Research Article

On the Bishop-Phelps-Bollobás Property for Numerical Radius

Sun Kwang Kim,[1] Han Ju Lee,[2] and Miguel Martín[3]

[1] *Department of Mathematics, Kyonggi University, Suwon 443-760, Republic of Korea*
[2] *Department of Mathematics Education, Dongguk University-Seoul, Seoul 100-715, Republic of Korea*
[3] *Departamento de Análisis Mátematico, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain*

Correspondence should be addressed to Han Ju Lee; hanjulee@dongguk.edu

We study the Bishop-Phelps-Bollobás property for numerical radius (in short, BPBp-nu) and find sufficient conditions for Banach spaces to ensure the BPBp-nu. Among other results, we show that $L_1(\mu)$-spaces have this property for every measure μ. On the other hand, we show that every infinite-dimensional separable Banach space can be renormed to fail the BPBp-nu. In particular, this shows that the Radon-Nikodým property (even reflexivity) is not enough to get BPBp-nu.

1. Introduction

Let X be a (real or complex) Banach space and X^* its dual space. The unit sphere of X will be denoted by S_X. We write $\mathscr{L}(X)$ for the space of all bounded linear operators on X. For $T \in \mathscr{L}(X)$, its *numerical radius* is defined by

$$v(T) = \sup\left\{\left|x^*Tx\right| : (x, x^*) \in \Pi(X)\right\}, \qquad (1)$$

where $\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. It is clear that v is a seminorm on $\mathscr{L}(X)$. We refer the reader to the monographs [1, 2] for background. An operator $T \in \mathscr{L}(X)$ *attains its numerical radius* if there exists $(x_0, x_0^*) \in \Pi(X)$ such that $v(T) = |x_0^*Tx_0|$.

In this paper we will discuss the density of numerical radius attaining operators, actually on a stronger property called Bishop-Phelps-Bollobás property for numerical radius. Let us present first a short account on the known results about numerical radius attaining operators. Motivated by the study of norm attaining operators initiated by J. Lindenstrauss in the 1960s, Sims [3] asked in 1972 whether the numerical radius attaining operators are dense in the space of all bounded linear operators on a Banach space. Berg and Sims [4] gave a positive answer for uniformly convex spaces and Cardassi showed that the answer is positive for ℓ_1, c_0, $C(K)$ (where K is a metrizable compact), $L_1(\mu)$, and uniformly smooth spaces [5–7]. Acosta showed that the numerical radius attaining operators are dense in $C(K)$ for every

compact Hausdorff space K [8]. Acosta and Payá showed that numerical radius attaining operators are dense in $\mathscr{L}(X)$ if X has the Radon-Nikodým property [9]. On the other hand, Payá [10] showed in 1992 that there is a Banach space X such that the numerical radius attaining operators are not dense in $\mathscr{L}(X)$, which gave a negative answer to Sims' question. Some authors also paid attention to the study of denseness of numerical radius attaining nonlinear mappings [11–14].

Motivated by the work [15] of Acosta et al. on the Bishop-Phelps-Bollobás property for operators, Guirao and Kozhushkina [16] introduced very recently the notion of Bishop-Phelps-Bollobás property for numerical radius.

Definition 1 (see [16]). A Banach space X is said to have the *Bishop-Phelps-Bollobás property for numerical radius* (in short, *BPBp-nu*) if, for every $0 < \varepsilon < 1$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathscr{L}(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $v(T) = 1$ and $|x^*Tx| > 1 - \eta(\varepsilon)$, there exit $S \in \mathscr{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = \left|y^*Sy\right| = 1, \qquad \|T - S\| < \varepsilon,$$
$$\|x - y\| < \varepsilon, \qquad \|x^* - y^*\| < \varepsilon. \qquad (2)$$

Notice that if a Banach space X has the BPBp-nu, then the numerical radius attaining operators are dense in $\mathscr{L}(X)$. One of the main results of this paper is to show that the converse result is no longer true (Section 5).

It is shown in [16] that the real or complex spaces c_0 and ℓ_1 have the BPBp-nu. This result has been extended to the real space $L_1(\mathbb{R})$ by Falcó [17]. Aviles et al. [18] give sufficient conditions on a compact space K for the real space $C(K)$ to have the BPBp-nu which, in particular, include all metrizable compact spaces.

The content of this paper is the following. First, we introduce in Section 2 a modulus of the BPBp-nu analogous to the one introduced in [19] for the Bishop-Phelps-Bollobás property for the operator norm, and we will use it as a tool in the rest of the paper. As easy applications, we prove that finite-dimensional spaces always have the BPBp-nu and that a reflexive space has the BPBp-nu if and only if its dual does. Next, Section 3 is devoted to prove that Banach spaces which are both uniformly convex and uniformly smooth satisfy a weaker version of the BPBp-nu and to discuss such weaker version. In particular, it is shown that $L_p(\mu)$ spaces have the BPBp-nu for every measure μ when $1 < p < \infty$, $p \neq 2$. We show in Section 4 that, given any measure μ, the real or complex space $L_1(\mu)$ has the BPBp-nu. Finally, we prove in Section 5 that every separable infinite-dimensional Banach space can be equivalently renormed to fail the BPBp-nu (actually, to fail the weaker version). In particular, this shows that reflexivity (or even superreflexivity) is not enough for the BPBp-nu, while the Radon-Nikodým property was known to be sufficient for the density of numerical radius attaining operators.

Let us introduce some notations for later use. The n-dimensional space with the ℓ_1 norm is denoted by $\ell_1^{(n)}$. Given a family $\{X_k\}_{k=1}^\infty$ of Banach spaces, $[\bigoplus_{k=1}^\infty X_k]_{c_0}$ (resp., $[\bigoplus_{k=1}^\infty X_k]_{\ell_1}$) is the Banach space consisting of all sequences $(x_k)_{k=1}^\infty$ such that each x_k is in X_k and $\lim_{k \to \infty} \|x_k\| = 0$ (resp., $\sum_{k=1}^\infty \|x_k\| < \infty$) equipped with the norm $\|(x_k)_{k=1}^\infty\| = \sup_k \|x_k\|$ (resp., $\|(x_k)_{k=1}^\infty\| = \sum_{k=1}^\infty \|x_k\|$).

2. Modulus of the Bishop-Phelps-Bollobás for Numerical Radius

Analogously to what is done in [19] for the BPBp for the operator norm, we introduce here a modulus to quantify the Bishop-Phelps-Bollobás property for numerical radius.

Notation 1. Let X be a Banach space. Consider the set

$$
\begin{aligned}
\Pi_{\text{nu}}(X) = \{(x, x^*, T) : (x, x^*) \in \Pi(X), \\
T \in \mathcal{L}(X), \ \nu(T) = 1 = |x^* T x|\},
\end{aligned}
\tag{3}
$$

which is closed in $S_X \times S_{X^*} \times \mathcal{L}(X)$ with respect to the following metric:

$$
\begin{aligned}
\text{dist}\left((x, x^*, T), (y, y^*, S)\right) \\
= \max \{\|x - y\|, \|x^* - y^*\|, \|T - S\|\}.
\end{aligned}
\tag{4}
$$

The *modulus of the Bishop-Phelps-Bollobás property for numerical radius* is the function defined by

$$
\eta_{\text{nu}}(X)(\varepsilon)
$$
$$
= \inf \{1 - |x^* T x| : (x, x^*) \in \Pi(X), \ T \in \mathcal{L}(X),
\tag{5}
$$
$$
\nu(T) = 1, \ \text{dist}\left((x, x^*, T), \Pi_{\text{nu}}(X)\right) \geqslant \varepsilon\}
$$

for every $\varepsilon \in (0, 1)$. Equivalently, $\eta_{\text{nu}}(X)(\varepsilon)$ is the supremum of those scalars $\eta > 0$ such that, whenever $T \in \mathcal{L}(X)$ and $(x, x^*) \in \Pi(X)$ satisfy $\nu(T) = 1$ and $|x^* T x| > 1 - \eta$, there exist $S \in \mathcal{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$
\nu(S) = |y^* S y| = 1, \qquad \|T - S\| < \varepsilon,
$$
$$
\|x - y\| < \varepsilon, \qquad \|x^* - y^*\| < \varepsilon.
\tag{6}
$$

It is immediate that a Banach space X has the BPBp-nu if and only if $\eta_{\text{nu}}(\varepsilon) > 0$ for every $0 < \varepsilon < 1$. By construction, if a function $\varepsilon \mapsto \eta(\varepsilon)$ is valid in the definition of the BPBp-nu, then $\eta_{\text{nu}}(\varepsilon) \geqslant \eta(\varepsilon)$.

An immediate consequence of the compactness of the unit ball of a finite-dimensional space is the following result. It was previously known to A. Guirao (private communication).

Proposition 2. *Let X be a finite-dimensional Banach space. Then X has the Bishop-Phelps-Bollobás property for numerical radius.*

Proof. Let $K = \{S \in \mathcal{L}(X) : \nu(S) = 0\}$. Then K is a norm-closed subspace of $\mathcal{L}(X)$. Hence $\mathcal{L}(X)/K$ is a finite-dimensional space with two norms:

$$
\nu([T]) := \inf \{\nu(T - S) : S \in K\} = \nu(T),
$$
$$
\|[T]\| := \inf \{\|T - S\| : S \in K\},
\tag{7}
$$

where $[T]$ is the class of T in the quotient space $\mathcal{L}(X)/K$. Hence there is a constant $0 < c \leqslant 1$ such that

$$
c \|[T]\| \leqslant \nu(T) \leqslant \|[T]\|.
\tag{8}
$$

Suppose that X does not have the BPBp-nu. Then, there is $0 < \varepsilon < 1$ such that $\eta_{\text{nu}}(X)(\varepsilon) = 0$. That is, there are sequences $(x_n, x_n^*) \in \Pi(X)$ and $(T_n) \in \mathcal{L}(X)$ with $\nu(T_n) = 1$ such that

$$
\text{dist}\left((x_n, x_n^*, T_n), \Pi_{\text{nu}}(X)\right) \geqslant \varepsilon \quad (n \in \mathbb{N}),
$$
$$
\lim_n |x_n^* T_n x_n| = 1.
\tag{9}
$$

By compactness, we may assume that $\lim_n \|[T_n] - [T_0]\| = 0$ for some $T_0 \in \mathcal{L}(X)$ and $\nu(T_0) = 1$. Hence there exists a sequence $\{S_n\}_n$ in K such that $\lim_n \|T_n - (T_0 + S_n)\| = 0$. Observe that $\nu(T_0 + S_n) = \nu(T_0) = 1$ for every $n \in \mathbb{N}$.

By compactness again, we may assume that (x_n, x_n^*) converges to $(x_0, x_0^*) \in X \times X^*$. This implies that $(x_0, x_0^*) \in \Pi(X)$, and $|x_0^*(T_0 + S_n) x_0| = \nu(T_0 + S_n) = 1$, that is, $(x_0, x_0^*, T_0 + S_n) \in \Pi_{\text{nu}}(X)$ for all n. This is a contradiction with the fact that

$$
0 = \lim_n \text{dist}\left((x_n, x_n^*, T_n), (x_0, x_0^*, T_0 + S_n)\right)
$$
$$
\geqslant \lim_n \text{dist}\left((x_n, x_n^*, T_n), \Pi_{\text{nu}}(X)\right) \geqslant \varepsilon. \qquad \square
\tag{10}
$$

We may also give the following easy result concerning duality.

Proposition 3. *Let X be a reflexive space. Then*

$$\eta_{nu}(X)(\varepsilon) = \eta_{nu}(X^*)(\varepsilon) \tag{11}$$

for every $\varepsilon \in (0,1)$. In particular, X has the BPBp-nu if and only if X^* has the BPBp-nu.

We will use that $\nu(T^*) = \nu(T)$ for all $T \in \mathscr{L}(X)$, where T^* denotes the adjoint operator of T. This result can be found in [1], but it is obvious if X is reflexive.

