FATIGUE PROGNOSIS IN COMPOSITES:

A BAYESIAN FRAMEWORK

by

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ABSTRACT

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Anticipating and characterizing the damage induced by fatigue loadings is a challenging problem in the composites science and technology. In contrast to metals, fatigue degradation of composites is presented since the initial stages of the process leading to a decrease in the mechanical performance. The literature covers a large number of fatigue models for composites, the majority of them are only valid in their experimental conditions. In this thesis a novel bayesian inverse strategy to reconstruct fatigue damage over lifetime is proposed. This model has been developed to be extensible to different material configurations and loading conditions, in a coherent statistical sense. Finally this result has led to a bayesian model class selection, by which it is possible to select the most plausible model parameterization. The proposed methodology has been validated against fatigue damage data. This bayesian framework has shown versatility to take into account all possible information about data, models and the relation between them. The updated information inserted into the reliability problem is shown to confer a way to considered the long term reliability without the need to make hypothesis for time to failure.

RESUMEN

PRONÓSTICO DE FATIGA EN COMPOSITES: UNA VISIÓN BAYESIANA

Anticipar y reconstruir el daño por fatiga en materiales compuestos es todavía un desafío. A diferencia de los materiales metálicos, el daño por fatiga se manifiesta desde los primeros ciclos de carga en forma de fallos locales progresivos que van reduciendo las propiedades mecánicas iniciales del material. Frente a los numerosos modelos de fatiga presentes en la literatura, en esta tesina se propone una metodología basada en el problema inverso bayesiano para la reconstrucción del daño por fatiga en toda la vida útil recogida por los datos; así como un método de selección bayesiana de modelos más evidentes según las observaciones. Este método se ha aplicado a datos experimentales de fatiga en forma de reducción de rigidez sobre laminados de fibra de vidrio tipo E-glass/polyester. Los resultados permiten obtener el modelo que mejor ajusta y predice los datos disponibles de fatiga. La aplicación de la información estadística procedente de la reconstrucción del daño para obtener los valores de fiabilidad a lo largo de la vida útil es una de las principales conclusiones del trabajo. La consideración de la vida útil en el pronóstico de daño y su relación con la fiabilidad, supone un paso importante para el desarrollo sostenible de materiales compuestos, y posibilitará la viabilidad de estos materiales en aplicaciones fuera de las aeroespaciales.

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Contents

Chapter 1

Introduction

1.1 Motivation and Objectives

As an emerging field, such as nanotechnology and biotechnology today, the composite technology in the seventies saw a boost in research and development. The result was important advances in mechanics of laminates as well as in composites failure criteria, some of which form standard topics for composite designers today. Initially, the strength and failure of composites materials were treated as a extension of metals theories. Experimental observations put in question this approach, showing that the accumulation of series of multiple local damage events, led to failure that was unpredictable by available theories.

Damage in composites become instead the focus of approaches in late seventies and early 80s and fatigue was appointed as important source of cumulative damage, since early stages in the lifetime. During 80s and 90s, fatigue damage covered an important area of the composites research topic and nowadays there are a wide spread of fatigue models available, all of them valid in its range of application [1].

Already since more than three decades in use, the applications have moved towards primary structures not only in aircraft, but also in in civil engineering, architecture and energetic infrastructures [2–4]. However, fatigue damage is still an open question, overall in those new applications combining long term behavior and difficulty or absence of maintenance like bridges, off-shore and underground structures, etc.

In aerospace industry, fatigue is overcome through high quality fibers and matrices that minimize the effect of fatigue, and also by standardized manufacturing processes and maintenance programs that reduce the uncertainty associated to fatigue modeling. The same cannot be apply in other applications out of aerospace industry to be of an inadmissible cost. It is one of the main reasons for today difficulties in developing new composites applications.

Moreover, the emerging challenges of the 21th century lie in assuring the sustainability of technological developments of the global community, that requires a paradigm out of the posture of only using high quality materials and processes. In a sustainable development scheme, the objective is to minimize the impact of today's products and processes, and so to suit the needed requirements for a specific performance during a established operational lifetime.

In this scheme the material damage prognosis plays an important role to make predictions of the performance over lifetime, and so a cost-performance trade-off exercise may be conducted to optimize the required material and processes parameters. This may view as a part of a bigger picture including the lifetime dimension that entails additional life cycle considerations that accounts for the materials and energy inputs in all stages of the life, as shown in Figure 1.1. In this approach, a statistical framework is imperative to account the unavoidable uncertainty into the global cost evaluation and also to establish levels of plausibility over lifetime predictions. This task firstly requires the consideration of evolutive phenomena as stochastic processes,

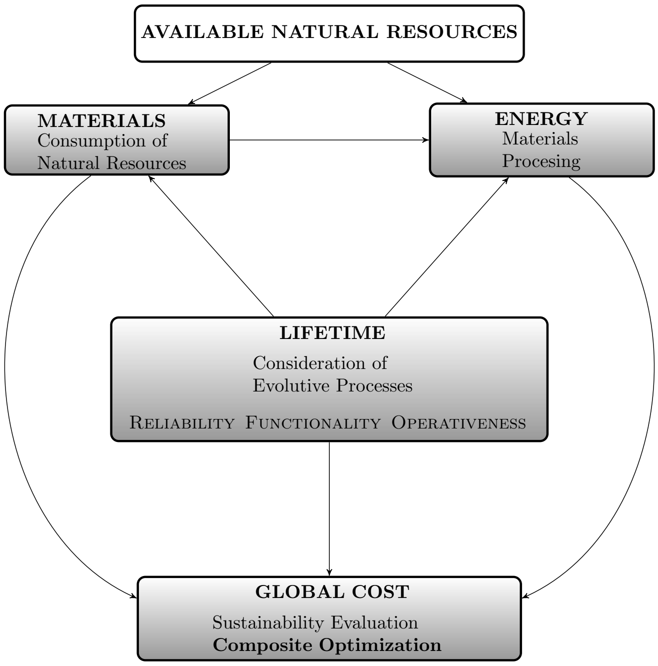


Figure 1.1: Sustainable Development of Composite Materials by consideration lifetime evolutive processes.

and then to deal with a formulation framework capable of consider full information over lifetime parameters.

In this thesis, a bayesian framework to infer damage in composites is presented, conferring the main step to achieve fatigue prognosis in composites materials. In this work, the likelihood function for parameterized Markov processes is imported from mathematical literature and a bayesian model class selection is originally proposed as a conjunction of states information [5]. This formulation has been validated against experimental fatigue damage data from the literature [6]. As a result, *a posteriori* distributions of Markov model parameters are obtained leading to a selection of the most plausible model parameterization.

The statistical information derived from the bayesian formulation is proposed to be used to derive long term reliability, and hence to make predictions for lifetime subjected to uncertainty. Previously to this work, a reasonable large spread in the methods to derive reliability in composites was observed in literature, and an up-to-date review of the existing methodologies has been required. The last part of this thesis, is advocated to serve as a review in reliability of composites.

1.2 Thesis Organization

This thesis is organized as follows. The present chapter remarks the motivation for working in fatigue modeling in composites. The main research objectives are also highlighted. In Chapter 2, the bayesian framework is presented in the format of a scientist paper prepared to be submitted to *Journal of Composite Science and Technology*. Chapter 3 deals with the review paper on reliability in composites, which is also prepared to be submitted to *Composite Part B: Engineering*.

Finally, the document is closed with appendices and lists of tables and figures, that helps the document to be easily read.

Chapter 2

A Bayesian Framework to Infer Fatigue in Composites

In this paper a bayesian inference of the complete stochastic damage process is proposed for fiber composites subjected to fatigue loading. A general bayesian inverse problem is formulated for a parametrized Markov chain model as a conjunction of states of information, leading to a general way to incorporate full information from data, models and the relation between them. This methodology is applied to data of stochastic evolution of damage considered as a stiffness reduction over open hole quasi-isotropic glass-fiber composite coupons subject to tension fatigue. As a result, the posterior information about model parameters from two parametrized nonstationary fatigue models is obtained. This approach confers an efficient way to update the initial believe on a particular fatigue model using measured data and, in general, to treat cumulative process in composites.

2.1 Introduction

In composites science, fatigue damage represents one of the most important sources of uncertainty for in service behavior leading to conservative designs and extra costs in the manufacture and maintenance [7]. Throughout decades of investigation, numerous fatigue models have been proposed and a large amount of data has been derived from expensive experimental programs.

The vast majority of fatigue models are deterministic approaches, and hence they can hardly account for the variability of test results or monitoring data, so more than often they are not practical for engineering purposes.

Recently, probabilistic damage approaches are emerging as a suitable tool for fatigue in composites materials, but the extension of such methods is not as mature as deterministic ones [1]. Among them, Discrete Time Markov Chain (DTMC) models are recently used by its ability to account the variability of the process [8]. In the classic work of Bogdanoff and Kozin [9], Markov chains are used to model the fatigue degradation of engineering materials and a time transformation-condensation method is proposed by which it's also possible to account the non-stationarity of the fatigue processes. Based on it, several authors have applied them to composites materials to model the compliance and lifetime evolution due to fatigue damage accumulatiom [6, 10–12].

However, the increasing need to improve the model predictions for accounting the complexity of damage accumulation during service condition together with the need to provide new models that best adjust to test data, makes necessary to update the initial belief on particular stochastic models using field data [13].

In this paper, a bayesian framework to infer the cumulative fatigue process is proposed for composites materials.

Other proposals have addressed the fatigue problem in a bayesian framework although by adopting some assumptions over the random vars [14] or over the relations between model and data [13]; and all of them are applied for isotropic materials or steel alloys [15, 16].

In our work, the inference has been formulated into a bayesian inverse problem of Tarantola [5] intended as a conjunction of states of information for giving the more general framework to incorporate all possible information about data, model and relation between both. In view of an industrial application, this method allows to treat a large range of problems in composites involving cumulative process due to fatigue, without modifying the formulation framework.

In Markov processes, the relation between model and data for inference purposes have been extensively studied by mathematical literature [17, 18] and more specializes in DTMC by [19, 20]. In our work, these concepts have been imported for engineering purposes and inserted into a inverse problem leading to a versatile way to make prognosis with parametrized models of fatigue.

As an example, this formulation has been applied to a stochastic damage data for sixteen quasi-isotropic open hole S2-glass laminates subjected to a constant amplitude tension-tension fatigue loading. Due to complexity of the posterior PDF of parameters, stochastic simulation technique Markov Chain Monte Carlo (MCMC) was used for its efficiency to draw samples of a PDF known up to a constant [21].

The results show this framework as a versatile tool to update the initial believe on a particular fatigue model using measured data, and hence, to treat damage evolution in composite materials.

The first part of this paper is devoted to provide a concise basis about fatigue based damage accumulation modeled as DTMC and also to present the parametrized fatigue models as forward problem. In Section 2.3 the problem of inference of DTMC formulated into a bayesian inverse problem framework, is addressed. In Section 2.4, our formulation is applied to a set of fatigue data and serve as a example of actualization of model parameters and class selection. Section 2.5 briefly concludes.

Through this paper, the path of simplifications required to resolve common model updating subjected to fatigue data, are highlighted with the objective to suit a common engineering necessities into the proposed generalized formulation, and so to attend not only to scientist community but also to industry and practitioners.

2.2 Stochastic fatigue model

The evolution of fatigue damage as a function of cycles is proposed to be modeled by Markov chains, under the main hypothesis established by the *Markov property*, which states that the *future* of the process only depends on its *present* state, which is independent of the *past*. This phenomenological stochastic approach is based on the theory of Markov Chains [22] and assumes the following underlying assumptions [9, 12]:

1. Damage is a nondecreasing random variable and it passes through an integer and finite number of states, $j = 1, 2, ..., s$, until the "absorbing" state s is reached.

2. The time period N over which damage may accumulate is discretized in integer units of duty cycles (DCs), $n = 0, 1, ..., N$.

3. Damage is only considered at the beginning and the end of a DC, without taking into account what is happening within a DC.

4. Damage can only increase within a DC from the state at the start of that DC to the next state.

It follows from the previous remarks that the proposed model is a finite-state (1), discrete-time (2), embedded (3) Markov process in which the damage accumulation mechanism is of the unit-jump type (4). At each integer time n, there is an integer-valued random variable (rv) D_n called the *damage state* at time n and the damage process is family of rv's $\{D_n; n \geqslant 0\}$.

Let then the rv D_n represents the damage state at time or duty cycle n. Thus the probability of D_n to be in state j at time n is denoted by

$$\mathrm{P}\left[D_n = j\right] = p_n(j) \tag{2.1}$$

The probability mass function of the rv D_n at time n is given by the vector

$$\mathbf{p_n} = \{p_n(1), p_n(2), ..., p_n(s)\} \tag{2.2}$$

where

$$\sum_{j=1}^{s} p_n(j) = 1 \tag{2.3}$$

From the theory of stochastic processes, the probability density function (PDF) of damage after a given number of duty cycles N, $\mathbf{p}_N$, is determined by the PDF of the initial damage state, $\mathbf{p}_0$, and the probability transition matrices (PTM), P_n, as

$$\mathbf{p}_N = \mathbf{p}_0 \prod_{n=0}^{N} P_n \tag{2.4}$$

The PTM summarizes the allowed transitions between damage states. Thus they adopt the form:

$$\mathbf{P}_n = \begin{pmatrix} p_1(n) & p_1(n) & & & \\ & p_2(n) & p_2(n) & & \\ & & \ddots & \ddots & \\ & & & p_{s-1}(n) & p_{s-1}(n) \\ & & & & 1 \end{pmatrix} \tag{2.5}$$

where the $p_j(n)$ and $q_j(n)$ are conditional probabilities that determine if the current damage state remains or proceeds to the next state at time n, respectively.

2.2.1 Forward problem

For the purpose of inference, the fatigue model described above must be parametrized by setting the transition probability matrix $\mathbf{P}^{\mathcal{M}}$ dependent on a vector $\mathcal{M}$ of model parameters. So, let define $\mathbf{P}^{\mathcal{M}} = p_{ij}^{\mathcal{M}}(n)(i,j = 1,\ldots,s; n = 0,\ldots,N)$ the probability of state j at time n given state i at time $n-1$, and $\mathcal{C}$ the manifold of possibles parameterized models for stochastic fatigue modeling.

In this work, two nonstationary models $\mathcal{C}_1$ and $\mathcal{C}_2$ have been defined with five and three parameters respectively as below:

Model $\mathcal{C}_1$: $\mathcal{M} = \{m_1, m_2, m_3, m_4, m_5\}$

$$\mathbf{P}^{\mathcal{M}} = \mathbf{p}_0 \begin{pmatrix} m_5 & 1-m_5 & & & \\ & m_5 & 1-m_5 & & \\ & & \ddots & \ddots & \\ & & & m_5 & 1-m_5 \\ & & & & 1 \end{pmatrix}^{\alpha} \tag{2.6a}$$

$$\alpha = n \times PMS(m_1, m_2, m_3, m_4) \tag{2.6b}$$

$$0 \leqslant m_j \leqslant 1 \quad j = 1 \ldots 5 \tag{2.6c}$$

Model $\mathcal{C}_2$: $\mathcal{M} = \{m_1, m_2, m_3\}$

$$\mathbf{P}^{\mathcal{M}} = \mathbf{p}_0 \begin{pmatrix} m_3 & 1-m_3 & & & \\ & m_3 & 1-m_3 & & \\ & & \ddots & \ddots & \\ & & & m_3 & 1-m_3 \\ & & & & 1 \end{pmatrix}^{\alpha} \tag{2.7a}$$

$$\alpha = n \times PMS(m_1, m_2) \tag{2.7b}$$

$$0 \leqslant m_j \leqslant 1 \quad j = 1 \ldots 3 \tag{2.7c}$$

In both models, the nonstationarity is accounted by means of an unitary time transformation while the probabilities of transition between states $p_{ij}^{\mathcal{M}}$ remain time-invariants.

