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# New fixed point theorem under $R$ -contractions

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## Abstract

In this manuscript we introduce the notions of  $R$ -function and  $R$ -contractions, and we show an *ad hoc* fixed point theorem. We prove that this new kind of contractions properly includes the family of all Meir-Keeler contractions and other well-known classes of contractions that have been given very recently (for instance, those using simulation functions and manageable functions). As a consequence, our approach turns out to be appropriate to unify the treatment of different kinds of contractive nonlinear operators.

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## 1 Introduction

Fixed point theory is a branch of nonlinear analysis that can be applied successfully to a wide range of contexts in social and natural sciences. Although some results had been introduced before, it is usually considered that this field of study was born in 1922, when Banach presented a celebrated theorem in order to guarantee that a nonlinear operator had a fixed point. After the appearance of the Banach contractive mapping principle, lots of generalizations, in many different frameworks, have been done. In many cases, new results that currently are being obtained involve contractivity conditions that depend on auxiliary functions (comparison functions, Geraghty functions, altering distance functions, Bianchini-Grandolfi gauge functions, *etc.*).

One of the extensions that have attracted much attention over the last years was due to Meir and Keeler (see [1]) who introduced in 1969 a family of contractive mappings in a new sense. Although their original notion did not depend on auxiliary functions, Lim [2] proved that a self-mapping was a Meir-Keeler contraction if, and only if, it satisfied a contractivity condition in a classical sense depending on a new class of functions (that he called *L-functions*). After that, several extensions of Meir-Keeler contractions have appeared (see, for instance, [3–11]).

Very recently, Khojasteh *et al.* (see [12]) introduced the notion of *simulation function*, which was later modified by Roldán-López-de-Hierro *et al.* in a subtle way (see [13]). The main difference with respect to previous approaches was that simulation functions depend on two variables rather than on a unique variable. And, in order to extend some results in the field of multi-valued maps, Du and Khojasteh presented the very close (but

independent) notion of *manageable function* (see [14]). Surprisingly, contractions that use simulation functions turned out to be Meir-Keeler contractions (see [15]).

This fact points up the difficulty in finding true extensions of Meir-Keeler contractions when we use a simple contractivity condition only involving classical terms as  $d(x, y)$  and  $d(Tx, Ty)$ , where  $d$  is the distance and  $T : X \rightarrow X$  is the nonlinear operator.

The main aim of the present manuscript is to give a set of auxiliary functions that let us consider a true extension of Meir-Keeler contractions. To do that, we present the notion of *R-contraction*, which permits us to introduce such a large family of contractions that includes not only Meir-Keeler contractions, but Geraghty contractions, contractions depending on simulations functions and manageable functions, *etc.* We illustrate such kind of contractions with an example in which previous results are not applicable.

## 2 Preliminaries

In the sequel,  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$  stands for the set of all nonnegative integers. Throughout this manuscript, let  $A \subseteq \mathbb{R}$  be a nonempty subset of real numbers, let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be a mapping from  $X$  into itself. We denote the range of  $d$  by

$$\text{ran}(d) = \{d(x, y) : x, y \in X\} \subseteq [0, \infty).$$

We say that a sequence  $\{x_n\} \subseteq X$  is *asymptotically regular on  $(X, d)$*  if  $\{d(x_n, x_{n+1})\} \rightarrow 0$ .

A *fixed point of  $T$*  is a point  $x \in X$  such that  $Tx = x$ . The *iterates* of the self-mapping  $T$  are the mappings  $\{T^n : X \rightarrow X\}_{n \in \mathbb{N}}$  defined by

$$T^0 = I_X, \quad T^1 = T, \quad T^2 = T \circ T, \quad T^{n+1} = T \circ T^n \quad \text{for all } n \geq 2.$$

Given a point  $x_0 \in X$ , the *Picard sequence of  $T$  based on  $x_0$*  is the sequence  $\{x_n\}_{n \in \mathbb{N}}$  given by  $x_{n+1} = Tx_n$  for all  $n \in \mathbb{N}$ . Notice that  $x_n = T^n x_0$  for all  $n \in \mathbb{N}$ .

Following [16, 17], we say that  $T$  is a *weakly Picard operator* if, for all  $x_0 \in X$ , the Picard sequence of  $T$  based on  $x_0$  converges to a fixed point of  $T$ . Furthermore,  $T$  is a *Picard operator* if it is a weakly Picard operator and it has a unique fixed point. In such a case, if  $z_0$  is the unique fixed point of  $T$ , then  $\{T^n x_0\} \rightarrow z_0$  for all  $x_0 \in X$ .

A function  $\phi : [0, \infty) \rightarrow [0, \infty)$  is *lower semi-continuous* if

$$\phi(L) \leq \liminf_{t \rightarrow L} \phi(t) \quad \text{for all } L \in [0, \infty).$$

A *Geraghty function* is a function  $\phi : [0, \infty) \rightarrow [0, 1)$  such that if  $\{t_n\} \subset [0, \infty)$  and  $\{\phi(t_n)\} \rightarrow 1$ , then  $\{t_n\} \rightarrow 0$ . A *Geraghty contraction* (see [18]) is a mapping  $T : X \rightarrow X$  such that

$$d(Tx, Ty) \leq \phi(d(x, y))d(x, y) \quad \text{for all } x, y \in X,$$

where  $\phi$  is a Geraghty function.

### 2.1 Simulation functions and manageable functions

The notion of *simulation function* was introduced by Khojasteh *et al.* in [12] as follows.

**Definition 1** (Khojasteh *et al.* [12]) A *simulation function* is a mapping  $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  satisfying the following conditions:

- ( $\zeta_1$ )  $\zeta(0, 0) = 0$ ;
- ( $\zeta_2$ )  $\zeta(t, s) < s - t$  for all  $t, s > 0$ ;
- ( $\zeta_3$ ) if  $\{t_n\}, \{s_n\}$  are sequences in  $(0, \infty)$  such that  $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$ , then

$$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

The third condition is symmetric in both arguments of  $\zeta$  but, in proofs, this property is not necessary. In fact, in practise, the arguments of  $\zeta$  have different meanings and they play different roles. Then, Roldán-López-de-Hierro *et al.* slightly modified the previous definition in order to highlight this difference and to enlarge the family of all simulation functions.

**Definition 2** (Roldán-López-de-Hierro *et al.* [13]) A *simulation function* is a mapping  $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  satisfying the following conditions:

- ( $\zeta_1$ )  $\zeta(0, 0) = 0$ ;
- ( $\zeta_2$ )  $\zeta(t, s) < s - t$  for all  $t, s > 0$ ;
- ( $\zeta_3$ ) if  $\{t_n\}, \{s_n\}$  are sequences in  $(0, \infty)$  such that  $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$  and  $t_n < s_n$  for all  $n \in \mathbb{N}$ , then

$$\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0.$$

Let  $\mathcal{Z}$  be the family of all simulation functions  $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ .

Every simulation function in the original Khojasteh *et al.*'s sense (Definition 1) is also a simulation function in our sense (Definition 2), but the converse is not true (see [13]).

**Definition 3** (Khojasteh *et al.* [12], Roldán-López-de-Hierro *et al.* [13]) Let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be a self-mapping. We say that  $T$  is a  $\mathcal{Z}$ -*contraction* if there exists  $\zeta \in \mathcal{Z}$  such that

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \quad \text{for all } x, y \in X \text{ such that } x \neq y.$$

In 2014, Du and Khojasteh [14] introduced the concept of manageable functions. They showed that many known results can be deduced of some local constraints related to manageable functions.