Proof. By reflexivity, it is enough to show that $\eta_{nu}(X)(\varepsilon) \leqslant \eta_{nu}(X^*)(\varepsilon)$. Let $\varepsilon \in (0,1)$ be fixed. If $\eta_{nu}(X)(\varepsilon) = 0$, there is nothing to prove. Otherwise, consider $0 < \eta < \eta_{nu}(X)(\varepsilon)$. Suppose that $T_1 \in \mathscr{L}(X^*)$ and $(x_1^*, x_1) \in \Pi(X^*)$ satisfy

$$\nu(T_1) = 1, \qquad |x_1 T_1 x_1^*| > \nu(T_1) - \eta. \tag{12}$$

By considering $T_1^* \in \mathscr{L}(X)$, we may find $S_1 \in \mathscr{L}(X)$ and $(y_1, y_1^*) \in \Pi(X)$ such that

$$\begin{aligned} |y_1^* S_1 y_1| = \nu(S_1) = 1, \qquad \|y_1 - x_1\| < \varepsilon, \\ \|y_1^* - x_1^*\| < \varepsilon, \qquad \|T_1^* - S_1\| < \varepsilon. \end{aligned} \tag{13}$$

Then $S_1^* \in \mathscr{L}(X^*)$ and $(y_1^*, y_1) \in \Pi(X^*)$ satisfy

$$\begin{aligned} |\langle y_1, S_1^* y_1^* \rangle| = \nu(S_1) = 1, \qquad \|y_1^* - x_1^*\| < \varepsilon, \\ \|y_1 - x_1\| < \varepsilon, \qquad \|T_1 - S_1^*\| < \varepsilon. \end{aligned} \tag{14}$$

This implies that $\eta_{nu}(X^*)(\varepsilon) \geqslant \eta$. We finish by just taking supremum on η. $\qquad\square$

We do not know whether the result above is valid in the nonreflexive case.

3. Spaces Which Are Both Uniformly Convex and Uniformly Smooth

For a Banach space which is both uniformly convex and uniformly smooth, we get a property which is weaker than BPBp-nu. This result was known to A. Guirao (private communication).

Proposition 4. *Let X be a uniformly convex and uniformly smooth Banach space. Then, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T_0 \in \mathscr{L}(X)$ with $\nu(T_0) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfy $|x_0^* T_0 x_0| > 1 - \eta(\varepsilon)$, there exist $S \in \mathscr{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that*

$$\begin{aligned} \nu(S) = |y^* S y|, \qquad \|x - y\| < \varepsilon, \\ \|x^* - y^*\| < \varepsilon, \qquad \|S - T_0\| < \varepsilon. \end{aligned} \tag{15}$$

Proof. Notice that the uniform smoothness of X is equivalent to the uniform convexity of X^*. Let $\delta_X(\varepsilon)$ and $\delta_{X^*}(\varepsilon)$ be the moduli of convexity X and X^*, respectively. Given $0 < \varepsilon < 1$, consider

$$\eta(\varepsilon) = \frac{\varepsilon}{4} \min \left\{ \delta_X \left(\frac{\varepsilon}{4} \right), \delta_{X^*} \left(\frac{\varepsilon}{4} \right) \right\} > 0. \tag{16}$$

Consider $T_0 \in \mathscr{L}(X)$ with $\nu(T_0) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfying $|x_0^* T_0 x_0| > 1 - \eta(\varepsilon)$. Define $T_1 \in \mathscr{L}(X)$ by

$$T_1 x = T_0 x + \lambda_1 \left(\frac{\varepsilon}{4} \right) x_0^*(x) x_0 \tag{17}$$

for all $x \in X$, where λ_1 is the scalar satisfying $|\lambda_1| = 1$ and $|x_0^* T_0 x_0 + \lambda_1(\varepsilon/4)| = |x_0^* T_0 x_0| + \varepsilon/4$. Now, choose $x_1 \in S_X$ and $x_1^* \in S_{X^*}$ such that $|x_1^*(x_1)| = 1$, $x_1^*(x_0) = |x_1^*(x_0)|$, and

$$|x_1^* T_1 x_1| \geqslant \nu(T_1) - \eta\left(\frac{\varepsilon^2}{4^2} \right). \tag{18}$$

Now we define a sequence (x_n, x_n^*, T_n) in $S_X \times S_{X^*} \times \mathscr{L}(X)$ inductively. Indeed, suppose that we have a defined sequence (x_j, x_j^*, T_j) for $0 \leqslant j \leqslant n$ and let

$$T_{n+1} x = T_n x + \lambda_{n+1} \frac{\varepsilon^{n+1}}{4^{n+1}} x_n^*(x) x_n. \tag{19}$$

Then choose $x_{n+1} \in S_X$ and $x_{n+1}^* \in S_{X^*}$ such that $|x_{n+1}^*(x_{n+1})| = 1$ and $|x_{n+1}^*(x_n)| = x_{n+1}^*(x_n)$:

$$|x_{n+1}^* T_{n+1} x_{n+1}| \geqslant \nu(T_{n+1}) - \eta\left(\frac{\varepsilon^{n+2}}{4^{n+2}} \right). \tag{20}$$

Notice that, for all $n \geqslant 0$, we have

$$\begin{aligned} \|T_{n+1} - T_n\| \leqslant \frac{\varepsilon^{n+1}}{4^{n+1}}, \\ |\nu(T_{n+1}) - \nu(T_n)| \leqslant \frac{\varepsilon^{n+1}}{4^{n+1}}. \end{aligned} \tag{21}$$

This implies that (T_n) is a Cauchy sequence and assume that it converges to $S \in \mathscr{L}(X)$. Then we have

$$\begin{aligned} \lim_n T_n = S, \qquad \|T_0 - S\| < \varepsilon \\ \lim_n |x_n^* T_n x_n| = \lim_n \nu(T_n) = \nu(S). \end{aligned} \tag{22}$$

We will show that both sequences (x_n) and (x_n^*) are Cauchy. From the definition, we have

$$\nu(T_{n+1}) - \eta\left(\frac{\varepsilon^{n+2}}{4^{n+2}} \right)$$

$$\leqslant |x_{n+1}^* T_{n+1} x_{n+1}|$$

$$\leqslant \left| x_{n+1}^* T_n x_{n+1} + \lambda_{n+1} \frac{\varepsilon^{n+1}}{4^{n+1}} x_n^* (x_{n+1}) x_{n+1}^* (x_n) \right|$$

$$\leqslant v(T_n) + \frac{\varepsilon^{n+1}}{4^{n+1}} x_{n+1}^* (x_n),$$

$$v(T_{n+1})$$

$$\geqslant \left| x_n^* T_{n+1} x_n \right| = \left| x_n^* T_n x_n + \lambda_{n+1} \frac{\varepsilon^{n+1}}{4^{n+1}} \right|$$

$$= \left| x_n^* T_n x_n \right| + \frac{\varepsilon^{n+1}}{4^{n+1}} \geqslant v(T_n) - \eta \left(\frac{\varepsilon^{n+1}}{4^{n+1}} \right) + \frac{\varepsilon^{n+1}}{4^{n+1}}. \tag{23}$$

In summary, we have

$$v(T_n) + \frac{\varepsilon^{n+1}}{4^{n+1}} x_{n+1}^* (x_n)$$
$$\geqslant v(T_n) - \eta \left(\frac{\varepsilon^{n+1}}{4^{n+1}} \right) + \frac{\varepsilon^{n+1}}{4^{n+1}} - \eta \left(\frac{\varepsilon^{n+2}}{4^{n+2}} \right). \tag{24}$$

Hence

$$x_{n+1}^* (x_n) \geqslant 1 - 2 \frac{4^{n+1}}{\varepsilon^{n+1}} \eta \left(\frac{\varepsilon^{n+1}}{4^{n+1}} \right)$$

$$= 1 - \frac{1}{2} \min \left\{ \delta_X \left(\frac{\varepsilon^{n+1}}{4^{n+2}} \right), \delta_{X^*} \left(\frac{\varepsilon^{n+1}}{4^{n+2}} \right) \right\},$$

$$\left\| \frac{x_n + x_{n+1}}{2} \right\| \geqslant x_{n+1}^* \left(\frac{x_n + x_{n+1}}{2} \right) \geqslant 1 - \delta_X \left(\frac{\varepsilon^{n+1}}{4^{n+2}} \right),$$

$$\left\| \frac{x_n^* + x_{n+1}^*}{2} \right\| \geqslant \frac{x_n^* + x_{n+1}^*}{2} (x_n) \geqslant 1 - \delta_{X^*} \left(\frac{\varepsilon^{n+1}}{4^{n+2}} \right). \tag{25}$$

This means that

$$\|x_n - x_{n+1}\| \leqslant \varepsilon^{n+1}/4^{n+2},$$
$$\|x_n^* - x_{n+1}^*\| \leqslant \varepsilon^{n+1}/4^{n+2} \tag{26}$$

for all n. So (x_n) and (x_n^*) are Cauchy. Let $x_\infty = \lim_n x_n$ and $x_\infty^* = \lim_n x_n^*$. Then $\|x_0 - x_\infty\| < \varepsilon/4$ and $\|x_0^* - x_\infty^*\| < \varepsilon/4$. Hence, $|x_\infty^*(x_\infty)| = \lim_n |x_n^*(x_n)| = 1$ and

$$v(S) = \lim_n v(T_n) = \lim_n |x_n^* T_n x_n| = |x_\infty^* S x_\infty|. \tag{27}$$

Let $\alpha = x_\infty^*(x_\infty)$, $y^* = \bar{\alpha} x_\infty^*$, and $y = x_\infty$. Then we have $y^*(y) = 1$, $v(S) = |y^* Sy|$, and $\|y - x_0\| < \varepsilon$. Notice that

$$|\alpha - 1| = |x_\infty^*(x_\infty) - x_0^*(x_0)|$$
$$\leqslant |(x_\infty^* - x_0^*)(x_\infty)| + |x_0^*(x_\infty) - x_0^*(x_0)| < \frac{\varepsilon}{2}. \tag{28}$$

Therefore

$$\|y^* - x^*\| \leqslant \|\bar{\alpha} y^* - y^*\| + \|y^* - x^*\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon. \tag{29}$$

This completes the proof. □

Let us discuss a little bit about the equivalence between the property in the result above and the BPBp-nu. For convenience, let us introduce the following definition.

Definition 5. A Banach space X has the *weak Bishop-Phelps-Bollobás property for the numerical radius* (in short *weak-BPBp-nu*); if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T_0 \in \mathscr{L}(X)$ with $v(T_0) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfy $|x_0^* T_0 x_0| > 1 - \eta(\varepsilon)$, there exist $S \in \mathscr{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^* Sy|, \qquad \|x - y\| < \varepsilon,$$
$$\|x^* - y^*\| < \varepsilon, \qquad \|S - T\| < \varepsilon. \tag{30}$$

Notice that the only difference between this concept and the BPBp-nu is the normalization of the operator S by the numerical radius. Of course, if the numerical radius and the operator norm are equivalent, these two properties are the same. This equivalence is measured by the so-called numerical index of the Banach space, as follows. For a Banach space X, the *numerical index* of X is defined by

$$n(X) = \inf \{v(T) : T \in \mathscr{L}(X), \|T\| = 1\}. \tag{31}$$

It is clear that $0 \leqslant n(X) \leqslant 1$ and $n(X)\|T\| \leqslant v(T) \leqslant \|T\|$ for all $T \in \mathscr{L}(X)$. The value $n(X) = 1$ means that v equals the usual operator norm. This is the case of $X = L_1(\mu)$ and $X = C(K)$, among many others. On the other hand, $n(X) > 0$ if and only if the numerical radius is equivalent to the norm of $\mathscr{L}(X)$. We refer the reader to [20] for more information and background.

The following result is immediate. We include a proof for the sake of completeness.

