J. of Inequal. & Appl., 2002, Vol. 7(5), pp. 701–725

Taylor & Francis
Taylor & Francis Group

A Hartman–Nagumo inequality for the vector ordinary p-Laplacian and Applications to Nonlinear Boundary Value Problems

JEAN MAWHIN[1] and ANTONIO J. UREÑA*[2]

[1] *Institut de Mathématique Pure et Appliquée, Université de Louvain, B-1348 Louvain-la-Neuve, Belgium; and* [2] *Departamento de Análisis Matemático, Universidad de Granada, 18071 Granada, Spain*

(Received 1 March, 2001)

A generalization of the well-known Hartman–Nagumo inequality to the case of the vector ordinary p-Laplacian and classical degree theory provide existence results for some associated nonlinear boundary value problems.

Keywords: Hartman–Nagumo inequality; p-Laplacian; Boundary value problems; Periodic solutions.

Classification: *AMS Subject Classification*: 26D10, 34B15, 47H 11

1 INTRODUCTION

In 1960, Hartman [1] (see also [2]) showed that the second order system in $\mathbb{R}^N$

$$u'' = f(t, u, u') \tag{0.1}$$

$$u(0) = u_0, \quad u(1) = u_1, \tag{0.2}$$

*This work was realized during a stay of the second author in Louvain-la Neuve, being supported by the Ministry of Education and Culture (Spain). He wants to thank here the Département de Mathématique of the UCL, and in a special way, the members of the Research Team 'Analyse non linéaire et équations différentielles', for its hospitality through this time and several fruitful discussions with its various members.

ISSN 1025-5834 print; ISSN 1029-242X.
DOI: 10.1080/1025583021000022469

with $f : [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ continuous, has at least one solution u such that $\|u(t)\| \leq R$ for all $t \in [0, 1]$ when there exists $R > 0$, a continuous function $\varphi : [0, +\infty[\to \mathbb{R}^+$ such that

$$\int_0^{+\infty} \frac{s}{\varphi(s)} ds = +\infty \tag{1}$$

and nonnegative numbers γ, C such that the following conditions hold:

(i) $\langle x, f(t, x, y)\rangle + \|y\|^2 \geq 0$ for all $t \in [0, 1]$ and $x, y \in \mathbb{R}^N$ such that $\|x\| = R$, $\langle x, y\rangle = 0$.

(ii) $\|f(t, x, y)\| \leq \varphi(\|y\|)$ and $\|f(t, x, y)\| \leq 2\gamma(\langle x, f(t, x, y)\rangle + \|y\|^2) + C$ for all $t \in [0, 1]$
and $x, y \in \mathbb{R}^N$
such that $\|x\| \leq R$.

(iii) $\|u_0\|$, $\|u_1\| \leq R$.

In 1971, Knobloch [3] proved that under conditions (*i*) and (*ii*) on the (locally Lipschitzian in u, u') nonlinearity f, the existence of a solution for the periodic problem arising from equation (0.2) was also ensured. The local Lipschitz condition was shown to be superfluous in [8]. A basic ingredient in those proofs is the so-called *Hartman–Nagumo inequality* which tells that:

If $x \in C^2([0, 1], \mathbb{R}^N)$ *is such that*

$$\|x(t)t \leq R, \|x''(t)\| \leq \varphi(\|x'(t)\|), \text{ and } \|x''(t) \\ \leq \gamma(\|x(t)\|^2)'' + C, \quad (t \in [0, 1]),$$

for some φ *satisfying* (1), *some* $R > 0, \gamma \geq 0, C \geq 0$, *then there exists some* $K > 0$ *(only depending on* R, φ, γ *and* C*), such that*

$$\|x'(t)\| < K \qquad (t \in [0, 1]).$$

Recently, Mawhin [6,7] extended the Hartman–Knobloch results to nonlinear perturbations of the ordinary vector p-laplacian of the form

$$(\|u'\|^{p-2}u')' = f(t, u).$$

His approach was based upon the application of the Schauder fixed point theorem to a suitable modification of the original problem, whose solutions coincide with those of the original one.

Our aim here is to extend, at the same time, the Hartman–Knobloch results to nonlinear perturbations of the ordinary p-Laplacian and Mawhin's results to derivative-depending nonlinearities. In the case of Dirichlet boundary conditions, we use the Schauder fixed point theorem to find solutions to a modified problem, while in dealing with periodic ones, our main tool is the continuation theorem proved in [6]. Both procedures strongly depend on the
extension of the Hartman–Nagumo inequality developed in Section 2.

For $N \in \mathbb{N}$ and $1 < p < +\infty$ fixed, we denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^N$ and by $|\cdot|$ the absolute value in $\mathbb{R}$, while $\langle \cdot, \cdot \rangle$ stands for the Euclidean inner product in $\mathbb{R}^N$. By p' we mean the Hölder conjugate of p (given by $1/p + 1/p' = 1$). For $q > 1$, the symbol ϕ_q is used to represent the mapping

$$\phi_q : \mathbb{R}^N \to \mathbb{R}^N, \qquad x \mapsto \begin{cases} \|x\|^{q-2}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then, it is clear that ϕ_p and $\phi_{p'}$ are mutually inverse homeomorphisms from $\mathbb{R}^N$ to itself, and mutually inverse analytic diffeomorphisms from $\mathbb{R}^N \setminus \{0\}$ to itself. Furthermore, an elementary computation shows that

$$\phi_q'(x)v = \|x\|^{q-2}\left((q-2)\left\langle \frac{x}{\|x\|}, v\right\rangle \frac{x}{\|x\|} + v\right) \qquad (\natural)$$

for all $x \in \mathbb{R}^N \setminus \{0\}$, all $v \in \mathbb{R}^N$ and $q > 1$.

2 A HARTMAN–NAGUMO-TYPE INEQUALITY FOR THE *p*-LAPLACIAN

In this section, we extend the Hartman–Nagumo inequality [1,2] associated to the second order differential operator $x \to x''$ to the p-Laplacian case $x \to (\phi_p(x'))'$. We first need a preliminary result giving an estimate on the L^{p-1} norm of x' when x is bounded in the uniform norm and some differential inequalities involving $(\phi_p(x'))'$ hold. Let us call, for brevity, *p-admissible* any C^1 mapping $x : [0, 1] \to \mathbb{R}^N$ such that $\phi_p(x') : [0, 1] \to \mathbb{R}^N$ is of class C^1.

