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Čebyšëv subspaces of JBW*-triples

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Abstract

We describe the one-dimensional Čebyšëv subspaces of a JBW*-triple M by showing that for a non-zero element x in M, $\mathbb{C}x$ is a Čebyšëv subspace of M if and only if x is a Brown-Pedersen quasi-invertible element in M. We study the Čebyšëv JBW*-subtriples of a JBW*-triple M. We prove that for each non-zero Čebyšëv JBW*-subtriple N of M, exactly one of the following statements holds:

(a) N is a rank-one JBW*-triple with $\dim(N) \geq 2$ (*i.e.*, a complex Hilbert space regarded as a type 1 Cartan factor). Moreover, N may be a closed subspace of arbitrary dimension and M may have arbitrary rank;
(b) $N = \mathbb{C}e$, where e is a complete tripotent in M;
(c) N and M have rank two, but N may have arbitrary dimension ≥ 2;
(d) N has rank greater than or equal to three, and $N = M$.

We also provide new examples of Čebyšëv subspaces of classic Banach spaces in connection with ternary rings of operators.

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1 Introduction

Let V be a subspace of a Banach space X. The subspace V is called a *Čebyšëv (Chebyshev) subspace* of X if and only if for each $x \in X$ there exists a unique point $x_0 \in V$ such that $\operatorname{dist}(x, V) = \|x - x_0\|$. The uniqueness of x_0 plays a key role in this paper (see, for example, Lemma 3 and Proposition 9).

Let K be a compact Hausdorff space. A classical theorem due to Haar establishes that an n-dimensional subspace V of the space $C(K)$, of all continuous complex-valued functions on K, is a Čebyšëv subspace of $C(K)$ if and only if any non-zero $f \in V$ admits at most $n - 1$ zeros (*cf.* [1] and the monograph [2], p.215). Having in mind the Riesz representation theorem and the characterization of the extreme points of the closed unit ball in the dual space of $C(K)$, we can easily see that, in the above conditions, V is an n-dimensional Čebyšëv subspace of $C(K)$ if and only if for every set $\{\delta_{t_1}, \ldots, \delta_{t_n}\}$ of n-mutually orthogonal pure states, we have $V \cap \bigcap_{i=1}^{n} \ker(\delta_{t_i}) = \{0\}$. This result implies that any non-zero f in $C(K)$ spans a Čebyšëv subspace of the latter space if and only if f is invertible in the algebra $C(K)$.

Later on, Stampfli proved in [3], Theorem 2, that the scalar multiples of the unit element in a von Neumann algebra M is a Čebyšëv subspace of M. In [4], Legg *et al.* characterize the semi-Čebyšëv and finite dimensional Čebyšëv subspaces of $K(H)$, the algebra of compact operators on an infinite-dimensional Hilbert space H. They conclude that, for a separable Hilbert space H, there exist Čebyšëv subspaces of every finite dimension in $K(H)$ [4], Theorem 3, when H is not separable, $K(H)$ has no finite-dimensional Čebyšëv subspaces [4], Corollary 2.

Robertson continued with the study on Čebyšëv subspaces of von Neumann algebras in [5], where he established the following results.

Theorem 1 ([5], Theorem 6) *Let x be a non-zero element in a von Neumann algebra M. Then the one-dimensional subspace $\mathbb{C}x$ is a Čebyšëv subspace of M if and only if there is a projection p in the center of M such that px is left invertible in pM and $(1-p)x$ is right invertible in $(1-p)M$.*

Theorem 2 ([5], Theorem 6) *Let N be a finite dimensional* **-subalgebra of an infinite dimensional von Neumann algebra M. Suppose that N has dimension* > 1. *Then N is not a Čebyšëv subspace of M.*

Robertson and Yost prove in [6], Corollary 1.4, that an infinite dimensional C*-algebra A admits a finite dimensional *-subalgebra B which is also a Čebyšëv in A if and only if A is unital and $B = \mathbb{C}1$.

The results proved by Robertson and Yost were complemented by Pedersen, who shows that if A is a C*-algebra without unit and B is a Čebyšëv C*-subalgebra of A, then $A = B$ (compare [7], Theorem 4).

The previous results of Robertson [5] and Pedersen [7], Theorem 2 (see also [8]) also prove the following which leads immediately to Theorem 1: for each non-zero element x in a von Neumann algebra M, the following statements are equivalent:

(a) $\mathbb{C}x$ is a Čebyšëv subspace of M;
(b) x is Brown-Pedersen quasi-invertible in M (see page 6 for the precise definition of this notion);
(c) For each pure state (*i.e.*, for each extreme point of the positive part of the closed unit ball of M^*) $\varphi \in M^*$, and for each unitary $u \in M$, we have $\varphi(x^*x) + \varphi(uxx^*u) > 0$.

A renewed interest in Čebyšëv subspaces of C*-algebras has led Namboodiri, Pramod and Vijayarajan to revisit and generalize the previous contributions of Robertson, Yost and Pedersen in [9].

On the other hand, C*-algebras can be regarded as elements in a strictly wider class of complex Banach spaces called JB*-triples (see Section 2 for the detailed definitions). Many geometric properties studied in the setting of C*-algebras have been also explored in the bigger class of JB*-triples. However, Čebyšëv subspaces and the theory of best approximations remains unexplored in the class of JB*-triples. In this note we present the first results about Čebyšëv subspaces and Čebyšëv subtriples in Jordan structures.

In Section 2 we prove that for a non-zero element x in a JBW*-triple M, $\mathbb{C}x$ is a Čebyšëv subspace of M if and only if x is a Brown-Pedersen quasi-invertible element in M (see Theorem 6). This theorem generalizes the result established by Robertson in Theorem 1 (*cf.* [5]), but it also adds a new perspective from an independent argument.

In Section 3 we establish a precise description of the JBW*-subtriples of a JBW*-triple M which are Čebyšëv subspaces in M. We should remark that in the setting of von Neumann algebras and C*-algebras, the scarcity of non-trivial Čebyšëv *-subalgebras is endorsed by Theorems 1 and 2 and [6, 7]. The first main difference in the setting of JB*-triples is the existence of Čebyšëv JB*-subtriples with arbitrary dimensions; complex Hilbert spaces and spin factors give a complete list of examples (compare Remark 7 and comments before it).