2.3 Bayesian Inverse Problem

Following the probabilistic formulation of the model reconstruction inverse problem established by Tarantola [5], the solution is, not a single-valued, set of model parameters $\mathcal{M}$. Their definition is given by providing their probability density function (PDF) $p(\mathcal{M})$ of the values of the model parameters $\mathcal{M}$ within the manifold $\mathfrak{M}$ of possible values. The PDF is assigned the sense of plausibility of the model values to be true.

Since an absolute probability cannot be computed, statistical inference theory is used to incorporate to the *a priori* available or observed information about the measured observations $\mathcal{D}$, the model parameters $\mathcal{M}$ and the model class $\mathcal{C}$, the information of idealized relationship between them $\mathcal{D} = \mathcal{D}(\mathcal{M})$ computed by a numerical model pertaining to a model class $\mathcal{C}$. The former are defined by the probability densities to prior (labelled ^{0}) data $p^0(\mathcal{D})$, $p^0(\mathcal{M})$ and $p^0(\mathcal{C})$ respectively, whereas the additional information about relationship (labelled ^{m}) between observations and model provided by the model class $\mathcal{C}$, is given by the PDF $p^m(\mathcal{D}, \mathcal{M}, \mathcal{C})$. The conjunction operator (preferred over the Bayesian formulation of statistical inference for being more general, see [5]) over probability densities, combines them to yield the *a posteriori* probability $p(\mathcal{D}, \mathcal{M}, \mathcal{C})$ of the hypothetical model $\mathcal{M}$ jointly with the observations $\mathcal{D}$ and class $\mathcal{C}$,

$$p(\mathcal{D}, \mathcal{M}, \mathcal{C}) = k_1 \frac{p^0(\mathcal{D}, \mathcal{M}, \mathcal{C}) p^m(\mathcal{D}, \mathcal{M}, \mathcal{C})}{\mu(\mathcal{D}, \mathcal{M}, \mathcal{C})} \tag{2.8}$$

where $\mu(\mathcal{D}, \mathcal{M}, \mathcal{C})$ is the uniform or noninformative density function and k_1 is a normalization constant.

In this specific case where we are interested in translating information from the data space $\mathfrak{D}$ into the model space $\mathfrak{M}$ and also where both spaces have different physical meanings, some assumptions will be made at this point. First, $\mathcal{D}$, $\mathcal{M}$ and $\mathcal{C}$ are assumed to be independent *a priori*, which allows to split the joint prior information $p^0(\mathcal{D}, \mathcal{M}, \mathcal{C}) = p^0(\mathcal{D})p^0(\mathcal{M})p^0(\mathcal{C})$ and the uniform distribution $\mu(\mathcal{D}, \mathcal{M}, \mathcal{C}) = \mu(\mathcal{D})\mu(\mathcal{M})\mu(\mathcal{C})$.

Second, the probabilistic model can be represented under the form of a probability density for $\mathcal{D}$ given any possible $\mathcal{M}$, which yields $p^m(\mathcal{D}, \mathcal{M}, \mathcal{C}) = p^m(\mathcal{D}|\mathcal{M}, \mathcal{C})p^m(\mathcal{M}, \mathcal{C})p^m(\mathcal{C})$. Third, the model is not assumed to provide conditional information between model and class, i.e. $p^m(\mathcal{M}, \mathcal{C}) = \mu(\mathcal{M})$, $p^m(\mathcal{C}) = \mu(\mathcal{C})$ are noninformative. This specialise the general expression for the inverse problem (2.8) to:

$$p(\mathcal{D}, \mathcal{M}, \mathcal{C}) = k_1 \frac{p^0(\mathcal{D})p^0(\mathcal{M})p^0(\mathcal{C})p^m(\mathcal{D}|\mathcal{M}, \mathcal{C})}{\mu(\mathcal{D})} \tag{2.9}$$

The posterior information of the model parameters $\mathcal{M}$ is obtained from the joint probability $p(\mathcal{D}, \mathcal{M}, \mathcal{C})$ by extracting the marginal probability $p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i}$ for all possible observations $\mathcal{D} \in \mathfrak{D}$ and provided the model class $\mathcal{C}_i \in \mathfrak{C}$ is assumed to be true $\Rightarrow p^0(\mathcal{C} = \mathcal{C}_i) = 1$,

$$\begin{aligned} p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} &= \int_{\mathcal{C}=\mathcal{C}_i} \int_{\mathfrak{D}} p(\mathcal{D}, \mathcal{M}, \mathcal{C}) d\mathcal{D} d\mathcal{C} \\ &= k_1 p^0(\mathcal{M}) \int_{\mathfrak{D}} \frac{p^0(\mathcal{D})p^m(\mathcal{D}|\mathcal{M}, \mathcal{C}_i)}{\mu(\mathcal{D})} d\mathcal{D} = k_2 p^0(\mathcal{M}) L(\mathcal{M}) \end{aligned} \tag{2.10}$$

being $L(\mathcal{M})$ the likelihood function.

2.3.1 Likelihood Function for Markov Chains

A general approach to maximum-likelihood estimation of the transition probabilities of a DTMC is based on the so-called Whittle formula [19].

In our special case of DTMC of damage states, $\mathfrak{D}$ is a linear space and so $\mu(\mathcal{D}) = const.$ Additionally, for the particularity of Whittle formula based on observed sequence of discrete states as will exposed bellow, $L(\mathcal{M})$ can be restricted to a subset $\mathcal{O} \subseteq \mathcal{D}$ representing the observed data which implies to assum the hypothesis of negligible observational uncertainties with respect to modelization uncertainties $\Rightarrow p^0(\mathcal{D}) = \delta(\mathcal{D} - \mathcal{O})$.

So, the factor $p^0(\mathcal{D})p^m(\mathcal{D}|\mathcal{M},\mathcal{C})$ from (2.10) derives to $p^m(\mathcal{O}|\mathcal{M},\mathcal{C})$ and the Likelihood function results then:

$$L(\mathcal{M}) = \int_{\mathfrak{O}} p^m(\mathcal{O}|\mathcal{M},\mathcal{C})d\mathcal{O} \tag{2.11}$$

Once established this framework, let define the states $i = 1, 2, \ldots, s$ and the time of observation $n = 0, 1, \ldots, N$. Let $\mathbf{P}^{\mathcal{M}} = p_{ij}^{\mathcal{M}}(n)(i, j = 1, \ldots, s; n = 0, \ldots, N)$ be the probability of state j at time n given state i at time $n-1$, as a function of model parameters $\mathcal{M} \in \mathfrak{M}$. And let $\mathcal{S}$ be the set $\{i, j \,/\, p_{ij}^{\mathcal{M}}(n) > 0\}$.

Given a sequence of observations $\mathcal{O}$ materialized by a l-dimensional vector $\mathbf{x} = (x_0, \ldots, x_l) \in I^{l+1}$ from an s-state Markov chain (X_m) with transition probability matrix $\mathbf{P}$, the *n-step transition count matrix* $\mathbf{F}(\mathbf{x}, n) \,/\, \mathbf{F} = f_{ij}(\mathbf{x}, n)$ is defined as the matrix that accounts the number of observed transitions $i_{(n-1)} \to j_{(n)} \in \mathcal{S}$ [20], i.e. the the number of times m, $0 \leqslant m < l$, for which $x_{m+1} = j$ at time n given $x_m = i$ at time $n-1$.

By Whittle's formula [19], the probability describing any particular sequence $\mathbf{x} = \mathbf{x}(n)$ with transition count $\mathbf{F}(\mathbf{x}, n)$ and PTM $\mathbf{P}^{\mathcal{M}}$ is given by:

$$p^m(\mathcal{O}|\mathcal{M},\mathcal{C}) = \prod_{n=0}^{N} \prod_{i,j=1}^{s} \left(p_{ij}^{\mathcal{M}}(n)\right)^{f_{ij}(n)} \tag{2.12}$$

It's known that both relations implies the transition count $\mathbf{F}(\mathbf{x}, n)$ on its own or together with the initial state $\mathbf{p}_0$, forms a sufficient statistic [18], in the sense that for any parametrization of the PTM $\mathbf{P}^{\mathcal{M}}, \mathcal{M} \in \mathfrak{M}$; $\mathbf{F}(\mathbf{x}, n)$ does not depend on $\mathcal{M}$, which also means that $\mathfrak{M}$ and $\mathfrak{O}$ are independent.

When all possible sequences $\mathbf{x} \in \mathcal{O}$ following a trajectory compatible with a given n-step transition count $\mathbf{F}$ are accounted, is demonstrated [19] that Equation 2.11 is equivalent to multiply Equation 2.12 by respective appropriate functions of factorials obtaining a Whittle's distribution [20]. This distribution assigns to $\mathbf{F}$ a probability, acting so as the likelihood function of the Markov Chain as follows:

$$L(\mathcal{M}) = \prod_{n=0}^{N} \left[\prod_{i=1}^{s} \left(\frac{f_{i+}(n-1)!}{\prod_{j=1}^{s} f_{ij}(n)!} \prod_{j=1}^{s} \left(p_{ij}^{\mathcal{M}}(n)\right)^{f_{ij}(n)} \right) \right] \tag{2.13}$$

with $f_{i+} = \sum_{j=1}^{s} f_{ij}$ and $f_{i+}(n-1) = \sum_{j=1}^{s} f_{ij}(n)$ respectively. Equation 2.13 represents how good a Markov model $\mathcal{M}|_{\mathcal{C}_i}$ is explaining the observed sequence $\mathbf{x} \in \mathcal{O}$.

2.3.2 Model-class selection

Let model class $\mathcal{C} \in \mathfrak{C}$ denote an idealized mathematical model hypothesized to simulate the experimental system, whereas model $\mathcal{M}$ denotes the set of constants of physical parameters that the model-class depends on. Different model classes can be formulated and hypothesized to idealize the experimental system, and each of them can be used to solve the probabilistic inverse problem in the previous section, yielding different values of model parameters but also physically different sets of parameters. To select among the infinitely many possible model classes that can be defined, user judgement is a criteria, but a probabilistic one can also be defined based on their compatibility between prior information $p^0(\mathcal{O}, \mathcal{M}, \mathcal{C})$ on observations $\mathcal{O}$, model parameters $\mathcal{M}$ and model class $\mathcal{C}$, and probabilistic model information given by $p^m(\mathcal{O}, \mathcal{M}, \mathcal{C})$. The conjunction of probabilities established in Equation 2.9 will be adopted instead of Bayes' theorem, for its generality.

The goal is to find the probability $p(\mathcal{C}|\mathcal{O})$, understood as a measure of the probability of a class of model conditional on the set of observed data $\mathcal{O}$ [23]. It can be derived as the marginal probability of the posterior probability $p(\mathcal{D}, \mathcal{M}, \mathcal{C})$ as defined in Equation 2.9 given the observed data $\mathcal{O} \subseteq \mathcal{D}$ as follows:

$$
\begin{aligned}
p(\mathcal{C}|\mathcal{O}) &= \int_{\mathfrak{D}\subseteq\mathfrak{D}} \int_{\mathfrak{M}} p(\mathcal{D},\mathcal{M},\mathcal{C}) d\mathcal{M} d\mathcal{D} && (2.14)\\
&= k_1 p^0(\mathcal{C}_i) \int_{\mathfrak{M}} p^0(\mathcal{M}) \underbrace{\int_{\mathfrak{O}} p^m(\mathcal{O}|\mathcal{M},\mathcal{C}_i) d\mathcal{O}}_{L(\mathcal{M})} d\mathcal{M} \\
&= k_2 p^0(\mathcal{C}_i) \int_{\mathfrak{M}} p^0(\mathcal{M}) L(\mathcal{M}) d\mathcal{M} = k_2 p^0(\mathcal{C}_i) \underbrace{p(\mathcal{O}|\mathcal{C}_i)}_{\text{evidence}} && (2.15)
\end{aligned}
$$

The same expression for the probability of a class model provided $\mathcal{O}$, can be obtained by using the Bayes' theorem:

$$
p(\mathcal{C}_i|\mathcal{O}) = \frac{p(\mathcal{O}|\mathcal{C}_i)p^0(\mathcal{C}_i)}{p(\mathcal{O})} \tag{2.16}
$$

where $p(\mathcal{O})$ can be solved from the theorem of total probability over all model classes $\mathfrak{C}$,

$$
p(\mathcal{O}) = \int_{\mathfrak{C}} p(\mathcal{O}|\mathcal{C})p^0(\mathcal{C}) \tag{2.17}
$$

The factor $p(\mathcal{O}|\mathcal{C}_i)$ is called the *evidence* for the model class $\mathcal{C}_i$ provide by the observed data $\mathcal{O}$. It expresses how likely the data are obtained if the model class $\mathcal{C}_i$ is assumed.

If no prior information is provided by the user about the class or the a priori information available for each class is the same, $p^0(\mathcal{C}_i) = \mu(\mathcal{C}_i)$ $\Rightarrow$ $p(\mathcal{C}_i|\mathcal{O}) = k_3 p(\mathcal{O}|\mathcal{C}_i)$. Note that the evidence is equal to the reciprocal of the normalizing constant k_2 in establishing the posterior PDF in Equation 2.10.

Once the evidence $p(\mathcal{O}|\mathcal{C})$ is computed for every class, its value allows to rank the models accordingly to how compatible they are with the observations [24]. This also allows to find a correct trade-off between model simplicity and fitting to observations, as explained in the following subsection.

Interpretation of class evidence

In the case of globally identifiable models and informative prior data, the posterior PDF may be accurately approximated by a Gaussian distribution and the evidence term can be obtained by an asymptotic expansion [25]. The general case where the posterior PDF may not be approximated by Gaussian distribution, an information point of view may be used for interpretation of the evidence for a model class [26], as follows:

$$\ln p(\mathcal{O}|\mathcal{C}) = [\ln p(\mathcal{O}|\mathcal{C})] \underbrace{\int_{\mathfrak{M}} p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} d\mathcal{M}}_{=1} \tag{2.18}$$

Since the evidence is independent of $\mathcal{M}$, we can bring it inside the integral and then make substitutions according to Bayes' Theorem in Equation 2.10 to expand the log-evidence as:

$$\ln p(\mathcal{O}|\mathcal{C}) = \ln L(\mathcal{M}) - \ln \frac{p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i}}{p^0(\mathcal{M})} \tag{2.19}$$

Substituting in Equation (2.18), the log-evidence results as difference of two terms:

$$\ln p(\mathcal{O}|\mathcal{C}) = \int_{\mathfrak{M}} [\ln L(\mathcal{M})]\, p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} d\mathcal{M} - \int_{\mathfrak{M}} \left[\ln \frac{p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i}}{p^0(\mathcal{M})}\right] p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} d\mathcal{M} \tag{2.20}$$

The first term is a measure of the average log-goodness of fit of the model class $\mathcal{C}$. It accounts for the log-goodness of fit for different combinations of the parameters, weighted by the posterior PDF [27]. The second term is the relative entropy between the posterior and prior PDF of parameters. This term is a measure of the information gained about the parameters given the observations $\mathcal{O}$.