LEMMA 2.1 *Let $B > 0$ be given. Then, there exists a positive number $M > 0$ (depending only on B) such that for each p-admissible mapping x verifying, for some C^1 and convex $r : [0, 1] \to \mathbb{R}$, the following inequalities*:

(*i*) $\|x(t)\|, |r(t)| \leq B$ *for all* $t \in [0, 1]$;
(*ii*) $\|(\phi_p(x'))'\| \leq r''$ *a.e. on* $[0, 1]$,

one has

$$\int_0^1 \|x'(t)\|^{p-1} dt < M.$$

Proof Condition (ii) implies that

$$\|\phi_p(x'(s)) - \phi_p(x'(t))\| \leq r'(s) - r'(t), \quad (0 \leq t \leq s \leq 1), \tag{2}$$

and hence

$$\|\phi_p(x'(s))\| \leq \|\phi_p(x'(t))\| + r'(s) - r'(t), \quad (0 \leq t \leq s \leq 1), \tag{2.1}$$

$$\|\phi_p(x'(s))\| \leq \|\phi_p(x'(t))\| + r'(t) - r'(s), \quad (0 \leq s \leq t \leq 1). \tag{2.2}$$

Integrating inequality (2.1) with respect to s, we find

$$\int_t^1 \|x'(s)\|^{p-1}ds \leq (1-t)\|x'(t)\|^{p-1} + r(1) - r(t) - (1-t)r'(t)$$
$$\leq (1-t)\|x'(t)\|^{p-1} + 2B - (1-t)r'(t), \quad (0 \leq t \leq 1) \tag{3.1}$$

while integrating inequality (2.2) with respect to s we get

$$\int_0^t \|x'(s)\|^{p-1}ds \leq t\|x'(t)\|^{p-1} - r(t) + r(0) + tr'(t)$$
$$\leq t\|x'(t)\|^{p-1} + 2B + tr'(t), \quad (0 \leq t \leq 1). \tag{3.2}$$

Adding expressions (3.1) and (3.2), we find

$$\int_0^1 \|x'(s)\|^{p-1}ds \leq \|x'(t)\|^{p-1} + (2t-1)r'(t) + 4B, \quad (0 \leq t \leq 1), \tag{4}$$

and we deduce that

$$\int_0^1 \|x'(s)\|^{p-1}ds \leq \|x'(t)\|^{p-1} + |r'(t)| + 4B, \quad (0 \leq t \leq 1). \tag{5}$$

Now, the convexity of r means that r' is increasing. Together with the bound $|r(t)| \leq B$ for all $t \in [0, 1]$, it implies that

$$|r'(t)| \leq 6B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right], \tag{6}$$

which, in combination with (5), gives us the inequality

$$\int_0^1 \|x'(s)\|^{p-1}ds \leq \|x'(t)\|^{p-1} + 10B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right]. \tag{7}$$

Following a similar process as before but integrating inequalities (2.1) and (2.2) with respect to t instead of s, we get

$$\|x'(s)\|^{p-1} \leq \int_0^1 \|x'(t)\|^{p-1}dt + |r'(s)| + 4B, \quad (0 \leq s \leq 1),$$

which, after changing the names of the variables s and t, is the same as

$$\|x'(t)\|^{p-1} \leq \int_0^1 \|x'(s)\|^{p-1} ds + |r'(t)| + 4B, \quad (0 \leq t \leq 1), \tag{8}$$

and, again, using (6), gives

$$\|x'(t)\|^{p-1} \leq \int_0^1 \|x'(s)\|^{p-1} ds + 10B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right]. \tag{9}$$

The information given by (7) and (9) can be written jointly as

$$\left| \|x'(t)\|^{p-1} - \int_0^1 \|x'(s)\|^{p-1} ds \right| \leq 10B \quad \text{for all} \quad t \in \left[\frac{1}{3}, \frac{2}{3}\right], \tag{10}$$

which clearly implies that

$$\left| \|x'(t)\|^{p-1} - \|x'(s)\|^{p-1} \right| \leq 20B \quad \text{for all} \quad t, s \in \left[\frac{1}{3}, \frac{2}{3}\right]. \tag{11}$$

Suppose now that the conclusion of Lemma 2.1 is not true. This will imply the existence of sequences $\{x_n\}$ in $C^1([0, 1], \mathbb{R}^N)$ and $\{r_n\}$ in $C^1([0, 1], \mathbb{R})$ such that x_n is p-admissible, r_n is convex for all $n \in \mathbb{N}$, and, furthermore,

$(\hat{\text{i}})_n$ $\|x_n(t)\|, |r_n(t)| \leq B$ for all $t \in [0, 1]$,

$(\hat{\text{ii}})_n$ $\|(\phi_p(x_n'))'(t)\| \leq r_n''(t)$ a.e. on $[0, 1]$,

$(\hat{\text{iii}})$ $\int_0^1 \|x_n'(t)\|^{p-1} dt \to +\infty$.

From $(\hat{iii})$ and (10) we deduce that $\|x_n'(\frac{1}{2})\|^{p-1} \to +\infty$ as $n \to +\infty$, or what is the same, that $\|x_n'(\frac{1}{2})\| \to +\infty$ as $n \to +\infty$. In particular, we can suppose, after taking apart a finite set of terms if necessary, that $x_n'(\frac{1}{2}) \neq 0$ for all $n \in \mathbb{N}$. From (11) we can conclude now that the sequence of continuous functions $\left\{\|x_n'(\cdot)\|^{p-1}/\|x_n'(\frac{1}{2})\|^{p-1}\right\}$ converges to 1 uniformly on $[\frac{1}{3}, \frac{2}{3}]$ as $n \to \infty$, or, what is the same, that the sequence of continuous functions $\left\{\|x_n'(\cdot)\|/\|x_n'(\frac{1}{2})\|\right\}$ converges to 1 uniformly on $[\frac{1}{3}, \frac{2}{3}]$ as $n \to \infty$.

Going back to (2) we can use (6) to obtain the inequalities

$$\left\| \phi_p(x_n'(t)) - \phi_p\left(x_n'\left(\frac{1}{2}\right)\right) \right\| \leq \left| r_n'(t) - r_n'\left(\frac{1}{2}\right) \right| \leq 12B \tag{12}$$

for all $t \in [\frac{1}{3}, \frac{2}{3}]$ and all $n \in \mathbb{N}$, and, if n is large enough so that $x_n'(t) \neq 0$ for all $t \in [\frac{1}{3}, \frac{2}{3}]$, dividing inequality (12) by $\|x_n'(\frac{1}{2})\|^{p-1}$ we obtain

$$\left\| \frac{\|x_n'(t)\|^{p-1}}{\|x_n'(\frac{1}{2})\|^{p-1}} \frac{x_n'(t)}{\|x'(n)(t)\|} - \frac{x_n'(\frac{1}{2})}{\|x_n'(\frac{1}{2})\|} \right\| \leq \frac{12B}{\|x_n'(\frac{1}{2})\|^{p-1}} \tag{13}$$

for all $t \in [\frac{1}{3}, \frac{2}{3}]$, and we deduce that

$$\left\{ \frac{x_n'(t)}{\|x_n'(t)\|} - \frac{x_n'(\frac{1}{2})}{\|x_n'(\frac{1}{2})\|} \right\} \to 0 \quad \text{as} \quad n \to \infty \tag{14}$$

