$

where π_I and π_F denote the projections onto I and F, respectively. This Lorentzian metric is clearly time orientable because the coordinate vector field $\partial_t := \partial/\partial_t$ is a (globally defined) timelike vector field. Thus, $(M, \overline{g})$ is a spacetime, which we will denote by $M := I \times_f F$. In fact, M is a warped product in the sense of [83, Chap. 7], with base $(I, -dt^2)$, fiber (F, g_F) and warping function f. Agreeing with the terminology introduced in [9], we will refer to M as a *Generalized Robertson-Walker (GRW) spacetime*[1]. This family of spacetimes properly extends to the classical Robertson-Walker spacetimes, which appear when the fiber has dimension three and constant sectional curvature.

A GRW spacetime is not necessarily spatially homogeneous. Remember that spatial homogeneity seems appropriate just as a rough approach to consider the universe on a large scale (see [78, Ch. 30], for instance). However, this assumption could not be physically realistic when the universe is considered in a more accurate scale. Hence, the family of GRW spacetimes could be suitable to shape universes with inhomogeneous spacelike geometry [87].

On the other hand, notice that a conformal change of (2.1), such that the conformal factor depends only on t, produces a new GRW spacetime. Furthermore, small deformations of the metric on a Robertson-Walker spacetime's fiber also fit into the class of GRW spacetimes. This suggests that GRW spacetimes may be useful to analyze the stability of the properties of a Robertson-Walker spacetime.

[1]To be fair, it should be named Generalized Friedman-Lemaître-Robertson-Walker (GFLRW) spacetime. However, when GRW spacetimes were introduced, the name RW spacetime was most common in the literature.

Example 2.1.1. We provide some classical spacetimes which admit a splitting like GRW spacetimes,

- Lorentz-Minkowski spacetime, $\mathbb{L}^{n+1}$, appears when $f = 1$ and the fiber is the Euclidean space, $\mathbb{R}^n$.

- De Sitter spacetime, $\mathbb{S}_1^{n+1}(c)$, $n \geq 2$ and $c > 0$, (see [109, Section 2.4]). This spacetime has positive constant sectional curvature c. The only global decompositions of $\mathbb{S}_1^{n+1}(c)$ as a GRW spacetime are obtained taking as fiber any usual round n-sphere of curvature $c_{_F} > 0$ and warping function $f(t) = \sqrt{c_{_F}/c}\,\cosh(\sqrt{c}\,t + b)$, for any $b \in \mathbb{R}$ [105, Cor. 2.1].

- Friedmann cosmological models, exact solutions to Einstein field equations (see, for instance, [83]). Particularly, they are Robertson-Walker spacetimes.

A GRW spacetime, M, is said to be *static* provided that its warping function is constant, i.e., M is, actually, a Lorentzian product. Note that under the assumption of completeness of F, a static GRW spacetime is complete if and only if its base is $\mathbb{R}$. On the contrary, if the warping function f is not locally constant (i.e., there is no open subinterval $J(\neq \emptyset)$ of I such that $f|_J$ is constant) then the GRW spacetime M is said to be *proper*. This assumption implies that there is no open subset of the GRW spacetime M, such that the sectional curvature in M of any plane tangent to a slice, $\{t_0\} \times F$, is equal to the sectional curvature of that plane in the inner geometry of the slice.

On any GRW spacetime, $M = I \times_f F$, there is a distinguished vector field $\xi := f(\pi_I)\,\partial_t$, which is timelike and, from the relationship between the Levi-Civita connections of M and those of the base and the fiber [83, Cor. 7.35], it satisfies

$$\overline{\nabla}_X \xi = f'(\pi_I)\,X, \tag{2.2}$$

for any $X \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the Levi-Civita connection of the metric (2.1).

Thus, ξ is conformal with $\mathcal{L}_\xi \overline{g} = 2 f'(\pi_I) \overline{g}$ and its metrically equivalent 1-form is closed. Then, any GRW spacetime has an infinitesimal symmetry provided by the above timelike conformal vector field, apart from those that the fiber could have. Let us recall that symmetries are a useful simplification so as to obtain exact solutions to Einstein equations. In some situations, they are assumed a priori, [36] and [37]. Besides, the use of affine and affine conformal vector fields gives raise to spacetimes that conform new exact solutions [41]. Remember that a spacetime which admits a timelike conformal vector field is said to be conformally stationary. Roughly speaking, a conformally stationary spacetime happens to be stationary when equipped with a conformal metric (see, for instance, [10]). When a GRW spacetime admits a non-trivial Killing vector field (in the sense of [105, p. 2]), then the warping function is determined [105, Thm. 4.1].

On the other hand, there exist several criteria to decide whether a given Lorentzian manifold is (locally or globally) a GRW spacetime (see [27], [104] and [31]).

According to Causality Theory, any GRW spacetime is stably causal [14, p. 64]. Moreover, it is globally hyperbolic if and only if its fiber is complete [14, Thm. 3.66]. In this case, any spacelike slice constitutes a Cauchy hypersurface. Note that, in a GRW spacetime, the integral curves of ∂_t are timelike geodesics and the coordinate t is, in point of fact, a universal time function.

In Riemannian Geometry, a complete Riemannian manifold is also geodesically connected. However, there exist complete spacetimes which are not geodesically connected, for instance, de Sitter spacetime [14]. However, the geodesic connectedness of a GRW spacetime can be assured in some cases. In fact, a GRW spacetime whose fiber is weakly convex (i.e., any two points can be joined by a minimizing geodesic) and whose warping function satisfies $\int_a^c f^{-1} = \int_c^b f^{-1} = \infty$, where $c \in (a, b) = I$, must be geodesically connected [104, Thm. 3.2]. Following the same reference, on a GRW spacetime, define a static trajectory, R_p, as $R_p = \{(t, p) : t \in I\}$, $p \in F$. Clearly, any static trajectory defines a timelike geodesic. Then, we can express the geodesic connectedness for a GRW spacetime in an equivalent form as follows. A

GRW spacetime whose fiber is weakly convex is geodesically connected provided that any point in the spacetime can be joined with any static trajectory by means of both future-directed and past-directed causal curves [104, Cor. 3.3]. On the other hand, let the future arrival time function $T_0 : (I \times F) \times F \to [0, 1)$ be

$$T_0((t_1, p), q) = \inf \big(t - t_1 : (t, q) \in J^+(t_1, p), t \in I \big) \ ,$$

where $J^+(t_1, p)$ denotes the causal future of (t_1, p), that is, the set of points which can be joined with (t_1, p) by means of a non-spacelike future oriented curve with the starting point (t_1, p). Analogously, $J^-(t_1, p)$ is the set of points liable to be joined by a non-spacelike past oriented curve starting at (t_1, p). The past arrival time function can be similarly considered. It is proved that every GRW spacetime with weakly convex fiber and finite future and past arrival functions must be geodesically connected [104, Cor. 3.5]. However, some extensions of this result to standard static spacetimes (i.e., warped products with fiber $(I, -dt^2)$ and base $(F, g_{_F})$) do not work (see the counterexamples in [104, p. 925]).

It is well known that in Lorentzian Geometry there is no analogous to the Hopf-Rinow theorem. On the other hand, the completeness of a Lorentzian manifold splits into spacelike, lightlike or timelike completeness, which are, in general, logically inequivalent. Whenever the fiber of a GRW spacetime is incomplete, so is the spacetime in the three causal senses [99]. On the contrary, let us assume now completeness on the fiber. In this case, the GRW spacetime is timelike complete towards the past (resp. towards the future) if and only if

$$\int_a^c \frac{f}{\sqrt{1+f^2}} = \infty \quad (\text{resp.} \int_c^b \frac{f}{\sqrt{1+f^2}} = \infty) \, ,$$

where $c \in (a, b) = I$, [104]. It will be timelike complete if it holds in both previous times senses. The lightlike completeness towards the past (resp. towards the future) is equivalent to

$$\int_a^c f = \infty \quad (\text{resp.} \int_c^b f = \infty) \, .$$

If it holds in both conditions, then the spacetime is lightlike complete. Finally, the

GRW spacetime is spacelike complete if and only if either f satisfies the previous assumptions or, when $\int_a^c f < \infty$ (resp. $\int_c^b f < \infty$), then f is unbounded in (a, c) (resp. (c, b)) [99]. Obviously, when $I \neq \mathbb{R}$, then the GRW spacetime is timelike incomplete. However, if $I = \mathbb{R}$, the fiber is complete and the warping function obeys $\inf f > 0$, and consequently, the GRW spacetime is complete in the three causal senses. On the other hand, the Ricci tensor can also determine the completeness or incompleteness of a GRW spacetime. Finally, if the Ricci tensor of a GRW spacetime satisfies $\overline{\mathrm{Ric}}(\partial_t, \partial_t) \geq 0$, then, either the GRW spacetime is static, or incomplete in all causal senses [104].

2.2 Energy curvature conditions

Coming from General Relativity, there exist some curvature assumptions with physical meaning. Let us recall that a Lorentzian manifold $(M, \overline{g})$ obeys the Timelike Convergence Condition (TCC) providing its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0,$$

for all timelike vector Z. It is commonly accepted that the TCC is the mathematical concept to express the physical idea that gravity, on average, attracts [102, Sec. 2.3]. A weaker energy condition is the Null Convergence Condition (NCC), which reads

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0\,,$$

for any null vector Z, i.e., $Z \neq 0$ satisfying $\overline{g}(Z, Z) = 0$. Doubtless, a continuity argument shows that TCC implies NCC. Furthermore, NCC holds in a spacetime whenever it satisfies the Einstein equation with a realistic stress-energy tensor [102, Ex. 4.3.7]. On the other hand, a spacetime obeys the ubiquitous energy condition if

$$\overline{\mathrm{Ric}}(Z, Z) > 0\,,$$

for all timelike vector Z. Doubtless, this last energy condition is stronger than the TCC and it shows the real presence of matter at any point in the spacetime.

From [83, Cor. 7.43], we obtain that the Ricci tensor of the GRW spacetime M is

$$\begin{aligned}\overline{\mathrm{Ric}}(X,Y) &= \mathrm{Ric}^F(X^F,Y^F) + \left(\frac{f''}{f} + \frac{(n-1)\,(f')^2}{f^2}\right)\overline{g}(X^F,Y^F) \\ &\quad -\frac{nf''}{f}\overline{g}(X,\partial_t)\,\overline{g}(Y,\partial_t),\end{aligned} \tag{2.3}$$

for any tangent vectors X, Y to M, where $X^F := X + \overline{g}(X,\partial_t)\,\partial_t$ and $Y^F := Y + \overline{g}(Y,\partial_t)\,\partial_t$ stand for the components of X and Y, respectively, on the fiber F, and Ric^F denotes the Ricci tensor of the fiber.

From the previous formula, it is clearly seen that a GRW spacetime obeys the NCC if the Ricci tensor of its fiber satisfies $\mathrm{Ric}^F \geq (n-1)f^2(\log f)''g_{_F}$. Moreover, it obeys the TCC (resp. the ubiquitous energy condition) if the NCC remains and $f'' \leq 0$ (resp. $f'' < 0$). Note that, in the static case, the NCC holds if and only if the fiber has non-negative Ricci curvature. Finally, in the case of a 2-dimensional fiber, we find that the NCC is satisfied if and only if the warping function obeys

$$\frac{K^F(\pi_{_F})}{f^2} - (\log f)'' \geq 0\,, \tag{2.4}$$

where K^F denotes the Gauss curvature of the fiber. Notice that, under the assumption on the Gauss curvature of the fiber $K^F \leq 0$, the inequality $(\log f)'' \leq 0$ is equivalent to the NCC.

2.3 Spacelike hypersurfaces

An immersion of an n-dimensional manifold $x : S \to M$ is said to be *spacelike* if the induced metric g on S is Riemannian. In this case, we will refer to S as a spacelike hypersurface. Since every GRW spacetime M is time-orientable, for each spacelike hypersurface S in M we can take $N \in \mathfrak{X}^{\perp}(S)$ as the only globally defined unit timelike vector field normal to S in the same time-orientation of the vector field $-\partial_t$ (i.e., such that $\overline{g}(N, -\partial_t) < 0$). From the wrong-way Cauchy-Schwarz inequality (see [83, Prop. 5.30], for instance), we have $\overline{g}\,(N, \partial_t) \geq 1$, and the equality holds at a point p in the hypersurface if and only if $N(p) = -\partial_t(p)$. In fact, $\overline{g}\,(N(p), \partial_t(p)) = \cosh\theta$, where θ is the *hyperbolic angle* between S and $-\partial_t$ at p.

The hyperbolic angle admits a physical interpretation. In a GRW spacetime M, the integral curves of the timelike unit vector field $-\partial_t$ are comoving observers and $-\partial_t(q)$, $q \in M$, is an instantaneous observer [102, p. 43]. Thus, at a point p in a spacelike hypersurface in M, there two distinguished instantaneous observers exist, $-\partial_t(p)$ and $N(p)$, where $N(p)$ is the normal vector at p in the same time-orientation than $-\partial_t$. Let us decompose orthogonally $N(p)$ as follows $N(p) = e(p)\ (-\partial_t(p)) + N^F(p)$. Then, the quantities $e(p) = \cosh\theta(p)$ and $v(p) = \frac{1}{\cosh\theta(p)} N^F(p)$ represent the energy and the velocity that $-\partial_t(p)$ measures for $N(p)$. Moreover, the relative speed function is $|v| = \tanh\theta$. Then, the boundedness of the hyperbolic angle assures that this relative speed function does not approach to the light speed in vacuum [102, pp. 45, 67].

For any spacelike hypersurface S in M, the restrictions to S from the natural projections of M onto I and F will be denoted by $\tau := \pi_{_I} \circ x$ and $\pi := \pi_{_F} \circ x$, respectively. Let $\partial_t^\top := \partial_t + \overline{g}(N, \partial_t)\, N$ be the tangential component of ∂_t along S. It is not difficult to obtain,

$$\nabla\tau = -\partial_t^\top\,, \tag{2.5}$$

where ∇ denotes here the gradient on S. From this equation, we get

$$g\,(\nabla\tau, \nabla\tau) = \sinh^2\theta\,. \tag{2.6}$$

The Levi-Civita connection of M is denote by $\overline{\nabla}$. From the Gauss and Weingarten formulas we have

$$\overline{\nabla}_X Y = \nabla_X Y - g\,(\mathrm{A}X, Y)\,N\,, \tag{2.7}$$

for all $X, Y \in \mathfrak{X}(S)$, where ∇ is the Levi-Civita connection on S and A is the shape operator associated to N,

$$\mathrm{A}X := -\overline{\nabla}_X N\,.$$

Let us recall that the *mean curvature function* relative to N is $H := -(1/n)\mathrm{tr}\,(\mathrm{A})$.[2] A spacelike hypersurface is said to be *maximal* when $H = 0$. From a variational point of view, maximal hypersurfaces appear as critical points of the volume functional for normal variations with compact support [21]. This terminology is derived from the fact that, in some cases such as the Lorentz-Minkowski spacetime, these hypersurfaces locally maximize the volume [77].

As we mentioned before, a spacelike hypersurface $x : S \to I \times_f F$ is a spacelike slice provided that τ is constant. It can be easily spotted that a spacelike hypersurface is a spacelike slice if and only if its hyperbolic angle identically vanishes. Physically, each spacelike slice represents the physical space at one instant of the universal time for the family of observers associated to $-\partial_t$. Notice that the spacelike slice $t = t_0$ is totally umbilical $\mathrm{A} = f'(t_0)/f(t_0)\,\mathbb{I}$, where $\mathbb{I}$ stands for the identity operator, and it has constant mean curvature $H = -f'(t_0)/f(t_0)$. Therefore, the spacelike slice $t = t_0$ is maximal provided that $f'(t_0) = 0$. Note that any maximal spacelike slice is totally geodesic.

A spacelike hypersurface $x : S \to M$ is said to be bounded from below (resp. from above) if there exists $t_0 \in I$ (resp. $t_1 \in I$) such that $\inf \tau \geq t_0$ (resp. $\sup \tau \leq t_1$). If both boundedness assumptions remain, we will say that S lies between two spacelike slices.

Example 2.3.1. [28] The assumption of boundedness of the hyperbolic angle is independent from lying between two spacelike slices. On the one hand, the non-horizontal spacelike hyperplanes in $\mathbb{L}^n$ have constant hyperbolic angle and are unbounded by

[2] The minus sign is taken in order to write that the mean curvature vector field satisfies $\vec{H} = H\,N$.

spacelike slices. On the other hand, let us consider the smooth function u on $\mathbb{R}^2$ given by $u(x,y) = \frac{2}{\pi}\sin y \arctan x$. It is easy to see that it is bounded and its gradient satisfies $|Du| < 1$. This implies that u defines a spacelike graph in $\mathbb{L}^3$ such that it lies between two spacelike slices (see Section 1.2 below). Furthermore, on the curve $\alpha(s) = (s,0)$, it can be computed that $\lim_{s\to\infty} |Du(\alpha(s))|^2 = 1$. Therefore, this spacelike graph has unbounded hyperbolic angle.

Given a spacelike hypersurface S, we can take tangential components from equation (2.2) to obtain

$$\nabla_Y \xi^\top + f(\tau)\,\overline{g}\,(N,\partial_t)\ \mathrm{A}Y = f'(\tau)\,Y\,, \tag{2.8}$$

where

$$\xi^\top := f(\tau)\,\partial_t^\top = \xi + \overline{g}\,(\xi,N)\,N \tag{2.9}$$

is the tangential component of ξ along S, $f(\tau) := f\circ\tau$ and $f'(\tau) := f'\circ\tau$.

From (2.8), we have

$$f(\tau)\,\mathrm{div}\left(\partial_t^\top\right) + \overline{g}\left(\nabla f(\tau),\partial_t^\top\right) + f(\tau)\,\overline{g}\,(N,\partial_t)\ \mathrm{tr}\,(\mathrm{A}) = nf'(\tau)\,,$$

where div denotes the divergence operator on S. Then, taking into account (2.5), we deduce

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\left\{n + |\nabla\tau|^2\right\} - nH\,\overline{g}\,(N,\partial_t)\ , \tag{2.10}$$

where Δ denotes the Laplacian operator on S. Therefore, we have the following equation for a maximal hypersurface,

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\left\{n + |\nabla\tau|^2\right\}\ . \tag{2.11}$$

A straightforward computation from (2.11) gives

$$\Delta f(\tau) = -n\frac{f'(\tau)^2}{f(\tau)} + f(\tau)\,(\log f)''\,(\tau)|\nabla\tau|^2\,. \tag{2.12}$$

Now, consider the function $\overline{g}\,(N,\xi)$ on a spacelike hypersurface S. It is easy to obtain, using (2.2),

$$\nabla\overline{g}\,(N,\xi) = -\mathrm{A}\xi^{\top}\,. \tag{2.13}$$

The curvature tensors of S and M are denoted by R and $\overline{\mathrm{R}}$, respectively. The Gauss equation is

$$g(\mathrm{R}(X,Y)U,V) = g(\overline{\mathrm{R}}(X,Y)U,V) - g(\mathrm{A}Y,U)\,g(\mathrm{A}X,V) + g(\mathrm{A}X,U)\,g(\mathrm{A}Y,V), \tag{2.14}$$

where $X,Y,U,V \in \mathfrak{X}(S)$. From the previous equation, it is deduced that

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) + g(\overline{\mathrm{R}}(N,X)Y,N) + n\,H\,g(\mathrm{A}X,Y) + g(\mathrm{A}^2X,Y), \tag{2.15}$$

where Ric denotes the Ricci tensor of S.

Now, the Codazzi equation, noticing that the normal bundle of the spacelike hypersurface is negative definite, is expressed as follows,

$$\overline{\mathrm{R}}(X,Y)N = -(\nabla_X\mathrm{A})Y + (\nabla_Y\mathrm{A})X\,, \tag{2.16}$$

for all $X,Y \in \mathfrak{X}(S)$, where $\overline{\mathrm{R}}$ denotes the Riemannian curvature tensor of M. From (2.13) and (2.16), we deduce that, for a maximal hypersurface,

$$\Delta\overline{g}\,(N,\xi) = \overline{\mathrm{Ric}}(N,\xi^{\top}) + \mathrm{tr}(\mathrm{A}^2)\,\overline{g}(N,\xi)\,. \tag{2.17}$$

2.4 The maximal hypersurface equation

Let (F, g_F) be an $n(\geq 2)$-dimensional Riemannian manifold and let $f : I \to \mathbb{R}^+$ be a smooth function. For each $u \in C^\infty(F)$ such that $u(F) \subseteq I$, we can consider its associated graph $\Sigma_u = \{(u(p), p) : p \in F\}$ in the GRW spacetime $M = I \times_f F$. The graph of u is endowed with the inherited metric, that is represented on F by

$$g_u = -du^2 + f(u)^2 \, g_F \,,$$

which is Riemannian (i.e., positive definite) if and only if u satisfies $|Du| < f(u)$ everywhere on F, where $|Du|^2 = g_F(Du, Du)$ and Du denotes the gradient of u in (F, g_F). The functions u and τ are naturally identified considering $\tau(u(p), p) = u(p)$, for any $p \in F$.