2.3.3 Stochastic Simulation

Due to difficulties with analytically evaluate Equation 2.13 for obtaining the posterior parameters, the stochastic simulation like Markov Chain Monte Carlo (MCMC) methods, are feasible alternatives for exploring the parameter space [28]. The goal of the stochastic simulation methods is to generate samples which are distributed according to a unknown probability density function (PDF). For this task, several algorithms have been proposed in literature such as the Metropolis-Hastings, Gibbs Sampler and Hybrid Monte Carlo algorithms.

Among them, Metropolis Hastings (MH) is widely used for its versatility [29]. This algorithm generates samples from a Markov Chain whose stationary distribution is any specified target PDF $\pi(\mathbf{y})$, known up to a constant. By sampling a *candidate point* $\mathbf{y}$ from a *proposal distribution* $q(.|\mathbf{x}_s)$, the MH obtains the state of the chain at $s+1$ given the state at s, known by the previous value $\mathbf{x}_s$.

To ensure the convergence to target distribution, some regularity conditions for $q(.|\mathbf{x}_s)$ are required, which are usually satisfied when $q(.|\mathbf{x}_s)$ has a positive density on the same support as the target distribution, like multivariate gaussian and uniform [30]. The candidate point $\mathbf{y}$ is accepted as the next state of the chain with probability given by:

$$\alpha(\mathbf{x}_s, \mathbf{y}) = min\left\{1, \frac{\pi(\mathbf{y})q(\mathbf{x}_s|\mathbf{y})}{\pi(\mathbf{x}_s)q(\mathbf{y}|\mathbf{x}_s)}\right\} \tag{2.21}$$

Appendix A provides a description of the implemented algorithm to explore the posterior $L(\mathcal{M})|_{\mathcal{C}=\mathcal{C}_i}$ as target distribution.

As a drawback, for high-dimensional parameter spaces, it may still be very difficult to draw samples that cover all the regions of high-probability content [26], so recently new improvements to the original algorithm have been provided, like Adaptive [31] and Transitional [32] Markov Chain Montecarlo Methods, AMCMC and TMCMC respectively.

In this work, the dimensional space is known to be in the interval $[0,1]^n$, with $n = \{3,5\}$ so the MH have been used without restrictions, with the only caution of adjusting by training the covariance matrix of $q(.|\mathbf{x}_s)$ to ensure the convergence for an adequate number of trials, once discounted the burn-in period . The chosen proposal distribution has been a multivariate gaussian with identical standard deviation in each dimension, which derives to the *Random Walk* version of the algorithm [33].

2.4 Application-Numerical Example

The proposed framework is illustrated in an example considering stochastic damage data from literature [6] for sixteen quasi-isotropic open-hole S2-glass laminates subjected to a constant amplitude tension-tension fatigue loading ($R = 0.1, f = 5Hz, \sigma_{max} = 0.5 \times \sigma_u$), as shown in Table 2.1. In this experiment, the observed data $d^{(k)} \in \mathcal{O}$ came from measurements of relative stiffness decreases for each k laminate defined as follows:

$$d^{(k)}(n) = \frac{E_0^{(k)} - E^{(k)}(n)}{0.6E_0^{(k)}} \tag{2.22}$$

E_0 is the initial longitudinal stiffness, $E(n)$ is a stiffness sample measurement in n. For this data, the most suitable value for duty cycle n was considered to be 500 load cycles with a DTMC assembly of $s = 25$ states, as was reported in J. chiachio et al through a sensitivity study. The total number of duty cycles results in $N = {}^{213900}/_{500} = 428$.

Note that in the experiment some measurements were taken out of $[0,1]$ interval, and so to ensure the existence of an absorbent state for the study conditions of a stochastic process, was required either to interpolate [6] or to consider as absorbent state the last duty cycle that $d^{(k)} \geqslant 1$. In this example, this last option was adopted to minimizes the alteration in the statistical information because only affects to a small portion of measurements near the absorbent state.

Table 2.1: Stiffness reduction stochastic data [6].

Cycles	$d^{(1)}$	$d^{(2)}$	$d^{(3)}$	$d^{(4)}$	$d^{(5)}$	$d^{(6)}$	$d^{(7)}$	$d^{(8)}$	$d^{(9)}$	$d^{(10)}$	$d^{(11)}$	$d^{(12)}$	$d^{(13)}$	$d^{(14)}$	$d^{(15)}$	$d^{(16)}$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6200	0,11	0,03	0,03	0,21	0,08	0,1	0,08	0,26	0,09	0,1	0,06	0,06	0,06	0,06	0,1	0,15
9300	0,19	0,11	0,06	0,26	0,09	0,11	0,11	0,3	0,1	0,12	0,1	0,06	0,13	0,09	0,16	0,19
12400	0,21	0,16	0,12	0,3	0,09	0,13	0,13	0,34	0,15	0,13	0,15	0,11	0,16	0,11	0,18	0,22
15500	0,26	0,18	0,13	0,34	0,1	0,14	0,13	0,37	0,18	0,13	0,17	0,14	0,18	0,17	0,2	0,25
18600	0,3	0,19	0,15	0,35	0,11	0,18	0,14	0,4	0,19	0,14	0,21	0,16	0,21	0,18	0,22	0,31
21700	0,33	0,2	0,16	0,36	0,12	0,19	0,14	0,4	0,21	0,17	0,21	0,17	0,21	0,19	0,25	0,35
27900	0,39	0,23	0,2	0,38	0,14	0,22	0,18	0,43	0,24	0,18	0,21	0,2	0,25	0,25	0,26	0,41
34100	0,44	0,26	0,23	0,4	0,17	0,25	0,28	0,46	0,27	0,25	0,25	0,22	0,28	0,26	0,27	0,43
40300	0,47	0,27	0,25	0,42	0,22	0,26	0,34	0,49	0,27	0,26	0,26	0,25	0,3	0,29	0,28	0,45
46500	0,51	0,31	0,27	0,43	0,26	0,28	0,43	0,5	0,34	0,27	0,29	0,27	0,33	0,36	0,3	0,46
52700	0,6	0,34	0,3	0,44	0,29	0,3	0,63	0,5	0,41	0,3	0,32	0,3	0,33	0,38	0,31	0,47
65100	0,66	0,36	0,43	0,46	0,38	0,35	1,04	0,54	0,42	0,38	0,35	0,34	0,41	0,91	0,38	0,48
77500	0,91	0,39	0,56	0,47	0,46	0,37		0,59	0,47	0,48	0,44	0,4	0,45	1,5	0,44	0,5
89900	1,16	0,48	0,75	0,49	0,56	0,41		0,61	0,53	0,56	0,47	0,65	0,49		0,45	0,52
102300		0,58	0,95	0,52	0,93	0,5		0,65	0,69	0,63	0,8	0,9	0,82		0,54	0,55
114700		0,73	1,14	0,55	1,33	0,58		0,68	0,92	1,07	1,23	1,1	1,1		0,79	0,56
127100		0,81		0,59		0,67		0,72	1,25						1,09	0,57
139500		0,98		0,62		0,79		0,84								0,6
151900		1,01		0,64		0,95		0,98								0,62
154986				0,67		0,98		0,98								0,63
164300				0,71		1,1		1,09								0,65
176700				0,74												0,72
189100				0,83												1
201500				0.94												
213900				1.15												

For simplification in obtaining the posterior of model parameters, the non-informative distribution $p^0(\mathcal{M}) = \mu(\mathcal{M})$ is assumed, which can in turn be dropped as follows:

$$p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} = \underbrace{k_2 p^0(\mathcal{M})}_{\text{constant}} \int_{\mathfrak{O}} p^m(\mathcal{O}|\mathcal{M},\mathcal{C})d\mathcal{O} = k_3 L(\mathcal{M}) \tag{2.23}$$

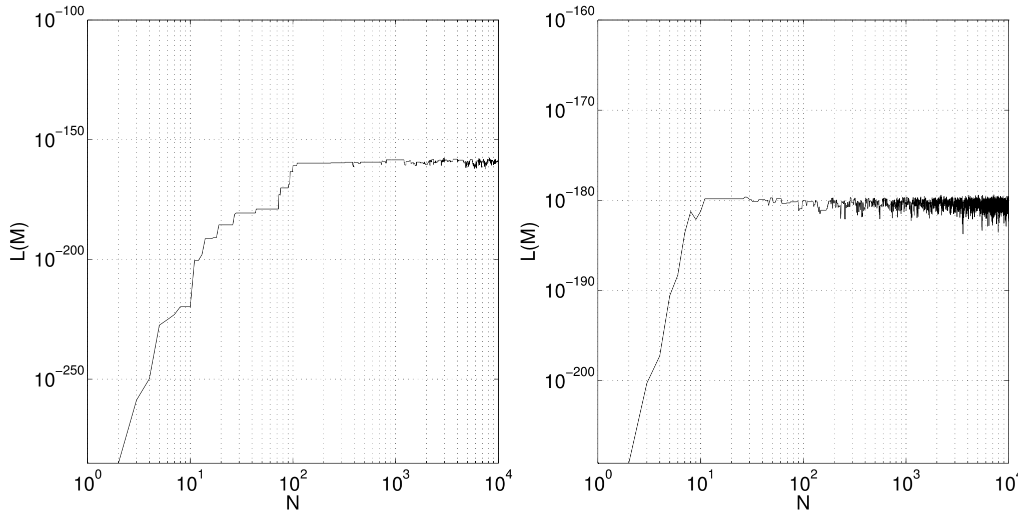
So, without lack of generality, in this example the parametric inference is contributed solely by the likelihood functions (2.13).

Additionally, by the fact that given an observed sequence, the relation between Equations 2.13 and 2.12 is a constant, Equation 2.23 can also be simplified to:

$$p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} = k_4 p^m(\mathcal{O}|\mathcal{M},\mathcal{C}_i) = k_4 \prod_{n=0}^{N} \prod_{i,j=1}^{s} \left(p_{ij}^{\mathcal{M}}\right)^{f_{ij}(n)} \tag{2.24}$$

with $N = 428$, $s = 25$ and k_4 is a normalization constant that is needed for $p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i}$ to fulfill the theorem of total probability $\int_{\mathfrak{M}} p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} d\mathcal{M} = 1$. This simplification substantially reduces the computational cost in the stochastic simulation.

To simulate the process, the MH algorithm has been implemented with $N = 10^4$ trials and $\sigma = 0.02$ for model class $\mathcal{C}_1$ and $\sigma = 0.01$ for model class $\mathcal{C}_2$. Each simulation required $2,4 \cdot 10^3$ and $2,7 \cdot 10^3$ [sec] respectively in a 2.4 [GHz] double core computer. The algorithm configuration was verified to ensure the chain is ergodic and, hence, converges to Equation 2.24 by choosing the first sample distributed according the target PDF [34]. And also, by observing that the sample stabilizes (to the expectation to target disribution) after the burn-in period, in this case of 115 and 11 samples for $\mathcal{C}_1$ and $\mathcal{C}_2$ respectivelly, as shown in figure 2.1.



(a) Model 1 (b) Model 2

Figure 2.1: Sample history of Equation 2.24 for both model classes. It's shown in logarithmic scale to correctly detect the burn-in period after which the expectance of the target distribution is stabilized. See also that model class 1 has larger likelihood value than model class 2.

The results for posterior parameters are summarized in Table 2.2 for respective models $\mathcal{C}_1$ and $\mathcal{C}_2$.

Table 2.2: Results for posterior parameters

	Parameter	Optimal m_i	Std. dev σ_{m_i}	C.O.V(%)	Burn-in period
	m_1	0.0741	0.0182	22.16	115
	m_2	0.0730-0.1	0.0308	30.46	115
$\mathcal{C}_1$	m_3	0.1139	0.0205	17.82	115
	m_4	0.3015	0.0378	12.16	115
	m_5	0.8592	0.096	1.12	115
	m_1	0.6	0.1375	19.55	11
$\mathcal{C}_2$	m_2	0.6335	0.1337	18.40	11
	m_3	0.89	0.0074	0.82	11

In figures 2.2 and 2.3 the scatterplot matrices for the simulation process are shown, with main diagonals representing marginal posterior PDF for model class $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

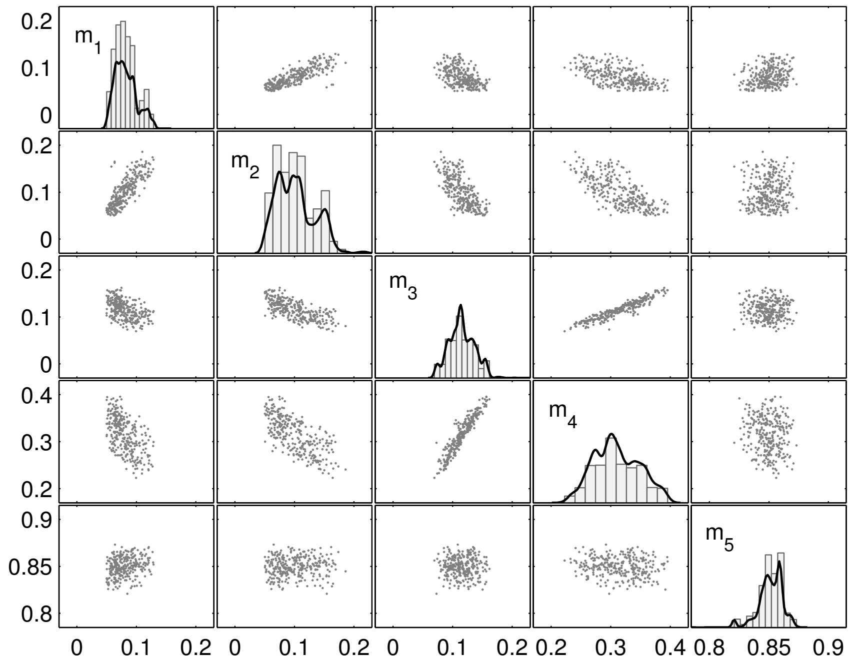


Figure 2.2: Plots of the samples in the $\mathcal{M}$ space when updating model class $\mathcal{C}_1$ with fatigue data $\mathcal{O}$. In diagonal histograms and kernel density estimate construction for parameters.

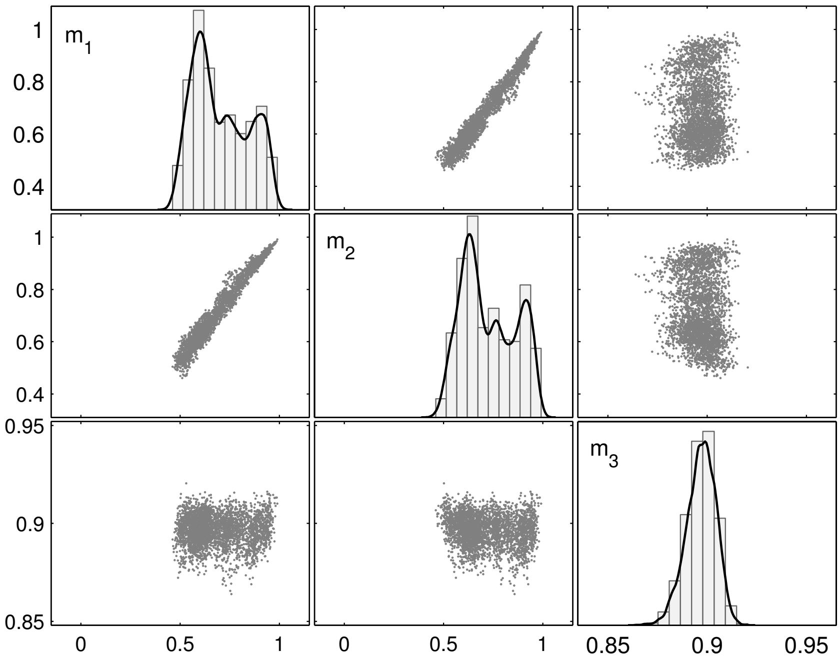


Figure 2.3: Plots of the samples in the $\mathcal{M}$ space when updating model class $\mathcal{C}_2$ with fatigue data $\mathcal{O}$. In diagonal histograms and kernel density estimate construction for parameters.

