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Multifractal Dimensional Dependence Assessment Based on Tsallis Mutual Information

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Abstract: Entropy-based tools are commonly used to describe the dynamics of complex systems. In the last few decades, non-extensive statistics, based on Tsallis entropy, and multifractal techniques have shown to be useful to characterize long-range interaction and scaling behavior. In this paper, an approach based on generalized Tsallis dimensions is used for the formulation of mutual-information-related dependence coefficients in the multifractal domain. Different versions according to the normalizing factor, as well as to the inclusion of the non-extensivity correction term are considered and discussed. An application to the assessment of dimensional interaction in the structural dynamics of a seismic real series is carried out to illustrate the usefulness and comparative performance of the measures introduced.

Keywords: complex system; generalized dimensions; multifractality; non-extensivity; seismicity; space-time dynamics; Tsallis entropy

1. Introduction

The term "complex system" is nowadays commonly present in the main branches of science. The increasing interest in the study of complexity has derived in a proper identification of a new field of knowledge, complex systems science [1–3]. There is not a uniquely accepted definition; however, it is assumed that complex systems consist of many linking components, can be characterized by long-range

interactions, chaotic behavior, self-similarity organization and may be far from equilibrium [4,5]. Methods based on entropy are among the most relevant and successfully applied to describe such systems. Multifractal measures are formally used to explain systems with scaling properties [6–10]. On the other hand, non-extensive statistics are used to characterize systems with long-range interactions, heavy-tail distributions and anomalous diffusion, among other aspects [11–14]. This paper is focused on entropy-based approaches for studying the dimensional interaction in the structural dynamics of complex systems in the multifractal domain.

Different formulations of Tsallis mutual information have been introduced as a generalization of Shannon mutual information, for quantification of the degree of dependence between components. Among other references, see the pioneering works by Yamano [15,16], based on a modified form of Tsallis entropy in a non-additive context, and Furuichi [17], who introduced "correlation coefficients" based on the normalization of "Tsallis mutual entropy", an expression used for a form of Tsallis mutual information that does not include the non-extensivity correction term. On the other hand, the scaling behavior present in complex systems led us to formulate in [18] a dependence coefficient in the multifractal domain based on Rényi generalized dimensions. In [19], we propose limiting forms of Furuichi's correlation coefficients in the multifractal domain, in terms of "generalized Tsallis dimensions" (also introduced in [18]). In this paper, we formulate new alternative versions of dependence coefficients considering different normalization factors, as well as the inclusion of the non-extensivity correction term.

For illustration, we analyze the interaction between the spatial coordinates and the magnitude and occurrence time in the structural dynamics of a real seismic series. The presence of scaling behavior and long-range interactions in the spatio-temporal distribution of earthquakes has been emphasized in many studies (e.g., [20–26]). The close relation between the study of earthquakes and statistics, as well as the interest in their direct environmental impact and derived consequences (floods, field contamination, volcanic eruptions, tsunamis, *etc*.) are highlighted in the works about "statistical seismology" by Vere-Jones [27–29]. In particular, entropy-based approaches to describe seismicity are used, for instance, in [30–35].

In the next section, we recall the preliminary concepts of Shannon, Rényi and Tsallis entropy, related formulations of divergence and mutual information and the definitions of generalized Rényi and Tsallis dimensions for multifractal measures. Section 3 is focused on the study of dimensional interaction. In particular, we introduce various versions of dependence coefficients and limiting versions in the multifractal domain. In Section 4, we study the spatio-temporal dynamics of a real seismic series using the multifractal dependence coefficients proposed. In concluding Section 5, we summarize and discuss the main aspects and related interpretations of the tools introduced and refer to continuing research in this context.

2. Preliminaries

In this section, we present a short review of the main approaches and tools based on entropy used to describe dynamical complex systems. Firstly, we summarize basic definitions of Shannon, Rényi and

Tsallis entropies and related versions of divergence and mutual information. Secondly, we introduce generalized Rényi and Tsallis dimensions adopting the partition function approach.

2.1. Entropy, Divergence and Mutual Information

2.1.1. Entropy

The concept of entropy was introduced by Shannon [36], with the aim to quantify the uncertainty in a dynamical system. The discrete form of Shannon entropy, for a discrete finite probability distribution with mass probability function $\bar{p} = (p_1, p_2, \ldots, p_N)$, is defined as:

$$H(\bar{p}) := -\sum_{i=1}^{N} p_i \ln(p_i) = -E[\ln(\bar{p})]. \tag{1}$$

The maximum possible entropy value is reached for the equiprobable probability distribution ($p_i = \frac{1}{N}$, $i = 1, ..., N$):

$$H^{max}(\bar{p}) = -\sum_{i=1}^{N} \frac{1}{N} \ln\left(\frac{1}{N}\right) = -\ln\left(\frac{1}{N}\right) = \ln(N).$$

Among many other definitions appearing in the literature, the most well-known generalizations are Rényi [37] and Tsallis [14] entropies, which depend on a deformation parameter, q, and both contain as a particular case Shannon entropy for $q = 1$. The discrete form of Rényi entropy of order q, $H_q(\bar{p})$, is defined as:

$$H_q(\bar{p}) := \frac{1}{1-q} \ln\left(\sum_{i=1}^{N} p_i^q\right) = \frac{1}{1-q} \ln\left(E[(\bar{p})^{q-1}]\right) \qquad (q \neq 1) \tag{2}$$

$$H_1(\bar{p}) := -\sum_{i=1}^{N} p_i \ln(p_i) = -E[\ln(\bar{p})] = \lim_{q \to 1} H_q(\bar{p}). \tag{3}$$

This generalized entropy has the additivity property for independent systems, and its maximum value coincides with the maximum Shannon entropy, for all values of q. Many statistical tools in the study of dynamical systems are based on this entropy; among them, we emphasize the multifractal formalism and some complexity measures.

Tsallis entropy, as an alternative, is becoming more and more used to describe complex dynamical systems. The discrete form of Tsallis entropy is defined as:

$$T_q(\bar{p}) := \frac{1}{q-1}\left(1 - \sum_{i=1}^{N} p_i^q\right) = \frac{1}{q-1}\left(1 - E[(\bar{p})^{q-1}]\right) \qquad (q \neq 1) \tag{4}$$

$$T_1(\bar{p}) := -\sum_{i=1}^{N} p_i \ln(p_i) = -E[\ln(\bar{p})] = \lim_{q \to 1} T_q(\bar{p}). \tag{5}$$

The maximum Tsallis entropy value is obtained for an equiprobable distribution, and in this case, it depends on q:

$$T_q^{max} = \frac{1 - \sum_{i=1}^{N}\left(\frac{1}{N}\right)^q}{q-1} = \frac{1 - N^{1-q}}{q-1}. \tag{6}$$

This entropy is non-additive for independent systems and has led to non-extensive statistical mechanics. It has been proven to be useful to study dynamical systems with long-range interactions or that can be far from equilibrium. In the last few years, more general parametric entropies have appeared, including Shannon and Tsallis entropies [38].