When Σ_u is spacelike, the unit normal vector field on Σ_u, N, that satisfies $\overline{g}(N, \partial_t) > 0$ is

$$N = -\frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}} \left(f(u)^2 \, \partial_t + Du \right) \,,$$

and its associated mean curvature function is

$$\begin{aligned} H(u) \;&=\; -\mathrm{div}\left(\frac{Du}{n \, f(u) \, \sqrt{f(u)^2 - |Du|^2}} \right) \\ &\quad -\frac{f'(u)}{n \, \sqrt{f(u)^2 - |Du|^2}} \left(n + \frac{|Du|^2}{f(u)^2} \right) . \end{aligned} \tag{2.18}$$

The differential equation $H(u) = 0$, under the constrain $|Du| < f(u)$ is known as the *maximal hypersurface equation* in M, and its solutions provide maximal graphs in M.

By a Calabi-Bernstein type problem, we intend to determine all the entire solutions (i.e., defined on all F) to the maximal hypersurface equation in some cases. In fact, we will focus here on Calabi-Bernstein results for the following elliptic PDE:

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2-\mid Du\mid^2}}\right)=-\frac{f'(u)}{\sqrt{f(u)^2-\mid Du\mid^2}}\Big(n+\frac{\mid Du\mid^2}{f(u)^2}\Big), \tag{E.1}$$

$$\mid Du\mid<\lambda f(u), \quad 0<\lambda<1. \tag{E.2}$$

Note that the constrain (E.2) assures that (E.1) is, actually, uniformly elliptic. Besides, this constrain is related to the boundedness of the hyperbolic angle of the graph Σ_u, namely, $\cosh\theta < 1/\sqrt{1-\lambda^2}$. Alternatively, we have $|Du|/f(u) = \tanh\theta$, where θ is the hyperbolic angle of the graph.

At this point, it is worth pointing out that an entire spacelike graph in a GRW spacetime may not be complete, as the following example shows,

Example 2.4.1. (See [68] and references therein) In the 2-dimensional Lorentz-Minkowski spacetime $\mathbb{L}^2 = (\mathbb{R}^2, -dt^2 + dx^2)$, we shall consider the graph of a smooth function u which satisfies

$$\begin{cases} u'(x) < 1 & \text{if } |x| < 1 \\ u'(x) = \sqrt{1-\exp(-|x|)} & \text{if } |x| \geq 1\,. \end{cases}$$

Undoubtly, it is a closed subset of $\mathbb{L}^2$, and its hyperbolic angle is not bounded. Since its length is finite, this entire spacelike graph is not complete.

Thus, completeness of spacelike graphs must be proved before applying uniqueness results to the parametric case. The following technical lemma provides sufficient conditions,

Lemma 2.4.2. *Let $M = I \times_f F$ be a GRW spacetime, whose fiber is a (non-compact)* *complete Riemannian manifold. Consider a function* $u \in C^\infty(M)$, *with* $\mathrm{Im}(u) \subseteq I$, *such that the entire graph* $\Sigma_u = \{(u(p), p) : p \in M\} \subset M$ *is spacelike. If the hyperbolic angle of* Σ_u *is bounded and* $\inf f(u) > 0$, *then the graph* (Σ_u, g) *is complete, or equivalently the Riemannian surface* (F, g_u) *is complete.*

Proof. The classical Schwartz inequality states

$$g(\nabla\tau, v)^2 \leq g(\nabla\tau, \nabla\tau)\, g(v, v), \quad \text{for all} \quad v \in T_q(\Sigma_u)$$

and therefore

$$g(v, v) \geq -g(\nabla\tau, \nabla\tau)\, g(v, v) + f(\tau)^2 g_F(d\pi_F(v), d\pi_F(v)),$$

which implies

$$g(v, v) \geq \frac{f(\tau)^2}{\cosh^2\theta}\, g_F(d\pi_F(v), d\pi_F(v)),$$

and $\sup(\cosh\theta) < \infty$. If $\mathcal{L}(\alpha)$ and $\mathcal{L}_u(\alpha)$ denote the lengths of a smooth curve α on F with respect to the metrics g_F and g_u, it is easily seen that

$$\mathcal{L}_u(\alpha) \geq B \inf(f(u)) \mathcal{L}_u(\alpha),$$

where $B = \frac{1}{\sup(\cosh\theta)}$. Therefore, since the Riemannian manifold (F, g_F) is complete and $\inf(f(u)) > 0$, then the metric g_u is also complete. $\square$

Finally, we may recall that (E.1), under (E.2), can be obtained considering critical points of the volume functional of a graph in a GRW spacetime,

$$\mathrm{vol}(\Sigma_u, K) = \int_K f(u)^{n-1} \sqrt{f(u)^2 - |\nabla u|^2}\, d\mu_{g_u}\,, \tag{2.19}$$

where $d\mu_{g_u}$ is the canonical measure associated to g_u and K is a compact subdomain of F. Some straightforward canonical computations drive us from (2.19) to (E.1) (see, for instance, [21]).

Chapter 3

Parabolicity of spacelike hypersurfaces

In order to introduce the notion of parabolicity, first we need to provide the definitions of superharmonic and subharmonic functions. A function u defined in the subdomain Ω of a Riemannian manifold M is said to be *superharmonic* providing it is continuous and if, for any relatively compact region $U \subset\subset \Omega$ and any harmonic function $v \in C^2(U) \cap C(\overline{U})$, $u \geq v$ on ∂U implies $u \geq v$ on U. If $u \in C^2(\Omega)$, then the superharmonicity of u is equivalent to

$$\Delta u \leq 0 \, ,$$

which comes from the maximum principle, where Δ denotes the Laplacian operator of M. A function u is said to be *subharmonic* provided that $-u$ is superharmonic. In all this thesis we will assume enough differentiability to take the last one as the definition of superharmonic function.

A complete (non-compact) Riemannian manifold is said to be *parabolic* if the only positive superharmonic functions are the constants (see, for instance, [62]).

The study of parabolicity can be approached from different points of view. From a physical one, for instance, it is necessary to understand the Brownian motion. This phenomenon describes the irregular motion of microscopic particles in a still liquid. It was already observed by the botanist R. Brown in 1828 with pollen grains in water. However, it was not completely explained until 1905, when A. Einstein described it as physical collisions of particles and molecules. This stochastic process was proved to satisfy a diffusion equation, and a certain diffusion coefficient was computed. This coefficient was experimentally confirmed by J. Perrin in 1908.

In a Riemannian manifold, parabolicity is equivalent to the recurrence of the Brownian motion (see, for instance, [51]). Roughly speaking, the Brownian motion is recurrent if any particle passes through any open set at an arbitrary large time.

On the other hand, parabolicity is also motivated from the heat equation. Let us recall that any function $p(t, x, y)$ on $(0, \infty) \times M \times M$ is a fundamental solution to the heat equation provided that

$$\frac{\partial p}{\partial t} - \frac{1}{2}\,\Delta p = 0\,,$$

in the (t, x) variables (taking fixed y) and, additionally, it satisfies the initial data

$$\lim_{t \to 0^+} p(t, \cdot, y) = \delta_y\,,$$

where δ_y is the delta function of Dirac [51]. The heat kernel is the smallest positive fundamental solution to the heat equation on a Riemannian manifold (M, g). J. Dodziuk proved that the heat kernel always exists (providing the Riemannian manifold is complete) (see [51] and references therein). It is linked to the Brownian motion as follows. Let $p(t, x, y)$ be the heat kernel of a Riemannian manifold (M, g). The probability of finding the particle in a measurable set $\Omega \subset M$ at the time t, when the motion started at the point x, is

$$\int_\Omega p(t, x, y) d\mu(y)\,,$$

where $d\mu(y)$ is the Riemannian measure of the slice $(0, +\infty) \times M \times \{y\}$.

In relation to Analysis on Riemannian manifolds, let us recall that the Green function $G(x, y)$, $x, y \in M$, $x \neq y$, is the smallest positive fundamental solution to the Laplace equation on M. When that function exists, it satisfies

$$\Delta G(\cdot, y) = -\delta_y \,.$$

Moreover, it can be related to the heat kernel of the heat equation. In fact, if the heat kernel $p(t, x, y)$ is known, then the Green function can be introduced by

$$G(x, y) := \frac{1}{2} \int_0^{\infty} p(t, x, y) \, dt \,.$$

The sign of the Green function decides about parabolicity. In fact, a Riemannian manifold that admits a positive Green function cannot be parabolic. But the converse also works: a complete (non-compact) Riemannian manifold is parabolic if and only if it does not admit a positive Green function [70].

In Potential Theory and Theory of Electricity, parabolicity plays also a central role [51]. Let Ω be an open set on a Riemannian manifold (M, g) and C be a compact set in Ω. The capacity $\mathrm{cap}(C, \Omega)$ is defined by

$$\mathrm{cap}(C, \Omega) = \inf_{\phi \in \mathcal{L}(C, \Omega)} \int_{\Omega} |\nabla \phi|^2 \, d\mu \,,$$

where $\mathcal{L}(C, \Omega)$ is the set of locally Lipschitz functions on M with compact support in $\overline{\Omega}$ and which satisfies $0 \leq \phi \leq 1$ and $\phi|_C = 1$. If $\Omega = M$, then an exhaustion sequence can be used to define the capacity. When Ω is relatively compact, the infimum in the previous definition is attained by the harmonic function which satisfies the following Dirichlet problem in $\Omega \setminus C$,

$$\begin{cases} \Delta u = 0 \,, \\ u|_{\partial \Omega} = 0 \,, \\ u|_{\partial C} = 1 \,. \end{cases}$$

The function u is known as the equilibrium potential. This terminology is inhereted from Electricity Theory. In fact, in physical terms, considering M is made of a conducting material, and that there exists a potential difference of 1 between $\partial\Omega$ and ∂C (as the previous problem shows), then $\mathrm{cap}(C, \Omega)$ is the conductivity of the piece of M between ∂C and $\partial\Omega$. Hence, $\mathrm{cap}(C, \Omega)^{-1}$ is the resistance of that piece (from Ohm's law), and the function u is the electrostatic potential. In this environment, the parabolicity of a Riemannian manifold is equivalent to the feature of having infinite resistance to the current flow (into infinity) [85].

A similar definition of capacity of annulus in a Riemannian manifold can be provided. If B_r and B_R $(0 < r < R)$ denote geodesic balls centered at the point p in a Riemannian manifold, we shall recall that

$$\frac{1}{\mu_{r,R}} := \int_{A_{r,R}} \mid \nabla\omega_{r,R} \mid^2 dV$$

is the capacity of the annulus $A_{r,R} := B_R \setminus \overline{B}_r$, being $\omega_{r,R}$ the harmonic measure of ∂B_R with respect to $A_{r,R}$, i.e., $\omega_{r,R}$ is the solution to the previous Dirichlet problem, when $\Omega = B_R$ and $C = \overline{B_r}$, (for more details, see also [69, Section 2]). A complete (non-compact) Riemannian manifold is parabolic if and only if $\frac{1}{\mu_{r,R}} \to 0$ as $R \to \infty$ [69].

The following technical fact will be useful for some of our purposes, [91, Lemma 2.2] (which is a reformulation of [8, Lemma 2.1]),

Lemma 3.0.3. *Let S be an $n(\geq 2)$-dimensional Riemannian manifold and let $v \in C^2(S)$ which satisfies $v\Delta v \geq 0$. Let B_R be a geodesic ball of radius R in S. For any r such that $0 < r < R$ we have*

$$\int_{B_r} |\nabla v|^2 \, dV \leq \frac{4 \, \mathrm{Sup}_{B_R} v^2}{\mu_{r,R}},$$

where B_r denotes the geodesic ball of radius r around p in S and $\frac{1}{\mu_{r,R}}$ is the capacity of the annulus $B_R \setminus \overline{B}_r$.

In a different environment, we may remark that there exist some other equivalent definitions for parabolicity which are closely related to the previous ones. In fact, in [51, Thm. 5.1], parabolicity of a Riemannian manifold is proved to be equivalent to several conditions, such as: the recurrence of the Brownian motion, the capacity of a Riemannian manifold, the non-finiteness of the Green function, etc.

Parabolicity of 2-dimensional Riemannian manifolds is very close to the classical parabolicity of Riemann surfaces. In point of fact, the Riemannian version of the classical uniformization theorem states that the universal Riemannian covering of a 2-dimensional Riemannian manifold is conformally equivalent to either the Euclidean plane, or the unit disk or a round sphere. The first case corresponds to the notion of parabolicity in the 2-dimensional case.

From a mathematical perspective, the study of parabolicity has been really fruitful. Its utility to clarify the behavior of the solutions to certain PDEs is well-known.

In relation to Riemannian Geometry, many authors paid attention to these kind of problems, for instance, in the search for conditions under which parabolicity can be stated on a Riemannian manifold, or Maximum Principles or Liouville properties (see, for instance, [51] and [62] for n-dimensional Riemannian manifolds and [4] and references therein for the case of surfaces). Parabolicity of n-dimensional Riemannian manifolds allows us to extend several classical results in the realm of analisys on $\mathbb{R}^n$ to a wider range of applicability. For instance, the classical Liouville theorem holds true on any parabolic Riemannian manifold.

In the 2-dimensional case, parabolicity is close to the behavior of the Gauss curvature. In this sense, an early result by Ahlfors and Blanc-Fiala-Huber, [57], stated,

> *A complete 2-dimensional Riemannian manifold with non-negative Gauss curvature must be parabolic.*

As clear consequence, it brings along the parabolicity of any elliptic paraboloid of type $z = a^2\,x^2 + b^2\,y^2$, $z, b \in \mathbb{R}$, $ab \neq 0$, in the Euclidean space $\mathbb{R}^3$.

There exist several results which point in the same direction. Namely, the following one, stated by R.E. Grenne and H. Wu [50], that generalizes the previous one,

> *If the Gauss curvature, K, of a complete Riemannian surface satisfies $K \geq \frac{-1}{r^2 \log r}$, for r, the distance to a fixed point, sufficiently large, then the surface must be parabolic.*

Furthermore, in the same reference there is also a close criterion which complements the previous one,

> *If the Gauss curvature of a simply-connected complete Riemannian surface satisfies $K \leq \frac{-(1+\epsilon)}{r^2 \log r}$, for some $\epsilon > 0$, and for r, the distance to a fixed point, sufficiently large, then the surface must be non-parabolic.*

The integrability of the Gauss curvature of a Riemannian surface can also determine parabolicity. Let us recall that a complete Riemannian surface (Σ, g_{Σ}) is said to have *finite total curvature* provided that the negative part of its Gauss curvature is integrable. More precisely, if K denotes the Gauss curvature of Σ, then Σ has finite total curvature when

$$\int_{\Sigma} \max\{0, -K\}\, d\mu_{\Sigma} < \infty\,. \tag{3.1}$$

It is well-known that, (see [69, Sec. 10]),

> *A complete Riemannian surface with finite total curvature must be parabolic.*

An easy consequence is that a complete Riemannian surface whose Gauss curvature is non-negative outside a compact set, must be parabolic.

By means of computing the total curvature, it can be found that, in Euclidean space $\mathbb{R}^3$, any hyperboloid $a^2\,x^2+b^2\,y^2-c^2\,z^2=d$, $a,b,c,d\in\mathbb{R}^+$, and any hyperbolic paraboloid $z=a\,x^2+b\,y^2$, $ab<0$, must be parabolic.

In the n-dimensional case, parabolicity has no clear relationship with sectional curvature. Indeed, the Euclidean space $\mathbb{R}^n$ is parabolic if and only if $n\leq 2$. Moreover, there exist parabolic Riemannian manifolds whose sectional curvature is not bounded from below, as Example 3.0.4.

Nevertheless, parabolicity can be related with other geometrical properties of a (complete, non-compact) Riemannian manifold (M,g), for example, the behavior of the volume growth of geodesic balls. Denote by $V(p,r)$ the volume of a geodesic ball of radius r centered at $p\in M$. We have, [52], [53], [61] and [108],

Let (M,g) be a complete Riemannian manifold. If, for some point $p\in M$, it holds

$$\int_1^\infty \frac{r\,dr}{V(p,r)}=\infty\,,$$

then M is parabolic.

Notice that the integral assumption in previous result holds if $V(p,r)\leq Cr^2$, $C\in\mathbb{R}^+$. Particularly, any complete Riemannian manifold with quadratic volume growth must be parabolic.

Closely related, the behavior of the area of the boundary of geodesic balls can also establish parabolicity. Denote by $S(p,r)$ the area of the boundary of a geodesic ball with radius r and center $p\in M$. Then, it can be asserted, [52], [53] and [74],

Let (M,g) be a complete Riemannian manifold. If, for some $p\in M$, it holds

$$\int_1^\infty \frac{dr}{S(p,r)}=\infty\,,$$

then M is parabolic.

Furthermore, the previous result is a characterization of parabolicity in spherically symmetric Riemannian manifolds (see [51] and references therein). In the 2-dimensional case, it was previously proved by Ahlfors [1]. It is worth mentioning that Ahlfors was one of the first authors to obtain faithful criteria for parabolicity.

We are now in position to provide an example of a parabolic Riemannian manifold whose curvature is not bounded from below.

Example 3.0.4. [94] Let us consider the hemisphere

$$\mathbb{S}^2_- := \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 = 1,\, z \leq 0\}$$

in $\mathbb{R}^3$ and the surface of revolution R given by

$$\mathbf{x}(z, \theta) = (f(z)\cos\theta,\, f(z)\sin\theta,\, z)\,,$$

where $\theta \in [0, 2\pi)$, $z \in [0, \infty)$, and $f(z)$ is a positive smooth function given by

$$f(z) = \begin{cases} h(z) & 0 \leq z \leq 1 \\ e^{-z^2} & z \geq 1 \end{cases}$$

where h is suitably chosen and such that $h(0) = 1$, $\frac{d^k}{dz^k} \mid_{z=0} h(z) = \frac{d^k}{dz^k} \mid_{z=0} \sqrt{1 - z^2}$ for all k. Construct a regular surface S in $\mathbb{R}^3$ by joining $\mathbb{S}^2_-$ and R according to $(x, y, 0) \equiv (\cos\theta,\, \sin\theta,\, 0)$, $\theta \in [0, 2\pi)$ unique such that $x = \cos\theta$ and $y = \sin\theta$. The surface S is complete. In order to obtain that, note that the induced metric g on the surface satisfies $g \geq dz^2$, for $z \geq 1$. Therefore, if γ is a divergent curve on S, a simple computation shows that its length is not bounded. On the other hand, the area of S is finite since the area of $\{(x, y, z) \in S \,:\, z \geq 1\}$ is finite. Hence, the surface S is parabolic. A straightforward computation shows that the Gauss curvature at $(x, y, z) \in S$, $z \geq 1$, only depends on z, $K(z)$, and $K(z) \to -\infty$ as z approaches ∞.

New parabolic Riemannian manifolds may be built from the previous criteria. For example, it is clearly seen that the Riemannian product of a compact Riemannian manifold and a parabolic one is also parabolic [62]. In particular, the product manifold of the real line $\mathbb{R}$ and any round sphere $\mathbb{S}^n$, $\mathbb{R} \times \mathbb{S}^n$, is parabolic. The same happens if $\mathbb{R}$ is replaced by $\mathbb{R}^2$.

Something important to decide whether a Riemannian manifold is parabolic or not is the notion of quasi-isometry. The next section is devoted to give a clear exposition about this topic.