In our main result we give a complete description of all Čebyšëv JBW*-subtriples of an arbitrary JBW*-triple (see Theorem 14). We provide examples of infinite dimensional proper Čebyšëv JBW*-subtriples of JBW*-triples (see Remark 7). We apply the solution of the minimum covering sphere problem in the Euclidean space ℓ_2^m to present new examples of Čebyšëv subspaces of classical Banach spaces (*cf.* Remark 12) and to construct an example of a rank-one Hilbert space which is a Čebyšëv JBW*-subtriple of a rank-n JBW*-triple, where n is an arbitrary natural number (*cf.* Remark 13).

It should be remarked at this point that the techniques applied by Robertson, Yost [5, 6] and Pedersen [7] in the setting of von Neumann algebras do not make any sense in the wider setting of JBW*-triples. The techniques developed in this paper are completely independent and provide new arguments to understand the Čebyšëv von Neumann subalgebras of a von Neumann algebra (Corollary 15).

2 One-dimensional Čebyšëv subspaces of JBW*-triples

A complex Jordan triple system is a complex linear space E equipped with a triple product $\{x,y,z\}$ which is bilinear and symmetric in the external variables and conjugate linear in the middle one and satisfies the Jordan identity

$$L(x,y)\{a,b,c\} = \{L(x,y)a,b,c\} - \{a,L(y,x)b,c\} + \{a,b,L(x,y)c\} \tag{2.1}$$

for all $x,y,a,b,c \in E$, where $L(x,y) : E \to E$ is the linear mapping given by $L(x,y)z = \{x,y,z\}$.

A *JB*-triple* is a complex Jordan triple system E which is a Banach space satisfying the additional '*geometric*' axioms:

(a) For each $x \in E$, the operator $L(x,x)$ is hermitian with non-negative spectrum;

(b) $\|\{x,x,x\}\| = \|x\|^3$ for all $x \in E$.

Every C*-algebra is a JB*-triple with respect to the triple product given by

$$\{a,b,c\} = \frac{1}{2}\bigl(ab^*c + cb^*a\bigr). \tag{2.2}$$

Every JB*-algebra (*i.e.*, a complex Jordan Banach *-algebra with product denoted by $x \circ y$ satisfying

$$\bigl\|U_a\bigl(a^*\bigr)\bigr\| = \|a\|^3$$

for every element a, where $U_a(x) := 2(a \circ x) \circ a - a^2 \circ x$, *cf.* [10], Section 3.8) is a JB*-triple under the triple product defined by

$$\{x,y,z\} = \bigl(x \circ y^*\bigr) \circ z + \bigl(z \circ y^*\bigr) \circ x - (x \circ z) \circ y^*. \tag{2.3}$$

The space $B(H,K)$ of all bounded linear operators between complex Hilbert spaces, although rarely is a C*-algebra, is a JB*-triple with the product defined in (2.2). In particular, every complex Hilbert space is a JB*-triple.

Other examples of JB*-triples are given by the so-called *Cartan factors*. A Cartan factor of type 1 is a JB*-triple which coincides with the Banach space $B(H,K)$ of bounded linear operators between two complex Hilbert spaces, H and K, where the triple product is defined by (2.2). Cartan factors of types 2 and 3 are JB*-triples which can be identified with the subtriples of $B(H)$ defined by $II^{\mathbb{C}} = \{x \in B(H) : x = -jx^*j\}$ and $III^{\mathbb{C}} = \{x \in B(H) : x = jx^*j\}$, respectively, where j is a conjugation on H. A Cartan factor of type 4 is a spin factor, that is, a complex Hilbert space provided with a conjugation $x \mapsto \overline{x}$, where the triple product and the norm are defined by

$$\{x,y,z\} = \langle x/y\rangle z + \langle z/y\rangle x - \langle x/\bar{z}\rangle \bar{y},$$

and $\|x\|^2 = \langle x/x\rangle + \sqrt{\langle x/x\rangle^2 - |\langle x/\overline{x}\rangle|^2}$, respectively. The Cartan factors of types 5 and 6 consist of finite dimensional spaces of matrices over the eight-dimensional complex Cayley division algebra $\mathbb{O}$; the type VI is the space of all hermitian 3×3 matrices over $\mathbb{O}$, while the type V is the subtriple of 1×2 matrices with entries in $\mathbb{O}$ (compare [11, 12] and [13], Section 2.5).

A JB*-triple W is called a *JBW*-triple* if it has a predual W_*. It is known that a JBW*-triple admits a unique isometric predual, and its triple product is separately $\sigma(W, W_*)$-continuous (see [14]). The second dual E^{**} of a JB*-triple E is a JBW*-triple with respect to a triple product which extends the triple product of E (*cf.* [15]).

For more details of the properties of JB*-triples and JBW*-triples, the reader is referred to the monographs [13] and [16].

Given an element a in a JB*-triple E, the symbol $Q(a)$ will denote the conjugate linear operator on E defined by $Q(a)(x) = \{a,x,a\}$.

An element $e \in E$ is called *a tripotent* when $\{e,e,e\} = e$. Each tripotent $e \in E$ induces a decomposition of E, called *the Peirce decomposition*, in the form $E = E_2(e) \oplus E_1(e) \oplus E_0(e)$, where $E_i(e)$ is the $\frac{i}{2}$ eigenspace of the operator $L(e,e)$, $i = 0,1,2$. This decomposition satisfies the following *Peirce rules:*

$$\big\{E_2(e), E_0(e), E\big\} = \big\{E_0(e), E_2(e), E\big\} = 0$$

and

$$\big\{E_i(e), E_j(e), E_k(e)\big\} \subseteq E_{i-j+k}(e),$$

when $i - j + k \in \{0,1,2\}$ and is zero otherwise. The projection $P_k(e)$ of E onto $E_k(e)$ is called the *Peirce k-projection*. It is known that Peirce projections are contractive (*cf.* [17], Corollary 1.2) and satisfy

$$P_2(e) = Q(e)^2, \qquad P_1(e) = 2\big(L(e,e) - Q(e)^2\big),$$

and

$$P_0(e) = Id_E - 2L(e,e) + Q(e)^2.$$

The separate weak*-continuity of the triple product of a JBW*-triple M implies that Peirce projections associated with a tripotent e in M are weak*-continuous.

It is known that the Peirce-2 subspace $E_2(e)$ is a JB*-algebra with unit e, Jordan product $x \circ_e y := \{x, e, y\}$ and involution $x^{*_e} := \{e, x, e\}$, respectively. Since surjective linear isometries and triple isomorphisms on a JB*-triple coincide (*cf.* [18], Proposition 5.5), the triple product in $E_2(e)$ is uniquely given by

$$\{x, y, z\} = \left(x \circ_e y^{*_e}\right) \circ_e z + \left(z \circ_e y^{*_e}\right) \circ_e x - (x \circ_e z) \circ_e y^{*_e},$$

$x, y, z \in E_2(e)$.