2.1.2. Divergence

The concept of divergence was originally introduced to quantify dissimilarity or departure between two distributions. Among the most used divergences are, in particular, the ones related to the above entropy measures. Specifically, the divergence based on Shannon entropy is known as Kullback–Leibler divergence, information divergence or relative entropy [39]. Formally, given two discrete finite probability distributions with probability mass functions $\bar{p}_1 = (p_{11}, p_{12}, ..., p_{1N})$ and $\bar{p}_2 = (p_{21}, p_{22}, ..., p_{2N})$, Kullback–Leibler divergence, denoted $D(\bar{p}_1 \| \bar{p}_2)$, is defined as:

$$D(\bar{p}_1 \| \bar{p}_2) := \sum_{i=1}^{N} p_{1i} \ln \frac{p_{1i}}{p_{2i}} = E_{\bar{p}_1} \left[\ln \frac{\bar{p}_1}{\bar{p}_2} \right]. \tag{7}$$

Rényi divergence [37] is defined as:

$$D_q^R(\bar{p}_1 \| \bar{p}_2) := \frac{1}{q-1} \ln \left(\sum_{i=1}^{N} \frac{p_{1i}^q}{p_{2i}^{q-1}} \right) = \frac{1}{q-1} \ln \left(E_{\bar{p}_1} \left[\left(\frac{\bar{p}_1}{\bar{p}_2} \right)^{q-1} \right] \right) \qquad (q \neq 1) \tag{8}$$

$$D_1^R(\bar{p}_1 \| \bar{p}_2) := D(\bar{p}_1 \| \bar{p}_2). \tag{9}$$

Similarly, Tsallis divergence [40] is defined as:

$$D_q^T(\bar{p}_1 \| \bar{p}_2) := \frac{1}{1-q} \left(1 - \sum_{i=1}^{N} \frac{p_{1i}^q}{p_{2i}^{q-1}} \right) = \frac{1}{1-q} \left(1 - E_{\bar{p}_1} \left[\left(\frac{\bar{p}_1}{\bar{p}_2} \right)^{q-1} \right] \right) \qquad (q \neq 1) \tag{10}$$

$$D_1^T(\bar{p}_1 \| \bar{p}_2) := D(\bar{p}_1 \| \bar{p}_2). \tag{11}$$

The three divergences defined above are non-symmetrical. Symmetrization can be achieved averaging the two divergence values obtained by interchanging the distributions $\bar{p}_1$ and $\bar{p}_2$.

2.1.3. Mutual Information

Mutual information measures are intended to quantify the amount of information that two random variables or vectors share. A natural definition of mutual information between X and Y is based on the divergence of the product of their marginal distributions with respect to the joint distribution of (X, Y). Here, we introduce the usual definitions of mutual information based on Shannon, Rényi and Tsallis entropies. For $X \sim \bar{p}_X$, $Y \sim \bar{p}_Y$ and $(X, Y) \sim \bar{p}_{XY}$, Shannon mutual information is given by:

$$I(X, Y) := D(\bar{p}_{XY} \| \bar{p}_X \bar{p}_Y) = H(X) + H(Y) - H(X, Y). \tag{12}$$

The last equality has a complementary interest for computation and for predictive interpretations. However, such an equivalence does not hold for the cases of Rényi and Tsallis divergence-based mutual

information, and then, alternative formulations can be given in each case. For Rényi mutual information (see, for example, [41]), we denote:

$$I_q^R(X,Y) := D_q^R(\bar{p}_{XY} \| \bar{p}_X \bar{p}_Y) \tag{13}$$

$$\tilde{I}_q^R(X,Y) := H_q(X) + H_q(Y) - H_q(X,Y). \tag{14}$$

Similarly, for Tsallis mutual information (see [17,42–44]), we denote:

$$I_q^T(X,Y) := D_q^T(\bar{p}_{XY} \| \bar{p}_X \bar{p}_Y) \tag{15}$$

$$\tilde{I}_q^T(X,Y) := T_q(X) + T_q(Y) - T_q(X,Y) \tag{16}$$

$$\widehat{I}_q^T(X,Y) := T_q(X) + T_q(Y) + (q-1)T_q(X)T_q(Y) - T_q(X,Y). \tag{17}$$

The latter measure includes the non-extensivity correction term.

2.2. Scaling Behavior and Multifractality

As mentioned above, complex systems may exhibit a self-similar structure or scaling properties. Distributions related to such systems are formally described in terms of the generalized Rényi dimensions and the singularity spectrum. Firstly, we introduce the generalized Rényi dimensions using the partition function formalism by means of the box-counting algorithm (see [45,46]). Secondly, we formulate the generalized dimensions based on Tsallis entropy introduced in [18].

2.2.1. Generalized Rényi Dimensions

Generalized Rényi dimensions reflect global scaling behavior based on average quantities, defined by the rate of divergence of Rényi entropy of the partition probability distribution P_ε of a multifractal measure μ with respect to decreasing box width ε, in the following sense:

$$D_1 := \lim_{\varepsilon \to 0} \frac{\sum_{k \in K_\varepsilon} \mu[B_\varepsilon(k)] \ln \mu[B_\varepsilon(k)]}{\ln \varepsilon} \tag{18}$$

$$D_q := \lim_{\varepsilon \to 0} \frac{1}{q-1} \frac{\ln \sum_{k \in K_\varepsilon} \mu^q[B_\varepsilon(k)]}{\ln \varepsilon} \qquad (q \neq 1), \tag{19}$$

where K_ε denotes the set of non-μ−null boxes of width ε. Thus, D_q defines the scaling power-law of Rényi entropy of order q of P_ε with respect to ε,

$$e^{-H_q(P_\varepsilon)} \sim \varepsilon^{D_q} \qquad (\forall q). \tag{20}$$

D_q is a monotonically non-increasing function of q, and varying the "deformation parameter" q, one can extract different features of the distribution of the multifractal measure μ on its support. In particular, based on their meaning and interpretation, there are four quantities of particular interest in practice:

- "Capacity" dimension, D_0: This shows how the points of a multifractal pattern fill the domain under study. The larger the value of this dimension, the better the space is covered.

- "Entropy" dimension, D_1: This is a measure of order-disorder of the points in the domain under study. Larger values indicate higher disorder.

- "Correlation" dimension, D_2: This quantifies the degree of clustering/inhibition. Lower values correspond to a higher level of clustering.

- "Multifractal step", $D_{-\infty} - D_{\infty}$: This indicates the degree of multifractality. Larger values correspond to a stronger multifractal behavior. The multifractal step is zero for monofractal behavior.