3.1 Quasi-isometries

Let us recall that, given (P, g) and (P', g') two Riemannian manifolds, a diffeomorphism ϕ from P onto P' is called a *quasi-isometry* provided that there exists a constant $c \geq 1$ such that

$$c^{-1}|v|_g \;\leq\; |d\phi(v)|_{g'} \;\leq\; c\,|v|_g$$

for all $v \in T_pP$, $p \in P$ (for more details see [58] and [59]). If there exists a quasi-isometry from (P, g) onto (P', g') we will say that (P, g) is quasi-isometric to (P', g'). Obviously, to be quasi-isometric is an equivalence relation and isometric manifolds are also quasi-isometric. Two quasi-isometric Riemannian manifolds are simultaneously complete or incomplete. Even more, we have, [51, Cor. 5.3], [59] and [100],

> *Parabolicity is invariant under quasi-isometries. That is, two quasi-isometric Riemannian manifolds are simultaneously parabolic or non-parabolic.*

Remark 3.1.1. a) The universal Riemannian covering map $\mathbb{R}^3 \to \mathbb{S}^1 \times \mathbb{R}^2$ is a local isometry. Note that $\mathbb{S}^1 \times \mathbb{R}^2$ is parabolic, whereas $\mathbb{R}^3$ is not. Therefore, in the notion of quasi-isometry, the diffeomorphism cannot be relaxed to be a local diffeomorphism. However, observe that if a Riemannian covering $\widetilde{M}$ of a Riemannian manifold M is parabolic, then M is also parabolic. In order to see that, we shall consider that there exists a non-constant positive superharmonic function on M. Then, the composition of this function with the projection $\widetilde{M} \to M$ results in another function with the same properties on $\widetilde{M}$, which leads to a contradiction in the parabolity of $\widetilde{M}$. **b)** The previous result also holds if the exterior of some compact subset in M is quasi-isometric to the exterior of a compact subset in another Riemannian manifold M' [51, Cor. 5.3]. **c)** There exists a much weaker notion than quasi-isometry: the so-called rough isometry (roughly isometric manifolds are not homeomorphic, in general). Under this hypothesis, it is necessary to impose extra geometric assumptions (in terms of the Ricci curvature and the injectivity radius) to obtain that parabolicity is preserved by rough isometries [59].

Quasi-isometries can be used to construct new parabolic Riemannian manifolds from some others previously given. Let us consider a parabolic Riemannian manifold (M, g) and a bounded function $f \in C^\infty(M)$ such that $\inf(f) > 0$. Since the identity map is a quasi-isometry, the Riemannian manifold $(M, f^2\, g)$ is also parabolic. In the same direction, suppose that the Riemannian product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) is parabolic, and consider $h \in C^\infty(M_1)$ such that $\inf(h) > 0$ and $\sup(h) < \infty$. Then, the warped product $(M_1 \times M_2, g_1 + h^2\, g_2)$ is a new parabolic Riemannian manifold. This follows from

$$\begin{aligned}(g_1 + h^2\, g_2)(v, v) &\leq g_1(v_1, v_1) + \sup(h^2)\, g_2(v_2, v_2) \\ &\leq (1 + \sup(h^2))(g_1 + g_2)(v, v)\,,\end{aligned}$$

$$\begin{aligned}(g_1 + h^2\, g_2)(X, X) &\geq g_1(v_1, v_1) + \inf(h^2)\, g_2(v_2, v_2) \\ &\geq \min\{1, \inf(h^2)\}\, (g_1 + g_2)(v, v)\,,\end{aligned}$$

where $v = (v_1, v_2)$.

As an adaptation of the previous procedure, it can be proved other specific cases. Let $(N, g_{_N})$ be a compact Riemannian manifold, $(M, g_{_M})$ a parabolic Riemannian manifold and let $f \in C^\infty(N)$ satisfy $\min f > 0$. Then, the warped product Riemannian manifold $N \times_f M$ is a new parabolic Riemannian manifold. On the other hand, take a function h such that $\sup h < \infty$ and $\inf h > 0$ on a parabolic Riemannian manifold, (M, g_M), and consider a compact Riemannian manifold, (N, g_N). Then the warped product Riemannian manifold $M \times_h N$ is parabolic.

Remark 3.1.2. There exists a family of 3-dimensional quasi-spherical Riemannian manifolds which has a high interest for General Relativity [13]. Namely, on $\mathbb{R}^+ \times \mathbb{S}^2$ any metric

$$g = u^2\, dr^2 + (\beta_1\, dr + r\, d\theta)^2 + (\beta_2\, dr + r\, \sin\theta\, d\phi)^2\,, \tag{3.2}$$

where u, β_1 and β_2 are unspecified metric components. Let us select $u = u(r) \geq c > 0$, $c \in \mathbb{R}$, $\beta_1 = \beta_1(r)$ and $\beta_2 = 0$. It is easily proved that

$$\frac{1}{2}\, g(v, v) \leq g_0(v, v) \leq 2\, g(v, v)\,, \tag{3.3}$$

for any tangent vector v, where $g_0 = (u^2 + \beta_1^2)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$. Notice that g_0 is spherically symmetric [51, p. 146]. Therefore, the quasi-isometry obtained from (3.3) provides us a criterium to decide when a metric in (3.2), under our hypothesis, must be parabolic.

3.2 Parabolicity of a complete spacelike hypersurface

When dealing with spacelike surfaces in certain 3-dimensional spacetimes, the parabolicity is sometimes attained as an intermediate step prior to its classification. For instance, any maximal surface S in $\mathbb{L}^3$ has non-negative Gauss curvature. Therefore, if, in addition, the completeness of S is assumed, then S must be parabolic according to the Ahlfors Blanc-Fiala-Huber theorem. On the other hand, any maximal surface admits a positive harmonic function, which is constant if and only if it is a portion of a plane [89]. Therefore, we end in the parametric version of the classical

Calabi-Bernstein's theorem [43] and [63]. More generally, some authors have obtained parabolicity on spacelike surfaces in certain GRW spacetimes. For instance, in the study of complete maximal surfaces in a Lorentzian product $\mathbb{R} \times F$, where F has non-negative Gauss curvature, the parabolicity of the maximal surface can be attained [4]. Analogously, the same happens for GRW spacetimes under certain energy condition [25], [90].

Our approach will be valid not only for surfaces, but also in arbitrary dimensions. We will adopt a completely different approach than previous ones so as to get parabolicity of an $n(\geq 2)$-dimensional complete spacelike hypersurface in certain GRW spacetimes.

In the Riemannian case, some authors have studied the case of parabolicity in submanifolds. For instance, in a series of papers, [42], [75] and [76] (see also the survey [85]), parabolicity is achieved by means of comparing a given Riemannian manifold with a spherically symmetric Riemannian manifold, and using suitable estimates of sectional curvatures.

In our environment, neither the intrinsic nor the extrinsic curvature assumption is needed to hold parabolicity on a complete spacelike hypersurface. The fact that parabolicity is preserved by quasi-isometries will be a key factor in order to state that kind of results (see the previous section).

Let $x : S \longrightarrow M$ be a spacelike hypersurface in a GRW spacetime $(M, \overline{g})$ and assume the induced metric g on S is complete. Suppose also that there exists a positive constant c such that $f(\tau) \leq \sqrt{c}$. Note that c can be used providing it satisfies $c \geq 1$. Under these hypotheses, we obtain that the projection of S on the fiber F, $\pi := \pi_I \circ x$, is a covering map [9, Lemma 3.1].

Now, for any tangent vector v at a point $p \in S$ we have

$$\begin{aligned} g(v,v) &= -g(\nabla\tau, v)^2 + f(\tau)^2 g_F(d\pi(v), d\pi(v)) \\ &\leq f(\tau)^2 g_F(d\pi(v), d\pi(v)) \\ &\leq c\, g_F(d\pi(v), d\pi(v))\,. \end{aligned}$$

On the other hand, the classical Schwartz inequality results in

$$g(\nabla\tau, v)^2 \leq g(\nabla\tau, \nabla\tau)\, g(v,v)\,,$$

and therefore,

$$g(v,v) \geq -g(\nabla\tau, \nabla\tau)\, g(v,v) + f(\tau)^2 g_F(d\pi(v), d\pi(v)),$$

which implies

$$g(v,v) \;\geq\; \frac{f(\tau)^2}{\cosh^2\theta}\, g_F(d\pi(v), d\pi(v)).$$

Taking into account all these considerations, we obtain the following technical result,

Lemma 3.2.1. *Let S be a spacelike hypersurface in a GRW spacetime M, whose hyperbolic angle is bounded. If the warping function on S satisfies:*

i) $\sup f(\tau) < \infty$, *and*

ii) $\inf f(\tau) > 0$,

then, there is a constant $c \geq 1$ *such that*

$$c^{-1}\, g_{_F}(d\pi(v), d\pi(v)) \,\leq\, g(v,v) \,\leq\, c\, g_{_F}(d\pi(v), d\pi(v)), \tag{3.4}$$

for all $v \in T_pS$, $p \in S$.

Now, recall a standard topological fact (see [54], for instance),

Lemma 3.2.2. *Suppose given a covering map* $\rho : (\widetilde{E}, \widetilde{x}_0) \longrightarrow (E, x_0)$ *and a continuous map* $h : (W, y_0) \longrightarrow (E, x_0)$, *where* W *is a path connected and locally path connected topological space. Then, there exists a lift* $\widetilde{h} : (W, y_0) \longrightarrow (\widetilde{E}, \widetilde{x}_0)$ *of* h *if only if* $h_*(\pi_1(W, y_0)) \subset \rho_*(\pi_1(\widetilde{E}, \widetilde{x}_0))$.

Denote by $(\widetilde{F}, g_{_{\widetilde{F}}})$ the universal Riemannian covering of $(F, g_{_F})$. We have,

Proposition 3.2.3. *Suppose that a GRW spacetime* M *admits a simply connected parabolic spacelike hypersurface* S *such that* $\sup f(\tau) < \infty$, $\inf f(\tau) > 0$ *and whose hyperbolic angle is bounded. Then* $(\widetilde{F}, g_{_{\widetilde{F}}})$ *is also parabolic.*

Proof. From Lemma 3.2.2, we get a lift $\widetilde{\pi} : S \longrightarrow \widetilde{F}$ of the mapping $\pi : S \longrightarrow F$. Note that $\widetilde{\pi}$ is in fact a diffeomorphism, and, from Lemma 3.2.1, we see that $\widetilde{\pi}$ is a quasi-isometry, leading to the parabolicity of $(\widetilde{F}, g_{\widetilde{F}})$ and, particularly, that (F, g_F) is parabolic. □

The previous proposition allows us to introduce the following notion,

Definition 3.2.4. A generalized Robertson-Walker spacetime is said to be *spatially parabolic* provided that the universal Riemannian covering of its fiber is parabolic.

Note that the previous definition implies that the fiber of a spatially parabolic GRW spacetime is also parabolic.

We are now in position to state,

Theorem 3.2.5. *Let S be a complete spacelike hypersurface in a spatially parabolic GRW spacetime. If the hyperbolic angle of S is bounded and the warping function on S satisfies:*

i) $\sup f(\tau) < \infty$*, and*

ii) $\inf f(\tau) > 0$*,*

then, S is parabolic.

Proof. First of all, under these hypotheses, we have that π is a covering map. Moreover, inequalities (3.4) remain as stated in Lemma 3.2.1.

Let $(\widetilde{S}, \widetilde{g})$ be the Riemannian universal covering of (S, g) and denote by $\widetilde{\pi}_S : \widetilde{S} \longrightarrow S$ the corresponding Riemannian covering map. Now, let us consider the Riemannian universal covering $(\widetilde{F}, g_{\widetilde{F}})$ of the fiber $(F, g_{_F})$. Then, Lemma 3.2.2 can be claimed to get a lift $\widetilde{h} : \widetilde{S} \longrightarrow \widetilde{F}$ of the map $h := \pi \circ \widetilde{\pi}_S : \widetilde{S} \longrightarrow F$. It is easy to see that $\widetilde{h}$ is in fact a diffeomorphism from $\widetilde{S}$ onto $\widetilde{F}$. Note that (3.4) results now in

$$c^{-1}\, g_{\widetilde{F}}(d\widetilde{h}(\widetilde{v}), d\widetilde{h}(\widetilde{v})) \;\leq\; \widetilde{g}(\widetilde{v}, \widetilde{v}) \;\leq\; c\, g_{\widetilde{F}}(d\widetilde{h}(\widetilde{v}), d\widetilde{h}(\widetilde{v})), \tag{3.5}$$

for any $\widetilde{v} \in T_{\widetilde{p}}\widetilde{S}$, $\widetilde{p} \in \widetilde{S}$, which means that $\widetilde{h}$ is a quasi-isometry form $(\widetilde{S}, \widetilde{g})$ onto $(\widetilde{F}, g_{\widetilde{F}})$.

Finally, from the parabolicity of the universal Riemannian covering of S, we obtain that S is also parabolic. □

Remark 3.2.6. The hypotheses on f and on the hyperbolic angle in Theorem 3.2.5 automatically hold true if the spacelike hypersurface S is assumed to be compact (consequently, the fiber should be compact) (compare with [9, Prop. 3.2]). On the other hand, if S lies between two spacelike slices, then the assumptions on f are also satisfied.

As a direct consequence,

Corollary 3.2.7. *Let S be a complete spacelike hypersurface in a static spatially parabolic GRW spacetime. If S has bounded hyperbolic angle, then it must be parabolic.*

Remark 3.2.8. **a)** The boundedness on the hyperbolic angle cannot be dropped. In fact, the hyperbolic plane $\mathbb{H}^2$ in $\mathbb{L}^3$ has unbounded hyperbolic cosine, and $\mathbb{H}^2$ is not parabolic. **b)** On the other hand, if only parabolicity on the fiber is assumed (not the parabolicity of its universal Riemannian covering), then the conclusion is not attained as in Theorem 3.2.5 in general. Even the remainder of the hypotheses hold true. For instance, consider the static GRW spacetime with fiber $\mathbb{S}^1 \times \mathbb{R}^2$ and base $\mathbb{R}$ (see Remark 3.1.1). Clearly, $\mathbb{R}^3$ can be seen as a (complete maximal) spacelike hypersurface with constant hyperbolic angle.

However, a natural, physically realistic characteristic in a spacetime is the presence of an initial singularity of type Big-Bang, or a final one of type Big-Crunch. As previously agreed, $-\partial_t$ determines the future in M. Let C be a compact subset of a spacelike slice $t = t_0$. The family of observers given by the vector field $-\partial_t$ on C can bring C into the past or the future by means of geodesic transport. The assumption $\inf f > 0$ prevents the volume of C (as a function of the time function t) from decreasing arbitrarily. This fact does not seem to be consistent with the notion of an initial or final singularity. Therefore, in order to shape more physically realistic GRW spacetimes, it may be convenient avoid this hypothesis. Moreover, in this setting, the assumption $\sup f < \infty$ guarantees that the volume of C does not increases arbitrarily. On the other hand, from Chapter 1, we shall recall that if the fiber of a GRW spacetime is complete (this is the case whenever it is parabolic) and $\inf f > 0$, with $I = \mathbb{R}$, then the GRW spacetime is complete (Section 2.1). This fact avoids the existence of a singularity (singularities are normally regarded as incompleteness of timelike or lightlike inextendible geodesics).

In a larger class of ambient spacetimes, we will deal with parabolicity of a certain conformally related metric to the induced one on the spacelike hypersurface, as the following result shows,

Theorem 3.2.9. *Let S be a complete spacelike hypersurface in a spatially parabolic GRW spacetime. If $\sup f(\tau) < \infty$ and the hyperbolic angle of S is bounded, then S, endowed with the conformal metric $\hat{g} = \frac{1}{f(\tau)^2}\, g$, is parabolic.*

Proof. As in the proof of Theorem 3.2.5, we have that the universal Riemannian covering of S, $\widetilde{S}$, is diffeomorphic to the universal Riemannian covering of the fiber, $\widetilde{F}$. Now, on S, we shall consider the conformal metric $\hat{g} = \frac{1}{f(\tau)^2}\, g$. As in the proof of Lemma 3.2.1, the same reasoning leads us to

$$\hat{g}(v,v) \leq g_{_F}(d\pi(v), d\pi(v))\,, \tag{3.6}$$

and

$$\hat{g}(v,v) \geq \frac{1}{\cosh^2\theta}\, g_{_F}(d\pi(v), d\pi(v))\,. \tag{3.7}$$

From previous inequalities and the boundedness of the hyperbolic angle, a real number $c \geq 1$ can be used such that

$$\frac{1}{c}\, g_{_F}(d\pi(v), d\pi(v)) \leq \hat{g}(v,v) \leq c\, g_{_F}(d\pi(v), d\pi(v))\,,$$

to build a quasi-isometry from the universal Riemannian covering of $(S, \hat{g})$ onto the universal Riemannian covering of $(F, g_{_F})$. Now, the proof ends applying the invariance of parabolicity under quasi-isometries. $\square$

Remark 3.2.10. a) Under the assumptions in the previous theorem, g-completeness on S implies $\hat{g}$-completeness. The hypothesis of g-completeness may be actually weakened to the $\hat{g}$-completeness one, and the conclusion of the above theorem remains. **b)** On the other hand, we note that, in terms of the conformal metric, a similar result to Prop. 3.2.3, [95, Prop. 7] can be stated.

Remark 3.2.11. For the case $n = 2$, notice that superharmonic functions of (S, g) and of $(S, \hat{g})$ are the same. Owing to this fact, as (S, g) is complete, its parabolicity comes from the $\hat{g}$-parabolicity. Therefore, Theorem 3.2.9 properly extends Theorem 3.2.5 for spacelike surfaces.

According to the conformal factor in previous theorem, a different geometry is conferred to the spacelike hypersurface. To illustrate this, note that any non-complete Riemannian manifold admits a conformal metric which is complete [80]. On the other hand, in relation to parabolicity,

Example 3.2.12. Let us consider, on $\mathbb{R}^3$, a spherically symmetric Riemannian metric that, in polar coordinates, it is expressed as $dr^2 + \sigma^2(r) g_{\mathbb{S}^2}$. Take $\sigma^2 = 1/r$ outside the compact set $r \leq 1$. As it was pointed out in the previous section, there exists a characterization result for parabolicity in this kind of manifolds (see [51, Cor. 5.6]). Using that result, it can be proved that this Riemannian manifold is parabolic. Now, consider the conformal metric obtain from the conformal factor

$$\phi = \begin{cases} 1/r & \text{for } r \geq 1 \\ \xi(r) & \text{for } r < 1\,, \end{cases}$$

where $\xi(r) > 0$ is such that ϕ is smooth. It is easily seen that the conformal metric is complete. Furthermore, applying again the same characterization, a straightforward computation shows that this conformal metric is not parabolic. Hence, parabolicity of a Riemannian manifold does not hold true under a conformal change, in general.

We end this chapter with a sharp version of Theorem 3.2.9 for the case of an entire spacelike graph Σ_u in a spatially parabolic GRW spacetime (in fact, this is one of the motivations for the introduction of this new technique, specially so as to get the results in Section 5.2).

First, note that the boundedness assumption on $f(\tau)$ is only used in Theorem 3.2.9 in order to assert that π is a diffeomorphism (which is automatically true now). Thus, we obtain

Theorem 3.2.13. *Let* Σ_u *be an entire spacelike graph in a spatially parabolic GRW spacetime. If the hyperbolic angle of* Σ_u *is bounded, then the conformal metric* $\widehat{g} = \frac{1}{f(u)^2}\, g$ *on* Σ_u *is parabolic.*

Chapter 4

Uniqueness of maximal hypersurfaces

To start with, we must say that, throughout this chapter it will be frequently assumed that $(\log f)'' \leq 0$ (i.e., the convexity of $-\log f$) on the warping function f of a GRW spacetime M. From a mathematical point of view, this hypothesis has been widely used to obtain uniqueness results for certain type of spacelike hypersurfaces (see, for instance, [7], [25] and [27] and references therein). On the other hand, this inequality has a curvature meaning which is relative to the ambient spacetime, [26], [67] and [90]. In fact, as it has been pointed out in Chapter 2, if the GRW spacetime obeys the TCC, then $f'' \leq 0$, which is stronger than this condition.

Moreover, this inequality admits a reasonable physical interpretation. Note that the divergence of the vector field $\mathcal{T} = -\partial_t$ in M is actually obtained from (2.2), resulting in

$$\operatorname{div}(\mathcal{T}) = -n\,\frac{f'}{f}\,.$$

In consequence, if $f' < 0$, then $\operatorname{div}(\mathcal{T}) > 0$, and so, the comoving observers (i.e., the integral curves of $\mathcal{T}$) in M are, on average, spreading apart. Note that the assumption $(\log f)'' \leq 0$ implies that

$$\frac{d}{ds}(\mathrm{div}(\mathcal{T}) \circ \gamma)(s) \geq 0$$

for any observer $\gamma = \gamma(s)$ in the reference frame $\mathcal{T}$. In addition, suppose that there exists a proper time s_0 of γ such that $\mathrm{div}(\mathcal{T})(\gamma(s_0)) > 0$. Then $\mathrm{div}(\mathcal{T})(\gamma(s)) > 0$ for any proper time $s > s_0$ of γ. Therefore, the assumption $(\log f)'' \leq 0$ favors the fact that M models an expanding universe.