We shall make use of the following property: given a tripotent $e \in E$ and an element λ in the unit sphere of $\mathbb{C}$, the mapping

$$S_\lambda(e) : E \to E, \qquad S_\lambda(e) = \lambda^2 P_2(e) + \lambda P_1(e) + P_0(e) \tag{2.4}$$

is a surjective linear isometry on E and a triple isomorphism (compare [17], Lemma 1.1).

A tripotent $e \in E$ is said to be *unitary* if the operator $L(e, e)$ coincides with the identity map I_E on E; that is, $E_2(e) = E$. We shall say that e is *complete* or *maximal* when $E_0(e) = E$. When $E_2(e) = P_2(e)(E) = \mathbb{C}e \neq \{0\}$, we say that e is *minimal*.

The complete tripotents of a JB*-triple E coincide with the real and complex extreme points of its closed unit ball E_1 (*cf.* [19], Lemma 4.1 and [20], Proposition 3.5 or [13], Theorem 3.2.3). Consequently, the Krein-Milman theorem assures that every JBW*-triple admits an abundant set of complete tripotents [13], Corollary 3.2.4.

Let a be an element in a JB*-triple E. It is known that the JB*-subtriple E_a generated by a identifies with some $C_0(L)$, where $\|a\| \in L \subseteq [0, \|a\|]$ with $L \cup \{0\}$ compact (*cf.* [18], Corollary 1.15). Moreover, there exists a triple isomorphism $\Psi : E_a \to C_0(L)$ such that $\Psi(a)(t) = t$.

When a is an element in a JBW*-triple M, the sequence $(a^{\frac{1}{2n-1}})$ converges in the weak*-topology of M to a tripotent, denoted by $r(a)$, called the *range tripotent of a*. The tripotent $r(a)$ is the smallest tripotent $e \in M$ satisfying that a is positive in the JBW*-algebra $M_2(e)$ (see [21], p.322). Clearly, the range tripotent $r(a)$ can be identified with the characteristic function $\chi_{(0,\|a\|]\cap L} \in C_0(L)^{**}$ (see [22], beginning of Section 2).

We recall that an element x in a Jordan algebra $\mathcal{J}$ with unit e is called *invertible* if there exists an element y such that $x \circ y = e$ and $x^2 \circ y = x$. The element y is called *the inverse* of x and is denoted by x^{-1}. The inverse of any element x in a Jordan algebra $\mathcal{J}$ is unique whenever it exists. The set of all invertible elements in $\mathcal{J}$ is denoted by $\mathcal{J}^{-1}$.

An element a in a JB*-triple E is called *von Neumann regular* if and only if there exists $b \in E$ such that

$$Q(a)(b) = a, \qquad Q(b)(a) = b, \quad \text{and} \quad \left[Q(a), Q(b)\right] := Q(a)Q(b) - Q(b)Q(a) = 0.$$

When a is von Neumann regular, the (unique) element $b \in E$ satisfying the above conditions is called *the generalized inverse of a* and is denoted by $a^\dagger$. It is known that an element $a \in E$ is von Neumann regular if and only if $Q(a)$ has norm-closed image if and only if the range tripotent $r(a)$ of a lies in E and a is a positive and invertible element of the JB*-algebra $E_2(r(a))$ (compare [23]). Furthermore, when a is von Neumann regular, $Q(a)Q(a^\dagger) = Q(a^\dagger)Q(a) = P_2(r(a))$ and $L(a, a^\dagger) = L(a^\dagger, a) = L(r(a), r(a))$ [23], p.192.

Given a pair of elements a, b in a JB*-triple E, *the Bergmann operator* associated to a and b is the mapping $B(a,b): E \to L(E)$ defined by $B(a,b) = Id_E - 2L(a,b) + Q(a)Q(b)$ (*cf.* [13], p.22).

An element a in a JB*-triple E is said to be *Brown-Pedersen quasi-invertible* (*BP-quasi-invertible* for short) when it is von Neumann regular with generalized inverse b such that the Bergmann operator $B(a,b)$ vanishes; in such a case, b is called *the BP-quasi-inverse* of a. The set of BP-quasi-invertible elements in E is denoted by E_q^{-1} (see [24]). It is established in [24] that an element $a \in E$ is BP-quasi-invertible if and only if one of the following equivalent statements holds:

(i) a is von Neumann regular, and its range tripotent $r(a)$ is an extreme point of the closed unit ball E_1 of E (*i.e.*, $r(a)$ is a complete tripotent of E);
(ii) There exists a complete tripotent $e \in E$ such that a is positive and invertible in the JB*-algebra $E_2(e)$.

We recall that two elements a, b in a JB*-triple E are said to be *orthogonal* (written $a \perp b$) if $L(a,b) = 0$. Lemma 1 in [25] shows that $a \perp b$ if and only if one of the following nine statements holds:

$$\begin{aligned} &\{a,a,b\} = 0; \qquad a \perp r(b); \qquad r(a) \perp r(b); \\ &E_2^{**}\bigl(r(a)\bigr) \perp E_2^{**}\bigl(r(b)\bigr); \qquad r(a) \in E_0^{**}\bigl(r(b)\bigr); \qquad a \in E_0^{**}\bigl(r(b)\bigr); \\ &b \in E_0^{**}\bigl(r(a)\bigr); \qquad E_a \perp E_b; \qquad \{b,b,a\} = 0. \end{aligned} \tag{2.5}$$

Let e be a tripotent in a JB*-triple E. Lemma 1.3(a) in [17] shows that

$$\|x_2 + x_0\| = \max\bigl\{\|x_2\|, \|x_0\|\bigr\}$$

for every $x_2 \in E_2(e)$ and every $x_0 \in E_0(e)$. Combining this result with the equivalences in (2.5), we see that

$$\|a + b\| = \max\bigl\{\|a\|, \|b\|\bigr\}, \tag{2.6}$$

whenever a and b are orthogonal elements in a JB*-triple.

Given a subset $M \subseteq E$, we write $M_E^{\perp}$ (or simply $M^{\perp}$) for the (orthogonal) annihilator of M defined by $M_E^{\perp} = \{y \in E : y \perp x, \forall x \in M\}$. If $e \in E$ is a tripotent, then $\{e\}^{\perp} = E_0(e)$ and $\{a\}^{\perp} = (E^{**})_0(r(a)) \cap E$ for every $a \in E$ (*cf.* [26], Lemma 3.2).