This results show that posterior parameters from model $\mathcal{C}_1$ have small uncertainty than those from model $\mathcal{C}_2$. However, this does not necessary give a good model class for future prediction because the model output may be too sensitive to parametric and modeling error [35]. The best choice between both class will be those with the higher evidence value.

Due to difficulties associate with analytically evaluate the evidence, an approximation based on stochastic simulation is used. Since $p^0(\mathcal{M})$ is a true PDF, the probability integral in Equation 2.15 can be viewed as an expectation of $L(\mathcal{M})$ over samples drawn from the prior $p^0(\mathcal{M})$, as follows:

$$p(\mathcal{O}|\mathcal{C}) = \int_{\mathfrak{M}} L(\mathcal{M}) p^0(\mathcal{M}) d\mathcal{M} \approx \sum_{i=1}^{N} L(\mathcal{M}_i) \tag{2.25}$$

with $\mathcal{M}_i \sim p^0(\mathcal{M}) = \mathcal{U}(0,1)$.

Since $L(\mathcal{M})$ is concentrated in a small volume of the parameter space (see Figures 2.2 and 2.3), $2 \cdot 10^4$ samples (in this case independents) were required to achieve the convergence of the estimator.

The results, given experimental data in Table 2.1, show an evidence of $2,84 \cdot 10^{-4}$ for $\mathcal{C}_1$ and $4,64 \cdot 10^{-4}$ for $\mathcal{C}_2$. This means that class $\mathcal{C}_1$ has better fitting capability due its largest maximum likelihood value, although $\mathcal{C}_2$ is better for robust prediction, as shown in Figure 2.4. The same result was obtained in J.Chiachio et al, through a cross-validation.

However both classes has similar evidence values, so that a good choice for damage prediction may be through a model averaging as follows:

$$p(d|\mathcal{O}) \approx \sum_{j=1}^{N_c} p(d|\mathcal{O}, \mathcal{C}_j)p(\mathcal{C}_j|\mathcal{O}) = 0,38p(d|\mathcal{O}, \mathcal{C}_1) + 0,62p(d|\mathcal{O}, \mathcal{C}_2) \quad (2.26)$$

where $p(d|\mathcal{O})$ is PDF of damage given the observed data $\mathcal{O}$ and $p(\mathcal{C}_j|\mathcal{O})$ obtained as in Equation 2.16 by assuming a equally weighted class prior $p^0(\mathcal{C}) = 1/2$. This approximation is exact only if the N_c models classes provides independent prediction for the damage or by accounting all possible correlations among all model classes. However, even discarding the correlations, this approximation still performs better than using the most plausible model class alone [27].

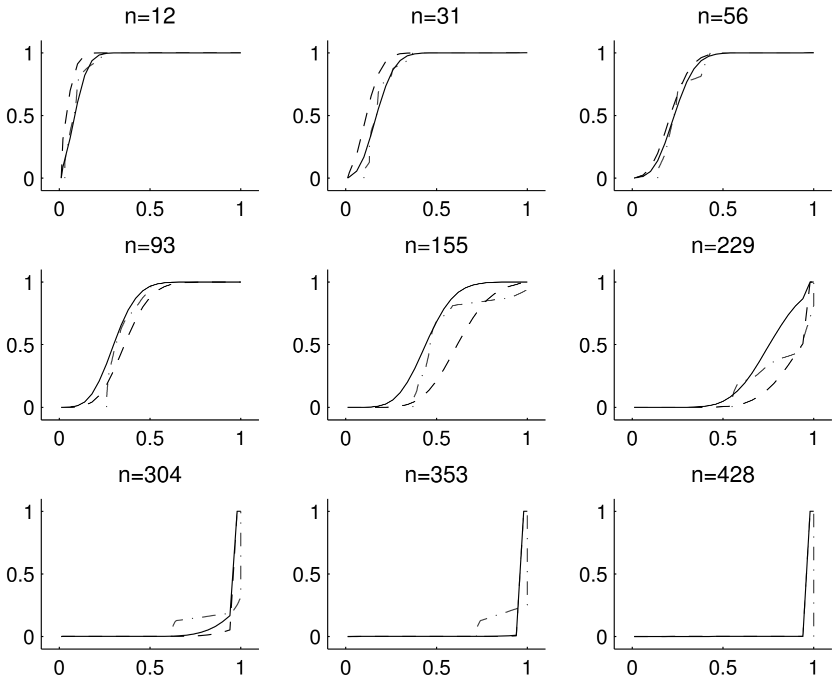


Figure 2.4: Plots of CDF of damage in nine equally spaced duty cycles covering the full time, for model class 1 and 2, solid line and dashed line respectivelly. In grey color (dashed-dot line), the empirical CDF of damage.

Effect of prior information

Due to the evidence integral is the inner product of the prior distribution and the likelihood of a model, the choice of prior distribution $p^0(\mathcal{M})$ is important for model class selection because it offers a reference for comparison in quantifying the information gain from the data. In this work, prior information about model parameters is not available, so the choice of uniform distribution is an appropriate option for parameter updating. However this option imply to absorb the prior distribution into the normalizing constant for model class at the expense of loose the information reference.

In this section, six gaussian priors are adopted to serve as an example of quantifying the goodness of fit and the information gain, as in Equation 2.20. All parameter are assumed to follow the same gaussian prior, centered each one in $\mu = \{0.25, 0.75\}$ with standard deviation of $\sigma = \{0.1, 0.2, 0.4\}$, obtaining six prior combinations (see Table 2.3).

To resolve the probability integrals in Equation 2.20, an approximation based on simulation with dependent samples drawn from $L(\mathcal{M})$ by the MH algorithm is adopted as follows:

$$p(\mathcal{C}|\mathcal{O}) = \int_{\mathfrak{M}} L(\mathcal{M}) \underbrace{p^0(\mathcal{M})}_{g} d\mathcal{M} \approx \sum_{i=1}^{N} g(\mathcal{M}_i) \tag{2.27}$$

$$\begin{aligned} \int_{\mathfrak{M}} L(\mathcal{M}) p(\mathcal{M})\big|_{\mathcal{C}=\mathcal{C}_i} d\mathcal{M} &= \frac{1}{p(\mathcal{O}|\mathcal{C})} \int_{\mathfrak{M}} \underbrace{[\ln L(\mathcal{M})]\, p^0(\mathcal{M})}_{h} L(\mathcal{M}) d\mathcal{M} \\ &\approx \sum_{i=1}^{N} h(\mathcal{M}_i) \end{aligned} \tag{2.28}$$

with $\mathcal{M}_i \sim L(\mathcal{M})$.

Results of Bayesian model class selection are shown in Table 2.3. The log-evidence and average log-likelihood function over the posterior PDF that are estimated from stochastic simulation are shown, along with the information gain, which is not directly estimated but rather calculated from the other two quantities using Equation 2.20.

Table 2.3: Results for model class selection using example gaussian priors.

	Prior PDF	$\log p(\mathcal{O}\|\mathcal{C})$	log Average fit	Information gained
	$\mu = 0.25$, $\sigma = 0.4$	-1.37	-366.53	-365.15
	$\mu = 0.25$, $\sigma = 0.2$	-5.45	-366.65	-361.20
$\mathcal{C}_1$	$\mu = 0.25$, $\sigma = 0.1$	-21.48	-367.26	-345.77
	$\mu = 0.75$, $\sigma = 0.4$	-4.61	-366.58	-361.97
	$\mu = 0.75$, $\sigma = 0.2$	-18.27	-366.99	-348.72
	$\mu = 0.75$, $\sigma = 0.1$	-70.82	-368.71	-297.88
	$\mu = 0.25$, $\sigma = 0.4$	-2.50	-415.26	-412.76
	$\mu = 0.25$, $\sigma = 0.2$	-8.69	-415.25	-406.55
$\mathcal{C}_2$	$\mu = 0.25$, $\sigma = 0.1$	-30.48	-415.21	-384.72
	$\mu = 0.75$, $\sigma = 0.4$	-0.18	-415.31	-415.12
	$\mu = 0.75$, $\sigma = 0.2$	-0.69	-415.31	-414.62
	$\mu = 0.75$, $\sigma = 0.1$	-2.21	-415.34	-413.12

This table remarks the effect of prior on the evidence of a model class. It is shown that irrespectively the prior, the model class $\mathcal{C}_1$ is the best for fitting capabilities but not necessarily the best for robust prediction because of its larger model complexity, which is indicated by the larger information gain. This also concord with the conclusion obtained when improper prior was adopted.

2.5 Conclusions

A new methodology is proposed to infer the fatigue-based damage evolution in composites, as solution of a general bayesian inverse problem. This framework has the versatility for accounting all possible information about data, model and relation between both. Given the required simplifications, this capability allows to be used for parameter estimation in fatigue testing, for model updating with monitoring damage data, or even for selecting the model classes with best evidence for a specific material data set. The methodology has been validated on a example for obtaining the posterior PDF of model parameters from two nonstationary models, in terms of fitting to stochastic damage data. It has been shown that the posterior PDF and the associated model evidence can be obtained using a Markov Chain Monte Carlo method like Metropolis Hastings algorithm with a moderate computational cost.

As a drawback, this methodology does not allow the duty cycle (Dc) and number of states (s) to be incorporated as model parameters because it would make $\mathbf{F}$ not to be a sufficient statistic, which is mandatory in the definition of the likelihood function. Depending on the specific set of data, this parameters can influence on fatigue stochastic modeling, so a previous sensitivity study for setting their values is recommended.

Other phenomena in composites like porous density, crack growing intensity, etc., that imply cumulative processes can be benefited by applying this method by only obtaining a set of data from a state variable observed through time.

Further work is needed to extrapolate this method to Continuos Time Markov Process that would allow to incorporate whatever heterogeneous set of data and hence would confer independence on (Dc) and (s).

Chapter 3

Reliability in Composites-a selective review and survey of current development

As a response to the rampant increase in research activity within reliability in the past few decades, and to the lack of a conclusive framework for composite applications, this article attempts to identify the most relevant reliability topics to composite materials and provide a selective review. Available reliability assessment methods are briefly explained, referenced and compared within an unified formulation. Recent developments to confer efficiency in computing reliability in large composite structures are also highlighted. Finally, some general conclusions are derived along with an overview of future directions of research within reliability of composite materials and their influence on design and optimization.

3.1 Introduction

The need to incorporate uncertainties in engineering design has long been recognized. In contrast to the traditional approach of using safety coefficients, the probabilistic design allows the estimation of reliability by considering the stochastic variability of the data for which designs are qualified to have a given reliability value [36]. The performance is generally evaluated by means of a variable such as the displacement of a point, the maximum stress, etc., or by a set of them. Variability in the performance of composite materials arises mainly from the variability in constituent properties, fibre distribution, structural geometry, loading conditions and also manufacturing process. As an orthotropic material, this variability can lead to a catastrophic failure mainly when inaccuracy arises in loading direction or fiber orientation, while the traditional approach of safety factors could result in a costly and unnecessary conservatism [37], which is a serious drawback for making composites competitive and sustainable.

In the recent decades, a large number of articles have been reported to cover probabilistic failure and reliability in composites. The first contributions were in the form of probabilistic strength over aircraft applications [38, 39]. Shortly later, the β-method by Hasofer Lind [40] was applied to laminated plates [41]. Wetherhold and Ucci [42] evaluated reliability methods used in composites through an example and Soares [43] made an overview and gave a perspective about deriving reliability from ply to laminate level.

However, due to the inherent variability in the material behavior, reliability in composites requires that several decisions are adopted. The reasons for that are multiple: 1) there are a wide range of possibles failure functions to adopt, 2) numerous influencing random variables need being incorporated, 3) several reliability methods arise and 4) there are different ways to consider reliability for a laminate, as shown in Figure 3.1.

According to Soares [43], several results have been reported, but unfortunately, a lack of consensual framework is observed in literature for the use of

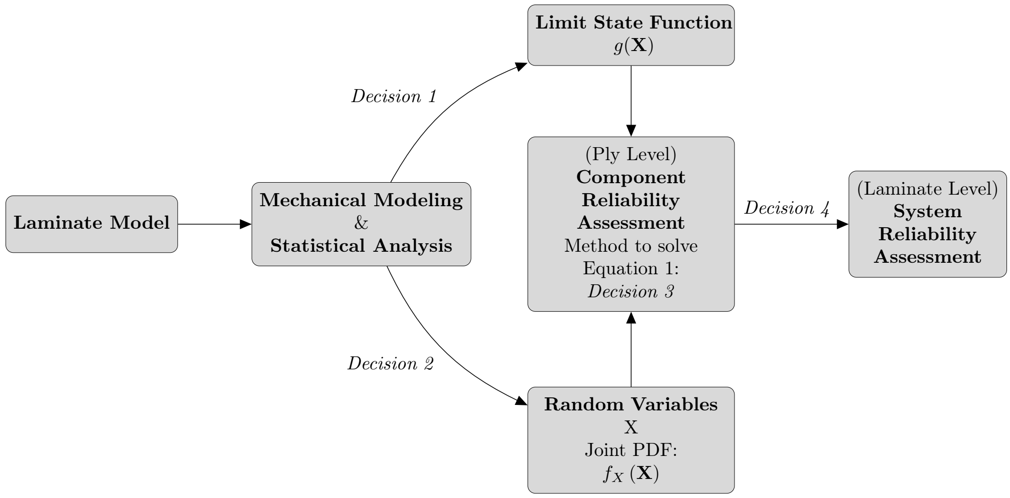


Figure 3.1: Schematic representation of a reliability problem in composites.

methods, failure criteria, statistical description of mechanical variables and even for conclusions. These, together with new trends to confer efficiency in reliability calculation require the need for a thorough and up-to-date review of the literature in this area.

Hence, as a first step to provide a basis for a discussion about this claim, the present paper reviews some fundamental concepts of reliability from an orthotropic material perspective. This work highlights the results where connections between reliability and failure criteria in composites are most striking. It also gives a concise background of reliability methods with special emphasis to those that already have a fruitful impact on composite applications, and identify results which evaluate the influence of such variability in methodology. Section 3.3 gives a set of examples where ideas of reliability in composite laminates have demonstrated advantages for laminate design and optimization, and identifies areas of particular potential for further development. In Section 3.4, some basic notions of techniques to confer computational efficiency are recalled. It is also shown how they provide a framework for reliability assessment of large structural composites systems. Section 3.5 briefly concludes.

In Table **??**, additional information related to the decision topics is provided, that helps to derive a perspective of reliability in composites.

This work is not only focused on reliability procedures but also in reliability based design and safety factor calibration, which are topics where reliability calculation is crucial.

Throughout the paper, methods and techniques to assess reliability from literature are expressed within an unified formulation which helps this review to be read with independence of the references.

Due to the large number of articles involved and the lack of electronic access to many conference proceedings, the emphasis of this review is on the more accessible refereed journal articles.

