From Plurality Rule to Proportional Representation*

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Abstract

I consider the decision of a parliament that might change the electoral system for the forthcoming elections from plurality rule to proportional representation. Parties are office-motivated. They care about winning and about the share of seats obtained. I consider two different scenarios of how parties in the government share the spoils of office: Equally or proportionally to their share of seats. If the government is formed by a single party and parties expect that each party will obtain the same share of votes in the next election the electoral rule will never be changed. That is, for a change to occur the government should be formed by a coalition. I find that a change is more likely to occur when the number of parties is larger and also when the spoils of office are shared equally among the members in the governing coalition. I extend these results to analyze the decision of a change from a less proportional rule to a more proportional one.

KEYWORDS: Electoral systems, Plurality, Proportional Representation, Coalitions

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1 Introduction

For political parties whose main objective is to win the elections, that is, to be a part of the government, there are two main decisions that might be important in order to achieve this goal. The first one is their campaigning activities and the policy promises made to the electorate before the election, because those might affect the decision of voters at the time of casting a ballot. Secondly, parties might make strategic constitutional choices in order to increase their future benefits. This last question is of great importance as it is noted by Lijphart (1992) who states that "among the most important -and, arguably, the most important-of all constitutional choices that have to be made in democracies is the choice of electoral system, especially majoritarian methods vs proportional representation...". This paper aims to focus on decisions of political parties of this second type, that is, on their choice of the electoral rule. Naturally, a choice of this type might have different effects on the welfare of parties and voters. Therefore, the optimal choice of an electoral rule might be different for different objectives. This paper assumes that parties, when deciding whether to change the electoral rule, are only considering their own interests. That is, I assume that parties do not consider voters' evaluation of rules when they take their decision.

I analyze a model of electoral system change where the parliament might decide to change the electoral system for the forthcoming elections. A simple definition of an electoral system would be: "Given a set of votes, an electoral system determines the composition of the parliament (or assembly, council and so on)".[1] More broadly, as defined by Bogdanor (1991), an electoral system can be analyzed in three dimensions: (1) The method of calculating the votes, or in other words the "electoral formula" (Rae 1971) where plurality, majority and proportional representation are the main ones; (2) the district magnitude, that is, the number of representatives or parliamentarians elected in each district, and lastly (3) the degree of choice a voter can face. This paper focuses mainly on the first of these three aspects, the electoral formula. The structure of the model build up, also refers implicitly to the second aspect.

I consider a situation where the current electoral rule under which the parliament is shaped, is plurality rule. Under plurality rule, which is sometimes also denoted as First-Past-The-Post (FPTP) system the winner of the election is the candidate who obtains the highest amount of votes among all candidates. Plurality rule is generally used in single member districts although there are some exceptions as it is the case of the election of the Electoral College members in the US (Blais and Massicotte 2002) or in Mauritius where the legislators are elected from multimember districts (Lijphart 1999). Plurality rule is being used in countries such as the US, UK, Canada, India and other small ex-British colonies. One

[1] Gallagher and Mitchell (2005) p.3.

important characteristic of plurality rule is that it tends to lead to over-representation of larger parties and under-representation of smaller parties. Since in most of the cases in each district there is a single winner, a party has to come first in a district to win the seat in this district. Clearly, for a party who has less support it is much less probable that it comes first in a certain district compared to a party with much higher support. Thus, a small party that collects a significant amount of votes spread out among different districts, may obtain a very reduced (or zero) amount of seats. This is why plurality rule tends to under represent smaller parties. As a striking example consider the 1974 British election results[2]. The two larger parties, Labor and Conservatives, obtained 39.3% and 35.8% of the total votes respectively, whereas the third party, the Liberals, obtained 18.3% of votes. While Labor obtained 50.2% of the seats and the Conservatives 43.6% the Liberals only obtained 2% of the seats. That is, the largest party was overrepresented as it obtained more than half of the seats with a vote share of less than 40% whereas the third party faced a large degree of under-representation as it obtained only 2% of the seats with about one fifth of the total votes.

Although there exists a huge variety of alternative electoral systems, I assume that the parliament, given that the proposal for electoral system change reaches the required level of support from its members, might only decide on the switch from plurality rule to proportional representation rule. Indeed, Colomer (2005) counts 37 changes during the last century from plurality/majority rules to proportional/mixed rules among which are the changes occurred in Germany in 1918, Norway in 1919, New Zealand in 1993 and Japan in 1994.

Proportional Representation is used in multimember districts and as described by Taagepera and Shugart (1989) it "refers to electoral laws that use some mathematical formula for the allocation of seats to parties in approximation to vote shares". Currently, most of Western European countries use different types of Proportional Representation. Proportional Representation rules differ in their quotas, that is the amount of votes obtained by a party which would be worth a seat. One of the main proportional representation formulas is the Hamilton-Hare exact quota, where a vote share of $1/M$, where M is the district magnitude is worth one seat and the remaining seats are distributed according to the largest remainders, which is used in Denmark and Costa Rica (Norris 1997). Another formula is the Webster-Sainte Lagüe formula where the quota is $1/M+1$ and the remaining seats are given to parties who reach half of the quota. Another commonly used formula is the Jefferson-d'Hondt quota of series of divisors, where the quota for party i is $\frac{v_i}{s+1}$ where v_i is the vote share of party i and s is the amount of seats party i has been assigned so far. In each round, the party that has the highest quota at the moment is assigned the corresponding seat. Each of these formulas leads to a different degree of proportionality. Consider

[2] Source: http://www.election.demon.co.uk/

the following example taken from Colomer (2004) where there are four parties (W, X, Y, Z), the district magnitude is 6 and there are 100 voters. The votes obtained by each party are $W = 40$, $X = 30$, $Y = 20$ and $Z = 10$ respectively. The seat allocation for each party under these three rules are shown in the following table where it can be seen that the d'Hondt rule over represents the highest voted party since party W obtains half of the seats although it only obtains 40% of the votes:

		Seat Allocation			
		W	**X**	**Y**	**Z**
Hamilton-Hare	By Quota	2	1	1	0
Exact quota: 100/6 =16.6	By Largest Remainders		1		1
	Total	2	2	1	1
Jefferson-d'Hondt					
Sufficient quota	**Total**	3	2	1	0
Webster-Sainte-Laguë	By Quota	2	2	1	0
Modified quota: 100/6 +1 =14	By Half Quota (=7)				1
	Total	2	2	1	1

For the Hamilton-Hare rule the quota is $100/6 = 16.6$ as the district size is 6. Therefore, W gets 2 seats and X and Y one seat. The remaining two seats are assigned to the parties with the highest remainders which are X with a remainder of 13.4 (30-16.6) and Z with a remainder of 10. For the Webster-Sainte Lagüe formula the quota is $100/6 + 1 = 14$. So, W and X get 2 seats and Y one seat. Since Z's vote share reaches the half quota (=7) it is given the remaining seat. In the Jefferson-d'Hondt quota of series of divisors method the first seat is given to the largest party, W. Now the quota of W becomes 20. So, the second seat is given to the party with largest quota, namely X. Then, the quota of X becomes 15 and the next two seats are given to W and Y as they have the highest quota, namely 20. Then, the quota of W becomes $40/(2 + 1) = 13.3$ and the quota of Y becomes 10. So, the next seat is given to X as it has the highest quota and its new quota is $30/(2 + 1) = 10$. So, the last seat is given to W as it has the highest quota.

Another important aspect that contributes to the degree of disproportionality is the district size. As argued by Lijphart (1999), the degree of proportionality increases with the district size, i.e. the number of seats available in a district. "For instance a party representing a 10 percent minority is unlikely to win a seat in a five-member district but will be successful in a ten-member district"[3]. Empirically the district size varies from country to country and region to region.

In the following analysis, I assume that under proportional representation each party obtains a share of seats equal to the share of votes obtained in the elections. This could be considered as "ideal proportional

[3] Lijphart (1999), p.152.

representation" which might need very large district magnitudes and can be thought as an approximation to the case where the whole country constitutes a single electoral district electing a large enough number of representatives. An example of this case would be Israel and the Netherlands in which the whole country is a single electoral district with the size of 120 and 150 members respectively. In this model, I assume that by switching from plurality rule to proportional representation a country would also switch from single member districts to a single nationwide district.

For plurality rule there does not exist a formula which would give the share of seats of parties taking as input only the nationwide share of votes each party obtains without taking as input their share of votes in each district separately. The only possibility is to find a theoretical approximation of the seat/vote ratio of parties in a parliament under plurality. A widely known analytical tool used as an approximation for plurality rule in single member districts is the so-called "cube law" which was first proposed by Parker Smith, a British mathematician, in 1909. This law states that for the two major parties, the ratio of their share of seats obtained under plurality rule is approximately the cube of the ratio of the share of votes obtained. It was found that the "Cube Law" was fitting well to British election results at that time. This formula was extended by Qualter (1968) to include more than two parties and it was applied to Canadian election results between 1921 and 1965 where the fit of the "law" was satisfactory. In my analysis, I assume that the distribution of seats in the parliament will be in line with the prediction of the "cube law".

That is, for plurality I assume that "cube law" applies whereas for proportional representation I assume "ideal" proportionality. Later, I relax the assumption on the vote to seat to transformation (cube law) and I assume that the ratio of the share of seats of any two parties obtained under plurality rule is the m-th power of the ratio of their share of votes obtained where m is strictly bigger than one.

The aim of this paper is to show under which conditions the parliament of a country would decide to switch from plurality rule to proportional representation and more generally from an electoral rule that is not proportional to a more proportional one. A party would be in favor of a change in the electoral rule only if it would increase its future payoffs. It can be argued that a party might have two different kinds of incentives to favor a change in the electoral rule. First, the new electoral rule might increase that party's amount of representation in the parliament, that is, its share of seats. The representation of a party depends on the amount of votes it and its opponents obtain. Therefore, a party would accept or reject an electoral system change proposal in accordance with its expected share of votes in the next elections. Secondly, the new electoral rule might increase a party's probability of being a member of the new government that will be formed after the electoral system change. I assume that parties have lexicographic preferences on forming part of a government and on the share of seats they obtain. A party's principal aim is to form part of a government with the highest share of seats possible. If it cannot form part of a government its objective would be simply to obtain the highest share of seats possible. As stated before parties build their decisions upon their expected share of votes.

Clearly, the probability of forming part of a government does not only depend on the share of seats obtained but also on the underlying process of coalition formation. Therefore, I consider two different specifications on how the spoils of office will be distributed among coalition members. First, I assume that coalition members share the spoils of office equally and then I assume that they share them in a manner proportional to the share of seats of each coalition member.

An important aspect of this analysis is the rule used to decide on the change of the electoral system. I assume that a certain threshold of votes in favor of the change by parliamentarians is needed. I first consider the case where this threshold is the simple majority and then I generalize it to consider any threshold larger than absolute majority. In the real world different countries have different thresholds. For instance, in France up to 1985 the threshold to change the electoral rule was simple majority and in 1985 the Socialist government switched from two-round majority to PR as it was in its interest. One year later the right-wing coalition reestablished the previous rule (Tsebelis 1990). Hungary is an example of countries that have a threshold larger than the absolute majority where the threshold is a two-thirds majority (Benoit 2004). I do not consider the possibility that a popular referendum is needed, as it is the case in Ireland (Benoit 2004).

I first consider a political environment with only two parties where parties expect to obtain the same share of votes in the next election. The main result I obtain is that for any threshold equal or higher than simple majority, the electoral rule will never be changed. This is so because the larger party is in office and it is overrepresented. Therefore, it is never in the interest of the larger party to change the rule. Since, the threshold to change the rule is assumed to be at least the simple majority no change is possible. On the other hand, the opposition party would benefit from a change in the electoral system as it would increase its share of seats yet it is never able to get it done. If the assumption on the expectation of the future share of votes is relaxed, and parties are allowed to have any expectation then if the larger party's share of seats is larger then the threshold the electoral rule will be changed if and only if this party expects to lose the next election. If the share of seats of the larger party is smaller than the threshold the electoral rule will be changed if and only if both parties expect to lose the next election.

Then, I consider an electoral environment with three parties and assume that parties expect that each party obtains the same share of votes in the next election. If the government is formed by a single party, then the electoral rule is not changed as the party in power is always against the change. The smallest party is always in favor of a change, whereas the preferences of the second largest party depend on the share of votes of his opponents.

In a three party environment when the government is formed by a coalition and the spoils of office are shared equally, the electoral rule change will occur if and only it is in the interest of the second largest party whose preference depends also on the share of votes of his opponents. This result holds whether the second largest party forms part of the government or not. As in the case of a single party government, it is never in the interest of the largest party to change the rule. The smallest party, on the other hand, is always in favor of a change. If spoils of office are shared proportionally, however, a change never occurs as the two largest parties are always better-off under plurality.