Lemma 3 *Let V be a non-zero Čebyšëv subspace of a JBW*-triple M. Then $V \cap M_q^{-1} \neq \emptyset$, where M_q^{-1} denotes the set of BP-quasi-invertible elements of M.*

Proof Arguing by contradiction, we suppose that $V \cap M_q^{-1} = \emptyset$.

Let us take $x \in V$ with $\|x\| = 1$. By assumptions, $x \notin M_q^{-1}$. By [27], Lemma 3.12, there exists a complete tripotent e in M such that $r(x) \leq e$, where $r(x)$ denotes the range tripotent of x.

We shall identify the JB*-subtriple M_x of M generated by x with some $C_0(L)$, where $1 = \|x\| \in L \subseteq [0, \|x\|]$ with $L \cup \{0\}$ compact (*cf.* [18], Corollary 1.15). We further know that there exists a triple isomorphism $\Psi : M_x \to C_0(L)$ such that $\Psi(x)(t) = t$, and the range tripotent $r(x)$ identifies with the characteristic function $\chi_{(0,\|x\|]\cap L} \in C_0(L)^{**}$ (see page 2). It is clear that, under this identification,

$$\left\| r(x) - \lambda x \right\| \leq 1 \quad \text{if } \Re\mathrm{e}(\lambda) \geq \frac{1}{2} \text{ and } |\lambda| = 1.$$

If $e = r(x)$, since the element x is not invertible in the JBW*-algebra $M_2(r(x))$, 0 lies in the closure of L, and hence $\|e - \lambda x\| = \|r(x) - \lambda x\| = 1$ for every $\lambda \in \mathbb{C}$ with $\Re\mathrm{e}(\lambda) \geq \frac{1}{2}$ and $|\lambda| = 1$.

When $e \gneqq r(x)$, we have $\|e - r(x)\| = 1$. Thus, applying $e - r(x) \perp r(x)$ and (2.6), we further know that for $\Re\mathrm{e}(\lambda) \geq \frac{1}{2}$ and $|\lambda| = 1$,

$$\|e - \lambda x\| = \left\| e - r(x) + r(x) - \lambda x \right\| = \max\left\{ \left\| e - r(x) \right\|, \left\| r(x) - \lambda x \right\| \right\} = 1.$$

We observe that, since e is a complete tripotent, $e \in M_q^{-1}$, and hence $e \notin V$. Since V is a Čebyšëv subspace, there exists a unique best approximation $c_V(e) \in V$ of e in V satisfying $\mathrm{dist}(e, V) = \|e - c_V(e)\| > 0$.

If $\mathrm{dist}(e, V) = \|e - c_V(e)\| \geq 1$, we would have $1 = \|e\| \geq \mathrm{dist}(e, V) = 1$, and

$$1 = \left\| e - c_V(e) \right\| = \mathrm{dist}(e, V) = \|e - \lambda x\|$$

for at least two values of λ, contradicting the uniqueness of the best approximation of e in V. We can therefore assume that $\mathrm{dist}(e, V) < 1$. Consequently, there exits $y \in V$ with $\|e - y\| < 1$. Corollary 2.4 in [28] implies that $y \in M_q^{-1} \cap V$, which is impossible. □

Let e be a tripotent in a JB*-triple E. Let us recall that e is a tripotent in the JBW*-triple E^{**}, and that Peirce projections associated with e on E^{**} are weak*-continuous. Goldstine's theorem assures that E is weak*-dense in E^{**}, and hence $E_k^{**}(e)$ coincides with the weak*-closure of $E_k(e)$ in E^{**} for every $k = 0, 1, 2$. In particular, e is complete in E^{**} whenever e is a complete tripotent in E. Moreover, since the orthogonal complement of a tripotent e in a JB*-triple F coincides with $F_0(e)$, we have the following.

Lemma 4 *Let e be a complete tripotent in a JB*-triple E. Then* $\{e\}_{E^{**}}^{\perp} = \{0\}$, *that is, e is not orthogonal to any non-zero element in* E^{**}.

The following technical result is part of the folklore in the theory of best approximation (see [5], Lemma 3 or [2], Theorem 2.1).

Lemma 5 ([5], Lemma 3) *Let x be an element in a complex Banach space X such that* $\mathbb{C}x$ *is not a Čebyšëv subspace of X. Then there exists an extreme point* ϕ *of the closed unit ball of* X^*, *a vector* $y \in X$ *and a scalar* $\lambda \in \mathbb{C}\backslash\{0\}$ *such that*

(a) $\phi(x) = 0$;

(b) $\phi(y) = \|y\| = \|y - \lambda x\|$.

We can characterize now the one-dimensional Čebyšëv subspaces of a JBW*-triple.

Theorem 6 *Let x be a non-zero element in a JBW*-triple M. The following statements are equivalent*:

(a) *$\mathbb{C}x$ is a Čebyšëv subspace of M*;
(b) *x is a Brown-Pedersen quasi-invertible element in M*.

Proof The implication (a) $\Rightarrow$ (b) follows from Lemma 3.

(b) $\Rightarrow$ (a) Suppose that x is BP-quasi-invertible in M. We note that the support tripotent $r(x)$ of x is complete in M, and hence a complete tripotent in M^{**} (*cf.* Lemma 4 and comments before it).

Suppose that $\mathbb{C}x$ is not a Čebyšëv subspace of M. By Lemma 5 there exists an extreme point ϕ of the closed unit ball of M^*, $\lambda \in \mathbb{C}\backslash\{0\}$, and $y \in M$ such that $\phi(x) = 0$ and $\phi(y) = \|y\| = \|y - \lambda x\|$.

The support tripotent $\upsilon = s(\phi)$ of ϕ in M^{**} is a (non-zero) minimal tripotent in M^{**} satisfying $\phi = P_2(\upsilon)^*\phi = \phi P_2(\upsilon)$ and $\phi(z)\upsilon = P_2(\upsilon)(z)$, $\forall z \in M^{**}$ (*cf.* [17], Proposition 4). Therefore, $P_2(\upsilon)(x) = \phi(x)\upsilon = 0$.