uniformly on $[\frac{1}{3}, \frac{2}{3}]$. We can find, therefore, an integer $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$\left\langle \frac{x_n'(t)}{\|x_n'(t)\|}, \frac{x_n'(\frac{1}{2})}{\|x_n'(\frac{1}{2})\|} \right\rangle \geq \frac{1}{2}$$

for all $t \in [\frac{1}{3}, \frac{2}{3}]$, what is the same as

$$\left\langle x_n'(t), \frac{x_n'(\frac{1}{2})}{\|x_n'(\frac{1}{2})\|} \right\rangle \geq \frac{\|x_n'(t)\|}{2} \tag{15}$$

for all $t \in [\frac{1}{3}, \frac{2}{3}]$ and all $n \geq n_0$. To end the proof, fix any $n_1 \geq n_0$ such that

$$\|x_{n_1}'(t)\| > 12B \tag{16}$$

for all $t \in [\frac{1}{3}, \frac{2}{3}]$, and verify that, because of (15),

$$\left\langle x_{n_1}'(t), \frac{x_{n_1}'(\frac{1}{2})}{\|x_{n_1}'(\frac{1}{2})\|} \right\rangle > 6B \tag{17}$$

for all $t \in [\frac{1}{3}, \frac{2}{3}]$. This inequality, integrated between $\frac{1}{3}$ and $\frac{2}{3}$, gives

$$\left\langle x_{n_1}\left(\frac{2}{3}\right) - x_{n_1}\left(\frac{1}{3}\right), \frac{x_{n_1}(\frac{1}{2})}{\|x_{n_1}(\frac{1}{2})\|}\right\rangle > 2B. \tag{18}$$

Hence, using the Cauchy-Schwarz inequality, we obtain the contradiction

$$\left\langle x_{n_1}\left(\frac{2}{3}\right) - x_{n_1}\left(\frac{1}{3}\right), \frac{x_{n_1}(\frac{1}{2})}{\|x_{n_1}(\frac{1}{2})\|}\right\rangle \le \left\| x_{n_1}\left(\frac{2}{3}\right) - x_{n_1}\left(\frac{1}{3}\right)\right\| \le 2B. \tag{19}$$

The following lemma provides, under an extra Nagumo-type hypothesis, an estimate for the uniform norm of x'.

LEMMA 2.2 *Let $B > 0$ be given and choose the corresponding $M > 0$ according to Lemma 2.1. Let $\varphi : [M, +\infty[\to \mathbb{R}^+$ be continuous and such that*

$$\int_M^{+\infty} \frac{s}{\varphi(s)}\, ds > M.$$

Then, there exists a positive number $K > 0$ (depending only on B, M and φ) such that for each p-admissible mapping x satisfying, for some C^1 convex function $r : [0, 1] \to \mathbb{R}$, the following conditions:

1. $\|x(t)\|, |r(t)| \le B$ *for all* $t \in [0, 1]$;
2. $\|(\phi_p(x'))'\| \le r''$ *a.e. on* $[0, 1]$;
3. $\|(\phi_p(x'))'(t)\| \le \varphi(\|x'(t)\|^{p-1})$ *for all* $t \in [0, 1]$ *with* $\|x'(t)\|^{p-1} \ge M$, *one has*

$$\|x'(t)\| < K \quad \textit{for all} \quad t \in [0, 1].$$

Proof Choose $K > \sqrt[p-1]{M}$ such that

$$\int_M^{K^{p-1}} \frac{s}{\varphi(s)}\, ds = M.$$

We show that the thesis holds for this K. To this aim, fix any x, r verifying the hypothesis of Lemma 2.2, and suppose that there exists some $t_0 \in [0, 1]$ such that $\|x'(t_0)\| \geq K$, and hence $\|x'(t_0)\|^{p-1} \geq K^{p-1} > M$. By definition of the constant M we have $\int_0^1 \|x'(t)\|^{p-1} dt < M$, so that there must exist some $t_1 \in [0, 1]$ (we pick the closest one to t_0), such that $\|x'(t_1)\|^{p-1} = M$.

Define

$$\Phi : [M, +\infty[\to [0, +\infty[, \quad t \mapsto \int_M^t \frac{s}{\varphi(s)}\, ds, \tag{20}$$

and notice that Φ is continuous, $\Phi(M) = 0$, Φ is strictly increasing and $\Phi(K^{p-1}) = M$. Now,

$$M = \Phi(K^{p-1}) \leq \Phi(\|x'(t_0)\|^{p-1}) = |\Phi(\|x'(t_0)\|^{p-1})|$$

$$= \left| \int_M^{\|x'(t_0)\|^{p-1}} \frac{s}{\varphi(s)}\, ds \right| = \left| \int_{\|x'(t_0)\|^{p-1}}^{\|x'(t_1)\|^{p-1}} \frac{s}{\varphi(s)}\, ds \right| = \left| \int_{\|\phi_p(x'(t_0))\|}^{\|\phi_p(x'(t_1))\|} \frac{s}{\varphi(s)}\, ds \right|. \tag{21}$$

Using the change of variables $s = \|\phi_p(x'(t))\|$, $t \in [\min\{t_0, t_1\}, \max\{t_0, t_1\}]$, (which is absolutely continuous because $\phi_p(x')$ is C^1 and $\|\cdot\|$ is Lipschitzian), we obtain, from hypothesis *3*

$$M \leq \left| \int_{\|\phi_p(x'(t_0))\|}^{\|\phi_p(x'(t_1))\|} \frac{s}{\varphi(s)}\, ds \right|$$

$$= \left| \int_{t_0}^{t_1} \frac{\|\phi_p(x'(t))\|}{\varphi\Big(\|\phi_p(x'(t))\|\Big)} \frac{\langle \phi_p(x'(t)), (\phi_p(x'))'(t) \rangle}{\|\phi_p(x'(t))\|}\, dt \right|$$

$$\leq \left| \int_{t_0}^{t_1} \|\phi_p(x'(t))\| \cdot \frac{\|(\phi_p(x'))'(t)\|}{\phi\Big(\|\phi_p(x'(t))\|\Big)}\, dt \right|$$

$$\leq \left| \int_{t_0}^{t_1} \|\phi_p(x'(t))\| dt \right| = \left| \int_{t_0}^{t_1} \|x'(t)\|^{p-1}\, dt \right|$$

so that

$$M \leq \int_{\min\{t_0,t_1\}}^{\max\{t_0,t_1\}} \|x'(t)\|^{p-1} dt \leq \int_0^1 \|x'(t)\|^{p-1} dt < M, \qquad (22)$$

a contradiction.

The following elementary result of real analysis is used in the proof of the next Theorem.