**Definition 4** (Du and Khojasteh [14]) A function  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is called *manageable* if the following conditions hold:

- ( $\eta_1$ )  $\eta(t, s) < s - t$  for all  $s, t > 0$ .
- ( $\eta_2$ ) For any bounded sequence  $\{t_n\} \subset (0, \infty)$  and any nonincreasing sequence  $\{s_n\} \subset (0, \infty)$ , it holds

$$\limsup_{n \rightarrow \infty} \frac{t_n + \eta(t_n, s_n)}{s_n} < 1.$$

We denote the set of all manageable functions by  $\widehat{\text{Man}}(\mathbb{R})$ .

Several examples of simulation functions and manageable functions can be found on [12–14].

**Example 5** If  $k \in (0, 1)$ , then the function  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by  $\eta(t, s) = ks - t$  for all  $t, s \in \mathbb{R}$ , is a manageable function, and its restriction to  $[0, \infty) \times [0, \infty)$  is a simulation function.

The notion of manageable function was introduced in order to study multi-valued contractions. Next, we particularize such a notion to single-valued mappings.

**Definition 6** (Khojasteh *et al.* [12]) Let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be a self-mapping. We say that  $T$  is a  $\widehat{\text{Man}}(\mathbb{R})$ -contraction if there exists  $\eta \in \widehat{\text{Man}}(\mathbb{R})$  such that  $\eta(d(Tx, T^2x), d(x, Tx)) \geq 0$  for all  $x \in X$ .

### 2.2 Meir-Keeler contractions

Meir and Keeler generalized the Banach theorem in the following way.

**Definition 7** (Meir and Keeler [1]) A *Meir-Keeler contraction* is a mapping  $T : X \rightarrow X$  from a metric space  $(X, d)$  into itself such that for all  $\varepsilon > 0$ , there exists  $\delta > 0$  verifying that if  $x, y \in X$  and  $\varepsilon \leq d(x, y) < \varepsilon + \delta$ , then  $d(Tx, Ty) < \varepsilon$ .

Meir-Keeler contractions have attracted much attention in the last years (see, for instance, [3–11]). Lim characterized this kind of mappings in terms of a contractivity condition using the following class of auxiliary functions.

**Definition 8** (Lim [2]) A function  $\phi : [0, \infty) \rightarrow [0, \infty)$  will be called an *L-function* if

- (a)  $\phi(0) = 0$ ,
- (b)  $\phi(t) > 0$  for all  $t > 0$ , and
- (c) for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\phi(t) \leq \varepsilon$  for all  $t \in [\varepsilon, \varepsilon + \delta]$ .

**Theorem 9** (Lim [2], Theorem 1) *Let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be a self-mapping. Then  $T$  is a Meir-Keeler mapping if, and only if, there exists an (nondecreasing, right-continuous) L-map  $\phi$  such that*

$$d(Tx, Ty) < \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ verifying } d(x, y) > 0. \tag{1}$$

Using a result of Chu and Diaz [19], Meir and Keeler [1] proved that every Meir-Keeler contraction from a complete metric space into itself has a unique fixed point.

For our purposes, we highlight the following properties of L-functions and Meir-Keeler contractions.

**Lemma 10** *Every L-function  $\phi$  satisfies the following properties:*

- (1)  $\phi(t) \leq t$  for all  $t \in [0, \infty)$ .
- (2) For all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\phi(t) \leq \varepsilon$  for all  $t \in [0, \varepsilon + \delta]$ .

*Proof* (1) If  $t = 0$ , then  $\phi(0) = 0$ . And if  $t > 0$ , using  $\varepsilon = t > 0$ , we deduce, from (c), that  $\phi(t) = \phi(\varepsilon) \leq \varepsilon = t$ .

(2) Let  $\varepsilon > 0$  be arbitrary and let  $\delta > 0$  be given by (c). Then, for all  $t \in [0, \varepsilon)$ , we have that  $\phi(t) \leq t < \varepsilon$ . □

The following result is useful to guarantee that a self-mapping is not a Meir-Keeler contraction.

**Proposition 11** *Let  $T : X \rightarrow X$  be a self-mapping from a metric space  $(X, d)$  into itself and assume that there exist  $L > 0$  and two sequences  $\{x_n\}, \{y_n\} \subseteq X$  such that*

$$d(Tx_n, Ty_n) = L < d(x_n, y_n) \quad \text{for all } n \in \mathbb{N} \text{ and}$$

$$\{d(x_n, y_n)\} \rightarrow L.$$

*Then  $T$  is not a Meir-Keeler contraction.*

*Proof* Reasoning by contradiction, assume that  $T$  is a Meir-Keeler contraction. By Theorem 9, there exists an  $L$ -map  $\phi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  such that

$$d(Tx, Ty) < \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ verifying } d(x, y) > 0.$$

As a consequence,

$$\phi(L) \leq L = d(Tx_n, Ty_n) < \phi(d(x_n, y_n)) \leq d(x_n, y_n) \quad \text{for all } n \in \mathbb{N}.$$

Using  $\varepsilon = L > 0$ , taking into account that  $T$  is a Meir-Keeler contraction, there exists  $\delta > 0$  such that

$$\phi(t) < L \quad \text{for all } t \in [L, L + \delta].$$

As  $\{d(x_n, y_n)\} \rightarrow L$  and  $d(x_n, y_n) > L$  for all  $n \in \mathbb{N}$ , there exists  $n_0 \in \mathbb{N}$  such that  $L < d(x_{n_0}, y_{n_0}) < L + \delta$ . Then  $\phi(d(x_{n_0}, y_{n_0})) < L$ , which contradicts the fact that  $L < \phi(d(x_{n_0}, y_{n_0}))$ . As a result,  $T$  cannot be a Meir-Keeler contraction. □

### 3 R-Functions and R-contractions

In this section we introduce the family of auxiliary functions we will use to present a new kind of contractive mappings. We will also show that this family contains several classes of contractive mappings, including Meir-Keeler contractions.

#### 3.1 The family of R-functions

**Definition 12** Let  $A \subseteq \mathbb{R}$  be a nonempty subset and let  $\varrho : A \times A \rightarrow \mathbb{R}$  be a function. We say that  $\varrho$  is an *R-function* if it satisfies the following two conditions.

- ( $\varrho_1$ ) If  $\{a_n\} \subset (0, \infty) \cap A$  is a sequence such that  $\varrho(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ , then  $\{a_n\} \rightarrow 0$ .
- ( $\varrho_2$ ) If  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  are two sequences converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\varrho(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ , then  $L = 0$ .

We denote by  $R_A$  the family of all *R-functions* whose domain is  $A \times A$ .

In some cases, given a function  $\varrho : A \times A \rightarrow \mathbb{R}$ , we will also consider the following property.

( $\varrho_3$ ) If  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  are two sequences such that  $\{b_n\} \rightarrow 0$  and  $\varrho(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ , then  $\{a_n\} \rightarrow 0$ .

**Remark 13** Notice that conditions ( $\varrho_1$ ), ( $\varrho_2$ ) and ( $\varrho_3$ ) establish that if there exist sequences verifying some assumptions, then a thesis must hold. However, we point out that if such kind of sequences does not exist, then conditions ( $\varrho_1$ ), ( $\varrho_2$ ) and ( $\varrho_3$ ) hold.