2.2.2. Generalized Tsallis Dimensions

Generalized Tsallis dimensions [18] are formulated, following the structure of generalized Rényi dimensions, as:

$$DT_1 := \lim_{\varepsilon \to 0} \frac{\sum_{k \in K_\varepsilon} \mu[B_\varepsilon(k)] \ln \mu[B_\varepsilon(k)]}{\ln \varepsilon} \tag{21}$$

$$DT_q := \lim_{\varepsilon \to 0} \frac{1 - \sum_{k \in K_\varepsilon} \mu^q[B_\varepsilon(k)]}{1 - \varepsilon^{q-1}} \qquad (q \neq 1). \tag{22}$$

In this case, we directly have, for $q \neq 1$, the approximation:

$$T_q(P_\varepsilon) \sim DT_q \frac{1 - \varepsilon^{q-1}}{q - 1}. \tag{23}$$

In particular, for $q > 1$, we can just write:

$$DT_q = (q - 1) \lim_{\varepsilon \to 0} T_q(P_\varepsilon). \tag{24}$$

3. Dependence Analysis

3.1. Generalized Dependence Coefficients

Based on Tsallis mutual information (16) (also named "Tsallis mutual entropy"), Furuichi [17] proposed the following expressions of dependence coefficients ("correlation coefficients") to quantify the degree of association between two random variables X and Y:

$$TDC1_q(X, Y) := \frac{\tilde{I}_q^T(X, Y)}{T_q(X, Y)} = \frac{T_q(X) + T_q(Y) - T_q(X, Y)}{T_q(X, Y)} \tag{25}$$

$$TDC2_q(X, Y) := \frac{\tilde{I}_q^T(X, Y)}{\max\{T_q(X), T_q(Y)\}} = \frac{T_q(X) + T_q(Y) - T_q(X, Y)}{\max\{T_q(X), T_q(Y)\}}, \tag{26}$$

for $T_q(X) > 0$, $T_q(Y) > 0$ and $q > 1$.

Here, we propose an alternative version of a dependent coefficient, considering as the normalizing factor the average of marginal Tsallis entropies,

$$TDC3_q(X, Y) := \frac{\tilde{I}_q^T(X, Y)}{1/2(T_q(X) + T_q(Y))} = \frac{T_q(X) + T_q(Y) - T_q(X, Y)}{1/2(T_q(X) + T_q(Y))}, \tag{27}$$

for $T_q(X) > 0$, $T_q(Y) > 0$ and $q > 1$. This is a particular case of a wider class based on generalized means, as discussed below in this section.

These measures do not include the non-extensivity correction term. We also formulate a "non-extensive dependence coefficient" based on the form (17) of Tsallis mutual information as follows:

$$
\begin{aligned}
TDC4_q(X,Y) &:= \frac{\widehat{I}_q^T(X,Y)}{1/2(T_q(X) + T_q(Y)) + (q-1)T_q(X)T_q(Y)} \\
&= \frac{T_q(X) + T_q(Y) + (q-1)T_q(X)T_q(Y) - T_q(X,Y)}{1/2(T_q(X) + T_q(Y)) + (q-1)T_q(X)T_q(Y)},
\end{aligned}
\tag{28}
$$

for $T_q(X) > 0$, $T_q(Y) > 0$ and $q > 1$.

3.1.1. A Formal Justification and Discussion

For $\tilde{I}_q^T(X,Y) = T_q(X) + T_q(Y) - T_q(X,Y)$, we can write:

$$
\begin{aligned}
0 \leq &\tilde{I}_q^T(X,Y) \leq \min\{T_q(X), T_q(Y)\} \leq M[T_q(X), T_q(Y)] \\
&\leq \max\{T_q(X), T_q(Y)\} \leq T_q(X,Y),
\end{aligned}
\tag{29}
$$

where $M[\cdot]$ represents any mean defined on $(\mathbb{R}^+)^n$ (here, $n = 2$), including $\min$ and $\max$ as special limit cases.

In principle, any of the last four terms in (29) might be considered as a candidate normalizing factor for $\tilde{I}_q^T(X,Y)$. Following [17], assume that we are interested in dependence coefficients ρ_q of this $q-$parametric quotient form satisfying the following properties: for $q > 1$, $T_q(X) > 0$ and $T_q(Y) > 0$,

(i) $\rho_q(X,Y) = \rho_q(Y,X)$,

(ii) $0 \leq \rho_q(X,Y) \leq 1$,

(iii) $\rho_q(X,Y) = 0$ if and only if X and Y are independent and $q = 1$,

(iv) $\rho_q(X,Y) = 1$ if and only if $X \equiv Y$.

In (iv), the equivalence $X \equiv Y$ is specifically understood in the sense that $T_q(X|Y) = T_q(Y|X) = 0$, considering Tsallis conditional entropy defined as $T_q(X|Y) = -\sum_{x,y} p(x,y)^q \ln_q p(x|y)$, $q \neq 1$, with $\ln_q(x) \equiv (x^{1-q} - 1)/(1 - q)$.

From Propositions 5.2 and 5.4 in [17], the coefficients based on the normalizing factors $\max\{T_q(X), T_q(Y)\}$ and $T_q(X,Y)$ satisfy the above properties. For $\min\{T_q(X), T_q(Y)\}$, Properties (i)–(iii) hold; however, regarding (iv), we can only ensure that if $\rho_q(X,Y) = 1$, then either $T_q(X|Y)$ or $T_q(Y|X)$ must be equal to zero.

Further, for any mean $M[\cdot]$ having the properties of symmetry (*i.e.*, invariance under the permutation of arguments), value preservation (*i.e.*, $M[(x,\ldots,x)] = x$) and strict monotonicity (*i.e.*, $M[\mathbf{x}] \leq M[\mathbf{x}']$ if $x_i \leq x_i'$, for all $i = 1,\ldots,n$, with $M[\mathbf{x}] < M[\mathbf{x}']$ if $x_i < x_i'$ for at least one index i), we can prove that the above Properties (i)–(iv) hold. Cases (i)–(iii) are trivial, and the argument for (iv) is as follows: we first note that:

$$
\tilde{I}_q^T(X,Y) = T_q(X) - T_q(X|Y) = T_q(Y) - T_q(Y|X).
$$

Hence, by value preservation, we have:

$$\tilde{I}_q^T(X,Y) = M[\tilde{I}_q^T(X,Y), \tilde{I}_q^T(X,Y)] = M[T_q(X) - T_q(X|Y), T_q(Y) - T_q(Y|X)].$$

Now,

$$\frac{\tilde{I}_q^T(X,Y)}{M[T_q(X), T_q(Y)]} = 1 \Leftrightarrow \tilde{I}_q^T(X,Y) = M[T_q(X), T_q(Y)]$$

$$\Leftrightarrow M[T_q(X) - T_q(X|Y), T_q(Y) - T_q(Y|X)] = M[T_q(X), T_q(Y)]$$

$$\Leftrightarrow T_q(X|Y) = T_q(Y|X) = 0 \Leftrightarrow X \equiv Y,$$

with strict monotonicity being applied for the third equivalence.

A significant case is the family of Hölder means or finite order $p \in \mathbb{R}$,

$$M_p[\mathbf{x}] = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p},$$

ranging continuously within the limits $\min\{x_1, \ldots, x_n\} = \lim_{p \to -\infty} M_p[\mathbf{x}]$ and $\max\{x_1, \ldots, x_n\} = \lim_{p \to \infty} M_p[\mathbf{x}]$. In particular, the three classical Pythagorean means, namely "arithmetic" (for $p = 1$), "geometric" (for $p \to 0$) and "harmonic" (for $p = -1$), are of special interest, with:

$$H[\mathbf{x}] \leq G[\mathbf{x}] \leq A[\mathbf{x}].$$

In this paper, we consider the specific case of coefficient $TDC3_q(X,Y)$, defined in terms of the arithmetic mean of the marginal Tsallis entropies as the normalizing factor.

