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On some fixed point theorems under (α, ψ, ϕ)-contractivity conditions in metric spaces endowed with transitive binary relations

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Abstract

After the appearance of Nieto and Rodríguez-López's theorem, the branch of fixed point theory devoted to the setting of partially ordered metric spaces have attracted much attention in the last years, especially when coupled, tripled, quadrupled and, in general, multidimensional fixed points are studied. Almost all papers in this direction have been forced to present two results assuming two different hypotheses: the involved mapping should be continuous or the metric framework should be regular. Both conditions seem to be different in nature because one of them refers to the mapping and the other one is assumed on the ambient space. In this paper, we unify such different conditions in a unique one. By introducing the notion of continuity of a mapping from a metric space into itself depending on a function α, which is the case that covers the partially ordered setting, we extend some very recent theorems involving control functions that only must be lower/upper semi-continuous from the right. Finally, we use metric spaces endowed with transitive binary relations rather than partial orders.

1 Introduction

In recent times, some extensions of the Banach contractive mapping principle have been introduced using a contractivity condition that involves two different functions. For instance, in 2008, Dutta and Choudhury presented the following generalization.

Theorem 1.1 (Dutta and Choudhury [1]) *Let (X,d) be a complete metric space and let $F : X \to X$ be a mapping such that*

$$\psi\big(d(Fx,Fy)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big) \quad \textit{for all } x,y \in X,$$

where $\psi, \varphi : [0,\infty) \to [0,\infty)$ are continuous, nondecreasing and $\psi^{-1}(\{0\}) = \varphi^{-1}(\{0\}) = \{0\}$. Then F has a unique fixed point.

Functions like ψ verifying the previous properties are known in the literature as *altering distance functions* (see [2]).

Remark 1.1 Notice that Theorem 1.1 remains true if φ only satisfies the following assumptions: φ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}$ (see, for instance, Abbas and Đorić [3] and Đorić [4]).

The above remark yields the following statement.

Theorem 1.2 *Let (X,d) be a complete metric space and let $F : X \to X$ be a mapping such that for each pair of points $x, y \in X$,*

$$\psi\big(d(Fx,Fy)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big),$$

where $\psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, $\psi^{-1}(\{0\}) = \{0\}$, and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}$. Then F has a unique fixed point.

We have also an ordered version of Theorem 1.2 (see [3–5]).

Theorem 1.3 *Let $(X,\preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \to X$ be a continuous nondecreasing mapping such that*

$$\psi\big(d(Fx,Fy)\big) \leq \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big)$$

for all $x, y \in X$ with $x \preccurlyeq y$, where $\psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, $\psi^{-1}(\{0\}) = \{0\}$, and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}$. If there exists $x_0 \in X$ such that $x_0 \preccurlyeq Fx_0$, then F has a fixed point.

The previous results were extended to the case of a contractivity condition involving three different functions. For instance, Eslamian and Abkar [6] established the following result.

Theorem 1.4 (Eslamian and Abkar [6]) *Let (X,d) be a complete metric space and $f : X \to X$ be such that*

$$\psi\big(d(fx,fy)\big) \leq \alpha\big(d(x,y)\big) - \beta\big(d(x,y)\big)$$

for all $x, y \in X$, where $\psi, \alpha, \beta : [0,\infty) \to [0,\infty)$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semi-continuous,

$$\psi(t) = 0 \quad \text{if and only if} \quad t = 0, \qquad \alpha(0) = \beta(0) = 0 \quad \text{and}$$
$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all } t > 0.$$

Then f has a unique fixed point.

Some of the previous results became equivalent.

Theorem 1.5 (Aydi *et al.* [7]) *Theorem* 1.4 *and Theorem* 1.2 *are equivalent.*

Choudhury and Kundu [8] also extended Theorem 1.4 to the ordered case.

Theorem 1.6 (Choudhury and Kundu [8]) *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous nondecreasing mapping such that*

$$\psi\big(d(fx, fy)\big) \le \alpha\big(d(x, y)\big) - \beta\big(d(x, y)\big)$$

for all $x, y \in X$ with $x \preccurlyeq y$, where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semi-continuous,

$$\psi(t) = 0 \quad \textit{if and only if} \quad t = 0, \qquad \alpha(0) = \beta(0) = 0 \quad \textit{and}$$
$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \textit{for all } t > 0.$$

If there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$, then f has a fixed point.

Following similar arguments as in the proof of Theorem 1.5, the following result was obtained.

Theorem 1.7 (Aydi *et al.* [7]) *Theorem* 1.3 *and Theorem* 1.6 *are equivalent.*

In a very recent paper, Shaddad *et al.* proved the following result, which is a generalization of the previous ones.

Theorem 1.8 (Shaddad *et al.* [9], Theorem 2.5) *Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space. Let $f : X \to X$ be a mapping which obeys the following conditions:*

1. *There exist an altering distance function ψ, an upper semi-continuous function $\theta : [0, \infty) \to [0, \infty)$, and a lower semi-continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$\psi\big(d(fx, fy)\big) \le \theta\big(d(x, y)\big) - \varphi\big(d(x, y)\big) \quad \textit{for all } x \succcurlyeq y,$$

   where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$.
2. *There exists $x_0 \in X$ such that $x_0 \asymp f(x_0)$;*
3. *f is nondecreasing;*
4. *At least, one of the following conditions holds:*
   (a) *f is continuous or*
   (b) *if $\{x_n\} \to x$ when $n \to \infty$ in X, then $x_n \asymp x$ for all n.*

Then f has a fixed point. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then the fixed point is unique.

The condition '$\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$' is not new because, as we have just commented, under some weak continuity conditions, it was firstly considered in Choudhury and Kundu [8], and it can also be found in Razani and Parvaneh [10]. As a consequence of the previous theorem, the authors obtained the following result.

Corollary 1.1 (Shaddad *et al.* [9], Corollary 2.6) *Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space. Let $f : X \to X$ be a mapping which obeys the following conditions:*

1. *There exist an altering distance function* ψ *and a lower semi-continuous function* $\varphi : [0,\infty) \to [0,\infty)$ *such that*

$$\psi\big(d(fx,fy)\big) \le \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big) \quad \textit{for all } x \succcurlyeq y, \tag{1}$$

where $\varphi(0) = 0$;
2. *There exists* $x_0 \in X$ *such that* $x_0 \asymp f(x_0)$;
3. f *is nondecreasing*;
4. *At least, one of the following conditions holds*:
   (a) f *is continuous or*
   (b) *if* $\{x_n\} \to x$ *when* $n \to \infty$ *in* X, *then* $x_n \asymp x$ *for all* n.

Then f *has a fixed point. Moreover, if for each* $x, y \in X$ *there exists* $z \in X$ *which is comparable to* x *and* y, *then the fixed point is unique.*

Two remarks must be done concerning the previous statements. On the one hand, condition 4(b) in Theorem 1.8 is unclear. We suppose that it means that $(X, d, \preccurlyeq)$ is a regular partially ordered metric space, that is, it verifies the following two properties:

- if $\{x_m\} \subseteq X$ is such that $\{x_m\} \to x \in X$ and $x_m \preccurlyeq x_{m+1}$ for all $m \in \mathbb{N}$, then we have that $x_m \preccurlyeq x$ for all $m \in \mathbb{N}$;
- if $\{x_m\} \subseteq X$ is such that $\{x_m\} \to x \in X$ and $x_m \succcurlyeq x_{m+1}$ for all $m \in \mathbb{N}$, then we have that $x_m \succcurlyeq x$ for all $m \in \mathbb{N}$.

On the other hand, in hypothesis 1 of Corollary 1.1, the condition '$\varphi(t) > 0$ for all $t > 0$' is necessary. For instance, consider the following example.

Example 1.1 Let $X = \mathbb{N} \setminus \{0\}$ be endowed with the usual partial order $\le$ and the Euclidean metric $d(x,y) = |x - y|$ for all $x, y \in X$. Then $(\mathbb{N} \setminus \{0\}, d, \le)$ is a complete partially ordered metric space. Let define $f : X \to X$ by $f(n) = n + 1$ for all $n \in X$. Then f does not have any fixed point on X. However, if we define

$$\psi(t) = \begin{cases} 2t & \text{if } 0 \le t \le 1/2, \\ 1 & \text{if } t > 1/2, \end{cases}$$

then ψ is an altering distance function satisfying assumption (1), where $\varphi(t) = 0$ for all $t \in [0,\infty)$ (because $\psi(n) = 1$ for all $n \in X$). Hence, Corollary 1.1 is false if we omit the condition '$\varphi(t) > 0$ for all $t > 0$', that is, $\varphi(t) = 0$ if, and only if, $t = 0$.