LEMMA 2.3 *Let $\alpha, h : [0,1] \to \mathbb{R}$ be continuous functions, α non decreasing. Suppose that h' exists and is nonnegative in the open set $\{t \in]0,1[: h(t) \neq \alpha(t)\}$. Then h is non decreasing on $[0,1]$.*

Proof Suppose, by contradiction, that there exist $s < t$ in [0,1] such that $h(s) > h(t)$. There must be some $x \in]s,t[$ such that $h(x) = \alpha(x)$ (otherwise, the Lagrange mean value theorem would give us the inequality $h(s) \leq h(t)$). Define

$$a := \min\{x \in [s,t] : h(x) = \alpha(x)\}, b := \max\{x \in [s,t] : h(x) = \alpha(x)\}.$$

Again, by the Lagrange mean value theorem, we have the inequalities

$$h(s) \leq h(a) = \alpha(a) \leq \alpha(b) = h(b) \leq h(t),$$

a contradiction.

We can now prove the proposed extension of the Hartman–Nagumo inequality.

THEOREM 2.4 *Let $R > 0, \gamma \geq 0, C \geq 0$ be given and choose $M > 0$ as associated by Lemma 2.1 to $B := \max\{R, \gamma R^2 + C/2\}$. Let $\varphi : [M, +\infty[\to \mathbb{R}^+$ be continuous and such that*

$$\int_M^{+\infty} \frac{s}{\varphi(s)} ds > M.$$

Then, there exists a positive number $K > 0$ (depending only on R, p, γ, C, M and φ) such that, for any p-admissible mapping x satisfying the following conditions:

(*i*) $\|x(t)\| \leq R, \quad (0 \leq t \leq 1)$;

(*ii*) $\|(\phi_p(x'))'(t)\| \leq \gamma(\|x(t)\|^2)'' + C$ *for all* $t \in [0, 1]$ *such that* $x'(t) \neq 0$;

(*iii*) $\|(\phi_p(x'))'(t)\| \leq \varphi(\|x'(t)\|^{p-1})$ *for all* $t \in [0, 1]$ *such that* $\|x'(t)\|^{p-1} \geq M$, *one has*

$$\|x'(t)\| < K \quad (t \in [0, 1]).$$

Proof From the chain rule we know that $x' = \phi_{p'}(\phi_p(x'))$ is a C^1 mapping on the set $\{t \in [0, 1] : x'(t) \neq 0\}$. Let us define

$$r : [0, 1] \to \mathbb{R}, \quad t \mapsto \gamma m(t) + C\,\frac{t^2}{2}, \tag{23}$$

where $m(t) = \|x(t)\|^2$. It is clear that r is a C^1 function. Moreover,

$$r'(t) = 2\gamma\langle x(t), x'(t)\rangle + Ct \quad (t \in [0, 1]). \tag{24}$$

It means that x' does not vanish on the set $\{t \in [0, 1] : r'(t) \neq Ct\}$, and then, on this set, r is C^2 and $r''(t) = \gamma m''(t) + C \geq \|(\phi_p(x'))'(t)\| \geq 0$.

By Lemma 2.3 we deduce that r' is non decreasing, what is equivalent to say that r is convex. Also, it is clear that

$$\|x(t)\|, |r(t)| \leq B \quad (t \in [0, 1]), \tag{$\tilde{1}$}$$

and, to be able to apply Lemma 2.2 we only have to check that inequality

$$\|(\phi_p(x'))'(t)\| \leq r''(t) \tag{$\tilde{2}$}$$

holds for almost every t in [0,1].

Notice, firstly, that our hypothesis (*ii*) says that $(\tilde{2})$ is true for all t in [0,1] such that $x'(t) \neq 0$. Secondly, in the interior of the set $\{t \in [0, 1] : x'(t) = 0\}$ we have

$$\|(\phi_p(x'))'(t)\| = 0 \leq r''(t) = C.$$

It remains to see what happens on $A := \partial\{t \in [0, 1] : x'(t) = 0\}$. We will prove that at every point $t \in A \cap]0, 1[$ such that $(r')'(t) = r''(t)$ exists we have the inequality

$$\|(\phi_p(x'))'(t)\| \leq r''(t). \tag{25}$$

Pick some point $t_0 \in A \cap]0, 1[$ such that $r''(t)$ exists. If t_0 is an isolated point of A, there exists some $\varepsilon > 0$ such that $]t_0, t_0 + \varepsilon[\subset]0, 1[\backslash A$. Then, r' and $\phi_p(x')$ are both of class C^1 on $]t_0, t_0 + \varepsilon[$ and we have the inequality

$$\|(\phi_p(x'))'(t)\| \leq r''(t) \quad (t \in]t_0, t_0 + \varepsilon[). \tag{26}$$

It follows that $\|\phi_p(x'(t)) - \phi_p(x'(s))\| \leq r'(t) - r'(s)$ for all s, t with $t_0 < s < t < t_0 + \varepsilon$, and letting $s \to t_0$, that

$$\|\phi_p(x'(t)) - \phi_p(x'(t_0))\| \leq r'(t) - r'(t_0) \quad (t \in]t_0, t_0 + \varepsilon[), \tag{27}$$

from which we deduce that $\|\phi_p(x'))'(t_0)\| \leq r''(t_0)$. If, otherwise, t_0 is an accumulation point of A, there exists a sequence $\{a_n\}$ of points from $A\backslash\{t_0\}$ converging to t_0. But $x'(a_n) = 0$ for all $n \in \mathbb{N}$, which implies that $\phi_p(x'(a_n)) = 0$ and $r'(a_n) = Ca_n$ for all $n \in \mathbb{N}$. We conclude then that $(\phi_p(x'))'(t_0) = 0 \leq C = r''(t_0)$.

Theorem 2.4. is now a simple consequence of Lemma 2.2.

3 NONLINEAR PERTURBATIONS OF THE *p*-LAPLACIAN

Let $f : [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be continuous, and consider the following system of differential equations

$$(\phi_p(x'))' = f(t, x, x'), \quad (0 \leq t \leq 1). \tag{28}$$

Our goal in the remaining part of this work is to develop some existence results for the solutions of (28) verifying either the *periodic boundary conditions*:

$$x(0) = x(1), \quad x'(0) = x'(1), \tag{P}$$

or the *Dirichlet boundary conditions*

$$x(0) = x_0, \quad x(1) = x_1, \tag{D}$$

where x_0 and x_1 are some given points of $\mathbb{R}^N$. We need the following two easy facts.

LEMMA 3.1 *Let x be a p-admissible mapping. For each $t_0 \in]0,1[$ such that $\|x(t_0)\| = \max_{t\in[0,1]} \|x(t)\|$, one has*

$$\langle x(t_0), x'(t_0)\rangle = 0 \quad \textit{and} \quad \langle x(t_0), (\phi_p(x'))'(t_0)\rangle + \|x'(t_0)\|^p \leq 0.$$

Furthermore, the same conclusion remains true when $t_0 = 0$ or 1 if x is assumed to verify the periodic boundary conditions (**P**).