**Proposition 14** *If  $\varrho(t, s) \leq s - t$  for all  $t, s \in A \cap (0, \infty)$ , then ( $\varrho_3$ ) holds.*

*Proof* Assume that  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  are two sequences such that  $\{b_n\} \rightarrow 0$  and  $\varrho(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . Since  $a_n, b_n \in (0, \infty) \cap A$ , then  $0 < \varrho(a_n, b_n) \leq b_n - a_n$  for all  $n \in \mathbb{N}$ . As a consequence,  $0 < a_n < b_n$  for all  $n \in \mathbb{N}$ , which means that  $\{a_n\} \rightarrow 0$ . □

Firstly, we show some examples.

**Lemma 15** *Every simulation function is an R-function that also verifies ( $\varrho_3$ ).*

*Proof* Let  $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be a simulation function.

( $\varrho_1$ ) Let  $\{a_n\} \subset (0, \infty)$  be a sequence such that  $\zeta(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore,  $0 < \zeta(a_{n+1}, a_n) < a_n - a_{n+1}$  for all  $n \in \mathbb{N}$ . Hence,  $\{a_n\}$  is a strictly decreasing sequence of positive real numbers, so it is convergent. Let  $L \geq 0$  be its limit. To prove that  $L = 0$ , assume, by contradiction, that  $L > 0$ . Let  $t_n = a_{n+1}$  and  $s_n = a_n$  for all  $n \in \mathbb{N}$ . Then  $\{t_n\}$  and  $\{s_n\}$  are convergent to  $L$  and  $t_n < s_n$  for all  $n \in \mathbb{N}$ . By condition ( $\zeta_3$ ),

$$0 \leq \limsup_{n \rightarrow \infty} \zeta(a_{n+1}, a_n) = \limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0,$$

which is a contradiction. Then  $\{a_n\} \rightarrow L = 0$ .

( $\varrho_2$ ) Let  $\{a_n\}, \{b_n\} \subset (0, \infty)$  be two sequences converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\zeta(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . To prove that  $L = 0$ , assume, by contradiction, that  $L > 0$ . On the one hand, by ( $\zeta_2$ ),  $0 < \zeta(a_n, b_n) = b_n - a_n$ , so  $L < a_n < b_n$  for all  $n \in \mathbb{N}$ . On the other hand, by condition ( $\zeta_3$ ), we have that

$$0 \leq \limsup_{n \rightarrow \infty} \zeta(a_n, b_n) < 0,$$

which is a contradiction. Then  $L = 0$ .

( $\varrho_3$ ) Let  $\{a_n\}, \{b_n\} \subset (0, \infty)$  be two sequences such that  $\{b_n\} \rightarrow 0$  and  $\zeta(a_n, b_n) \geq 0$  for all  $n \in \mathbb{N}$ . Since  $\zeta$  is a simulation function,  $0 \leq \zeta(a_n, b_n) < b_n - a_n$  for all  $n \in \mathbb{N}$ . Hence,  $0 < a_n < b_n$  for all  $n \in \mathbb{N}$ , which means that  $\{a_n\} \rightarrow 0$ . □

**Lemma 16** *Every manageable function is an R-function that also verifies ( $\varrho_3$ ).*

*Proof* Let  $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a manageable function.

( $\varrho_1$ ) Let  $\{a_n\} \subset (0, \infty)$  be a sequence such that  $\eta(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore,  $0 < \eta(a_{n+1}, a_n) < a_n - a_{n+1}$  for all  $n \in \mathbb{N}$ . Hence,  $\{a_n\}$  is a strictly decreasing sequence of positive real numbers, so it is convergent. Let  $L \geq 0$  be its limit. To prove that  $L = 0$ , assume, by contradiction, that  $L > 0$ . Let  $t_n = a_{n+1}$  and  $s_n = a_n$  for all  $n \in \mathbb{N}$ . Then  $\{t_n\}$  is a bounded

sequence (in fact, it is convergent) and  $\{s_n\}$  is a nonincreasing sequence (convergent to  $L$ ). By condition  $(\eta_2)$ ,

$$\limsup_{n \rightarrow \infty} \frac{a_{n+1} + \eta(a_{n+1}, a_n)}{a_n} = \limsup_{n \rightarrow \infty} \frac{t_n + \eta(t_n, s_n)}{s_n} < 1.$$

Taking into account that

$$\frac{a_{n+1}}{a_n} \leq \frac{a_{n+1} + \eta(a_{n+1}, a_n)}{a_n} \quad \text{for all } n \in \mathbb{N},$$

we deduce that

$$1 = \frac{L}{L} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1} + \eta(a_{n+1}, a_n)}{a_n} < 1,$$

which is a contradiction. Then  $\{a_n\} \rightarrow L = 0$ .

$(Q_2)$  Let  $\{a_n\}, \{b_n\} \subset (0, \infty)$  be two sequences converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\zeta(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . To prove that  $L = 0$ , assume, by contradiction, that  $L > 0$ . On the one hand, by  $(\eta_1)$ ,  $0 < \eta(a_n, b_n) = b_n - a_n$ , so  $L < a_n < b_n$  for all  $n \in \mathbb{N}$ . As  $\{b_n\} \rightarrow L$  and  $L < b_n$  for all  $n \in \mathbb{N}$ , then there exists a partial subsequence  $\{b_{n(k)}\}_{k \in \mathbb{N}}$  of  $\{b_n\}$  such that  $\{b_{n(k)}\}$  is strictly decreasing, that is,

$$L < b_{n(k+1)} < b_{n(k)} \quad \text{for all } k \in \mathbb{N}.$$

Hence  $\{a_{n(k)}\}$  is a bounded sequence (in fact, it is convergent to  $L$ ) and  $\{b_{n(k)}\}$  is a nonincreasing sequence. By condition  $(\eta_2)$ , we have that

$$\limsup_{n \rightarrow \infty} \frac{b_{n(k)} + \eta(a_{n(k)}, b_{n(k)})}{a_{n(k)}} < 1.$$

Taking into account that

$$\frac{b_{n(k)}}{a_{n(k)}} \leq \frac{b_{n(k)} + \eta(a_{n(k)}, b_{n(k)})}{a_{n(k)}} \quad \text{for all } n \in \mathbb{N},$$

we deduce that

$$1 = \frac{L}{L} = \lim_{n \rightarrow \infty} \frac{b_{n(k)}}{a_{n(k)}} = \limsup_{n \rightarrow \infty} \frac{b_{n(k)}}{a_{n(k)}} \leq \limsup_{n \rightarrow \infty} \frac{b_{n(k)} + \eta(a_{n(k)}, b_{n(k)})}{a_{n(k)}} < 1,$$

which is a contradiction. Then  $L = 0$ .

$(Q_3)$  Let  $\{a_n\}, \{b_n\} \subset (0, \infty)$  be two sequences such that  $\{b_n\} \rightarrow 0$  and  $\eta(a_n, b_n) \geq 0$  for all  $n \in \mathbb{N}$ . Since  $\eta$  is a manageable function,  $0 \leq \eta(a_n, b_n) < b_n - a_n$  for all  $n \in \mathbb{N}$ . Hence,  $0 < a_n < b_n$  for all  $n \in \mathbb{N}$ , which means that  $\{a_n\} \rightarrow 0$ . □

Lemmas 15 and 16 provide us with a wide range of  $R$ -functions taking into account examples given in [12, 14]. In the following examples we show that the notion of  $R$ -function is more general than the previous ones.