This paper has three main aims. On the one hand, we generalize Theorem 1.1 introducing a contractivity condition involving control function that does not have to be continuous nor monotone. In fact, this new kind of control functions only have to verify sequential properties. We will show that two functions are powerful enough to handle contractivity conditions as in hypothesis 1 of Theorem 1.8, in which three functions appear. The new class of control functions includes pairs that are not necessary altering distance functions, and the semi-continuity is imposed only from the right side and on the interval $(0,\infty)$. On the other hand, the second objective is to describe a unified condition to handle two independent hypotheses (the continuity of a mapping and the regularity of the partially ordered metric space), which were initially introduced from Ran and Reurings' theorem and Nieto and Rodríguez-López's theorem. Finally, we show that many results obtained in the setting of a partially ordered metric space do not need a partial order, but only a transitive binary relation on a subset of the metric space.

2 Preliminaries

In the sequel, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denotes the set of all nonnegative integers and $\mathbb{R}$ denotes the set of all real numbers. Henceforth, X and Y will denote nonempty sets. Elements of X are usually called *points*.

Let $T : X \to Y$ be a mapping. The *domain of* T is X and it is denoted by $\operatorname{Dom} T$. Its *range*, that is, the set of values of T in Y, is denoted by $T(X)$. A mapping T is completely characterized by its domain, its range, and the manner in which each *origin* $x \in \operatorname{Dom} T$ is applied to its *image* $T(x) \in T(X)$. For simplicity, we denote, as usual, $T(x)$ by Tx. For any set X, we denote the *identity mapping on* X by $I_X : X \to X$, which is defined by $I_X x = x$ for all $x \in X$.

Given two self-mappings $T, g : X \to X$, we will say that a point $x \in X$ is a *coincidence point of* T *and* g if $Tx = gx$. We will denote by $\operatorname{Coin}(T, g)$ the set of all coincidence points of T and g. If x is a coincidence point of T and g, then the point $\omega = Tx = gx$ is called a *point of coincidence of* T *and* g. A *common fixed point of* T *and* g is a point $x \in X$ such that $Tx = gx = x$. Given a self-mapping $T : X \to X$, we will say that a point $x \in X$ is a *fixed point of* T if $Tx = x$. We will denote by $\operatorname{Fix}(T)$ the set of all fixed points of T.

Given two mappings $T : X \to Y$ and $S : Y \to Z$, the *composite of* T *and* S is the mapping $S \circ T : X \to Z$ given by

$$(S \circ T)x = STx \quad \text{for all } x \in \operatorname{Dom} T.$$

We say that two self-mappings $T, S : X \to X$ are *commuting* if $TSx = STx$ for all $x \in X$ (that is, $T \circ S = S \circ T$).

The *iterates* of a self-mapping $T : X \to X$ are the mappings $\{T^n : X \to X\}_{n\in\mathbb{N}}$ defined by

$$T^0 = I_X, \qquad T^1 = T, \qquad T^2 = T \circ T, \qquad T^{n+1} = T \circ T^n \quad \text{for all } n \geq 2.$$

The notion of *metric space* and the concepts of *convergent sequence* and *Cauchy sequence* in a metric space can be found, for instance, in [11]. We will write $\{x_n\} \to x$ when a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X converges to $x \in X$ in the metric space (X, d). A metric space (X, d) is *complete* if every Cauchy sequence in X converges to some point of X. The limit of a convergent sequence in a metric space is unique.

In a metric space (X, d), a mapping $T : X \to X$ is *continuous at a point* $z \in X$ if $\{Tx_n\} \to Tz$ for all sequence $\{x_n\}$ in X such that $\{x_n\} \to z$. And T is *continuous* if it is continuous at every point of X.

A *binary relation on* X is a nonempty subset $\mathcal{R}$ of $X \times X$. For simplicity, we denote $x \prec y$ if $(x, y) \in \mathcal{R}$, and we will say that $\prec$ is the binary relation on X. This notation lets us write $x \precneq y$ when $x \prec y$ and $x \neq y$. We write $y \succ x$ when $x \prec y$. We will say that x and y are $\prec$*-comparable*, and we will write $x \asymp y$ if $x \prec y$ or $y \prec x$. A binary relation $\prec$ on X is *reflexive* if $x \prec x$ for all $x \in X$; it is *transitive* if $x \prec z$ for all $x, y, z \in X$ such that $x \prec y$ and $y \prec z$; and it is *antisymmetric* if $x \prec y$ and $y \prec x$ imply $x = y$.

A reflexive and transitive relation $\preccurlyeq$ on X is a *preorder* (or a *quasiorder*) on X. In such a case, $(X, \preccurlyeq)$ is a *preordered space*. If a preorder $\preccurlyeq$ is also antisymmetric, then $\preccurlyeq$ is called a *partial order*, and $(X, \preccurlyeq)$ is a *partially ordered space* (or a *partially ordered set*). We will use the symbol $\prec$ for a general binary relation on X, and the symbol $\preccurlyeq$ for a reflexive binary relation on X (for instance, a preorder or a partial order).

The usual order of the set of all real numbers $\mathbb{R}$ is denoted by $\leq$. In fact, this partial order can be induced on any nonempty subset $A \subseteq \mathbb{R}$. Let $\preccurlyeq$ be the binary relation on $\mathbb{R}$ given by

$$x \preccurlyeq y \quad \Leftrightarrow \quad (x = y \text{ or } x < y \leq 0).$$

Then $\preccurlyeq$ is a partial order on $\mathbb{R}$, but it is different from $\leq$. Any equivalence relation is a preorder.

An *ordered metric space* is a triple $(X, d, \preccurlyeq)$ where (X, d) is a metric space and $\preccurlyeq$ is a partial order on X. And if $\preccurlyeq$ is a preorder on X, then $(X, d, \preccurlyeq)$ is a *preordered metric space*.

Definition 2.1 Let (X, d) be a metric space, let $A \subseteq X$ be a nonempty subset, and let $\prec$ be a binary relation on X. Then $(A, d, \prec)$ is said to be:

- *nondecreasing-regular* if for all sequence $\{x_m\} \subseteq A$ such that $\{x_m\} \to a \in A$ and $x_m \prec x_{m+1}$ for all $m \in \mathbb{N}$, we have that $x_m \prec a$ for all $m \in \mathbb{N}$;
- *nonincreasing-regular* if for all sequence $\{x_m\} \subseteq A$ such that $\{x_m\} \to a \in A$ and $x_m \succ x_{m+1}$ for all $m \in \mathbb{N}$, we have that $x_m \succ a$ for all $m \in \mathbb{N}$;
- *regular* if it is both nondecreasing-regular and nonincreasing-regular.

Some authors called *ordered complete* to a regular ordered metric space (see, for instance, [12]). Furthermore, Roldán *et al.* called *sequential monotone property* to nondecreasing-regularity (see [13]).

Definition 2.2 Let $\prec$ be a binary relation on X and let $T, g : X \to X$ be two mappings. We say that T is $(g, \prec)$ -nondecreasing if $Tx \prec Ty$ for all $x, y \in X$ such that $gx \prec gy$. And T is *$\prec$-nondecreasing* if $Tx \prec Ty$ for all $x, y \in X$ such that $x \prec y$.

Let us consider the following families of control functions.

$$\mathcal{F}_{\text{alt}} = \{\phi : [0, \infty) \to [0, \infty) : \phi \text{ is continuous, nondecreasing}, \phi(t) = 0 \Leftrightarrow t = 0\},$$
$$\mathcal{F}'_{\text{alt}} = \{\phi : [0, \infty) \to [0, \infty) : \phi \text{ is lower semi-continuous}, \phi(t) = 0 \Leftrightarrow t = 0\}.$$

Functions in $\mathcal{F}_{\text{alt}}$ are called *altering distance functions* (see [2, 14–16]).

Remark 2.1 If $\phi : [0, \infty) \to [0, \infty)$ is nondecreasing (for instance, if $\phi \in \mathcal{F}_{\text{alt}}$) and $t, s \in [0, \infty)$ verify $\phi(t) < \phi(s)$, then $t < s$.

To prove it, assume that $t \geq s$. As ϕ is nondecreasing, then $\phi(s) \leq \phi(t) < \phi(s)$, which is impossible.

Definition 2.3 Let $s \in [0, \infty)$ be a point and let $A \subseteq [0, \infty)$ be a nonempty subset. We will say that a function $\phi : [0, \infty) \to [0, \infty)$ is

- *lower semi-continuous from the right at $s \in [0, \infty)$* if

$$\phi(s) \leq \liminf_{t \to s^+} \phi(t);$$

- *lower semi-continuous from the right on A* if it is lower semi-continuous from the right at every $s \in A$;
- *lower semi-continuous from the right* if it is lower semi-continuous from the right on $[0, \infty)$.

Similarly, we will say that ϕ is *upper semi-continuous from the right at $s \in [0, \infty)$* if

$$\phi(s) \geq \limsup_{t \to s^+} \phi(t),$$

and ϕ is *upper semi-continuous from the right* if it is upper semi-continuous from the right at every $s \in [0, \infty)$.