Finally I consider the case of four parties and I assume that parties expect that each party obtains the same share of votes in the next election. As in the previous cases, when the government is formed by a single party the rule is not changed. The second largest party still plays a key role in some cases. If spoils of office are shared proportionally, for certain distributions of vote shares a change in the electoral system that is approved by a absolute majority leads to different possible coalition government candidates. If this is the case, the change occurs irrespective of whether the second largest party is underrepresented or not.

In the literature there exist at least two papers that share similar characteristics with my analysis. The first, set up by Benoit (2004) is a general model of electoral system change. He assumes that the parties' objective is to maximize their share of seats, and he gives some real-life examples of electoral system change and discusses some empirical implications of his model. My analysis could be considered as an application of this general model to some concrete situations which allows me to solve the model explicitly and obtain concrete results. Benoit's theory predicts that the electoral rule will be changed when a coalition of parties, who have sufficient power to change the rule, exists such that each party in this coalition would gain more seats under the new rule. My model, on the other hand, suggests that, the situation described by Benoit would not necessarily lead to a change if the probability of being part of the government for one of these parties is negatively affected. Moreover, the possibility of obtaining a higher share of seats is not a necessary condition for a party to favor a change.

Boix (1999) analyzes the electoral system change from plurality/majority rule to proportional representation as a result of the entry of new voters (assumed to be leftists) and a new party (socialist) at the turn of the 20th century in the Western world. He finds that a change in the electoral rule occurs if and only if it is in the interest of the ruling rightist parties. My model instead is intended to characterize the cases in which the decision of electoral system change will be taken without assuming any significant change in the underlying political situation of a country, that is, where there is no threat of new parties and where parties do not expect huge changes in their share of votes.

The outline of the paper is as follows. In the next section I build up the model. In Section 3, I describe some general characteristics and results of the model which will be used during the whole analysis. In Section 4, I analyze the model for two parties. In Section 5, I analyze the case of single party governments for more than two parties. In Section 6, I analyze the model for three parties and in Section 7 for four parties under coalition governments. In Section 8, I discuss some implications of the model and in Section 9, I analyze some extensions of the model. In the last section, I reach some general conclusions and describe the future research to be done with this model.

2 Model

There exist a number of parties, p. The set of parties is denoted by $P = \{1, 2, .., p\}$. Each party j has already competed in the past election and obtained a certain amount of votes v_j, where $v_j \in [0, 1], \forall j \in P$ and $\sum_{j=1}^{p} v_j = 1$ where $v_i \geq v_j$ for $i < j$. According to the share of votes obtained, the electoral rule determines the share of seats of each party in the parliament. The share of seats of party j when rule k is applied is denoted by $s_j^k \in [0, 1], \forall j \in P$ and $\sum_{j=1}^{p} s_j^k = 1$ where k represents the electoral rule being used, either plurality rule or proportional representation respectively. That is, $k \in \{PL, PR\}$.

The important aspect here is how votes are transformed into seats. Clearly, this transformation is a result of the electoral rule used. I assume first that, under plurality rule this transformation would be according to the "cube law". It states that if two parties (a and b) obtain vote shares of v_a and v_b respectively, then the ratio of parliamentary seats will be $\frac{s_a^{PL}}{s_b^{PL}} = (\frac{v_a}{v_b})^3$. This formula can be applied to more than two parties in the same manner. That is, for the case of p parties, $\frac{s_i^{PL}}{s_j^{PL}} = (\frac{v_i}{v_j})^3$, $\forall i, j \in P$. Then, I relax this assumption and assume that $\frac{s_i}{s_j} = (\frac{v_i}{v_j})^m$, $\forall i, j \in P$ where $m > 1$. When the electoral rule is switched form plurality rule to proportional representation, I assume that each party's share of seats is equal to its share of votes obtained in the election, that is, $s_j^{PR} = v_j$, $\forall j \in P$. As stated before, this transformation could be considered as the ideal proportional representation.

Moreover, I assume that parliamentary members are taking their decisions in line with the interests of their parties rather than their individualist interests of being reelected or of being a member of the future governments. That is, there exists full party discipline in the decision of electoral system change.

Parties care about being in government and the share of votes they obtain. The total amount of office spoils shared among the governing parties is fixed and equal to $\overline{U}$. As stated before I consider two different specifications z ($z \in \{1, 2\}$) on how office spoils are shared among parties. I denote the utility obtained by party j from forming part of a certain government under specification z as $U_{j,z}$. Under the first specification ($z = 1$), I assume that parties forming part of the government share the spoils of office equally, i.e. each party j forming part of the government receives $U_{j,1} = \overline{U}/|C|$ where C is the governing coalition. Under the second specification ($z = 2$), I assume that parties forming part of the government share the spoils of the office proportional to their share of votes i.e. a party i forming part of the government receives $U_{i,2} = \frac{s_i * \overline{U}}{\sum_{j \in C} s_j}$ where C is the governing coalition.

These two specifications described above might lead to more than one possible winning coalition. If there exists more than one possible winning coalition, I assume that each of these coalitions has equal probability of being formed. The utility party j obtains under specification z, $U_{j,z}$, takes a different value for every different winning coalition that contains party j. Thus I define a vector $\overrightarrow{U}_{j,z} \in \mathbb{R}^{2^p-1}$ where $2^p - 1$ is the total number of possible coalitions given p parties and leaving out the empty set. Notice that the vector $\overrightarrow{U}_{j,z}$ has a component for each possible coalition, but party j only obtains a positive pay-off for those winning coalitions of which it is a member. Therefore, only those components of the vector that correspond to a winning coalition of which party j is a member, will take a positive value, whereas the remaining components will take the value 0. Similarly, q_j denotes a vector of dimension $2^P - 1$ representing the probability of formation of a certain governing coalition that includes party j. Again, q_j has a component for each possible coalition, but only those components of the vector that correspond to the probability of formation of a winning coalition of which party j is a member will take a positive value, whereas the remaining components will take the value 0. Then, if we multiply these vectors ($(\overrightarrow{U}_{j,z})^T * q_j$) we obtain the expected utility from being in the government for party j.

Parties care not only about being in government but also about the share of votes they obtain. That is, the utility of party j obtained from its seats is $f(s_j)$ where $f(s_j)$ is assumed to be an increasing function in s_j. I assume that parties have lexicographic preferences. That is the primary objective of a party is to maximize its expected utility of forming part of the government. Thus, when comparing two different electoral rules, a party will first compare the expected utility it obtains from being in the government under each rule, and it will choose the rule from which it derives a higher expected utility. In case of indifference, the party will choose the rule that would provide it a larger share of seats. In other words, a party would prefer PR to PL iff 1) its expected utility of office spoils is higher under PR or 2) in case it expects the same utility of office spoils under both rules then it prefers PR to PL if its seat share is higher under PR.

Moreover, I assume that in order to form a government, a coalition must have more than half of the seats in the parliament, that is, I rule out the possibility of minority governments. I also assume that the parliament consists of an odd number of seats in order to avoid having to deal with ties.

Once a government is in power, the government, the opposition, or some members of the government together with some parties of the opposition might decide to change the electoral rule for the forthcoming elections. The threshold of share of votes in the parliament needed to change the electoral rule is denoted by T and I assume that $T \geq \frac{1}{2}$. During the whole analysis, if not otherwise stated, I assume that T is the simple majority. Moreover, I assume that if a party is indifferent between the two electoral rules, it always opposes the change.

Since parties are office-motivated and self-interested, the electoral rule will be changed only if it is in the interest of at least the absolute majority of the parliament in terms of their expected utilities. Parties' expected utilities depend on their expected share of votes. Therefore, first of all, parties should have expectations about their own and about their opponents' future share of votes. I denote party $i's$ expectations by the vector $v_i^e = (v_{i1}^e, v_{i2}^e, ...v_{ip}^e)$ where v_{ij}^e denotes the share of votes that party i expects party j would obtain and where $\sum_{j=1}^{p} v_{ij}^e = 1$ for all $i \in P$. First, I assume that parties expect that each party gets the same share of votes in the forthcoming election as they got in the last election, i.e. $v_i^e = (v_{1,}v_2, ...v_p)$ for any $i \in P$. Later, I relax this assumption and allow for different expectations over share of votes.

3 Preliminary Results

This section aims to describe some general characteristics of the model and to obtain some general results under plurality both under the "cube law" and also for any $1 < m$ in general which would hold for any number of parties. The first result is as follows:

Lemma 1: If $v_i > v_j$, then $s_i^{PL} > s_j^{PL}$.

Proof: If $v_i > v_j$ i.e. $\frac{v_i}{v_j} > 1$ we have that $\frac{s_i^{PL}}{s_j^{PL}} = (\frac{v_i}{v_j})^m > 1$. Therefore, if $v_i > v_j$ we have that $s_i^{PL} > s_j^{PL}$. #

That is, the cube law suggests that the share of votes are transformed monotonically into share of seats. Trivially, this result also holds for PR as we have $m = 1$. We can also define a party's share of seats under the plurality rule in terms of the share of votes of all parties. The result is as follows:

Lemma 2: Under plurality rule $s_i^{PL} = \frac{v_i^m}{\sum_{j=1}^{p} v_j^m}$ for any $i \in P$.

Proof: Since $\frac{s_j^{PL}}{s_P^{PL}} = (\frac{v_j}{v_P})^m$ for any $j \in P$, that is $s_j^{PL} = (\frac{v_j}{v_P})^m s_P^{PL}$ and $\sum_{j=1}^{p} s_j^{PL} = 1$, we have that $s_p^{PL}((\frac{v_1}{v_p})^m + (\frac{v_2}{v_p})^m + ... + (\frac{v_{p-1}}{v_p})^m + 1) = 1$ implying that $s_p^{PL} = \frac{1}{(\frac{v_1}{v_p})^m + (\frac{v_2}{v_p})^m + ... + (\frac{v_{p-1}}{v_p})^m + 1}$. So we have that $s_j^{PL} = \frac{v_j^m}{\sum_{j \in P} v_j^m}$ for all $j \in P$. #

So, under the cube law we would have $s_i^{PL} = \frac{v_i^3}{\sum_{j=1}^{p} v_j^3}$ for any $i \in P$. Notice that for PR we have that $s_i^{PR} = v_i$ for all $i \in P$. We can also find some general results on which party(ies) would be underrepresented or overrepresented. The result is as follows:

Lemma 3: i. Under plurality rule party 1 is always overrepresented and party p is always underrepresented.

ii. Under plurality rule, if party i is underrepresented, then any party j with $j > i$ and $i, j \in P$ is also underrepresented.

iii. Under plurality rule, if party i is overrepresents, then any party j with $j < i$ and $i, j \in P$ is also overrepresented.

Proof: i. We know that $\sum_{j=1}^{p} s_j^{PL} = \sum_{j=1}^{p} v_j = 1$. Suppose that $s_1^{PL} \leq v_1$, then since $\frac{s_1^{PL}}{s_2^{PL}} = (\frac{v_1}{v_2})^m > \frac{v_1}{v_2}$ we have $s_2^{PL} < v_2$. So, for a similar reasoning $s_3^{PL} < v_3$ and so on... Then, $\sum_{j=1}^{p} s_j^{PL} < \sum_{j=1}^{p} v_j$. Contradiction. Therefore, $s_1^{PL} > v_1$. Now, suppose that $s_p^{PL} \geq v_p$. Since $\frac{s_{p-1}^{PL}}{s_p^{PL}} = (\frac{v_{p-1}}{v_p})^m > \frac{v_{p-1}}{v_p}$, we have that $s_{p-1}^{PL} > v_2$. So, for a similar reasoning $s_{p-2}^{PL} > v_{p-2}$ and so on... Then, $\sum_{j=1}^{p} s_j^{PL} > \sum_{j=1}^{p} v_j$ Contradiction. Therefore, $s_p^{PL} < v_p$.

ii. Under plurality rule, for parties i and j with $j > i$ we have that $\frac{s_i^{PL}}{s_j^{PL}} = (\frac{v_i}{v_j})^m > \frac{v_i}{v_j}$. So, if $s_i^{PL} < v_i$ we must have $s_j^{PL} < v_j$.

iii. Under plurality rule, for parties i and j with $j < i$ we have that $\frac{s_i^{PL}}{s_j^{PL}} = (\frac{v_i}{v_j})^m < \frac{v_i}{v_j}$. So, if $s_i^{PL} > v_i$ we must have $s_j^{PL} > v_j$. #

That is, we have that, under plurality rule the largest party will always be overrepresented and the smallest party will always be underrepresented. If party i is underrepresented than all parties whose share of votes are smaller than party i's are also necessarily underrepresented. We can also describe when a party will be underrepresented under plurality rule in terms of its and its opponents' share of votes. The result is as follows:

Lemma 4: Under plurality rule party i is underrepresented if $\sum_{j=1}^{p} v_j^m > v_i^{m-1}$.