3.2 Reliability formulation. Ply level

The essence of the structural reliability problem is the probability integral:

$$P_f = \int_{\mathbf{X}|g(\mathbf{X})\leq 0} f_X(\mathbf{X}) d(\mathbf{X}) \tag{3.1}$$

where $\mathbf{X} = \{x_1, \ldots, x_n\}^T$ is a vector of random variables that represent uncertain quantities influencing the state of the structure, $f_X(\mathbf{X})$ is the probability density function (PDF) and $g(\mathbf{X}) \leq 0$ denotes a subset of the outcome space where failure occurs [44].

For a mathematical analysis, is necessary to describe the failure domain $g(\mathbf{X}) \leq 0$ in an analytical form, which is widely named as limit state function (LSF). The next section 3.2.1 is dedicated to expose different formulations of the LSF used for reliability in composites. Methods of resolving the integral in Equation 3.1 will be commented in section 3.2.2.

Both mentioned topics about Equation 3.1, together with the discussion about what to consider as random variables, cover almost all of the literature discussion on composites reliability.

3.2.1 Concept of failure

Failure criteria used in probabilistic analysis are the same as used in a deterministic approach, so the accuracy of reliability analysis is critically dependent on an appropriate criterion for the study conditions. Composite materials display a wide variety of failure mechanisms as a result of their complex structure and manufacturing processes. So, in literature, a wide spread of possibilities for LSF have been developed, all apparently valid depending on each specific problem [45–47]. Recently, a comprehensive review of failure theories is given by Orifici et al. [48], in which a concise way to classify them is also proposed according to whether they are based on strength or fracture mechanics theories, whether they predict failure in a general sense or are specific to a particular failure mode and whether they focus on in-plane or inter-laminar failure. Following this classification, the in-plane general strength failure criteria ranges almost all the literature in reliability, although important contributions have also been derived in composites reliability based on other LSF like damage based criteria [49], crack initiation over pipe surfaces [50, 51] and buckling failure [37, 52].

In relation to the scale level, although recent advances in multiscale failure have been reported [53, 54], the body of reliability literature takes a mesoscale or macroscopic approach to the failure as the phenomenological model to analytically describe the reliability of composites. An interesting approach which seems to be a first step to multiscale reliability evaluation of composites have been recently reported [55]. In these study, a micro and macro-scale evaluations of the Tsai-Hill LSF are critically compared in a reliability framework showing good agreement and conclude that reliability analysis starting from micro level would help benchmarking corresponding macro-level analyses.

In reliability literature, due to the complexity of the failure concept, a step by step approximation to the subject is observed, from uniaxial tension reliability [39, 56] to a more general multiaxial case in recent years. In the latter multiaxial case, two main approaches have been proposed: the interactive and non-interactive, depending on the stress working or not collectively towards the failure of the element [57].

The non-interactive case considers reliability at each stress direction independently [57] or exclusively the most stressed direction [58, 59], in conjunction to Max Stress, Max Strain or Max Work criteria as LSF. This approach has not been extensively used in reliability due to its well-known insecure position for certain stress combinations [60].

Among the interactive failure criteria, Quadratic Failure Criteria, are the most used in reliability mainly because a mature knowledge has been achieved in considering quadratic functions as LSF for reliability [61]. This criteria takes into account the interactions between different stress components. The LSF for the Quadratic Failure Criteria in the component orientation for one ply is expressed by:

$$g(\mathbf{X}) = 1 - (F_{ij}\sigma_i\sigma_j + F_i\sigma_i) \leqslant 0 \tag{3.2}$$

where $F_{ij} = F_{ij}(\mathbf{X})$, $F_i = F_i(\mathbf{X})$ are the strength parameters, $\sigma_i = \sigma_i(\mathbf{X})$ the stress in the tensor component i, with $i, j = 1, 2, 6$ the stress or strain tensor components [60]; and $\mathbf{X} = \{x_1, \ldots, x_n\}^T$ the random variables written in matricial notation.

Particularly, the quadratic Tsai's criterion has been fairly used in literature motivated by being one of the existing mature theories [62–64]. The main contributions in reliability have used the Tsai's criterion, although not exclusively, as shown in Table **??**.

Under such variability of failure criteria to define the LSF, certain authors [42, 58, 65–67] declined to probe with several possibles and compare to experimental or reference reliability data when available. In Nakayasu and Maekawa [68] a quantitative trade-off for six different failure criteria from the viewpoint of reliability-oriented design of composite materials was carried out. This work yielded an important conclusion about the need to verify the criterion suitability under specific load combinations, which also agrees with Lin [69].

3.2.2 Reliability methods used in composites

Methods used in literature for computation of the probability integral in Equation 3.1, are reviewed in subsequence chapters. To avoid duplication in the current review but conferring a sufficient conceptual framework, the methods have been presented in a concise way.

3.2.2.1 Fast probability integration methods (FPI)

FPI methods rely on approximating the failure surface by a predetermined geometric form for which evaluation of the integral is practical [44].

A most probable point (MPP) is searched during the evaluation, over which the failure surface is approximated by such geometric form. The distance between the origin and the MPP corresponds to the radius β of a n-sphere beside the failure domain and tangent with it, in the MPP. In literature, this β value is called as *Reliability Index* and means the distance from MPP to the origin in units of standard deviation, as shown in Figure 3.2.

In FPI methods, first order reliability methods (FORM) and second order reliability methods (SORM) are included.

First order reliability methods The well known technique FORM uses a linear approximation of the LSF in the vicinity of the design point to evalu-

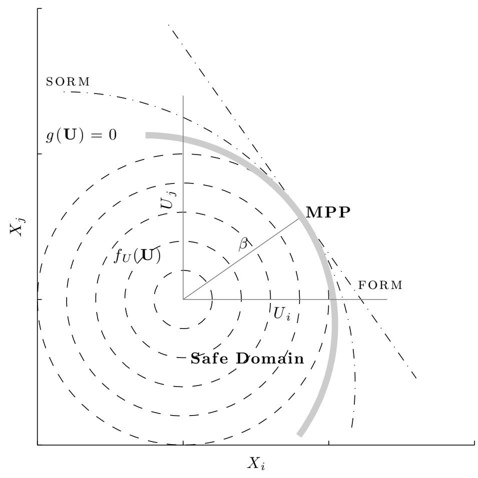


Figure 3.2: Schematic representation of FORM/SORM approximations

ate the β index [40]. This method requires standard normal non-correlated variables, so the vector of random variables $\mathbf{X}$ must be transformed into standard non-correlated variables vector $\mathbf{U}$ taking,

$$\mathbf{U} = \phi^{-1}(F_X(\mathbf{X})) \tag{3.3}$$

where $F_X(\mathbf{X})$ and ϕ^{-1} are the cumulative distribution function and the inverse of the standard cumulative distribution function for the vector of normal variables $\mathbf{X}$, respectively.

The reliability index β is then calculated by:

$$\beta = min(\mathbf{U} \cdot \mathbf{U}^T)^{\frac{1}{2}} \tag{3.4}$$

which represents an Euclidean distance between the origin and the failure function $g(\mathbf{U})$, in the non-correlated normal standard space U, as shown in Figure 3.2. If any correlation exists in the random variables, a Cholesky decomposition of the covariance matrix may be used to transform from the real space to the non-correlated standard space [70]. In case of non normal variables, Rackwitz-Fiessler Method [61] can be employed. In case of correlated and non-normal variables, the Rosenblatt transformation is recommend [71, 72].

The value of the density function integrated over the hyper volume is found to be equal to the standard normal integral (distribution function) at β, and so, the reliability R can be expressed as,

$$R = \phi(\beta) \tag{3.5}$$

while the probability of failure is the complement,

$$P_f = 1 - R = 1 - \phi(\beta) = \phi(-\beta) \tag{3.6}$$

Second order reliability methods To improve the approximation of the failure surface beyond the level employed in FORM, additional information about the failure surface is required [44]. The SORM use the β value in conjunction with the second derivatives of $g(\mathbf{X})$ at MPP. The method is based on a general quadratic expansion by expanding the failure surface $g(\mathbf{X})$, into a second order Taylor series about the MPP. Since the curvatures may have positive, negative and zero values; parabolic, elliptic, or hyperbolic forms may result. These methodology requires complicated integrations that restrict the applicability in the study of reliability [73]. Two simpler forms are extensively used in literature for the quadratic approximation that are relatively simple for use: the rotational paraboloid and non-central hyphersphere forms based on a predetermined axis [61].

Since only one curvature is used with the predetermined forms, a method for determining that one curvature must be selected. For conservatism, the largest positive curvature κ it is used, and hence the smallest radius of curvature since $r = {}^1\!/\!_\kappa$.

The rotational paraboloid approximation gives,

$$P_f = \int_0^\infty \phi \left[\beta - \frac{t}{2r}\right] f_{\chi^2_{n-1}}(t)dt \tag{3.7}$$

where $f_{\chi^2_{n-1}}$ is the chi-square density function with n degrees of freedom.

Analogously, the non-central hypersphere approximation gives,

$$P_f = 1 - \chi^2_{n,\delta}(r^2) \tag{3.8}$$

where $\chi^2_{n,\delta}(r^2)$ is the non-central Chi-Squared distribution with non-centrality parameter $\delta = [r - \beta]^2$.

3.2.2.2 Monte Carlo methods (MCM)

Monte Carlo method is a very simple and accurate approach mainly used as reference or exact method [44, 74, 75].

Given the joint probability density function $f_X(\mathbf{X})$ of $\mathbf{X}$, then the failure probability in Equation 3.1 can be alternatively written as,

$$P_f = \int_{\mathbf{X}|g(\mathbf{X})\leq 0} f_X(\mathbf{X})d(\mathbf{X}) = \int_{\mathbf{X}} I\left[g(\mathbf{X})\right] f_X(\mathbf{X})d(\mathbf{X}) \tag{3.9}$$

where $I\left[g(\mathbf{X})\right]$ is an indicative function defined by:

$$I[g(\mathbf{X})] = \begin{cases} 1 & \text{if } g(\mathbf{X}) \leq 0 \\ 0 & \text{if } g(\mathbf{X}) > 0 \end{cases} \tag{3.10}$$

Using the indicative function, it is possible to evaluate the probability integral in Equation 3.1 over the whole domain and not only over the failure domain. This probability integral in Equation 3.9 can be viewed as a mathematical expectation of $I\left[g(\mathbf{X})\right]$ with $\mathbf{X}$ distributed as $f_X(\mathbf{X})$, and this perspective leads to the direct Monte Carlo method, where P_f is estimated as a sample average of $I\left[g(\mathbf{X})\right]$ over independent and identically distributed samples of $\mathbf{X}$ drawn from the PDF $f_X(\mathbf{X})$, as follows:

$$P_f = E\left[I[g(\mathbf{X}^j)]\right] \simeq \frac{1}{n_s}\sum_{j=1}^{n_s} I\left[g(\mathbf{X}^j)\right] \tag{3.11}$$

where n_s is the number of simulations, $\mathbf{X}^j$ the vector of random variables of the j$^{\text{th}}$ sample. The error of this method is only dependent on n_s and so it is extremely robust with respect to applications. The term $\sum_{n_s}^{j} I\left[g(\mathbf{X}^j)\right]$ represents the sum of the number of simulations (n_f) in the failure domain, and so Equation 3.11 may be also be written as,

$$P_f \simeq \frac{n_f}{n_s} \tag{3.12}$$

This method has a serious drawback in cases of small failure probabilities, by the fact that the total number of required simulations increases drastically. Hence, attention has been focused on developing more efficient simulation methods.

For the structural reliability problem, the most promising technique appears to be the importance sampling method (MC-IS) [76]. This method reduces the variance of the estimate by sampling more frequently from inside the failure domain.

Following the same concept of failure probability as a mathematical expectation, Equation 3.9 may be also written as follows:

$$P_f = \int_{\mathbf{X}|g(\mathbf{X})\leq 0} f_X(\mathbf{X})d(\mathbf{X}) = \int_{\mathbf{X}} \underbrace{\frac{I[g(\mathbf{X})]f_X(\mathbf{X})}{h(\mathbf{X})}}_{H(\mathbf{X})} h(\mathbf{X})d(\mathbf{X}) = E\left[H(\mathbf{X}^j)\right] \tag{3.13}$$

where $H(\mathbf{X})$ is called the *importance sampling quotient* and $\mathbf{X}^j$ distributed as $h(\mathbf{X})$. h can be selected to shift and spread the simulations close to the failure domain. h is assumed to be appropriately chosen such that H has finite variance under h.

3.2.2.3 Analytical methods

In order to confer more simplicity in reliability calculations, some analytical approaches have appeared for composites applications. Only few of this approaches have been successfully developed, and in their range of application, they have been demonstrated good agreement as compared to MCM, taken as a reference.

Edgeworth expansion method (EDW) and Pearson's empirical distribution (PRS) In Philippidis and Lekou [77] two analytical approaches, namely a functional expansion technique and the introduction of Pearson's semi-empirical distribution function, were developed for off-axis UD FRP composites for the general plane stress. In that work, only strength parameters were considered as random variables, each following a Weibull distribution.

The quadratic version of the failure tensor polynomial in the principal material coordinate system under plane stress conditions, was considered as follows:

$$g(\mathbf{X}) = 1 - (F_{ij}\sigma_i\sigma_j + F_i\sigma_i) \tag{3.14}$$

with $\mathbf{X} = \mathbf{X}^T$ the strength random variables, $F_{ij} = F_{ij}(\mathbf{X}^T)$, $F_i = F_i(\mathbf{X}^T)$ the strength parameters [60] for one ply and σ_i stress tensor components, considered as deterministic values.

The purpose of this two analytical approaches, was to determine the CDF (F_g) of the failure condition $g(\mathbf{X})$, by which the failure probability $P(g \leqslant 0)$ can be obtained.

The EDW, that was previously introduced in off-axis composites for the case of uniaxial tension [78, 79], was used to predict the cumulative probability of complex systems in terms of individual component moments

[80]. The failure function in Equation 3.14, was expanded in a multivariable Taylor series in term of central moments of the random variable, g. This is given by:

$$F(g) = \Phi(g) - \frac{1}{3!}\frac{\mu_3}{\mu_2^{3/2}}\Phi^3(g) + \frac{1}{4!}\frac{\mu_4}{\mu_2^2}\Phi^4(g) + \frac{10}{6!}\frac{\mu_3}{\mu_2^{3/2}}\Phi^6(g) + \ldots \tag{3.15}$$

where μ_k are the central k-moments of the LSF g and $\Phi^n(g)$ is the n$^{\text{th}}$ derivate of the normal CDF $\Phi(g)$.

This method was further developed for the case of a laminate in a plane stress state considering the strength properties as stochastic variables [81], and in a more recently work [82] by considering the elastic and thermal properties as random too. In the latter work, it was demonstrated over wind turbine blades, that the stochastic nature of the material elastic properties drastically affects the failure locus, whereas, on the contrary, the effect of the material thermal properties is minimal within the temperature range met during operation of wind turbine rotor blades.

In PRS method, the unknown CDF of the failure condition is alternatively fitted by empirical statistical distributions once the central moments of g are calculated. As an example in Philippidis and Lekou [77], the group of distribution families proposed by Pearson, called as Pearson Families generated as a solution to the differential Equation 3.16 [83], were considered by proper choice of the parameters λ and b_i $(i = 0, 1, 2)$.

$$\frac{df(g)}{dg} = \frac{(g - \lambda)}{b_0 + b_1 g + b_2 g^2} f(g) \tag{3.16}$$

The Pearson distribution families include the Normal, Beta (Pearson Type I), and Gamma Distribution (Pearson Type III). From Equation 3.16, after some detailed algebraic manipulations, the constant parameters can be expressed in terms of the central moments of the distribution function.