Example 2.1 The function $\phi : [0, \infty) \to [0, \infty)$ defined, for all $t \in [0, \infty)$, by

$$\phi(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ 2 & \text{if } t = 1, \\ t + 2 & \text{if } t > 1, \end{cases}$$

is strictly increasing and lower semi-continuous from the right, but it is not lower semi-continuous at $t = 1$.

Lemma 2.1 *Let (X, d) be a metric space and let $\{x_n\} \subseteq X$ be a sequence which is not Cauchy in (X, d). Then there exist $\varepsilon_0 > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that*

$$k \leq n(k) < m(k) < n(k+1) \quad \textit{and} \quad d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) \quad \textit{for all } k \in \mathbb{N}.$$

Furthermore, if $\{d(x_n, x_{n+1})\} \to 0$, then

$$\lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k \to \infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon_0.$$

3 Main results

In this section, we present some results that extend and unify all theorems given in Introduction. To do that, some notions are introduced. In the sequel, X will denote a nonempty set, d will be a metric on X, $T, g : X \to X$ will be arbitrary self-mappings and $\alpha : X \times X \to [0, \infty)$ will denote a function.

Definition 3.1 Given a metric space (X, d), a point $z_0 \in X$, a function $\alpha : X \times X \to [0, \infty)$ and two mappings $T, g : X \to X$, we say that T is *(d, g, α)-right-continuous at z_0* if we have that $\{Tx_n\}$ converges to Tz_0 for all sequence $\{x_n\} \subseteq X$ such that $\{gx_n\}$ is convergent to gz_0 and verifying that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. And T is *(d, g, α)-right-continuous* if it is (d, g, α)-right-continuous at every point of X. If g is the identity mapping on X, we say that T is *(d, α)-right-continuous.*

Similarly, a mapping $T : X \to X$ is *(d, g, α)-left-continuous at z_0* if we have that $\{Tx_n\}$ converges to Tz_0 for all sequence $\{x_n\}$ such that $\{gx_n\}$ is convergent to gz_0 and verifying that $\alpha(gx_{n+1}, gx_n) \geq 1$ for all $n \in \mathbb{N}$. And T is *(d, g, α)-left-continuous* if it is (d, g, α)-left-continuous at every point of X. If g is the identity mapping on X, we say that T is *(d, α)-left-continuous.*

A mapping $T : X \to X$ is (d, g, α)-*continuous* if it is both (d, g, α)-right-continuous and (d, g, α)-left-continuous.

If $\alpha(x, y) = 1$ for all $x, y \in X$, we say that T is (d, g)-*continuous* (notice that both sides lead to the same condition) if $\{gx_n\} \to gz_0$ implies that $\{Tx_n\} \to Tz_0$, whatever the point $z_0 \in X$ and the sequence $\{x_n\} \subseteq X$.

It is obvious that every continuous mapping from X into itself is also (d, I_X, α)-right-continuous and (d, I_X, α)-left-continuous whatever α, but the converse is false.

Definition 3.2 Given a function $\alpha : X \times X \to [0, \infty)$, we say that two points $x, y \in X$ are (g, α)-*comparable* if $\alpha(gx, gy) \geq 1$ or $\alpha(gy, gx) \geq 1$.

Definition 3.3 We say that a function $\alpha : X \times X \to [0, \infty)$ is *transitive* if, for all $x, y, z \in X$, we have

$$\alpha(x, y) \geq 1, \qquad \alpha(y, z) \geq 1 \quad \Rightarrow \quad \alpha(x, z) \geq 1.$$

Similarly, we say that α is g-*transitive* if, for all $x, y, z \in X$, we have

$$\alpha(gx, gy) \geq 1, \qquad \alpha(gy, gz) \geq 1 \quad \Rightarrow \quad \alpha(gx, gz) \geq 1.$$

Obviously, every transitive mapping α is also g-transitive, whatever the mapping g.

Definition 3.4 Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. We say that T is (g, α)-*admissible* if, for all $x, y \in X$, we have

$$\alpha(gx, gy) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

Example 3.1 If $\alpha(x, y) \geq 1$ for all $x, y \in X$, then α is transitive and g-transitive, and every self-mapping $T : X \to X$ is (g, α)-admissible.

Remark 3.1 Given a binary relation $\prec$ on X, let $\alpha_{\prec} : X \times X \to [0, \infty)$ be the function defined by

$$\alpha_{\prec}(x, y) = \begin{cases} 1 & \text{if } x \prec y, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

- The binary relation $\prec$ is transitive if, and only if, $\alpha_{\prec}$ is transitive.
- A self-mapping $T : X \to X$ is $(g, \prec)$-nondecreasing if, and only if, T is $(g, \alpha_{\prec})$-admissible.

In the next definition, we present the kind of control functions we will involve in the contractivity condition.

Definition 3.5 Let $\mathcal{F}_{\mathcal{A}}$ be the family of all pairs (ψ, ϕ) where $\psi, \phi : [0, \infty) \to [0, \infty)$ are two functions verifying the following two conditions:

$(\mathcal{F}^1_{\mathcal{A}})$ If $\{a_n\} \subset (0, \infty)$ is a sequence such that $\psi(a_{n+1}) \leq \phi(a_n)$ for all $n \in \mathbb{N}$, then $\{a_n\} \to 0$;

($\mathcal{F}_{\mathcal{A}}^2$) If $\{a_n\},\{b_n\}\subset[0,\infty)$ are two sequences converging to the same limit L and such that $L<a_n$ and $\psi(b_n)\leq\phi(a_n)$ for all $n\in\mathbb{N}$, then $L=0$.

Notice that the previous conditions do not impose any constraint about the continuity nor the monotony of the functions ψ and ϕ, as in the following example.

Example 3.2 Let $\psi,\phi:[0,\infty)\to[0,\infty)$ be the functions given, for all $t\in[0,\infty)$, by

$$\psi(t)=\begin{cases} t/2, & \text{if } 0\leq t\leq 1,\\ t/4, & \text{if } t>1;\end{cases}\qquad \phi(t)=\frac{\psi(t)}{2}.$$

Then ψ and ϕ are not continuous nor monotone in $[0,\infty)$. However, $(\psi,\phi)\in\mathcal{F}_{\mathcal{A}}$. To prove it, notice that

$$\frac{1}{4}<\psi(t)\quad\text{for all } t>\frac{1}{2}. \tag{3}$$

Let $\{a_n\}\subset(0,\infty)$ be a sequence such that $\psi(a_{n+1})\leq\phi(a_n)$ for all $n\in\mathbb{N}$. Therefore,

$$\psi(a_{n+1})\leq\phi(a_n)=\frac{\psi(a_n)}{2}\quad\text{for all } n\in\mathbb{N}.$$

Hence, $\{\psi(a_n)\}\to 0$. It follows that there exists $n_0\in\mathbb{N}$ such that $\psi(a_n)<1/4$ for all $n\geq n_0$. By (3), $a_n\leq 1/2$ for all $n\geq n_0$. In such a case, $a_n=2\psi(a_n)$ for all $n\geq n_0$, so $\{a_n\}\to 0$. As a consequence, ($\mathcal{F}_{\mathcal{A}}^1$) holds.

Next, assume that $\{a_n\},\{b_n\}\subset[0,\infty)$ are two sequences converging to the same limit L and such that $L<a_n$ and $\psi(b_n)\leq\phi(a_n)$ for all $n\in\mathbb{N}$. We distinguish two cases.

- Suppose that $L=1$. As $1=L<a_n$ for all $n\in\mathbb{N}$, then $\psi(a_n)=a_n/4$ and $\{\psi(a_n)\}\to L/4=1/4$. On the other hand, as $1/4\leq\psi(t)$ for all $t\in(0.9,1.1)$ and $\{b_n\}\to 1$, then there exists $n_0\in\mathbb{N}$ such that

$$\frac{1}{4}\leq\psi(b_n)\leq\phi(a_n)=\frac{\psi(a_n)}{2}=\frac{a_n}{8}\quad\text{for all } n\geq n_0,$$

  which is a contradiction because $\{a_n/8\}\to 1/8$. Hence, the case $L=1$ is impossible.
- Suppose that $L\neq 1$. As ψ and ϕ are continuous at L, then the inequality

$$\psi(b_n)\leq\phi(a_n)=\frac{\psi(a_n)}{2}\quad\text{for all } n\in\mathbb{N}$$

  implies that $\psi(L)\leq\psi(L)/2$. Hence, $\psi(L)=0$ and $L=0$.

Let us show that the class $\mathcal{F}_{\mathcal{A}}$ includes several types of functions.

Lemma 3.1 *If $\psi,\phi:[0,\infty)\to[0,\infty)$ are two functions such that ψ is continuous on $(0,\infty)$, ϕ is upper semi-continuous from the right on $(0,\infty)$, and $\phi<\psi$ on $(0,\infty)$, then (ψ,ϕ) verifies ($\mathcal{F}_{\mathcal{A}}^2$).*

Furthermore, if, additionally, ψ is nondecreasing on $(0,\infty)$, then $(\psi,\phi)\in\mathcal{F}_{\mathcal{A}}$.