Proof: If party i is underrepresented, we have that $s_i^{PL} < v_i$. From, Lemma 2, this implies that $\frac{v_i^m}{\sum_{j=1}^{p} v_j^m} < v_i$ which can be rearranged as $\sum_{j=1}^{p} v_j^m > v_i^{m-1}$. #

So, under cube law, party i is underrepresented if $\sum_{j=1}^{p} v_j^3 > v_i^2$. Notice that, trivially, under PR each party is "ideally" represented.

4 Two party model

In this section I analyze the case where there are only two parties and the electoral rule currently used is plurality. The share of votes obtained of these two parties are v_1 and v_2 respectively. First, I assume that $v_i^e = v_i \ \forall i$. That is, each party expects to obtain the same share of votes in the next election. In this case, would it be in the interest of one or both parties to change the electoral rule and would they be able to implement it? The first result shows that the electoral rule will not change for any threshold larger than the absolute majority:

Proposition 1: If $v_i^e = v_i \ \forall i$, then for any $T \geq \frac{1}{2}$ the electoral rule will not be changed.

Proof: From Lemma 1 we know that if $v_1 > v_2$ we have that $s_1^{PL} > s_2^{PL}$. From Lemma 3 we know that the smallest party is underrepresented under plurality. So, we have $s_1^{PL} > v_1 > v_2 > s_2^{PL}$ and party 1 has the majority and will form the government. So, party 1 would be against electoral rule change since under PR, $s_1^{PR} = v_1 < s_1^{PL}$. Therefore, the electoral rule will never change. #

However, the next proposition shows that it would always be in the interest of the smaller party to change the electoral system.

Proposition 2: If $v_i^e = v_i \ \forall i$, the smaller party would always be in favor of switching from plurality rule to PR.

Proof: From Proposition 1 we know that $s_2^{PR} = v_2 > s_2^{PL}$. Therefore, the smaller party, party 2, would be in favor of an electoral rule change since it increases its share of votes. #

These results imply that, if parties expect to get the same share of votes in the forthcoming election, then the system would never change from plurality rule to proportional representation.

For all the results above, it was assumed that $v_i^e = v_i$ for all $i \in P$. How would these results be affected if we relax this assumption and let each party have any kind of expectation about their share of votes in the next election? Notice that, since only two parties are competing, the expectations of a party about its future share of votes necessarily imply its expectation about the share of votes of its opponent. The following proposition states the results:

Proposition 3: For any $T > \frac{1}{2}$, the electoral rule will switch from plurality if and only if either
(1) $s_1^{PL} > T$ and $v_1^e < \frac{1}{2}$ for any v_2^e or
(2) $s_1^{PL} < T$, $v_1^e < \frac{1}{2}$ and $v_2^e < \frac{1}{2}$

Proof: From Lemma 1 and Proposition 1, we know that if $v_1 > v_2$, party 1 would have the majority in parliament and $s_1^{PL} > v_1 > v_2 > s_2^{PL}$. So, the electoral can only be changed if it in the interest of party 1. If $s_1^{PL} > T$, then party 1 can change the electoral rule by its own. If $s_1^{PL} \leq T$ then both parties should be in favor of an electoral system change. If $v_1^e > \frac{1}{2}$, party 1 would expect to obtain the majority in the next elections which implies that it would expect to win the next elections and obtain more seats under plurality rule than under PR. So, it would be against a change. Therefore, for a change to occur party 1's expectation should be $v_1^e < \frac{1}{2}$, given that $v_1^e = \frac{1}{2}$, party 1 would be indifferent between both rules. So, if $s_1^{PL} > T$ and $v_1^e < \frac{1}{2}$, party 1 would change the rule alone for any expectation of the other party. What if $v_1^e < \frac{1}{2}$ but $s_1^{PL} < T$? Then, the electoral rule can be changed iff both parties support the change. For the same reasoning as for party 1, party 2 would favor a change if $v_2^e < \frac{1}{2}$. #

The proposition states that if the larger party's share of seats is smaller than the necessary threshold, the electoral rule will be changed only when both parties expect to lose the forthcoming election. However, it is very unlikely to occur that both parties expect to lose the election. If the share of seats of the larger party exceeds the threshold then it is enough that the larger party expects to lose the forthcoming election. This result implies that under some conditions there might occur changes in the electoral rule for two party competition which seems contradictory to the reality, as for example in the US, where two parties compete, the electoral rule has not been changed. However, the results are based on the assumption that parties can have any expectation about the future which is also not that realistic, as they would certainly depend also on previous results. Moreover, the model at hand assumes that the change of rule has no cost for parties in terms of loss of credibility or the effort to pass the change which are certainly factors that would make a change more difficult. As argued by Benoit (2004), a party that changes the electoral rule too frequently might be discredited due to the manipulation of the rules for its own interest.

In the analysis above I have not made an explicit assumption about the vote to seat transformation under plurality. As it can be seen the results hold for the "cube law" and any $m > 1$ in general.

5 Single party government

In this section I consider the case where the government is formed by a single party under plurality rule. I consider directly the more general case with $m > 1$ rather than the cube law and do the analysis for any threshold higher than the absolute majority. From Lemma 1, we know that if $v_i > v_j$ we have that $s_i^{PL} > s_j^{PL}$ since the votes are transformed into seats monotonically. Therefore we can obtain directly the following result which holds for any number of parties:

Proposition 4: If $v_i^e = (v_1, v_2, .., v_p)$ $\forall i$ and the government is formed by a single party then for any $T \geq \frac{1}{2}$ the electoral rule will not be changed.

Proof: If the government is formed by a single party (party 1), we have $s_1^{PL} > 0.5$. Since parties expect to get the same share of votes in the forthcoming elections, party 1 will be against any change as from Lemma 3, $s_1^{PL} > v_1 = s_1^{PR}$. Therefore, the opposition parties will never be able to change the system as their share of seats is necessarily smaller than the simple majority.#

That is, when the government is formed by a single party, for any degree of disproportionality and any threshold higher or equal to the absolute majority the electoral rule will not change. In the next two sections I will analyze more thoroughly the case of three and four parties. Knowing that, for a change to occur we need a coalition government it would be wise to describe when this would happen. The following proposition describes the case of three parties under the cube law:

Proposition 5: If the vote share of the leading party in the elections is smaller than 0.387, the government will be formed for sure by more than one party. If the vote share of the leading party is between 0.387 and 0.5 then the government is formed by a single party if and only if $v_1^3 > v_2^3 + v_3^3$.

Proof: If the share of votes of the three parties are $v_1 > v_2 > v_3$ under plurality we must have $s_1^k > s_2^k > s_3^k$ ($k \in \{PL, PR\}$). From Lemma 2, we know that $s_1^{PL} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3}$. To have a one party government, $s_1^{PL} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3} > \frac{1}{2}$ i.e. $v_1^3 > v_2^3 + v_3^3$ should be satisfied subject to the constraints $v_1 > v_2 > v_3$ and $v_1 + v_2 + v_3 = 1$. So, whenever this inequality holds, the largest party can form the government alone. It can be shown that the lower bound of v_1 satisfying the inequality above can be obtained when $v_2 = v_3$. In this case, $v_2 = v_3 = \frac{1-v_1}{2}$. Therefore, the inequality takes the form of $v_1^3 > \frac{(1-v_1)^3}{4}$ which would be satisfied if $v_1 > 0.386488$. #

That is, when the vote share of the largest party is above 0.387, which should be expected to happen quite often for three parties, it might well be the case that it gets more than half of the seats and forms the government by its own. Notice that, the higher v_1 (given that it is between 0.387 and 0.5) the more it is probable that the largest party would obtain the majority. For a given value of v_1 between 0.387 and 0.5, the closer v_2 and v_3 are, the higher the probability that the largest party would obtain the majority.

A similar analysis for the case of four parties is shown in the next proposition:

Proposition 6: If the vote share of the leading party in the elections is smaller than 0.3247, the government will be formed for sure by more than one party. If the vote share of the leading party is between 0.3247 and 0.5 then the government is formed by a single party if and only if $v_1^3 > v_2^3 + v_3^3 + v_4^3$.

Proof: If the share of votes of the four parties are $v_1 > v_2 > v_3 > v_4$ we know from Lemma 1 that we have $s_1^{PL} > s_2^{PL} > s_3^{PL} > s_4^{PL}$. From Lemma 2, we know that $s_1^{PL} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3+v_4^3}$. To have a one party government, $s_1^{PL} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3+v_4^3} > \frac{1}{2}$ i.e. $v_1^3 > v_2^3 + v_3^3 + v_4^3$ should be satisfied subject to the constraints $v_1 > v_2 > v_3 > v_4$ and $v_1 + v_2 + v_3 + v_4 = 1$. So, whenever this inequality holds, the largest party can form the government alone. It can be shown that the lower bound of v_1 satisfying the inequality above can be obtained when $v_2 = v_3 = v_4$. In this case, $v_2 = v_3 = v_4 = \frac{1-v_1}{3}$. Therefore, the inequality takes the form of $v_1^3 > \frac{(1-v_1)^3}{9}$ which would be satisfied if $v_1 > 0.3247$. #

Notice that, the higher v_1 (given that it is between 0.3247 and 0.5) the more it is probable that the largest party would obtain the majority. For a given value of v_1 between 0.3247 and 0.5, the closer v_2, v_3 and v_4 are, the higher the probability that the largest party would obtain the majority.

6 Three party model

In this section, having already discussed before that under a one party government there would be no change in the electoral rule I consider now a model of electoral system change for three parties where the government is formed by a coalition. As before, they compete under plurality rule, having the opportunity to switch to proportional representation. I assume that $v_i^e = (v_1, v_2, v_3)$ for all $i \in P$. I analyze the two specifications for the utility functions of parties separately. First, I consider the case where government parties share the utility obtained from forming part of the government equally.

6.1 Electoral System Change Under Equally Shared Spoils

If the spoils are shared equally, the government will be formed by the least number of parties possible. Therefore, $C_1(1,\ 2)$, $C_2(2,\ 3)$, $C_3(1,\ 3)$ would be the three candidates to form the coalition. If a change in the electoral rule occurs, this does not affect the candidates for forming the government. That is, before and after the change the set of coalitions that might form the government would be the same. Now we can proceed to analyze when a change would occur. The first thing one should notice is that under equally shared spoils if the government is formed before and after the change by the same parties, the utility they obtain from being part of the government will not be affected as it does not depend on the relative size of each party in the coalition. The following proposition shows that for three parties when the government is necessarily formed by a coalition and the spoils of office are shared equally, the electoral rule will be changed from plurality rule to proportional representation if and only if it is in the interest of the second largest party. It also shows that it is always in the interest of the smallest party to change the electoral rule whereas the largest party would always oppose to a change. The results are as follows:

Proposition 7: If $v_i^e = (v_1, v_2, v_3)$ $\forall i$, spoils of office are shared equally, $T = \frac{1}{2}$ and the government is formed by a coalition, then the electoral rule will be changed if and only if $v_1^3 + v_2^3 + v_3^3 > v_2^2$.

Proof: Given that no party can form the government alone ($v_1 < \frac{1}{2}$ and $s_1^{PL} < \frac{1}{2}$), any coalition of two parties would be a candidate to form government. Given that each of these coalitions can be formed with equal probability and the relative size of each coalition party does not affect the utility obtained from forming part of the government, a party will be in favor of an electoral rule change if and only if this change would increase its share of seats. From Lemma 3, $s_1^{PL} > v_1 = s_1^{PR}$ and $s_3^{PR} = v_3 > s_3^{PL}$. Therefore, the smallest party would favor a change and the largest party will be against. So, the electoral system change will occur if and only if the second largest party is in favor of it, i.e. iff $s_2^{PR} = v_2 > s_2^{PL}$. So, when does $s_2^{PR} = v_2 > s_2^{PL}$ hold? From Lemma 4, we need $v_1^3 + v_3^3 + v_2^3 > v_2^2$. #

The proposition states that if office spoils are shared equally the electoral rule is changed if and only if such a change would increase the share of seats of the two smallest parties. From the analysis above we can deduce that if the share of votes of the two smallest parties are equal then both of them will be underrepresented. So, the electoral rule will be changed for sure as the change would increase the share of seats of both parties. Since any coalition is assumed to be formed with equal probability, the change might be supported by both coalition partners or by one of the coalition partners and the opposition party.

An important point to notice is that the assumption on the distribution of office spoils leads to the same set of coalition candidates as the minimal winning coalition theory in number of parties suggested by Leiserson (1966) does. This theory suggests that the government would be formed by a minimal winning coalition with the minimum of number of parties in it. A winning coalition is a minimal winning coalition (MWC) if the defection of any of his members (in this model parties) turns the coalition into a loosing one. This theory is based on the bargaining proposition which states that the smaller the number of parties in a coalition the easier they will find it to reach an agreement.