Proof Suppose first that $t_0 \in]0,1[$. The equality

$$\|x(t_0)\|^2 = \max_{t\in[0,1]} \|x(t)\|^2 \tag{29}$$

implies that

$$2\langle x(t_0), x'(t_0)\rangle = \frac{d}{dt} \|x(t)\|^2_{|t=t_0} = 0. \tag{30}$$

Next, suppose by contradiction that

$$\langle x(t_0), (\phi_p(x'))'(t_0)\rangle + \|x'(t_0)\|^p > 0, \tag{31}$$

what is the same as

$$\frac{d}{dt} \langle x(t), \phi_p(x'(t))\rangle_{|t=t_0} > 0. \tag{32}$$

As

$$\langle x(t_0), \phi_p(x'(t_0))\rangle = \|x'(t_0)\|^{p-2}\langle x(t_0), x'(t_0)\rangle = 0,$$

we deduce the existence of some $\varepsilon > 0$ such that $]t_0 - \varepsilon, t_0 + \varepsilon[\subset [0, 1]$ and

$$\langle x(t), \phi_p(x'(t))\rangle < 0, \quad t \in]t_0 - \varepsilon, t_0[\tag{33}$$

$$\langle x(t), \phi_p(x'(t))\rangle > 0, \quad t \in]t_0, t_0 + \varepsilon[. \tag{34}$$

Equivalently,

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle x(t), x'(t)\rangle < 0, \quad t \in]t_0 - \varepsilon, t_0[\tag{35}$$

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle x(t), x'(t)\rangle > 0, \quad t \in]t_0, t_0 + \varepsilon[, \tag{36}$$

which implies that $\|x(t)\|$ attains a strict local minimum at $t = t_0$. Of course, this is not compatible with our hypothesis and this first case is proved.

If now x verifies the periodic boundary conditions (**P**) and

$$\|x(0)\| = \|x(1)\| = \max_{t\in[0,1]} \|x(t)\|,$$

define $y : [0, 1] \to \mathbb{R}^N$ by $y(t) := x(t + \frac{1}{2})$ if $0 \le t \le \frac{1}{2}$, $y(t) := x(t - \frac{1}{2})$ if $\frac{1}{2} \le t \le 1$ and apply the above result to y (at $t_0 = \frac{1}{2}$) to obtain the desired result.

LEMMA 3.2. *Let $f_i : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, (i = 1,2,3,\ldots)$, be a sequence of continuous mappings, converging uniformly on compact sets to $f : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$. Suppose that there exist positive numbers $R, K > 0$ such that for every $i \in \mathbb{N}$ there exist a solution x_i of the system*

$$(\phi_p(x'))' = f_i(t, x, x')$$

with

$$\|x_i(t)\| \le R, \ \|x_i'(t)\| \le K \qquad (t \in [0,1]).$$

Then there exists a subsequence of $\{x_i\}$ *converging in the space* $C^1[0,1]$ *to some p-admissible mapping* $\bar{x} : [0,1] \to \mathbb{R}^N$, *which is a solution of (28).*

Proof The two sequences of continuous mappings $\{x_i\}$ and $\{\phi_p(x_i')\}$ are uniformly bounded together with its derivatives, so that, by the Ascolí-Arzelà Lemma, we can find a subsequence $\{z_i\}$ of $\{x_i\}$ uniformly converging on [0,1] and such that the sequence $\{\phi_p(z_i')\}$ is also uniformly converging on [0,1]. As ϕ_p is an homeomorphism from $\mathbb{R}^N$ to itself, we deduce that both $\{z_i\}$ and $\{z_i'\}$ are uniformly converging on [0,1]. Finally, from the equalities

$$(\phi_p(x_i'))' = f_i(t, x_i, x_i') \quad (i = 1, 2, 3, \ldots,)$$

we deduce that also the sequence $\{(\phi_p(z_i'))'\}$ converges uniformly on [0,1]. The result now follows.

The following set of hypothesis on the nonlinearity f will be widely used in the remaining of this work and will be denoted by **(H)**:

There exist $R > 0, \gamma \ge 0, C \ge 0, M > 0$ *as associated by Lemma 2.1 to* $B := \max\{R, \gamma R^2 + C/2\}$ *and* $\varphi : [M, +\infty[\to \mathbb{R}^+$ *continuous with*

$$\int_M^{+\infty} \frac{s}{\varphi(s)}\, ds > M$$

such that

(a) For any $t \in [0,1], x, y \in \mathbb{R}^N$ *such that* $\|x\| = R, \langle x, y\rangle = 0$, *we have*

$$\langle x, f(t,x,y)\rangle + \|y\|^p \ge 0;$$

(b) For any $t \in [0,1], x, y \in \mathbb{R}^N$ *such that* $\|x\| \le R$ *and* $\|y\|^{p-1} \ge M$,

$$\|f(t,x,y)\| \le \varphi(\|y\|^{p-1});$$

(c) For any $t \in [0, 1], x \in \mathbb{R}^N$ *with* $\|x\| \leq R$ *and* $y \in \mathbb{R}^N$,

$$\|y\|^p \|f(t, x, y)\| \leq 2\gamma((p' - 2)\langle y, f(t, x, y)\rangle\langle x, y\rangle + \|y\|^2 \langle x, f(t, x, y)\rangle + \|y\|^{p+2}) + C\|y\|^p.$$

As we will see next, these assumptions on f will be sufficient to ensure the existence of a solution for both the periodic and the Dirichlet problems associated to equation (28). However, in our approach to these problems, we will have to assume first a slightly stronger set of hypothesis, consisting in replacing

(a) by $(\tilde{a})$ *For any* $t \in [0, 1], x, y \in \mathbb{R}^N$ *such that* $\|x\| = R$, $\langle x, y\rangle = 0$, *we have*

$$\langle x, f(t, x, y)\rangle + \|y\|^p > 0.$$

The new set of hypothesis will be denoted by $(\tilde{\mathbf{H}})$.

Notice, furthermore, that if there exist numbers $R > 0, \gamma \geq 0, C \geq 0$ and a continuous function $\varphi : [0, +\infty[\to \mathbb{R}^+$ verifying the classical Nagumo condition

$$\int_0^{+\infty} \frac{s}{\varphi(s)}\, ds = +\infty,$$

such that (a), (b) and (c) are still satisfied for some $M > 0$, then, the whole set of hypothesis **(H)** is ensured.

4 THE PERIODIC PROBLEM

We prove in this section the existence of a solution for the periodic problem associated to equation (28).

THEOREM 4.1 *Let* $f : [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ *be a continuous mapping satisfying* **(H)**. *Then, the periodic boundary value problem* **(P)** *for equation (28) has at least one solution* $x : [0, 1] \to \mathbb{R}^N$ *such that* $\|x(t)\| \leq R$ *for all* $t \in [0, 1]$.