Since for $n = 2$ and for a symmetric mean $M[\cdot]$ we have $M[x,y] = M[\min\{x,y\}, \max\{x,y\}]$, considering also that the linear scaling property $cM[\mathbf{x}] = M[c\mathbf{x}] = M[cx_1, \ldots, x_n]$ is satisfied by Hölder means for $c > 0$, it is noteworthy, in particular, the relation:

$$TDC3_q(X,Y) := \frac{\tilde{I}_q^T(X,Y)}{A[T_q(X), T_q(Y)]} = H \left[\frac{\tilde{I}_q^T(X,Y)}{\min\{T_q(X), T_q(Y)\}}, \frac{\tilde{I}_q^T(X,Y)}{\max\{T_q(X), T_q(Y)\}} \right].$$

Regarding the formulation of the dependence coefficient $TDC4_q(X,Y)$, which is based on the form of Tsallis mutual information $\widehat{I}_q^T(X,Y) := T_q(X) + T_q(Y) + (q-1)T_q(X)T_q(Y) - T_q(X,Y)$, it is clear, by construction, that the simultaneous addition of the non-extensivity correction term $(q-1)T_q(X)T_q(Y)$ to both the numerator and the denominator of any dependence coefficient of the form:

$$\frac{\tilde{I}_q^T(X,Y)}{M[T_q(X), T_q(Y)]}$$

satisfying Properties (i)–(iv) leads to a new dependence coefficient,

$$\frac{\widehat{I}_q^T(X,Y)}{M[T_q(X), T_q(Y)] + (q-1)T_q(X)T_q(Y)},$$

which also satisfies such properties. An interesting information-related rationale can be found when, in such a construction, Tsallis entropies are replaced by the "normalized" Tsallis entropies:

$$\overline{T}_q(\bar{p}) = \frac{T_q(\bar{p})}{\sum_{i=1}^{n} p_i^q}$$

considered in [15,16]. In such a case, it can be proven that the modified dependence coefficient:

$$\overline{TDC4}_q(X,Y) := \frac{\overline{T}_q(X) + \overline{T}_q(Y) + (q-1)\overline{T}_q(X)\overline{T}_q(Y) - \overline{T}_q(X,Y)}{1/2(\overline{T}_q(X) + \overline{T}_q(Y)) + (q-1)\overline{T}_q(X)\overline{T}_q(Y)}$$

satisfies the harmonic mean relation:

$$\overline{TDC4}_q(X,Y) = H\left[\frac{\overline{I}_q(X;Y)}{\overline{T}_q(X)}, \frac{\overline{I}_q(Y;X)}{\overline{T}_q(Y)}\right],$$

where $\overline{I}_q(X;Y)$ is the non-symmetric form of Tsallis mutual information defined in [15,16] as:

$$\overline{I}_q(X;Y) := \frac{\overline{T}_q(X) + \overline{T}_q(Y) + (q-1)\overline{T}_q(X)\overline{T}_q(Y) - \overline{T}_q(X,Y)}{1 + (q-1)\overline{T}_q(X)}.$$

3.2. Generalized Dependence Coefficients in the Multifractal Domain

As mentioned in the Introduction, this paper is focused on the assessment of dimensional interactions in the multifractal domain. From the limiting relation between generalized Tsallis dimensions and Tsallis entropy, we can write, for $q > 1$,

$$\lim_{\varepsilon \to 0} \tilde{I}_q^T(X,Y) = \frac{1}{q-1}\left(DT_q(X) + DT_q(Y) - DT_q(X,Y)\right) \tag{30}$$

$$\lim_{\varepsilon \to 0} \widehat{I}_q^T(X,Y) = \frac{1}{q-1}\left(DT_q(X) + DT_q(Y) + DT_q(X)DT_q(Y) - DT_q(X,Y)\right). \tag{31}$$

Thus, the multifractal domain limiting extensions of the dependence coefficients (25, 26, 27, 28) are given by:

$$MTDC1_q(X,Y) := \frac{DT_q(X) + DT_q(Y) - DT_q(X,Y)}{DT_q(X,Y)} \tag{32}$$

$$MTDC2_q(X,Y) := \frac{DT_q(X) + DT_q(Y) - DT_q(X,Y)}{\max\{DT_q(X), DT_q(Y)\}} \tag{33}$$

$$MTDC3_q(X,Y) := \frac{DT_q(X) + DT_q(Y) - DT_q(X,Y)}{1/2(DT_q(X) + DT_q(Y))} \tag{34}$$

$$MTDC4_q(X,Y) := \frac{DT_q(X) + DT_q(Y) + DT_q(X)DT_q(Y) - DT_q(X,Y)}{1/2(DT_q(X) + DT_q(Y)) + DT_q(X)DT_q(Y)}, \tag{35}$$

for $DT_q(X) > 0$, $DT_q(Y) > 0$ and $q > 1$. According to the discussion in the final part of Section 3.1, different related formulations might be considered, for instance with alternative means of the marginal Tsallis entropies for the normalizing factor, as well as using modified forms of Tsallis entropy.

4. Application to a Real Seismic Series

To illustrate the usefulness and comparative performance of the multifractal dependence coefficients proposed, we analyze a real seismic series, consisting of 3.214 earthquakes registered in an area near the village of Lorca (Murcia, Spain), delimited by longitude $2.5°W - 0.05°W$ and latitude $37°N - 38.1°N$, during the period between 1 January 2000 and 31 December 2014. The dataset contains four periods

with a high concentration of seismic activity, denoted here as Series A, B, C and D, which are clearly visible in the daily frequencies depicted in Figure 1. The three first periods, known as the "Bullas 2002", "La Paca 2005" and "Lorca 2011" seismic series, have been studied and compared in [47], and they are composed by 290, 605 and 143 events, respectively. It is important to highlight that during the Lorca 2011 series, a catastrophic event occurred, causing nine casualties and significant localized damage in the region; the time of this event is marked in the different plots with a green star. The fourth period with a high concentration of events corresponds to the last months of 2014. Figure 2 shows that the events of significant magnitude are temporally located within the higher frequency episodes in Series A, B and C and not in Series D.
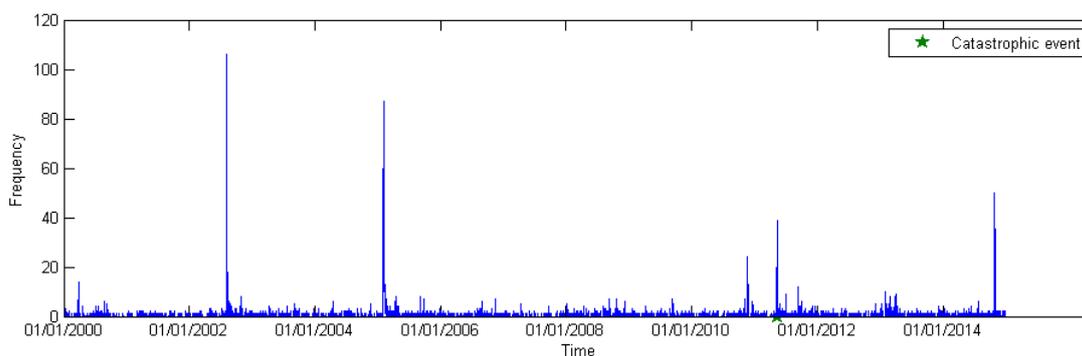