**Example 17** Given  $\lambda \in (0, 1)$ , let  $\varrho : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  be the function given by  $\varrho(t, s) = \lambda s - t$  for all  $t, s \in [0, 1]$ . Then  $\varrho$  is an  $R$ -function, but it is not a simulation function neither a manageable function because its domain is neither  $[0, \infty) \times [0, \infty)$  nor  $\mathbb{R} \times \mathbb{R}$ .

**Example 18** Let  $\varrho : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be the function defined, for all  $t, s \in [0, \infty)$ , by

$$\varrho(t, s) = \begin{cases} \frac{1}{2}s - t, & \text{if } t < s, \\ 0, & \text{if } t \geq s. \end{cases}$$

Let us show that  $\varrho$  is an  $R$ -function on  $[0, \infty)$  which also satisfies condition  $(\varrho_3)$ .

$(\varrho_1)$  Assume that  $\{a_n\} \subset (0, \infty)$  is a sequence such that  $\varrho(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore  $a_{n+1} < a_n$  and

$$0 < \varrho(a_{n+1}, a_n) = \frac{a_n}{2} - a_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

As a consequence,  $0 < a_{n+1} < a_n/2$  for all  $n \in \mathbb{N}$ , which implies that  $\{a_n\} \rightarrow 0$ .

$(\varrho_2)$  Assume that  $\{a_n\}, \{b_n\} \subset (0, \infty)$  are two sequences converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\varrho(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . Hence

$$a_n < b_n \quad \text{and} \quad 0 < \varrho(a_n, b_n) = \frac{b_n}{2} - a_n \quad \text{for all } n \in \mathbb{N}.$$

As a consequence,  $2a_n < b_n$  for all  $n \in \mathbb{N}$ . Letting  $n \rightarrow \infty$ , we deduce that  $0 \leq 2L \leq L$ , which means that  $L = 0$ .

$(\varrho_3)$  Let  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  be two sequences such that  $\{b_n\} \rightarrow 0$  and  $\varrho(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore,  $0 < a_n < b_n$ , which implies that  $\{a_n\} \rightarrow 0$ .

As a consequence,  $\varrho$  is an  $R$ -function on  $[0, \infty)$  which also satisfies condition  $(\varrho_3)$ . However,  $\varrho$  is not a simulation function because if we take  $t_n = s_n = 1$  for all  $n \in \mathbb{N}$ , then  $\{t_n\} \rightarrow 1, \{s_n\} \rightarrow 1$  but  $\varrho(t_n, s_n) = 0$  for all  $n \in \mathbb{N}$ , which implies that  $\varrho$  does not verify condition  $(\zeta_3)$ . The same argument guarantees that  $\varrho$ , defined from  $\mathbb{R} \times \mathbb{R}$  into  $\mathbb{R}$ , is not a manageable function.

**Proposition 19** *If  $\varrho \in R_A$ , then  $\varrho(a, a) \leq 0$  for all  $a \in (0, \infty) \cap A$ .*

*Proof* By contradiction, assume that there exists  $a \in (0, \infty) \cap A$  such that  $\varrho(a, a) > 0$ . Let us define  $a_n = a$  for all  $n \in \mathbb{N}$ . Therefore  $\varrho(a_{n+1}, a_n) = \varrho(a, a) > 0$  for all  $n \in \mathbb{N}$ . Condition  $(\varrho_1)$  implies that  $\{a_n\} \rightarrow 0$ , which contradicts the fact that  $a > 0$ . □

Functions taking values greater than or equal to an  $R$ -function can be an  $R$ -function.

**Proposition 20** *If  $\varrho \in R_A$  and  $\lambda > 0$ , then  $\varrho_\lambda : A \times A \rightarrow \mathbb{R}$ , defined by  $\varrho_\lambda(t, s) = \lambda\varrho(t, s)$  for all  $t, s \in A$ , is also an  $R$ -function. And if  $\varrho$  satisfies  $(\varrho_3)$ , then  $\varrho_\lambda$  also satisfies it.*

An interesting subclass of the family of  $R$ -functions can be considered involving  $L$ -functions as follows.

**Theorem 21** *Given an  $L$ -function  $\phi : [0, \infty) \rightarrow [0, \infty)$ , let  $\varrho_\phi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be the function defined by*

$$\varrho_\phi(t, s) = \phi(s) - t \quad \text{for all } t, s \in [0, \infty).$$

*Then  $\varrho_\phi$  is an  $R$ -function on  $[0, \infty)$ . Furthermore,  $\varrho_\phi$  satisfies condition  $(\varrho_3)$ .*



*Proof* (Q<sub>1</sub>) Assume that  $\{a_n\} \subset (0, \infty)$  is a sequence such that  $\varrho_\phi(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore,

$$0 < \varrho_\phi(a_{n+1}, a_n) = \phi(a_n) - a_{n+1},$$

which means that  $a_{n+1} < \phi(a_n)$  for all  $n \in \mathbb{N}$ . Using item (1) of Lemma 10,

$$a_{n+1} < \phi(a_n) \leq a_n \quad \text{for all } n \in \mathbb{N}.$$

Hence,  $\{a_n\}$  is a strictly decreasing sequence of positive real numbers. Let  $L \geq 0$  be its limit. In order to prove that  $L = 0$ , assume that  $L > 0$ . Therefore,

$$L < a_{n+1} < \phi(a_n) \leq a_n \quad \text{for all } n \in \mathbb{N}.$$

As  $\phi$  is an  $L$ -function, using  $\varepsilon = L > 0$ , there exists  $\delta > 0$  such that

$$\phi(t) \leq \varepsilon = L \quad \text{for all } t \in [\varepsilon, \varepsilon + \delta] = [L, L + \delta]. \tag{2}$$

Since  $\{a_n\} \searrow L^+$ , there exists  $n_0 \in \mathbb{N}$  such that  $a_{n_0} < L + \delta$ . Since  $L < a_{n_0} < L + \delta$ , then (2) implies that  $\phi(a_{n_0}) \leq L$ , which yields the contradiction

$$\phi(a_{n_0}) \leq L < a_{n_0+1} < \phi(a_{n_0}).$$

As a consequence,  $L = 0$ .

(Q<sub>2</sub>) Assume that  $\{a_n\}, \{b_n\} \subset (0, \infty)$  are two sequences converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\varrho_\phi(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . In order to prove that  $L = 0$ , assume that  $L > 0$ . Therefore,

$$0 < \varrho_\phi(a_n, b_n) = \phi(b_n) - a_n.$$

As a consequence,

$$L < a_n < \phi(b_n) \leq b_n \quad \text{for all } n \in \mathbb{N}.$$

As  $\phi$  is an  $L$ -function, using  $\varepsilon = L > 0$ , there exists  $\delta > 0$  such that

$$\phi(t) \leq \varepsilon = L \quad \text{for all } t \in [\varepsilon, \varepsilon + \delta] = [L, L + \delta]. \tag{3}$$

Since  $\{b_n\} \searrow L^+$ , there exists  $n_0 \in \mathbb{N}$  such that  $L < b_{n_0} < L + \delta$ . And, by using (3), we deduce that  $\phi(b_{n_0}) \leq L$ , which is a contradiction because  $\phi(b_{n_0}) \leq L < a_{n_0} < \phi(b_{n_0})$ . Thus,  $L = 0$ .