Proof The theorem will be proved in two steps. In the first one, we assume that the set of hypothesis ($\tilde{\mathbf{H}}$) holds. To prove the theorem in this more restrictive case, choose $K > 0$ as given by Theorem 2.4 for R, γ, C, M and φ, and define

$$\Omega := \{x \in C^1_T\ ([0, 1]) : \|x(t)\| < R,\ \|x'(t)\| < K \quad \text{for all} \quad t \in [0, 1]\}. \tag{37}$$

Our aim is to apply the continuation theorem 5.1 from [6] in our case. First we have to prove is that for each $\lambda \in]0, 1[$, the problem

$$(P_\lambda) \equiv \begin{cases} (\phi_p(x'))' = \lambda f(t, x, x') \\ x(0) = x(1),\ x'(0) = x'(1) \end{cases} \tag{38}$$

has no solutions on $\partial\Omega$. Indeed, notice that

$$\bar{\Omega} = \{x \in C^1_T[0, 1] : \|x(t)\| \le R,\ \|x'(t)\| \le K \quad \text{for all} \quad t \in [0, 1]\}. \tag{39}$$

Now, fix any $\lambda \in]0, 1[$ and let $\bar{x} \in \bar{\Omega}$ be a solution of (P_λ). Our hypothesis (b) tells us that

$$\begin{aligned} \|(\phi_p(\bar{x}'))'(t)\| &= \lambda \| f(t, \bar{x}(t), \bar{x}'(t))\| \\ &\le \| f(t, \bar{x}(t)), \bar{x}'(t))\| \le \varphi(\|\bar{x}'(t)\|^{p-1}) = \varphi(\|\phi_p(\bar{x}'(t))\|) \end{aligned} \tag{40}$$

for every $t \in [0, 1]$ such that $\|\bar{x}'(t)\|^{p-1} \ge M$. That is the third hypothesis needed in Theorem 2.4. The first one is obviously satisfied. Let us check the second one. We can use (♮) to find that

$$z, v \in \mathbb{R}^N,\ z \ne 0 \Rightarrow \phi'_{p'}(\phi_p(z))v = \|z\|^{-p}((p' - 2)\langle z, v\rangle z + \|z\|^2 v). \tag{41}$$

In our context, it means that, for each $t \in [0, 1]$ such that $\bar{x}'(t) \neq 0, \bar{x}''(t)$ exists, and furthermore,

$$\begin{aligned}\bar{x}''(t) &= (\phi_{p'}(\phi_p(\bar{x}')))'(t) = \phi'_{p'}(\phi_p(\bar{x}'(t)))(\phi_p(\bar{x}'))'(t)\\ &= \phi'_{p'}(\phi_p(\bar{x}'(t)))(\lambda f(t, \bar{x}(t), \bar{x}'(t)))\\ &= \lambda\|\bar{x}'(t)\|^{-p}((p'-2)\langle\bar{x}'(t), f(t, \bar{x}(t), \bar{x}'(t))\rangle\bar{x}'(t)\\ &\quad + \|\bar{x}'(t)\|^2 f(t, \bar{x}(t), \bar{x}'(t))),\end{aligned} \tag{42}$$

and consequently,

$$\begin{aligned}&2(\langle\bar{x}(t), \bar{x}''(t)\rangle + \|\bar{x}'(t)\|^2)\\ &\quad\geq 2(\langle\bar{x}(t), \bar{x}''(t)\rangle + \lambda\|\bar{x}'(t)\|^2)\\ &\quad= 2\lambda\|\bar{x}'(t)\|^{-p}((p'-2)\langle\bar{x}'(t), f(t, \bar{x}(t), \bar{x}'(t))\rangle\langle\bar{x}(t), \bar{x}'(t)\rangle\\ &\qquad + \|\bar{x}'(t)\|^{p+2} + \|\bar{x}'(t)\|^2\langle\bar{x}(t), f(t, \bar{x}(t), \bar{x}'(t))\rangle)\end{aligned} \tag{43}$$

for all $t \in [0, 1]$ with $\bar{x}'(t) \neq 0$. It turns out that, if we define $r : [0, 1] \to \mathbb{R}$ by $r(t) = \|\bar{x}(t)\|^2$, for each $t \in [0, 1]$ such that $\bar{x}'(t) \neq 0$, we can write, using hypothesis (c),

$$\begin{aligned}\|(\phi_p(\bar{x}'))'(t)\| &= \lambda\|f(t, \bar{x}(t), \bar{x}'(t))\|\\ &\leq 2\lambda\gamma\|\bar{x}'(t)\|^{-p}((p'-2)\langle\bar{x}'(t), f(t, \bar{x}(t), \bar{x}'(t))\rangle\langle\bar{x}(t), \bar{x}'(t)\rangle\\ &\quad + \|\bar{x}'(t)\|^2\langle\bar{x}(t), f(t, \bar{x}(t), \bar{x}'(t))\rangle + \|\bar{x}'(t)\|^{p+2}) + \lambda C\\ &\leq \gamma r''(t) + \lambda C \leq \gamma r''(t) + C.\end{aligned} \tag{44}$$

Now, Theorem 2.4 tells us that

$$\|\bar{x}'(t)\| < K \qquad (t \in [0, 1]), \tag{45}$$

and therefore, in order to see that $\bar{x} \in \Omega$, it only remains to prove the inequality

$$\|\bar{x}(t)\| < R \qquad (t \in [0, 1]). \tag{46}$$

Suppose, otherwise, that there exists some point $t_0 \in [0, 1]$ such that $\|\bar{x}(t_0)\| = R$. Then, $\|\bar{x}(t_0)\| = \max_{t\in[0,1]} \|\bar{x}(t)\|$, and from Lemma 3.1 we should have

$$\langle \bar{x}(t_0), (\phi_p(\bar{x}'))'(t_0)\rangle + \|\bar{x}'(t_0)\|^p = \langle \bar{x}(t_0), f(t_0, \bar{x}(t_0), \bar{x}'(t_0))\rangle + \|\bar{x}'(t_0)\|^p \\ \leq 0,$$

contradicting our hypothesis $(\tilde{a})$.

Finally, it remains to check that the equation

$$\mathcal{F}(a) := \int_0^1 f(t, a, 0)dt = 0 \tag{47}$$

has no solutions on $\partial\Omega \cap \mathbb{R}^N = \{a \in \mathbb{R}^N : \|a\| = R\}$, and that the Brouwer degree

$$deg_B[\mathcal{F}, \Omega \cap \mathbb{R}^N, 0] = deg_B[\mathcal{F}, \mathbb{B}_R(0), 0]$$

is not zero. But from hypothesis $(\tilde{a})$ (taking $y = 0$) we deduce

$$\langle a, f(t, a, 0)\rangle > 0 \quad \text{for all} \quad a \in \mathbb{R}^N, \ \|a\| = R, \quad \text{and all} \quad t \in [0, 1], \tag{48}$$

and, integrating from 0 to 1 we get

$$\left\langle a, \int_0^1 f(t, a, 0)dt\right\rangle = \langle a, \mathcal{F}(a)\rangle > 0 \quad \text{for all} \quad a \in \partial\mathbb{B}_R(0), \tag{49}$$

which, effectively, implies that $\deg_B[\mathcal{F}, \mathbb{B}_R(0), 0] = 1$. This concludes our first step. The theorem is proved assuming $(\tilde{a})$ instead of (a). And the whole theorem follows now from a simple approximation argument that we sketch below.