We may suppose that $\|y\| = 1$. Since $P_2(\upsilon)(y) = \phi(y)\upsilon = \upsilon$, Lemma 1.6 in [17] implies that $P_1(\upsilon)(y) = 0$, which shows that $y = \upsilon + P_0(\upsilon)y$. We similarly get $P_1(\upsilon)(y - \lambda x) = 0$ (we simply observe that $\phi(y - \lambda x) = \|y\| = \|y - \lambda x\| = 1$). Therefore, $P_1(\upsilon)(x) = 0$, and $x = P_0(\upsilon)x \in (M^{**})_0(\upsilon) = ((M^{**})_2(\upsilon))^{\perp}$, implying that $x \perp \upsilon$. The equivalent statements in (2.5) prove that $r(x) \perp \upsilon$, which contradicts Lemma 4. □

The above Theorem 6 generalizes the previously commented results obtained by Robertson in [5] (compare Theorem 1). We have been unable to find a triple version of the reformulation established by Pedersen in [7], Theorem 2, stated as statement (c) on page 2. However, we do have a partial result in that direction.

For each functional φ in the predual of a JBW*-triple W, and for each z in W with $\varphi(z) = \|\varphi\|$ and $\|z\| = 1$, the mapping $x \mapsto \|x\|_\varphi := (\varphi\{x, x, z\})^{1/2}$ defines a pre-Hilbertian semi-norm on W. Moreover, $\varphi\{x, x, w\} = \varphi\{x, x, z\}$ whenever $w \in W$ with $\varphi(w) = \|\varphi\|$ and $\|w\| = 1$ (*cf.* [29], Proposition 1.2). It is known that

$$\big|\varphi(x)\big| \leq \|x\|_\varphi \tag{2.7}$$

for every $x \in W$ (see [30], p.258).

The inequality in (2.7) together with Lemma 5 imply the following property: Let x be a non-zero element in a JBW*-triple M such that $\mathbb{C}x$ is a Čebyšëv subspace of M. Then, for each extreme point φ of the closed unit ball of M^*, we have $\|x\|_\varphi \gneqq 0$. It would be interesting to know under what additional hypothesis the condition $\|x\|_\varphi \gneqq 0$ for every extreme point φ of the closed unit ball of M^* implies that x is BP-quasi-invertible.

3 Čebyšëv subtriples of JBW*-triples

In this section, we shall determine the JBW*-subtriples of a JBW*-triple M which are Čebyšëv subspaces in M. The scarcity of non-trivial Čebyšëv C*-subalgebras in general C*-algebras can be better understood with the following result due to Pedersen: If A is a C*-algebra without unit and B is a Čebyšëv C*-subalgebra of A, then $A = B$ (compare [7], Theorem 4).

The first main difference in the setting of JB*-triples is the existence of Čebyšëv JB*-subtriples with arbitrary dimensions. For example, let $E = H$ be a complex Hilbert space regarded as a type 1 Cartan factor with the Hilbert norm and the product

$$\{x,y,z\} = \frac{1}{2}\big(\langle x,y\rangle z + \langle z,y\rangle x\big), \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of H. It is known that elements in the unit sphere of a complex Hilbert H space regarded as a type 1 Cartan factor are precisely the complete tripotents of H. The *orthogonal projection theorem* tells that any closed subspace of H is a Čebyšëv subspace of H and clearly a JB*-subtriple.

The following remark provides an additional example.

Remark 7 Let E be a spin factor with triple product and norm, equivalent to the Hilbert norm, given by

$$\{x,y,z\} = \langle x/y\rangle z + \langle z/y\rangle x - \langle x/\bar{z}\rangle \bar{y},$$

and $\|x\|^2 = \langle x/x\rangle + \sqrt{\langle x/x\rangle^2 - |\langle x/\overline{x}\rangle|^2}$, respectively, where $x \mapsto \overline{x}$ is a conjugation on E, and $\langle\cdot/\cdot\rangle$ denotes the inner product of E. Let K be a closed subspace of E with $\overline{K} = K$. Clearly, K is a JB*-subtriple of E. Since K is a closed subspace of the complex Hilbert space E, there exists an orthogonal projection P of E onto K and $E = K \oplus H$, where $H = (I - P)(E)$ with $\langle K/H\rangle = 0$. Since $\overline{K} = K$, we also have $\overline{H} = H$. Given $\eta \in K$ and $\xi \in H$, since $|\langle \xi/\overline{\xi}\rangle| \leq \langle \xi/\xi\rangle$, it is easy to check that

$$\begin{aligned}
\|\eta + \xi\|^2 &= \langle \eta + \xi/\eta + \xi\rangle + \sqrt{\langle \eta + \xi/\eta + \xi\rangle^2 - \big|\langle \eta + \xi/\overline{\eta} + \overline{\xi}\rangle\big|^2} \\
&= \langle \eta/\eta\rangle + \langle \xi/\xi\rangle + \sqrt{\langle \eta/\eta\rangle^2 - \big|\langle \eta/\overline{\eta}\rangle\big|^2 + \langle \xi/\xi\rangle^2 - \big|\langle \xi/\overline{\xi}\rangle\big|^2} \\
&\geq \langle \eta/\eta\rangle + \sqrt{\langle \eta/\eta\rangle^2 - \big|\langle \eta/\overline{\eta}\rangle\big|^2} = \|\eta\|^2.
\end{aligned}$$

Moreover, $\|\eta + \xi\| = \|\eta\|$ if and only if $\xi = 0$. We also have $\|\eta + \xi\| \geq \|\xi\|$, and $\|\eta + \xi\| = \|\xi\|$ if and only if $\eta = 0$. Thus, if $x = \eta + \xi$, $\operatorname{dist}(x,K) = \inf_{\eta'\in K} \|\eta + \xi - \eta'\| \geq \|\xi\| = \|x - P(x)\|$, showing that $P(x)$ is a best approximation to x. Moreover, if for some $\eta' \in K$, $\|\xi\| = \|x - P(x)\| = \|x - \eta'\| = \|(\eta - \eta') + \xi\|$, then $\eta' = \eta = P(x)$. Therefore, K is a Čebyšëv JB*-subtriple of E. We observe that the dimensions of E and K can be arbitrarily big.

We can present now our conclusions on Čebyšëv JB*-subtriples.

The next property of Čebyšëv subspaces is probably part of the folklore in the theory of best approximation in normed spaces, but we could not find an exact reference.

Lemma 8 *Let V be a Čebyšëv subspace of a normed space X. For each $x \in X$, we denote by $c_V(x)$ the unique element in V satisfying $\|x - c_V(x)\| = \operatorname{dist}(x, V)$. Let $P : X \to X$ be a contractive projection such that $P(V) \subseteq V$. Then*

$$P\big(c_V\big(P(x)\big)\big) = c_V\big(P(x)\big)$$

for every $x \in X$. Furthermore, $P(V)$ is a Čebyšëv subspace of the normed space $P(X)$, and for each $x \in X$, $c_{P(V)}(P(x)) = P(c_V(x))$.