How would the results change if we would consider a threshold higher than the absolute majority and if we would assume that $\frac{s_i}{s_j} = (\frac{v_i}{v_j})^m \; \forall i, j \in P$ with $3 \geq m > 1$? That is, what would happen for a higher range of thresholds and a generalization of the cube law? The result is as follows:

Proposition 8: If $v_i^e = (v_1, v_2, v_3) \; \forall i$, spoils of office are shared equally and $\frac{1}{2} \leq T < \frac{2}{3}$, the electoral rule will be changed iff $v_1^m + v_2^m + v_3^m > v_2^{m-1}$ and $s_2^{PL} + s_3^{PL} \geq T$. If $T \geq \frac{2}{3}$ the electoral rule will never be changed.

Proof: From Proposition 7 we know that the electoral rule will be changed iff it is in the interest of the second largest party, that is when it is underrepresented. From Lemma 4 we know that this holds iff $v_1^m + v_2^m + v_3^m > v_2^{m-1}$. Since, party 1 is always against a change and party 3 is always in favor $s_2^{PL} + s_3^{PL} \geq T$ should be satisfied for a change. However, since $s_1^{PL} > \frac{1}{3}$ for sure, we have that $s_2^{PL} + s_3^{PL} < \frac{2}{3}$. So, if $T \geq \frac{2}{3}$ the rule will never be changed as party 2 and party 3 won't have enough votes to pass the change. #

The proposition states that if $T \geq \frac{2}{3}$, the electoral rule will never be changed as the sum of share of seats of the two smallest parties will necessarily be smaller than the threshold. On the other hand, if the threshold is between $\frac{1}{2}$ and $\frac{2}{3}$, then the electoral rule will be changed if and only if the second largest party is underrepresented and the sum of share of votes of the two smallest parties is larger than the necessary threshold. So, as before, a change occurs if and only if it is in the interest of the second largest party.

The minimal winning coalition theory in number of parties is a refinement of the more general minimal winning coalition theory (MWC) proposed by von Neumann and Morgenstern (1953). However, for three parties the predictions of the two theories coincide. Therefore, if we were to do the above analysis under MWC theory we would reach exactly to the same results.

6.2 Electoral System Change Under Proportionally Shared Spoils

If the spoils of office are shared proportionally to the share of seats of the coalition members, the government would be formed by the coalition of the smallest total weight of seats possible. Therefore, we would have a unique candidate, namely, $C_1(2, 3)$. So, even when a change occurs, the government will still be formed by those two parties. Under this specification the utility obtained by a party of forming part of the government may not be the same under both rules as the relative size of each party in the coalition determines the utility obtained by each coalition party and the change in the electoral rule might change the relative size of coalition partners. So, assuming that the government is formed by a coalition, when would a change occur? I consider directly the more general case for any $3 \geq m > 1$ and any $T \geq \frac{1}{2}$. The result is as follows:

Proposition 9: If $v_i^e = (v_1, v_2, v_3)$ $\forall i$, spoils of office are shared proportionally and the government is formed by a coalition, then for any $T \geq \frac{1}{2}$ the electoral rule will never change.

Proof: Under this specification the coalition is formed by party 2 and party 3. Party 1 would be against a change as it is overrepresented. What happens with party 2? Under plurality its utility from forming part of the government is $\frac{s_2^{PL}}{s_2^{PL}+s_3^{PL}}\overline{U} = \frac{v_2^m}{v_2^m+v_3^m}\overline{U}$ and under PR it is $\frac{v_2}{v_2+v_3}\overline{U}$. $\frac{v_2^m}{v_2^m+v_3^m} > \frac{v_2}{v_2+v_3}$. Why? If it is true than, $\frac{v_2^m}{v_2^m+v_3^m} - \frac{v_2}{v_2+v_3} > 0$ which is the same as $v_2^m v_3 - v_3^m v_2 > 0$ i.e. $v_2 v_3 (v_2^{m-1} - v_3^{m-1}) > 0$ which always holds as $v_2 > v_3$. So, party 2 is always against a change whereas party 3 is always in favor. Therefore a change never occurs. #

That is, under this specification a change in the electoral rule never occurs for any threshold larger than the absolute majority. The largest and second largest party would be against a change whereas the smallest party would be in favor. On the other hand if we would consider the situation where the share of votes of the two smallest parties are equal than a change occurs for sure as long as $T \leq \frac{2}{3}$ as both parties' utility of forming part of the government will be the same under both rules and from Lemma 3, both will be underrepresented.

The assumption on the proportional distribution of office spoils points the same coalition candidate as the minimum winning coalition theory proposed by Riker (1962) does. This theory suggests that the government would be formed by a MWC which has the smallest total weight of seats. Using the assumption that each party expects to receive a larger share of the payoff the greater the weight it brings to a winning coalition, Riker predicted that minimum winning coalitions would form as they maximize the expectations of each coalition member. Generically, as in the case of three parties, this theory predicts a unique coalition. So, the results in this section apply in fact for Riker's theory.

7 Four Party Model

In this section, I consider a model of electoral system change with four parties. I assume that $v_i^e = (v_1, v_2, v_3, v_4)$ for all i. From the previous analysis, we know that if the government is formed by a single party the electoral rule will not change as all parties expect to obtain the same share of votes in the next elections. Therefore, the electoral rule can be changed only if the government is formed by a coalition. As in the case of three parties, the two specifications on the distribution of office spoils will be analyzed separately. Before doing so, I obtain some results that would hold under both specifications:

Lemma 5: i. Under both plurality rule and PR, $C_1(3, 4)$ and $C_2(2, 4)$ can never be winning coalitions.

ii. Under both rules, $C_1(1, 4)$ and $C_2(2, 3)$ cannot be winning coalitions at the same time.

Proof: i. If party 3 and party 4 form a winning coalition, then for $k \in \{PL, PR\}$, $s_3^k + s_4^k > \frac{1}{2}$ which implies that $s_1^k + s_2^k < \frac{1}{2}$. Contradiction, as from Lemma 1 $s_1^k > s_3^k$ and $s_2^k > s_4^k$. A similar reasoning applies for $C_2(2, 4)$.

ii. Suppose that party 1 and party 4 form a winning coalition. Then $s_1^k + s_4^k > \frac{1}{2}$ for $k \in \{PR, PL\}$. So, $s_2^k + s_3^k < \frac{1}{2}$ which means that party 2 and party 3 cannot form a winning coalition. #

From parts *i.* and *ii.* we can deduce that party 1 and party 2 or party 1 and party 3 can always form a winning coalition except the case with $v_1 = v_2 = v_3 = v_4$ which is ruled out as the number of seats in the parliament is assumed to be an odd number. The reasoning in *ii.* implies that both $C_1(1, 4)$ and $C_2(2, 3)$ will not be a winning coalition iff $s_1 + s_4 = s_2 + s_3 = \frac{1}{2}$ which is also ruled out by the assumption stated before. Notice that, we know from Lemma 3 that party 2 and party 3 might be under or overrepresented depending on the share of votes of all parties.

Having found some common properties that would hold under both specifications, in the next section I analyze when an electoral system change occurs with equally shared office spoils.

7.1 Electoral System Change Under Equally Shared Spoils

In this section I consider the case where office spoils are shared equally and no party obtains the absolute majority. First we have to find which coalitions would be possible under this specification. Disregarding the case where $v_1 = v_2 = v_3 = v_4$ and given that the parliament consists of an odd number of seats, we would have the following cases:

Lemma 6: For four parties, when no party obtains the absolute majority alone and office spoils are shared equally the following possible coalitions might be formed ($k \in \{PR, PL\}$):

Case 1: If $s_1^k + s_4^k < \frac{1}{2}$ then $C_1(1, 2)$, $C_2(2, 3)$, $C_3(1, 3)$

Case 2: If $s_1^k + s_4^k > \frac{1}{2}$ then $C_1(1, 2)$, $C_2(1, 3)$, $C_3(1, 4)$

Proof: First of all, we can discard coalitions of three parties as the spoils of office are shared equally. So, we have to find the possible coalitions of two parties. If $s_1^k + s_4^k < \frac{1}{2}$, then $s_2^k + s_3^k > \frac{1}{2}$ so party 2 and party 3 form a winning coalition. Since $s_1^k \geq s_2^k \geq s_3^k$, $C_1(1, 2)$ and $C_3(1, 3)$ would also be winning coalitions. Notice that there exists no other winning coalition of two parties in this case. If $s_1^k + s_4^k > \frac{1}{2}$, then instead of party 2 and party 3, party 1 and party 4 would form a winning coalition. The other two coalitions could still be formed for a similar reasoning as before. #

Compared to the case of three parties, different coalitions might occur for different distributions of share of votes. Moreover, the change of the electoral rule might lead to a change in the possible coalitions. For instance, under plurality rule we might be in Case 1 whereas with the same share of votes under proportional representation we might reach Case 2. So, to find the conditions under which the electoral rule might be changed, first we should check under which of the four cases a change would be feasible considering all possibilities in terms of a switch from one case to another and the possibility of staying in the same case. The analysis is as follows where it is assumed that the electoral rule might be changed from plurality to PR and no party obtains an absolute majority under plurality[4]:

1) *From Case 1 to Case 2:* Party 1 and party 4 would be in favor of a change as their probability of being part of the government would increase but party 2 and party 3 would be against as their probability of being part of the government would decrease. In Case 1, $s_1^{PL} + s_4^{PL} < \frac{1}{2}$, so no change occurs.

2) *From Case 2 to Case 1:* Party 1 and party 4 would be against a change as their probability of being part of the government would decrease. So, no change occurs as $s_1^{PL} + s_4^{PL} > \frac{1}{2}$.

3) *From Case 1 to Case 1:* The probabilities of being part of a government will not be affected. From Lemma 3, we know that party 1 would be against and party 4 in favor of a change. So, if both party 2 and party 3 are underrepresented under plurality a change will occur.

4) *From Case 2 to Case 2:* For the same reasoning as in the previous case iff both party 2 and party 3 are underrepresented under plurality a change will occur.

From the analysis above we obtain that the electoral rule will be changed in only two of the four possible cases. In both cases the change does not affect the composition of the government. Notice that different than the case for two or three parties, it might well be the case that the largest party would be in favor of a change in the electoral rule. Yet we have still to find the exact conditions for these two cases under which the electoral will be changed. The result is as follows:

Proposition 10: If $v_i^e = (v_1, v_2, v_3, v_4)$ $\forall i$, $T = \frac{1}{2}$ and spoils of office are shared equally, then the electoral rule will be changed if and only if:

i. $s_2^{PL} + s_3^{PL} > \frac{1}{2}$, and $\sum_{j=1}^{4} v_j^3 > v_2^2$ or

[4]This implies that no party obtains the absolute majority under PR neither. Notice also that it was assumed that each possible coalition under a specific case has the same probability of occuring.

ii. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$ and $\sum_{j=1}^{4} v_j^3 > v_2^2$

Proof: We had obtained before that the electoral rule will be changed iff this change does not affect the future candidates of coalitions. Therefore, for a change $s_2^{PL} + s_3^{PL} > \frac{1}{2}$, or $s_2^{PL} + s_3^{PL} < \frac{1}{2}$ and $v_2 + v_3 < \frac{1}{2}$ is needed. Moreover, both party 2 and party 3 should be in favor of the change, i.e. they should be underrepresented. From Lemma 3, we know that if party 2 is underrepresented, then also party 3 is underrepresented. So, it is sufficient to have $s_2^{PL} = \frac{v_2^3}{v_1^3+v_2^3+v_3^3+v_4^3} < v_2$ i.e. $\sum_{j=1}^{4} v_j^3 > v_2^2$. If both parties are underrepresented we have in the first case $v_2 + v_3 > s_2^{PL} + s_3^{PL} > \frac{1}{2}$. #

The results suggest that a change in the electoral rule occurs only if this change does not affect the set of possible coalitions. However, this is not a sufficient condition. As in the case of three parties, the second largest party should be in favor of a change, that is, it should be underrepresented. If this happens, then all parties except the largest one will be in favor of the change. As it can be seen from the possible coalitions, it might well be the case that all parties in the government (for example if the government is formed by party 2 and party 3 in Case 1), or only the smaller coalition partner (for example if the government is formed by party 1 and party 2 in Case 2) would be in favor of the change. The following two examples report possible share of votes for which an electoral system change would occur for both cases.