(Q<sub>3</sub>) Let  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  be two sequences such that  $\{b_n\} \rightarrow 0$  and  $\varrho_\phi(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore, by item (1) of Lemma 10,  $0 < \varrho_\phi(a_n, b_n) = \phi(b_n) - a_n \leq b_n - a_n$ , so  $0 < a_n < b_n$  for all  $n \in \mathbb{N}$ , which implies that  $\{a_n\} \rightarrow 0$ . □

**Theorem 22** Let  $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$  be two functions such that  $\psi$  is nondecreasing and continuous from the right,  $\varphi$  is lower semi-continuous and  $\varphi^{-1}(\{0\}) = \{0\}$ . Let  $\varrho_{\psi, \varphi} :$

$[0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be the function defined by

$$\varrho_{\psi,\varphi}(t, s) = \psi(s) - \varphi(s) - \psi(t) \quad \text{for all } t, s \in [0, \infty).$$

Then  $\varrho_{\psi,\varphi}$  is an R-function on  $[0, \infty)$ . Furthermore,  $\varrho_{\psi,\varphi}$  satisfies condition  $(\varrho_3)$ .

*Proof* First of all, we show that, for  $t, s \in [0, \infty)$ ,

$$s > 0, \quad \varrho_{\psi,\varphi}(t, s) \geq 0 \quad \Rightarrow \quad t < s. \tag{4}$$

Indeed, assume that  $\varrho_{\psi,\varphi}(t, s) \geq 0$  and  $t \geq s$ , and we are going to show that  $s = 0$ . As  $\psi$  is nondecreasing,

$$\psi(s) \leq \psi(t) \leq \psi(s) - \varphi(s) \leq \psi(s),$$

which implies that  $\varphi(s) = 0$ . Hence  $s = 0$ , which contradicts the fact that  $s > 0$ . Thus,  $t < s$ .

$(\varrho_1)$  Assume that  $\{a_n\} \subset (0, \infty)$  is a sequence such that  $\varrho_{\psi,\varphi}(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ . By (4),  $a_{n+1} < a_n$  for all  $n \in \mathbb{N}$ . Hence,  $\{a_n\}$  is a strictly decreasing sequence of positive real numbers. Let  $L \geq 0$  be its limit. Then  $L < a_n$  for all  $n \in \mathbb{N}$ . In order to prove that  $L = 0$ , assume that  $L > 0$ . Therefore, for  $n \in \mathbb{N}$ ,

$$\begin{aligned} 0 < \varrho_{\psi,\varphi}(a_{n+1}, a_n) &= \psi(a_n) - \varphi(a_n) - \psi(a_{n+1}) \\ \Rightarrow \quad 0 &\leq \varphi(a_n) < \psi(a_n) - \psi(a_{n+1}). \end{aligned}$$

As  $\psi$  is continuous from the right and  $\{a_n\} \searrow L^+$ , letting  $n \rightarrow \infty$  we deduce that  $\lim_{n \rightarrow \infty} \varphi(a_n) = 0$ . And as  $\varphi$  is lower semi-continuous,

$$0 \leq \varphi(L) \leq \liminf_{r \rightarrow L} \varphi(r) = \lim_{n \rightarrow \infty} \varphi(a_n) = 0.$$

Hence  $\varphi(L) = 0$ , so  $L = 0$ .

$(\varrho_2)$  Assume that  $\{a_n\}, \{b_n\} \subset (0, \infty)$  are two sequences converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\varrho_{\psi,\varphi}(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . By (4),  $L < a_n < b_n$  for all  $n \in \mathbb{N}$ . In order to prove that  $L = 0$ , assume that  $L > 0$ . Therefore,

$$0 < \varrho_{\psi,\varphi}(a_n, b_n) = \psi(b_n) - \varphi(b_n) - \psi(a_n) \quad \Rightarrow \quad 0 \leq \varphi(b_n) < \psi(b_n) - \psi(a_n).$$

As  $\psi$  is continuous from the right and  $\{b_n\} \searrow L^+$ , letting  $n \rightarrow \infty$  we deduce that  $\lim_{n \rightarrow \infty} \varphi(b_n) = 0$ . And as  $\varphi$  is lower semi-continuous,

$$0 \leq \varphi(L) \leq \liminf_{r \rightarrow L} \varphi(r) = \lim_{n \rightarrow \infty} \varphi(b_n) = 0.$$

Hence  $\varphi(L) = 0$ , so  $L = 0$ .

(Q<sub>3</sub>) Let  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  be two sequences such that  $\{b_n\} \rightarrow 0$  and  $\varrho_\phi(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . By condition (4),  $0 < a_n < b_n$  for all  $n \in \mathbb{N}$ , which implies that  $\{a_n\} \rightarrow 0$ .  $\square$

Another example, involving Geraghty functions, is the next statement.

**Lemma 23** *If  $\phi : [0, \infty) \rightarrow [0, 1)$  is a Geraghty function, then  $\varrho'_\phi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ , defined by*

$$\varrho'_\phi(t, s) = \phi(s)s - t \quad \text{for all } t, s \in [0, \infty),$$

*is an R-function on  $[0, \infty)$  satisfying condition (Q<sub>3</sub>).*

*Proof* (Q<sub>1</sub>) Assume that  $\{a_n\} \subset (0, \infty)$  is a sequence such that  $\varrho'_\phi(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore,

$$0 < \varrho'_\phi(a_{n+1}, a_n) = \phi(a_n)a_n - a_{n+1}.$$

As  $a_n > 0$  and  $\phi(a_n) < 1$  for all  $n \in \mathbb{N}$ , then

$$a_{n+1} < \phi(a_n)a_n < a_n \quad \text{for all } n \in \mathbb{N}.$$

Hence,  $\{a_n\}$  is a strictly decreasing sequence of positive real numbers. Let  $L \geq 0$  be its limit. In order to prove that  $L = 0$ , assume that  $L > 0$ . Therefore,

$$0 < L < a_{n+1} < \phi(a_n)a_n < a_n \quad \text{for all } n \in \mathbb{N}.$$

Letting  $n \rightarrow \infty$  in the last inequalities, we deduce that  $\lim_{n \rightarrow \infty} \phi(a_n) = 1$ . As  $\phi$  is a Geraghty function, it follows that  $L = \lim_{n \rightarrow \infty} a_n = 0$ , which contradicts  $L > 0$ . Then necessarily  $L = 0$ .

(Q<sub>2</sub>) Assume that  $\{a_n\}, \{b_n\} \subset (0, \infty)$  are two sequences converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\varrho'_\phi(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . In order to prove that  $L = 0$ , assume that  $L > 0$ . Therefore,

$$0 < \varrho'_\phi(a_n, b_n) = \phi(b_n)b_n - a_n \quad \text{for all } n \in \mathbb{N}.$$

As a consequence,

$$L < a_n < \phi(b_n)b_n \leq b_n \quad \text{for all } n \in \mathbb{N}.$$

Letting  $n \rightarrow \infty$  in the last inequalities, we deduce that  $\lim_{n \rightarrow \infty} \phi(b_n) = 1$ . As  $\phi$  is a Geraghty function, it follows that  $L = \lim_{n \rightarrow \infty} b_n = 0$ , which contradicts  $L > 0$ . Then necessarily  $L = 0$ .