Fix some $\varepsilon_* > 0$ small enough so that, after defining

$$\varphi_* := \varphi + \varepsilon_* R, \ R_* := R, \ C_* := C + R\varepsilon_*, \gamma_* := \gamma,$$

$$B_* := \max\left\{R_*, \gamma R_*^2 + \frac{C_*}{2}\right\}, \ M_* := \max\left\{\left(\frac{B_*}{B}\right)^{1/p-1}, \frac{B_*}{B}\right\} M,$$

(where, as the reader can easily check, M_* has been carefully chosen so that it satisfies the conditions of Lemma 2.1 for the parameter B_*), we still have the inequality

$$\int_{M_*}^{+\infty} \frac{s}{\varphi_*(s)}\, ds > M_* \tag{50}$$

Next, choose a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}} \to 0$ with $0 < \varepsilon_i < \varepsilon_*$ ($i \in \mathbb{N}$), and define

$$f_i : [0,1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N, \quad (t,x,y) \mapsto f(t,x,y) + \varepsilon_i x \quad (i \in \mathbb{N}). \tag{51}$$

Now, it is clear that, for each $i \in \mathbb{N}$,

$(a_i)_*$ For any $t \in [0,1], x, y \in \mathbb{R}^N$, such that $\|x\| = R_*$, $\langle x, y\rangle = 0$, we have $\langle x, f_i(t,x,y)\rangle + \|y\|^p > 0$.

$(b_i)_*$ For any $t \in [0,1], x, y \in \mathbb{R}^N$ such that $\|x\| \le R_*$ and $\|y\|^{p-1} \ge M_*$, $\|f_i(t,x,y)\| \le \varphi_*(\|y\|^{p-1})$.

$(c_i)_*$ For any $t \in [0,1], x \in \mathbb{R}^N$ such that $\|x\| \le R_*$ and $y \in \mathbb{R}^N$,

$$\begin{aligned}
\|y\|^p \| f_i(t,x,y)\| &\le \|y\|^p \|f(t,x,y)\| + \|y\|^p \varepsilon_* R_* \\
&\le 2\gamma_*((p'-2)\langle y, f(t,x,y)\rangle\langle x,y\rangle \\
&\quad + \|y\|^2\langle x, f(t,x,y)\rangle + \|y\|^{p+2}) + C_*\|y\|^p \\
&= 2\gamma_*((p'-2)\langle y, f_i(t,x,y)\rangle\langle x,y\rangle \\
&\quad + \|y\|^2\langle x, f_i(t,x,y)\rangle + \|y\|^{p+2}) + C_*\|y\|^p \\
&\quad - 2\gamma_*((p'-2)\varepsilon_i\langle x,y\rangle^2 + \varepsilon_i\|x\|^2\|y\|^2) \\
&\le 2\gamma_*((p'-2)\langle y, f(t,x,y)\rangle\langle x,y\rangle \\
&\quad + \|y\|^2\langle x, f_i(t,x,y)\rangle + \|y\|^{p+2}) + C_*\|y\|^p.
\end{aligned} \tag{52}$$

(because $p' - 2 > -1$).

We deduce, by the first step proved above, the existence for each $i \in \mathbb{N}$ of a solution $x_i : [0,1] \to \mathbb{R}^N$ of the periodic boundary value problem

$$(P_i) \equiv \begin{cases} (\phi_p(x'))' = f(t,x,x') + \varepsilon_i x \\ x(0) = x(1), \quad x'(0) = x'(1), \end{cases} \tag{53}$$

verifying $\|x_i(t)\| < R_*$, $\|x_i'(t)\| < K_*$ for all $t \in [0,1]$. (K_* being given by Theorem 2.4 for R_*, γ_*, C_* and M_*).

The existence of a solution to our problem is now a consequence of Lemma 3.2.

5 THE DIRICHLET PROBLEM

Consider now the boundary value problem arising from equation (28) together with the Dirichlet boundary conditions (**D**). For the reader's convenience, we reproduce here a result of [5].

LEMMA 5.1 *Let $x_0, x_1 \in \mathbb{R}^N$ be fixed. Then, for each $h \in C[0,1]$ there exists a unique solution $x_h \in C^1[0,1]$ to the problem*

$$(D_h) \equiv \begin{cases} (\phi_p(x'))' = h \\ x(0) = x_0, \ x(1) = x_1 \end{cases} \tag{54}$$

Furthermore, if we define $\mathcal{K} : C[0,1] \to C^1[0,1]$ by $h \mapsto x_h$, the mapping $\mathcal{K}$ is completely continuous.

Proof Integrating the differential equation in (54) from 0 to t we find that a C^1 mapping $x : [0,1] \to \mathbb{R}^N$ is a solution to this equation if and only if there exist some $a \in \mathbb{R}^N$
(necessarily unique) such that

$$\phi_p(x'(t)) = a + \mathcal{H}(h)(t) \quad (t \in [0,1]), \tag{55}$$

where $\mathcal{H}(h)(t) := \int_0^t h(s)ds$. This formula can be rewritten as

$$x'(t) = \phi_p^{-1}(a + \mathcal{H}(h)(t)) \quad (t \in [0,1]). \tag{56}$$

Now, the boundary conditions imply that

$$x(t) = x_0 + \int_0^t \phi_p^{-1}(a + \mathcal{H}(h)(s))ds \quad (t \in [0, 1]), \tag{57}$$

$$\int_0^T \phi_p^{-1}(a + \mathcal{H}(h)(s))ds = x_1 - x_0. \tag{58}$$

We therefore conclude that there exists a bijective correspondence between the set of solutions to (54) and the set of points $a \in \mathbb{R}^N$ verifying (58), given by $x \mapsto \phi_p(x'(0))$.

Following a completely analogous reasoning to that carried out in Proposition 2.2 from [4], we find that

(i) For each $h \in C[0, 1]$ there exists an unique solution $a(h)$ of (58).
(ii) The function $a : C[0, 1] \to \mathbb{R}^N$ defined in (i) is continuous and maps bounded sets into bounded sets.

We deduce that for every $h \in C[0, 1]$, there exists a unique solution $\mathcal{K}(h)$ of (D_h), given by the formula

$$\mathcal{K}(h)(t) = x_0 + \int_0^t \phi_p^{-1}(a(h) + \mathcal{H}(h)(s))ds \quad (t \in [0, 1]). \tag{59}$$

The continuity of the mapping a allows us to deduce the continuity of $\mathcal{K}$. The boundedness of a on bounded sets of $C[0, 1]$ has as a consequence the compactness of $\mathcal{K}$ on bounded sets of $C[0, 1]$.