**Figure 1.** Frequency distribution over time.
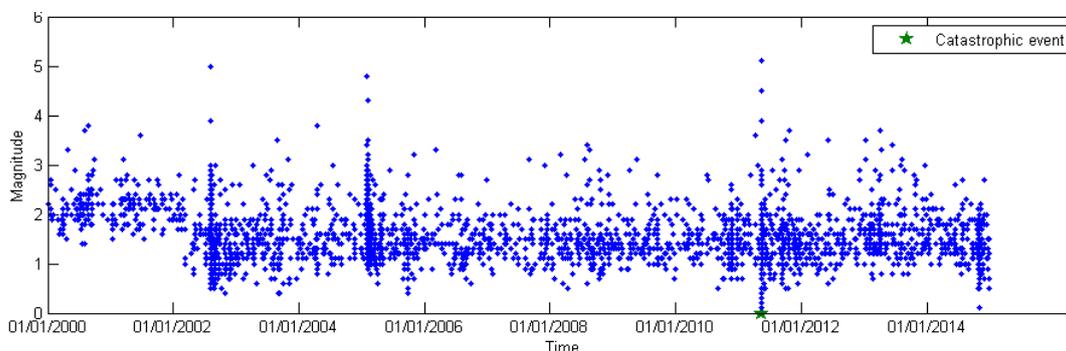
Figure 2. Magnitude distribution over time.

Considering the singularities of the dataset, we focus on the study of the distribution variations around the catastrophic event of 2011. The main objective is to assess the degree of dependence between the spatial locations and both the associated magnitudes and occurrence times. As mentioned above, the series involves three critical periods (A, B and C), although we are mainly interested in the third one, when the catastrophic event occurred. We calculate the dependence coefficients proposed based on data subsets from sliding windows with the same number of events and a fixed overlapping. In particular, we perform the analysis considering a window size of 120 and 240 events, for a comparison of the results under different levels of aggregation. Diverse interpretations can be derived varying the value of the deformation parameter q. To study the dimensional interaction without significant deformation, we start considering a value of q close to one ($q = 1.0001$), and gradually increase this value (in particular, the results are shown for $q = 1.3$ and $q = 1.6$) to assess the dependence between components, emphasizing

the zones with a larger representation of events, until reaching $q = 2$, which is associated with the degree of clustering/inhibition in the system.
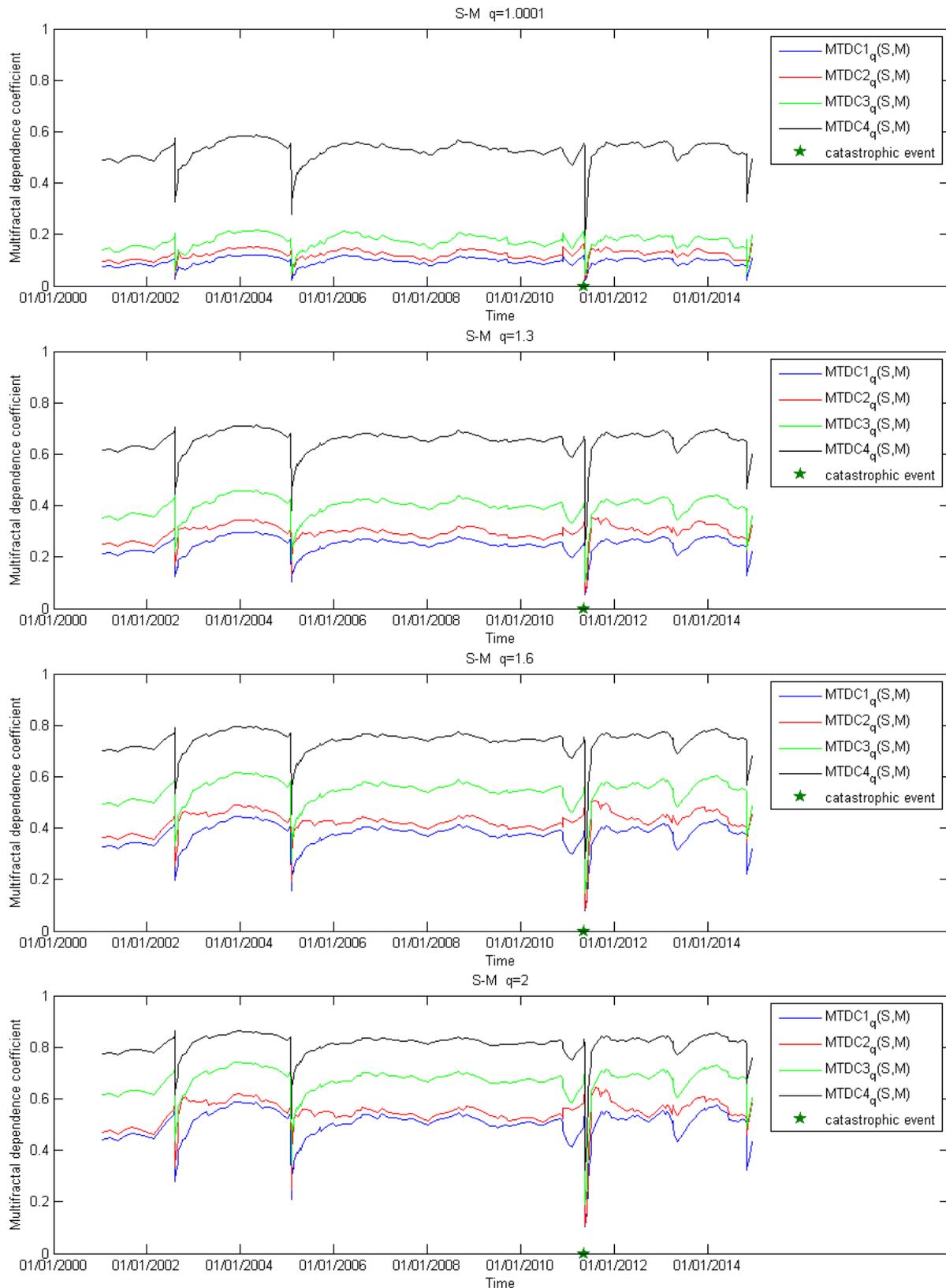
Figure 3. Temporal dynamics of the values of multifractal dependence coefficients for $S \leftrightarrow M$ based on sliding windows (size 120 events, overlapping 10 percent).
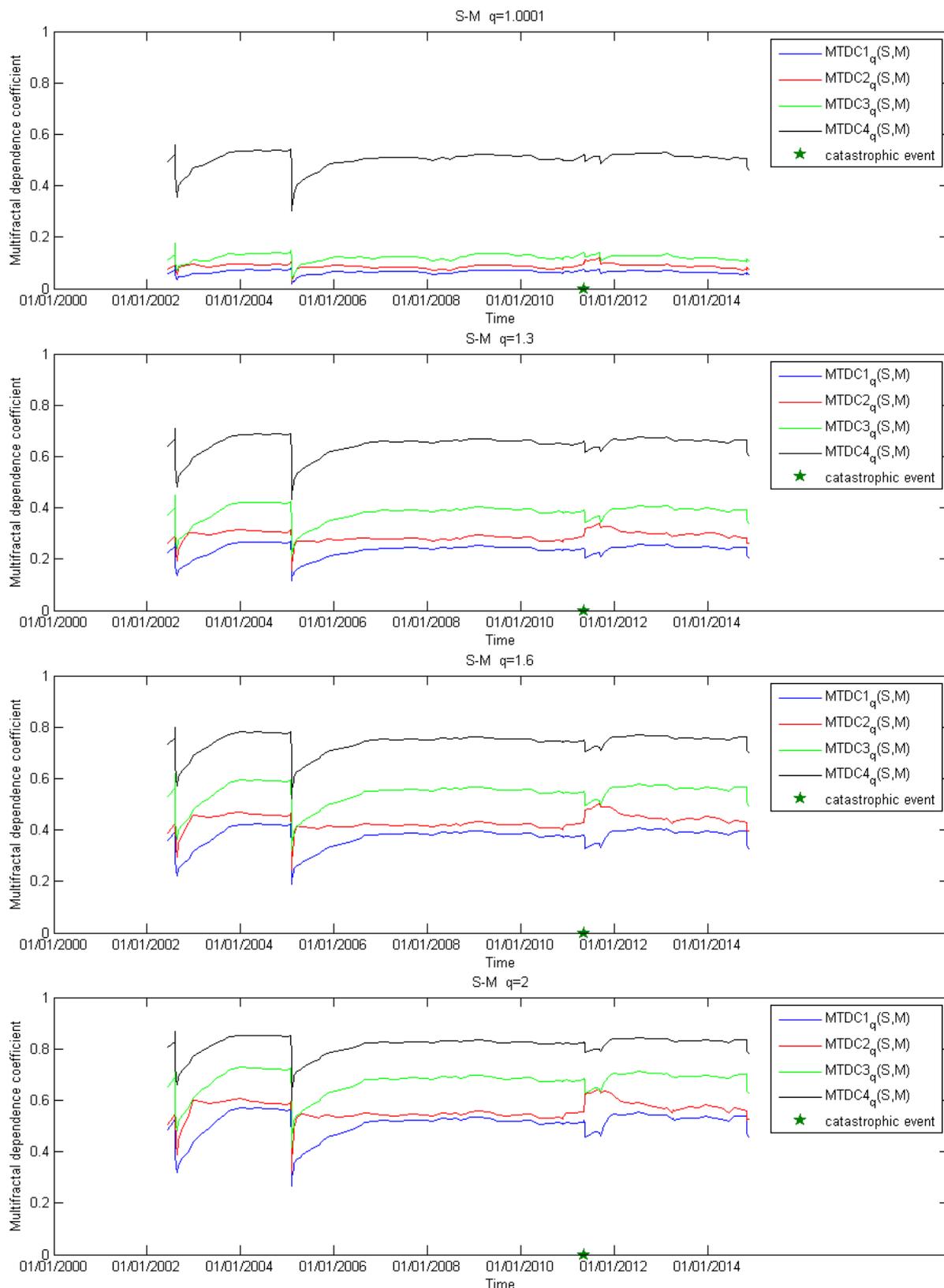
Figure 4. Temporal dynamics of the values of multifractal dependence coefficients for $S \leftrightarrow M$ based on sliding windows (size 240 events, overlapping 10 percent).