By using the coordinate transformation $k = g - \lambda$, Equation 3.16 reads:

$$\frac{df(g)}{dk} = \frac{k}{B_0 + B_1 k + B_2 k^2} f(g) \tag{3.17}$$

where B_i are certain algebraic linear combinations of b_i and λ for simplicity. If the roots of the polynomial in the denominator of Equation 3.17 are real and of the opposite sign, the distribution $f(g)$ reduces to Beta distribution $B(p, q)$, whit parameters p, q found by equating the Pearson distribution's moments with that of the failure function.

Finally, for evaluating the cumulative distribution function by which can be derived the failure probability, was used the next expression:

$$\frac{1}{B(p,q)} \int_0^z z^{p-1}(1-z)^{q-1} dz \tag{3.18}$$

with $(p, q > 0, 0 \leq z \leq 1)$ and z as a algebraic function of roots of the polynomial in the denominator of Equation 3.17.

In this work, several comparisons between analytical EDW, PRS, MCM and a semi-determinisitic failure analyses, were made considering different fibre angle and assumptions for the Tsai-Wu failure domain. The results obtained with the analytical approaches were shown to be in excellent agreement with experimental or Monte Carlo data.

Generalization of LSF Another relevant result in analytical methods for reliability in composites comes from Gurvich and Pipes [84]. A new approach considering the LSF in the form of a random linear function of products of applied random stresses is presented, in stead of the traditional consideration of the LSF as a random non-linear function of the stresses (see Equation 3.2). This approach allows to obtain exact evaluation of the main statistical parameters (moments) of the LSF considered as a random function. The starting point is the consideration of a deterministic 3-D framework of the LSF in a more general formulation as follows,

$$g(\mathbf{X}) = 1 - \left(\prod_{ij} \sigma_{ij} + \prod_{ijkl} \sigma_{ij}\sigma_{kl} + \dots \right)$$
$$i, j, k, \dots = x, y, z, \dots ; \tag{3.19}$$

where $\mathbf{X} = \left(\prod_{ij}, \prod_{ijkl}, \sigma_{ij}, \sigma_{ijkl} \right)$; with $\prod_{ij}$, $\prod_{ijkl}$,.. the strength tensors and σ_{ij}, σ_{ijkl},..., the tensor of the applied stress state.

The following matrix columns were introduced by the rules,

$$[\mathbf{s}_t] = [s_1, s_2, \dots, s_n] = [\sigma_{ij}, \sigma_{ij}\sigma_{kl} \dots]$$
$$[\boldsymbol{\rho}_t] = [\rho_1, \rho_2, \dots, \rho_n] = \left[\prod_{ij}, \prod_{ijkl} \cdots \right] \tag{3.20}$$

where s_m are components characterizing all necessary combinations of the stresses in increasing order, ρ_m are the strength characteristics and n is the number of elements in the matrices.

Thus, Equation 3.19 may be presented as,

$$g(\mathbf{X}) = 1 - \left(\sum_{m=1}^{n} \rho_m s_m \right) \tag{3.21}$$

which is useful in a probabilistic framework, since this allows one to consider g as a linear function of random parameters of the problem as follows:

$$g = 1 - [\tilde{\mathbf{p}}_t] [\tilde{\mathbf{s}}] = 1 - \left(\sum_{m=1}^{n} \tilde{p}_m \tilde{s}_m \right) \tag{3.22}$$

In this formulation, the random matrices $[\tilde{s}]$, $[\tilde{p}]$ may be determined by the mean matrices-column $[\bar{s}]$, $[\bar{p}]$ and the correlation matrices $[K_s]$, $[K_\rho]$, respectively; all of them considered as initial data.

Therefore, basic statistical characteristics of g, such the first two moments: μ_1 and μ_2, can be obtained as,

$$\mu_1 = 1 - \left(\sum_{m=1}^{n} \bar{p}_m \bar{s}_m \right) \tag{3.23}$$

$$\mu_2 = \sum_{m'=1}^{n} \sum_{m''=1}^{n} \{K_{sm',m''}\bar{p}_{m'}\bar{p}_{m''} + K_{pm',m''}\bar{s}_{m'}\bar{s}_{m''} + K_{sm',m''}K_{pm',m''}\} \quad (3.24)$$

where $K_{sm',m''}$, $K_{pm',m''}$ are the correlations between random variables $s_{m'}$, $s_{m''}$ and $p_{m'}$, $p_{m''}$ respectively; with $(m', m'' = 1, \ldots, n)$. The possibility of considering all possible correlations between random variables is an important advantage of this method [84]. Finally, reliability R was proposed to be calculated as a probability of the condition $g(\mathbf{X}) \leqslant 0$,

$$R = P\{g \leqslant 0\} = \int_{-\infty}^{0} f_g(g)dg \quad (3.25)$$

where f_g is the probability density function of g. The only assumption of this approach is connected with a type of distribution g: Normal, Weibull, Gamma Function, etc. In all of the remaining methods cited above, reliability calculation requires an assumption regarding the type of the distributions for strength and/or stress, whereas Gurvich's method requires those in the type of distribution g. An interesting discussion between this analytical method in relation to the others is done at the end of Gurvich's work.

3.2.2.4 Numerical methods

In a numerical scheme, particularly in the context of finite element modeling, the stochastic finite element modeling (SFEM) are receiving special attention for reliability, due to the technological advances in the available computational power [85]. SFEM involves finite elements whose properties are random. These new advances have been carried out in an effort to generate statistics from a response vector for each node [86, 87].

There are three main variants of SFEM in the literature: a) the perturbation approach [88] which is based on a Taylor series expansion of the response vector, b) the spectral stochastic finite element method (SSFEM) [89] where each response quantity is represented using a series of random

Hermite polynomials and c) Monte Carlo simulations (MCS) [90–92] based on independent sampling of the response vector.

In composites applications, Lin [69] used the stochastic finite element method (SFEM) to predict the reliability of angle-ply laminates with different types of buckling failure modes subject to in-plane edge random loads. This author also provides a comparison of different reliability methods and different failure criteria using (SFEM) to derive for the statistics of the First-Ply-Failure (FPF) load by mean-centered second-order perturbation technique. The results were compared with experimental FPF load data of centrally loaded composite plates with different lamination arrangements to study the accuracy of the methods.

Onkar et al. [67] used SFEM by the first order perturbation techniques and studied the form to generate statistics for the failure load index using Tsai-Wu and Hoffman as failure criterion in orthotropic plates with random material properties and random loads. In this case, the results were compared with analytical solutions.

Ngah and Young [36] demonstrated an application of SSFEM in a composite panel subject to random loads and constitutive properties. Covariance and probability density functions were derived for different approximation schemes. A comparative study of accuracy and computationally effort of SSFEM versus MCS, was also presented.

3.2.2.5 Comparison between reliability methods

Due to the wide range of reliability approaches and the lack of results coincidence when they are applied to composites, several authors have declined to contrast different well accepted reliability methods to a specific composite application or to check one proposed method to a experimental data. All examples encountered in literature, use at least MCM as a reference.

In Ucci [73] the FPI methods and MCM was presented, and a comparison between them was done considering both Tsai-Wu and Tsai-Hill as failure criteria in different loading levels and ply angles. A sensitivity study was done to evaluate the influence of each stochastic variable in the reliability calculation. The comparisons were performed over three main fields: accuracy, conservatism and computational speed.

For accuracy, FPI was observed to derive satisfactory accuracy in cases of low stresses and moderate fibre angle (it is pointed out the interval $30° - 40°$), when preferably using Tsai-Wu as failure criteria. In extremely low or high orientation angles, near 0° and 90°, planar FPI were seem to be quite accurate.

When studied the conservatism, the report concluded the need to consider the curvature in the MPP. Particularly, for planar FPI, independently of the accuracy, the conservatism would be depend upon the curvature is safe or unsafe.

In computational speed, this work does not give substantial conclusions as compared to others [93] cited in section 3.4. However, an interesting result about computational cost as compared to MCM was implicitly derived through reduction of variables to be sampled in MC-IS by a sensitivity analyses, by the fact that depending on each specific case, the bulk of the reliability value depends upon several localized stochastic variables.

That conclusion was later explicitly pointed out by Di Sciuva and Lomario [37], who compared FORM methods with MCM and explicitly pointed out for Directional Cosines, using important factors, as an efficient method to reduce the stochastic variables to be sampled in MCM without significant less of accuracy. In this work, a laminated composite flat plate loaded by compressive distributed forces acting in its mid-plane was studied, with the LSF defined analytically for buckling load. The results showed acceptable level of accuracy when FORM methods were used in this specific case, in which the buckling LSF fits well to linear. Directional Cosines were pointed out to be efficient for this calculation.

In Lin [69] three different methods, MCM, FORM and first-order second moment method, were used to calculate the reliability and compared to experimental FPF of centrally loaded laminated composite plates with different lay-ups. In the first-order second moment method, the SFEM was used to derive for the statistics of the FPF load from those of the baseline random variables. The LSF and baseline for load values, were also took as variables for comparison. As conclusion, this work also pointed out to FORM together with Tsai-Wu for obtaining reasonably good result. However according to [42], this conclusion may be erroneous with different tensional ranges and fiber orientations than used for the study.

In [82] the EDW previously introduced by Philippidis [77], was compared to MCM and FORM with Tsai-Hahn as failure function for FPF noting that the EDW estimation overrate the structural load carrying capacity of the laminated plate.

3.3 Reliability and design of composites laminates

Since a laminate can be viewed as a mechanical set of plies, whole laminate reliability may consider systems reliability.

An accurate evaluation of laminate reliability is essential almost all in those areas where reliability determines the final composite design, like reliability based design and safety factor calibration, which are designing tools fully used in research and industry.

3.3.1 Laminate reliability

In composites, Soares [43] presented an overview of methods used for laminates and pointed out two main approaches: the bounding and system reliability formulation [57]. The former establishes an interval in which relies the actual reliability, while in system reliability is considered the progressive failure process. The vast majority of authors use bounding formulation for laminate failure consideration in reliability subject. Most of them, for simplification in a safe position, propose lower bound reliability with FPF as LSF, which implies the ply considered as failure unit. For this reason and to provide a basis for a discussion about this claim, its timely to consider the subject again in the form of fundamental concepts.

3.3.1.1 Bounding formulation

The starting point for such bounding formulation is the definition of the unit of failure as the unit statistically homogeneous for the failure. Two such units have been proposed: the ply units and modal units [57]. The first one assumes that individual plies are the failure units while the modal failure units allow the recognition of three potential modes of failure within each ply: longitudinal, transverse and shear; resulting in $3n$ failure units for an n-ply laminate. Obviously that last failure unit implies non interaction between longitudinal, transverse and shear effects which assumes non-interactive failure, exposed in Section 3.2.1.

The upper bound reliability limit, considers that ultimate failure of the laminate will not occur until every individual unit had failed. Thus, the probability of failure for the laminate is given by the product of probabilities of failure for the individual units. In terms of reliabilities, this gives the following expressions:

$$R_{Uply} = 1 - \prod_{i=1}^{n}(1 - R_i) \qquad \text{Non-Interactive} \qquad (3.26a)$$

$$R_{Umodal} = 1 - \prod_{i=1}^{n} \prod_{j=1,2,6} R_{ij} \qquad \text{Interactive} \qquad (3.26b)$$

where R_i is the reliability of i$^{\text{th}}$ ply, and R_{ij} is the reliability of the j$^{\text{th}}$ mode of layer i.

As lower bound reliability, a series system formulation is proposed, so that the failure of the whole laminate is subject to the failure of the weakest unit. In reliability terms,

$$R_{Lply} = \prod_{i=1}^{n} R_i \qquad \text{Non-Interactive} \qquad (3.27a)$$

$$R_{Lmodal} = \prod_{i=1}^{n} \prod_{j=1,2,6} R_{ij} \qquad \text{Interactive} \qquad (3.27b)$$

whit the same meaning for R_i and R_{ij} as described above.

The most representative works that belong to bounding approach are cited by Soares [43] review. Those up to Soares [43] are nextly introduced in which interesting conclusions about composites design are also highlighted.

Kam and Chang [66] used experimental distributions of FPF load for validation of different types of baselines probability density functions on the bounding failure probability over centrally loaded graphite-epoxy laminated composite plates with different lamination arrangements. The failure data were compared with those obtained analytically with a F.E.A for stress calculations, in both interactive and non interactive failure criteria. Results showed that, in general, differences between the experimental and theory are small (less than 12%) irrespective to the types of probability distributions used for modeling the lamina strength parameters and FPF load.

More recently, Frangopol and Recek [94] presented a benchmark study of laminate failure probability by MCM considering random loads with Tsai-Wu as failure criterion. Two main cases were studied: uniaxial loaded single-layer laminate plate of graphite/epoxy and two layers laminate plate of glass epoxy, each one subjected to uniaxial and biaxial tension. In such two cases, the material strength parameters were considered as deterministic, and stresses as lognormal distributed random variables since no information on the type of distribution for principal stresses was available for this study. As a first conclusion of these work, the importance of the mean value of the principal stress, specially in tension-tension case, was shown and the low influence of coefficient of correlation between principal stresses on the probability of failure, was also highlighted.

Another important conclusion was pointed out about the effects on reliability of additional layers in a composite laminate. In presence of new layers, the plate does not necessarily increases the reliability but it's depends on the fibre orientation and its thickness ratios. The special case of two orthogonal layers was studied, showing that the weakest more stressed lamina approximately determines the whole reliability, which implicitly supports the weakest link hypothesis in this specific case.

Others results encountered up to Soares [43] review also use the bounding approach for system reliability calculation in composites, particularly FPF [49, 65, 67, 69, 82, 93]; which are commented in more suitable chapters of this review.

3.3.1.2 System reliability formulation

In system reliability formulation, the approach consists in considering the step by step failure process of the laminate. The bounding formulation just described, does not attempt to represent the whole collapse process of the laminate. Indeed, such approach establishes an interval in which relies the desired reliability value. Although an attempt to precisely describe probabilistic failure of a laminate would be really impacting and necessary, the methodology of system reliability has been shortly explored in literature.

In Yang and Ma [39] was derived the full quantity loading method for reliability analysis of a composite structural system with consideration of stiffness degradation process of set of whole plies.

Gurvich and Pipes [95] also utilized a mesoscale approach for progressive failure of composite laminates with both in plane and bending loads which call attention the search for computational efficiency by agreeing individual plies into sublaminates as whole units for the step-by-step failure. This author also made a comparative study contrasted with experimental data considering step-by-step failure process over weakest link assumption, and concluded the weakest link assumption lead to lower failure results with increasing the material strength scatter.

Wu and Robinson [96] proposed a micromechanical approach in which the laminate is treated as a mechanical system and accounted local load sharing and sizing effects.

In system reliability, the scale of the approach influences the reliability, so exploring multiscale probabilistic failure seems to be an interesting way to derive a robust framework for progressive failure of composites. Recent works about uncertainty quantification at different scales [53–55] and propagation of uncertainties from micro-to-macroscale [97] in composites, provide a basis for this claim.