Throughout this Chapter the main procedure to achieve uniqueness results will become Theorem 3.2.5. We refer to [94] in this part of the thesis.

4.1 The parametric case

Theorem 4.1.1. *Let S be a complete maximal hypersurface of a proper spatially parabolic GRW spacetime whose warping function f satisfies* $(\log f)''(t) \leq 0$*. If the hyperbolic angle of S is bounded,* $\sup f(\tau) < \infty$ *and* $\inf f(\tau) > 0$*, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Proof. From Theorem 3.2.5, the spacelike hypersurface S must be parabolic. On the other hand, equation (2.12) implies that $f(\tau)$ is superharmonic. Therefore, $f(\tau)$ must be constant, which leads, due to the properness condition, to $t = t_0$, where $f'(t_0) = 0$. □

Remark 4.1.2. In the previous result and in the following, the assumption $(\log f)'' \leq 0$ may be relaxed to $(\log f)''(\tau) \leq 0$, since it is only required on S.

The family of non-horizontal spacelike hyperplanes in $\mathbb{L}^3$ (note that it is a spatially parabolic GRW spacetime) shows that the properness assumption is necessary. Hence, for more general GRW spacetimes, an extra condition may be assumed to get the same conclusion as in the previous theorem.

On the other hand, boundedness assumptions on $f(\tau)$ remain if we assume that S lies between two spacelike slices (see Remark 3.2.6). Hence, we have,

Theorem 4.1.3. *Let S be a complete maximal hypersurface of a spatially parabolic GRW spacetime whose warping function satisfies* $(\log f)''(t) \leq 0$. *If the hyperbolic angle of S is bounded and S lies between two spacelike slices, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Proof. As in the proof of the previous result, we arrive to $f(\tau)$ constant. Using (2.12) we obtain $f'(\tau) = 0$. This implies that τ is harmonic taking into consideration (2.11), concluding the proof. □

Remark 4.1.4. In previous results, no assumption on the curvature of M is needed. Other techniques to obtain uniqueness results of maximal hypersurfaces make use of some curvature assumption on the GRW spacetime, i.e., on the curvature of the fiber and on the warping function. Among the most relevant techniques, the Omori-Yau generalized maximum principle [82], [111] needs the Ricci curvature of the (complete) spacelike hypersurface be bounded from below [79], [111]. In Chapter 7 we will see that this property comes from an analogous one on the fiber of the GRW spacetime. Then, another kind of uniqueness results will be achieved. Nevertheless, there exist Riemannian manifolds which are parabolic and whose Ricci curvature is not bounded from below (see Example 3.0.4, [94]).

Remark 4.1.5. In order to illustrate the range of application of theorems 4.1.1 and 4.1.3, note that F may be taken as $\mathbb{S}^{n-1} \times \mathbb{R}$, $n \geq 2$, endowed with the product metric $g + ds^2$, where g is an arbitrary Riemannian metric on $\mathbb{S}^{n-1}$. The Riemannian manifold F is parabolic (see Chapter 3). Among the GRW spacetimes constructed with this fiber and whose warping function satisfies $(\log f)'' \leq 0$, there is a relevant subfamily. Let us assume that the metric g on $\mathbb{S}^{n-1}$ has non-negative Ricci curvature. In this case, $g + ds^2$ also has non-negative Ricci curvature. Taking also into account that $(\log f)'' \leq 0$, we obtain that the Ricci tensor of the spacetime M is positive semi-definite on lightlike vectors. Hence, M satisfies the Null Convergence Condition. On the other hand, it should be also highlighted that class of spacetimes has reasonable spacelike symmetries apart from the timelike conformal one which any GRW spacetime possesses. In fact, ∂_s is a Killing vector field. This makes any member of this family of spacetimes a suitable candidate to represent an exact solution to the Einstein equation.

Now we will focus on the special case f = constant, i.e., the GRW spacetime is static. We will change the assumption of boundedness between two spacelike slices in Theorem 4.1.3 to the non-negativity of the Ricci curvature of the fiber. However, in order to succeed, first we need a lower estimate of the Laplacian of $\sinh^2\theta$. Although we will provide a direct proof of this estimate, it can be also deduced from [68, Formulas 16, 17].

Lemma 4.1.6. *Let S be a maximal hypersurface in a static GRW spacetime $I \times F$. The hyperbolic angle of S satisfies the following differential inequality*

$$\frac{1}{2}\Delta \sinh^2\theta \geq \cosh^2\theta \,\mathrm{trace}(A^2) + \cosh^2\theta \,\mathrm{Ric}^F(N^F, N^F)\,, \tag{4.1}$$

where $N^F := N + \overline{g}(N, \partial_t)\,\partial_t$ is the projection of N on the fiber and Ric^F *denotes the Ricci tensor of F.*

Proof. We begin from (2.6) which suggests applying the Bochner-Lichnerowicz formula, which holds true on any Riemannian manifold [30, p. 83] or [68, Sec. 3], to the function τ. Taking into account (2.11), the Bochner-Lichnerowicz formula results in

$$\frac{1}{2}\Delta \sinh^2\theta = |\,\mathrm{Hess}(\tau)\,|^2 + \mathrm{Ric}(\nabla\tau, \nabla\tau)\,, \tag{4.2}$$

where the first term of the right hand is the square length of the Hessian of τ and Ric is the Ricci tensor of S. Now, from (2.8), we obtain

$$|\,\mathrm{Hess}(\tau)\,|^2 = \cosh^2\theta \,\mathrm{trace}(A^2)\,. \tag{4.3}$$

On the other hand, from (2.15) and [83, Props. 7.42, 7.43] we obtain

$$\mathrm{Ric}(\nabla\tau, \nabla\tau) = \cosh^2\theta\, \mathrm{Ric}^F(N^F, N^F) + g(A^2\nabla\tau, \nabla\tau)\,. \tag{4.4}$$

Finally, (4.1) directly follows (4.2) using (4.3) and (4.4). □

Theorem 4.1.7. *Let S be a complete maximal hypersurface in a static spatially parabolic GRW spacetime $I \times F$. If the Ricci curvature of the fiber is non-negative and the hyperbolic angle of S is bounded, then S must be totally geodesic.*

Proof. First, we obtain that S is parabolic according to Theorem 3.2.5. On the other hand, previous Lemma asserts that $\sinh^2\theta$ is subharmonic and we may assume it is bounded. Therefore, this function is constant. Finally, using again (4.1) we get $A \equiv 0$, which concludes the proof. □

As a concrete application of the previous result we have,

Corollary 4.1.8. *The only complete maximal hypersurfaces with bounded hyperbolic angle in the static GRW spacetime $M = \mathbb{R} \times F$, where $F = \mathbb{S}^{2m} \times \mathbb{R}$ is endowed with a product Riemannian metric $g + ds^2$, where g is a Riemannian metric on $\mathbb{S}^{2m}$ with non-negative Ricci curvature, are the hypersurfaces*

$$\{\,(t, x, s) \in \mathbb{R} \times \mathbb{S}^{2m} \times \mathbb{R} \;:\; a_1 t + a_2 s + a_3 = 0\,\},$$

where $a_1, a_2, a_3 \in \mathbb{R}$ satisfy $-a_1^2 + a_2^2 < 0$.

Proof. Taking into account Theorem 4.1.7, we only have to find all the complete totally geodesic spacelike hypersurfaces in M. Since the unit normal vector field N is parallel on S, its projection N^F onto the fiber is also parallel. On the other hand, the projection of N^F on $(\mathbb{S}^{2m}, g)$ must be parallel and, therefore, with constant g-norm, which has to be zero. Otherwise, $\mathbb{S}^{2m}$ will support a nowhere zero vector field. Consequently, N^F must be collinear with $\partial/\partial s$, which ends the proof. □

Remark 4.1.9. An analogous result to the previous one can be stated if the fiber there is replaced by the parabolic Riemannian manifold $F = \mathbb{S}^{2m} \times \mathbb{R}^2$. More generally, if the fiber consists of a Riemannian product of a parabolic Riemannian manifold and $\mathbb{S}^{2m}$ under the assumptions of previous result, then we arrive to $\pi_{_{\mathbb{S}^{2m}}} \circ x =$ constant, that is, S has no dependence on the coordinates of $\mathbb{S}^{2m}$.

Remark 4.1.10. It should be recalled that S. Nishikawa [79] proved that a complete maximal hypersurface in a locally symmetric Lorentzian manifold M whose Ricci tensor satisfies $\overline{\mathrm{Ric}}(X, X) \geq 0$ for any timelike tangent vector X to M must be totally geodesic. Note that the spacetime M in Corollary 4.1.8 is not locally symmetric, generally speaking.

4.2 Calabi-Bernstein type problems

Here, we are going to present several uniqueness results for entire solutions of certain PDEs as an application of the previous section.

As in Section 2.3, let us consider a complete Riemannian manifold $(F, g_{_F})$ and a positive smooth function $f : I \to \mathbb{R}$. Any $u \in C^\infty(F)$ such that $\operatorname{Im}\tau \subseteq I$ defines a graph Σ_u in the GRW spacetime $I \times_f F$. Denote by g_u the inherited metric on the graph, which is represented on F as $g_u = -du^2 + f(u)^2 g_{_F}$. Let us assume u satisfies (E.2). Recall that this assumption implies that Σ_u is spacelike and has bounded hyperbolic angle.

Clearly enough, the map π is always a diffeomorphism in the non-parametric case (see Section 3.2). Furthermore, now Theorem 3.2.5 reads that every entire spacelike graph with bounded hyperbolic angle and such that $\inf f(u) > 0$, must be parabolic providing the fiber is so.

Note that if the graph is defined by a bounded function u, then the assumption

$\inf f(u) > 0$ is trivially satisfied (see also Remark 3.2.6).

From Theorem 4.1.1 we obtain,

Theorem 4.2.1. *Let $f : I \longrightarrow \mathbb{R}$ be a non-locally constant positive smooth function. Assume f satisfies $(\log f)'' \leq 0$, $\sup f < \infty$ and $\inf f > 0$. The only entire solutions to the equation* (E) *on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

Now, as a direct consequence of Theorem 4.1.3 we obtain,

Theorem 4.2.2. *Let $f : I \longrightarrow \mathbb{R}$ be a positive smooth function. Assume $(\log f)'' \leq 0$. The only bounded entire solutions to the equation* (E) *on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

And from Corollary 4.1.8,

Theorem 4.2.3. *The only entire solutions to the equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1- \mid Du \mid^2}}\right) = 0$$

$$\mid Du \mid < \lambda, \quad 0 < \lambda < 1,$$

on $(\mathbb{S}^{2m} \times \mathbb{R}, g + ds^2)$ where g is a Riemannian metric on $\mathbb{S}^{2m}$ with non-negative Ricci curvature, are the functions $u(x,s) = as + b$, with $a, b \in \mathbb{R}$, $a^2 < 1$. □

To conclude, tet us remark that any function u on $\mathbb{S}^{2m}$ may be naturally extended to a function $\widetilde{u}$ on $\mathbb{S}^{2m} \times \mathbb{R}$. A natural consequence of Theorem 4.2.3 is that if such a $\widetilde{u}$ is a solution to the previous equation, then u must be a constant.

Chapter 5

Uniqueness of maximal hypersurfaces: another more general approach

Our starting point now will be Theorem 3.2.9, instead of Theorem 3.2.5 (the key procedure throughout the previous Chapter). The advantage of this new approach is that the assumption $\inf f > 0$ can be dropped. Hence, GRW spacetimes are allowed to have some kind of singularities, extending the class of GRW spacetimes that we considered before (for a more detailed discussion see Section 3.2).

However, rather than dealing with the geometry of the induced metric, we will use a certain pointwise conformal metric related to the induced one. The methods of proof here are inspired from the developed ones in the previous Chapter. Again, most uniqueness results have suppositions neither on the curvature of the GRW spacetime nor on the maximal hypersurface. We refer to [95] for the results of this part of the thesis.

5.1 The parametric case

In order to being able to use Theorem 3.2.9, we may rewrite equations (2.11), (2.12) and (2.17) in terms of the Laplacian $\hat{\Delta}$ of the conformal metric, $\hat{g} = \frac{1}{f(\tau)^2}\, g$, respectively as follows,

$$\hat{\Delta}\tau = -f(\tau)f'(\tau)\left\{n + \frac{n-1}{f(\tau)^2}|\hat{\nabla}\tau|^2_{\hat{g}}\right\}, \tag{5.1}$$

$$\hat{\Delta}f(\tau) = -nf'(\tau)^2 f(\tau) + \left[(\log f)''(\tau) - (n-2)\,\frac{f'(\tau)^2}{f(\tau)^2}\right] f(\tau)\,|\hat{\nabla}\tau|^2_{\hat{g}}, \tag{5.2}$$

$$\begin{aligned} \hat{\Delta}\overline{g}(\xi, N) \;&=\; f(\tau)^3\left[\mathrm{tr}(\mathrm{A}^2)\,\cosh\theta + (n-2)\,\frac{f'(\tau)}{f(\tau)}\,g(\mathrm{A}\nabla\tau, \nabla\tau)\right] + \\ & f(\tau)^2\,\overline{\mathrm{Ric}}(\xi^\top, N)\,. \end{aligned} \tag{5.3}$$

We are now in a position to state,

Theorem 5.1.1. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime which is not a complete static one. Suppose that* $\sup f(\tau) < \infty$ *and there exists a positive constant σ such that* $(\log f)''(\tau) \leq (n-2+\sigma\, f(\tau))\,(\log f)'(\tau)^2$. *If the hyperbolic angle of S is bounded, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Proof. Let us consider the function $v = -\exp(-\sigma f(\tau))/\sigma + C$ on S, where $C \in \mathbb{R}$ is taken in order to ensure that $v > 0$. From (5.2), the $\hat{g}$-Laplacian of this function is

$$\begin{aligned} \hat{\Delta}v \;&=\; e^{-\sigma f(\tau)}\, f(\tau)\left((\log f)''(\tau) - (n-2+\sigma f(\tau))\,\frac{f'(\tau)^2}{f(\tau)^2}\right)\,|\hat{\nabla}\tau|^2_{\hat{g}} - \\ & n\, e^{-\sigma f(\tau)}\, f(\tau)\, f'(\tau)^2\,, \end{aligned} \tag{5.4}$$

leading to $\hat{\Delta} v \leq 0$.

On the other hand, $(S, \hat{g})$ is parabolic according to Theorem 3.2.9. Therefore, v must be constant and, consequently, $f(\tau)$ is also constant. Regarding again (5.4), we obtain that $f'(\tau) = 0$, and as a consequence of (5.1), τ is $\hat{g}$-harmonic. If f is constant, then I must be a proper interval of $\mathbb{R}$. Hence, τ is bounded from below or from above, which means that τ is constant. In case f is not constant, there are $t_1, t_2 \in I$, $t_1 \neq t_2$, with $f(t_1) \neq f(t_2)$. It cannot hold that t_1 and t_2 belong simultaneously to $\tau(S)$. On the contrary, there exist $p_1, p_2 \in S$ such that $\tau(p_1) = t_1$ and $\tau(p_2) = t_2$, and therefore $f(\tau(p_1)) = f(t_1) \neq f(t_2) f(\tau(p_2))$, which actually contradicts the fact that $f(\tau)$ is constant. Therefore, we deduce that t_1 or t_2 does not lie in the interval $\tau(S)$. In particular, τ is bounded from below or from above, reaching again the conclusion that τ is constant. $\square$

The previous result should be compared with Theorem 4.1.1. Here, the properness assumption on the warping function is weakened so as to fulfill the requirement that the spacetime is not simultaneously static and complete.

Remark 5.1.2. The condition required on the warping function is weaker than $(\log f)''(\tau) \leq 0$, which is satisfied whenever the GRW spacetime obeys the Timelike Convergent Condition (see Section 2.2). Uniqueness results of maximal hypersurfaces in GRW spacetimes obeying some energy condition will be developed in Section 5.1.2.

Remark 5.1.3. In [55, p. 58], it is discussed that no physical inconvenience appears providing the spacetime is analytic. For a GRW spacetime, this argument supports that the warping function f is assumed to be analytic. In this case, the inequality involving the derivatives of f in theorem above implies that it can attain a maximum value at the most. Indeed, it is easily proved that, under that inequality, f cannot have any minimum value (see Theorem 5.1.14 and Remark 5.1.15). On the other hand, suppose that f has an inflection point at t_c. That condition is tantamount to the existence of a positive upper bound of $(\log f)''/(\log f)'^2$ in a neighborhood of t_c. Therefore, it is satisfied $(\log f)''(t_c) = 0$, and an iterative process gives that

$(\log f)^{i)}(t_c) = 0$, for all derivation order i. Since $\log f$ is analytical, f must be constant. Consequently, if f is analytic (and not constant), there must exist, at the most, a maximal spacelike slice. Furthermore, notice that if such spacelike slice exists, then supposition $\sup f < \infty$ can be dropped.

Remark 5.1.4. If the warping function does not obey the assumptions required, some counterexamples can be found. Namely, consider the spatially parabolic GRW spacetime $(-\sqrt{3}, \sqrt{3}) \times_f \mathbb{R}^2$, where $\mathbb{R}^2$ is the Euclidean plane and the warping function is given by

$$f(t) = 2\sqrt{2 + \sqrt{-t^4 + 2t^2 + 3}}\,,$$

and the entire spacelike graph $S = \{(u(x,y), x, y) : (x,y) \in \mathbb{R}^2, u(x,y) = \tanh x\}$. It is easily seen that there is a constant λ, $0 < \lambda < 1$, such that $|Du|_{\mathbb{R}^2} < \lambda\, f(u)$, where D denotes the gradient on $\mathbb{R}^2$ and $|Du|^2_{\mathbb{R}^2} = g_{\mathbb{R}^2}(Du, Du)$. Consequently, the graph is complete and its hyperbolic angle is bounded. On the other hand, u is a solution to the maximal hypersurface equation (E). Note that $\inf f(u) > 0$, $\sup f(u) < \infty$ and the GRW spacetime is proper.

In order to include the complete static GRW spacetimes to achieve a uniqueness result, an extra hypothesis must be added (compare with Theorem 4.1.3).

Theorem 5.1.5. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime, M. Suppose that* $\sup f(\tau) < \infty$ *and there exists a positive constant σ such that* $(\log f)''(\tau) \leq (n-2+\sigma f)\ (\log f)'(\tau)^2$. *Assume also that the base I of M is a proper interval or that S is bounded from below or from above. If the hyperbolic angle of S is bounded, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Proof. The same reasoning as the one in the proof of Theorem 5.1.1 shows that $f'(\tau) = 0$. From the supposition on I or its boundedness counterpart hypothesis on S, we can consider $C \in \mathbb{R}$ such that $\tau + C$ is signed. On the other hand, we have that τ is $\hat{g}$-harmonic from (5.1). Making use of Theorem 3.2.9, we get to know that $\hat{g}$ is parabolic, putting an end to the proof of the theorem. □

Remark 5.1.6. The boundedness of the hypersurface cannot be dropped as the non-horizontal spacelike planes in $\mathbb{L}^3$ show.

Remark 5.1.7. If only the parabolicity of the fiber is required, the conclusion in Theorem 5.1.5 cannot be reached (hence, our Definition 3.2.4 is accurate for our purposes). In order to support this assertion, consider a compact 2-dimensional Riemannian manifold of Gauss curvature -1, T. Let $\phi : \mathbb{H}^2 \to T$ be the universal Riemannian covering map, where $\mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$, endowed with $g_{_{\mathbb{H}^2}} = (dx_1^2 + dx_2^2)/x_2^2$, is the hyperbolic plane of Gauss curvature -1. Now, we will use the entire complete maximal graph defined in [3, Ex. 5.2]. Explicitly, this graph is given by the function

$$w(x_1, x_2) = i\,\frac{2}{\sqrt{5}}\,F\left(\arcsin\left(i\,\frac{x_1}{x_2}\right), \frac{1}{\sqrt{5}}\right) + c\,,$$

where c is a real constant, i stands for the imaginary unit and $F(\xi, k)$ stands for the elliptic integral of the first kind with elliptic modulus k and Jacobi amplitude ξ. In the same paper it is shown that this graph has bounded hyperbolic angle. To prove that the graph lies between two spacelike slices, first we should make a change of parameters, taking $x_1 := r\,\cos\theta$ and $x_2 := r\,\sin\theta$, where $r > 0$ and $\theta \in (0, \pi)$. In these new variables, the following derivatives can be computed,

$$\frac{\partial w}{\partial r} = 0 \qquad \frac{\partial w}{\partial \theta} = \frac{1}{\sqrt{1 + 4\sin^2\theta}}\,.$$

Hence, what follows is the boundedness of the function. From this surface we will build a complete maximal hypersurface in a GRW spacetime with parabolic fiber such that its hyperbolic angle is bounded and lies between two spacelike slices. Concretely, we take the GRW spacetime $M' = \mathbb{R} \times_1 (T \times \mathbb{R})$. Note that the fiber endowed with the product Riemannian metric is parabolic. In order to provide the cited example, consider the maximal graph S in the GRW spacetime $\mathbb{R} \times_1 (\mathbb{H}^2 \times \mathbb{R})$ of the function $\overline{w}$ given by $\overline{w}(x_1, x_2, s) = w(x_1, x_2)$. The maximal hypersurface $x : S \to \mathbb{R} \times_1 (\mathbb{H}^2 \times \mathbb{R})$, defined by the graph of $\overline{w}$, naturally results in the desired maximal hypersurface $\overline{x} : S \to \mathbb{R} \times_1 (T \times \mathbb{R})$, using the Riemannian covering map ϕ.