Firstly, we analyze the temporal variations in the degree of association between the spatial location of the events, $S = (X, Y)$, and their magnitude, M ($S \leftrightarrow M$); see Figure 3 (windows size 120 events)

and Figure 4 (windows size 240 events). For the results considering a windows size of 120 events, the values obtained for all coefficients show trend changes corresponding to the four periods of high concentration of events. During Series A, B, C and D, drastic decreases are visible, which are interpreted as a significant descent in the degree of association between the spatial coordinates and the magnitude of the events during the high concentration episodes, indicating that the physical state of the system is not in equilibrium. Regarding the comparison between the coefficients considered, $MTDC4_q$ takes higher values and seems to enhance the system dynamics structural changes. As for $MTDC2_q$, rapid increases in the degree of dependence are visible in the periods subsequent to the drops, compared to the other coefficients. This coefficient also presents clear differences in the trend with respect to the other ones in the period previous to the Lorca catastrophic event as q is increased, indicating a lower departure between $DT_q(X,Y)$ and $DT_q(M)$ during these days. In contrast, for the results obtained considering a window size of 240 events, the drop observed in the values obtained associated with the Lorca series (Period C) is less marked. It must be noted that this series is composed by 143 events. The comparison between the coefficients shows, as in the case of a windows size of 120 events, a change of trend in $MTDC2_q$ around the Lorca catastrophic event. In both analyses, visible changes in the dynamics during the previous days deserve special attention, since they might indicate a precursory behavior.

Next, we perform a similar analysis to study the structural dynamics of the dimensional interaction between the spatial coordinates and the occurrence time ($S \leftrightarrow T$); results are depicted in Figures 5 and 6 considering window sizes of 120 and 240 events, respectively. In comparison with the analysis of the interaction $S \leftrightarrow M$, we observe two main differences: firstly, the degree of dependence is higher in this case; secondly, before, during and after the Lorca catastrophic event, significant changes in the dynamics are particularly enhanced for lower values of q.

In general, regarding the variations in the results when we take different values of the deformation parameter q, we can observe that the values obtained for the multifractal dependence coefficients considered increase with q. However, as can be seen in the figures displayed, the increase is not homogeneous and clearly depends on the physical state of the system, as well as on the election of the window size (discrimination of local *vs.* global effects) and the particular structure and meaning of the coefficient considered. Some example plots are included in Figure 7 for better visualization of this aspect. Specifically, the different increase patterns in the values of the dependence coefficients considered, applied to assess the interaction $S \leftrightarrow T$, are displayed for q varying from 1.0001–5, based on data segments of sizes of 120 and 240 events, in the two periods immediately before and after the catastrophic event and in a period of stability of the system.