Proposition 6. *Let X be a Banach space with $n(X) > 0$. Then, X has the BPBp-nu if and only if X has the weak-BPBp-nu.*

Proof. The necessity is clear. For the converse, assume that we have $\eta(\varepsilon) > 0$ satisfying the conditions of the weak-BPBp-nu for all $0 < \varepsilon < 1$. If $T \in \mathscr{L}(X)$ with $v(T) = 1$ and $(x_0, x_0^*) \in \Pi(X)$ satisfy $|x_0^* Tx_0| > 1 - \eta(\varepsilon)$ for $0 < \varepsilon < 1$, then there exist $S \in \mathscr{L}(X)$ and $(y, y^*) \in \Pi(X)$ such that

$$v(S) = |y^* Sy|, \qquad \|S - T\| < \varepsilon,$$
$$\|x - y\| < \varepsilon, \qquad \|x^* - y^*\| < \varepsilon. \tag{32}$$

As $v(S) > 0$ by the above, let $S_1 = (1/v(S))S$. Then we have

$$1 = v(S_1) = |y^* S_1 y|, \qquad \|x - y\| < \varepsilon$$
$$\|x^* - y^*\| < \varepsilon. \tag{33}$$

Finally, we have

$$
\begin{aligned}
\|S_1 - T\| &\leq \left\| \frac{1}{\nu(S)} S - S \right\| + \|S - T\| \\
&= \frac{\|S\|}{\nu(S)} |\nu(S) - 1| + \|S - T\| \\
&\leq \frac{1}{n(X)} |\nu(S) - \nu(T)| + \|S - T\| \\
&\leq \left(\frac{1}{n(X)} + 1 \right) \|S - T\| < \frac{n(X) + 1}{n(X)} \varepsilon.
\end{aligned}
\tag{34}
$$

An obvious change of parameters finishes the proof. □

We do not know whether the hypothesis of $n(X) > 0$ can be omitted in the above result.

Putting together Propositions 4 and 6, we get the following.

Corollary 7. *Let X be a uniformly convex and uniformly smooth Banach space with $n(X) > 0$. Then X has the BPBp-nu.*

Let us comment that every complex Banach space X satisfies $n(X) \geq 1/e$, so the above corollary automatically applies in the complex case. In the real case, this is no longer true, as the numerical index of a Hilbert space of dimension greater than or equal to two is 0. On the other hand, it is proved in [21] that real $L_p(\mu)$ spaces have nonzero numerical index for every measure μ when $p \neq 2$. Therefore, we have the following examples.

Example 8. (a) Complex Banach spaces which are uniformly smooth and uniformly convex satisfy the BPBp-nu.

(b) In particular, for every measure μ, the complex spaces $L_p(\mu)$ have the BPBp-nu for $1 < p < \infty$.

(c) For every measure μ, the real spaces $L_p(\mu)$ have the BPBp-nu for $1 < p < \infty$, $p \neq 2$.

Note Added in Revision. Very recently, H. J. Lee, M. Martín, and J. Merí have proved that Proposition 6 can be extended to some Banach spaces with numerical index zero as, for instance, real Hilbert spaces. Hence, they have shown that Hilbert spaces have the BPBp-nu. These results will appear elsewhere.

4. L_1 Spaces

In this section, we will show that $L_1(\mu)$ has the BPBp-nu for every measure μ. In the proof, we are dealing with complex integrable functions since the real case is followed easily by applying the same proof. Our main result here is the following.

Theorem 9. *Let μ be a measure. Then $L_1(\mu)$ has the Bishop-Phelps-Bollobás property for numerical radius. More precisely, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ (which does not depend on μ) such that whenever $T_0 \in \mathcal{L}(L_1(\mu))$ with $\nu(T_0) = 1$ and*

$(f_0, g_0) \in \Pi(L_1(\mu))$ *satisfy* $|\langle T_0 f_0, g_0 \rangle| > 1 - \eta(\varepsilon)$, *then there exist* $T \in \mathcal{L}(L_1(\mu))$, $(f_1, g_1) \in \Pi(L_1(\mu))$ *such that*

$$
\begin{aligned}
|\langle T f_1, g_1 \rangle| = \nu(T) = 1, \qquad \|f_0 - f_1\| < \varepsilon, \\
\|g_0 - g_1\| < \varepsilon, \qquad \|T - T_0\| < \varepsilon.
\end{aligned}
\tag{35}
$$

As a first step, we have to start dealing with finite regular positive Borel measures, for which a representation theorem for operators exists.

Proposition 10. *Let m be a finite regular positive Borel measure on a compact Hausdorff space Ω. Then $L_1(m)$ has the Bishop-Phelps-Bollobás property for numerical radius. More precisely, given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ (which is independent of the measure m) such that if a norm-one element T in $\mathcal{L}(L_1(m))$ and $(f_0, g_0) \in \Pi(L_1(m))$ satisfy $|\langle T f_0, g_0 \rangle| > 1 - \eta(\varepsilon)$, then there exist an operator $S \in \mathcal{L}(L_1(m))$, $(f_1, g_1) \in \Pi(L_1(m))$ such that*

$$
\begin{aligned}
|\langle S f_1, g_1 \rangle| = \|S\| = 1, \qquad \|f_0 - f_1\| \leq \varepsilon, \\
\|g_0 - g_1\| \leq \varepsilon, \qquad \|T - S\| \leq \varepsilon.
\end{aligned}
\tag{36}
$$

To prove this proposition, we need some background on representation of operators on Lebesgue spaces on finite regular positive Borel measures and several preliminary lemmas.

Let m be a finite regular positive Borel measure on a compact Hausdorff space Ω. If μ is a complex-valued Borel measure on the product space $\Omega \times \Omega$, then define their marginal measures μ^i on Ω ($i = 1, 2$) as follows: $\mu^1(A) = \mu(A \times \Omega)$ and $\mu^2(B) = \mu(\Omega \times B)$, where A and B are Borel measurable subsets of Ω.

Let $M(m)$ be the complex Banach lattice of measures consisting of all complex-valued Borel measures μ on the product space $\Omega \times \Omega$ such that $|\mu|^i$ are absolutely continuous with respect to m for $i = 1, 2$, endowed with the norm

$$
\left\| \frac{d|\mu|^1}{dm} \right\|_\infty.
\tag{37}
$$

Each $\mu \in M(m)$ defines a bounded linear operator T_μ from $L_1(m)$ to itself by

$$
\langle T_\mu(f), g \rangle = \int_{\Omega \times \Omega} f(x) g(y) \, d\mu(x, y),
\tag{38}
$$

where $f \in L_1(m)$ and $g \in L_\infty(m)$. Iwanik [22] showed that the mapping $\mu \mapsto T_\mu$ is a lattice isometric isomorphism from $M(m)$ onto $\mathcal{L}(L_1(m))$. Even though he showed this for the real case, it can be easily generalized to the complex case. For details, see [22, Theorem 1] and [23, IV Theorem 1.5(ii), Corollary 2].

We will also use that, given an arbitrary measure μ, every $T \in \mathcal{L}(L_1(\mu))$ satisfies $\nu(T) = \|T\|$ [24] (that is, the space $L_1(\mu)$ has numerical index 1).

Lemma 11 (see [15, Lemma 3.3]). *Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leq 1$ for every n, and let $\eta > 0$ such*

that, for a convex series $\sum \alpha_n$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A := \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$ satisfies the estimate

$$\sum_{i \in A} \alpha_i \geqslant 1 - \frac{\eta}{1 - r}. \tag{39}$$

From now on, m will be a finite regular positive Borel measure on the compact Hausdorff space Ω.

Lemma 12. *Suppose that there exist a nonnegative simple function $f \in S_{L_1(m)}$ and a function $g \in S_{L_\infty(m)}$ such that*

$$\operatorname{Re} \langle f, g \rangle > 1 - \frac{\varepsilon^3}{16}. \tag{40}$$

Then there exist a nonnegative simple function $f_1 \in S_{L_1(m)}$ and a function $g_1 \in S_{L_\infty(m)}$ such that

$$g_1(x) = \chi_{\operatorname{supp}(f_1)}(x) + g(x)\chi_{\Omega \setminus \operatorname{supp}(f_1)}(x),$$

$$\langle f_1, g_1 \rangle = 1, \qquad \|f - f_1\|_1 < \varepsilon, \tag{41}$$

$$\|g - g_1\|_\infty < \sqrt{\varepsilon}, \qquad \operatorname{supp}(f_1) \subset \operatorname{supp}(f).$$

Proof. Let $f = \sum_{j=1}^m (\beta_j/m(B_j))\chi_{B_j}$ for some (β_j) such that $\beta_j \geqslant 0$ for all j and $\sum_{j=1}^m \beta_j = 1$, and B_j's are mutually disjoint. By the assumption, we have

$$\operatorname{Re} \langle f, g \rangle = \sum_{j=1}^n \beta_j \frac{1}{m(B_j)} \int_{B_j} \operatorname{Re} g(x)\,dm(x) > 1 - \frac{\varepsilon^3}{16}, \tag{42}$$

and letting

$$J = \left\{ j : 1 \leqslant j \leqslant n, \frac{1}{m(B_j)} \int_{B_j} \operatorname{Re} g(x)\,dm(x) > 1 - \frac{\varepsilon^2}{4} \right\}, \tag{43}$$

we have by Lemma 11

$$\sum_{j \in J} \beta_j > 1 - \frac{\varepsilon}{4}. \tag{44}$$

For each $j \in J$, we have

$$1 - \frac{\varepsilon^2}{4} < \frac{1}{m(B_j)} \int_{B_j} \operatorname{Re} g(x)\,dm(x)$$

$$= \frac{1}{m(B_j)} \int_{B_j \cap \{\operatorname{Re} g \leqslant 1 - \varepsilon\}} \operatorname{Re} g(x)\,dm(x)$$

$$+ \int_{B_j \cap \{\operatorname{Re} g > 1 - \varepsilon\}} \operatorname{Re} g(x)\,dm(x)$$

$$\leqslant \frac{1}{m(B_j)} \left((1 - \varepsilon) m(B_j \cap \{\operatorname{Re} g \leqslant 1 - \varepsilon\}) \right.$$

$$\left. + m(B_j \cap \{\operatorname{Re} g > 1 - \varepsilon\}) \right)$$

$$= 1 - \varepsilon \frac{m(B_j \cap \{\operatorname{Re} g \leqslant 1 - \varepsilon\})}{m(B_j)}. \tag{45}$$

This implies that

$$\frac{m(B_j \cap \{\operatorname{Re} g \leqslant 1 - \varepsilon\})}{m(B_j)} < \frac{\varepsilon}{4}. \tag{46}$$

Define $\widetilde{B}_j = B_j \cap \{\operatorname{Re} g > 1 - \varepsilon\}$ for all $j \in J$,

$$f_1 = \left(1/\sum_{j \in J} \beta_j \right) \sum_{j \in J} \beta_j \left(\chi_{\widetilde{B}_j}/m(\widetilde{B}_j) \right), \tag{47}$$

and $g_1(x) = 1$ on $\operatorname{supp}(f_1)$ and $g_1(x) = g(x)$ elsewhere. Then it is clear that $\operatorname{supp}(f_1) \subset \operatorname{supp}(f)$, $\|g - g_1\|_\infty < \sqrt{\varepsilon}$, and $\langle f_1, g_1 \rangle = 1$. Finally we will show that $\|f - f_1\| < \varepsilon$. Notice first that

$$\left\| \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(\widetilde{B}_j)} - \sum_{j \in J} \beta_j \frac{\chi_{B_j}}{m(B_j)} \right\|$$

$$\leqslant \left\| \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(\widetilde{B}_j)} - \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(B_j)} \right\|$$

$$+ \left\| \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(B_j)} - \sum_{j \in J} \beta_j \frac{\chi_{B_j}}{m(B_j)} \right\| \tag{48}$$

$$= 2 \sum_{j \in J} \beta_j \frac{m(B_j \setminus \widetilde{B}_j)}{m(B_j)} < \frac{\varepsilon}{2}.$$

Hence

$$\|f - f_1\|$$

$$\leqslant \left\| \frac{1}{\sum_{j \in J} \beta_j} \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(\widetilde{B}_j)} - \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(\widetilde{B}_j)} \right\|$$

$$+ \left\| \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(\widetilde{B}_j)} - f \right\|$$

$$\leqslant \frac{1 - \sum_{j \in J} \beta_j}{\sum_{j \in J} \beta_j} \left\| \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(\widetilde{B}_j)} \right\| \tag{49}$$

$$+ \left\| \sum_{j \in J} \beta_j \frac{\chi_{\widetilde{B}_j}}{m(\widetilde{B}_j)} - \sum_{j \in J} \beta_j \frac{\chi_{B_j}}{m(B_j)} \right\| + \frac{\varepsilon}{4}$$

$$= \left(1 - \sum_{j \in J} \beta_j \right) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}$$

$$\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

$\square$

Lemma 13 (see [25, Lemma 3.3]). *Suppose that T_μ is a norm-one element in $\mathscr{L}(L_1(m))$ for some $\mu \in M(m)$ and there is a nonnegative simple function f_0 such that f_0 is a norm-one element of $L_1(m)$ and $\|T_\mu f_0\| \geqslant 1 - \varepsilon^3/2^6$ for some $0 < \varepsilon < 1$.*

Then there exist a norm-one bounded linear operator T_ν for some $\nu \in M(m, m)$ and a nonnegative simple function f_1 in $S_{L_1(m)}$ such that $\|T_\mu - T_\nu\| \leq \varepsilon$, $\|f_1 - f_0\| \leq 3\varepsilon$, and $(d|\nu|^1/dm)(x) = 1$ for all $x \in \operatorname{supp}(f_1)$.