3.3.2 Reliability based design

Due to the well-known high specific stiffness, strength and corrosion resistance, composite laminates are often selected for high-responsibility structural applications like aircraft, automobile, machinery and marine. Nowadays new applications in all-composite bridges [98], off-shore and civil engineering are emerging [2, 4, 99]. In these applications requiring big amount of composites materials, design optimization plays an important role through providing tools to rationally select the best over a wide range of choices in enhancing the structure's performance [100–103]. Over such named conventional optimization problem, the probabilistic optimum design is an increasing issue, in which how to obtain the best laminate structure under a reliability constraint or how to get the maximum reliability under the constraint of structure cost is the key question. This problem is called the Reliability Based Design Optimization (RBDO) [104], in which an accurate calculation of reliability is crucial in final composite design, as follows in next equation:

$$\begin{aligned} &\min_{\mathbf{X},\boldsymbol{\pi}} F(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\pi}) \text{ s.t:} \\ &\beta(\mathbf{X}, \boldsymbol{\pi}) \leqslant \beta_t \\ &\boldsymbol{\pi}^l \leqslant \boldsymbol{\pi} \leqslant \boldsymbol{\pi}^u \end{aligned} \tag{3.28}$$

where β_t is the target reliability index, $\boldsymbol{\pi} \in \mathbb{R}^n$ is a vector of deterministic design variables and $\boldsymbol{\mu}_{\mathbf{X}}$ is the realization of the vector of random design variables $\mathbf{X} \in \mathbb{R}^m$. $F(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\pi})$ is the function describing the structural performance, which is usually considered a structural weight or cost function; although some recent works have also considered others like frequency response [105], structural efficiency [106] or even statistical robustness [107].

The first efforts to apply RBDO in laminate design, derived results that clearly remark the difference between deterministic and probabilistic designs [56, 108].

An open question remains about which random variables $\mathbf{X}$ to be considered into the optimization problem, specifically those in relation to laminate design like fiber orientation, ply ratios, laminate arrangement, etc. In Eamon and Rais-Rohani [109], a probabilistic sensitivity analysis was derived to determine the influence of uncertainty in each candidate variable on β. Other related works have declined to include these variables as uncertain parameters into the optimization problem [49, 110]. In Miki et al. [111], a simultaneous optimization of fiber angles and ply ratios was corroborated, which concords with Frangopol and Recek [94]. The cross-ply configuration was pointed out to be optimal or near optimal for the case where does not exist uncertainty in shear stress.

The above cited works consider the formulation of the reliability-based optimum under a hard constraint, in the sense that constraints are clearly specified and if the solution is outside the constraint range, even if the deviation is very little, an unacceptable solution is derived. A recent approach that complements this work is the soft constraint RBDO, by which fuzzy reliability optimum models are established. This method provides with an especially useful tool in designing optimum laminated composites, owing to the fact that due to its complex manufacture process, a laminate can be influenced by many factors including probabilistic variables and also fuzzy ones [112].

3.3.3 Reliability based safety factors

Because of possible lack of statistical data from the strength of materials used and the applied loads, design concepts based on traditionally safety factors have also been studied. In this approach, the effects E of actions on a structure and the resistance S to these effects, verify a criterion in the form:

$$E < \frac{S}{\gamma} \tag{3.29}$$

Several authors made a direct comparison between probabilistic and safety factor based deterministic design [82, 108, 111] where important differences in failure prediction, sometimes in a insecure position, are highlighted.

One successful approach to minimize that differences leads to obtain safety factor from probabilistic previous calibration, which is frequently named *reliability based safety factor*.

Zhu [113] proposed a first approach to reliability based safety factor for aircraft composite structures and a method was presented to compare such safety factor to those used in metallic aircraft design.

Boyer et al. [65] presented a method of safety factor calibration from the probabilistic method to achieve a specific reliability level. In this work, an interesting discussion about sensitivity of safety factors with stochastic parameters, was also carried out.

Richard and Perreux [50] utilized the same concept as describe above for safety factor calibration, but in a damaged elasto-viscoplastic model for composites in a thermodynamic framework for long term applications over a pipe for fluid transportation.

An extension of this work for strongly non linear behavior caused by damage, was done by Carbillet et al. [114] who also took into account for possible correlations between the different variables and spatial variability of material properties for a $[0°, 90°]_S$ composite plate, showing up an important effect on safety factor calibration.

3.4 Computational efficiency

The structural integrity analysis of composite structures based on probabilistic concepts is a time consuming process unless inaccuracy FPI methods were employed, and the problem can be exacerbated by the convergence difficulties associated to the non-linearity or complex non explicit LSF. Other methods employing simulation procedures, such as MCM or MC-IS, may have a prohibitive computational cost in large structural systems even if the structural evaluation is accelerated by a vectorized manner, by techniques such as Neumann Series Expansion [87, 115] or by reducing the stochastic variables to be sampled, as previously mentioned [37, 42].

In literature, there have been advised two efficient ways to reduce the computational cost: a) by using new efficient reliability algorithms and b) by reducing the effort of evaluation the LSF. In the former, new reliability algorithms have proved to save great amount of computation time. Special attention require SUBSET Simulation [116] and ^{2}SMART algorithms [117], which confer large efficiency as compared to crude MCM, overall for small failure probabilities and high dimension problems [118]. Nowadays they appear integrated on a OpenSees computational platform called FERUM, as acronym of Finite Element Reliability using Matlab® [119], that is a high versatile reliability tool. Unfortunately, these algorithms have not been sufficiently exploited in composites.

In relation to the second approaches, the Response Surface Method (RSM), and more recently, Artificial Neuronal Networks (ANN), have also emerged as feasible alternatives. The next chapters are dedicated to application of this techniques in composites reliability.

Evolutionary strategies like Genetic Algorithms (GA) are also computation techniques fully employed nowadays in reliability although their well-known high computational cost, which contrasts with the aim of this chapter. However, the existence of multiple design points MPP in the LSF, especially when linking reliability and optimal design, makes necessary the employ GA. Certain authors have provided genetic algorithm strategies in application to composites by which the efficacy of the reliability design problem has also largely improved [120].

3.4.1 Response surface methods (RSM)

In Response Surface Methods, the LSF is substituted or sampled to improve the computational effort. The principle consists in the substitution of the real LSF by approximate simple functions or sampled data, at the neighborhood of the design points where their contribution to the total failure probability is more important [121]. As a consequence, the computational cost can be reduced with respect to the cost required when the full LSF is used or when it is necessary to evaluate the LSF by Finite Element Method (FEM) runs.

When the LSF is substituted by simple functions, generally by explicit polynomial expressions, the method is called Polynomial Based Response Surface Method or simply RSM. Those that the LSF is approximated with training sampling data in contrast to the last one, are called Artificial Neuronal Network (ANN)-based response surface methods [59].

3.4.1.1 Polynomial based response surface

In the original conceptual form of the Response Surface technique, polynomials are used to approximate real LSF. So an important requirement for the LSF is to be smooth around the area of interest. In order to obtain the Response Surface, some regression analysis (for instance the Least Square Method) must be accomplished. As states in Gomes and Awruch [115], the main point resides in to adjust the polynomials to the L.S.F using the sample points, by using some of the several fitting techniques such as a) the central composite design [122, 123] b) the fractional factorial design [124], c) the random design, d) the partially balanced incomplete box design [125] and e) Bucher and Bourgund's [126] proposal.

With this method, the L.S.F is assimilated as follows:

$$g(\mathbf{X}) = a + \sum_{i=1}^{n} b_i x_i + \sum_{i=1}^{n} c_i x_i^2 \tag{3.30}$$

with a, b_i and c_i the polynomial constants to be calculated.

As a consequence of Equation 3.30, only $2n + 1$ samples must be taken along the coordinates axes of each variable at a distance $x_i = U_i(\pm h)$, where U_i is the probabilistic transformation of the variable x_i from the real space to the non-correlated Gaussian space, with h being an arbitrary factor.

In composites, Chen et al. [52] derived the longitudinal ultimate compressive strength of a composite stiffened ship's hull, by a polynomial type with quadratic terms RSM. The reliability analysis was carried out by FORM, and interesting conclusions about ship hull compression dimensioning was derived with the help of a sensitivity analyses.

In the same way, but in an effort to confer computational efficiency in a RBDO problem, Young et al. [106] have recently proposed the polynomial RSM by regression analysis in a complex LSF with Eulerian fluid interaction of a Hexcel (IM7-8552) CFRP marine propeller. A FORM was used to evaluate the influence of uncertainties in material and load parameters and thus to optimize the design parameters, obtaining in this case high accuracy contrasted to MCM.

3.4.1.2 ANN based response surface

As described in previous sections, when reliability analysis is applied to a complicated structural system, the responses of the structure need to be calculated by sophisticated numerical methods. In those cases, sampling the LSF by a trained ANN in substitution of MCM or direct FEA sampling points, is achieved conferring large efficiency [127]. ANN-based response surface emerges in reliability applications to solve the main limitation of polynomial-based response surface methods about the need to increase the number of deterministic analysis when the number of random variables is high, thus making them no as efficient as desirable [59]. Several authors have compared between both methods, showing that the ANN-based response surface method is more efficient than polynomial-based response surface method [115].

ANN are computational models based in parallel distributed processing with interesting properties such as the ability to learn, to generalize, to classify and to organize data. There are two main models developed for different specific computational tasks: those with a supervised training and networks without a supervised training. Networks may be also divided in feed forward, feedback architectures and a combination of both architectures. In reliability, Perceptron Multilayer Neural Networks and Neural Networks with Radial Basis Functions are mostly used. Both types of Networks have a supervised training, feed forward architecture and are universal tools for function approximation. To avoid duplication in literature, a concise introduction of ANN in reliability, done by Hosni Elhewy et al. [59], is recommended. More details about different aspects of Neural Networks are given in the work of Haykin [128].

In composites, ANNs have been used in a wide range of applications like fatigue life prediction, dynamic mechanical properties, processing optimization, numerical modeling, damage detection, delamination, among others [129–132]. But only few works have been encountered in reliability applications for composites, precisely where the computational efficiency of using ANNs can be fully amortized.

Recently, Lopes et al. [93] use artificial neural network (ANN) to generate sample data for the LSF (Tsai-Wu) in stead of FEA, in which high computational efficiency is demonstrated, particularly for low failure probability values regardless the method employed for reliability evaluation. In this work were used two ANN for comparison: the Multilayer Perceptron Network and the Radial Basis Network. The results demonstrated that only 0.02% of MCM using FE as reference CPU time is required for reliability calculation employing an ANN with high accuracy.

3.5 Concluding remarks

In the past few decades, numerous studies have been conducted on the reliability of composite materials and the corresponding applications. The inherent statistical scatter in the material properties together with their complex mechanical performance, makes reliability in composites to be a question of decisions.

Methods, assumptions and applications of reliability of composites have been reviewed to confer a perspectival framework that helps to adopt these decisions. Both, traditional approaches and new trends in reliability computation, have been exposed. Following, general concluding remarks are made:

- In contrast to the deterministic approaches, probabilistic failure and reliability in composites have demonstrated a prolific framework over a design viewpoint to make composites competitive, sustainable and secure.

- Due to the large number of variables involved in the mechanical description of composites as compared to traditional materials, importance measures related to input parameters is a necessary exercise to derive an adequate reliability result. Particularly important is the influence of stiffness randomness description over reliability based design, as recent results demonstrate. Those cases in which stochastic description of certain mechanical variables are not available or incomplete, statistical uncertainty analysis by incorporation available prior or interval probability [110, 133] are prolific ways to carry out the problem.

- Several works remark the convenience of studying the suitability of reliability method over the failure criterion chosen for a specific situation and compare to experimental or reference reliability data when available. Certain stress levels and fiber orientations require a specific reliability method to ensure accuracy. In case of utilization of safety factors in stead of a reliability method, a reliability based calibration may warrant good results.

- More research effort is need about the progressive failure of composite laminates and its relationship with reliability, in order to help to optimize composite design in a probabilistic framework. In this scenario, the consideration of others failure modes than fracture, like stiffness and/or strength reduction by mechanical damage and delamination, is also necessary. This framework would help to derive a reliability formulation over the lifetime of composites.

- Large composite structures require efficient techniques for reliability computation. Recent studies have proved Artificial Neuronal Networks (ANN's) as an advantageous technique. Genetic Algorithms (GA) are also relevant tools for those cases where reliability is inside on a complex design optimization problem. New reliability algorithms available on OpenSees computation platforms like FERUM, should also be explored in composite reliability. These new algorithms together with ANN's for LSF evaluation, is a suggestion that may drastically reduce the computational cost for large composite structures systems and provide sufficient accuracy for small probabilities cases.

Appendix A

Metropolis Hastings Algorithm

The Metropolis-Hastings algorithm, is expressed through a iterative scheme which can be easily implement by computer.

Chosen a proposal PDF $q(.|\mathbf{x}_s)$ with $cov\,(q(.|\mathbf{x}_s)) = \sigma^2 \cdot I_{n \cdot n}$ (a multivariate normal in this work), the chain follows sequence of steps through N iterations as described below:

1.- Randomly initialize $\mathbf{x}_0$ and set $cov\,(q(.|\mathbf{x}_s)) = \sigma^2 \cdot I_{n \cdot n}$ $s = 1$ to N 2.- Generate $\mathbf{y} \sim q(.|\mathbf{x}_s)$, $\mathbf{y} \in \mathcal{M}$. 3.- Generate $u \sim \mathcal{U}(0, 1)$
$u \geq \alpha(\mathbf{x}_s, \mathbf{y})$ eq.(2.21) $\mathbf{x}_{s+1} = \mathbf{y}$ $\mathbf{x}_{s+1} = \mathbf{x}_s$
5.- Set $s = s + 1$

Samples obtained with this algorithm are correlated (the next sample depends on the previous one) but follow the target distribution after a burn-in period, i.e. after the Markov chain reaches stationarity. When N is large enough, it is generated sample that are effectively independent sample from $\pi(\mathbf{y}) = L(\mathcal{M})|_{\mathcal{C}=\mathcal{C}_i}$.

In order to adjust the cov$(q(.|\mathbf{x}_s))$, a small value in comparison with the length scale of $\mathcal{M}$ is preferable. If σ is large, movement around the state space will only occur when a transition to a state with low probability is accepted or when the step chances to land in another probable state. So the convergence will be reached only when the chain makes a big amount of steps.

The disadvantage of small values for σ is that the algorithm will explore the target $\pi(\mathbf{y})$ slowly by a random walk, and it takes time.

Although several rule of thumbs have been proposed in literature, training between N and σ and examining the convergence of each parameter by locating the burn-in period, have demonstrate to be a effective way to obtain a reasonable configuration of the algorithm in cases of complex target distributions, as matter.