The same conclusion as the one in Theorem 5.1.5 is attained without needing the boundedness on S whenever the warping function satisfies a stronger assumption,

Theorem 5.1.8. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime. Suppose that* $\sup f(\tau) < \infty$ *and there exists a positive constant σ such that* $(\log f)''(\tau) < (n-2+\sigma f)\,(\log f)'(\tau)^2$. *If the hyperbolic angle of S is bounded, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Proof. From (5.4), it follows that $\hat{\nabla}\tau$ vanishes. □

Remark 5.1.9. Under the hypothesis of the previous result, if such a maximal slice $t = t_0$ exists, then the warping function $f(t)$ must attain a local maximum value at t_0. On the other hand, note that when $(\log f)''(\tau) < 0$, or in particular when $f''(\tau) < 0$, the inequality in the result above is automatically fulfilled. Moreover, note that, in any case, the GRW spacetime is proper.

5.1.1 Monotonicity of the warping function

Now, we will focus on the case when the warping function is monotone. From a physical point of view, it can be interpreted as the fact that the expansion or contraction of the universe never ceases.

Theorem 5.1.10. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime whose warping function is monotone. Suppose that* $\sup f(\tau) < \infty$ *and $f \in \mathrm{L}^1(I)$. If the hyperbolic angle of S is bounded, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Proof. Consider a primitive $\mathcal{F}$ of f and define on S

$$\mathcal{F}(\tau(p)) = \int_{s_0}^{\tau(p)} f(s)\, ds\,,$$

for all $p \in S$, where $s_0 = \inf(\tau)$. Clearly, $\mathcal{F}$ is bounded and so $\mathcal{F}(\tau)$, and its Laplacian on $(S, \hat{g})$ is found to be

$$\hat{\Delta}\mathcal{F}(\tau) = -f'(\tau)\left\{n\, f(\tau)^2 + (n-2)|\hat{\nabla}\tau|^2_{\hat{g}}\right\}\,. \tag{5.5}$$

Making use of the parabolicity of $(S, \hat{g})$, obtained from Theorem 3.2.9, we conclude that $\mathcal{F}(\tau)$ must be constant, leading to $\hat{\nabla}\tau = 0$, which ends the proof. □

Remark 5.1.11. Note, here and from now on, that the integral condition is only needed on the hypersurface, then it is enough to assume $f \in \mathrm{L}^1(\mathrm{Im}\,\tau)$.

On the other hand, as the proof shows, if the warping function is non-decreasing (resp. non-increasing), then the hypothesis of integrability can be weaken to $\int_{\inf(I)}^{a} f(s)\, ds < \infty$ (resp. $\int_{a}^{\sup(I)} f(s)\, ds < \infty$), for some $a \in I$. From our chosen time-orientation, when $\sup(I) \in \mathbb{R}$ (resp. $\inf(I) \in \mathbb{R}$) the GRW spacetime M could shape an expanding (resp. contracting) universe from an initial (resp. to a final) singularity.

As a direct consequence of the theorem above,

Corollary 5.1.12. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime such that $f'(\tau)$ is signed. If S has bounded hyperbolic angle and lies between two spacelike slices, then S must be a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

The same characterization as in Theorem 5.1.10 is proved if the integrability assumption on f is replaced by some boundedness of S.

Theorem 5.1.13. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime. Suppose that f is non-decreasing (resp. non-increasing) and the hypersurface is bounded from below or* $\inf(I) > -\infty$ *(resp. from above or* $\sup(I) < \infty$ *). If* $\sup f(\tau) < \infty$ *and the hyperbolic angle of S is bounded, then S must be a spacelike slice $t = t_0$ with $f'(t_0) = 0$.*

Proof. Consider f to be non-decreasing. Up to an additive term, we may take $\tau \geq 0$. From equation (5.1), we find that τ is $\hat{g}$-superharmonic. Hence, the $\hat{g}$-parabolicity of S makes τ constant. The other case proceeds analogously. □

In the previous results, the conclusion $f'(t_0) = 0$ does not mean that t_0 is a local extreme, in general, because of the monotonicity of f. From now on, f is not supposed to be monotone. Instead of that, we will assume that f has no local minimum, which leads us to the following nice geometrical interpretation. The volume element of any spacelike slice $t = t_0$ satisfies $dV_{\{t_0\}\times F} = f(t_0)^n \, dV_F$. If f attains a local minimum at $t = t_0$, then the spacelike slice $t = t_1$ with t_1 close to t_0 have a bigger volume element. This behavior is far from the geometrical notion of the area's maximization of a maximal hypersurface, which occurs in some spacetimes though [77].

Theorem 5.1.14. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime. Suppose that* $\sup f(\tau) < \infty$*, f has no local minimum and* $f \in \mathrm{L}^1(\mathrm{Im}\,\tau)$*. If the hyperbolic angle of S is bounded, then S must be a spacelike slice $t = t_0$ with $f'(t_0) = 0$.*

Proof. Under these assumptions, there are two cases for f: f has no local maximum, or f has just one. Using Theorem 5.1.10, the only case we have to deal with is that when f has a local maximum. Let $c \in I$ be such that point. Define the function

$$\mathcal{G}(z) = \int_c^z f(s) \, ds \, .$$

Note that $\mathcal{G} \geq 0$ when $z \geq c$, and $\mathcal{G} \leq 0$ when $z \leq c$. Now, using (5.5), we obtain,

$$\mathcal{G}(\tau)\,\hat{\Delta}\mathcal{G}(\tau) = -\mathcal{G}(\tau)\,f'(\tau)\,f(\tau)^2\left\{n + (n-2)\,|\hat{\nabla}\tau|^2_{\hat{g}}\right\} \geq 0\,.$$

Lemma 3.0.3 can be stated, and, from the boundedness of $\mathcal{G}$ and $\hat{g}$-parabolicity, we conclude that $\mathcal{G}$ is constant, putting and end to the proof. $\square$

Remark 5.1.15. Let us consider $I = (a,b)$, $a < b$, $a, b \in \mathbb{R}$ and $f \in C^\infty(I)$, $f > 0$, such that $\lim_{t\to a} f(t) = \lim_{t\to b} f(t) = 0$. It is clear that f attains a global maximum at $t_0 \in (a,b)$. Suppose that t_0 is the only critical point of f. Note that, from the previous result, in any spatially parabolic GRW spacetime with warping function f, the unique complete maximal hypersurface with bounded hyperbolic is the spacelike slice $t = t_0$.

Notice that the example in Remark 5.1.3 shows that if the supposition that the existence of a local minimum point is dropped, then the same conclusion is not attained.

5.1.2 GRW spacetimes obeying certain energy condition

We will begin the uniqueness results of this subsection coming back to Theorem 3.2.5, and assuming some energy condition on the GRW spacetime. First, the following extension of Theorem 4.1.7 is proved,

Theorem 5.1.16. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime, M. Suppose that M obeys NCC,* $\sup f(\tau) < \infty$ *and* $\inf f(\tau) > 0$. *If S has bounded hyperbolic angle, then S must be totally geodesic.*

Proof. At each $p \in S$, we can state

$$N_p = -\cosh\theta\,\partial_t(p) + \sinh\theta\, y$$

where $y \in T_{x(p)}M$, $y \perp \partial_t(p)$ and $\overline{g}(y,y) = 1$. Using this formula, we rewrite (2.9) as follows

$$\xi_p^\top = -f(\tau)\,\sinh^2\theta\,\partial_t(p) + f(\tau)\cosh\theta\sinh\theta\,y\,.$$

Using [83, Cor. 7.43], it is satisfied

$$\overline{\mathrm{Ric}}\left(N_p, \xi_p^\top\right) = \frac{\cosh\theta\sinh^2\theta}{f(\tau)}\left[\mathrm{Ric}^F(y,y) - (n-1)f(\tau)^2(\log f)''(\tau)\right], \tag{5.6}$$

and, therefore $\overline{\mathrm{Ric}}\left(N, \xi^\top\right) \geq 0$ on S.

Now, having in mind (2.17), we get that $\overline{g}\,(N,\xi)$ is subharmonic. On the other hand, S is parabolic as a consequence of Theorem 3.2.5. Therefore, the bounded function $\overline{g}\,(N,\xi)$ must be constant. Using again (2.17), we conclude that the shape operator of S vanishes identically. □

Note that the curvature hypothesis in the theorem above is automatically satisfied if M obeys the TCC. On the other hand, it is also fulfilled under the hypothesis in the following consequence,

Corollary 5.1.17. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime. Suppose that the Ricci curvature of the fiber is non-negative,* $\sup f(\tau) < \infty$, $\inf f(\tau) > 0$ *and* $(\log f)''(\tau) \leq 0$. *If S has bounded hyperbolic angle, then S must be totally geodesic.*

In the particular case that f is constant, S is a spacelike slice or the universal Riemannian covering $\widetilde{F}$ of F isometrically splits as $\mathbb{R}\times\widetilde{F}'$, in which case the lift $\widetilde{x}$ of the immersion $x : S \to M$ satisfies

$$\widetilde{x}(S) = \left\{(t,s,p) \in \mathbb{L}^2 \times \widetilde{F}' : (\cosh\theta)\ t + (\sinh\theta)\ s = c\right\},$$

where θ and c are constants.

Proof. If f is constant and $I = \mathbb{R}$, then the projection N^F of N on the fiber is parallel. If $N^F = 0$, there is nothing to prove. Otherwise, N^F is a non-zero parallel vector field on F, which can be lifted to $\widetilde{F}$ keeping this property. Now, the de Rham decomposition theorem [38] can be claimed to obtain the splitting $\widetilde{F} = \mathbb{R} \times \widetilde{F}'$. A straightforward computation ends the proof. □

Remark 5.1.18. a) The case in which f is constant and I is a proper interval of $\mathbb{R}$ can be treated claiming Theorem 5.1.5 to get that the maximal hypersurface S is a spacelike slice. On the other hand, if f is constant and $I = \mathbb{R}$, the existence of a point in the fiber at which the Ricci curvature is positive implies that S must be a spacelike slice. **b)** Note that the last assertion in the previous result gives a wide generalization of Corollary 4.1.8.

Theorem 5.1.19. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime M. Suppose that M obeys the TCC,* $\sup f(\tau) < \infty$, $\inf f(\tau) > 0$, *and there exists a point in F where* $\mathrm{Ric}^F > (n-1)f^2 (\log f)'' g_F$ *holds. If S has bounded hyperbolic angle, then S is a spacelike slice $t = t_0$, with $f'(t_0) = 0$.*

Proof. From Theorem 5.1.16, we obtain that $\overline{g}(N, \xi) = f(\tau) \cosh\theta$ is constant and $\overline{\mathrm{Ric}}\left(N, \xi^\top\right) = 0$. According to (5.6), there exists a point where the hyperbolic angle of S vanishes. As the GRW spacetime obeys TCC, Theorem 4.1.1 can be recalled to get that $f(\tau)$ is constant. Therefore, the hyperbolic angle identically vanishes. □

Remark 5.1.20. It should be noted that Theorems 5.1.16 and 5.1.19 hold true when $n = 2$ without the supposition $\inf f(\tau) > 0$ (see Remark 3.2.11).

We finish this section with another characterization of totally geodesic spacelike hypersurfaces. The energy assumption on the ambient spacetime will remain, and a new one will be taken for granted.

Given a spacelike hypersurface $x : S \to M$ in a GRW spacetime M consider, at any $p \in S$, the greatest eigenvalue in absolute value of the shape operator A,

$||\mathrm{A}||_\infty(p)$ and also $|f'(\tau(p))|/f(\tau(p))$, i.e., the same quantity associated with the spacelike slice which contains $x(p)$. It is natural to wonder under what hypothesis involving $||\mathrm{A}||_\infty(p)$ and $|f'(\tau(p))|/f(\tau(p))$ we can deduce that S is totally geodesic in M. In this direction we prove,

Theorem 5.1.21. *Let S be a complete maximal hypersurface in a spatially parabolic GRW spacetime, M. Suppose that M obeys the NCC and* $\sup f(\tau) < \infty$. *Assume*

$$||\mathrm{A}||_\infty(p) \geq (n-2)\,\sinh\theta(p)\,|f'(\tau(p))|/f(\tau(p))\,, \tag{5.7}$$

for all $p \in S$. If S has bounded hyperbolic angle, then S is totally geodesic.

Proof. First, $(S, \hat{g})$ is parabolic from Theorem 3.2.9.

On the other hand, at any point $p \in S$, it is not difficult to see that

$$\mathrm{tr}\left(\mathrm{A}^2\right)\,\cosh\theta \geq ||\mathrm{A}||_\infty^2\,\sinh\theta\,.$$

From these suppositions, this inequality can be expressed as

$$\mathrm{tr}\left(\mathrm{A}^2\right)\cosh\theta \geq (n-2)\sinh^2\theta\,||\mathrm{A}||_\infty\,\frac{|f'(\tau)|}{f(\tau)}\,,$$

which clearly implies,

$$\mathrm{tr}\left(\mathrm{A}^2\right)\cosh\theta + (n-2)\frac{f'(\tau)}{f(\tau)}\,g(\mathrm{A}\nabla\tau, \nabla\tau) \geq 0\,.$$

Making use again of (5.6), the NCC and the previous inequality gives that $\overline{g}(N,\xi)$ is $\hat{g}$-subharmonic, according to (5.3). The proof ends resulting that $\mathrm{A}=0$ holds as a consequence of equation (5.3). □

Remark 5.1.22. The assumption (5.7) on the theorem above holds true providing the following inequality is satisfied,

$$\mathrm{tr}(\mathrm{A}^2)(p) \geq \frac{(n-2)^2 \sinh^2\theta(p)}{n^2}\,\mathrm{tr}(\mathrm{A}^2_{t(p)})\,,$$

where $\mathrm{A}_{t(p)}$ denotes the shape operator of the spacelike slice $t = t(p)$. On the other hand, note that in the case of 2-dimensional fiber, the hypothesis (5.7) is always satisfied (compare with [25]). For the case $n > 2$, we may rewrite the previous inequality as follows,

$$\frac{n^2}{(n-2)^2}\,|\sigma|^2 + |\sigma_{_{t(p)}}|^2 \leq -n\cosh^2\theta(p)\frac{f'(\tau(p))^2}{f(\tau(p))^2}$$

at any $p \in S$, where $|\sigma|^2 := -\mathrm{tr}(\mathrm{A}^2)$ and $|\sigma_{_{t(p)}}|^2 := -\mathrm{tr}(\mathrm{A}^2_{t(p)})$ are the squared lengths of the second fundamental forms of S at p and of the spacelike slice $t = t(p)$ (note that the normal bundle of a spacelike hypersurface is negative definite). Notice that the right member in this inequality is the squared length of the second fundamental form of the spacelike slice $t = t(p)$ projected onto the N_p-direction of $T_{x(p)}M$, which may be denoted by $|\sigma^N_{_{t(p)}}|^2$. Finally, a sufficient condition for the inequality (5.7) to be held is

$$|\sigma|^2(p) + |\sigma_{_{t(p)}}|^2 \leq |\sigma^N_{_{t(p)}}|^2\,.$$

Corollary 5.1.23. *Under the same assumptions as those in Theorem* 5.1.21*, for* $n \geq 3$*,*

i) If M is a Lorentzian product $\mathbb{R}\times F$ with the universal Riemannian covering $\widetilde{F}$ of F satisfying $\widetilde{F} = \mathbb{R}\times F'$, then

$$x(S) = \{(t, s, x) \in \mathbb{R}\times\mathbb{R}\times F' : t = a\,s + b\}\,,$$

for some $a, b \in \mathbb{R}$ such that $|a| < 1$.

ii) Otherwise, S is a spacelike slice.

Proof. (i) Let S be a totally geodesic complete hypersurface. Some straightforward computations show that N^F is a parallel vector field on F. If $|N^F| \neq 0$, then we have a parallel vector field globally defined on F. Hence, the De Rham decomposition theorem can be used to end this case.

(ii) From (5.7), we have $\sinh\theta\, f'(\tau) = 0$. We state that $S \subset \{t : f'(t) = 0\} \times F$. In fact, let us assume that there exists a point $p \in S$ such that $f'(\pi_I(x(p))) \neq 0$. We may find a neighbourhood of p such that $f'(x(S|_U)) \neq 0$. Hence, $\sinh^2\theta|_U = 0$. This implies that S must be a portion of a spacelike slice. Then, we find a contradiction with the maximality of S. Now, notice that the geodesics of S (also geodesics of M) may be written as $(a\,s + b, \sigma(s))$, with $a, b \in \mathbb{R}$, $|a| < 1$ and $\sigma(s)$ a geodesic of F. From the completeness of S, we have that if S is not a spacelike slice, then M is complete and static. Contradiction. □

5.2 Calabi-Bernstein type problems

The key technical factor to be used throughout this section is Theorem 3.2.13.

The first non-parametric uniqueness result for the maximal hypersurface equation (E) in Section 2.4 follows from Theorem 5.1.1,

Theorem 5.2.1. *Let $f : I \to \mathbb{R}^+$ be a non-constant smooth function. Assume f satisfies $(\log f)'' \leq (n-2+\sigma f)\ (\log f)'^2$, for some $\sigma \in \mathbb{R}^+$. The only entire solutions to equation* (E) *on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

Remark 5.2.2. Consider the family of spatially parabolic GRW spacetimes with fiber $\mathbb{R}^2$ and warping function $f_m : (0, \infty) \to \mathbb{R}$, $f_m(t) = t^m$, $m \in \mathbb{R}$, $m \neq 0$. Note that the classical 3-dimensional Einstein-de Sitter spacetime is included in this family taking $m = 2/3$. The previous result may be interpreted as the fact that none of the GRW spacetimes of this family admits any maximal entire graph with bounded hyperbolic angle.

As a direct consequence of Theorem 5.1.5 we obtain,

Theorem 5.2.3. *Let $f : I \to \mathbb{R}^+$, $I \neq \mathbb{R}$ (resp. $I = \mathbb{R}$) be a smooth function. Assume $(\log f)'' \leq (n-2+\sigma\, f)\ (\log f)'^2$, for some $\sigma \in \mathbb{R}^+$. The only entire solutions (resp. bounded from below or from above entire solutions) to equation (E) on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

Under a sharper assumption on f, from Theorem 5.1.8, we obtain,

Theorem 5.2.4. *Let $f : I \to \mathbb{R}^+$ be a positive smooth function. Assume f satisfies $(\log f)'' < (n-2+\sigma f)\ (\log f)'^2$, for some $\sigma \in \mathbb{R}^+$. The only entire solutions to equation (E) on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

Another result, drawn from Theorem 5.1.10,

Theorem 5.2.5. *Let $f : I \to \mathbb{R}^+$ be a positive monotone smooth function which satisfies $f \in \mathrm{L}^1(I)$. The only entire solutions to equation (E) on a parabolic Riemannian manifold F are the constants $u = c$, with $f'(c) = 0$.*

The integrability supposition on f may be replaced by the boundedness of the solution, as in Theorem 5.1.13,

Theorem 5.2.6. *Let $f : \mathbb{R} \to \mathbb{R}^+$ be a non-increasing (resp. non-decreasing) smooth function. The only bounded from below (resp. from above) entire solutions to equation (E) on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

Theorem 5.2.7. *Let f be a a positive smooth function defined on a proper interval I,* $\sup I < \infty$ *(resp.* $\inf I > -\infty$ *) non-increasing (resp. non-decreasing). The only entire solutions to equation* (E) *on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

Requiring some global behavior on f, from Theorem 5.1.14 we arrive to

Theorem 5.2.8. *Let $f : I \to \mathbb{R}^+$ be a smooth function with no local minimum points and such that $f \in \mathrm{L}^1(I)$. The only entire solutions to equation* (E) *on a parabolic Riemannian manifold F are the constant functions $u = c$, with $f'(c) = 0$.*

We conclude this Chapter with the non-parametric versions of Corollary 5.1.17 and Theorem 5.1.19,

Theorem 5.2.9. *The only entire solutions to*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1- \mid Du \mid^2}}\right) = 0$$

$$\mid Du \mid < \lambda, \quad 0 < \lambda < 1,$$

on a parabolic Riemannian manifold F with non-negative Ricci curvature are:

i) If the universal Riemannian covering $\tilde{F}$ of F is reducible and satisfies $\tilde{F} = \mathbb{R} \times \tilde{F}'$, $g_{\tilde{F}} = ds^2 + g_{\tilde{F}'}$, then $u = \tanh\theta\, s + d$, where θ and d are constants.

ii) Otherwise, u is the constant functions.