Lemma 14. *Suppose that $T_\nu \in \mathcal{L}(L_1(m))$ is a norm-one operator, $f = \sum_{i=1}^n \beta_i(\chi_{B_i}/m(B_i))$, where $m(B_j) > 0$ for all $1 \leq j \leq n$ and $\{B_j\}_{j=1}^n$ are mutually disjoint Borel subsets of Ω, is a norm-one nonnegative simple function, and g is an element of $S_{L_\infty(m)}$ such that*

$$\operatorname{Re} \langle g, T_\nu f \rangle \geq 1 - \frac{\varepsilon^6}{2^7} \qquad (50)$$

for some $0 < \varepsilon < 1$ and

$$\frac{d|\nu|^1}{dm}(x) = 1, \qquad g(x) = 1 \qquad (51)$$

for all x in the support of f.

Then there exist a nonnegative simple function $\tilde{f} \in S_{L_1(m)}$, a function $\tilde{g} \in S_{L_\infty(m)}$, and an operator $T_{\tilde{\nu}}$ in $\mathcal{L}(L_1(m), L_1(m))$ such that

$$\langle \tilde{g}, T_{\tilde{\nu}} \tilde{f} \rangle = \|T_{\tilde{\nu}}\| = 1, \qquad \|T_\nu - T_{\tilde{\nu}}\| \leq 2\varepsilon,$$

$$\|f - \tilde{f}\| \leq 3\varepsilon, \qquad \|g - \tilde{g}\| \leq \sqrt{\varepsilon}, \qquad \langle \tilde{f}, \tilde{g} \rangle = 1. \qquad (52)$$

Proof. Since

$$\operatorname{Re} \langle g, T_\nu f \rangle \geq 1 - \frac{\varepsilon^6}{2^7}, \qquad (53)$$

we have

$$1 - \frac{\varepsilon^6}{2^7} < \operatorname{Re} \langle g, T_\nu f \rangle$$

$$= \int_{\Omega \times \Omega} f(x) \operatorname{Re} g(y) \, d\mu(x, y) \qquad (54)$$

$$= \sum_{j=1}^n \beta_j \int_{\Omega \times \Omega} \frac{\chi_{B_j}(x)}{m(B_j)} \operatorname{Re} g(y) \, d\nu(x, y).$$

Let

$$J = \left\{ j : \int_{\Omega \times \Omega} \left(\frac{\chi_{B_j}(x)}{m(B_j)} \right) \operatorname{Re} g(y) \, d\nu(x, y) > \frac{1 - \varepsilon^3}{2^6} \right\} \qquad (55)$$

Then from Lemma 11 we have $\sum_{j \in J} \beta_j > 1 - \varepsilon^3/2$. Let $f_1 = \sum_{j \in J} \tilde{\beta}_j(\chi_{B_j}/m(B_j))$, where $\tilde{\beta}_j = \beta_j/(\sum_{j \in J} \beta_j)$ for all $j \in J$. Then

$$\|f_1 - f\| \leq \left\| \sum_{j \in J} (\tilde{\beta}_j - \beta_j) \frac{\chi_{B_j}}{m(B_j)} \right\| + \sum_{j \in J} \beta_j \leq \varepsilon^3 \leq \varepsilon. \quad (56)$$

Note that there is a Borel measurable function h on $\Omega \times \Omega$ such that $d\nu(x, y) = h(x, y) \, d|\nu|(x, y)$ and $|h(x, y)| = 1$ for all $(x, y) \in \Omega \times \Omega$. Let

$$C = \left\{ (x, y) : |g(y) h(x, y) - 1| < \frac{\sqrt{\varepsilon}}{2^{3/2}} \right\}. \qquad (57)$$

Define two measures ν_f and ν_c as follows:

$$\nu_f(A) = \nu(A \setminus C), \qquad \nu_c(A) = \nu(A \cap C) \qquad (58)$$

for every Borel subset A of $\Omega \times \Omega$. It is clear that

$$d\nu = d\nu_f + d\nu_c, \qquad d|\nu_f| = \bar{h} \, d\nu_f,$$

$$d|\nu_c| = \bar{h} \, d\nu_c, \qquad d|\nu| = d|\nu_f| + d|\nu_c|. \qquad (59)$$

Since $(d|\nu|^1/dm_1)(x) = 1$ for all $x \in \bigcup_{j=1}^n B_j$, we have

$$1 = \frac{d|\nu|^1}{dm_1}(x) = \frac{d|\nu_f|^1}{dm_1}(x) + \frac{d|\nu_c|^1}{dm_1}(x) \qquad (60)$$

for all $x \in B = \bigcup_{j=1}^n B_j$, and we deduce that $|\nu|^1(B_j) = m_1(B_j)$ for all $1 \leq j \leq n$.

We claim that $|\nu_f|^1(B_j)/m_1(B_j) \leq \varepsilon^2/2^2$ for all $j \in J$. Indeed, if $|g(y) h(x, y) - 1| \geq \sqrt{\varepsilon}/2^{3/2}$, then

$$\operatorname{Re} (g(y) h(x, y)) \leq 1 - \frac{\varepsilon}{2^4}. \qquad (61)$$

So we have

$$1 - \frac{\varepsilon^3}{2^6}$$

$$\leq \frac{1}{m_1(B_j)} \operatorname{Re} \int_{\Omega \times \Omega} \chi_{B_j(x)} g(y) \, d\nu(x, y)$$

$$= \frac{1}{m_1(B_j)} \int_{\Omega \times \Omega} \chi_{B_j(x)} \operatorname{Re} (g(y) h(x, y)) \, d|\nu|(x, y)$$

$$= \frac{1}{m_1(B_j)} \int_{\Omega \times \Omega} \chi_{B_j(x)} \operatorname{Re} (g(y) h(x, y)) \, d|\nu_f|(x, y)$$

$$+ \frac{1}{m_1(B_j)} \int_{\Omega \times \Omega} \chi_{B_j(x)} \operatorname{Re} (g(y) h(x, y)) \, d|\nu_c|(x, y)$$

$$\leq \frac{1}{m_1(B_j)} \left(\left(1 - \frac{\varepsilon}{2^4}\right) |\nu_f|^1(B_j) + |\nu_c|^1(B_j) \right)$$

$$= 1 - \frac{\varepsilon}{2^4} \frac{|\nu_f|^1(B_j)}{m_1(B_j)}. \qquad (62)$$

This proves our claim.

We also claim that, for each $j \in J$, there exists a Borel subset $\tilde{B}_j$ of B_j such that

$$\left(1 - \frac{\varepsilon}{2}\right) m_1(B_j) \leq m_1(\tilde{B}_j) \leq m_1(B_j),$$

$$\frac{d|\nu_f|^1}{dm_1}(x) \leq \frac{\varepsilon}{2} \qquad (63)$$

for all $x \in \widetilde{B}_j$. Indeed, set $\widetilde{B}_j = B_j \cap \{x \in \Omega : (d|\nu_f|^1/dm_1)(x) \leqslant \varepsilon/2\}$. Then

$$\int_{B_j \setminus \widetilde{B}_j} \frac{\varepsilon}{2} dm_1(x) \leqslant \int_{B_j} \frac{d|\nu_f|^1}{dm_1}(x) \, dm_1(x)$$

$$= |\nu_f^1|(B_j) \leqslant \frac{\varepsilon^2}{2^2} m_1(B_j). \tag{64}$$

This shows that $m_1(B_j \setminus \widetilde{B}_j) \leqslant (\varepsilon/2)m_1(B_j)$. This proves our second claim.

Now, we define $\widetilde{g}$ by $\widetilde{g}(y) = g(y)/|g(y)|$ if $|g(y)| \geqslant 1 - \sqrt{\varepsilon}/2^{3/2}$ and $\widetilde{g}(y) = g(y)$ if $|g(y)| < 1 - \sqrt{\varepsilon}/2^{3/2}$, and we write $\widetilde{f} = \sum_{j \in J} \widetilde{\beta}_j(\chi_{\widetilde{B}_j}/m_1(\widetilde{B}_j))$. It is clear that $\widetilde{g} \in S_{L_\infty(m)}$, $\|g - \widetilde{g}\| < \sqrt{\varepsilon}$, and $\widetilde{g}(y) = 1$ for all $x \in \text{supp } \widetilde{f}$.

Finally, we define the measure

$$d\widetilde{\nu}(x, y)$$

$$= \sum_{j \in J} \chi_{\widetilde{B}_j}(x) \overline{\widetilde{g}(y)h(x, y)} d\nu_c(x, y) \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} \tag{65}$$

$$+ \chi_{J_1 \setminus \widetilde{B}}(x) \, d\nu(x, y),$$

where $\widetilde{B} = \bigcup_{j \in J} \widetilde{B}_j$. It is easy to see that $(d|\widetilde{\nu}|^1/dm_1)(x) = 1$ on $\widetilde{B}$ and $(d|\widetilde{\nu}|^1/dm_1)(x) \leqslant 1$ elsewhere. Note that

$$d(\widetilde{\nu} - \nu)(x, y)$$

$$= \sum_{j \in J} \chi_{\widetilde{B}_j}(x) \left[\overline{\widetilde{g}(y)h(x, y)} \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} - 1 \right] d\nu_c(x, y)$$

$$- \sum_{j \in J} \chi_{\widetilde{B}_j}(x) \, d\nu_f(x, y). \tag{66}$$

If $(x, y) \in C$, then $|g(y)| \geqslant 1 - \sqrt{\varepsilon}/2^{3/2} \geqslant 1 - 1/2^{3/2}$ and

$$\left| \overline{\widetilde{g}(y)h(x, y)} - 1 \right|$$

$$= \left| \overline{\frac{g(y)}{|g(y)|}h(x, y)} - 1 \right|$$

$$\leqslant \frac{|g(y)h(x, y) - 1|}{|g(y)|} + \frac{|1 - |g(y)||}{|g(y)|} \tag{67}$$

$$\leqslant 2 \frac{|g(y)h(x, y) - 1|}{|g(y)|} \leqslant 2 \frac{\sqrt{\varepsilon}}{2^{3/2}} \frac{2^{3/2}}{2^{3/2} - 1} \leqslant 2\sqrt{\varepsilon}.$$

Hence, for all $(x, y) \in C$, we have

$$\left| \overline{\widetilde{g}(y)h(x, y)} \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} - 1 \right|$$

$$\leqslant \left| \overline{\widetilde{g}(y)h(x, y)} - 1 \right| \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1}$$

$$+ \left| \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} - 1 \right| \tag{68}$$

$$\leqslant 2\sqrt{\varepsilon} \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} + \left| \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} - 1 \right|.$$