(Q<sub>3</sub>) Let  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  be two sequences such that  $\{b_n\} \rightarrow 0$  and  $\varrho'_\phi(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . Therefore  $0 < \varrho'_\phi(a_n, b_n) = \phi(b_n)b_n - a_n \leq b_n - a_n$ , so  $0 < a_n < b_n$  for all  $n \in \mathbb{N}$ , which implies that  $\{a_n\} \rightarrow 0$ .  $\square$

### 3.2 R-Contractions

In this section we introduce the notion of *R-contraction* and we show several examples of such kind of contractions.

**Definition 24** Let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be a mapping. We will say that  $T$  is an *R-contraction* if there exists an *R-function*  $\varrho : A \times A \rightarrow \mathbb{R}$  such that  $\text{ran}(d) \subseteq A$  and

$$\varrho(d(Tx, Ty), d(x, y)) > 0 \quad \text{for all } x, y \in X \text{ such that } x \neq y. \tag{5}$$

In such a case, we will say that  $T$  is an *R-contraction with respect to*  $\varrho$ . We denote the family of all *R-contractions* from  $(X, d)$  into itself with respect to  $\varrho$  by  $R_{X,d,A,\varrho}$  or, when no confusion is possible, by  $R_\varrho$ .

The following result shows an extensive family of *R-contractions*.

**Theorem 25** *Every Meir-Keeler contraction is an R-contraction with respect to an R-function  $\varrho$  which satisfies  $(\varrho_3)$ .*

In the last result, the *R-function*  $\varrho$  may be chosen satisfying the following properties.

- $\varrho(t, s) < s - t$  for all  $t, s \in (0, \infty)$ .
- $\varrho(t, s) \leq s - t$  for all  $t, s \in [0, \infty)$ .

*Proof* Let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be a Meir-Keeler contraction. By Theorem 9, there exists an *L-function*  $\phi$  such that

$$d(Tx, Ty) < \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ verifying } d(x, y) > 0.$$

Theorem 21 guarantees that  $\varrho_\phi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ , defined by

$$\varrho_\phi(t, s) = \phi(s) - t \quad \text{for all } t, s \in [0, \infty),$$

is an *R-function* on  $[0, \infty)$  which satisfies  $(\varrho_3)$ . Moreover, for all  $x, y \in X$  such that  $x \neq y$ , we have that

$$\varrho_\phi(d(Tx, Ty), d(x, y)) = \phi(d(x, y)) - d(Tx, Ty) > 0,$$

which means that  $T$  is an *R-contraction*. □

The previous statement implies that every fixed point theorem that can be proved for *R-contractions* (such as Theorem 27) also holds for Meir-Keeler contractions. However, the converse is false as we shall see in the next section.

**Corollary 26** *Every Geraghty contraction is an R-contraction with respect to an R-function  $\varrho$  which satisfies  $(\varrho_3)$ .*

*Proof* Let  $(X, d)$  be a metric space and let  $T : X \rightarrow X$  be a Geraghty contraction. Then there exists a Geraghty function  $\phi : [0, \infty) \rightarrow [0, 1)$  such that

$$d(Tx, Ty) \leq \phi(d(x, y))d(x, y) \quad \text{for all } x, y \in X.$$

Let  $\varphi : [0, \infty) \rightarrow [0, 1)$  be the function defined by  $\varphi(t) = (1 + \phi(t))/2$  for all  $t \in [0, \infty)$ . Then  $\varphi$  is also a Geraghty function and it verifies  $\phi(t) < \varphi(t) < 1$  for all  $t \in [0, \infty)$ . Therefore, if  $x \neq y$ , then

$$d(Tx, Ty) \leq \phi(d(x, y))d(x, y) < \varphi(d(x, y))d(x, y).$$

Lemma 23 guarantees that  $\varrho'_\varphi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ , defined by

$$\varrho'_\varphi(t, s) = \varphi(s)s - t \quad \text{for all } t, s \in [0, \infty),$$

is an  $R$ -function on  $[0, \infty)$  which satisfies  $(\varrho_3)$ . Moreover, for all  $x, y \in X$  such that  $x \neq y$ , we have that  $\phi(d(x, y)) < 1$  and

$$\varrho'_\varphi(d(Tx, Ty), d(x, y)) = \varphi(d(x, y))d(x, y) - d(Tx, Ty) > 0,$$

which means that  $T$  is an  $R$ -contraction. □

#### 4 Some fixed point theorems under $R$ -contractivity conditions

This section is dedicated to obtaining fixed point theorems under  $R$ -contractivity conditions. Later, we will show that some well-known results can be deduced as simple consequences of our main result, which is the following one.

**Theorem 27** *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow X$  be an  $R$ -contraction with respect to  $\varrho \in R_A$ . Assume that, at least, one of the following conditions holds:*

- (a)  $T$  is continuous.
- (b) The function  $\varrho$  satisfies condition  $(\varrho_3)$ .
- (c)  $\varrho(t, s) \leq s - t$  for all  $t, s \in A \cap (0, \infty)$ .

*Then  $T$  is a Picard operator. In particular, it has a unique fixed point.*

*Proof* Let  $x_0 \in X$  be an arbitrary point and let  $\{x_n\}_{n \geq 0}$  be the Picard sequence of  $T$  based on  $x_0$ , that is,  $x_{n+1} = Tx_n$  for all  $n \in \mathbb{N}$ . If there exists some  $n_0 \in \mathbb{N}$  such that  $x_{n_0+1} = x_{n_0}$ , then  $x_{n_0}$  is a fixed point of  $T$ . In the contrary case, assume that  $x_n \neq x_{n+1}$  for all  $n \in \mathbb{N}$ . Let  $\{a_n\} \subset (0, \infty)$  be the sequence defined by  $a_n = d(x_n, x_{n+1}) > 0$  for all  $n \in \mathbb{N}$ . Taking into account that  $T$  is an  $R$ -contraction with respect to  $\varrho$ , then

$$\varrho(a_{n+1}, a_n) = \varrho(d(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) = \varrho(d(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})) > 0.$$

Applying  $(\varrho_1)$  we deduce that  $\{d(x_n, x_{n+1})\} = \{a_n\} \rightarrow 0$ , that is,  $\{x_n\}$  is an asymptotically regular sequence.

Next we show that  $\{x_n\}$  is a Cauchy sequence reasoning by contradiction. If  $\{x_n\}$  is not a Cauchy sequence, then there exist  $\varepsilon_0 > 0$  and two subsequences  $\{x_{n(k)}\}$  and  $\{x_{m(k)}\}$  of  $\{x_n\}$

such that

$$k \leq n(k) < m(k), \quad d(x_{n(k)}, x_{m(k)-1}) \leq \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) \quad \text{for all } k \in \mathbb{N},$$

$$\lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k \rightarrow \infty} d(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon_0.$$

Let  $L = \varepsilon_0 > 0$ ,  $\{a_k = d(x_{n(k)}, x_{m(k)})\} \rightarrow L$  and  $\{b_k = d(x_{n(k)-1}, x_{m(k)-1})\} \rightarrow L$ . Since  $L = \varepsilon_0 < d(x_{n(k)}, x_{m(k)}) = a_k$  and

$$\begin{aligned} \varrho(a_k, b_k) &= \varrho(d(x_{n(k)}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)-1})) \\ &= \varrho(d(Tx_{n(k)-1}, Tx_{m(k)-1}), d(x_{n(k)-1}, x_{m(k)-1})) > 0 \end{aligned}$$

for all  $k \in \mathbb{N}$ , condition  $(\varrho_2)$  guarantees that  $\varepsilon_0 = L = 0$ , which is a contradiction. As a consequence,  $\{x_n\}$  is a Cauchy sequence. Since  $(X, d)$  is complete, there exists  $z \in X$  such that  $\{x_n\} \rightarrow z$ . Let us show that  $z$  is a fixed point of  $T$  distinguishing two cases.