Proof Let $\{a_n\},\{b_n\}\subset[0,\infty)$ be two sequences converging to the same limit $L\in[0,\infty)$ and such that $L<a_n$ and $\psi(b_n)\leq\phi(a_n)$ for all $n\in\mathbb{N}$. We are going to show that $L=0$ reasoning

by contradiction. Assume that $L > 0$. As $a_n > L > 0$ for all $n \in \mathbb{N}$, then $\phi(a_n) < \psi(a_n)$. Hence,

$$\psi(b_n) \leq \phi(a_n) < \psi(a_n) \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

Since ψ is continuous at $L \in (0,\infty)$, then $\{\psi(a_n)\} \to \psi(L)$ and $\{\psi(b_n)\} \to \psi(L)$. Therefore, by (4), we also have that $\{\phi(a_n)\} \to \psi(L)$. Since $\{a_n\} \to L^+$, the upper semi-continuity from the right of ϕ in $(0,\infty)$ yields

$$\phi(L) \geq \limsup_{t \to L^+} \phi(t).$$

As a consequence,

$$\psi(L) = \lim_{n\to\infty} \phi(a_n) \leq \limsup_{t\to L^+} \phi(t) \leq \phi(L) < \psi(L),$$

which is a contradiction. Thus, $L = 0$.

Next, assume that ψ is nondecreasing on $(0,\infty)$ and let $\{a_n\} \subset (0,\infty)$ be a sequence such that $\psi(a_{n+1}) \leq \phi(a_n)$ for all $n \in \mathbb{N}$. Since $a_n > 0$, then

$$\psi(a_{n+1}) \leq \phi(a_n) < \psi(a_n) \quad \text{for all } n \in \mathbb{N}.$$

By Remark 2.1, $a_{n+1} < a_n$ for all $n \in \mathbb{N}$. As $\{a_n\}$ is a decreasing sequence of positive real numbers, then it is convergent. Let L be its limit. We are going to show that $L = 0$ reasoning by contradiction. Assume that $L > 0$. Hence, $L < a_{n+1} < a_n$ for all $n \in \mathbb{N}$, which means that $\{a_n\} \to L^+$. Repeating, point by point, the previous arguments, we deduce that

$$\psi(L) = \lim_{n\to\infty} \phi(a_n) \leq \limsup_{t\to L^+} \phi(t) \leq \phi(L) < \psi(L),$$

which is a contradiction. As a consequence, $L = 0$ and $\{a_n\} \to 0$. □

Corollary 3.1 *If $\psi \in \mathcal{F}_{\mathrm{alt}}$ and $\phi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous from the right function such that $\phi(0) = 0$ and $\phi(t) < \psi(t)$ for all $t > 0$, then $(\psi,\phi) \in \mathcal{F}_{\mathcal{A}}$.*

Corollary 3.2 *If ψ, θ and φ are three functions as in hypothesis* 1 *of Theorem* 1.8, *then $(\psi, \phi = \theta - \varphi) \in \mathcal{F}_{\mathcal{A}}$.*

The main result of the present paper is the following one.

Theorem 3.1 *Let (X,d) be a metric space, let $\alpha : X \times X \to [0,\infty)$ be a function, and let $T, g : X \to X$ be two mappings such that the following conditions are fulfilled:*

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A \subseteq g(X)$ and (A,d) is complete;*
2. *α is g-transitive and T is (g,α)-admissible;*
3. *There exists $(\psi,\phi) \in \mathcal{F}_{\mathcal{A}}$ such that*

$$\alpha(gx,gy)\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(gx,gy)\big) \quad \textit{for all } x,y \in X; \tag{5}$$

4. *At least, one of the following conditions holds:*

(a) *there exists* $x_0 \in X$ *such that* $\alpha(gx_0, Tx_0) \geq 1$ *and* T *is* (d,g,α)*-right-continuous*;
(b) *there exists* $x_0 \in X$ *such that* $\alpha(Tx_0, gx_0) \geq 1$ *and* T *is* (d,g,α)*-left-continuous*.

Then T *and* g *have, at least, a coincidence point.*

Proof Starting from the point $x_0 \in X$ given by hypothesis 4, condition 1 guarantees that there exists a Picard-Jungck sequence of (T,g) based on x_0, that is, a sequence $\{x_n\} \subseteq X$ which verifies $gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists some $n_0 \in \mathbb{N}$ such that $gx_{n_0} = gx_{n_0+1}$, then $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$, so x_{n_0} is a coincidence point of T and g, and the proof is finished. On the contrary, assume that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$, that is,

$$d(gx_n, gx_{n+1}) > 0 \quad \text{for all } n \in \mathbb{N}.$$

We are going to show that $\{gx_n\}$ converges to a point of coincidence of T and g assuming hypothesis 4(a), that is, supposing that $\alpha(gx_0, Tx_0) \geq 1$ and T is (d,g,α)-right-continuous (the other case is similar). As $\alpha(gx_0, gx_1) = \alpha(gx_0, Tx_0) \geq 1$ and T is (g,α)-admissible, then $\alpha(gx_1, gx_2) = \alpha(Tx_0, Tx_1) \geq 1$. By induction, we obtain that

$$\alpha(gx_n, gx_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{6}$$

Moreover, as α is g-transitive, then, if $n < m$,

$$\alpha(gx_n, gx_{n+1}) \geq 1, \qquad \alpha(gx_{n+1}, gx_{n+2}) \geq 1, \qquad \ldots, \qquad \alpha(gx_{m-1}, gx_m) \geq 1$$
$$\Rightarrow \quad \alpha(gx_n, gx_m) \geq 1.$$

As a result,

$$\alpha(gx_n, gx_m) \geq 1 \quad \text{for all } n, m \in \mathbb{N} \text{ with } n < m. \tag{7}$$

Applying the contractivity condition (5) to $x = x_n$ and $y = x_{n+1}$, we obtain that

$$\begin{aligned}
\psi\big(d(gx_{n+1}, gx_{n+2})\big) &\leq \alpha(gx_n, gx_{n+1})\psi\big(d(gx_{n+1}, gx_{n+2})\big) \\
&= \alpha(gx_n, gx_{n+1})\psi\big(d(Tx_n, Tx_{n+1})\big) \\
&\leq \phi\big(d(gx_n, gx_{n+1})\big)
\end{aligned}$$

for all $n \in \mathbb{N}$. Since $(\psi, \phi) \in \mathcal{F}_{\mathcal{A}}$, then $\{d(gx_n, gx_{n+1})\} \to 0$ by property $(\mathcal{F}^1_{\mathcal{A}})$.

Next, we show that $\{gx_n\}$ is a Cauchy sequence in (X,d) reasoning by contradiction. Assume that $\{gx_n\}$ is not Cauchy. By Lemma 2.1, there exists $\varepsilon_0 > 0$ and two subsequences $\{gx_{n(k)}\}$ and $\{gx_{m(k)}\}$ of $\{gx_n\}$ such that

$$k \leq n(k) < m(k) < n(k+1) \quad \text{and}$$
$$d(gx_{n(k)}, gx_{m(k)-1}) \leq \varepsilon_0 < d(gx_{n(k)}, gx_{m(k)}) \quad \text{for all } k \in \mathbb{N},$$

and also

$$\lim_{n\to\infty} d(gx_{n(k)}, gx_{m(k)}) = \lim_{n\to\infty} d(gx_{n(k)+1}, gx_{m(k)+1}) = \varepsilon_0.$$

As a consequence, there exists $k_0 \in \mathbb{N}$ such that

$$d(gx_{n(k)}, gx_{m(k)}) > \varepsilon_0/2 \quad \text{and} \quad d(gx_{n(k)+1}, gx_{m(k)+1}) > \varepsilon_0/2 \quad \text{for all } k \geq k_0.$$