So, we have, for all $x \in J_1$,

$$\frac{d|\widetilde{\nu} - \nu|^1}{dm_1}(x)$$

$$\leqslant \sum_{j \in J} \chi_{\widetilde{B}_j}(x) \left[2\sqrt{\varepsilon} \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} \right.$$

$$\left. + \left| \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} - 1 \right| \right] \frac{d|\nu_c|^1}{dm_1}(x)$$

$$+ \sum_{j \in J} \chi_{\widetilde{B}_j}(x) \frac{d|\nu_f|^1}{dm_1}(x) \tag{69}$$

$$\leqslant \sum_{j \in J} \chi_{\widetilde{B}_j}(x) \left(2\sqrt{\varepsilon} + \left(1 - \frac{d|\nu_c|^1}{dm_1}(x) \right) \right)$$

$$+ \sum_{j \in J} \chi_{\widetilde{B}_j}(x) \left(\frac{d|\nu_f|^1}{dm_1}(x) \right)$$

$$\leqslant 2\sqrt{\varepsilon} + \varepsilon < 3\sqrt{\varepsilon}.$$

This gives that $\|T_\nu - T_{\widetilde{\nu}}\| < 3\sqrt{\varepsilon}$. Note also that, for all $j \in J$,

$$\left\langle T_{\widetilde{\nu}} \frac{\chi_{\widetilde{B}_j}}{m_1(\widetilde{B}_j)}, \widetilde{g} \right\rangle$$

$$= \int_{\Omega \times \Omega} \frac{\chi_{\widetilde{B}_j}(x)}{m_1(\widetilde{B}_j)} \widetilde{g}(y) \, d\widetilde{\nu}(x, y)$$

$$= \int_{\Omega \times \Omega} \frac{\chi_{\widetilde{B}_j}(x)}{m_1(\widetilde{B}_j)} \overline{h(x, y)} \left(\frac{d|\nu_c|^1}{dm_1}(x) \right)^{-1} d\nu_c(x, y)$$

$$= \int_\Omega \frac{\chi_{\widetilde{B}_j}(x)}{m_1(\widetilde{B}_j)} \left(\frac{d|v_c|^1}{dm_1}(x)\right)^{-1} d|v_c|^1(x)$$

$$= \int_\Omega \frac{\chi_{\widetilde{B}_j}(x)}{m_1(\widetilde{B}_j)} dm_1(x) = 1. \tag{70}$$

Hence we get $\langle T_{\widetilde{\gamma}}\widetilde{f}, \widetilde{g}\rangle = 1$, which implies that $\|T_{\widetilde{\gamma}}\widetilde{f}\| = \|T_{\widetilde{\gamma}}\| = 1$. Finally,

$$\|\widetilde{f} - f\|$$

$$\leqslant \|\widetilde{f} - f_1\| + \|f_1 - f\|$$

$$= \left\|\sum_{j\in J}\widetilde{\beta}_j \frac{\chi_{\widetilde{B}_j}}{m_1(\widetilde{B}_j)} - \sum_{j\in J}\widetilde{\beta}_j \frac{\chi_{B_j}}{m_1(B_j)}\right\| + \varepsilon$$

$$\leqslant \sum_{j\in J}\widetilde{\beta}_j \left(\left\|\frac{\chi_{\widetilde{B}_j}}{m_1(\widetilde{B}_j)} - \frac{\chi_{B_j}}{m_1(\widetilde{B}_j)}\right\|\right.$$

$$\left. + \left\|\frac{\chi_{B_j}}{m_1(\widetilde{B}_j)} - \frac{\chi_{B_j}}{m_1(B_j)}\right\|\right) + \varepsilon \tag{71}$$

$$= 2\sum_{j\in J}\widetilde{\beta}_j \frac{m_1(B_j \setminus \widetilde{B}_j)}{m_1(\widetilde{B}_j)} + \varepsilon$$

$$\leqslant 2\sum_{j\in J}\widetilde{\beta}_j \frac{(\varepsilon/2)m_1(B_j)}{m_1(\widetilde{B}_j)} + \varepsilon \leqslant \frac{\varepsilon}{1 - \varepsilon/2} + \varepsilon < 3\varepsilon. \quad \square$$

We are now ready to present the proof of the main result in the case of finite regular positive Borel measures.

Proof of Proposition 10. Let $\delta_1 = \delta_2^3/(5\cdot 2^4)$, $\delta_2 = \delta_3^{12}/(3^2\cdot 2^{14})$, and $\delta_3 = (\varepsilon/10)^2$ for some $0 < \varepsilon < 1$. Suppose that $T \in \mathscr{L}(L_1(m))$ with $\|T\| = 1$ and that there is an $f_0 \in S_{L_1(m)}$ and $g_0 \in S_{L_\infty(m)}$ such that $\langle f_0, g_0\rangle = 1$ and $|\langle Tf_0, g_0\rangle| > 1 - \delta_1^3/2^6$. Then there is an isometric isomorphism Ψ from $L_1(m)$ onto itself such that $\Psi(f_0) = |f_0|$ and there is a scalar number α in $S_\mathbb{R}$ such that $|\langle Tf_0, g_0\rangle| = |\langle \alpha Tf_0, g_0\rangle|$. Then letting $f_1 = \Psi f_0$, $g_1 = (\Psi^{-1})^* g_0$, and $T_1 = \alpha\Psi T\Psi^{-1}$, we have

$$\langle Sf_1, g_1\rangle = \left\langle \alpha\Psi T_0\Psi^{-1}\Psi f_0, (\Psi^{-1})^* g_0\right\rangle$$

$$= \langle \alpha Tf_0, g_0\rangle > 1 - \frac{\delta_1^3}{2^6}, \tag{72}$$

$$\langle f_1, g_1\rangle = \left\langle \Psi f_0, (\Psi^{-1})^* g_0\right\rangle = 1.$$

Since $\|T_1 f_1\| > 1 - \delta(\delta_1^3/2^6)$, by Lemma 13, there exists a norm-one bounded operator T_γ and a nonnegative simple function $f_2 \in S_{L_1(m)}$ such that $\|T_1 - T_\gamma\| \leqslant \delta_1$, $\|f_2 - f_1\| \leqslant 3\delta_1$, and $(d|v|^1/dm_1)(x) = 1$ for all $x \in \text{supp}(f_2)$. Then

$$\langle T_\gamma f_2, g_1\rangle$$

$$= \langle T_1 f_1, g_1\rangle - \langle T_1 f_1 - T_1 f_2, g_1\rangle - \langle T_1 f_2 - T_\gamma f_2, g_1\rangle$$

$$\geqslant \langle T_1 f_1, g_1\rangle - \|f_1 - f_2\| - \|T_1 - T_\gamma\|$$

$$\geqslant 1 - \frac{\delta_1^3}{2^6} - 3\delta_1 - \delta_1 \geqslant 1 - 5\delta_1 = 1 - \frac{\delta_2^3}{16}. \tag{73}$$

Notice also that

$$\langle f_2, g_1\rangle = \langle f_1, g_1\rangle - \langle f_1 - f_2, g_1\rangle \geqslant 1 - \|f_1 - f_2\|$$

$$\geqslant 1 - 3\delta_1 \geqslant 1 - 5\delta_1 = 1 - \frac{\delta_2^3}{16}. \tag{74}$$

By Lemma 12 there are a nonnegative simple function $f_3 \in S_{L_1(m)}$ and a function $g_3 \in S_{L_\infty(m)}$ such that

$$g_3(x) = \chi_{\text{supp } f_3}(x) + g_2(x)\chi_{\Omega\setminus\text{supp } f_3}(x)$$

$$\|f_2 - f_3\| \leqslant \delta_2, \qquad \|g_3 - g_1\| \leqslant \sqrt{\delta_2}, \qquad \langle f_3, g_3\rangle = 1. \tag{75}$$

So we have

$$\langle T_\gamma f_3, g_3\rangle = \langle T_\gamma f_2, g_1\rangle - \langle T_\gamma f_2 - T_\gamma f_3, g_1\rangle$$

$$- \langle T_\gamma f_3, g_1 - g_3\rangle \tag{76}$$

$$\geqslant 1 - \frac{\delta_2^3}{16} - 2\sqrt{\delta_2} \geqslant 1 - 3\sqrt{\delta_2} = 1 - \frac{\delta_3^6}{2^7}.$$

By Lemma 14, there exist $f_4 \in S_{L_1(m)}$ and $g_4 \in S_{L_\infty(m)}$ and an operator T_4 such that

$$\langle g_4, T_4 f_4\rangle = 1 = \|T_4\|, \qquad \|T_4 - T_\gamma\| \leqslant 2\delta_3,$$

$$\|f_4 - f_3\| \leqslant 3\delta_3, \qquad \|g_4 - g_3\| \leqslant \sqrt{\delta_3}, \qquad \langle f_4, g_4\rangle = 1. \tag{77}$$

So we have

$$\|T_4 - T_1\| \leqslant \|T_4 - T_\gamma\| + \|T_\gamma - T_1\|$$

$$\leqslant \delta_1 + 2\delta_3 \leqslant 3\delta_3,$$

$$\|f_1 - f_4\| \leqslant \|f_1 - f_2\| + \|f_2 - f_3\| + \|f_3 - f_4\|$$

$$\leqslant 3\delta_1 + \delta_2 + 3\delta_3 \leqslant 10\delta_3, \tag{78}$$

$$\|g_1 - g_4\| \leqslant \|g_1 - g_3\| + \|g_3 - g_4\|$$

$$\leqslant \delta_2 + \sqrt{\delta_3} \leqslant 2\sqrt{\delta_3}.$$

Let $S = \alpha\Psi^{-1}T_4\Psi$, $\tilde{f} = \Psi^{-1}f_4$, and $\tilde{g} = \Psi^*g_4$; then we have

$$\|T - S\| = \left\|T - \alpha\Psi^{-1}T_4\Psi\right\| = \left\|\alpha\Psi T\Psi^{-1} - T_4\right\|$$

$$= \|T_1 - T_4\| \leq 3\delta_3,$$

$$\left\|f_0 - \tilde{f}\right\| = \left\|f_0 - \Psi^{-1}f_4\right\| = \|f_1 - f_4\| \leq 10\delta_3,$$

$$\left\|g_0 - \tilde{f}\right\| = \left\|g_0 - \Psi^*g_4\right\| = \left\|\left(\Psi^{-1}\right)^*g_0 - g_4\right\| \tag{79}$$

$$= \|g_1 - g_4\| \leq 2\sqrt{\delta_3},$$

$$\left\langle \tilde{f}, \tilde{g} \right\rangle = \left\langle \Psi^{-1}f_4, \Psi^*g_4 \right\rangle = \left\langle f_4, g_4 \right\rangle = 1,$$

$$\left|\left\langle S\tilde{f}, \tilde{g} \right\rangle\right| = \left|\left\langle \alpha\Psi^{-1}T_4\Psi\Psi^{-1}f_4, \Psi^*g_4 \right\rangle\right| = |\alpha| = 1.$$

This completes the proof. □

Finally, we may give the proof of the main result in full generality.

Proof of Theorem 9. Notice that the Kakutani representation theorem (see [26] for a reference) says that, for every σ-finite measure ν, the space $L_1(\nu)$ is isometrically isomorphic to $L_1(m)$ for some positive Borel regular measure on a compact Hausdorff space. Then, by Proposition 10, there is a universal function $\varepsilon \mapsto \eta(\varepsilon) > 0$ which gives the BPBp-nu for $L_1(\nu)$ for every σ-finite measure ν.