This lemma is now used to prove the following existence theorem for the Dirichlet problem associated to (28).

THEOREM 5.2 *Let $f : [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be a continuous mapping verifying* (**H**). *Let $x_0, x_1 \in \mathbb{R}^N$ with $\|x_0\|, \|x_1\| \leq R$. Then, the boundary value problem* (28)-(**D**), *has at least one solution $x : [0, 1] \to \mathbb{R}^N$ such that $\|x(t)\| \leq R$ for all $t \in [0, 1]$.*

Proof Define $\mathcal{F} : C^1[0, 1] \to C[0, 1]$ by

$$\mathcal{F}(x)(t) := f(t, x(t), x'(t)), \quad (t \in [0, 1]), \tag{60}$$

so that our problem can be rewritten as

$$x = \mathcal{KF}(x), \quad (x \in C^1[0, 1]). \tag{61}$$

Notice that $\mathcal{KF} : C^1[0, 1] \to C^1[0, 1]$ is a completely continuous mapping, so that if f were bounded, $\mathcal{F}$ and $\mathcal{KF}$ would be bounded and the Schauder fixed point theorem would give us the existence of a solution of our problem. Thus, our problem is now reduced to finding some $f_* : [0, 1] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ continuous, bounded and such that every solution to the equation

$$(\phi_p(x'))' = f_*(t, x, x') \tag{62}$$

verifying the boundary conditions (**D**) is also a solution to (28).

The following construction is essentially taken from [2]. As in the periodic case, Lemma 3.2 can be used to see that it suffices to prove the theorem assuming that f actually verifies the more restrictive set of hypothesis ($\tilde{\mathbf{H}}$). Let $K > 0$ be as given by Theorem 2.4 for R, γ, C, M and φ. Choose some continuous function

$$\rho : [0, \infty[\to \mathbb{R}^+ \tag{63}$$

such that

$$\rho(t) = 1, \quad (0 \le t \le K), \tag{64}$$

and

$$\sup\{\rho(\|y\|)\|f(t, x, y)\| : t \in [0, 1], \ \|x\| \le R, \ y \in \mathbb{R}^N\} < +\infty \tag{65}$$

For instance, ρ could be chosen as

$$\rho(t) := \begin{cases} 1 \quad \text{if} \quad 0 \leq t \leq K \\ \dfrac{1}{1 + \max\{\| f(t, x, y)\| : t \in [0, 1], \|x\| \leq R, \| y\| \leq t\}} \\ \hfill \text{if} \quad t \geq K + 1 \\ (1 + K - t)\rho(K) + (t - K)\rho(1 + K) \\ \hfill \text{if} \quad K \leq t \leq K + 1 \end{cases} . \tag{66}$$

Define

$$f_*(t, x, y) := \begin{cases} \rho(\| y\|)f(t, x, y) & \text{if} \quad \|x\| \leq R \\ \rho(\| y\|)f\left(t, R\dfrac{x}{\|x\|}, y\right) & \text{if} \quad \|x\| \geq R. \end{cases} \tag{67}$$

It is easy to check that f_* is still a continuous bounded function satisfying not only the same set $(\tilde{\mathbf{H}})$ of hypothesis (for the same parameters R, γ, C, M), but, moreover, (a_*) For any $t \in [0, 1], x, y \in \mathbb{R}^N$ such that $\|x\| \geq R$, $\langle x, y\rangle = 0$, we have

$$\langle x, f_*(t, x, y)\rangle + \|y\|^p > 0.$$

Also, it is clear that $f_*(t, x, y) = f(t, x, y)$ if $\|x\| \leq R$ and $\|y\| \leq K$.

So, let $\bar{x} : [0, 1] \to \mathbb{R}^N$ be a solution to (62) verifying the boundary conditions (**D**), where $\|x_0\|, \|x_1\| \leq R$. Let us show that $\|\bar{x}(t)\| \leq R$, $\|\bar{x}'(t)\| \leq K$ for all $t \in [0, 1]$. First suppose that there exists some point $t_0 \in [0, 1]$ such that $\|\bar{x}(t_0)\| > R$. This point t_0 can be taken so as $\|\bar{x}(t_0)\| = \max_{t \in [0,1]} \|\bar{x}(t)\|$. As $\|\bar{x}(0)\| = \|x_0\| \leq R$, $\|\bar{x}(1)\| = \|x_1\| \leq R$, we see that $t_0 \in]0, 1[$. Now, using Lemma 3.1, we deduce that $\langle \bar{x}(t_0), \bar{x}'(t_0)\rangle = 0$ and

$$\begin{aligned} \langle \bar{x}(t_0), (\phi_p(\bar{x}'))'(t_0)\rangle + \|x'(t_0)\|^p = {} & \langle \bar{x}(t_0), f_*(t_0, \bar{x}(t_0), \bar{x}'(t_0))\rangle \\ & + \|\bar{x}'(t_0)\|^p \leq 0, \end{aligned}$$

which contradicts (a_*). It means that $\|\bar{x}(t)\| \leq R$ for all $t \in [0, 1]$. And, in the same way as happened in the proof of Theorem 4.1, our hypothesis (b) and (c) on f (translated to f_*) make $\bar{x}$ verify the second and third hypothesis of Theorem 2.4. Applying it we obtain that $\|\bar{x}'(t)\| \leq K$ for all $t \in [0, 1]$, so that $\bar{x}$ is in fact solution to the system (28-(**D**)). The theorem is proved.

References

[1] Hartman, P., *On boundary value problems for systems of ordinary nonlinear second order differential equations*, Trans. Amer. Math. Soc. **96** (1960), 493–509.
[2] Hartman, P., *Ordinary Differential Equations*, Wiley, New York, 1964.
[3] Knobloch, H.W., *On the existence of periodic solutions for second order vector differential equations*, J. Differential Equations **9** (1971), 67–85.
[4] Manásevich, R. and Mawhin, J., *Periodic solutions for nonlinear systems with p-Laplacian-like operators*, J. Differential Equations **145**, (1998), 367–393.
[5] Manásevich, R. and Mawhin, J., *Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators*, J. Korean Math. Soc., **37** (2000), 665–685.
[6] Mawhin, J., *Periodic solutions of systems with p-Laplacian-like operators*, in Nonlinear Analysis and its Applications to Differential Equations, Lisbon, 1998, Birkhäuser, Basel, 2001, 37–63.
[7] Mawhin, J., *Some boundary value problems for Hartman-type perturbations of the ordinary vector p-Laplacian*, Nonlinear Anal. **40**, (2000), 497–503.
[8] Rouche, N., and Mawhin, J., *Equations différentielles ordinaires*, vol. 2, Masson, Paris, 1973. English translation: Pitman, Boston, 1980.