Proof Let x be an element in X. The condition $\|P\| \leq 1$ implies that

$$\big\|P(x) - P\big(c_V\big(P(x)\big)\big)\big\| \leq \big\|P(x) - c_V\big(P(x)\big)\big\| = \operatorname{dist}\big(P(x), V\big).$$

The element $P(c_V(P(x))) \in P(V) \subseteq V$. Thus, the uniqueness of the best approximation in V proves that $P(c_V(P(x))) = c_V(P(x))$. The rest is clear. □

Proposition 9 *Let F be a Čebyšëv JB***-subtriple of a JB***-triple E. Suppose that e is a non-zero tripotent in F. Then $E_0(e) = F_0(e)$. Consequently, every complete tripotent in F is complete in E.*

Proof Since e is a tripotent in F and the latter is a JB*-subtriple of E, e is a tripotent in E and $F_0(e) \subseteq E_0(e)$. Arguing by contradiction, let us assume that there exists $b \in E_0(e)\backslash F_0(e) = E_0(e)\backslash F \neq \emptyset$. Since $\operatorname{dist}(b,F) > 0$ and F is a Čebyšëv subspace, there exists a unique $c_F(b) \in F$ such that $\|b - c_F(b)\| = \operatorname{dist}(b,F)$.

Since $P_0(e)(F) \subseteq F$ and $P_0(e)(b) = b$, Lemma 8 implies that

$$P_0(e)\big(c_F(b)\big) = c_F(b) \in F_0(e).$$

Having in mind that $e \in E_2(e) \perp E_0(e) \ni b - c_F(b)$, we deduce, via (2.6), that

$$\big\|b - c_F(b) - \lambda e\big\| = \max\big\{\big\|b - c_F(b)\big\|, |\lambda|\big\} = \big\|b - c_F(b)\big\| = \operatorname{dist}(b,F)$$

for every $|\lambda| \leq \operatorname{dist}(b,F)$. This contradicts the uniqueness of the best approximation $c_F(b)$ of b in F because $c_F(b) + \lambda e \in F$ for every $|\lambda| \leq \operatorname{dist}(b,F)$. □

Proposition 10 *Let F be a Čebyšëv JB***-subtriple of a JB***-triple E. Suppose that e is a tripotent in F with $F_0(e) = \{e\}_F^{\perp} \neq 0$. Then $E_2(e) = F_2(e)$.*

Proof Clearly $F_j(e) \subseteq E_j(e)$ for $j = 0,1,2$. We have to show that $E_2(e) \subseteq F_2(e)$. Suppose, on the contrary, that $E_2(e)\backslash F_2(e) = E_2(e)\backslash F \neq \emptyset$. Pick $b \in E_2(e)\backslash F$. Since F is a Čebyšëv subspace of E, there exists a unique $c_F(b) \in F$ satisfying $\|b - c_F(b)\| = \operatorname{dist}(b,F) > 0$.

By Lemma 8 applied to $P = P_2(e)$, $X = E$ and $V = F$, we deduce that $P_2(e)(c_F(b)) = c_F(b)$.

By hypothesis, $F_0(e) = \{e\}_F^{\perp} \neq 0$. So, there exists a norm-one element $z \in F_0(e)$. The conditions $b \in E_2(e)$, $c_F(b) \in F_2(e)$ and $z \in F_0(e)$ combined with (2.6) give

$$\big\|b - c_F(b) - \lambda z\big\| = \max\big\{\big\|b - c_F(b)\big\|, |\lambda|\big\} = \big\|b - c_F(b)\big\| = \operatorname{dist}(b,F)$$

for every $|\lambda| \leq \operatorname{dist}(b,F)$, which contradicts the uniqueness of the best approximation of b in F because $c_F(b) - \lambda z \in F$ for every λ. □

Let e and v be tripotents in a JB*-triple E. We shall say that $v \leq e$, when $e - v$ is a tripotent in E with $e - v \perp v$ (compare the notation in [17]).

Let E be a JB*-triple. A subset $S \subseteq E$ is said to be *orthogonal* if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. The minimal cardinal number r satisfying $\operatorname{card}(S) \leq r$ for every orthogonal subset $S \subseteq E$ is called the *rank* of E (and will be denoted by $r(E)$). Given a tripotent $e \in E$, the rank of the Peirce-2 subspace $E_2(e)$ will be called the rank of e.

Theorem 3.1 in [31] combined with Proposition 4.5(iii) in [32] assures that a JB*-triple is reflexive if and only if it is isomorphic to a Hilbert space if and only if it has finite rank.

Suppose that E is a rank-one JB*-triple. The above comments show that E is reflexive and hence a JBW*-triple. Let e be a complete tripotent in E. Since the rank of e is smaller than the rank of E, we deduce that e is a minimal tripotent in E. Proposition 3.7 in [26] and its proof show that $E = \{e\}^{\perp\perp} = \{0\}^{\perp}$ is a rank-one Cartan factor of the form $L(H,\mathbb{C})$, where H is a complex Hilbert space or a type 2 Cartan factor II_3 (it is known that II_3 is JB*-triple isomorphic to a three-dimensional complex Hilbert space). We have proved the following.

Lemma 11 *Every JB*-triple of rank one is JB*-isomorphic* (*and hence isometric*) *to a complex Hilbert space regarded as a type* 1 *Cartan factor.* □

The above result is also stated in [33], Corollary in p.308.

We have already commented that orthogonal elements are M-orthogonal in the sense of the geometric theory of Banach spaces (see (2.6)). We shall state next other results of a geometric nature. Let u and v be two non-zero tripotents in a JB*-triple E. We recall that u and v are *colinear* (written $u \top v$) when $u \in E_1(v)$ and $v \in E_1(u)$ (*cf.* [33], p.296). Suppose $u \top v$ in E. Clearly, the JB*-subtriple $E_{u,v}$ of E generated by u and v is algebraically isomorphic to $\mathbb{C}u \oplus \mathbb{C}v$. We observe that u and v are minimal colinear tripotents in $E_{u,v}$. It follows from [17], Proposition 5, that $E_{u,v}$ is JB*-triple isomorphic and hence isometric to $M_{1,2}(\mathbb{C})$ (regarded as a type 1 Cartan factor). We, consequently, have

$$\|\lambda u + \mu v\| = \left(|\lambda|^2 + |\mu|^2\right)^{\frac{1}{2}} \tag{3.2}$$

for every $\lambda, \mu \in \mathbb{C}$. It should be also noted here that, in a Hilbert space F regarded as a type 1 Cartan factor with product given in (3.1), the tripotents in F are precisely the elements in its unit sphere, and the relation of being Hilbert-orthogonal is exactly the relation of colinearity in terms of the triple product.