Fix $\varepsilon > 0$. Suppose that $T_0 \in \mathcal{L}(L_1(\mu))$ with $\nu(T_0) = 1$ and $(f_0, f_0^*) \in \Pi(L_1(\mu))$ satisfy

$$\left|\left\langle f_0^*, T_0f_0 \right\rangle\right| > 1 - \eta(\varepsilon). \tag{80}$$

Choose a sequence $\{f_n\}$ in $L_1(\mu)$ such that $\sup_n\|T_0f_n\| = 1$ and let G be the closed linear span of

$$\{T_0^nf_m : n, m \in \mathbb{N} \cup \{0\}\}. \tag{81}$$

As G is separable, there is a dense subset $\{g_n : n \in \mathbb{N}\}$ of G and let $E = \bigcup_{n=1}^{\infty} \operatorname{supp} g_n$, where $\operatorname{supp} g_n$ is the support of g_n. Then the measure $\mu|_E$ is σ-finite. Let

$$Y = \{f \in L_1(\mu) : \operatorname{supp}(f) \subset E\} \tag{82}$$

be a closed subspace of $L_1(\mu)$. It is clear that $L_1(\mu) = Y \oplus_1 Z$ and Y is isometrically isomorphic to $L_1(\mu|_E)$. So Y has the BPBp-nu with $\eta(\varepsilon)$.

Now, write $S_0 = T_0|_Y : Y \to Y$, consider $y_0 = f_0 \in S_Y$, $y_0^* = f_0^*|_Y \in S_{Y^*}$, and observe that $y_0^*(y_0) = 1$ and $|y_0^*(S_0y_0)| = |f_0^*(T_0f_0)| > 1 - \eta(\varepsilon)$. Hence, there exist $S \in \mathcal{L}(Y)$ and $(\tilde{y}_0, \tilde{y}_0^*) \in \Pi(Y)$ such that

$$\left|\tilde{y}_0^*(S\tilde{y}_0)\right| = 1 = \nu(S), \qquad \|S - S_0\| < \varepsilon,$$

$$\|y_0 - \tilde{y}_0\| < \varepsilon, \qquad \|y_0^* - \tilde{y}_0^*\| < \varepsilon. \tag{83}$$

Finally consider the operator $T \in \mathcal{L}(L_1(\mu))$ given by

$$T(y, z)$$
$$= (Sy, 0) + T_0(0, z) \left((y, z) \in L_1(\mu) \equiv Y \oplus_1 Z\right). \tag{84}$$

We have $\|T\| = 1$ (and so $\nu(T) = 1$). Indeed,

$$\|T(y, z)\| = \|(Sy, 0)\| + \|T_0(0, z)\| \leq \|y\| + \|z\| = \|(y, z)\| \tag{85}$$

for all $(y, z) \in L_1(\mu)$ and $\|T(\tilde{y}_0, 0)\| = \|(S\tilde{y}_0, 0)\| = \|S\tilde{y}_0\| = 1$. Let $x = (\tilde{y}_0, 0)$ and $x^* = (\tilde{y}_0^*, f_0|_Z)$. Then $(x, x^*) \in \Pi(L_1(\mu))$. Moreover, we have

$$\left|x^*Tx\right| = \left|\tilde{y}_0^*S y_0\right| = 1 = \nu(T),$$

$$\|x - f_0\| = \|y - y_0\| < \varepsilon,$$

$$\|x_0^* - f_0^*\|$$

$$= \max\{\|y - f_0^*|_Y\|, \|f_0^*|_Z - f_0^*|_Z\|\}$$

$$= \|y^* - y_0^*\| < \varepsilon, \tag{86}$$

$$\|T - T_0\| = \sup_{\|y\| + \|z\| \leq 1} \|T(y, z) - T_0(y, z)\|$$

$$= \sup_{\|y\| \leq 1} \|Sy - S_0y\| = \|S - S_0\| < \varepsilon.$$

This completes the proof. □

5. Examples of Spaces Failing the Bishop-Phelps-Bollobás Property for Numerical Radius

Our goal here is to prove that the density of numerical radius attaining operators does not imply the BPBp-nu. Actually, we will show that, among separable spaces, there is no isomorphic property implying the BPBp-nu other than finite-dimensionality.

We need to relate the BPBp-nu to the Bishop-Phelps-Bollobás property for operators which, as mentioned in the introduction, was introduced in [15]. A pair (X, Y) of Banach spaces has the *Bishop-Phelps-Bollobás property for operators* (in short, *BPBp*); if given $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that, given $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x \in S_X$ such that $\|Tx\| > 1 - \eta(\varepsilon)$, then there exist $z \in S_X$ and $S \in \mathcal{L}(X, Y)$ satisfying

$$\|S\| = \|Sz\| = 1, \qquad \|x - z\| < \varepsilon,$$

$$\|T - S\| < \varepsilon. \tag{87}$$

We refer the reader to [15, 19, 25] and references therein for more information and background. Among the interesting results on the BPBp, we emphasize that a pair (X, Y) when X is finite-dimensional does not necessarily have the BPBp. For instance, if Y is a strictly convex space which is not uniformly convex, then the pair $(\ell_1^{(2)}, Y)$ fails to have the BPBp (this is contained in [15]; see [19, Section 3]).

The next result relates the BPBp-nu to the BPBp for operators in a particular case. We will deduce our example from it.

Theorem 15. *If $L_1(\mu)\oplus_1 X$ has the BPBp-nu, then the pair $(L_1(\mu), X)$ has the BPBp for operators.*

Before proving this proposition, we will use it to get the main examples of this section. The first example shows that the density of numerical radius attaining operators does not imply the BPBp-nu.

Example 16. There is a reflexive space (and so numerical radius attaining operators on it are dense) which fails to have the BPBp-nu. Indeed, let Y be a reflexive separable space which is not superreflexive and we may suppose that Y is strictly convex. Observe that Y cannot be uniformly convex since it is not superreflexive. Now, $X = \ell_1^{(2)}\oplus_1 Y$ is reflexive, but the pair $(\ell_1^{(2)}, Y)$ fails the BPBp since Y is strictly convex but not uniformly convex [19, Corollary 3.3]. Therefore, Theorem 15 gives us that X does not have the BPBp-nu.

The example above can be extended to get the result that every infinite-dimensional separable Banach space can be renormed to fail the BPBp-nu. This follows from the fact that every infinite-dimensional separable Banach space can be renormed to be strictly convex but not uniformly convex (this result can be proved "by hand"; an alternative categorical argument for it can be found in [27] and references therein). With a little more of effort, we may get the main result of the section.

Theorem 17. *Every infinite-dimensional separable Banach space can be renormed to fail the weak-BPBp-nu (and so, in particular, to fail the BPBp-nu).*

We need the following result which is surely well known. As we have not found a reference, we include a nice and easy proof kindly given to us by Vladimir Kadets. We recall that, given a Banach space Y, the set of all equivalent norms on Y can be viewed as a metric space using the Banach-Mazur distance.

Lemma 18. *Let Y be an infinite-dimensional separable Banach space. Then the set of equivalent norms on Y which are strictly convex and are not (locally) uniformly convex is dense in the set of all equivalent norms on Y (with respect to the Banach-Mazur distance).*

Proof. Fix $e \in S_Y$ and $e_1^* \in S_{Y^*}$ such that $e_1^*(e) = 1$. For a fixed $\varepsilon \in (0, 1/2)$, denote

$$q(y) = \max\left\{(1-\varepsilon)\|y\|, |e_1^*(y)|\right\} \quad (y \in Y). \tag{88}$$

Evidently, $(1-\varepsilon)\|y\| \leqslant q(y) \leqslant \|y\|$ for every $y \in Y$. Fix a sequence $\{e_k^* : k \geqslant 2\}$ of norm-one functionals separating the points of Y, and denote

$$p(y) = \sqrt{\sum_{k=1}^{\infty} \frac{1}{2^k}|e_k^*(y)|^2} \quad (y \in Y). \tag{89}$$

Then, p is a strictly convex norm on Y, $p(e) \geqslant 1/\sqrt{2}$, and $p(y) \leqslant \|y\|$ for all $y \in X$. Finally, write

$$\|y\|_1 = (1-\varepsilon)q(y) + \varepsilon\frac{p(y)}{p(e)} \quad (y \in Y). \tag{90}$$

Then, $\|\cdot\|_1$ is a strictly convex norm on Y and

$$(1-\varepsilon)^2\|y\| \leqslant \|y\|_1 \leqslant (1+\varepsilon)\|y\| \quad (y \in Y). \tag{91}$$

We will finish the proof by showing that $\|\cdot\|_1$ is not uniformly convex (actually, it is not locally uniformly convex). Indeed, for each $n \in \mathbb{N}$ we select $y_n \in \bigcap_{k=1}^n \ker e_k^*$ with $\|y_n\| = 1$ and consider $e_n = e + (\varepsilon/4)y_n$. Then, $q(e) = 1$, $q(e_n) = 1$, and $q(e+e_n) = 2$. At the same time, $p(y_n) \to 0$, so $p(e_n) \to p(e)$ and $p(e+e_n) \to 2p(e)$. Consequently,

$$\|e\|_1 = 1, \qquad \|e_n\|_1 \longrightarrow 1, \qquad \|e+e_n\|_1 \longrightarrow 2, \tag{92}$$

but $\|e - e_n\|_1 = (\varepsilon/4)\|y_n\|_1 \geqslant (1-\varepsilon)^2(\varepsilon/4)$, which means the absence of local uniform convexity at e. $\qquad\square$

Proof of Theorem 17. Let X be an infinite-dimensional separable Banach space. Take a closed subspace Y of X of codimension two. By [28, Proposition 2], the map carrying every equivalent norm on Y to its numerical index is continuous and so, the set of values of the numerical index of Y up to reforming is a nontrivial interval [28, Theorem 9]. Then Lemma 18 allows us to find an equivalent norm $|\cdot|$ on Y in such a way that $(Y, |\cdot|)$ is strictly convex and is not uniformly convex, and $n(Y, |\cdot|) > 0$. Now, the space $\widetilde{X} = \ell_1^{(2)}\oplus_1 (Y, |\cdot|)$ is an equivalent renorming of X which does not have the BPBp-*nu* (indeed, otherwise, the pair $(\ell_1^{(2)}, (Y, |\cdot|))$ would have the BPBp for the operator norm and so, $(Y, |\cdot|)$ would be uniformly convex by [19, Corollary 3.3], a contradiction.) Moreover, as

$$n(\widetilde{X}) = \min\left\{n(\ell_1^{(2)}), n(Y, |\cdot|)\right\} > 0 \tag{93}$$

(see [20, Proposition 2], for instance), $\widetilde{X}$ also fails the weak-BPBp-nu by Proposition 6. $\qquad\square$

To finish the section with the promised proof of Theorem 15, we first see the following stability result.

Lemma 19. *Let $X = [\bigoplus_{k=1}^{\infty} X_k]_{c_0}$ or $[\bigoplus_{k=1}^{\infty} X_k]_{\ell_1}$. If X has the Bishop-Phelps-Bollobás property for numerical radius with a function η, then each Banach space X_i has the Bishop-Phelps-Bollobás property for numerical radius with $\eta_{nu}(X_i) \geqslant \eta$. That is, $\inf_i \eta_{nu}(X_i)(\varepsilon) \geqslant \eta_{nu}(X)(\varepsilon)$ for all $0 < \varepsilon < 1$.*

Proof. Let $P_i : X \to X_i$ and $P_i' : X^* \to X_i^*$ be the natural projections, and let $Q_i : X_i \to X$ and $Q_i' : X_i^* \to X^*$ be the natural embeddings.