Theorem 5.2.10. *Let $f : I \to \mathbb{R}^+$ be a bounded smooth function such that* $\inf(f) >$

0. *Let* $(F, g_{_F})$ *be an* $n(\geq 2)$*-dimensional parabolic Riemannian manifold obeying* $\mathrm{Ric}^F > (n-1)f^2(\log f)''\, g_{_F}$*. Then, the only entire solutions to equation* (E)*, which are bounded from below or from above, are the constant functions* $u = c$*, with* $f'(c) = 0$.

Chapter 6

Spacelike surfaces with controlled mean curvature function

In this chapter we shall consider a special subclass of spatially parabolic GRW spacetimes. We focus on the case in which the spacetime has 2-dimensional fiber with finite total curvature. From a geometrical point of view, this kind of GRW spacetimes leads to a wide generalization of those Robertson-Walker spacetimes with fiber the Euclidean plane $\mathbb{R}^2$, altough their fibers keep properties of $\mathbb{R}^2$: its area growth is quadratic at most and they are parabolic Riemannian manifolds (see Chapter 3).

The results obtained here generalize those firstly studied in [92], where the fiber of the GRW spacetimes is $\mathbb{R}^2$, and [93], where the fiber is assumed to be compact.

We will follow here a different approach than the one in the previous chapters. First, we consider the non-parametric case and, later, the parametric one as an application.

Let $f : I \longrightarrow \mathbb{R}$ be a positive smooth function on an open interval $I = (a, b)$, $-\infty \leq a < b \leq \infty$, in the real line $\mathbb{R}$ and let Ω be an open domain of a Riemannian surface (F, g_F). For each $u \in C^\infty(\Omega)$ such that $|Du| < f(u)$, where $|Du|$ stands for the length of the gradient of u, we take into account the smooth function $H(u)$, which was given in (2.18) with $n = 2$. Recall that, geometrically, $H(u)$ is the mean curvature function of the spacelike graph given by u in the GRW spacetime $I \times_f F$.

We will consider here the following non-linear differential inequality

$$H(u)^2 \leq \frac{f'(u)^2}{f(u)^2} \qquad (\mathbf{I}.1)$$

$$| \, Du \, |< \lambda f(u), \quad 0 < \lambda < 1. \qquad (\mathbf{I}.2)$$

on a (non-compact) complete manifold.

The geometric meaning of (**I**.2) is that the graph of u is spacelike and its hyperbolic angle is bounded. From a physical point of view, it assures that the relative speed between the observers $N(p)$ and $-\partial_t(p)$ does not approach to the speed of light in vacuum (see Chapter 1). On the other hand, (**I**.1) means that at the point of the graph of u corresponding to p_0, $p_0 \in F$, the absolute value of the mean curvature is at the same quantity for the graph of constant function $u = u_0$ at most, where $u_0 = u(p_0)$.

Note that instead of assuming that H is constant, as in the case of the constant mean curvature spacelike graph equation, we only assume a natural comparison inequality between $H(u)$ and $f'(u)/f(u)$. From now on, inequality (**I**) will mean inequality (**I**.1) with the additional assumption (**I**.2).

Notice that a maximal surface trivially satisfies (**I**). Therefore, our results here can be restricted to that case. Even more, under reasonable assumptions on the ambient spacetime, a complete spacelike surface with constant mean curvature which lies between two spacelike slices must satisfy the inequality, [24] and [90].

It is clear that the constant functions are entire solutions to inequality (**I**). Our main aim in this chapter is to state several converses, that is, finding conditions under which the only entire solutions to (**I**) are the constant functions (the contents in this chapter were previously published in [96]).

6.1 The Gauss curvature of a spacelike surface

Let (S, g) be a spacelike surface in a GRW spacetime $(M, \overline{g})$. According to (2.15), using a local orthonormal frame field $\{E_1, E_2, E_3\}$ on M which is adapted to S (that is, $\{E_1, E_2\}$ are tangent to S and $E_3 = N$), we obtain

$$2K = \sum_{i=1}^{2} \mathrm{Ric}(E_i, E_i) = \sum_{i=1}^{2} \overline{\mathrm{Ric}}(E_i, E_i) + \sum_{i=1}^{2} g(\overline{\mathrm{R}}(N, E_i)E_i, N) - 4H^2 + \mathrm{trace}(\mathrm{A}^2),$$

where K is the Gauss curvature of S. Using [83, Prob. 7.13], the previous equation can be rewritten as follows

$$K = \frac{f'(\tau)^2}{f(\tau)^2} + \left\{ \frac{K^F(\pi_{_F})}{f(\tau)^2} - (\log f)''(\tau) \right\} \mid \nabla\tau \mid^2 + \frac{K^F(\pi_{_F})}{f(\tau)^2} - 2H^2 + \frac{1}{2}\mathrm{trace}(\mathrm{A}^2). \quad (6.1)$$

Note that, when the GRW spacetime obeys the NCC, then the inequality $H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}$ implies, taking into account (6.1), that $K \geq \frac{K^F(\pi_F)}{f(\tau)^2}$, i.e., at each $p \in S$, $K(p)$ is at least the Gauss curvature of the slice $t = \tau(p)$ at the point $\pi_{_F}(p)$.

6.2 The restriction of the warping function on a spacelike surface

Now, following the Gauss formula, taking into account $\xi^\top = f(t)\,\partial_t^\top$ and (2.5), the Laplacian of τ satisfies

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\Big\{2 + |\nabla t|^2\Big\} - 2H\,g(N,\partial_t)\,. \tag{6.2}$$

A direct computation obtained from (2.5) and (6.2) results in

$$\Delta f(\tau) = -2\,\frac{f'(\tau)^2}{f(\tau)} + f(\tau)(\log f)''(\tau)|\nabla\tau|^2 - 2f'(\tau)H\,g(N,\partial_t)\,, \tag{6.3}$$

for any spacelike surface of the GRW spacetime M.

Let us consider the function $\log f(\tau)$ defined on the surface S. The Laplacian of this function satisfies

$$\frac{\Delta f(\tau)}{f(\tau)} = \Delta\log f(\tau) + \frac{f'(\tau)^2}{f(\tau)^2}|\nabla\tau|^2.$$

On the other hand, from (6.3) we have

$$\frac{\Delta f(\tau)}{f(\tau)} = -\Big(\frac{f'(\tau)}{f(\tau)} + H\,g(N,\partial_t)\Big)^2 + \Big(H^2 - \frac{f'(\tau)^2}{f(\tau)^2}\Big)g(N,\partial_t)^2 + \frac{f''(\tau)}{f(\tau)}\,|\nabla\tau|^2 \tag{6.4}$$

and as consequence

$$\begin{aligned}\Delta\log f(\tau) &= -\Big(\frac{f'(\tau)}{f(\tau)} + H\,g(N,\partial_t)\Big)^2 + \Big(H^2 - \frac{f'(\tau)^2}{f(\tau)^2}\Big)g(N,\partial_t)^2 \\ &\quad + (\log f)''(\tau)|\nabla\tau|^2.\end{aligned} \tag{6.5}$$

Notice that if $(\log f)''(\tau) \leq 0$ and $H^2 \leq \frac{f'(\tau)2}{f(\tau)2}$, then $\Delta \log f(\tau) \leq 0$. Particularly, if M obeys the TCC (or the NCC with $K^F \geq 0$), then for any spacelike surface S in M such that $H^2 \leq \frac{f'(\tau)2}{f(\tau)2}$, we obtain $\Delta \log f(\tau) \leq 0$.

6.3 Uniqueness results for entire solutions to inequality (I)

First, let us recall that a spacelike graph in a GRW spacetime is complete because of Lemma 2.4.2. Then, we are in a position to state the first characterization result (compare with [92, Th. 4.8] and [90, Th. 4.1]),

Theorem 6.3.1. *Let $(F, g_{_F})$ be a complete Riemannian surface with finite total curvature and let $f : I \longrightarrow (0, \infty)$, $I \subset \mathbb{R}$ be a smooth function such that f is not locally constant,* $\inf f > 0$ *and* $(\log f)'' \leq 0$. *Then, the only entire solutions to inequality* (**I**) *are the constants.*

Proof. Let u be an entire solution to the inequality (**I**). From Lemma 2.4.2 we infer that the Riemannian surface (F, g_u) is complete. Making use of equality (6.1), we obtain

$$-f(u)^2 K_u \leq -K^F \cosh\theta, \tag{6.6}$$

where K_u denotes the Gauss curvature of the Riemannian surface (F, g_u).

We can write

$$\int_F \max\{0, -K_u\}\, dV_{g_u} = \int_F \max\{0, -K_u\} \frac{f(u)^2}{\cosh^2\theta}\, dV_{g_F} \leq \int_F \max\left\{0, -K^F\right\} dV_{g_F} < \infty,$$

thus Σ_u has finite total curvature.

Now, let us consider the function $f \circ \tau : \Sigma_u \longrightarrow (0, \infty)$. Since that $\inf f > 0$, there is a suitable constant D such that $\log f(\tau) + D \geq 0$ and let us define

$$v := \operatorname{arccot}(\log f(\tau) + D) : \Sigma_u \longrightarrow (\pi, 2\pi).$$

A direct computation drawn from (6.5) results in $v\Delta v \geq 0$ and Lemma 3.0.3 can be recalled. Therefore, if B_R denotes a geodesic disc of radius R around a fixed point p in Σ_u, then, for any r such that $0 < r < R$, there exists a positive constant $C = C(p, r)$ such that

$$\int_{B_r} |\nabla f(\tau)|^2 \, dV \leq \frac{C}{\mu_{r,R}}, \tag{6.7}$$

where B_r is the geodesic disc of radius r around p in Σ_u, and $\frac{1}{\mu_{r,R}}$ is the capacity of the annulus $B_R \setminus \overline{B}_r$. Now, the parabolicity of Σ_u implies $\frac{1}{\mu_{r,R}} \to 0$ as $R \to \infty$. Hence, if R approaches infinity from a fixed arbitrary point and a fixed r, we obtain that the function $f(\tau)$ must be constant, thus $\tau \equiv u$ is constant. □

In the case of a non-necessarily proper GRW spacetime, we obtain the following result.

Theorem 6.3.2. *Let (F, g_F) be a complete Riemannian surface with finite total curvature and let $f : I \longrightarrow (0, \infty)$, $I \subset \mathbb{R}$ be a smooth function such that* $\inf f > 0$ *and* $(\log f)'' \leq 0$*. Then, the only entire solutions to inequality* (**I**)*, which are bounded from above or from below are the constants.*

Proof. Following back on the previous result, we have $f(\tau) = \text{constant}$. Let us assume first that u is bounded from below and let us consider the non-negative function

$$\mathcal{F}(\tau) = \int_{\inf u}^{\tau} f(s)ds.$$

From (6.4) we obtain that the function $\mathcal{F}(\tau)$ is harmonic, thus $\mathcal{F}(\tau)$ is constant. Note that as

$$\nabla \mathcal{F}(\tau) = f(\tau)\nabla\tau,$$

consequently $\tau \equiv u$ must be constant.

When u is bounded from above, it is enough for our purpose to take the non-positive function

$$\mathcal{G}(\tau) = \int_{\sup u}^{\tau} f(s)ds$$

and the result is drawn. $\square$

Remark 6.3.3. The boundedness assumption on the solutions cannot be dropped in the previous Theorem. Indeed, consider $F = \mathbb{R}^2$ with its Euclidean metric and $f \equiv 1$. It is clear that every function $u(x, y) = a\,x + b\,y + c$, with $a, b, c \in \mathbb{R}$ such that $\sqrt{a^2 + b^2} < 1$ is a solution to inequality (**I**).

Taking into account Remark 3.2.6, we can state,

Corollary 6.3.4. *Let $(F, g_{_F})$ be a complete Riemannian surface with finite total curvature and let $f : I \longrightarrow (0, \infty)$ be a smooth function such that $(\log f)'' \leq 0$. Then, the only entire bounded solutions to inequality* (**I**) *are the constants.*

6.4 Applications to the parametric case

In order to apply the previous results to the parametric case, we need an extra topological hypothesis.

Let us consider a GRW spacetime $M = I \times_f F$, whose fiber is a 2-dimensional complete Riemannian manifold. Recall that if the warping function is bounded on a complete spacelike surface $x : S \longrightarrow M$, then

$$\tilde{\pi} := \pi_{_F} \circ x : S \longrightarrow F$$

is a covering map (Section 3.2).

Let us consider a point $p_0 \in F$ be and $\tilde{p}_0 \in S$ such that $\tilde{\pi}(\tilde{p}_0) = p_0$. Denote by

$$A = \frac{\pi_1(F,\, p_0)}{\tilde{\pi}_*(\pi_1(S,\, \tilde{p}_0))}$$

the set of all left cosets of $\tilde{\pi}_*(\pi_1(S,\, \tilde{p}_0))$ in $\pi_1(F,\, p_0)$. It is well-known that

$$\sharp(\tilde{\pi}^{-1}(p_0)) = \sharp(A).$$

Now, let us assume $\sharp(A) < \infty$. Thus, S covers $\sharp(A)$-times the fiber. Moreover, taking into account the reasoning in Theorem 6.3.1, it is not difficult to see that S also has finite total curvature.

Once we have assured that S has finite total curvature, from the results of the previous section we can obtain new uniqueness results. The first one is,

Theorem 6.4.1. *Let $M = I \times_f F$ be a proper GRW spacetime, whose 2-dimensional fiber has finite total curvature. Let S be a complete spacelike surface in M, such that function $f(\tau)$ is bounded on S, $(\log f)''(\tau) \leq 0$ and $\sharp(A) < \infty$. Suppose that the inequality*

$$H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}$$

holds on S, being H the mean curvature function of S. Then S is a spacelike slice.

Proof. Since the spacelike surface S is complete with finite total curvature, the reasoning in Theorem 6.3.1 can be applied so as to conclude that $f(\tau)$ is constant. □

Remark 6.4.2. a) Consider $F = \mathbb{S}^1 \times \mathbb{R}$ endowed with its canonical product metric, and f an arbitrary positive smooth function. Set $S = F$. For each positive integer m, let $x : S \to I \times_f F$ be the spacelike immersion given by $x(e^{i\theta}, s) = (t_0, e^{im\theta}, s)$. This example shows that there exist surfaces with arbitrary $\sharp(A)$. **b)** However, we cannot force the fact that the fundamental group of the fiber is finite unless it is trivial. This is due to the fact that the fundamental group of any non-compact surface must be free (see, for instance, [54]).

When $I = \mathbb{R}$, $F = \mathbb{R}^2$ and $f(t) = e^t$, the corresponding Robertson-Walker spacetime $\mathcal{N}$ is isometric to a proper open subset of the De Sitter spacetime of sectional curvature 1, which is known as the 3-dimensional steady state spacetime. Let us recall that a spacelike surface S in $\mathcal{N}$ is said to be bounded away from future infinity if $\sup \tau(S) < \infty$.

As an application of Theorem 6.4.1, we can give the following result, which extends [24, Thm. 6.20],

Corollary 6.4.3. *The only complete spacelike surfaces in the steady state spacetime whose mean curvature function satisfies $H^2 \leq 1$ and are bounded away from future infinity are the spacelike slices.*

As a consequence of Theorem 6.3.2 we obtain,

Theorem 6.4.4. *Let $M = I \times_f F$ be a GRW spacetime, whose 2-dimensional fiber has finite total curvature. Let S be a complete spacelike surface in M such that $\sharp(A) < \infty$ and*

a) S is bounded from above and $(\log f)''(\tau) \leq 0$, or

b) S is bounded from below, $f(\tau)$ is bounded on S and $(\log f)'' \leq 0$.

Suppose that the inequality

$$H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}$$

holds on S, being H the mean curvature function of S. Then S is a spacelike slice.

Compare Theorem 6.4.1 and 6.4.4 with [92, Cor. 4.3]. In the particular case of $H = 0$, we obtain,

Corollary 6.4.5. *Let $M = I \times_f F$ be a GRW spacetime, whose 2-dimensional fiber has finite total curvature. Let S be a complete maximal surface S in M such that $\sharp(A) < \infty$ and*

a) S is bounded from above and $(\log f)''(\tau) \leq 0$, or

b) S is bounded from below, $f(\tau)$ is bounded on S and $(\log f)'' \leq 0$.

Then S must be a spacelike slice $t = c$, with $f'(c) = 0$.

Remark 6.4.6. a) Note that in the previous Corollary, if the base of the spacetime is an interval bounded from below or from above, we can drop the timelike boundedness assumption on the spacelike surface. **b)** If $(\log f)'' \leq 0$ and there exists $t_0 \in I$ such that $f'(t_0) = 0$, then f is bounded. Moreover, if f is not locally constant then this zero of f' is unique.

We are now ready to state the following wider extension of [67, Cor. 5.1] and [92, Cor. 4.4].

Corollary 6.4.7. *Let $M = I \times_f F$ be a proper GRW spacetime, whose 2-dimensional fiber is simply connected and has finite total curvature and its warping function satisfies $(\log f)'' \leq 0$. If there exists a maximal slice in M, then it is the only complete maximal surface in M.*

If the boundedness assumptions on Theorem 6.4.4 are dropped, with extra hypotheses on H, we get the following result,

Theorem 6.4.8. *Let $M = I \times_f F$ be a GRW spacetime, whose 2-dimensional fiber* *has finite total curvature. Let S be a complete spacelike in M, such that the warping function $f(\tau)$ on S is bounded, $(\log f)''(\tau) \leq 0$ and $\sharp(A) < \infty$. Suppose that the inequality*

$$H^2 \leq \frac{f'(\tau)^2}{f(\tau)^2}$$

holds on S, being H the mean curvature function of S. If each zero of H is isolated (particularly, if H has no zero) then S is a spacelike slice.

Proof. Since $f(\tau)$ is constant and $\Delta \log f(\tau) = 0$, according to (6.5) we have $H = \frac{-f'(\tau)}{f(\tau)\, g(N,\partial_t)}$ and $|H| = \frac{f'(\tau)}{f(\tau)}$, thus $g(N, \partial_t) = 1$ on S. □

6.5 The total energy of a spacelike surface

Let us assume the $(n+1)$-dimensional GRW spacetime (M, g) is a perfect fluid with flow vector field $-\partial_t$, energy density function ρ and pressure p [83, Chap. 12]. From (2.3), it is not difficult to see that,

$$\overline{\mathrm{Ric}}(\partial_t, \partial_t) = -n\frac{f''}{f} \quad \text{and} \quad S = \frac{S^F}{f^2} + 2n\frac{f''}{f} + n(n-1)\frac{(f')^2}{f^2}, \tag{6.8}$$

where Ric denotes the Ricci tensor, S the scalar curvature of spacetime and S^F the scalar curvature of the fiber. If M obeys the Einstein field equation, from (6.8), we obtain

$$8\pi\rho = \frac{1}{2}\frac{S^F}{f^2} + \frac{n(n-1)}{2}\frac{(f')^2}{f^2}. \tag{6.9}$$

When $n = 2$, the Ricci tensor of the fiber is $\mathrm{Ric}^F = K^F g_F$ and $S^F = 2K^F$, thus

$$S = \frac{2K^F}{f^2} + 4\frac{f''}{f} + 2\frac{(f')^2}{f^2} \quad \text{and} \quad 8\pi\rho = \frac{K^F}{f^2} + \frac{(f')^2}{f^2}. \tag{6.10}$$

The spacetime must satisfy the NCC. If the mean curvature function satisfies $H^2 \leq$

$\frac{f'(\tau)^2}{f(\tau)^2}$, regarding (6.1) we have

$$K \geq 8\pi\rho - H^2, \tag{6.11}$$

on each spacelike surface Σ in M. Suppose that the spacelike surface is maximal with finite total curvature. Then if we denote the total energy on Σ by

$$E_\Sigma = \int_\Sigma \rho \, dV \, ,$$

making use of the Cohn-Vossen inequality, we have

$$8\pi E_\Sigma \leq \int_\Sigma K \, dV \leq 2\pi\mathcal{X}(\Sigma).$$

Notice that if Σ is a spacelike slice $t = t_0$ in a GRW spacetime with finite total curvature, the previous equation reads,

$$8\pi E_\Sigma \leq 2\pi\mathcal{X}(F).$$

Note that in the previous estimation, the total energy is bounded by a topological invariant, the Euler-Poincaré characteristic. Moreover, the same reasoning as that above works whenever the spacelike surface is assumed to be compact (without boundary). In this case, the inequality becomes just an equality from the Gauss-Bonnet theorem.