*Case 1. Assume that  $T$  is continuous.* In this case,  $\{x_{n+1} = Tx_n\} \rightarrow Tz$ , so  $Tz = z$ .

*Case 2. Assume that the function  $\varrho$  satisfies condition  $(\varrho_3)$ .* In this case, let  $a_n = d(x_{n+1}, Tz)$  and  $b_n = d(x_n, z)$  for all  $n \in \mathbb{N}$ . Then  $\{b_n\} \rightarrow 0$ . Moreover,

$$\varrho(a_n, b_n) = \varrho(d(x_{n+1}, Tz), d(x_n, z)) = \varrho(d(Tx_n, Tz), d(x_n, z)) > 0 \tag{6}$$

for all  $n \in \mathbb{N}$ . Furthermore, it is clear that

$$b_n = 0 \quad \Rightarrow \quad a_n = 0 \tag{7}$$

because

$$b_n = 0 \quad \Leftrightarrow \quad x_n = z \quad \Rightarrow \quad x_{n+1} = Tx_n = Tz \quad \Leftrightarrow \quad a_n = 0.$$

Let us consider the set

$$\Omega = \{n \in \mathbb{N} : a_n = 0\} = \{n \in \mathbb{N} : d(x_{n+1}, Tz) = 0\}.$$

*Subcase 2.1. Assume that  $\Omega$  is finite.* In this case, there exists  $n_0 \in \mathbb{N}$  such that  $d(x_{n+1}, Tz) = a_n > 0$  for all  $n \geq n_0$ . By (7),  $d(x_n, z) = b_n > 0$  for all  $n \geq n_0$ . Taking into account (6), condition  $(\varrho_3)$ , applied to  $\{a_n\}_{n \geq n_0}$  and  $\{b_n\}_{n \geq n_0}$ , implies that  $\{d(x_{n+1}, Tz) = a_n\} \rightarrow 0$ , which means that  $\{x_{n+1}\} \rightarrow Tz$ . By the uniqueness of the limit,  $Tz = z$ .

*Subcase 2.2. Assume that  $\Omega$  is not finite.* In this case, there exists a subsequence  $\{x_{n(k)}\}$  of  $\{x_n\}$  such that

$$d(x_{n(k)+1}, Tz) = 0 \quad \text{for all } k \in \mathbb{N}.$$

Hence  $x_{n(k)+1} = Tz$  for all  $k \in \mathbb{N}$ . Since  $\{x_n\} \rightarrow z$ , then  $Tz = z$ .

*Case 3. Assume that  $\varrho(t, s) \leq s - t$  for all  $t, s \in A \cap (0, \infty)$ .* Proposition 14 implies that Case 2 is applicable.

In any case,  $z$  is a fixed point of  $T$ . Then  $T$  is a weakly Picard operator.

Finally, let us show that  $z$  is the unique fixed point of  $T$ . Let  $x, y \in X$  be two arbitrary fixed points of  $T$ . We will deduce that  $x = y$  reasoning by contradiction. Assume that  $x \neq y$  and let us define  $a_n = d(x, y) > 0$  for all  $n \in \mathbb{N}$ . Therefore, for all  $n \in \mathbb{N}$ ,

$$\varrho(a_{n+1}, a_n) = \varrho(d(x, y), d(x, y)) = \varrho(d(Tx, Ty), d(x, y)) > 0.$$

Applying  $(\varrho_1)$ ,  $\{a_n\} \rightarrow 0$ , which contradicts the fact that  $d(x, y) \neq 0$ . As a consequence,  $x = y$  and  $T$  has a unique fixed point. □

**Corollary 28** *Every continuous  $R$ -contraction from a complete metric space into itself has a unique fixed point.*

**Corollary 29** *Every  $\mathcal{Z}$ -contraction from a complete metric space into itself has a unique fixed point.*

**Corollary 30** *Every  $\widehat{\text{Man}}(\mathbb{R})$ -contraction from a complete metric space into itself has a unique fixed point.*

Every manageable function leads to a simple corollary considering the corresponding  $R$ -function. For instance,

$$\varrho(t, s) = \varphi(s) - s, \quad \varrho(t, s) = \phi(s) - \psi(t), \quad \text{etc.}$$

**Corollary 31** *Let  $(X, d)$  be a complete metric space and let  $T : X \rightarrow X$  be a self-mapping. Assume that there exist two functions  $\psi, \varphi : [0, \infty) \rightarrow [0, \infty)$  such that*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \quad \text{for all } x, y \in X.$$

*If  $\psi$  is nondecreasing and continuous from the right,  $\varphi$  is lower semi-continuous and  $\varphi^{-1}(\{0\}) = \{0\}$ , then  $T$  has a unique fixed point.*

*Proof* It follows from Theorem 27 taking into account Theorem 22. □

**Corollary 32** *Every Geraghty contraction from a complete metric space into itself has a unique fixed point.*

*Proof* It follows from Theorem 27 taking into account Corollary 26. □

**Corollary 33** *Every Meir-Keeler contraction from a complete metric space into itself has a unique fixed point.*

*Proof* It follows from Theorem 27 taking into account Theorem 25. □

Next, we show an example of an  $R$ -contraction which is not a Meir-Keeler contraction. Let  $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}$  and let  $\{x_n\}$  and  $\{y_n\}$  be the sequences

$$x_n = 10n \quad \text{and} \quad y_n = 10n + 1 + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}^*.$$

Let  $X$  be the set

$$X = \{0, 1\} \cup \{x_n\}_{n=1}^\infty \cup \{y_n\}_{n=1}^\infty \subset [0, \infty)$$

endowed with the Euclidean metric  $d(x, y) = |x - y|$  for all  $x, y \in X$ . As  $X$  is discrete and

$$d(x, y) \geq 1 \text{ for all } x, y \in X \text{ such that } x \neq y, \tag{8}$$

then  $(X, d)$  is complete. Let  $T : X \rightarrow X$  be defined by

$$Tx = \begin{cases} 0, & \text{if } x \in \{0, 1\} \cup \{10n\}_{n=1}^\infty, \\ 1, & \text{if } x \in \{10n + 1 + \frac{1}{n}\}_{n=1}^\infty. \end{cases}$$

Notice that  $T$  satisfies the following properties:

- $T(X) = \{0, 1\};$  (9)

- If  $x, y \in X$  verify  $d(Tx, Ty) > 0$ , then  $d(Tx, Ty) = 1.$  (10)

We claim that  $T$  is an  $R$ -contraction but it is not a Meir-Keeler contraction. The second fact follows from Proposition 11 taking into account that

$$d(Tx_n, Ty_n) = 1 < 1 + \frac{1}{n} = d(x_n, y_n) \text{ for all } n \in \mathbb{N}^*.$$