As $n(k) < m(k)$, by (7) we have that $\alpha(gx_{n(k)}, gx_{m(k)}) \geq 1$ for all $k \geq k_0$. Hence, the contractivity condition (5) yields

$$\begin{aligned}\psi\left(d(gx_{n(k)+1}, gx_{m(k)+1})\right) &\leq \alpha(gx_{n(k)}, gx_{m(k)})\psi\left(d(gx_{n(k)+1}, gx_{m(k)+1})\right) \\ &= \alpha(gx_{n(k)}, gx_{m(k)})\psi\left(d(Tx_{n(k)}, Tx_{m(k)})\right) \leq \phi\left(d(gx_{n(k)}, gx_{m(k)})\right)\end{aligned}$$

for all $k \geq k_0$. Applying condition $(\mathcal{F}_{\mathcal{A}}^2)$ to

$$\left\{a_k = d(gx_{n(k)}, gx_{m(k)})\right\}_{k \geq k_0}, \qquad \left\{b_k = d(gx_{n(k)+1}, gx_{m(k)+1})\right\}_{k \geq k_0} \quad \text{and} \quad L = \varepsilon_0,$$

we deduce that $\varepsilon_0 = L = 0$, which is a contradiction. This contradiction guarantees that $\{gx_n\}$ is a Cauchy sequence in (X, d). Since $\{gx_{n+1}\}_{n\in\mathbb{N}} = \{Tx_n\}_{n\in\mathbb{N}} \subseteq T(X) \subseteq A$ and (A, d) is complete, there exists $u_0 \in A$ such that $\{gx_n\} \to u_0$. Moreover, as $u_0 \in A \subseteq g(X)$, then there exists $z_0 \in X$ such that $gz_0 = u_0$. Taking into account that T is (d, g, α)-right-continuous and (6), it follows that

$$\left.\begin{array}{l}\{gx_n\} \to u_0 = gz_0, \\ \alpha(gx_n, gx_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}\end{array}\right\} \quad \Rightarrow \quad \{Tx_n\} \to Tz_0.$$

But as $Tx_n = gx_{n+1}$ for all $n \in \mathbb{N}$ and $\{gx_n\} \to gz_0$, the uniqueness of the limit of a convergent sequence in a metric space allows us to conclude that $Tz_0 = gz_0$, that is, z_0 is a coincidence point of T and g. □

Theorem 3.2 *Under the hypothesis of Theorem* 3.1, *assume that* $\phi(0) = 0$ *and* $\psi^{-1}(\{0\}) = \{0\}$. *If x and y are two coincidence points of T and g for which there exists $z \in X$ such that z is, at the same time, (g, α)-comparable to x and to y, then $Tx = gx = gy = Ty$.*

Proof Let $x, y \in \operatorname{Coin}(T, g)$ be two coincidence points of T and g for which there exists $z_0 \in X$ such that z_0 is, at the same time, (g, α)-comparable to x and to y. Let $\{z_n\}$ be the Picard-Jungck sequence of (T, g) based on z_0, that is, $gz_{n+1} = Tz_n$ for all $n \in \mathbb{N}$. We are going to show that $\{gz_n\} \to gx$ and $\{gz_n\} \to gy$ so, by the uniqueness of the limit, we will conclude that $gx = gy$.

Firstly, we show that $\{gz_n\} \to gx$. Assume, for instance, that $\alpha(gz_0, gx) \geq 1$. As T is (g, α)-admissible, then $\alpha(gz_1, gx) = \alpha(Tz_0, Tx) \geq 1$. By induction, we deduce that $\alpha(gz_n, gx) \geq 1$ for all $n \in \mathbb{N}$. Using the contractivity condition (5),

$$\psi\left(d(gz_{n+1}, gx)\right) \leq \alpha(gz_n, gx)\psi\left(d(gz_{n+1}, gx)\right) = \alpha(gz_n, gx)\psi\left(d(Tz_n, Tx)\right) \leq \phi\left(d(gz_n, gx)\right)$$

for all $n \in \mathbb{N}$. Next, we distinguish two cases.

- If there exists some $n_0 \in \mathbb{N}$ such that $d(gz_{n_0}, gx) = 0$, then $\psi(d(gz_{n_0+1}, gx)) \leq \phi(0) = 0$, so $gz_{n_0+1} = gx$. In this case, by induction, we deduce that $gz_n = gx$ for all $n \geq n_0$, which implies that $\{gz_n\} \to gx$.

- On the contrary case, assume that $d(gz_n, gx) > 0$ for all $n \in \mathbb{N}$. In such a case, property $(\mathcal{F}^1_{\mathcal{A}})$ applied to $\{a_n = d(gz_n, gx)\}_{n\in\mathbb{N}} \subset (0,\infty)$ guarantees that $\{d(gz_n, gx)\} \to 0$, that is, $\{gz_n\} \to gx$.

If we had supposed that $\alpha(gx, gz_0) \geq 1$, we would have obtained the same conclusion. Then, in any case, $\{gz_n\} \to gx$. Changing the roles of x and y, we also have that $\{gz_n\} \to gy$. Therefore $gx = gy$. □

Theorem 3.3 *Under the hypothesis of Theorem* 3.1, *assume that T and g commute*, $\phi(0) = 0$, $\psi^{-1}(\{0\}) = \{0\}$, *and the following property holds*:

(U) *For all coincidence points x and y of T and g, there exists $z \in X$ such that z is, at the same time, (g,α)-comparable to x and to y.*

Hence T and g have a unique common fixed point $\omega \in X$. Furthermore, $\omega = gx$ for all $x \in \mathrm{Coin}(T, g)$.

Proof Let $x \in \mathrm{Coin}(T, g)$ be an arbitrary coincidence point of T and g and let $\omega = gx$. As T and g commute, then $T\omega = Tgx = gTx = g\omega$, so ω is another coincidence point of T and g. By hypothesis (U), there exists $z \in X$ such that z is, at the same time, (g,α)-comparable to x and to ω. Hence, Theorem 3.2 guarantees that $gx = g\omega$, which means that $\omega = gx = g\omega$. As a result, $\omega = g\omega = T\omega$, that is, ω is a common fixed point of T and g.

To prove the uniqueness, let $u, v \in X$ be two common fixed points of T and g. As u and v are coincidence points of T and g, hypothesis (U) implies that there exists $z \in X$ such that z is, at the same time, (g,α)-comparable to u and to v. Thus, Theorem 3.2 guarantees that $gu = gv$, which means that $u = gu = gv = v$. As a consequence, T and g have a unique common fixed point, which is ω. □

4 Consequences of the main results

The best advantage of the previous theorems is that they can be particularized in a wide variety of different results. This section is dedicated to deducing some direct consequences of them in the context of metric spaces. For instance, in the following statement we use Lemma 3.1.

Corollary 4.1 *Let (X, d) be a metric space, let $\alpha : X \times X \to [0,\infty)$ be a function, and let $T, g : X \to X$ be two mappings such that the following conditions are fulfilled*:

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A \subseteq g(X)$ and (A, d) is complete*;
2. *α is g-transitive and T is (g,α)-admissible*;
3. *There exist two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and nondecreasing on $(0,\infty)$, ϕ is upper semi-continuous from the right on $(0,\infty)$, $\phi < \psi$ on $(0,\infty)$, and the following inequality holds*:

$$\alpha(gx, gy)\psi\big(d(Tx, Ty)\big) \leq \phi\big(d(gx, gy)\big) \quad \textit{for all } x, y \in X;$$

4. *At least, one of the following conditions holds*:
   (a) *there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geq 1$ and T is (d, g,α)-right-continuous*;
   (b) *there exists $x_0 \in X$ such that $\alpha(Tx_0, gx_0) \geq 1$ and T is (d, g,α)-left-continuous.*

Then T and g have, at least, a coincidence point.

Additionally, assume that T and g commute, $\phi(0) = 0$, $\psi^{-1}(\{0\}) = \{0\}$, *and the following property holds*:

(U) *For all coincidence points x and y of T and g, there exists $z \in X$ such that z is, at the same time, (g,α)-comparable to x and to y.*

Then T and g have a unique common fixed point $\omega \in X$. Furthermore, $\omega = gx$ for all $x \in \operatorname{Coin}(T,g)$.

Proof It follows from Theorems 3.1 and 3.3 taking into account that, by Lemma 3.1, $(\psi,\phi) \in \mathcal{F}_{\mathcal{A}}$. □

In the following result, we use a different contractivity condition involving three control functions by decomposing $\phi = \theta - \varphi$.

Corollary 4.2 *Let (X,d) be a metric space, let $\alpha : X \times X \to [0,\infty)$ be a function, and let $T,g : X \to X$ be two mappings such that the following conditions are fulfilled:*

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A \subseteq g(X)$ and (A,d) is complete;*
2. *α is g-transitive and T is (g,α)-admissible;*
3. *There exist three functions $\psi,\theta,\varphi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and nondecreasing on $(0,\infty)$, θ is upper semi-continuous from the right on $(0,\infty)$, φ is lower semi-continuous from the right on $(0,\infty)$, $\theta - \varphi < \psi$ on $(0,\infty)$, and the following inequality holds:*

$$\alpha(gx,gy)\psi\big(d(Tx,Ty)\big) \le \theta\big(d(gx,gy)\big) - \varphi\big(d(gx,gy)\big) \quad \text{for all } x,y \in X;$$

4. *At least, one of the following conditions holds:*
   (a) *there exists $x_0 \in X$ such that $\alpha(gx_0,Tx_0) \ge 1$ and T is (d,g,α)-right-continuous;*
   (b) *there exists $x_0 \in X$ such that $\alpha(Tx_0,gx_0) \ge 1$ and T is (d,g,α)-left-continuous.*

Then T and g have, at least, a coincidence point.