Assume that an operator $T_i : X_i \to X_i$ and a pair $(x_i, x_i^*) \in \Pi(X_i)$ satisfy that

$$v(T_i) = 1, \qquad |x_i^* T_i x_i| > 1 - \eta(\varepsilon). \tag{94}$$

We define an operator $T : X \to X$ and $(x, x^*) \in \Pi(X)$ by

$$T = Q_i \circ T_i \circ P_i, \qquad (x, x^*) = (Q_i x_i, Q_i' x_i^*); \tag{95}$$

then clearly we see that

$$|x^*Tx| = |x_i^*T_ix_i| > 1 - \eta(\varepsilon). \tag{96}$$

From the assumption, there exist $S : X \rightarrow X$ and a pair $(y, y^*) \in \Pi(X)$ such that

$$|y^*Sy| = 1 = v(S), \qquad \|S - T\| < \varepsilon,$$
$$\|y^* - x^*\| < \varepsilon, \qquad \|y - x\| < \varepsilon. \tag{97}$$

Since this clearly shows that

$$\|P_i \circ S \circ Q_i - T_i\| < \varepsilon, \qquad \|P_i'y^* - x_i^*\| < \varepsilon,$$
$$\|P_iy - x_i\| < \varepsilon, \tag{98}$$

we only need to show that $|P_i'y^*(P_i \circ S \circ Q_i)P_iy| = 1$.

We first show the case of c_0 sum. Since

$$\|P_jy\| = \|P_jy - P_jx\| \leqslant \|y - x\| < \varepsilon \tag{99}$$

for every $j \neq i$, we have

$$1 = y^*(y) = \sum_{j \in \mathbb{N}} P_j'y^*(P_jy) \leqslant \sum_{j \in \mathbb{N}} \|P_j'y^*\| \|P_jy\|$$
$$\leqslant \|P_i'y^*\| + \varepsilon \sum_{j \in \mathbb{N}, j \neq i} \|P_j'y^*\| \leqslant \|y^*\| = 1. \tag{100}$$

This shows that $\|P_i'y^*\| = 1$ and $P_j'y^* = 0$ for every $j \neq i$. So $y^* = Q_i'P_i'y^*$ and $P_i'y^*(P_iy) = 1$. This and the fact that $\|y - Q_iP_iy\| < \varepsilon$ imply that

$$\left(Q_iP_iy + \left(\frac{1}{\varepsilon}\right)(y - Q_iP_iy), Q_i'P_i'y^*\right) \in \Pi(X). \tag{101}$$

So we get that $(Q_i'P_i'y^*)S(Q_iP_iy + (1/\varepsilon)(y - Q_iP_iy))v(S) = 1$. Hence, we have

$$1 = |y^*Sy| = \left|\left(Q_i'P_i'y^*\right)Sy\right|$$
$$= \left|(1 - \varepsilon)\left(Q_i'P_i'y^*\right)S\left(Q_iP_iy\right)\right.$$
$$\left.+\varepsilon\left(Q_i'P_i'y^*\right)S\left(Q_iP_iy + \frac{1}{\varepsilon}(y - Q_iP_iy)\right)\right| \leqslant 1, \tag{102}$$

and so we get $|P_i'y^*(P_i \circ S \circ Q_i)P_iy| = |(Q_i'P_i'y^*)S(Q_iP_iy)| = 1$.

We next show the case of ℓ_1 sum. The proof is almost the same as that of the c_0 case. However, for the sake of completeness, we provide it here.

Since $\|P_j'y^*\| = \|P_j'y^* - P_j'x^*\| \leqslant \|y^* - x^*\| < \varepsilon$ for every $j \neq i$, we have

$$1 = y^*(y) = \sum_{j \in \mathbb{N}} P_j'y^*(P_jy) \leqslant \sum_{j \in \mathbb{N}} \|P_j'y^*\| \|P_jy\|$$
$$\leqslant \|P_iy\| + \varepsilon \sum_{j \in \mathbb{N}, j \neq i} \|P_jy\| \leqslant \|y\| = 1, \tag{103}$$

which shows $\|P_iy\| = 1$ and $P_jy = 0$ for every $j \neq i$. Since this implies $(Q_iP_iy, Q_i'P_i'y^* + (1/\varepsilon)(y^* - Q_i'P_i'y^*)) \in \Pi(X)$, we get

that $|(Q_i'P_i'y^* + (1/\varepsilon)(y^* - Q_i'P_i'y^*))S(Q_iP_iy)| \leqslant v(S) = 1$. Hence, we have

$$1 = |y^*Sy| = |y^*S(Q_iP_iy)|$$
$$= \left|(1 - \varepsilon)\left(Q_i'P_i'y^*\right)S\left(Q_iP_iy\right)\right.$$
$$\left.+\varepsilon\left(Q_i'P_i'y^* + \frac{1}{\varepsilon}(y^* - Q_i'P_i'y^*)\right)S\left(Q_iP_iy\right)\right| \leqslant 1, \tag{104}$$

and so $|P_i'y^*(P_i \circ S \circ Q_i)P_iy| = |(Q_i'P_i'y^*)S(Q_iP_iy)| = 1$. □

Proof of Theorem 15. Note that $\eta_{\mathrm{nu}}(L_1(\mu)\oplus_1 X)(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Fix $0 < \varepsilon_0 < 1$ and choose $0 < \varepsilon < 1$ such that $6\varepsilon + \eta_{\mathrm{nu}}(L_1(\mu)\oplus_1 X)(\varepsilon) < \varepsilon_0$. Let $\eta(\varepsilon_0) = \eta_{\mathrm{nu}}(L_1(\mu)\oplus_1 X)(\varepsilon)$.

Suppose that $T_0 \in \mathcal{L}(L_1(\mu), X)$ with $\|T_0\| = 1$ and $f_0 \in S_{L_1(\mu)}$ satisfy

$$\|T_0f_0\| > 1 - \eta(\varepsilon_0). \tag{105}$$

For any measurable subset B, let

$$L_1(\mu|_B) = \{f|_B : f \in L_1(\mu)\} \tag{106}$$

with the norm $\|f|_B\| = \|f\chi_B\|_1$. Then it is easy to see that $L_1(\mu|_B)$ is isometrically isomorphic to a complemented subspace of $L_1(\mu)$. Let $P_B : L_1(\mu) \rightarrow L_1(\mu|_B)$ be the restriction defined by $P_B(f) = f|_B$ for all $f \in L_1(\mu)$ and let $J_B : L_1(\mu|_B) \rightarrow L_1(\mu)$ be the extension defined by $J_B(f)(\omega) = f(\omega)$ if $\omega \in B$ and $J_B(f)(\omega) = 0$ otherwise. It is clear that $P_BJ_B = \mathrm{Id}_{L_1(\mu|_B)}$ and $J_BP_B(f) = f\chi_B$ for all $f \in L_1(\mu)$. Notice also that $L_1(\mu)$ is isometrically isomorphic to $L_1(\mu|_B)\oplus_1 L_1(\mu|_{B^c})$.

Let $A = \mathrm{supp}\, f_0$ and $g_0 = P_Af_0$. Then

$$\|T_0J_Ag_0\| = \|T_0f_0\| > 1 - \eta(\varepsilon_0) > 0 \tag{107}$$

and define the operator $T_A : L_1(\mu|_A) \rightarrow X$ by $T_Af = T_0J_Af/\|T_0J_A\|$ for every $f \in L_1(\mu|_A)$. Then,

$$\|T_Ag_0\| \geqslant \|T_0f_0\| > 1 - \eta(\varepsilon_0). \tag{108}$$

Since $\mu|_A$ is σ-finite, $L_1(\mu|_A)^* = L_\infty(\mu|_A)$. Let $g_0^* \in S_{L_\infty(\mu|_A)}$ be a function such that $\langle g_0^*, g_0 \rangle = 1$, and choose $x_0^* \in S_{X^*}$ such that $x_0^*(T_Ag_0) = \|T_Ag_0\|$. Define the operator $S_0 \in \mathcal{L}(L_1(\mu|_A)\oplus_1 X)$ by

$$S_0(f, x) = (0, T_Af) \quad ((f, x) \in L_1(\mu|_A)\oplus_1 X) \tag{109}$$

and observe that $\|S_0\| = v(S_0) = 1$. Indeed,

$$\|S_0\| \leqslant 1 = \|T_A\|$$
$$= \sup\left\{|x^*T_Af| : x^* \in S_{X^*}, \ f \in S_{L_1(\mu|_A)}\right\}$$
$$= \sup\left\{|(f^*, x^*)S_0(f, x)| : \right.$$
$$\left.((f^*, x^*), (f, x)) \in \Pi(L_1(\mu|_A)\oplus_1 X)\right\} \tag{110}$$
$$= v(S_0) \leqslant \|S_0\|.$$

It is immediate that

$$(g_0^*, x_0^*) S_0 (g_0, 0) = x_0^* (T_A g_0) = \|T_A g_0\| > 1 - \eta(\varepsilon_0). \tag{111}$$

By Lemma 19, $L_1(\mu|_A) \oplus_1 X$ has the BPBp-nu with the function η. Therefore, there exist $S_1 \in \mathscr{L}(L_1(\mu|_A) \oplus_1 X)$, $(g_1, x_1) \in S_{L_1(\mu|_A) \oplus_1 X}$, and $(g_1^*, x_1^*) \in S_{L_\infty(\mu|_A) \oplus_\infty X^*}$ such that

$$\|(g_1, x_1) - (g_0, 0)\| < \varepsilon, \qquad \|(g_1^*, x_1^*) - (f_0^*, x_0^*)\| < \varepsilon,$$

$$\|S_1 - S_0\| < \varepsilon, \qquad \langle (g_1^*, x_1^*), (g_1, x_1) \rangle = 1,$$

$$|(g_1^*, x_1^*) S_1 (g_1, x_1)| = \nu(S_1) = 1. \tag{112}$$

Claim 1. We claim that $x_1 = 0$.

Otherwise,

$$1 = \mathrm{Re}\, \langle (g_1^*, x_1^*), (g_1, x_1) \rangle$$

$$= \|g_1\| \, \mathrm{Re} \left\langle (g_1^*, x_1^*), \frac{(g_1, 0)}{\|g_1\|} \right\rangle \tag{113}$$

$$+ \|x_1\| \, \mathrm{Re} \left\langle (g_1^*, x_1^*), \frac{(0, x_1)}{\|x_1\|} \right\rangle \leqslant 1.$$

We deduce that

$$\left(\frac{(g_1, 0)}{\|g_1\|}, (g_1^*, x_1^*) \right), \left(\frac{(0, x_1)}{\|x_1\|}, (g_1^*, x_1^*) \right)$$

$$\in \Pi \left(L_1(\mu|_A) \oplus_1 X \right). \tag{114}$$

Since $\|S_1((0, x_1)/\|x_1\|)\| = \|(S_1 - S_0)((0, x_1)/\|x_1\|)\| < \varepsilon$, we get that

$$1 = |\langle (g_1^*, x_1^*), S_1 (g_1, x_1) \rangle|$$

$$= \left| \|g_1\| \left\langle (g_1^*, x_1^*), S_1 \left(\frac{(g_1, 0)}{\|g_1\|} \right) \right\rangle \right. \tag{115}$$

$$\left. + \|x_1\| \left\langle (g_1^*, x_1^*), S_1 \left(\frac{(0, x_1)}{\|x_1\|} \right) \right\rangle \right|$$

$$\leqslant \|g_1\| \nu(S_1) + \varepsilon \|x_1\| < \|g_1\| + \|x_1\| = 1,$$

a contradiction. This proves the claim.

We define the operator $S_2 : L_1(\mu|_A) \oplus_1 X \to L_1(\mu|_A) \oplus_1 X$ by $S_2(f, x) = S_1(f, 0)$ for every $f \in L_1(\mu|_A)$ and for every $x \in X$. Then we have

$$\nu(S_2) = |(g_1^*, x_1^*) S_2 (g_1, 0)| = 1, \qquad \|S_1 - S_2\| \leqslant \varepsilon. \tag{116}$$

Indeed, from Claim 1, we have

$$\nu(S_1) = |(g_1^*, x_1^*) S_1 (g_1, x_1)| = |(g_1^*, x_1^*) S_1 (g_1, 0)|$$

$$= |(g_1^*, x_1^*) S_2 (g_1, 0)| \leqslant \nu(S_2). \tag{117}$$

On the other hand, we see that

$$|(f^*, x^*) S_2 (f, x)|$$

$$= |(f^*, x^*) S_1 (f, 0)| \leqslant \|f\| \nu(S_1) \leqslant \nu(S_1) \tag{118}$$

for every $((f^*, x^*), (f, x)) \in \Pi(L_1(\mu|_A) \oplus_1 X)$. Therefore, $\nu(S_2) \leqslant \nu(S_1)$. Also,

$$\|S_1 - S_2\| \leqslant \sup_{x \in S_X} \|S_1 (0, x)\|$$

$$= \sup_{x \in S_X} \|S_1 (0, x) - S_0 (0, x)\| \leqslant \varepsilon. \tag{119}$$

Claim 2. There exists $S_3 : L_1(\mu|_A) \oplus_1 X \to L_1(\mu|_A) \oplus_1 X$ such that $\|S_3(g_1, 0)\| = \|S_3\| = 1$, $S_3(0, x) = 0$, $S_3(f, x) \in \{0\} \oplus_1 X$ for every $(f, x) \in L_1(\mu|_A) \oplus_1 X$, and $\|S_3 - S_2\| < 4\varepsilon$.