The reverse inequality may be obtained using extra topological notions. A control on the topology of a (non-compact) Riemannian manifold is given by the requirement that the manifold is, outside a compact set C, diffeomorphic to $\partial C \times [1, \infty]$. If a Riemannian manifold satisfies this condition, then it is said to have finite topology. It can be proved that a Riemannian manifold with finite topology is homeomorphic to a closed surface with a finite number of points removed (these points are called ends). By means of this closed surface, the ideal boundary can be considered. In relation to the ideal boundary, it can be measured the length of the ideal boundary associated with each end, in the sense of [40].

Coming back to the estimation of the total energy of a maximal surface, if the hyperbolic angle is bounded, denoting $\cosh\theta^0 := \sup_S\{\cosh\theta\}$,

$$8\pi E_\Sigma \geq \frac{1}{\cosh\theta^0}\int_\Sigma K\,dV \geq \frac{1}{\cosh\theta^0}\Big(2\pi\mathcal{X}(\Sigma) - \sum_{i=1}^{k} l_i\Big), \tag{6.12}$$

where l_i stands for the length of the ideal boundary associated to each end of Σ (see [40] and references therein).

Chapter 7

Maximum principles and maximal hypersurfaces

In this chapter, we will use some classical maximum principles (the strong Liouville property and the Omori-Yau generalized maximum principle) in order to obtain several uniqueness results for complete maximal hypersurfaces. In contrast to previous chapters, now curvature assumptions will comein handy. Our research will also focus on getting non-existence results (Section 7.2). The contents of this chapter can be found in [97].

7.1 The parametric case

We shall begin with the special case of a static GRW spacetime. Let us recall that, in this class of spacetimes, the Gauss equation for a maximal hypersurface S reads (see (2.15))

$$\mathrm{Ric}(v,v) = \mathrm{Ric}^F(v^F,v^F) + g_{_F}\left(\mathrm{R}^F(N_p^F,v^F)v^F, N_p^F\right) + g\left(\mathrm{A}^2 v, v\right) ,$$

being v^F and N^F the projections onto F of v and N_p, respectively, and where R^F and Ric denote the curvature tensor of the fiber and the Ricci tensor of S, respectively.

It is obvious that whenever the sectional curvature of the fiber is non-negative, then the Ricci curvature of S is so. This fact implies that, on a complete maximal hypersurface, the following Liouville's type result remains [111],

Theorem 7.1.1. *If S is a complete Riemannian manifold with non-negative Ricci curvature, then S has the strong Liouville property; i.e., S does not admit any non-constant positive harmonic function.*

According to these considerations, we can state the first uniqueness result of this chapter (compare with [79, Thm. A], Theorem 4.1.7 and Corollary 5.1.17),

Theorem 7.1.2. *Let S be a complete maximal hypersurface in a static GRW spacetime whose fiber has non-negative sectional curvature. If S is bounded from below or from above, then S must be a spacelike slice.*

Proof. Taking into account equation (2.11), $\tau + C$ or $C - \tau$ is a positive harmonic function for some suitable constant $C \in \mathbb{R}$. Theorem 7.1.1 can be used to conclude that τ is constant. □

The boundedness assumption of S cannot be withdrawed in order to get the rigidity result as the non-horizontal spacelike hyperplanes in Lorentz-Minkowski spacetime $\mathbb{L}^n$ show.

In different environment, recall the well-known Omori-Yau Maximum Principle [82], [111],

Theorem 7.1.3. *Let S be a complete Riemannian manifold whose Ricci curvature* *is bounded from below and let* $u : S \to \mathbb{R}$ *be a smooth function bounded from below (resp. bounded from above). Then, for each* $\epsilon > 0$*, there exists a point* $p_\epsilon \in S$ *such that*

i) $|\nabla u(p_\epsilon)| < \epsilon$,

ii) $\Delta u(p_\epsilon) > -\epsilon$ *(resp.* $\Delta u(p_\epsilon) < \epsilon$ *),*

iii) $\inf u \leq u(p_\epsilon) < \inf u + \epsilon$ *(resp.* $\sup u - \epsilon < u(p_\epsilon) \leq \sup u$ *).*

The following lemma imposes some geometrical conditions which allow us to apply the previous generalized maximum principle.

Lemma 7.1.4. *Let* S *be a maximal hypersurface in a GRW spacetime whose fiber has sectional curvature bounded from below. Suppose that* $f''(\tau)/f(\tau)$ *and* $f'(\tau)/f(\tau)$ *are bounded and* $\inf f(\tau) > 0$*. If* S *has bounded hyperbolic angle, then its Ricci curvature must be bounded from below.*

Proof. Again, from the Gauss equation, the Ricci curvature of S satisfies

$$\mathrm{Ric}(v,v) = \overline{\mathrm{Ric}}(v,v) + \overline{g}\left(\overline{\mathrm{R}}(N_p,v)v, N_p)\right) + g\left(\mathrm{A}^2 v, v\right) , \tag{7.1}$$

for any unit $v \in T_pS$, $p \in S$, where $\overline{\mathrm{R}}$ denotes the curvature tensor of the GRW spacetime.

We will show that the right side of equation (7.1) is bounded from below. Let us write

$$v = v_{_I}\, \partial_t(p) + v_{_F}\, z\,, \tag{7.2}$$

where $z \in T_pF$, $z \perp \partial_t(p)$ and $\overline{g}(z,z) = 1$. Making use of the Schwarz inequality, $v_{_I}$ satisfies

$$v_{_I}^2 = \overline{g}(\partial_t, v)^2 = g(\nabla\tau, v)^2 \leq |\nabla\tau|^2\,.$$

Falling back on the boundedness of the hyperbolic angle and (2.6), we get that $v_{_I}^2$

is bounded. Since v is unit, the same conclusion remains for $v_{_F}^2$. Now, using the expressions [83, Cor. 7.43] together with (7.2), we reach the conclusion that

$$\overline{\mathrm{Ric}}(v,v) = v_{_F}^2\ \mathrm{Ric}^F(z,z) - (n-1)\, v_{_F}^2\, (\log f)''(\tau) + n\,\frac{f''(\tau)}{f(\tau)}$$

is bounded from below.

On the other hand, N_p may be expressed in a similar way to v, that is,

$$N_p = -\cosh\theta\,\partial_t(p) + \sinh\theta\, y\,,$$

where $y \in T_pF$, $y \perp \partial_t(p)$ and $\overline{g}(y,y) = 1$. From [83, Prop. 7.42], we obtain,

$$\begin{aligned}\overline{g}(\overline{R}(N_p,v)v,N_p) \;\;=\;\; & -\frac{f''(\tau)}{f(\tau)}\left\{2\, v_{_F}\, v_{_I}\, \cosh\theta\,\sinh\theta\,\overline{g}(y,z) + v_{_F}^2\,\cosh^2\theta - v_{_I}^2\,\sinh^2\theta\right\}\\ & + v_{_F}^2\,\sinh^2\theta\left\{\frac{1}{f(\tau)^2}\, g_{_F}(\mathrm{R}^F(z_{_F},y_{_F})y_{_F},z_{_F})\right.\\ & \left. -\frac{f'(\tau)^2}{f(\tau)^2}\,\left[\overline{g}(y,z)^2 - 1\right]\right\}, \qquad (7.3)\end{aligned}$$

where $y_{_F} = f(\tau)\,y$ and $z_{_F} = f(\tau)\,z$ are used in order to obtain $g_{_F}(y_{_F},y_{_F}) = g_{_F}(z_{_F},z_{_F}) = 1$. Thus, this term is bounded from below. Therefore, we conclude that the Ricci curvature of S is bounded from below. □

The previous lemma allows us to state,

Corollary 7.1.5. *Let S be a maximal hypersurface in a GRW spacetime whose fiber has sectional curvature bounded from below. If S has bounded hyperbolic angle and lies between two spacelike slices, then its Ricci curvature must be bounded from below.*

The boundedness assumption on the hyperbolic angle can be dropped provided that the spacetime obeys some stronger conditions.

Lemma 7.1.6. *Let S be a maximal hypersurface in a GRW spacetime whose fiber* *has non-negative sectional curvature. If the restriction of the warping function to S satisfies* $(\log f)'' \leq 0$, *then the Ricci curvature of S must be non-negative.*

Proof. Given $p \in S$, let us place a local orthonormal frame $\{U_1, \ldots, U_n\}$ around p. From the Gauss equation we get that the Ricci curvature of S satisfies

$$\mathrm{Ric}(Y,Y) \geq \sum_{i=1}^{n} \overline{g}(\overline{\mathrm{R}}(Y,U_i)U_i, Y)\,.$$

Now, according to [83, Prop. 7.42], we have

$$\begin{aligned}\sum_{i=1}^{n} \overline{g}(\overline{\mathrm{R}}(Y,U_i)U_i, Y) \;=\;& f(\tau)^2 \sum_{i=1}^{n} g_{_F}(\mathrm{R}^F(Y^F, U_i^F)U_i^F, Y^F) + (n-1)\frac{f'(\tau)^2}{f(\tau)^2}|Y|^2 \\ & -(n-2)(\log f)''(\tau) g(Y, \nabla\tau)^2 \\ & -(\log f)''(\tau)|\nabla\tau|^2|Y|^2\,, \qquad (7.4)\end{aligned}$$

where Y^F and U_i^F are the projections of Y and U_i on the fiber. From this equation, taking into account these assumptions, we obtain that the Ricci curvature of S must be non-negative. □

On the other hand, when a maximal hypersurface has its Ricci curvature bounded from below, the Omori-Yau Maximum Principle can be claimed to state,

Lemma 7.1.7. *Let S be a complete maximal hypersurface in a GRW spacetime. Suppose that the Ricci curvature of S is bounded from below and* $(\log f)''(\tau) \leq 0$*. If S lies between two spacelike slices, then* $f'(\tau) = 0$.

Proof. Let $\mathcal{F}$ be the following primitive function of f, $\mathcal{F}(\tau) = \int_{\inf \tau}^{\tau} f(s)\, ds$, which is bounded. We have

$$\nabla \mathcal{F}(\tau) = f(\tau)\nabla\tau\,,$$

and using (2.11),

$$\Delta \mathcal{F}(\tau) = -n f'(\tau)\,.$$

Notice that $\mathcal{F}(\tau)$ grows strictly with τ. From Theorem 7.1.3, we have that, for each real $\epsilon > 0$, there exists a $p_\epsilon \in S$ such that

$$|\nabla \mathcal{F}(\tau(p_\epsilon))| < \epsilon \,,$$

$$-\epsilon \leq \Delta \mathcal{F}(\tau(p_\epsilon)) = -nf'(\tau(p_\epsilon)) \,,$$

with $\inf \mathcal{F}(S) \leq \mathcal{F}(\tau(p_\epsilon)) \leq \inf \mathcal{F}(S) + \epsilon$.

Note that $\inf \mathcal{F}(S) = \mathcal{F}(\inf\, \tau(S))$. Hence, as a direct consequence,

$$0 \leq -nf'(\inf\, \tau(S)) \,,$$

and

$$\frac{f'(\inf\, \tau(S))}{f(\inf\, \tau(S))} \leq 0 \,.$$

With a similar reasoning, considering that $\mathcal{F}(S)$ is bounded from above, we conclude

$$\frac{f'(\sup\, \tau(S))}{f(\sup\, \tau(S))} \geq 0 \,.$$

Since $(\log f)''(\tau) \leq 0$, we obtain $f'(\tau) = 0$. □

Now, from the last result and from Corollary 7.1.5 we obtain (compare the following two results with Theorems 4.1.1, 4.1.3, 5.1.1 and Corollary 5.1.17),

Theorem 7.1.8. *Let S be a complete maximal hypersurface in a proper GRW spacetime whose fiber has sectional curvature bounded from below. Assume that $(\log f)''(\tau) \leq 0$ and S lies between two spacelike slices. If S has bounded hyperbolic angle, then S must be a spacelike slice.*

Proof. It is enough to observe that the warping function is not locally constant and $f'(\tau) = 0$, thus τ must be constant. □

Now, going back to Lemma 7.1.6, a similar result is obtained,

Theorem 7.1.9. *Let S be a complete maximal hypersurface in a GRW spacetime whose fiber has non-negative sectional curvature. Suppose that* $(\log f)''(\tau) \leq 0$*. If S lies between two spacelike slices, then S must be a spacelike slice.*

Proof. From Lemma 7.1.6 we have that S has non-negative Ricci curvature, which permits arriving to $f'(\tau) = 0$, taking into account Lemma 7.1.7. Therefore, τ is harmonic on S. Now, from Theorem 7.1.1 and a similar reasoning to the one in Theorem 7.1.2, the proof comes to an end. □

To conclude this section, we provide an application to an interesting family of Robertson-Walker spacetimes, which are well-known in Cosmology. They are the Friedmann cosmological models (see [83, Ch. 12] for instance). Recall that this family of spacetimes are exact solutions to the Einstein equations and they represent physically realistic universes whose matter content is dust. Since a Friedmann-Robertson-Walker spacetime satisfies the Timelike Convergent Condition, its warping function obeys $f''(t) \leq 0$. Thus, as a consequence of the previous theorem,

Remark 7.1.10. In the Friedmann-Robertson-Walker cosmological models, there is only one case where there exists a complete maximal hypersurface with bounded hyperbolic angle and which lies between two spacelike slices. This case corresponds to the spatially closed model and this spacelike hypersurface is unique.

7.2 Non-existence results

Here, we will apply our previous technical lemmas so as to state some non-existence results. We begin with the following,

Lemma 7.2.1. *Let M be a GRW spacetime whose fiber has sectional curvature bounded from below. Then, M admits no complete maximal hypersurface S with bounded hyperbolic angle, such that S is bounded from below* (*resp. from above*)*,* *with* $\inf f(\tau) > 0$, f *non-decreasing* (*resp. non-increasing*), $\inf f'(\tau) > 0$ (*resp.* $\sup f'(\tau) < 0$) *and having bounded the functions* $f''(\tau)/f(\tau)$ *and* $f'(\tau)/f(\tau)$.

Proof. Let us assume such a hypersurface S does exist. Taking into account our assumptions, from (2.11) we have $\Delta\tau \leq -\sigma^2 < 0$ or $\Delta\tau \geq \sigma^2 > 0$, for some $\sigma \in \mathbb{R}$. Using the Lemma 7.1.4, we know that the generalized maximum principle holds on S. After applying this principle to the function τ we obtain a contradiction. □

As a nice consequence,

Theorem 7.2.2. *Let* M *be a GRW spacetime whose fiber has sectional curvature bounded from below. Then,* M *admits no complete maximal hypersurface* S *with bounded hyperbolic angle such that* S *lies between two spacelike slices and* $\inf |f'(\tau)| > 0$.

Now, taking into account the Lemma 7.1.6, an identical reasoning leads us to the same conclusion.

Theorem 7.2.3. *Let* M *be a GRW spacetime whose fiber has non-negative sectional curvature and its warping function satisfies* $(\log f)''(t) \leq 0$. *Then,* M *admits no complete maximal hypersurface bounded from below* (*resp. bounded from above*), *with* $\inf f(\tau) > 0$, f *non-decreasing* (*resp. non-increasing*) *and* $\inf f'(\tau) > 0$ (*rep.* $\sup f'(\tau) < 0$).

Corollary 7.2.4. *Let* M *be a GRW spacetime whose fiber has non-negative sectional curvature and its warping function satisfies* $(\log f)''(t) \leq 0$. *Then,* M *admits no complete maximal hypersurface lying between two spacelike slices and satisfying* $\inf |f'(\tau)| > 0$.

Remark 7.2.5. The assumption $\inf |f'(\tau)| > 0$ in the previous results can be interpreted geometrically as the non-existence of any maximal slice in the timelike bounded region of the spacetime given by $\operatorname{Im}\tau$.

Again, from the Omori-Yau maximum principle and equation (2.12), we can assert,

Theorem 7.2.6. *Let M be a GRW spacetime whose fiber has sectional curvature bounded from below. It admits no complete maximal hypersurface such that* $\inf f(\tau) > 0$, $\sup f(\tau) < \infty$, $(\log f)''(\tau) \leq 0$, $\inf |f'(\tau)| > 0$, *and having bounded the functions $f''(\tau)/f(\tau)$ and $f'(\tau)/f(\tau)$.*

Proof. From Lemma 7.1.4, the generalized maximum principle remains. Taking into account equation (2.12), if such a spacelike hypersurface exists, then a contradiction is found. □

Finally, we put an end to this section with the following theorem, which is actually a reformulation of the previous result based on Lemma 7.1.6 instead of Lemma 7.1.4

Theorem 7.2.7. *Let M be a GRW spacetime whose fiber has non-negative sectional curvature and its warping function satisfies* $(\log f)''(t) \leq 0$. *Then M admits no complete maximal hypersurface such that* $\inf f(\tau) > 0$, $\sup f(\tau) < \infty$, *and* $\inf |f'(\tau)| > 0$.

7.3 Calabi-Bernstein type problems

As in previous chapters, we provide the associated Calabi-Bernstein type results from the theorems previously developed. The completeness of a spacelike graph is guaranteed by Lemma 2.4.2. Using Theorem 7.1.2, we arrive to (compare with Theorem 5.2.9),

Theorem 7.3.1. *The only entire solutions to*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-\mid Du\mid^2}}\right)=0\,,$$

$$\mid Du\mid<\lambda,\quad 0<\lambda<1,$$

on a complete Riemannian manifold F with non-negative sectional curvature are the constant functions.

Now, from Theorem 7.1.8, we obtain (compare with Theorems 4.2.2, 5.2.1 and 5.2.3),

Theorem 7.3.2. *Let $f : I \longrightarrow \mathbb{R}$ be a non-locally constant positive smooth function. Assume f satisfies $(\log f)'' \leq 0$, and $\inf f > 0$. The only bounded entire solutions to the equation* (E) *on a complete Riemannian manifold F with sectional curvature bounded from below are the constant functions $u = c$, with $f'(c) = 0$.*

Finally, Theorem 7.1.9 leads to (see also Theorem 4.2.1)

Theorem 7.3.3. *Let $f : I \longrightarrow \mathbb{R}$ be a positive smooth function. Assume f satisfies $(\log f)'' \leq 0$, and $\inf f > 0$. The only bounded entire solutions to the equation* (E) *on a complete Riemannian manifold F with non-negative sectional curvature are the constant functions $u = c$, with $f'(c) = 0$.*

Conclusions and future research

In this thesis we have introduced a new family of open spacetimes: the spatially parabolic GRW spacetimes. We have mentioned several of its properties and its suitability to potentially model a relativistic universe.

In Chapter 3, we have developed several formulitae which allow us to assure parabolicity on a complete spacelike hypersurface in this class of spacetimes. These procedures are, in principle, potentially applicable to many different problems. We have focused mainly on uniqueness results for maximal hypersurfaces. The two dimensional case was specially paid attention to in Chapter 6, where the notion of finite total curvature (only defined for Riemannian surfaces) was imperative. As an application of most of the results, the associated Calabi-Bernstein type problems have been solved.

The first technique (Theorem 3.2.5) states sufficient conditions under which a complete spacelike hypersurface (in a spatially parabolic GRW spacetime) is parabolic. In Chapter 4, this procedure is used to obtain different uniqueness results. The second technique (Theorem 3.2.9) assures parabolicity on a complete spacelike hypersurface whenever it is endowed with certain conformal metric. Then, in Chapter 5 other sort of uniqueness results are given. It should be pointed out that both approaches do not need any assumption on the mean curvature function.