Additionally, assume that T and g commute, $\theta(0) = \varphi(0)$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:

(U) *For all coincidence points x and y of T and g, there exists $z \in X$ such that z is, at the same time, (g,α)-comparable to x and to y.*

Then T and g have a unique common fixed point $\omega \in X$. Furthermore, $\omega = gx$ for all $x \in \operatorname{Coin}(T,g)$.

Proof We only have to use $\phi = \theta - \varphi$ in Corollary 4.1, because ϕ is upper semi-continuous from the right on $(0,\infty)$. Moreover, for all $t > 0$, we have that

$$\psi(t) - \phi(t) = \psi(t) - \big(\theta(t) - \varphi(t)\big) = \psi(t) - \theta(t) + \varphi(t) > 0,$$

so $\phi(t) < \psi(t)$ for all $t > 0$. □

Corollary 4.3 *Corollary* 4.2 *also holds if we replace the third condition with the following one:*

(3′) *There exist three functions $\psi \in \mathcal{F}_{\text{alt}}$ and $\theta,\varphi : [0,\infty) \to [0,\infty)$ such that θ is upper semi-continuous from the right on $(0,\infty)$, φ is lower semi-continuous from the right on $(0,\infty)$, $\theta - \varphi < \psi$ on $(0,\infty)$, and the following inequality holds:*

$$\alpha(gx,gy)\psi\big(d(Tx,Ty)\big) \le \theta\big(d(gx,gy)\big) - \varphi\big(d(gx,gy)\big) \quad \text{for all } x,y \in X.$$

It is also interesting to highlight the case in which $\theta = \psi$.

Corollary 4.4 *Let (X,d) be a metric space, let $\alpha : X \times X \to [0,\infty)$ be a function, and let $T,g : X \to X$ be two mappings such that the following conditions are fulfilled:*

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A \subseteq g(X)$ and (A,d) is complete;*
2. *α is g-transitive and T is (g,α)-admissible;*
3. *There exist two functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and nondecreasing on $(0,\infty)$, φ is lower semi-continuous from the right on $(0,\infty)$, $\varphi > 0$ on $(0,\infty)$, and the following inequality holds:*

$$\alpha(gx,gy)\psi\bigl(d(Tx,Ty)\bigr) \leq \psi\bigl(d(gx,gy)\bigr) - \varphi\bigl(d(gx,gy)\bigr) \quad \textit{for all } x,y \in X;$$

4. *At least, one of the following conditions holds:*
   (a) *there exists $x_0 \in X$ such that $\alpha(gx_0,Tx_0) \geq 1$ and T is (d,g,α)-right-continuous;*
   (b) *there exists $x_0 \in X$ such that $\alpha(Tx_0,gx_0) \geq 1$ and T is (d,g,α)-left-continuous.*

Then T and g have, at least, a coincidence point.

Additionally, assume that T and g commute, $\psi(0) = \varphi(0)$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:

(U) *For all coincidence points x and y of T and g, there exists $z \in X$ such that z is, at the same time, (g,α)-comparable to x and to y.*

Then T and g have a unique common fixed point $\omega \in X$. Furthermore, $\omega = gx$ for all $x \in \operatorname{Coin}(T,g)$.

If we use ψ as the identity mapping on $[0,\infty)$, we deduce the following statement.

Corollary 4.5 *Corollary* 4.4 *remains true if we replace the third condition with the following one:*

(3′) *There exists a lower semi-continuous from the right on $(0,\infty)$ function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi > 0$ on $(0,\infty)$, and the following inequality holds:*

$$\alpha(gx,gy)d(Tx,Ty) \leq d(gx,gy) - \varphi\bigl(d(gx,gy)\bigr) \quad \textit{for all } x,y \in X.$$

Furthermore, if $\lambda \in [0,1)$ and we use $\varphi(t) = (1-\lambda)t$ for all $t \in [0,\infty)$, then we derive the following result.

Corollary 4.6 *Corollary* 4.4 *remains true if we replace the third condition with the following one:*

(3″) *There exists $\lambda \in [0,1)$ such that*

$$\alpha(gx,gy)d(Tx,Ty) \leq \lambda d(gx,gy) \quad \textit{for all } x,y \in X.$$

In the following result, we use $\alpha(x,y) = 1$ for all $x,y \in X$.

Corollary 4.7 *Let (X,d) be a metric space and let $T,g : X \to X$ be two mappings such that the following conditions are fulfilled:*

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A \subseteq g(X)$ and (A,d) is complete;*

2. *There exists* $(\psi,\phi) \in \mathcal{F}_{\mathcal{A}}$ *such that*

$$\psi\big(d(Tx,Ty)\big) \le \phi\big(d(gx,gy)\big) \quad \text{for all } x,y \in X;$$

3. T *is* (d,g)*-continuous.*

Then T *and* g *have, at least, a coincidence point.*

Additionally, assume that T *and* g *commute,* $\phi(0) = 0$ *and* $\psi^{-1}(\{0\}) = \{0\}$. *Then* T *and* g *have a unique common fixed point* $\omega \in X$. *Furthermore,* $\omega = gx$ *for all* $x \in \mathrm{Coin}(T,g)$.

5 Some coincidence point theorems in metric spaces endowed with a binary relation

As we pointed out in Introduction, one of the branches that have attracted much attention in fixed point theory is dedicated to partially ordered metric spaces. However, some properties of a partial order are not necessary to prove some fixed/coincidence point theorem. In this section we show some consequences of our main results in the setting of metric spaces endowed with a binary relation which has only to be transitive on a subset of the metric space.

Definition 5.1 Let $\prec$ be a binary relation on a set X and let A be a nonempty subset of X. We say that $\prec$ is *transitive on* A if $a \prec c$ for all $a,b,c \in A$ such that $a \prec b$ and $b \prec c$.

Definition 5.2 Let (X,d) be a metric space and let $\prec$ be a binary relation on X. Given two mappings $T,g : X \to X$ and a point $z_0 \in X$, we say that T is $(d,g,\prec)$*-nondecreasing-continuous at* z_0 is $\{Tx_n\} \to Tz_0$ for all sequence $\{x_n\} \subseteq X$ such that $\{gx_n\} \to gz_0$ and $gx_n \prec gx_{n+1}$ for all $n \in \mathbb{N}$. And T is $(d,g,\prec)$*-nondecreasing-continuous in* $A \subseteq X$ if T is $(d,g,\prec)$-nondecreasing-continuous at every point of A.

Similarly, we say that T is $(d,g,\prec)$*-nonincreasing-continuous* at z_0 is $\{Tx_n\} \to Tz_0$ for all sequence $\{x_n\} \subseteq X$ such that $\{gx_n\} \to gz_0$ and $gx_{n+1} \prec gx_n$ for all $n \in \mathbb{N}$. And T is $(d,g,\prec)$*-nonincreasing-continuous* in $A \subseteq X$ if T is $(d,g,\prec)$-nonincreasing-continuous at every point of A.

The following result directly follows from the respective definitions.

Lemma 5.1 *Let* (X,d) *be a metric space, let* $T,g : X \to X$ *be two mappings, let* $\prec$ *be a binary relation on* X, *and let* $\alpha_\prec : X \times X \to [0,\infty)$ *be the function defined in* (2). *Then the following properties hold.*

1. *Given* $z_0 \in X$, *the mapping* T *is* $(d,g,\prec)$*-nonincreasing-continuous at* z_0 *if and only if it is* $(d,g,\alpha_\prec)$*-right-continuous at* z_0;
2. T *is* $(d,g,\prec)$*-nonincreasing-continuous if and only if* T *is* $(d,g,\alpha_\prec)$*-right-continuous*;
3. T *is* $(g,\alpha_\prec)$*-admissible if and only if* T *is* $(g,\prec)$*-nondecreasing*;
4. *The binary relation* $\prec$ *is transitive on* $g(X)$ *if and only if* $\alpha_\prec$ *is* g*-transitive.*

The main result of this section is the following one.

Theorem 5.1 *Let* (X,d) *be a metric space endowed with a binary relation* $\prec$ *and let* $T,g : X \to X$ *be two mappings such that the following conditions are fulfilled*:

1. *There exists a subset* $A \subseteq X$ *such that* $T(X) \subseteq A \subseteq g(X)$ *and* (A,d) *is complete*;

2. *The binary relation $\prec$ is transitive on $g(X)$ and T is $(g,\prec)$-nondecreasing*;
3. *There exists $(\psi,\phi) \in \mathcal{F}_\mathcal{A}$ such that*

$$\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(gx,gy)\big) \quad \textit{for all } x,y \in X \textit{ such that } gx \prec gy;$$

4. *At least, one of the following conditions holds*:
   (a) *there exists $x_0 \in X$ such that $gx_0 \prec Tx_0$ and T is $(d,g,\prec)$-nondecreasing-continuous*;
   (b) *there exists $x_0 \in X$ such that $Tx_0 \prec gx_0$ and T is $(d,g,\prec)$-nonincreasing-continuous.*

Then T and g have, at least, a coincidence point.