Then  $T$  is not a Meir-Keeler contraction. Next, we are going to show that  $T$  is an  $R$ -contraction. To prove it, let  $A = \text{ran}(d) \subset [0, \infty)$  be the range of the metric  $d$ . As  $X$  is numerable, then  $A = d(X \times X)$  is also a numerable set. In fact, if  $x, y \in X$  are such that  $x \leq y$ , then one, and only one, of the following cases holds:

$$\begin{cases} (c_1) & d(x, y) = 0 \text{ if, and only if, } x = y; \\ (c_2) & d(x, y) = 1 \text{ if, and only if, } x = 0 \text{ and } y = 1; \\ (c_3) & d(x, y) = 1 + \frac{1}{n} \text{ if, and only if, } x = x_n \text{ and } y = y_n \text{ for some } n \in \mathbb{N}^*; \\ (c_4) & d(x, y) \geq 5 \text{ in any other case.} \end{cases} \tag{11}$$

As a consequence, the range of  $d$  can be expressed as

$$A = \text{ran}(d) = \{0, 1\} \cup \left\{1 + \frac{1}{n}\right\}_{n=1}^\infty \cup \{\lambda_k\}_{k=0}^\infty,$$

where  $\lambda_k \geq 5$  for all  $k \in \mathbb{N}$ . In particular,

$$s \in A = d(X \times X), \quad s > 0 \quad \Rightarrow \quad s \geq 1. \tag{12}$$

Let  $\varrho : A \times A \rightarrow \mathbb{R}$  be the function given by

$$\varrho(t, s) = \begin{cases} \frac{s}{2}, & \text{if } t = 0, \\ \frac{s}{2} - t, & \text{if } s \geq 5 \text{ and } t > 0, \\ s - t, & \text{if } t = 1 \text{ and } s = 1 + \frac{1}{n} \text{ for some } n \in \mathbb{N}^*, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$



Let us show that  $\varrho$  is an  $R$ -function verifying condition  $(\varrho_3)$ . To do that, recall Remark 13 because we are going to show that there are not sequences as in the hypotheses of conditions  $(\varrho_1)$ ,  $(\varrho_2)$  and  $(\varrho_3)$ .

$(\varrho_1)$  We claim that it is impossible to have a sequence  $\{a_n\} \subset (0, \infty) \cap A$  such that  $\varrho(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ . To prove it, assume that such sequence exists. As  $a_n > 0$  for all  $n \in \mathbb{N}$ , description (11) leads to three cases.

*Case 1.* There exists  $n_0 \in \mathbb{N}$  such that  $a_{n_0} = 1$ . As  $t = a_{n_0+1} > 0$  and  $s = a_{n_0} = 1 < 5$ , then the inequality  $\varrho(a_{n_0+1}, 1) = \varrho(a_{n_0+1}, a_{n_0}) > 0$  is impossible following (13).

*Case 2.* There exists  $n_0 \in \mathbb{N}$  such that  $a_{n_0} = 1 + \frac{1}{m}$  for some  $m \in \mathbb{N}^*$ . As  $t = a_{n_0+1} > 0$  and  $s = a_{n_0} = 1 + \frac{1}{m} < 5$ , then necessarily  $a_{n_0+1} = 1$ , but this is impossible by Case 1.

*Case 3.*  $a_n \geq 3$  for all  $n \in \mathbb{N}$ . In this case, for all  $n \in \mathbb{N}$ ,

$$0 < \varrho(a_{n+1}, a_n) = \frac{a_n}{2} - a_{n+1} \Rightarrow a_{n+1} \leq \frac{a_n}{2}.$$

Hence  $\{a_n\} \rightarrow 0$  but this is impossible because, by (12),  $a_n \geq 1$  for all  $n \in \mathbb{N}$ .

In any case, it is impossible to have a sequence  $\{a_n\} \subset (0, \infty) \cap A$  such that  $\varrho(a_{n+1}, a_n) > 0$  for all  $n \in \mathbb{N}$ , which means that  $(\varrho_1)$  holds.

$(\varrho_2)$  We claim that it is impossible to have two sequences  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  converging to the same limit  $L \geq 0$  and verifying that  $L < a_n$  and  $\varrho(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . To prove it, assume that such sequences exist. Since  $a_n > 0$ , then  $a_n \geq 1$  for all  $n \in \mathbb{N}$  by (12). Hence  $1 \leq L < a_n$  for all  $n \in \mathbb{N}$ . The only case in which  $\varrho(t, s) > 0$  and  $t > 1$  occurs when  $s \geq 5$ . Thus  $\varrho(a_n, b_n) > 0$  implies that  $b_n \geq 5$  for all  $n \in \mathbb{N}$ . Moreover, for all  $n \in \mathbb{N}$ ,

$$0 < \varrho(a_n, b_n) = \frac{b_n}{2} - a_n \Rightarrow 2a_n < b_n.$$

Letting  $n \rightarrow \infty$ , we deduce that  $0 \leq 2L \leq L$ , so  $L = 0$ , which contradicts the fact that  $L \geq 1$ . This contradiction ensures that  $(\varrho_2)$  holds.

$(\varrho_3)$  We claim that it is impossible to have two sequences  $\{a_n\}, \{b_n\} \subset (0, \infty) \cap A$  such that  $\{b_n\} \rightarrow 0$  and  $\varrho(a_n, b_n) > 0$  for all  $n \in \mathbb{N}$ . To prove it, assume that such sequences exist. Since  $b_n > 0$ , then  $b_n \geq 1$  for all  $n \in \mathbb{N}$  by (12), which contradicts the fact that  $\{b_n\} \rightarrow 0$ . Hence  $(\varrho_3)$  holds.

As a consequence,  $\varrho$  is an  $R$ -function on  $A = \text{ran}(d)$ . Finally, we claim that  $T$  is an  $R$ -contraction with respect to  $\varrho$ . Let  $x, y \in X$  be such that  $x \neq y$ , that is,  $d(x, y) > 0$ . If  $d(Tx, Ty) = 0$ , then

$$\varrho(d(Tx, Ty), d(x, y)) = \varrho(0, d(x, y)) = \frac{d(x, y)}{2} > 0.$$

Suppose that  $d(Tx, Ty) > 0$ . In this case, by (9) and (10), we have that  $\{Tx, Ty\} = \{0, 1\}$  and  $d(Tx, Ty) = 1$ . Suppose that  $Tx = 0$  and  $Ty = 1$ . Hence  $x \in \{0, 1\} \cup \{10n\}_{n=1}^\infty$  and  $y \in \{10m + 1 + \frac{1}{m}\}_{m=1}^\infty$ . Therefore, only cases  $(c_3)$  and  $(c_4)$  of (11) are possible. In case  $(c_3)$ ,  $x = x_n$  and  $y = y_n$  for some  $n \in \mathbb{N}^*$ . Then

$$\varrho(d(Tx, Ty), d(x, y)) = \varrho\left(1, 1 + \frac{1}{n}\right) = \left(1 + \frac{1}{n}\right) - 1 = \frac{1}{n} > 0.$$

In case  $(c_4)$ ,  $d(x, y) \geq 5$ , so

$$\varrho(d(Tx, Ty), d(x, y)) = \frac{d(x, y)}{2} - d(Tx, Ty) \geq \frac{5}{2} - 1 > 1 > 0.$$

As a result, in any case,  $\varrho(d(Tx, Ty), d(x, y)) > 0$  for all  $x, y \in X$  such that  $x \neq y$ , so  $T$  is an  $R$ -contraction with respect to  $\varrho$ .

As a consequence, Theorem 27 guarantees that  $T$  has a unique fixed point. However, other previous results about Meir-Keeler contractions are not applicable.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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