To summarize, the results obtained show significant trend changes in correspondence with the periods of concentration of seismic activity, indicating that the system is not in equilibrium. The spatial coordinates present a higher degree of dependence with the occurrence time than with the magnitude. Further, the different analyses reflect a trend change in the period before the Lorca catastrophic event, which could be viewed as a precursory element. The coefficients considered provide certain differential specificities, particularly regarding the critical episodes. The importance of the election of the windows size is also reflected in the results obtained, in terms of the level of temporal aggregation to study global *vs.* local behavior. Finally, from the variations of the deformation parameter q, we can contrast the effects of enhancing the highly represented areas.
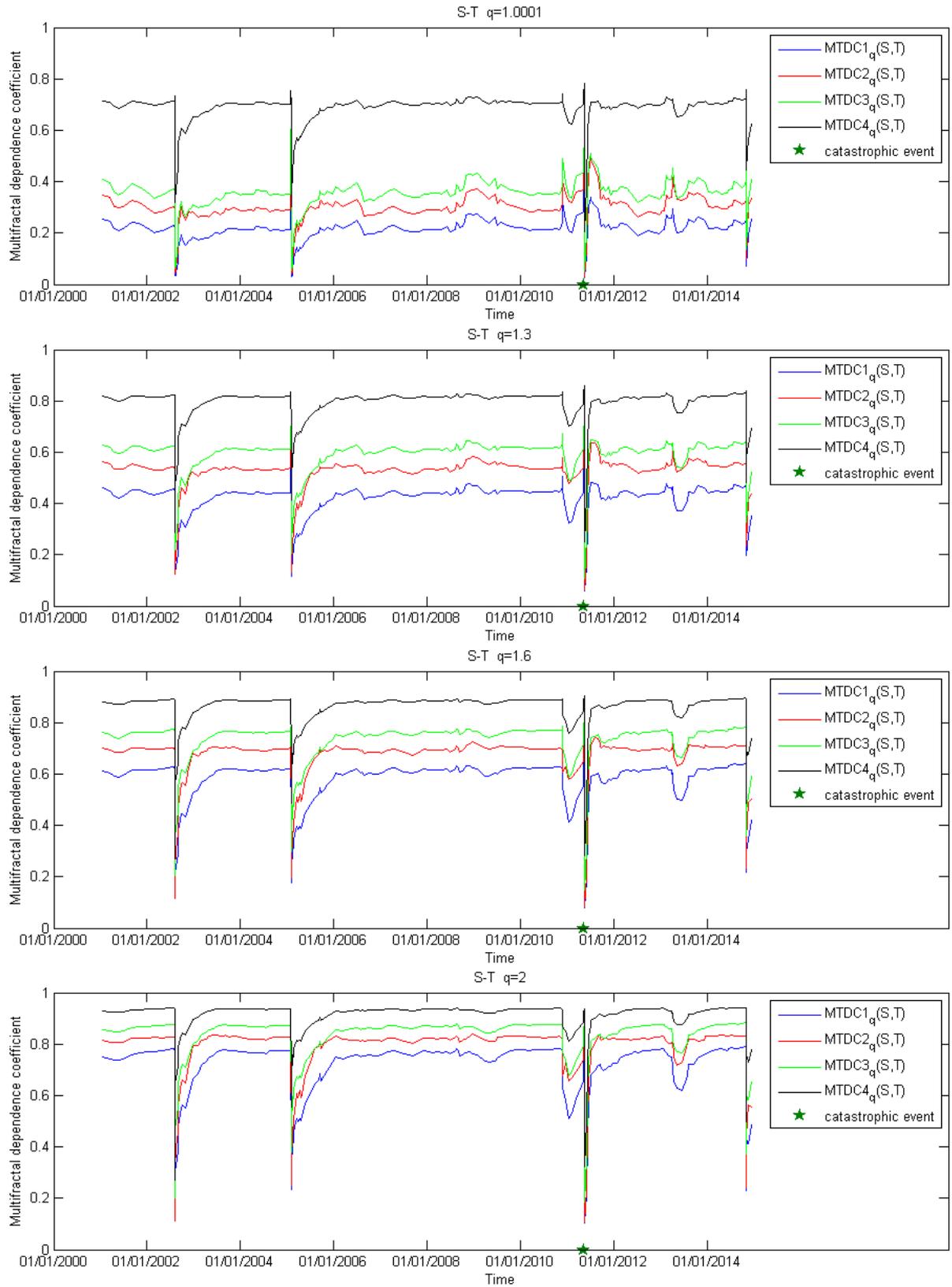
Figure 5. Temporal dynamics of multifractal dependence coefficients for $S \leftrightarrow T$ based on sliding windows (size 120 events, overlapping 10 percent).
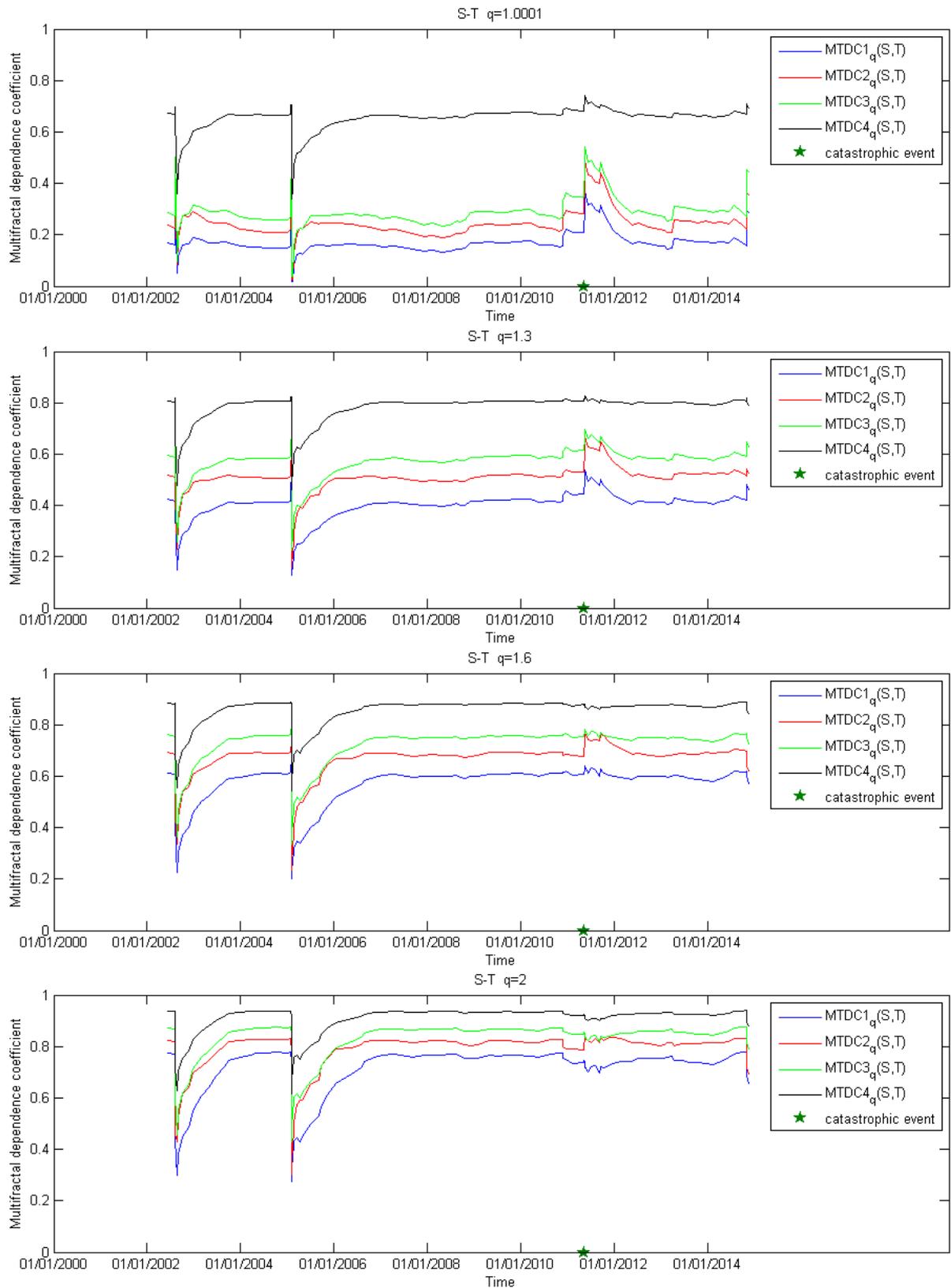
Figure 6. Temporal dynamics of multifractal dependence coefficients for $S \leftrightarrow T$ based on sliding windows (size 240 events, overlapping 10 percent).
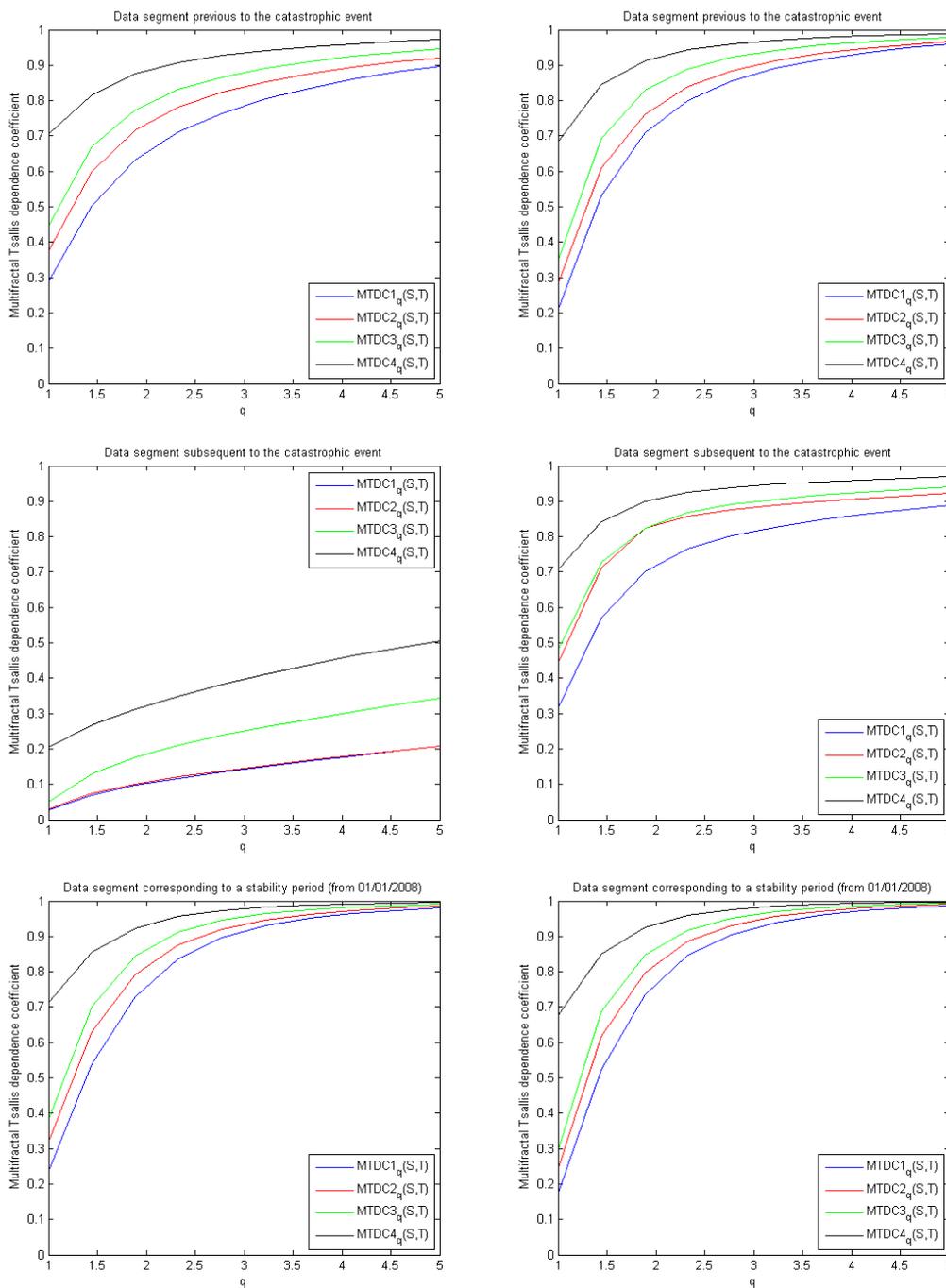
Figure 7. Values of multifractal dependence coefficients for $S \leftrightarrow T$ for q varying from 1.0001–5, based on data segments of size 120 (left plots) and 240 (right plots) events, in the two periods immediately before and after the catastrophic event and in a period of stability of the system (from top to bottom).