Additionally, assume that T and g commute, $\phi(0) = 0$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:

(U) *For all coincidence points x and y of T and g, there exists $z \in X$ such that gz is, at the same time, $\prec$-comparable to gx and to gy.*

Then T and g have a unique common fixed point $\omega \in X$. Furthermore, $\omega = gx$ for all $x \in \mathrm{Coin}(T,g)$.

Notice that, in the previous result, the binary relation $\prec$ must only be transitive on $g(X)$.

Proof It follows from Theorems 3.1 and 3.3 using the function $\alpha_\prec : X \times X \to [0,\infty)$ defined in (2) and taking into account the equivalences given in Lemma 5.1. □

We can repeat Corollaries 4.1-4.7 in this new framework. However, among them, we only highlight the following one.

Corollary 5.1 *Let (X,d) be a metric space endowed with a binary relation $\prec$ and let $T,g : X \to X$ be two mappings such that the following conditions are fulfilled*:

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A \subseteq g(X)$ and (A,d) is complete*;
2. *The binary relation $\prec$ is transitive on $g(X)$ and T is $(g,\prec)$-nondecreasing*;
3. *There exist three functions $\psi,\theta,\varphi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and nondecreasing on $(0,\infty)$, θ is upper semi-continuous from the right on $(0,\infty)$, φ is lower semi-continuous from the right on $(0,\infty)$, $\theta - \varphi < \psi$ on $(0,\infty)$, and the following inequality holds*:

$$\psi\big(d(Tx,Ty)\big) \leq \theta\big(d(gx,gy)\big) - \varphi\big(d(gx,gy)\big) \quad \textit{for all } x,y \in X \textit{ such that } gx \prec gy;$$

4. *At least, one of the following conditions holds*:
   (a) *there exists $x_0 \in X$ such that $gx_0 \prec Tx_0$ and T is $(d,g,\prec)$-nondecreasing-continuous*;
   (b) *there exists $x_0 \in X$ such that $Tx_0 \prec gx_0$ and T is $(d,g,\prec)$-nonincreasing-continuous.*

Then T and g have, at least, a coincidence point.

Additionally, assume that T and g commute, $\theta(0) = \varphi(0)$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:

(U) *For all coincidence points x and y of T and g, there exists $z \in X$ such that gz is, at the same time, $\prec$-comparable to gx and to gy.*

Then T and g have a unique common fixed point $\omega \in X$. Furthermore, $\omega = gx$ for all $x \in \mathrm{Coin}(T,g)$.

Proof It directly follows from Corollary 4.2 using the function $\alpha_{\prec} : X \times X \to [0,\infty)$ defined in (2). □

Corollary 5.2 *Theorem* 5.1 *and Corollary* 5.1 *also hold if $\prec$ is a transitive relation on X, or a preorder on X, or a partial order on X.*

6 Fixed point theorems

If we use g as the identity mapping on X, we obtain the following fixed point theorems in metric spaces, endowed with a binary relation or not.

Theorem 6.1 *Let (X,d) be a metric space, let $\alpha : X \times X \to [0,\infty)$ be a function and let $T : X \to X$ be a mapping such that the following conditions are fulfilled*:

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A$ and (A,d) is complete*;
2. *α is transitive and T is α-admissible*;
3. *There exists $(\psi,\phi) \in \mathcal{F}_{\mathcal{A}}$ such that*

$$\alpha(x,y)\psi\bigl(d(Tx,Ty)\bigr) \le \phi\bigl(d(x,y)\bigr) \quad \text{for all } x,y \in X;$$

4. *At least, one of the following conditions holds*:
   (a) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and T is (d,α)-right-continuous*;
   (b) *there exists $x_0 \in X$ such that $\alpha(Tx_0,x_0) \ge 1$ and T is (d,α)-left-continuous.*

Then T has, at least, a fixed point.
Additionally, assume that $\phi(0) = 0$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:
(U) *For all fixed points x and y of T, there exists $z \in X$ such that z is, at the same time, α-comparable to x and to y.*
Then T has a unique fixed point.

Proof It follows from Theorems 3.1 and 3.3 using g as the identity mapping on X. □

Corollary 6.1 *Let (X,d) be a metric space, let $\alpha : X \times X \to [0,\infty)$ be a function and let $T : X \to X$ be a mapping such that the following conditions are fulfilled*:

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A$ and (A,d) is complete*;
2. *α is transitive and T is α-admissible*;
3. *There exist two functions $\psi,\phi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and nondecreasing on $(0,\infty)$, ϕ is upper semi-continuous from the right on $(0,\infty)$, $\phi < \psi$ on $(0,\infty)$, and the following inequality holds*:

$$\alpha(x,y)\psi\bigl(d(Tx,Ty)\bigr) \le \phi\bigl(d(x,y)\bigr) \quad \text{for all } x,y \in X;$$

4. *At least, one of the following conditions holds*:
   (a) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and T is (d,α)-right-continuous*;
   (b) *there exists $x_0 \in X$ such that $\alpha(Tx_0,x_0) \ge 1$ and T is (d,α)-left-continuous.*

Then T has, at least, a fixed point.
Additionally, assume that $\phi(0) = 0$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:

(U) *For all fixed points x and y of T, there exists $z \in X$ such that z is, at the same time, α-comparable to x and to y.*

Then T has a unique fixed point.

Proof It follows from Corollary 4.1 using g as the identity mapping on X. □

In the context of metric spaces that are not endowed with binary relations, we can also highlight the following statement, in which we assume that $\alpha(x,y) = 1$ for all $x, y \in X$.

Corollary 6.2 *Let (X,d) be a complete metric space and let $T : X \to X$ be a continuous mapping. Assume that there exist an altering distance function ψ, an upper semi-continuous from the right function $\theta : [0,\infty) \to [0,\infty)$, and a lower semi-continuous from the right function $\varphi : [0,\infty) \to [0,\infty)$ such that*

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \theta\bigl(d(x,y)\bigr) - \varphi\bigl(d(x,y)\bigr) \quad \text{for all } x, y \in X,$$

where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

In the case of metric spaces endowed with binary relations, we have the following results.

Theorem 6.2 *Let (X,d) be a metric space endowed with binary relation $\prec$ and let $T : X \to X$ be a mapping such that the following conditions are fulfilled:*

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A$ and (A,d) is complete;*
2. *The binary relation $\prec$ is transitive and T is $\prec$-nondecreasing;*
3. *There exists $(\psi,\phi) \in \mathcal{F}_A$ such that*

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \phi\bigl(d(x,y)\bigr) \quad \text{for all } x, y \in X \text{ such that } x \prec y;$$

4. *At least, one of the following conditions holds:*
   (a) *there exists $x_0 \in X$ such that $x_0 \prec Tx_0$ and T is $(d,\prec)$-nondecreasing-continuous;*
   (b) *there exists $x_0 \in X$ such that $Tx_0 \prec x_0$ and T is $(d,\prec)$-nonincreasing-continuous.*

Then T has, at least, a fixed point.

Additionally, assume that $\phi(0) = 0$, $\psi^{-1}(\{0\}) = \{0\}$, and the following property holds:

(U) *For all fixed points x and y of T, there exists $z \in X$ such that z is, at the same time, $\prec$-comparable to x and to y.*

Then T has a unique fixed point.

Proof It follows from Theorem 5.1 using g as the identity mapping on X. □

Corollary 6.3 *Let (X,d) be a metric space endowed with a binary relation $\prec$ and let $T : X \to X$ be a mapping such that the following conditions are fulfilled:*

1. *There exists a subset $A \subseteq X$ such that $T(X) \subseteq A$ and (A,d) is complete;*
2. *The binary relation $\prec$ is transitive and T is $\prec$-nondecreasing;*
3. *There exist three functions $\psi, \theta, \varphi : [0,\infty) \to [0,\infty)$ such that ψ is continuous and nondecreasing on $(0,\infty)$, θ is upper semi-continuous from the right on $(0,\infty)$, φ is*

lower semi-continuous from the right on $(0,\infty)$, $\theta - \varphi < \psi$ *on* $(0,\infty)$, *and the following inequality holds*:

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \theta\bigl(d(x,y)\bigr) - \varphi\bigl(d(x,y)\bigr) \quad \textit{for all } x,y \in X \textit{ such that } x \prec y;$$

4. *At least, one of the following conditions holds*:
   (a) *there exists* $x_0 \in X$ *such that* $x_0 \prec Tx_0$ *and* T *is* $(d,\prec)$*-nondecreasing-continuous*;
   (b) *there exists* $x_0 \in X$ *such that* $Tx_0 \prec x_0$ *and* T *is* $(d,\prec)$*-nonincreasing-continuous*.

Then T *has, at least, a fixed point*.