Example 1: Suppose that $v_1 = 0.34$, $v_2 = 0.28$, $v_3 = 0.27$, and $v_4 = 0.11$. Under plurality rule we would have: $s_1^{PL} = 0.478$, $s_2^{PL} = 0.267$, $s_3^{PL} = 0.239$ and $s_4^{PL} = 0.016$. We would be in Case 1, as $s_2^{PL} + s_3^{PL} > \frac{1}{2}$ and party 2 would be underrepresented as $s_2^{PL} < v_2$. If the rule were changed we would still be in Case 1 as $v_2 + v_3 > \frac{1}{2}$. So, for the given share of votes, the electoral rule would be changed.

Example 2: Suppose that $v_1 = 0.32$, $v_2 = 0.25$, $v_3 = 0.24$, and $v_4 = 0.19$. Under plurality rule we would have: $s_1^{PL} = 0.475$, $s_2^{PL} = 0.226$, $s_3^{PL} = 0.2$ and $s_4^{PL} = 0.099$. We would be in Case 2, as $s_2^{PL} + s_3^{PL} < \frac{1}{2}$ and party 2 would be underrepresented as $s_2^{PL} < v_2$. If the rule were changed we would still be in Case 2 as $v_2 + v_3 < \frac{1}{2}$. So, for the given share of votes, the electoral rule would be changed.

In the analysis above it was assumed that T is the absolute majority and that the vote to seat transformation was according to the cube law. How would the results be affected if we consider higher thresholds and allow for any $3 > m > 1$? First of all, notice that Lemma 6 still holds, as it does not depend on the vote to seat transformation, that is, the coalition candidates are the same. Moreover, the analysis for the possible switches form one case to another is not affected neither as it does not depend on m as long as $m > 1$. Taking this fact into account, the result is as follows:

Proposition 11: If $v_i^e = (v_1, v_2, v_3, v_4)$ $\forall i$, $\frac{s_i^{PL}}{s_j^{PL}} = (\frac{v_i}{v_j})^m$, $\forall i, j \in P$ where $3 > m > 1$ and $T \geq \frac{3}{4}$ the electoral rule will never be changed. If $\frac{1}{2} \leq T < \frac{3}{4}$ the electoral rule will be changed iff:

i. $s_2^{PL} + s_3^{PL} > \frac{1}{2}$, $\sum_{j=1}^{4} v_j^m > v_2^{m-1}$ and $s_2^{PL} + s_3^{PL} + s_4^{PL} > T$ or

ii. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $\sum_{j=1}^{4} v_j^m > v_2^{m-1}$ and $s_2^{PL} + s_3^{PL} + s_4^{PL} > T$

Proof: From Proposition 10, we know that the rule will be changed iff it is in the interest of party 2. So, we have only to change the cubes with m. In that case, the change is supported by parties 2, 3 and 4. So, a change occurs iff $s_2^{PL} + s_3^{PL} + s_4^{PL} > T$ plus the conditions from Proposition 10 changed with m's. $s_1^{PL} > 0.25$ for sure, so $s_2^{PL} + s_3^{PL} + s_4^{PL} < 0.75$. Therefore, if $T \geq \frac{3}{4}$, a change never occurs. #

It can be found different distributions of vote shares such that a change occurs for any $\frac{1}{2} \leq T < \frac{3}{4}$. However, as T becomes closer to $\frac{3}{4}$ a change occurs only if the share of votes of all four parties are very close to each other. Given that $\frac{1}{2} \leq T < \frac{3}{4}$, as before, a change occurs if and only if it is 1) in the interest of the second largest party and 2) the change does not affect the coalition candidates.

Notice once again that the specification of utilities considered here lead to the same coalition candidates as MWC in number of parties. If we would simply consider MWC, then the only difference we would have in Lemma 6 would be an additional coalition candidate ($C(2, 3, 4)$) in Case 2. However, it can easily be shown that this additional candidate does not change the analysis in this section. That is, the implications of this model under MWC for four parties would be the same as those for equally shared office spoils.

7.2 Electoral System Change Under Proportionally Shared Spoils

In this section I consider the case where office spoils are shared proportionally and no party obtains the absolute majority. First we have to find which coalitions would be possible under this specification. Notice that as the office spoils are shared proportionally only the winning coalition(s) with the lowest total share will form. Disregarding the case where $v_1 = v_2 = v_3 = v_4$ and given that the parliament consists of an odd number of seats, we would have the following cases:

Lemma 7: For four parties, when no party obtains the absolute majority alone and spoils of office are shared proportionally the following coalitions might be formed ($k \in \{PR, PL\}$):

Case 1: If $s_1^k + s_4^k < \frac{1}{2}$ then $C_1(2, 3)$

Case 2: If $s_1^k + s_4^k > \frac{1}{2}$ then i) $C_1(1, 4)$ if $s_1^k < s_2^k + s_3^k$, ii) $C_1(2, 3, 4)$ if $s_1^k > s_2^k + s_3^k$ and iii) $C_1(1, 4)$, $C_2(2, 3, 4)$ if $s_1^k = s_2^k + s_3^k$

Proof: *Case 1:* If $s_1^k + s_4^k < \frac{1}{2}$, then $s_2^k + s_3^k > \frac{1}{2}$. So $C_1(2, 3)$ is a winning coalition. $C_2(p_1, p_2)$ and $C_3(p_1, p_3)$ would also be winning coalitions but with a higher total share. All winning coalitions with three or four parties will also have a higher total share than $C_1(2, 3)$. So, only $C_1(2, 3)$ might be formed. *Case 2:* If $s_1^k + s_4^k > \frac{1}{2}$, then $C_1(1, 4)$ is a winning coalition. We can immediately eliminate $C(1, 2)$ and $C(1, 3)$, the four party coalition and the three party coalitions including parties 1 and 4 as $s_2^k > s_3^k > s_4^k$. The only remaining candidate is $C_2(2, 3, 4)$. From these two candidates the smaller in terms of number of seats is $C(1, 4)$ if $s_1^k + s_4^k < s_2^k + s_3^k + s_4^k$ i.e. if $s_1^k < s_2^k + s_3^k$. So, if $s_1^k > s_2^k + s_3^k$ we have as only candidate $C(2, 3, 4)$ and if $s_1^k = s_2^k + s_3^k$ we have both. #

Notice that, Case 2 has now three subcases depending on the share of seats of the three largest parties. Now, as in the previous cases we should consider all possible movements from one case to another given that the rule might change from plurality to PR where it should be taken into account not only the change of the probability of winning and the degree of representation of a party but also the relative size of a party in the government which depends on the electoral rule as this affects the utility obtained from forming part of the government. From an analysis as in the proof of Proposition 9 we can obtain the following result that will be used throughout the section:

Lemma 8: i. If parties i and j with $i < j$ form the government under both rules, than party i obtains a higher utility of forming part of the government under plurality than under PR.

ii. If parties i, j and k with $i < j < k$ form the government under both rules, than party i obtains a higher utility of forming part of the government under plurality than under PR.

Proof: i. Under plurality party i's utility from forming part of the government is $\frac{s_i^{PL}}{s_i^{PL}+s_j^{PL}}\overline{U} = \frac{v_i^3}{v_i^3+v_j^3}\overline{U}$ and under PR it is $\frac{v_i}{v_i+v_j}\overline{U}$. $\frac{v_i^3}{v_i^3+v_j^3} > \frac{v_i}{v_i+v_j}$. Why? If it is true than, $\frac{v_i^3}{v_i^3+v_j^3} - \frac{v_i}{v_i+v_j} > 0$ which holds if $v_i^3 v_j - v_j^3 v_i > 0$ i.e. if $v_i v_j(v_i^2 - v_j^2) > 0$ which always holds as $v_i > v_j$. So, party i is better-off under plurality rule. **ii.** Under plurality party i's utility from forming part of the government is $\frac{s_i^{PL}}{s_i^{PL}+s_j^{PL}+s_k^{PL}}\overline{U} = \frac{v_i^3}{v_i^3+v_j^3+v_3^3}\overline{U}$ and under PR it is $\frac{v_i}{v_i+v_j+v_k}\overline{U}$. $\frac{v_i^3}{v_i^3+v_j^3+v_3^3} > \frac{v_i}{v_i+v_j+v_k}$. Why? If it is true than, $\frac{v_i^3}{v_i^3+v_j^3+v_3^3} - \frac{v_i}{v_i+v_j+v_k} > 0$ which holds if $v_i^3 v_j + v_i^3 v_k - v_j^3 v_i - v_k^3 v_i > 0$ i.e. if $v_i v_j(v_i^2 - v_j^2) + v_i v_k(v_i^2 - v_k^2) > 0$ which always holds as $v_i > v_j$ and $v_i > v_k$. So, party i is better-off under plurality rule. #

Notice that the first part of the Lemma implies that the smaller party would be better-off under PR and the second part implies that for a coalition of three parties the smallest party will be better off under PR. Under Lemma 7 and 8, the analysis of a possible change is as follows:

1) *From Case 1 to Case 1:* The probabilities of being part of a government will not be affected. From Lemma 3, we know that party 1 would be against and party 4 in favor of a change and from Lemma 8 we know that party 2 is also against. So, no change occurs.

2) *From Case 2 to Case 1:* In all three subcases of Case 2, party 4 forms part of the government but under Case 1 not. So, party 4 would be against a change. Party 1 would also be against a change since its share of seats would decrease under PR. In Case 2, $s_1^{PL} + s_4^{PL} > \frac{1}{2}$. So, no change occurs.

3) *From Case 1 to Case 2i:* Parties 2 and 3 would be against a change as their probability of being part of the government would decrease. So, no change occurs as $s_2^{PL} + s_3^{PL} > \frac{1}{2}$.

4) *From Case 1 to Case 2ii or 2iii:* Is not possible. Why? If under plurality we are in 1, we have $s_1^{PL} < s_2^{PL} + s_3^{PL}$ which implies that $\frac{v_2^3+v_3^3}{v_1^3+v_2^3+v_3^3+v_4^3} > \frac{v_1^3}{v_1^3+v_2^3+v_3^3+v_4^3}$ i.e. $v_2^3 + v_3^3 > v_1^3$. If after the change we should reach 2ii or 2iii, we need $v_1 \geq v_2 + v_3$ i.e. $v_1^3 \geq (v_2 + v_3)^3$. For $v_2, v_3 > 0$, $(v_2 + v_3)^3 > v_2^3 + v_3^3$. So, we have a contradiction.

5) *From Case 2i to Case 2i:* The probabilities of being part of a government will not be affected. From Lemma 8 we know that party 1 would be against and party 4 would be in favor. Parties 2 and 3 are in favor if they are underrepresented. Therefore, from Lemma 3 for a change to occur party 2 should be underrepresented.

6) *From Case 2i to Case 2ii:* Is not possible. Why? If under plurality we are in 2i, we have $s_1^{PL} < s_2^{PL} + s_3^{PL}$ and to reach 2ii. we need $v_1 > v_2 + v_3$ i.e. $v_1^3 > (v_2 + v_3)^3$. So, the argument is the same as in 4).

7) *From Case 2i to Case 2iii:* For the same reasoning as in the previous case, it is not possible.

8) *From Case 2ii to Case 2i:* Party 1 would be in favor of a change as after the change it would form part of the government and party 2 and party 3 would be against as they would be out of the government. What happens with party 4? Assuming the change occurs, under plurality its utility from forming part of the government is $\frac{s_2^{PL}}{s_2^{PL}+s_3^{PL}+s_4^{PL}}\overline{U} = \frac{v_4^3}{v_2^3+v_3^3+v_4^3}\overline{U}$ and under PR it is $\frac{v_4}{v_1+v_4}\overline{U}$. $\frac{v_4^3}{v_2^3+v_3^3+v_4^3} < \frac{v_4}{v_1+v_4}$. Why? If it is true than, $\frac{v_4}{v_1+v_4} - \frac{v_4^3}{v_2^3+v_3^3+v_4^3} > 0$ which is the same as saying $v_2^3 v_4 + v_3^3 v_4 - v_1^3 v_4 > 0$ i.e. $v_4(v_2^3 + v_3^3 - v_1 v_4^2) > 0$. So, we need $v_2^3 + v_3^3 - v_1 v_4^2 > 0$. In Case 2i. we should have $v_1 < v_2 + v_3$. So, $v_2^3 + v_3^3 - v_1 v_4^2 > v_2^3 + v_3^3 - (v_2 + v_3)v_4^2$ and it can easily be seen that $v_2^3 + v_3^3 - (v_2 + v_3)v_4^2 > 0$. So, party 4 is in favor of a change and the change occurs as $s_1^{PL} + s_4^{PL} > \frac{1}{2}$.