5. Conclusion

This paper is focused on the study of complex systems using entropy-based approaches. Specifically, the objective is the assessment of the structural dynamics of dimensional interactions in the multifractal domain.

New formulations of dependence coefficients based on Tsallis mutual information are introduced. First, a flexible normalization factor defined by the class of symmetric, value preserving and strictly monotonic generalized means applied to the marginal Tsallis entropies is considered, which guarantees the compliance of the reference properties stated in [17]. The special case of the arithmetic mean is considered in particular. Second, a related formulation involving the non-extensivity correction term in both the numerator and the denominator is proposed, also fulfilling the mentioned reference properties under similar conditions. Again, in particular, a normalization factor in terms of the arithmetic mean is specifically considered. In the case where Tsallis entropies are replaced by the modified Tsallis entropies considered in [15,16], such a construction is equivalent to a weighted harmonic mean of the reciprocal values of the non-symmetric Tsallis mutual information introduced in these references.

In a multifractal context, based on the partition function approach, limiting versions of dependence coefficients are derived in terms of the generalized Tsallis dimensions introduced in [18]. An application to seismic data is performed showing the usefulness of this approach to analyze the dimensional interaction in the structural dynamics of complex systems. Interpretations on different features are extracted, with a comparison of the results considering different windows sizes for aggregation, varying values of the deformation parameter q and alternative multifractal dependence coefficients. In particular, a non-homogeneous increase of the values of the coefficients with respect to the increasing value of the deformation parameter q depending on the physical state of the system and sensitivity to non-equilibrium regimes in the dynamics is emphasized.

Further study of the analytical properties of the different measures introduced, as well as an investigation of new related and generalized versions are considered for continuing research.

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Author Contributions

All authors contributed to the original ideas and technical content of the paper. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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