Additionally, assume that $\theta(0) = \varphi(0)$, $\psi^{-1}(\{0\}) = \{0\}$, *and the following property holds*:

(U) *For all fixed points* x *and* y *of* T, *there exists* $z \in X$ *such that* z *is, at the same time,* $\prec$*-comparable to* x *and to* y.

Then T *has a unique fixed point*.

Proof It follows from Corollary 5.1 using g as the identity mapping on X. □

7 A unified version of Ran and Reurings' theorem and Nieto and Rodríguez-López's theorem

After the appearance of Ran and Reurings' theorem [17] and Nieto and Rodríguez-López's theorem [18], many fixed point results were introduced in the ambient of metric spaces endowed with partial orders (see, for instance, [19] in which the authors introduced a close contractivity condition in L-spaces). Since them, many coincidence/fixed point theorems have been proved distinguishing between either the involved mappings are continuous or the ambient space is regular. In this section, we show a unified version of both theorems using a unique condition. The following one is the particularization of Definition 5.2 to the case in which g is the identity mapping on X and $\prec$ is an arbitrary binary relation on X.

Definition 7.1 Given a metric space (X,d) endowed with a binary relation $\prec$, a mapping $T: X \to X$ is $(d,\prec)$*-nondecreasing-continuous at* $z_0 \in X$ if we have that $\{Tx_n\}$ converges to Tz_0 for all $\prec$-nondecreasing sequence $\{x_n\}$ convergent to z_0. And T is $(d,\prec)$*-nondecreasing-continuous* if it is $(d,\prec)$-nondecreasing-continuous at every point of X.

Similarly, a mapping $T: X \to X$ is $(d,\prec)$*-nonincreasing-continuous at* $z_0 \in X$ if we have that $\{Tx_n\}$ converges to Tz_0 for all $\prec$-nonincreasing sequence $\{x_n\}$ convergent to z_0. And T is $(d,\prec)$*-nonincreasing-continuous* if it is $(d,\prec)$-nonincreasing-continuous at every point of X.

It is obvious that every continuous mapping is also nondecreasing-continuous, but the converse is false.

Example 7.1 If $\mathbb{R}$ is endowed with the Euclidean metric ($d_e(x,y) = |x-y|$ for all $x,y \in \mathbb{R}$) and its usual partial order $\leq$, then the mapping

$$Tx = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases}$$

is $(d_e,\leq)$-nondecreasing-continuous on $\mathbb{R}$, but it is not continuous at $x = 0$.

The following one is a particularization of Corollary 6.3 using altering distance functions.

Theorem 7.1 *Let $(X, d, \prec)$ be a metric space endowed with a transitive binary relation $\prec$ and let $T : X \to X$ be a mapping such that the following conditions are fulfilled*:

1. (X, d) *(or $(T(X), d)$) is complete*;
2. *T is nondecreasing (w.r.t. $\prec$)*;
3. *There exist an altering distance function ψ and an upper semi-continuous from the right function $\phi : [0,\infty) \to [0,\infty)$ such that*

$$\psi\big(d(Tx, Ty)\big) \le \phi\big(d(x, y)\big) \quad \textit{for all } x, y \in X \textit{ such that } x \prec y,$$

   where $\phi(0) = 0$ and $\phi(t) < \psi(t)$ for all $t > 0$;
4. *At least, one of the following conditions holds*:
   (a) *there exists $x_0 \in X$ such that $x_0 \prec Tx_0$ and T is $(d, \prec)$-nondecreasing-continuous*;
   (b) *there exists $x_0 \in X$ such that $x_0 \succ Tx_0$ and T is $(d, \prec)$-nonincreasing-continuous*;

Then T has, at least, a fixed point.

Furthermore, assume that the following property holds:

(U) *For each $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ which is $\prec$-comparable to x and y.*

Then T has a unique fixed point.

The previous result improves Theorem 1.8 in three senses: (1) the binary relation $\prec$ does not have to be a partial order, but a transitive binary relation; (2) ϕ has only to be upper semi-continuous from the right; (3) the mapping T must only be $(d, \prec)$-nondecreasing-continuous, which is a condition that unifies and extends hypotheses 4(a) and 4(b) of Theorem 1.8.

Theorem 7.2 *Theorem* 7.1 *also holds if we replace assumption* 3 *with the following one*:

1. *There exist an altering distance function ψ, an upper semi-continuous from the right function $\theta : [0,\infty) \to [0,\infty)$, and a lower semi-continuous from the right function $\varphi : [0,\infty) \to [0,\infty)$ such that*

$$\psi\big(d(Tx, Ty)\big) \le \theta\big(d(x, y)\big) - \varphi\big(d(x, y)\big) \quad \textit{for all } x \succ y, \tag{8}$$

   where $\theta(0) = \varphi(0) = 0$ and $\psi(t) - \theta(t) + \varphi(t) > 0$ for all $t > 0$.

Corollary 7.1 *Theorem* 7.2 *also holds if we replace assumption* 4 *with the following one*:

(3′) *There exists $x_0 \in X$ such that $x_0 \asymp Tx_0$ and T is continuous.*

Proof If follows from the fact that if T is continuous, then it is both $(d, \prec)$-nondecreasing-continuous and $(d, \prec)$-nonincreasing-continuous. □

Corollary 7.2 *Theorem* 7.2 *also holds if we replace assumption* 4 *with the following one*:

(3″) *There exists $x_0 \in X$ such that $x_0 \asymp Tx_0$ and $(X, d, \prec)$ is regular.*

Proof Following, point by point, the arguments of the proof of Theorem 3.1, we obtain that the Picard sequence $\{x_{n+1} = Tx_n\}$ is Cauchy and also it is $\prec$-monotone. As (X, d) (or

$(T(X), d))$ is complete, there exists $z \in X$ such that $\{x_n\} \to z$. Since $(X, d, \prec)$ is regular, then

$$x_n \prec z \quad \text{for all } n \in \mathbb{N} \quad \text{or} \quad x_n \succ z \quad \text{for all } n \in \mathbb{N}.$$

In any case, we can use the contractivity condition (8), which yields

$$\psi\big(d(x_{n+1}, Tz)\big) = \psi\big(d(Tx_n, Tz)\big) \le \theta\big(d(x_n, z)\big) - \varphi\big(d(z_n, z)\big) \tag{9}$$

for all $n \in \mathbb{N}$. Assume, for n arbitrarily large, that $x_n \neq z$. If $d(x_{n+1}, Tz) > d(x_n, z)$ for some (large) n, then $\psi(d(x_{n+1}, Tz)) \ge \psi(d(x_n, z)) > \theta(d(x_n, z)) - \varphi(d(x_n, z))$, which contradicts (9). As a result, we have $d(x_{n+1}, Tz) \le d(x_n, z)$ for n arbitrarily large. On taking limit as $n \to \infty$, we conclude that $\{d(x_{n+1}, Tz)\} \to 0$, that is, $\{x_{n+1}\} \to Tz$. By the uniqueness of the limit, $Tz = z$ and z is a fixed point of T. □

The following results are well known in the field of fixed point theory.

Corollary 7.3 (Ran and Reurings [17]) *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold*:

(a) *(X, d) is complete*;
(b) *T is nondecreasing (w.r.t. $\preccurlyeq$)*;
(c) *T is continuous*;
(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$*;
(e) *There exists a constant $\lambda \in (0, 1)$ such that $d(Tx, Ty) \le \lambda d(x, y)$ for all $x, y \in X$ with $x \succcurlyeq y$.*

Then T has a fixed point. Moreover, if for all $(x, y) \in X^2$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.

Proof It follows from Corollary 7.1. □

Corollary 7.4 (Nieto and Rodríguez-López [18]) *Let $(X, \preccurlyeq)$ be an ordered set endowed with a metric d and $T : X \to X$ be a given mapping. Suppose that the following conditions hold*:

(a) *(X, d) is complete*;
(b) *T is nondecreasing (w.r.t. $\preccurlyeq$)*;
(c) *If a nondecreasing sequence $\{x_m\}$ in X converges to some point $x \in X$, then $x_m \preccurlyeq x$ for all m*;
(d) *There exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$*;
(e) *There exists a constant $\lambda \in (0, 1)$ such that $d(Tx, Ty) \le \lambda d(x, y)$ for all $x, y \in X$ with $x \succcurlyeq y$.*

Then T has a fixed point. Moreover, if for all $(x, y) \in X^2$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we obtain uniqueness of the fixed point.

Proof It follows from Corollary 7.2. □

Finally, we also prove that the Shaddad *et al.* theorem is an easy consequence of our main results.

Corollary 7.5 *Theorem* 1.8 *immediately follows from Theorem* 7.1.

Proof It is only necessary to apply Corollaries 7.1 and 7.2 (which are immediate consequences of Theorem 7.1), which cover cases 4(a) and 4(b) in Theorem 1.8. □

Finally, we point out that the present techniques can be easily generalized to guarantee the existence and uniqueness of multidimensional coincidence/fixed points following the techniques described in [20–26].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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