Indeed, write $S_1 = (D_1, D_2)$, where $D_1 : L_1(\mu|_A) \oplus_1 X \to L_1(\mu|_A)$ and $D_2 : L_1(\mu|_A) \oplus_1 X \to X$. We have that

$$\sup \left\{ |g^* D_1 (g_1, 0) + x^* D_2 (g_1, 0)| : \right.$$

$$\left. x^* \in S_{X^*}, \ \langle g^*, g_1 \rangle = 1, \ g^* \in S_{L_\infty(\mu|_A)} \right\}$$

$$= \sup \left\{ |g^* D_1 (g_1, 0)| + \|D_2 (g_1, 0)\| : \right.$$

$$\left. \langle g^*, g_1 \rangle = 1, \ g^* \in S_{L_\infty(\mu|_A)} \right\} \tag{120}$$

$$= \sup \left\{ |g^* D_1 (g_1, 0)| : \right.$$

$$\left. \langle g^*, g_1 \rangle = 1, \ g^* \in S_{L_\infty(\mu|_A)} \right\} + \|D_2 (g_1, 0)\|$$

$$\leqslant \nu(S_2) = |(g_1^*, x_1^*) S_2 (g_1, 0)|$$

$$= |g_1^* D_1 (g_1, 0) + x_1^* D_2 (g_1, 0)|.$$

This implies that

$$|x_1^* D_2 (g_1, 0)| = \|D_2 (g_1, 0)\|,$$

$$|g_1^* D_1 (g_1, 0)|$$

$$= \sup \{|g^* D_1 (g_1, 0)| : \langle g^*, g_1 \rangle = 1, g^* \in L_\infty(\mu|_A)\}. \tag{121}$$

Therefore, $|g_1^*|$ equals 1 on the support of $D_1(g_1, 0)$. As $|\langle g_1^*, g_1 \rangle| = 1$, we also have that $|g_1^*|$ equals 1 on the support of g_1. Changing the values of g_1^* by the ones of f_0^* on $A \setminus (\mathrm{supp}(D_1(g_1, 0)) \cup \mathrm{supp}(g_1))$, we may and do suppose that $|g_1^*| = 1$ on the whole A.

We also have $\|D_2(g_1, 0)\| > 0$. Indeed,

$$\|S_2 (g_1, 0) - S_0 (g_0, 0)\|$$

$$\leqslant \|S_2 (g_1, 0) - S_0 (g_1, 0)\| + \|S_0 (g_1, 0) - S_0 (g_0, 0)\| \tag{122}$$

$$< 2\varepsilon + \varepsilon = 3\varepsilon.$$

So we have

$$\|D_2 (g_1, 0) - T_A g_0\|$$

$$\leqslant \|D_1 (g_1, 0)\| + \|D_2 (g_1, 0) - T_A g_0\|$$

$$= \left\| \left(D_1 \left(g_1, 0 \right), D_2 \left(g_1, 0 \right) \right) - \left(0, T_A g_0 \right) \right\|$$

$$= \left\| S_2 \left(g_1, 0 \right) - S_0 \left(g_0, 0 \right) \right\| < 3\varepsilon$$

$$(123)$$

and $\left\| D_2(g_1, 0) \right\| > \left\| T_A g_0 \right\| - 3\varepsilon \geqslant 1 - \eta(\varepsilon_0) - 3\varepsilon > 0$.

Finally define the operator S_3 by

$$S_3 \left(f, x \right) = \left(0, D_2 \left(f, 0 \right) + g_1^* \left(D_1 \left(f, 0 \right) \right) \frac{D_2 \left(g_1, 0 \right)}{x_1^* D_2 \left(g_1, 0 \right)} \right)$$

$$(124)$$

for $(f, x) \in L_1(\mu|_A) \oplus_1 X$. It is clear that

$$\left\| S_3 \right\| \leqslant \sup_{f \in S_{L_1(\mu|_A)}} \left(\left\| D_2 \left(f, 0 \right) \right\| + \left| g_1^* D_1 \left(f, 0 \right) \right| \right). \quad (125)$$

Notice also that

$$\left\| D_1 \left(f, 0 \right) \right\|$$

$$\leqslant \left\| D_1 \left(f, 0 \right) \right\| + \left\| D_2 \left(f, 0 \right) - T_A f \right\|$$

$$= \left\| \left(D_1 \left(f, 0 \right), D_2 \left(f, 0 \right) \right) - \left(0, T_A f \right) \right\| \quad (126)$$

$$= \left\| S_2 \left(f, x \right) - S_0 \left(f, x \right) \right\|$$

for all $(f, x) \in L_1(\mu|_A) \oplus_1 X$. Hence we have

$$\left\| S_3 - S_2 \right\|$$

$$= 2 \sup_{f \in S_{L_1(\mu|_A)}} \left\| D_1 \left(f, 0 \right) \right\| \leqslant 2 \left\| S_2 - S_0 \right\| < 4\varepsilon. \quad (127)$$

On the other hand, let $G : L_1(\mu|_A) \to L_1(\mu|_A)$ be defined by $G(f) = \overline{g_1^*} f$ for every $f \in L_1(\mu|_A)$. Then, we have

$$v \left(S_2 \right)$$

$$= \sup \left\{ \left| z^* S_2 z \right| : (z, z^*) \in \Pi \left(L_1 \left(\mu|_A \right) \oplus_1 X \right) \right\}$$

$$\geqslant \sup \left\{ \left| x^* D_2 \left(G \left(\frac{1}{\mu(C)} \chi_C \right), 0 \right) \right. \right.$$

$$+ g_1^* D_1 \left(G \left(\frac{1}{\mu(C)} \chi_C \right), 0 \right) \Big| :$$

$$x^* \in S_{X^*}, \ C \in \Sigma_A, \ \mu(C) > 0 \Big\} \quad (128)$$

$$= \sup \left\{ \left\| D_2 \left(G \left(\frac{1}{\mu(C)} \chi_C \right), 0 \right) \right\| \right.$$

$$+ \left| g_1^* D_1 \left(G \left(\frac{1}{\mu(C)} \chi_C \right), 0 \right) \right| :$$

$$C \in \Sigma_A, \ \mu(C) > 0 \Big\},$$

where Σ_A is the family of measurable subsets of A.

Hence, for any simple function $s = \sum_{i=1}^{n} ((\alpha_i / \mu(A_i)) \chi_{A_i}) \in S_{L_1(\mu|_A)}$, where $\{A_i\}_i$ is a family of disjoint measurable subsets with strictly positive measure, we have

$$v \left(S_2 \right)$$

$$\geqslant \sum_{i=1}^{n} \left| \alpha_i \right| \left(\left\| D_2 \left(G \left(\frac{1}{\mu(A_i)} \chi_{A_i} \right), 0 \right) \right\| \right.$$

$$+ \left| g_1^* D_1 \left(G \left(\frac{1}{\mu(A_i)} \chi_{A_i} \right), 0 \right) \right| \Big) \quad (129)$$

$$\geqslant \left\| D_2 \left(G(s), 0 \right) \right\| + \left| g_1^* D_1 \left(G(s), 0 \right) \right|.$$

Since $\left| g_1^* \right| = 1$, G is an isometric isomorphism, so for each $f \in S_{L_1(\mu|_A)}$ there exists a sequence of norm-one simple functions (s_k) such that $G(s_k)$ converges to f. Therefore,

$$v \left(S_2 \right) \geqslant \sup_{f \in S_{L_1(\mu|_A)}} \left(\left\| D_2 \left(f, 0 \right) \right\| + \left| g_1^* D_1 \left(f, 0 \right) \right| \right) \geqslant \left\| S_3 \right\|.$$

$$(130)$$

On the other hand, we see that

$$\left\| S_3 \right\| \geqslant \left| \left(g_1^*, x_1^* \right) S_3 \left(g_1, 0 \right) \right|$$

$$= \left| x_1^* D_2 \left(g_1, 0 \right) + g_1^* D_1 \left(g_1, 0 \right) \right| = v \left(S_2 \right) = 1. \quad (131)$$

Therefore, $1 = \left\| S_3 \right\| = \left\| S_3(g_1, 0) \right\|$ which proves Claim 2.

Finally, set $S_3 = (0, \widetilde{T})$ for a suitable $\widetilde{T} : L_1(\mu|_A) \oplus_1 X \to X$ and define the operator $T_1 : L_1(\mu) \to X$ by

$$T_1 \left(f \right) = T_0 \left(f \chi_{A^c} \right) + \widetilde{T} \left(P_A f, 0 \right) \quad (132)$$

for every $f \in L_1(\mu)$. Then, we have

$$\left\| T_1 \left(f \right) \right\| \leqslant \left\| T_0 \right\| \left\| f \chi_{A^c} \right\| + \left\| \widetilde{T} \right\| \left\| f \chi_A \right\| = \left\| f \right\| \quad (133)$$

for every $f \in L_1(\mu)$, so $\left\| T_1 \right\| \leqslant 1$. Also,

$$\left\| T_1 \left(J_A g_1 \right) \right\| = \left\| S_3 \left(g_1, 0 \right) \right\| = \left\| S_3 \right\| = 1, \quad (134)$$

so T_1 attains its norm at $J_A g_1 \in L_1(\mu)$, and

$$\left\| J_A g_1 - f_0 \right\| = \left\| g_1 - g_0 \right\| < \varepsilon. \quad (135)$$

We also have that, for any $f \in S_{L_1(\mu)}$,

$$\left\| T_0 \left(f \right) - T_1 \left(f \right) \right\| = \left\| T_0 \left(f \chi_A \right) - \widetilde{T} \left(P_A f, 0 \right) \right\|$$

$$\leqslant \left\| T_0 \left(J_A P_A f \right) - T_A \left(P_A f \right) \right\|$$

$$+ \left\| T_A \left(P_A f \right) - \widetilde{T} \left(P_A f, 0 \right) \right\| \quad (136)$$

$$\leqslant \left\| T_0 J_A - T_A \right\| + \left\| S_0 - S_3 \right\|$$

$$< \eta \left(\varepsilon_0 \right) + 6\varepsilon.$$

Hence $\left\| T_0 - T_1 \right\| \leqslant \eta(\varepsilon_0) + 6\varepsilon < \varepsilon_0$. $\qquad \square$

Conflict of Interests
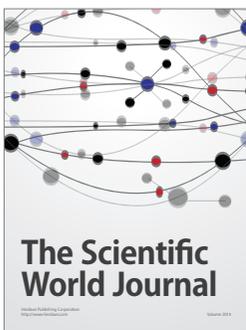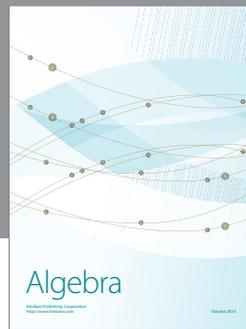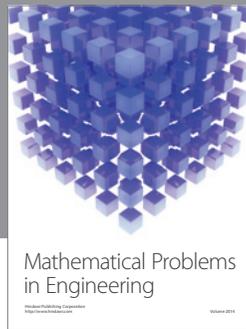
The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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