The (non-compact) complete Riemannian surfaces with finite total curvature have nice properties and naturally extend to the Euclidean plane. In particular, they are parabolic. In Chapter 6, we consider a GRW spacetime whose fiber is one of the previously mentioned surfaces. The main aim of this Chapter is to study a more general problem than the characterization of complete maximal surfaces. In particular, we study spacelike surfaces which obey a natural non-linear differential inequality involving its mean curvature function (automatically satisfied in the maximal case). Now, we consider complete spacelike graphs which obey that inequality. The main idea of the technique (which appear in the proof of Theorem 6.3.1) is to find when a complete spacelike graph has also finite total curvature. Then, we provide several answers to our problem. Here, the parametric case is obtained from the non-parametric case assuming an extra topological hypothesis on the fiber.

Finally, in Chapter 7 we use the strong Liouville property and the Omori-Yau generalized maximum principle in orde to prove, under some curvature assumptions, several uniqueness and non-existence results.

Considering a different environment, the techniques here developed might have several applications in the future. Although the most part of this thesis was devoted to the maximal case, the problem of controlling the mean curvature may be considered in arbitrary dimension. In [93, Remark 2.3], $\Delta \log f(\tau)$ is computed for a general spacelike hypersurface. It is likely that this formula would be helpful in order to lead to new uniqueness results for complete hypersurfaces with controlled mean curvature. Moreover, another kind of problems with similar geometric nature can be susceptible to be contemplated, for instance, problems involving the k-th mean curvature. Hence, the techniques here presented are potentially applicable to a wide set of open questions.

On the other hand, the notion of spatially parabolicity may be conveniently studied in another family of spacetimes. The (open) globally hyperbolic spacetimes are a class of spacetimes with nice geometrical and causality properties. It has been recently proved that any globally hyperbolic spacetime can be expressed as the product of an interval of the real line and a Riemannian manifold, endowed with certain Lorentzian metric, [16]. Moreover, the coordinate of the interval of the real line represents a universal time function. Thus, the topological structure is tantamount to that of a GRW spacetime. Another research line may consist in trying to establish similar techniques in order to characterize maximal hypersurfaces as level hypersurfaces of the universal time function. Analogously, new Calabi-Bernstein type problems may be expected to appear in this environment.

The open problems we will deal with in future do not restrict to Lorentzian Geometry. In Riemannian Geometry, the warped product manifolds are a family of paramount importance. The problem of studying hypersurfaces with zero mean curvature (minimal) is also natural and interesting. In this environment, when the fiber is parabolic, it can be proved that an entire graph, when it is endowed with a pointwise conformal metric, is also parabolic [98, Lemma 1] (compare with Theorem 3.2.9). This allows us to solve new Moser-Bernstein problems [98].

Recently, several ideas from these works have been successfully applied to study ϕ-minimal hypersurfaces, in the context of Riemannian manifolds with density, [103]. This paper has in common with Chapter 6 that it can be also regarded as characterization results of controlled mean curvature. However, in the former, the dimension is arbitrary.

To sum up, this work opens the door to new different research lines which can provide new interesting problems and which can help us to understand better the role of the geometric meaning of certain hypersurfaces in Lorentzian and Riemannian Geometry.

References

[1] L.V. Ahlfors, Sur le type d'une surface de Riemann, *C.R. Acad. Sci. Paris*, **201** (1935), 30–32.

[2] R. Aiyama, On the Gauss map of complete spacelike hypersurfaces of constant mean curvature in Minkowski space, *Tsukuba J. Math.*, **16** (1992), 353–361.

[3] A.L. Albujer and L.J. Alías, Calabi-Bernstein results for maximal surfaces in Lorentzian product spaces, *J. Geom. Phys.*, **56** (2009), 620–631.

[4] A.L. Albujer and L.J. Alías, Calabi-Bernstein results and parabolicity of maximal surfaces in Lorentzian product spaces, *Recent trends in Lorentzian geometry*, Edited by M. Sánchez, M. Ortega and A. Romero, Springer Proceedings in Mathematics and Statistics, **26** (2013), 49–85.

[5] J.A. Aledo and L.J. Alías, On the curvature of bounded complete spacelike hypersurfaces in the Lorentz-Minkowski space, *Manuscripta Math.*, **101** (2000), 401–413.

[6] L.J. Alías and P. Mira, On the Calabi-Bernstein theorem for maximal hypersurfaces in the Lorentz-Minkowski space, *Proc. of the meeting Lorentzian Geometry-Benalmádena 2001, Spain, Pub. RSME,* **5** (2003), 23–55.

[7] L.J. Alías and S. Montiel, Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized RobertsonâWalker spacetimes, *Differential geometry*, Valencia 2001, 59–69. World Science Publication, River Edge (2002).

[8] L.J. Alías and B. Palmer, Zero mean curvature surfaces with non-negative curvature in flat Lorentzian 4-spaces, *Proc. R. Soc. London A.* **455** (1999), 631–636.

[9] L.J. Alías, A. Romero and M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker Spacetimes, *General Relativity and Gravitation*, **27** (1995), 71–84.

[10] L.J. Alías, A. Romero and M. Sánchez, Spacelike hypersurfaces of constant mean curvature in spacetimes with symmetries, *VI Fall Workshop "Differential Geometry and its Application", Valencia Pub. RSME*, **1** (2000), 1–14.

[11] D. Bak and S-J Rey, Cosmic holography, *Class. Quantum Grav.*, **17** (2000), L83–L89.

[12] R. Bartnik, Existence of maximal surfaces in asymptotically flat spacetimes, *Comm. Math. Phys.*, **94** (1984), 155–175.

[13] R. Bartnik, Quasi-spherical metrics and prescribed scalar curvature, *J. Differential Geometry*, **37** (1993), 31–71.

[14] J.K. Beem, P.E. Ehrlich and K.L. Easley, *Global Lorentzian Geometry, second edition*, Pure and Applied Mathematics, **202**, Marcel Dekker New York, 1996.

[15] A.N. Bernal and M. Sánchez, On smooth Cauchy hypersurfaces and Geroch's splitting theorem, *Commun. Math. Phys.*, **243** (2003), 461–470.

[16] A.N. Bernal and M. Sánchez, Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes, *Com. Math. Phys.*, **257** (2005), 43–50.

[17] S. Bernstein, Sur une théorème de géometrie et ses applications aux équations dérivées partielles du type elliptique, *Comm. Soc. Math. Kharkov*, **15** (1914), 38–45.

[18] L. Bers, Isolated singularities of minimal surfaces, *Ann. of Math.*, **53** (1951), 364–386.

[19] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969), 1–49.

[20] R. Bousso, The holographic principle, *Rev. Mod. Phys.*, **74** (2002), 825–874.

[21] A. Brasil, A.G. Colares, On constant mean curvature spacelike hypersurfaces in Lorentz manifolds, *Mat. Contemp.*, **17** (1999) 99–136.

[22] D.R. Brill, Maximal surfaces in closed and open spacetimes, *Proc. of the first Marcel Grossman meeting on General Relativity*, Edited by R. Ruffini (North-Holland Pub. Comp., Amsterdam) (1977), 193–206.

[23] D. Brill and F. Flaherty, Isolated maximal surfaces in spacetime, *Commun. math. phys.*, **50** (1976), 157–165.

[24] M. Caballero, A. Romero and R. Rubio, Constant mean curvature spacelike surfaces in three-dimensional generalized Robertson-Walker spacetimes, *Lett. Math. Phys.*, **93** (2010), 85–105.

[25] M. Caballero, A. Romero and R.M. Rubio, Uniqueness of maximal surfaces in generalized Robertson-Walker spacetimes and Calabi-Bernstein type problems, *J. Geom. Phys.*, **60** (2010), 394–402.

[26] M. Caballero, A. Romero and R.M. Rubio, Complete CMC spacelike surfaces with bounded hyperbolic angle in Generalized Robertson-Walker spacetimes, *Int. J. Geom. Meth. Mod. Phys.*, **6** (2010), 961–978.

[27] M. Caballero, A. Romero and R.M. Rubio, Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds with a timelike gradient conformal vector field, *Class. Quantum Grav.*, **28** (2011), 145009–145022.

[28] M. Caballero, A. Romero and R.M. Rubio, New Calabi-Bernstein results for some elliptic nonlinear equations, *Anal. Appl.*, **11** (2013), 1–13.

[29] E. Calabi, Examples of Bernstein problems for some non-linear equations, *Proc. Sympos. Pure Math.*, **15** (1970), 223–230.

[30] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure Appl. Math. **115**, Academic Press, New York, 1984.

[31] B.Y. Chen, A simple characterization of generalized Robertson-Walker spacetimes, *General Relativity and Gravitation* (to appear).

[32] S.Y. Cheng, S.T. Yau, Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces, *Ann. of Math.*, **104** (1976), 407–419.

[33] H.Y. Chiu, A cosmological model for our universe, *Annals of Physics*, **43** (1967), 1–41.

[34] Y. Choquet-Bruhat, Quelques propriétés des sousvariétés maximales d'une variété lorentzienne, *C R Acad. Sci. Paris Ser. A B*, **281** (1975) 577–580.

[35] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations*, Oxford University Press, 2009.

[36] A.A. Coley and B.O.J. Tupper, Special conformal Killing vector spacetimes and symmetry inheritance, *J. Math. Phys.*, **30** (1989), 2616–2625.

[37] A.A. Coley and B.O.J. Tupper, Fluid Spacetimes admitting covariantly constant vectors and tensors, *General Relativity and Gravitation*, **23** (1991), 1113–1142.

[38] G. de Rham, Sur la réductibilité d'un espace de Riemann, *Comment. Math. Helv.*, **26** (1952), 328–344.

[39] U. Dierkes, A Bernstein result for enery minimizing hypersurfaces, *Calc. Var.*, **1** (1993), 37–54.

[40] F. Dillen and W. Kühnel, Total curvature of complete submanifolds of euclidean space, *Tohoku Math. J.*, **57** (2005), 171–200.

[41] D. Earley, J. Isenberg, J. Marsden and V. Moncrief, Homothetic and conformal symmetries of solutions to Einstein's equations, *Commun. Math. Phys.*, **106** (1986), 137–158.

[42] A. Esteve and V. Palmer, On the characterization of parabolicity and hyperbolicity of submanifolds, *J. London Math. Soc.*, **84** (2011), 120–136.

[43] F.J.M. Estudillo and A. Romero, Generalized maximal surfaces in the Lorentz-Minkowski space $\mathbb{L}^3$, *Math. Proc. Camb. Phil. Soc.*, **111** (1992), 515–524.

[44] A.E. Fischer and J.A. Wolf, The structure of compact Ricci-flat Riemannian manifolds, *J. Differential Geometry*, **18** (1975), 277–288.

[45] W.H. Fleming, On the oriented Plateau problem, *Rend. Circ. Mat. Palermo*, **11** (1962), 69–90.

[46] L.Z. Gao and S.T. Yau, The existence of negatively Ricci curved metrics on three manifolds, *Invent. Math.*, **85** (1986), 637–652.

[47] C. Gerhardt, *Curvature problems*, Series in Geometry and Topology, **39**, International Press, Somerville, MA, 2006.

[48] R. Geroch, Domain of dependence, *J. Mathematical Phys.*, **11** (1970), 437–449.

[49] E. De Giorgi, Una estensione del teorema di Bernstein, *Ann. Scuola Norm. Sup. Pisa*, **19** (1965), 79–85.

[50] R.E. Grenne and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Math., **699**, Springer-Verlag, New York, 1979.

[51] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.*, **36** (1999), 135–249.

[52] A. Grigor'yan, On the existence of a Green function on a manifold, *Uspekhi Matem. Nauk*, **38** (1983), 161–162. Engl. transl. *Russian Math. Surveys*, **38** (1983), 190–191.

[53] A. Grigor'yan, On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds, *Matem. Sbornik*, **128** (1985), 354–363. Engl. transl. *Math. USSR Sb.*, **56** (1987) 349–358.

[54] A. Hatcher, *Algebraic Topology*, Cambrige University Press, 2002.

[55] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press 1973.

[56] E. Hopf, On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature, *Proc. Amer. Math. Soc.*, **1** (1950), 80–85.

[57] A. Huber, On subharmonic functions and differential geomery in the large, *Comment. Math. Helv.*, **32** (1957), 13–72.

[58] M. Kanai, Rough isometries, and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan*, **37** (1985), 391–413.

[59] M. Kanai, Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan*, **38** (1986), 391–413.

[60] L. Karp, Subharmonic function on real and complex manifolds, *Math. Z.,* **179** (1982), 535–554.

[61] L. Karp, Subharmonic functions, harmonic mappings and isometric immersions, *in: Seminar on Differential Geometry, ed. S.T.Yau, Ann. Math. Stud.*, **102** (1982), 133–142.

[62] J.L. Kazdan, Parabolicity and the Liouville property on complete Riemannian manifolds, *Aspects of Math.*, **E10**, Ed. A.J. Tromba, Friedr. Vieweg and Sohn, Bonn (1987), 153–166.

[63] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space $\mathbb{L}^3$, *Tokyo J. Math.*, **6** (1983), 297–309.

[64] M. Kriele, Spacetime, Foundations of General Relativity and Differential Geometry, *Springer*, 2001.

[65] K. Land and J. Magueijo, Examination of evidence for a preferred axis in the cosmic radiation anisotropy, *Phys. Rev. Lett.*, **95** (2005), 071–301.

[66] K. Land and J. Magueijo, Is the universe odd?, *Phys. Rev. D*, **72** (2005), 101–302.

[67] J.M. Latorre and A. Romero, New examples of Calabi-Bernstein problems for some nonlinear equations, *Diff. Geom. Appl.*, **15** (2001), 153–163.

[68] J.M. Latorre and A. Romero, Uniqueness of Noncompact Spacelike Hypersurfaces of Constant Mean Curvature in Generalized Robertson-Walker Spacetimes, *Geom. Dedicata*, **93** (2002), 1–10.

[69] P. Li, Curvature and function theory on Riemannian manifolds, *Surveys in Differential Geometry*, **II**, International Press (2000), 375–432.

[70] P. Li, *Geometric analysis*, Cambridge studies in advance mathematics, **134**, Cambridge University Press, 2012.

[71] A. Lichnerowicz, L' integration des équations de la gravitation relativiste et le problème des n corps, *J. Math. Pures et Appl.*, **23** (1944), 37–63.

[72] C.C. Linder, *Particle physics and inflationary cosmology*, Harwood, 1990.

[73] J. Lohkamp, Metrics of negative Ricci curvature, *Ann. of Math.*, **140** (1994), 655–683.

[74] T. Lyons and D. Sullivan, Function theory, random paths and covering spaces, *J. Differential Geometry*, **19** (1984), 299–323.

[75] S. Markvorsen and V. Palmer, Transience and capacity of minimal submanifolds, *Geometric and Functional Analysis*, **13** (2003), 915–933.

[76] S. Markvorsen and V. Palmer, How to obtain transience from bounded radial mean curvature, *Trans. Amer. Math. Soc.*, **357** (2005), 3459–3479.

[77] J.E. Marsden and F.J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in General Relativity, *Physics Reports*, **66** (1980), 109–139.

[78] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman and Co, 1973.

[79] S. Nishikawa, On maximal spacelike hypersurfaces in a Lorentzian manifold, *Nagoya Math. J.*, **95** (1984), 117–124.

[80] K. Nomizu and H. Ozeki, The existence of complete Riemannian metrics, *Proc. Amer. Math. Soc.*, **12** (1961), 889–891.

[81] A. Oliveria-Costa, M. Tegmark, M. Zaldarriaga and A. Hamilton, Significance of the largest scale CMB fluctuations in WMAP, *Phys. Rev D*, **69** (2004), 063–516.

[82] H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan*, **19** (1967), 205–214.

[83] B. O'Neill, *Semi-Riemannnian geometry with applications to Relativity*, Pure Appl. Math. **103**, Academic Press, 1983.

[84] R. Osserman, The minimal surface equation, *Seminar on nonlinear partial differential equations*, *Math. Sci. Res. Inst. Publ.*, Springer, New York, **2** (1984), 237–259.

[85] V. Palmer, On deciding whether a submanifold is parabolic or hyperbolic using its mean curvature, *Topics in Modern Diff. Geom., Simon Stevin Trans. on Geom.* (2010), 131–158.

[86] S. Pigola, M. Rigoli and A.G. Setti, *Maximum principles on Riemannian manifolds and Applications*, Mem. Am. Math. Soc., **822**, 2005.

[87] M. Rainer and H-J Schmidt, Inhomogeneous cosmological models with homogeneous inner hypersurface geometry *General Relativity and Gravitation* **27** (1995), 1265–1293.

[88] H. Ringström, *The Cauchy problem in General Relativity*, ESI Lectures in Mathematics and Physics, 2009.

[89] A. Romero, Simple proof of Calabi-Bernstein's theorem on maximal surfaces, *Proc. Amer. Math. Soc.*, **124** (1996), 1315–1317.

[90] A. Romero and R.M. Rubio, On the mean curvature of spacelike surfaces in certain 3-dimensional Robertson-Walker spacetimes and Calabi-Bernstein's type problems, *Ann. Glob. Anal. Geom.*, **37** (2010), 21–31.

[91] A. Romero and R.M. Rubio, New proof of the Calabi-Bernstein theorem, *Geom. Dedicata*, **147** (2010), 173–176.

[92] A. Romero and R.M. Rubio, A nonlinear inequality arising in geometry and Calabi-Bernstein type problems, *J. Inequalities Appl.*, (2010), Art. ID 950380, 10 pp.

[93] A. Romero and R.M. Rubio, Compact spacelike surfaces whose mean curvature function satisfies a nonlinear inequality in a 3-dimensional generalized Robertson-Walker spacetime, *submitted*.

[94] A. Romero, R.M. Rubio and J.J. Salamanca, Uniqueness of complete maximal hypersurfaces in spatially parabolic Generalized Robertson-Walker spacetimes, *Class. Quantum Grav.*, **30** (2013), 115007, 13 pp.

[95] A. Romero, R.M. Rubio and J.J. Salamanca, A new approach for uniqueness of complete maximal hypersurfaces in spatially parabolic GRW spacetimes, *J. Math. Anal. Appl.*, **419** (2014), 355–372.

[96] A. Romero, R.M. Rubio and J.J. Salamanca, Spacelike graphs of finite total curvature in certain 3-dimensional generalized Robertson-Walker spacetimes, *Reports on Math. Phys.*, **73** (2014), 241–254.

[97] A. Romero, R.M. Rubio and J.J. Salamanca, Complete maximal hypersurfaces in certain spatially open generalized Robertson-Walker spacetimes, *R.A.C.S.A.M.*, (2014), DOI 10.1007/s13398–014-0195–1

[98] A. Romero, R.M. Rubio and J.J. Salamanca, New examples of Moser-Bernstein problems for some nonlinear equations, *submitted*.

[99] A. Romero and M. Sánchez, On the completeness of certain families of semi-Riemannian manifolds, *Geom. Dedicata*, **53** (1994), 103–117.

[100] H. Royden, Harmonic functions on open Riemann surfaces, *Trans. Amer. Math. Soc.*, **73** (1952), 40–94.

[101] R.M. Rubio, Complete constant mean curvature spacelike hypersurfaces in the Einstein-De Sitter spacetime, *Rep. Math. Phys.*, **74** (2014), 127–133.

[102] R.K. Sachs and H. Wu, *General Relativity for Mathematicians*, Grad. Texts Math., **48** Springer-Verlag, New York, 1977.

[103] J.J. Salamanca and I.M.C. Salavessa, Uniqueness of ϕ-minimal hypersurfaces in warped product manifolds, *J. Math. Anal. Appl.*, DOI: 10.1016/j.jmaa.2014.09.028.

[104] M. Sánchez, On the geometry of generalized Robertson-Walker spacetimes: geodesics, *General Relativity and Gravitation*, **30** (1998), 915–932.

[105] M. Sánchez, On the geometry of generalized Robertson-Walker spacetimes: curvature and Killing fields, *J. Geom. Phys.*, **31** (1999), 1–15.

[106] R. Schoen, L. Simon and S.T. Yau, Curvature estimates for minimal hypersurfaces, *Acta Math.*, **134** (1975), 275–288.

[107] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math.*, **88** (1968) 62–105.

[108] N.Th. Varopoulos, Potential theory and diffusion of Riemannian manifolds, *(Chicago, Ill., 1981) Wadsworth Math. Ser., Wadsworth, Belmont, CA*, (1983), 821–837.

[109] J.A. Wolf, *Spaces of constant curvature*, Publish or Perish Press, Boston, 1974.

[110] Y.L. Xin, On the Gauss image of a spacelike hypersurface with constant mean curvature in Minkowski space, *Comment. Math. Helv.*, **66** (1991), 590–598.

[111] S.T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.*, **28** (1975), 201–228.