9) *From Case 2ii to Case 2iii:* Is not possible. Why? We should have $v_2 + v_3 = v_1$ and $s_1^k < 0.5$ $k \in \{PR, PL\}$. We know that $s_1^{PL} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3+v_4^3}$. Suppose that $v_2 + v_3 = v_1$. So, we have that $v_4 = 1 - 2v_1$. So, $s_1^{PL} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3+(1-2v_1)^3}$. For $v_2, v_3, v_4 > 0$, $v_2^3 + v_3^3 + (1 - 2v_1)^3$ (given that $v_2 + v_3 = v_1$) would be maximum if $v_3 = 1 - 2v_1$ (the smallest value possible). So, s_1^{PL} would be minimum if $s_1^{PL} = \frac{v_1^3}{v_1^3+(3v_1-1)^3+(1-2v_1)^3+(1-2v_1)^3}$ which can be simplified as $s_1^{PL} = \frac{v_1^3}{1-3v_1-3v_1^2+12v_1{}^3}$. Moreover, since $v_2 \geq v_3$, we need $3v_1 - 1 \geq 1 - 2v_1$ i.e. $v_1 \geq 0.4$. So, if we minimize $s_1^{PL} = \frac{v_1^3}{1-3v_1-3v_1^2+12v_1{}^3}$ with respect to $0.5 > v_1 \geq 0.4$ we obtain that $s_1^{PL} < 0.5$ can never hold as the minimum it can obtain is 0.5.

10) *From Case 2ii to Case 2ii:* Is not possible. Given that the change in 9) is not possible, this change is not possible neither because it requires a even stronger condition, namely, $v_1 > v_2 + v_3$.

11) *From Case 2iii to Case 2i:* Party 2 and party 3 would be against as they would be out of the government after the change. Party 1 would be in favor iff his utility of being part of the government is higher after the change. Assuming the change occurs, under plurality its utility from forming part of the government is $\frac{1}{2}\frac{s_1^{PL}}{s_1^{PL}+s_4^{PL}}\overline{U} = \frac{1}{2}\frac{v_1^3}{v_1^3+v_4^3}\overline{U}$ and under PR it is $\frac{v_1}{v_1+v_4}\overline{U}$. $\frac{1}{2}\frac{v_1^3}{v_1^3+v_4^3} < \frac{v_1}{v_1+v_4}$. Why? If it is true then: $\frac{v_1}{v_1+v_4} - \frac{1}{2}\frac{v_1^3}{v_1^3+v_4^3} > 0$ which would hold if $v_1^4 + 2v_4^3 v_1 - v_1^3 v_4 > 0$. This inequality always holds as $v_1 > v_4$. So, party 1 is always in favor. From Lemma 8, party 4 would be in favor[5]. So, the change occurs as $s_1^{PL} + s_4^{PL} > \frac{1}{2}$.

12) *From Case 2iii to Case 2ii:* Is not possible. Why? If under plurality we are in 2iii, we have $s_1^{PL} = s_2^{PL} + s_3^{PL}$ which implies that $\frac{v_2^3+v_3^3}{v_1^3+v_2^3+v_3^3+v_4^3} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3+v_4^3}$ i.e. $v_2^3 + v_3^3 = v_1^3$. If after the change we should reach 2ii. we need $v_1 > v_2 + v_3$ i.e. $v_1^3 > (v_2 + v_3)^3$. For $v_2, v_3 > 0$, $(v_2 + v_3)^3 > v_2^3 + v_3^3$. So, we would have $v_2^3 + v_3^3 > (v_2 + v_3)^3$. Contradiction.

13) *From Case 2iii to Case 2iii:* It is not possible to reach the same situation under both rules. Why? Suppose we can reach it. Then we would have both $s_2^{PL} + s_3^{PL} = s_1^{PL}$ (which implies that $\frac{v_2^3+v_3^3}{v_1^3+v_2^3+v_3^3+v_4^3} = \frac{v_1^3}{v_1^3+v_2^3+v_3^3+v_4^3}$ i.e. $v_1^3 = v_2^3 + v_3^3$) and $v_2 + v_3 = v_1$. So, we should have $(v_2 + v_3)^3 = v_2^3 + v_3^3$ which never holds for $v_2, v_3 > 0$.

From Case 2i to Case 2i, where the change of rule does not affect the possible coalitions, for a change to occur the same conditions as for equally shared spoils are necessary. However, we would need as an additional condition that $s_1^k < s_2^k + s_3^k$ should also hold under both rules. Different than for the case of equally shared spoils, now we might have cases where a change in the electoral rule alters not only the degree of representation of each party but also the coalition that forms the government. A change of that type occurs if we have a switch from Case 2ii or Case 2iii to Case 2i. As in the case of three parties the change need not necessarily be supported by all parties that form the government. Formally, the result is as follows:

Proposition 12: If $v_i^e = (v_1, v_2, v_3, v_4)\ \forall i$, $T = \frac{1}{2}$ and spoils of office are shared proportionally, then the electoral rule will be changed if and only if:

i. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $s_1^k < s_2^k + s_3^k$ ($k \in \{PR, PL\}$) and $\sum_{j=1}^{4} v_j^3 > v_2^2$ or

ii. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $s_1^{PL} > s_2^{PL} + s_3^{PL}$, and $s_1^{PR} < s_2^{PR} + s_3^{PR}$ or

iii. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $s_1^{PL} = s_2^{PL} + s_3^{PL}$, and $s_1^{PR} < s_2^{PR} + s_3^{PR}$

Proof: *i.* In this situation we are initially and also after the change in Case 2i. So, we need that $s_2^{PL} + s_3^{PL} < \frac{1}{2}$ and $v_2 + v_3 < \frac{1}{2}$ to be in Case 2. We also need $s_1^k < s_2^k + s_3^k$ ($k \in \{PR, PL\}$) to be in Case

[5] Notice that the utility obtained by party 4 in Case 2iii. would be the same no matter which of the two coalitions would form.

2i. before and after the change. The rule will be changed iff party 2 is underrepresented i.e. $\sum_{j=1}^{4} v_j^3 > v_2^2$. *ii.* and *iii.* describe the conditions such that a change would move the environment from Case 2ii. to Case 2i. or from Case 2iii. to Case2i. respectively. So, we need the conditions that guarantee that we are in Case 2 which are $s_2^{PL} + s_3^{PL} < \frac{1}{2}$ and $v_2 + v_3 < \frac{1}{2}$ and before the change we should be in Case 2ii. or Case 2iii respectively, so we need $s_1^{PL} > s_2^{PL} + s_3^{PL}$ and $s_1^{PR} < s_2^{PR} + s_3^{PR}$ respectively. The last conditions in both cases guarantee that we end up in Case 2i. or in Case 2ii. respectively. #

In all the cases where the electoral rule changes, the smallest party forms part of the government before and after the change. Example 2 would still be valid for Proposition 12ii. The following two examples report possible share of votes for which an electoral system change would occur for cases as in Proposition 12i. and 12iii. respectively.

Example 3: Suppose that $v_1 = 0.3057$, $v_2 = 0.2539$, $v_3 = 0.2302$, and $v_4 = 0.2102$. Under plurality rule we would have: $s_1^{PL} = 0.4299$, $s_2^{PL} = 0.2466$, $s_3^{PL} = 0.1837$ and $s_4^{PL} = 0.1398$. We would be in Case 2i, as $s_2^{PL} + s_3^{PL} < \frac{1}{2}$ and $s_1^{PL} < s_2^{PL} + s_3^{PL}$; party 2 would be underrepresented as $s_2^{PL} < v_2$. If the rule were changed we would still be in case 2i as $v_2 + v_3 < \frac{1}{2}$ and $v_1 < v_2 + v_3$. So, for the given share of votes, the electoral rule would be changed, where the coalition candidate does not change.

Example 4: Suppose that $v_1 = 0.3480$, $v_2 = 0.3353$, $v_3 = 0.1640$, and $v_4 = 0.1527$. Under plurality rule we would have: $s_1^{PL} = 0.4797$, $s_2^{PL} = 0.4294$, $s_3^{PL} = 0.0503$ and $s_4^{PL} = 0.0406$. We would be in Case 2iii, as $s_2^{PL} + s_3^{PL} < \frac{1}{2}$ and $s_1^{PL} = s_2^{PL} + s_3^{PL}$. If the rule were changed we would be in case 2i as $v_2 + v_3 < \frac{1}{2}$ and $v_1 < v_2 + v_3$. So, for the given share of votes, the electoral rule would be changed, where the coalition candidates are changed.

As it was done for the other two theories of coalition formation, I consider now a threshold higher than the absolute majority. Moreover, once again, we could analyze the possibility of a change by relaxing the assumption on the cube law in the same manner as it was done before. Lemma 7 would still define the coalition candidates as they do not depend on m. Moreover, the result in Lemma 8 would also hold[6]. So, we have to check whether there would be some change in the possible movements. (The points from 1 to 13 analyzed above) It can easily be seen that the arguments in points 1,2, 3, and 5 do not depend on the value of m.[7] So they will still be valid. Now reconsider the remaining ones:

4') *From Case 1 to Case 2ii or 2iii:* Is not possible. Why? If under plurality we are in Case 1, we have $s_1^{PL} < s_2^{PL} + s_3^{PL}$ which implies that $\frac{v_2^m + v_3^m}{v_1^m + v_2^m + v_3^m + v_4^m} > \frac{v_1^m}{v_1^m + v_2^m + v_3^m + v_4^m}$ i.e. $v_2^m + v_3^m > v_1^m$. If

[6] These results can be obtained simply by changing 3 with m in the proof of this Lemma.

[7] For points 1 and 5 we know that Lemma 8 holds for any $m > 1$.

after the change we should reach 2ii or 2iii, we need $v_1 \geq v_2 + v_3$ i.e. $v_1^m \geq (v_2 + v_3)^m$. For $v_2, v_3 > 0$, $(v_2 + v_3)^m > v_2^m + v_3^m$. So, we have a contradiction.

6') *From Case 2i to Case 2ii:* Is not possible. Why? If under plurality we are in Case 2i, we have $s_1^{PL} < s_2^{PL} + s_3^{PL}$ and to reach 2ii. we need $v_1 > v_2 + v_3$ i.e. $v_1^m > (v_2 + v_3)^m$. So, the argument becomes the same as in 4').

7') *From Case 2i to Case 2iii:* For the same reasoning as in the previous case, it is not possible.

8') *From Case 2ii to Case 2i:* Party 1 would be in favor of a change as after the change it would form part of the government and party 2 and party 3 would be against as they would be out of the government. What happens with party 4? Assuming the change occurs, under plurality its utility from forming part of the government is $\frac{s_4^{PL}}{s_2^{PL}+s_3^{PL}+s_4^{PL}}\overline{U} = \frac{v_4^m}{v_2^m+v_3^m+v_4^m}\overline{U}$ and under PR it is $\frac{v_4}{v_1+v_4}\overline{U}$. $\frac{v_4^m}{v_2^m+v_3^m+v_4^m} < \frac{v_4}{v_1+v_4}$. Why? If it is true than, $\frac{v_4}{v_1+v_4} - \frac{v_4^m}{v_2^m+v_3^m+v_4^m} > 0$ which is the same as $v_4(v_2^m + v_3^m - v_1 v_4^{m-1}) > 0$. So, we need $v_2^m + v_3^m - v_1 v_4^{m-1} > 0$. In Case 2i. we should have $v_1 < v_2 + v_3$. So, $v_2^m + v_3^m - v_1 v_4^{m-1} > v_2^m + v_3^m - (v_2 + v_3)v_4^{m-1}$ and it can easily be seen that $v_2^m + v_3^m - (v_2 + v_3)v_4^{m-1} > 0$. So, party 4 is in favor of a change and the change occurs as $s_1^{PL} + s_4^{PL} > \frac{1}{2}$.

9') *From Case 2ii to Case 2iii:* Might only be possible if $m < 1.58496$. Why? We should have $v_2+v_3 = v_1$ and $s_1^k < 0.5\ k \in \{PR, PL\}$. We know that $s_1^{PL} = \frac{v_1^m}{v_1^m+v_2^m+v_3^m+v_4^m}$. Suppose that $v_2+v_3 = v_1$. So, we have that $v_4 = 1 - 2v_1$. So, $s_1^{PL} = \frac{v_1^m}{v_1^m+v_2^m+v_3^m+(1-2v_1)^m}$. For $v_2, v_3, v_4 > 0$, $v_2^m + v_3^m + (1 - 2v_1)^m$ (given that $v_2 + v_3 = v_1$) would be maximum (i.e. s_1^{PL} minimum) if $v_3 = 1 - 2v_1$ (the smallest value possible). So, s_1^{PL} would be minimum if $s_1^{PL} = \frac{v_1^m}{v_1^m+(3v_1-1)^m+2(1-2v_1)^m}$. Moreover, since $v_2 \geq v_3$, we need $3v_1 - 1 \geq 1 - 2v_1$ i.e. $v_1 \geq 0.4$. For $0.5 \geq v_1 \geq 0.4$, $\frac{v_1^m}{v_1^m+(3v_1-1)^m+2(1-2v_1)^m}$ is first strictly increasing and then strictly decreasing for a given m. Moreover, it is strictly increasing in m. For $v_1 = 0.5$ it takes the value 0.5 for any m. So, if for $v_1 = 0.4$ it takes a value higher than 0.5 then it never can take a smaller value. It can be shown that it would take a smaller value if and only if $m < 1.58496$. We need also $s_1^{PL} > s_2^{PL} + s_3^{PL}$ which always holds since if $v_2 + v_3 = v_1$ we have $(v_2 + v_3)^m = v_1^m > v_2^m + v_3^m$. Therefore, for sure $s_1^{PL} > s_2^{PL} + s_3^{PL}$. So, a change might occur only if $m < 1.58496$ which is supported by party 1 as it would enter the government after the change and from Lemma 8 also by party 4.