We have shown several examples of Hilbert spaces (regarded as a type 1 Cartan factor) which are Čebyšëv JB*-subtriples of JB*-triples of rank one and two. We present next more examples of Hilbert spaces which are Čebyšëv JB*-subtriples of JB*-triples having a bigger rank. The first example is a construction with classical Banach spaces and the second one is an isometric translation to the setting of JB*-triples.

Remark 12 Let H be complex Hilbert space of dimension two with norm denoted by $\|\cdot\|_2$. We consider the Banach space $X = \overbrace{H \oplus^{\ell_\infty} \cdots \oplus^{\ell_\infty} H}^{(n)}$ ($n \geq 2$). Let $\{\xi_1, \xi_2\}$ be an orthonormal basis of H. Each $h \in H$ writes uniquely in the form $h = \lambda_1\xi_1 + \lambda_2\xi_2$. Let V denote the two-dimensional subspace of X generated by the vectors $e_1 = (\xi_1, \ldots, \xi_1)$ and $e_2 = (\xi_2, \ldots, \xi_2)$. That is, every vector in V is of the form $\lambda e_1 + \mu e_2$. Clearly,

$$\begin{aligned}\|\lambda e_1 + \mu e_2\| &= \left\|\lambda(\xi_1, \ldots, \xi_1) + \mu(\xi_2, \ldots, \xi_2)\right\|_2 \\ &= \max_{i=1,\ldots,n} \|\lambda\xi_1 + \mu\xi_2\|_2 = \sqrt{|\lambda|^2 + |\mu|^2},\end{aligned}$$

and hence V is isometrically isomorphic to a Hilbert space.

We claim that V is a Čebyšëv subspace of X. Indeed, let $x = (h_1, \ldots, h_n)$ be an element in X and let $\lambda e_1 + \mu e_2 \in V$. We write $h_i = \lambda_1^i \xi_1 + \lambda_2^i \xi_2$. We write the formula for the distance from x to V in the form:

$$\begin{aligned}
\operatorname{dist}(x, V)^2 &= \inf_{\lambda,\mu\in\mathbb{C}} \big\| (h_1, \ldots, h_n) - \lambda e_1 - \mu e_2 \big\|^2 \\
&= \inf_{\lambda,\mu\in\mathbb{C}} \max_{i=1,\ldots,n} \big\| \lambda_1^i \xi_1 + \lambda_2^i \xi_2 - \lambda \xi_1 - \mu \xi_2 \big\|_2^2 \\
&= \inf_{\lambda,\mu\in\mathbb{C}} \max_{i=1,\ldots,n} \left(\left|\lambda_1^i - \lambda\right|^2 + \left|\lambda_2^i - \mu\right|^2 \right)^{\frac{1}{2}} \\
&= \inf_{\lambda,\mu\in\mathbb{C}} \max_{i=1,\ldots,n} \operatorname{dist}_{\mathbb{C}^2}\left(\left(\lambda_1^i, \lambda_2^i\right), (\lambda, \mu) \right).
\end{aligned}$$

Our problem is equivalent to determining a point $(\lambda, \mu) \in \mathbb{C}^2$ so that the maximum Euclidean distance from (λ, μ) to the points $(\lambda_1^i, \lambda_2^i) \in \mathbb{C}^2$ $(i = 1, \ldots, n)$ is minimized, where $\mathbb{C}^2$ is equipped with the Euclidean distance $\|(\lambda, \mu)\|_2 = \sqrt{|\lambda|^2 + |\mu|^2}$. This problem is commonly called '*the Euclidean delivery problem*' or '*the min-max location problem*' or '*the minimum covering sphere problem*'. It is known that an equivalent reformulation of the problem is

$$\operatorname{Min}\left\{ \rho : (\lambda, \mu) \in \mathbb{C}^2, \rho > 0, \left\| \left(\lambda_1^i, \lambda_2^i\right) - (\lambda, \mu) \right\|_2 \leq \rho, \forall i \right\}.$$

The goal is to find the circle of center $(\lambda, \mu) \in \mathbb{C}^2$ of smallest radius ρ that encloses all the points $(\lambda_1^i, \lambda_2^i) \in \mathbb{C}^2$ $(i = 1, \ldots, n)$.

It is well known that a solution to the minimum covering sphere problem always exists, the center (λ, μ) and the radius ρ are unique (*cf.* [34, 35]). This shows that every element $x = (\lambda_1^1 \xi_1 + \lambda_2^1 \xi_2, \ldots, \lambda_1^n \xi_1 + \lambda_2^n \xi_2)$ in X admits a unique best approximation in V, which proves the claim.

Remark 13 Let e and u be two colinear complete tripotents in a JB*-triple E. Let us assume that we can find two sets $\{e_1, \ldots, e_n\}$ and $\{u_1, \ldots, u_n\}$ of mutually orthogonal tripotents in $E_2(e)$ and $E_2(u)$, respectively, such that $e_i \top u_i$ for all i and $u_i \perp e_j$ for every $i \neq j$. Take, for example, $E = M_{n\times(2n)}(\mathbb{C})$, $e = \sum_{i=1}^n w_{i,i}$, $u = \sum_{i=1}^n w_{i,i+n}$, $e_i = w_{i,i}$ and $u_i = e = w_{i,i+n}$, where $w_{i,j}$ is the matrix with entry 1 at the position i, j and zero elsewhere.

Let F be the JB*-subtriple of E generated by $\{e_1, \ldots, e_n, u_1, \ldots, u_n\}$, and let W be the closed JB*-subtriple of F generated by $\{e, u\}$. For each $i \in \{1, \ldots, n\}$, $e_i \top u_i$ and hence

$$\|\lambda_i e_i + \mu_i u_i\| = \sqrt{|\lambda_i|^2 + |\mu_i|^2},$$

that is, the subtriple F_i generated by e_i and u_i is a two-dimensional complex Hilbert space (*cf.* (3.2)). Since, for each $i \neq j$, $\{e_i, u_i\} \perp \{e_j, u_j\}$, that is, $F_i \perp F_j$, we deduce from (2.6) that $\|x_i + x_j\| = \max\{\|x_i\|, \|x_j\|\}$ for every $x_i \in F_i$, $x_j \in F_j$, $i \neq j$. Having in mind that $F = F_1 \oplus^{\ell_\infty} \cdots \oplus^{\ell_\infty} F_n$ and $F_i \equiv \ell_2^2$, we can easily see that F is isometrically isomorphic to the space X in Remark 12. It is also easy to see that under the natural isometric identification of F and X in Remark 12, the JB*-subtriple W is identified with the subspace V in that remark. Therefore, it follows that W is a Čebyšëv JB*-subtriple of F. The JB*-triple F has been constructed to have rank n.

The theorem describing the Čebyšëv JBW*-subtriples of a JBW*-triple can be stated now. We shall show that the examples given in Remarks 7 and 13 are essentially the unique examples of non-trivial Čebyšëv JBW*-subtriples.

Theorem 14 *Let N be a non-zero Čebyšëv JBW*-subtriple of a JBW*-triple M. Then exactly one of the following statements holds*:

(a) *N is a rank-one JBW*-triple with* $\dim(N) \geq 2$ (*i.e., a complex Hilbert space regarded as a type* 1 *Cartan factor*). *Moreover, N may be a closed subspace of arbitrary dimension and M may have arbitrary rank*;
(b) $N = \mathbb{C}e$, *where e is a complete tripotent in M*;
(c) *N and M have rank two, but N may have arbitrary dimension* ≥ 2;
(d) *N has rank greater than or equal to three, and $N = M$.*

Proof We can always find a complete tripotent e in N (see the comments on page 5). Proposition 9 implies that e is complete in M (*i.e.*, $M_0(e) = \{0\}$). We have three possibilities:

(i) e has rank one in N;
(ii) e has rank two in N;
(iii) e has rank greater than or equal to three in N.

(i) Suppose first that e has rank one in N. In this case, e is a minimal and complete tripotent in N and a complete tripotent in M. Therefore, N is a complex Hilbert space regarded as a type 1 Cartan factor (*cf.* Lemma 11 or Proposition 3.7 in [26]). If $\dim N = 1$, then (b) holds. If $\dim N \geq 2$, (a) holds.

In the latter case, the examples given before Remark 7 and in Remark 13 show that N may have arbitrary dimension and M may have rank as big as desired.

(ii) We assume now that e has rank two in N. Then there exist two non-zero minimal, mutually orthogonal tripotents $e_1, e_2 \in N$ with $e = e_1 + e_2$. Propositions 9 and 10 show that $M_2(e_j) = N_2(e_j)$, and $M_0(e_j) = N_0(e_j) \neq \{0\}$ for every j in $\{1, 2\}$. Since $M_2(e_j) = N_2(e_j) = \mathbb{C}e_j$, we deduce that e_1 and e_2 are minimal tripotents in M. We also know that $e = e_1 + e_2$ is a complete tripotent in M (*i.e.*, $M = M_2(e) \oplus M_1(e)$), which proves that M has rank two. The statement concerning the dimension of N follows from the example in Remark 7. Thus (c) holds.

(iii) Suppose now that e has rank greater than or equal to three in N. We shall show that $M = N$. Under the present assumptions, we can find three non-zero mutually orthogonal tripotents e_1, e_2, e_3 with $e_1 + e_2 + e_3 = e$. Clearly, $N_0(e_j + e_k) \neq \{0\}$ for every $k \neq j$ in $\{1, 2, 3\}$. Propositions 9 and 10 assure that $M_2(e_j + e_k) = N_2(e_j + e_k)$, $M_0(e_j + e_k) = N_0(e_j + e_k)$, $M_2(e_j) = N_2(e_j)$, and $M_0(e_j) = N_0(e_j)$ for every $k \neq j$ in $\{1, 2, 3\}$. In the Peirce decomposition

$$M = M_2(e_1) \oplus M_1(e_1) \oplus M_0(e_1),$$

we have $M_2(e_1) = N_2(e_1)$ and $M_0(e_1) = N_0(e_1)$. We shall show that $M_1(e_1) \subseteq N$.

Pick $x \in M_1(e_1)$. Since $e_1 \perp e_j$ ($j = 2, 3$), we have $M_1(e_1) \cap M_2(e_j) = \{0\}$ for $j = 2, 3$. Therefore,

$$x = P_1(e_2)(x) + P_0(e_2)(x),$$

where $P_0(e_2)(x) \in M_0(e_2) = N_0(e_2) \subseteq N$.

We next show that $P_1(e_2)(x) \in N$. Since

$$\frac{1}{2}P_0(e_2)(x) + \frac{1}{2}P_1(e_2)(x) = \frac{1}{2}x = \{e_1, e_1, x\}$$
$$= \{e_1, e_1, P_0(e_2)(x)\} + \{e_1, e_1, P_1(e_2)(x)\},$$

it follows from Peirce rules that

$$\frac{1}{2}P_1(e_2)(x) = \{e_1, e_1, P_1(e_2)(x)\},$$

and hence $P_1(e_2)(x) \in M_1(e_1) \cap M_1(e_2)$. The condition $e_1 \perp e_2$ leads us to $\{e_1 + e_2, e_1 + e_2, P_1(e_2)(x)\} = P_1(e_2)(x)$, which means that

$$P_1(e_2)(x) \in M_2(e_1 + e_2) = N_2(e_1 + e_2) \subseteq N.$$

We have therefore shown that $x = P_1(e_2)(x) + P_0(e_2)(x) \in N$, which implies that $M_1(e_1) \subseteq N$ and, consequently, $M = N$. This concludes the proof. □

Let us recall that a C*-algebra is reflexive if and only if it is finite dimensional (*cf.* [36], Proposition 2). Consequently, a C*-algebra has finite rank if and only if it is finite dimensional. It is further known that a C*-algebra A has rank one if and only if $A = \mathbb{C}1$. In particular, the result established by Robertson in [5], Theorem 6 (see Theorem 2) is a direct consequence of our last theorem.

Corollary 15 *Let M be an infinite dimensional von Neumann algebra. Let N be a Čebyšëv von Neumann subalgebra of M. Then $N = \mathbb{C}1$ or $M = N$.* □

We have already seen that, for each natural n, we can find a complex Hilbert space (of dimension two) which is a Čebyšëv JB*-subtriple of a JB*-triple having rank n. It is natural to ask whether we can find a precise description of those complex Hilbert spaces which are Čebyšëv JBW*-subtriples of a JBW*-triple. Another general question that remains open in this paper is the following:

Problem 16 Determine the Čebyšëv JB*-subtriples of a general JB*-triple.

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The authors declare no conflict of interest in this article.

Authors' contributions
All authors contributed equally in writing this article and collaborated in its design in coordination. All authors read and approved the final paper.

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