10') *From Case 2ii to Case 2ii:* For a similar reasoning as in the previous case a change would be possible only if $m < 1.58496$. But then party 1 (as it is overrepresented under plurality) and party 2 (from Lemma 8) would be against a change. So, no change occurs.

11') *From Case 2iii to Case 2i:* Party 2 and party 3 would be against as they would be out of the government after the change. Party 1 would be in favor iff his utility of being part of the government is higher after the change. Assuming the change occurs, under plurality its utility from forming part of the government is $\frac{1}{2} * \frac{s_1^{PL}}{s_1^{PL}+s_4^{PL}}\overline{U} = \frac{1}{2}\frac{v_1^m}{v_1^m+v_4^m}\overline{U}$ and under PR it is $\frac{v_1}{v_1+v_4}\overline{U}$. $\frac{1}{2} * \frac{v_1^m}{v_1^m+v_4^m} < \frac{v_1}{v_1+v_4}$. Why? If it is true then: $\frac{v_1}{v_1+v_4} - \frac{1}{2} * \frac{v_1^m}{v_1^m+v_4^m} > 0$ which holds if $v_1^{m+1} + 2v_4^m v_1 - v_1^m v_4 > 0$. This inequality always holds as $v_1 > v_4$. So, party 1 is always in favor. Party 4 is in favor too as the argument in footnote 5 would also hold for any $m > 1$. So, the change occurs as $s_1^{PL} + s_4^{PL} > \frac{1}{2}$.

12') *From Case 2iii to Case 2ii:* Is not possible. Why? If under plurality we are in 2iii, we have $s_1^{PL} = s_2^{PL} + s_3^{PL}$ which implies that $v_2^m + v_3^m = v_1^m$. If after the change we should reach 2ii. we need $v_1 > v_2 + v_3$ i.e. $v_1^m > (v_2 + v_3)^m$. For $v_2, v_3 > 0$, $(v_2 + v_3)^m > v_2^m + v_3^m$. So, we would have $v_2^m + v_3^m > (v_2 + v_3)^m$. Contradiction.

13') *From Case 2iii to Case 2iii:* It is not possible to reach the same situation under both rules. Why? Suppose we can reach it. Then we would have both $s_2^{PL} + s_3^{PL} = s_1^{PL}$ (which implies that $v_1^m = v_2^m + v_3^m$) and $v_2 + v_3 = v_1$. So, we should have $(v_2 + v_3)^m = v_2^m + v_3^m$ which never holds for $v_2, v_3 > 0$.

As we can see from the analysis above, relaxing the assumption of cube law does not change the results except for point 9'. For this case, there might occur a change if $m < 1.58496$ which would never have occurred under the assumption of the cube law. Formally, the result is as follows:

Proposition 13: If $v_i^e = (v_1, v_2, v_3, v_4)$ $\forall i$, $T > \frac{1}{2}$, spoils of office are shared proportionally and $m \geq 1.58496$, then the electoral rule will be changed if and only if:

i. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $s_1^k < s_2^k + s_3^k$ ($k \in \{PR, PL\}$), $\sum_{j=1}^{4} v_j^m > v_2^{m-1}$, $s_2^{PL} + s_3^{PL} + s_4^{PL} > T$ and $T < \frac{3}{4}$ or

ii. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $s_1^{PL} > s_2^{PL} + s_3^{PL}$, $s_1^{PR} < s_2^{PR} + s_3^{PR}$, $s_1^{PL} + s_4^{PL} > T$ and $T < \frac{2}{3}$ or

iii. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $s_1^{PL} = s_2^{PL} + s_3^{PL}$, $s_1^{PR} < s_2^{PR} + s_3^{PR}$, $s_1^{PL} + s_4^{PL} > T$ and $T < \frac{3}{5}$

If $m < 1.58496$, then a change occurs iff either one of the four conditions above hold or

iv. $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, $v_2 + v_3 < \frac{1}{2}$, $s_1^{PL} > s_2^{PL} + s_3^{PL}$, $s_1^{PR} = s_2^{PR} + s_3^{PR}$ and either $s_1^{PL} + s_3^{PL} + s_4^{PL} > T$ and $T < \frac{5}{6}$ if party 3 is in favor of a change or $s_1^{PL} + s_4^{PL} > T$ and $T < \frac{2}{3}$ if it is not.

Proof: The conditions except those of the threshold of points i. to iii. is very similar to those of Proposition 12. Regarding the conditions on the threshold: *i.* The same argument as in Proposition 11. *ii.* In that case an electoral system change is favored by party 1 and party 4. So, besides the conditions from before we need $s_1^{PL} + s_4^{PL} > T$. Since we have a coalition government $s_1^{PL} < 0.5$. So, $s_2^{PL} + s_3^{PL} + s_4^{PL}$ is at least 0.5. So, the minimum value that can take $s_2^{PL} + s_3^{PL}$ is $\frac{1}{3}$ as they have to have at least the same amount of votes as party 4. Therefore, $s_1^{PL} + s_4^{PL} < \frac{2}{3}$. So, for any T higher than this value no change occurs. *iii.* In that case an electoral system change is favored by party 1 and party 4. So, besides the conditions from before we need $s_1^{PL} + s_4^{PL} > T$. For a given s_1^{PL}, $s_2^{PL} + s_3^{PL}$ would be minimum if $v_2 = v_3 = v_4$. Given that $s_1^{PL} = s_2^{PL} + s_3^{PL}$, we would have $5s_4^{PL} = 1$. So, $s_1^{PL} + s_4^{PL}$ can be maximum $\frac{3}{5}$. Therefore for $T > \frac{3}{5}$ a change never occurs. As was shown above the change in iv. is only possible if $m < 1.58496$. The change is favored by parties 1 and 4 for sure but not by party 2. Party 3 might be in favor or against depending on m and the vote shares. From *ii.* s_2^{PL} is at least $\frac{1}{6}$ and $s_2^{PL} + s_3^{PL}$ is at least $\frac{1}{3}$. Therefore, we need the conditions above. #

Notice once again, that the assumption of proportional distribution of office spoils leads to the same coalition candidates as the minimum winning coalition theory does. Therefore, we can conclude that the analysis in this section coincides with the result that we would obtain by considering this coalition theory.

8 Implications of the Results

8.1 Size of Government

An interesting aspect that should be analyzed is how the size of the government is affected if a change in the electoral rule occurs. When parties expect that each party obtains the same share of votes in the next election, we have seen that no change occurs for the case of two parties. So, the focus will be on the case of three and four parties. I compare the size of the government before and after the change for each case in which a change might occur.

For the case of three parties, under equally shared office spoils (MWC in number of parties), the set of possible coalition candidates do not change even if a change occurs. Under the assumption that each coalition candidate forms the government with equal probability, the expected size of the government would be:

$$\frac{(s_1^k + s_2^k) + (s_1^k + s_3^k) + (s_2^k + s_3^k)}{3} = \frac{2}{3} \qquad k \in \{PL, PR\}$$

If a change occurs, the decrease in the seat share of party 1 will be equal to the sum of the increase in the seat share of party 2 and party 3. Therefore, the expected size of the government will remain $\frac{2}{3}$ even if the rule changes. When office spoils are shared proportionally (MWC in number of seats), we had previously found that no change occurs.

If there are four parties, then for equally shared office spoils, different than the case of three parties, as the following analysis shows, if the electoral rule changes the size of the government goes down. Under this theory, as was shown in Section 7.1, we have two different cases in which a change occurs:

1) If $s_2^{PL} + s_3^{PL} < \frac{1}{2}$, then, given that a change occurs, the expected size of the government under plurality would be $\frac{(s_1^{PL}+s_2^{PL})+(s_1^{PL}+s_3^{PL})+(s_2^{PL}+s_3^{PL})}{3} = \frac{2(s_1^{PL}+s_2^{PL}+s_3^{PL})}{3} = \frac{2(1-s_4^{PL})}{3}$. Under PR, the expected size of the government would be $\frac{(v_1+v_2)+(v_1+v_3)+(v_2+v_3)}{3} = \frac{2(1-v_4)}{3}$ which is smaller than the size under plurality as from Lemma 3, $s_4^{PL} < v_4$.

2) If $s_2^{PL} + s_3^{PL} > \frac{1}{2}$, then, given that a change occurs, the expected size of the government under plurality would be $\frac{(s_1^{PL}+s_2^{PL})+(s_1^{PL}+s_3^{PL})+(s_1^{PL}+s_4^{PL})}{3} = \frac{3s_1^{PL}+s_2^{PL}+s_3^{PL}+s_4^{PL}}{3} = \frac{1+2s_1^{PL}}{3}$. Under PR, the expected size of the government would be $\frac{(v_1+v_2)+(v_1+v_3)+(v_1+v_4)}{3} = \frac{1+2v_1}{3}$ which is smaller than the size under plurality as $v_1 < s_1^{PL}$.

Different than above, if office spoils are shared proportionally given that a change in the electoral rule occurs, the size of the government might go up or down. Consider once again Example 2. Under plurality the government is formed by parties 2, 3 and 4 and its size would be 0.525. Under PR, the government would be formed by parties 1 and 4 with a size of 0.51. So, after the change the size of the government decreases. If on the other hand, we suppose that $v_1 = 0.3202$, $v_2 = 0.2493$, $v_3 = 0.2252$, and $v_4 = 0.2053$ the corresponding share of seats under plurality would be $s_1^{PL} = 0.4799$, $s_2^{PL} = 0.2264$, $s_3^{PL} = 0.1672$ and $s_4^{PL} = 0.1265$. We would be in the same situation as in Example 2, so a change would occur, under plurality the government is formed by parties 2, 3 and 4 with a size of 0.5201. Under PR, the government would be formed by parties 1 and 4 with a size of 0.5255. So, after the change the size of the government increases.

8.2 Degree of the Representation of the Government

A widely used argument by advocates of proportional representation is that under this rule the government would have a larger support in terms of its total vote share compared to governments formed under plurality/majoritarian rules. Do the above results confirm this hypothesis?

The approach I take is to compare the total amount of votes obtained by each party of the government whenever a change occurs. For the case of three parties, when a change occurs, it does not affect the set of coalitions who can form the government, therefore we can conclude that although the rule might change, the degree of the representation of the government will not change. For four parties with equally shared office spoils (MWC in number of parties) there will not be a change in the degree of the representation of the government as a change in the electoral rule does not affect the set of winning coalition candidates neither.

Under proportionally shared office spoils, a change in the electoral rule might affect the composition of the government. The previous results show that the government under plurality is formed by parties 2, 3 and 4 if $s_2^{PL} + s_3^{PL} < \frac{1}{2}$ and $s_1^{PL} > s_2^{PL} + s_3^{PL}$. If this is the case, a change occurs if $v_1 < v_2 + v_3 < \frac{1}{2}$ and under PR the government is formed by parties 1 and 4. So, the degree of the representation of the government under plurality would be $v_2 + v_3 + v_4$ which is greater than the degree of the representation of the government under PR ($v_1 + v_4$) as $v_1 < v_2 + v_3$. In the other two cases where a change occurs, the degree of representation of the government does not change[8]. So, although it seems counter-intuitive, a change in the electoral rule might decrease the degree of the representation of the government and it does never increase it for the case of two, three and four parties under the model at hand. Under proportionally shared office spoils, parties prefer to form government with smaller parties. So, as it was shown, this preference might lead to a government with a lower degree of representation. The intuition behind this observation is that if the office spoils are shared proportionally the government is formed by the parties with the lowest total seat share possible. A change in the electoral rule might lead to formation of an government with a smaller total seat share and degree of representation.

8.3 Effective Number of Parties and Electoral System Change

Colomer (2005) by using data of 219 elections in 87 countries runs a regression on data of electoral results where the dependent variable is the probability of change and the independent variable the effective number of parties, and he obtains that the probability of a switch from a majoritarian rule to a more proportional one would increase as the effective number of parties increases. More concretely, he obtains that only when the number of effective parties increases to 4, the probability of an electoral system change rises above half (61%). The definition of effective number of parties (ENP) used by Colomer is due to Laakso and Taagapera (1979) where $ENP = \frac{1}{\sum_{i=1}^{p} v_i^2}$. What do the results of our model suggest about the relationship between the ENP and a possible change in the electoral rule?

We know that no change will occur for the case of two parties. For the case of three parties, we need necessarily a coalition government for a change to occur and the office spoils should not be distributed proportionally. So, we need $v_1 < 0.5$. Therefore, for a change to occur ENP should be definitely larger than 2 (it would be closer to 2 as $v_2 \to 0.5$ and $v_3 \to 0$). For three parties ENP can be at maximum 3. So, we can conclude that for $2 < ENP < 3$ a change may occur as the following example shows:

Example 5: Suppose that $v_1 = 0.37$, $v_2 = 0.33$ and $v_3 = 0.30$, where $ENP = 2.81$ and the spoils of office are equally shared. Under plurality rule we would have: $s_1^{PL} = 0.4459$, $s_2^{PL} = 0.3164$ and $s_3^{PL} = 0.2377$. As party 2 is underrepresented, the electoral rule will change.

However, one can not say that for a given number of parties, if a change occurs for a certain ENP, that a change would definitely occur in every other case with a higher ENP. Consider, for instance,

[8] It does not change neither for the additional case that occurs if $m < 1.58496$.

Example 1 where a change occurs. In that example, ENP can be found as 3.58. Now, suppose that $v_1 = 0.33$, $v_2 = 0.3$, $v_3 = 0.23$, and $v_4 = 0.14$ where $ENP = 3.68$. Therefore, $s_2^{PL} = 0.3468$ and $s_3^{PL} = 0.1563$. We would be in the same case as in Example 1 but the change will not occur as party 2 is not underrepresented even though ENP is higher than in Example 1.

9 Extensions

In this section I consider three extensions of the model for the case of two and three parties. First, I consider the case where "ideal" proportionality is not achieved under PR. Then I consider a situation where the vote to seat transformation is different than as it was until now. Lastly, I relax the assumption that parties expect to obtain the same share of votes under both rules.

9.1 "Non-ideal" Proportionality

Until now I assumed that under PR the seat share of a party was equal to its vote share. In reality, however, this is, if it is really wanted, a goal hard to achieve. Generally, larger parties are overrepresented and smaller parties are underrepresented even under PR. To address to this phenomenon, I assume now that under PR there exists also a certain degree of disproportionality, i.e. $\frac{s_i^{PR}}{s_j^{PR}} = (\frac{v_i}{v_j})^n$ for all $i, j \in P$ with $3 > n > 1$. For plurality I assume that the cube law applies, that is, plurality is more disproportional. I consider the case of two and three parties.

As the seat to vote transformation is monotonic, the largest party will be better-off under plurality and the smallest party under PR. Therefore, for the case of two parties, as before, a change never occurs as the larger party forming the government alone would be against a change. Similarly, for any number of parties, if the government is formed by a single party the rule will never be changed.

For the case of three parties, where the government is formed by a coalition, as the following propositions show, the results are almost identical to the ones obtained before. First consider the scenario where office spoils are shared equally:

Proposition 14: If $v_i^e = (v_1, v_2, v_3)\ \forall i$, spoils of office are shared equally, $T = \frac{1}{2}$ and the government is formed by a coalition, then the electoral rule will be changed if and only if $\frac{v_2^n}{v_1^n + v_2^n + v_3^n} > \frac{v_2^3}{v_1^3 + v_2^3 + v_3^3}$.

Proof: Given that each of the coalitions with two parties can be formed with equal probability and the relative size of each coalition party does not affect the utility obtained from forming part of the government, a party will be in favor of an electoral rule change if and only if this change would increase its share of seats. We know that party 1 will be against a change and party 3 in favor. So, a change occurs iff party 2 is in favor i.e. if $s_2^{PR} > s_2^{PL}$ which from Lemma 3, holds iff$\frac{v_2^n}{v_1^n+v_2^n+v_3^n} > \frac{v_2^3}{v_1^3+v_2^3+v_3^3}$. #

The key party is still the second largest one. The only difference is that now it is in favor of a change iff it is less underrepresented or more overrepresented under PR. As before, if $T > \frac{2}{3}$ a change never occurs and if $\frac{1}{2} < T < \frac{2}{3}$ we need $s_2^{PL} + s_3^{PL} > T$. Notice that the above inequality does not necessarily hold for any n or any distribution of votes. If we assume that office spoils are shared proportionally the result is as follows:

Proposition 15: If $v_i^e = (v_1, v_2, v_3)$ $\forall i$, spoils of office are shared proportionally and the government is formed by a coalition, then for any $T \geq \frac{1}{2}$ the electoral rule will never change.

Proof: Under this specification the coalition is formed by party 2 and party 3. Party 1 would be against a change for sure. As a rule change affects the relative size of parties 2 and 3, the one whose relative size decreases will be against the change. So, no change occurs as $s_1^{PL} + s_2^{PL} > \frac{1}{2}$ and $s_1^{PL} + s_3^{PL} > \frac{1}{2}$. #

That is, as before a change is never possible. One important point to mention is that the characteristics of the results would not change if we take for plurality m with $3 > m > n$ rather than the cube law.

9.2 A More General Vote to Seat Transformation

Now I assume that the vote to seat transformation for three parties under plurality is as follows: $\frac{s_1^{PL}}{s_2^{PL}} = (\frac{v_1}{v_2})^m$ and $\frac{s_2^{PL}}{s_3^{PL}} = (\frac{v_2}{v3})^n$ where $m > 1$, $n > 1$ and $m \neq n$. Notice that this is the most general case possible. For PR, I assume that ideal proportionality applies. So, from an analysis similar to Lemma 2 we obtain that $s_1^{PL} = \frac{v_1^m v_2^{n-m}}{v_1^m v_2^{n-m}+v_2^n+v_3^n}$, $s_2^{PL} = \frac{v_2^n}{v_1^m v_2^{n-m}+v_2^n+v_3^n}$ and $s_3^{PL} = \frac{v_3^n}{v_1^m v_2^{n-m}+v_2^n+v_3^n}$. From an analysis very similar to Lemma 3i. we can obtain that party 1 is overrepresented and party 3 is underrepresented. Therefore, as before, under a single party government the rule will never change.

If we consider equally shared office spoils and assume that the government is formed by a coalition, party 1 will be against and party 3 in favor of a change. Therefore, the key party is as before party 2. A change will occur iff party 2 is underrepresented, i.e. iff $v_2 > \frac{v_2^n}{v_1^m v_2^{n-m}+v_2^n+v_3^n}$.

If we consider proportionally shared office spoils, the coalition is formed by parties 2 and 3. Party 1 would be against a change as it is overrepresented under plurality. From the same analysis as in Proposition 9, we obtain that party 2 would also be against a change. Therefore, a change never occurs. As it can be seen, the results have the same characteristics as before.

9.3 Expectations with Changing Vote Shares

Until now I assumed for the case of three and four parties that parties expect to obtain the same share of votes in the next election whether the electoral rule changes or not. However, it has been argued that voters who would prefer to vote for one of the smaller parties, end up voting for the "less evil" of the two leading parties in order not to waste their vote when the electoral rule is plurality. That is, smaller party would increase their vote share under PR. In order to address to this behavior I assume now that parties' expectations of their future vote share would be different for the two rules and I consider the case of three parties. If the electoral rule does not change, parties still expect to obtain the same share of votes as before, i.e. $v_i^e = (v_{1,}v_2, v_3)$ for any $i \in P$. If the electoral rule switches to PR however, I assume that parties expect that the share of votes of parties 1 and 2 will decrease whereas the share of votes of party 3 will increase, and that all parties have the same expectations, which is not too unrealistic as in the real world there exist quite accurate poll results. That is, $v_i^e = (v_1', v_2', v_3')$ for any $i \in P$ where $v_1 > v_1'$, $v_2 > v_2'$ and $v_3' > v_3$. Moreover, I assume that the order of parties in terms of their share of votes does not change, that is, $v_1' > v_2' > v_3'$. Notice that, if the government is formed by a single party the rule will not change.

Consider the case where office spoils are shared equally and the government is formed by a coalition. Party 1 who is overrepresented under plurality will be against a change. Party 3, as before, will be in favor of a change. Therefore, as before, the key party is the second largest one. If party 2 is overrepresented under plurality it will be against a change. So, for a change to occur we need party 2 to be underrepresented under plurality and that it expects to obtain more seats under PR, i.e. a change occurs iff $v_2' > \frac{v_2^3}{v_1^3+v_2^3+v_3^3}$[9]. Notice that, the result has the same characteristics as before but it is possible for a smaller range of vote shares as $v_2 > v_2'$.

Now suppose that spoils of office are shared proportionally among coalition members where the government is formed by parties 2 and 3. Party 1 would be against a change as its share of seats would decrease. Now suppose that one of the remaining two parties is in favor of a change. This means that this party's relative size in the government increases with the rule change. However, then the other party should be against a change as its relative size in the government would decrease. Therefore, as before, a change never occurs.

[9] This result would hold for any $m > 1$ by changing the cubes with m.

10 Discussion and Concluding Remarks

The analysis shows that for three or more parties the electoral rule cannot change if the government is formed by a single party and parties expect that each party obtains the same share of votes in the next election. This finding might help to understand why the electoral rule has never been changed in Britain where the current rule is plurality. Over the last decades, many times the debate of changing the electoral rule in Britain has been brought up, yet with no success. If the governments formed in Britain from 1945 up to today[10] are examined, we can see that it has never been formed by a coalition of parties.

The results obtained also indicate that the fact that the government is formed by a coalition does not necessarily imply that the electoral rule will be switched to proportional representation. For the case of three parties the key party is the second largest one independently of whether it forms part of the government or not. Considering different theories of coalition formation, I found that the strategic choices of parties with respect to a change in the electoral rule are not the same under all these theories. On the other hand, with a change in the electoral rule, the possible governing coalition candidates are still the same ones.

For the case of four parties if it is assumed that coalitions are formed according to MWC or MWC in number of parties (equally shared office spoils), the set of governing coalitions in not affected neither. If it is assumed that the underlying theory is MWC in number of seats (proportionally shared office spoils), then in the cases where the electoral rule might be changed without affecting the candidate(s) of governing coalitions, the key party is still the second largest party. However, there might well occur a change for some distributions of vote shares, which alter the candidate(s) for governing coalitions. In those cases, the second largest party does not play a key role. One interesting point to mention is that while in the case of three parties the largest party is always against a change and the smallest party always in favor, in the four party case this is not always the case. That is, the largest party although it is overrepresented under plurality might be in favor of a change as it might enter the government after the change while it would out of the government under plurality. Similarly, the smallest party, always underrepresented under plurality, might be against a change when it forms part of the government as the change might lead to a coalition of which it does not form part. The analysis shows that for four parties a change is possible for a higher range of thresholds compared to the case of three parties.

Clearly the results are obtained by maintaining some assumptions. One of these assumptions was about the assignment of seats according to the share of votes under plurality. First, I assumed that the

[10] Source: Web page of Richard Kimber, http://www.psr.keele.ac.uk/

share of seats are obtained according to the "cube law" where the share of seats ratio of two parties is the cube of the share of votes ratio. Laakso (1979) argues that it would be more appropriate to take a ratio of 2.5 rather than 3 as it fits better the British election results. On the other hand, Maloney et al. (2001) examining election results of six countries where non-proportional rules are used (Australia, Canada, France, New Zealand, UK and US) argue that the ratio would be between 2 and 3 for the two major parties. When this assumption was relaxed, I obtained that for the case of two and three parties (for all three theories of coalition formation) the general characteristics of the results do not change because they do not depend on the magnitude of the ratio. In the case of four parties, the results do not change neither for the theories of MWC and minimum winning coalitions in number of parties. In the case of minimum winning coalitions in number of seats, however, if the degree of disproportionality is sufficiently low, a change occurs under a wider range of conditions.

Another assumption made in the analysis is that under proportional representation each party's share of seats equals its share of votes. Obviously, none of the PR rules used today in elections can reach total proportionality. Yet, as the analysis above shows that relaxing this assumption for the case of three parties does not change the results. However, in the case of four parties, the degree of proportionality of the alternative rule might play an important role. This aspect needs a further analysis.

One further step that should be taken in the analysis is relaxing the assumption that parties expect that each party obtains the same share of votes in the future elections for four parties as it was done for two and three parties. It can easily be argued that a party's expectation would depend on his past share of votes and on the electoral rule at hand. It is a well-known argument that plurality rule leads to higher degree of strategic voting compared to proportional representation. Therefore it would be wise to take the expectations of parties in a more sophisticated manner as a function of their past share of votes and the electoral rule. I believe that this analysis would lead to interesting implications.

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