Appendix B

Summary of Bibliographic Survey in Composites-Reliability

Author	Failure Criteria	Methodology	Random Vars	Main Objetive	Level
Yang [38, 39]	TH	Others	Lds & Strn	RBDO	Ply
Cederbaum et al. [41]	H	FORM	Lds	Reliability	Ply
Thomas and Wetherhold [57]	Max-DEn	MCM	Strn	Reliability	Laminate
Kam et al. [58]	Max-S & Min-S & Max-W	Others	Lds & Strn	Reliability with damage	Laminate
Zhu [113]	Non-Interactive	FORM	Lds & Strn	Safety Factor Calibration	Ply
Wetherhold and Ucci [42]	TH,TW	Comparison	Lds & Strn	Reliability-Comparision	Ply
Murotsu et al. [108]	TW	AFOSM	Lds & Str & Geo	RBDO	Laminate
Gurvich and Pipes [95]	Baseline based Criteria	MCM	Lds & Strn	Probabilistic Strn	Laminate
Kam and Chang [66]	Max-S & TW	FORM	Strn	Validation FPF Reliability	Laminate
Miki et al. [111]	TW	AFOSM	Lds & Strn	RBDO	Laminate
Boyer et al. [65]	TH; TW, Max-S	FORM	Lds & Strn	Safety Factor Calibration	Laminate
Nakayasu and Maekawa [68]	Comparision	Comparison	Lds & Strn	Reliability-Comparison	Laminate
Soares [43]	TH-TW	FORM	Lds & Strn	State of the Art	Laminate
Philippidis and Lekou [77]	TH	Analytical	Lds & Strn	Reliability	Ply
Gurvich and Pipes [84]	TW or any	Analytical	Lds & Strn	Reliability	Laminate
Richard and Perreux [49]	damage	FORM	Lds & Strn	Reliability and RBDO with damage	Laminate
Richard and Perreux [50]	damage strain criteria	FORM	Lds & Strn & Geo	Safety Factor Calibration	Laminate
Lin [69]	TW,TH,H,Max-S	Comparison	Lds & Strn & Geo	Reliability	Laminate
António. [120]	TW & Buckling	FORM	Lds & Strn	Reliability & RBDO	Laminate
Di Sciuva and Lomario [37]	Bucling	Comparison	Lds & Strn & Stff & Geo	Reliability- Comparision	Laminate
Frangopol and Recek [94]	TW	MCM	Lds	Reliability- Comparision	Laminate
Chen et al. [52]	Buckling	FORM	Lds & Str & Stff & Geo	Reliability	Laminate
Onkar et al. [67]	TW,H	SFEA	Lds & Strn	Reliability	Laminate
Lekou and Philippidis [82]	T-HN	Comparison	Lds & Strn & Stff	Compare Methods	Laminate
Ge et al. [110]	TW	FORM	Strn	RBDO	Laminate
Carbillet et al. [114]	damage	FORM	Lds & Str & Stff & Geo	Safety factor Calibration	Laminate
António and Hoffbauer [97]	TW	FORM	Strn & Stff	RBDO	Laminate
Lopes et al. [93]	TW	Comparison	Lds & Strn	Reliability	Laminate
Young et al. [106]	Other(Fluid-Structure Interaction Failure	FORM	Geo	RBDO	Laminate

TW: Tsai-Wu H: Hasin TH: Tsai-Hahn Max-DEn: Max. Density Energy Max-S: Max. Stress Min-S: Min. Strain Max-W: Max. Work Lds: Loads Strn: Strength Str: Stress Geo: Geometry

Table B.1: Reliability bibliography Synoptic Table. Papers increasingly ordered by date of publication

Appendix C

Matlab® codes

This appendix provides a summary of the algorithms developed for the research work presented herein. The code consists of a collection of Matlab® files developed *ad hoc* in conjunction with other Matlab® functions. A description of the main part of the code is provided below.

```
%%%%%%%%%%%%%%%%%%%%%%MAIN_NSTT.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(Evaluates the Likelihood function of a Markov Chain from Model 1)

format compact;
clear all;
load newdata; %load  Young´s modulus experimental data (N/m2)
load datatime; %load the corresponding experimental times
norm_compl=newdata';
datatime;

global PMFe
global T
global D_e
```

```
global mdl
global R

%*******************input parameters**********************************
DC=500; %number of cycles in a DC
nx=2^7; %number of experimental points
tol=15;
%*******************************************************************

nzro_compl=[]; %time zero is avoided (trivial case)
for i=2:size(norm_compl,2)
    nzro_compl=cat(2,nzro_compl,norm_compl(:,i));
end
abs_st=1; %absorbing state
nzro_compl=absrvnt(nzro_compl,abs_st);
nzro_compl=treatdata(nzro_compl);
norm_compl=nzro_compl;

dutytime=datatime/DC;
Tmax=floor(max(dutytime));
T_e=dutytime(2:end);
T=T_e;
[D_e,PMFe]=non_smoothing(norm_compl,T,nx);
D_e=treatdata(D_e);
PMFe=treatdata(PMFe);
[PMFe]=adjs_zero(PMFe);
mu_samples=mean(norm_compl,1);
desv_samples=sqrt(var(norm_compl,1,1));

%model evaluation
[accPTMy,accQy,real_rt]=PTM_nstt(b,p,alfa1,beta1,alfa2,beta2,T);
%Likelihood function%
[Fnstt,F,P_mle]=Ftrnscnt_nnst(b);

```

```
L_t=[];
for tim=1:numel(real_rt)
    pval_Matrix=[];
    for i=1:b
        j_pval_Matrix=[];
        for j=i:b
            if accQy{tim,1}(i,j)==0
                j=j+1;
            else
                j_pval_stat=(accQy{tim,1}(i,j))^(Fnstt{tim,1}(i,j));
                j_pval_Matrix=[j_pval_Matrix,j_pval_stat];
            end
        end
        pval_stat=prod(j_pval_Matrix);
        pval_Matrix=[pval_Matrix,pval_stat];
    end
    L_aux=prod(pval_Matrix);
    L_t=[L_t;L_aux];
end
L_nstt=prod(L_t);
```

```
%%%%%%%%%%%%%%%%%%%%PTM_nnst.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(Calculates the PDF of damage by nostationary model 1)
function [accPTMy,accQy,real_rt]=PTM_nstt(b,p,alfa1,beta1,alfa2,beta2,T)
b=round(b);
alfa2=alfa1+alfa2*(1-alfa1);
beta2=beta1+beta2*(1-beta1);
q=1-p;
p0=zeros(1,b);
p0(1,1)=1;

for j=1:(b-1)
    P1(j,j)=p;
```

```
    P1(j,j+1)=q;
end

P1(b,b)=1;
xx=0:0.001:1;
x=[0 alfa1 alfa2 1];
y=[0 beta1 beta2 1];
yy=pchip(x,y,xx);
PTMy=eye(size(P1)); %Initialize
accPTMy={};accQy={};
PMFd=[];
X=max(T);
x_time=xx*(X);
y_time=yy*(X);
real_rt=interp1(y_time,x_time,T);
D_d=cat(2,0.01,1/b*((1:b)-0.5),1);
mu_d=[];
desv_d=[];

for i=1:numel(real_rt)
    if i==1
        n=floor(real_rt(i))-0;
    elseif real_rt(i-1)==0;
        n=floor(real_rt(i))-ceil(real_rt(i-1));
    else
        n=floor(real_rt(i))-ceil(real_rt(i-1))+1;
    end

    if n<0;
        disp('MCHR_error: fmodel, line 63')
        break
    else

        if n==0;
```

```
            Qy=eye(size(P1));
        elseif n==1
            Qy=P1;
        else
            Qy=binprod(P1,n);
        end

        accQy{i,1}=Qy;
        PTMy=PTMy*Qy;
        pt=p0*PTMy;
        pt=cat(2,0,pt,0);
        med=sum(D_d.*pt);
        stdev=sqrt(sum(((D_d-med).^2).*pt));
        mu_d=[mu_d,med];
        desv_d=[desv_d,stdev];
        CDF_D=pt(1);

        for n=2:numel(pt)
            CDF_D=[CDF_D,CDF_D(n-1)+pt(n)];
        end

        PMFd=[PMFd;CDF_D];
    end
    accPTMy{i,1}=PTMy;
end
```

```
%%%%%%%%%%%%%%%%%%%%%%MAIN_NSTT_II.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(Evaluates the Likelihood function of a Markov Chain from Model 2)
%Setting initial data (Wei'10)
format compact;
clear all;
load newdata; %load  Young´s modulus (N/m2)
load datatime; %load the time-cycle
```

```
norm_compl=newdata';
datatime;

global PMFe
global T
global D_e
global mdl
global R

%*******************input parameters*****************************
DC=500; %number of cycles in a DC (≤500)
nx=2^7; %number of experimental points
tol=15; % percentual range (100*1/tol) tolerance of data

%****************************************************************

nzro_compl=[]; %time zero is avoided (trivial case)
for i=2:size(norm_compl,2)
    nzro_compl=cat(2,nzro_compl,norm_compl(:,i));

end
abs_st=1; %absorbing state
nzro_compl=absrvnt(nzro_compl,abs_st);
nzro_compl=treatdata(nzro_compl);
norm_compl=nzro_compl;
dutytime=datatime/DC; %display time data
global dutytime
Tmax=floor(max(dutytime));
T_e=dutytime; %experimental time
T=T_e(2:end); %vector of time where model has to be evaluated.
%[D_e,PMFe,bndwth,dens]=smoothing(norm_compl,T,nx,tol);
[D_e,PMFe]=non_smoothing(norm_compl,T,nx);
D_e=treatdata(D_e);
PMFe=treatdata(PMFe);
```

```
[PMFe]=adjs_zero(PMFe);

mu_samples=mean(norm_compl,1);
desv_samples=sqrt(var(norm_compl,1,1));
median_samples=median(norm_compl,1);
%%
%%%%%%%%PTM for non stationary model
T=dutytime;
b=25; %Number of states
%%%%Model Parameters%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
p=0.8;
alfa1=0.01;
beta1=0.045;
%%%%%%%%Model evaluation%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[accPTMy,accQy,real_rt]=PTM_nstt_II(b,p,alfa1,beta1,T);
% %%
% %%%%%%%%%%Likelihood function%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[Fnstt,F,P_mle]=Ftrnscnt_nnst(b);
L_t=[];
for tim=1:numel(real_rt)

    pval_Matrix=[];
    for i=1:b
        j_pval_Matrix=[];
        for j=i:b
            if accQy{tim,1}(i,j)==0
            j=j+1;
            else
                j_pval_stat=(accQy{tim,1}(i,j))^(Fnstt{tim,1}(i,j));
                j_pval_Matrix=[j_pval_Matrix,j_pval_stat];
                 end
             end
        pval_stat=prod(j_pval_Matrix);
```

```
        pval_Matrix=[pval_Matrix,pval_stat];

    end
    L_aux=log10(prod(pval_Matrix));
    L_t=[L_t;L_aux];
end

L_nstt=sum(L_t);%%%Likelihood value
%%
%%%CONTOUR PLOT FOR SENSITIVITY ANALYSIS%%%%%%%%%%%%
BNpts=30;
ANpts=30;
lb=[0.1,0.1];
ub=[0.99,0.99];

alfa1=linspace(lb(1),ub(1),ANpts);
beta1=linspace(lb(2),ub(2),BNpts);

[Fnstt,F,P_mle]=Ftrnscnt_nnst(b);%%%DATA
cont=1; %To number the subplot
tic;
par1=[0.7,0.8,0.88,0.904];
 for k=1:numel(par1)
        %%%%Rest of Model Parameters%%%%%%%%%%%%%%%%%%%%%%%%
        p=par1(k);
            L=[];
for A=1:numel(alfa1)
L_A=[];
for B=1:numel(beta1)

%%%%%%%%Model evaluation%%%%%%%%%%%%%%%%%%%%%%%%%%
[accPTMy,accQy,real_rt]=PTM_nstt_II(b,p,alfa1(A),beta1(B),T);
 L_t=[]; %Partial Likelihood for diff. times step
     for tim=1:numel(real_rt)
```

```
            pval_Matrix=[];
            for i=1:b
                 j_pval_Matrix=[];
                  for j=i:b %Max size
                   if accQy{tim,1}(i,j)==0
                   j=j+1; %It discardes the zero elements
                   else
                      j_pval_stat=(accQy{tim,1}(i,j))^(Fnstt{tim,1}(i,j));
                      j_pval_Matrix=[j_pval_Matrix,j_pval_stat];
                      end
                      end
                     pval_stat=prod(j_pval_Matrix);
                     pval_Matrix=[pval_Matrix,pval_stat];
            end
            L_aux=log10(prod(pval_Matrix));

             L_t=[L_t;L_aux];
                 end

                 L_nstt=sum(L_t);%%%Likelihood value
                 L_A=[L_A;alfa1(A),beta1(B),L_nstt];
             end
             L=[L;L_A];

             sensimatrix=L;

         end
    %Contour plot
xlin=linspace(lb(1),ub(1),70);
ylin=linspace(lb(2),ub(2),70);
[X,Y]=meshgrid(xlin,ylin);
Z=griddata(sensimatrix(:,1),sensimatrix(:,2),...
    sensimatrix(:,3),X,Y,'cubic');
M=subplot(0.5*numel(par1) ,0.5*numel(par1),cont),
```

```
        [C,h]=contour(X,Y,Z,20);axis tight;...
            colormap ([0 0 0]);
        text_handle=clabel(C,h,'FontSize',6,'Interpreter','latex');
        Ax3=gca;
        set(Ax3,'Xlim',[0,1],'Ylim',[0,1], 'YGrid','off','XGrid',...
'off','FontName','latex')
        xlabel('$m_{1}$','Interpreter','latex','FontSize',8);...
            ylabel('$m_{2}$',...
'Interpreter','latex','FontSize',8);

        title(['States:',num2str(b) 'p:',num2str(par1(k))]...
            ,'FontName','latex','FontSize',8);

[.8 1 5 5]);
        cont=cont+1;

saveas(figure,['p',num2str(p),'stats_',num2str(b) '_.eps']);
 end
toc;
```

```
%%%%%%%%%%%%%%%%%%%%%PTM_nstt_II.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(Calculates the PDF of damage by nostationary model 2)
function [accPTMy,accQy,real_rt]=PTM_nstt_II(b,p,alfa1,beta1,T)

b=round(b);
q=1-p;
p0=zeros(1,b);
p0(1,1)=1;

for j=1:(b-1)
    P1(j,j)=p;
    P1(j,j+1)=q;
end
```

```

P1(b,b)=1;
xx=0:0.001:1;
x=[0 alfa1 1];
y=[0 beta1 1];
yy=pchip(x,y,xx);
PTMy=eye(size(P1));
accPTMy={};accQy={};
PMFd=[];
X=max(T);
x_time=xx*(X);
y_time=yy*(X);
real_rt=interp1(y_time,x_time,T);
D_d=cat(2,0.01,1/b*((1:b)-0.5),1);
mu_d=[];
desv_d=[];

for i=1:numel(real_rt)
    if i==1
        n=floor(real_rt(i))-0;
    elseif real_rt(i-1)==0;
        n=floor(real_rt(i))-ceil(real_rt(i-1));
    else
        n=floor(real_rt(i))-ceil(real_rt(i-1))+1;
    end

    if n<0;

        disp('MCHR_error: fmodel, line 63')
        break
    else
        if n==0;
            Qy=eye(size(P1));
        elseif n==1
```

```
            Qy=P1;
        else
            Qy=binprod(P1,n);
        end

        accQy{i,1}=Qy;
        PTMy=PTMy*Qy;
        pt=p0*PTMy;
        pt=cat(2,0,pt,0);
        med=sum(D_d.*pt);
        stdev=sqrt(sum(((D_d-med).^2).*pt));
        mu_d=[mu_d,med];
        desv_d=[desv_d,stdev];
        CDF_D=pt(1);

        for n=2:numel(pt)
            CDF_D=[CDF_D,CDF_D(n-1)+pt(n)];
        end
        PMFd=[PMFd;CDF_D];
    end
    accPTMy{i,1}=PTMy;
end
```

```
%%%%%%%%%%%%%%%%%%%%%Ftrnscnt_nnst.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(Calculates the transition count matrix)

function [Fnstt,F,P_mle]=Ftrnscnt_nnst(s)

load newdata;
load datatime;
norm_compl=newdata';
datatime;
abs_st=1;
```

```
norm_compl=absrvnt(norm_compl,abs_st);
norm_compl=treatdata(norm_compl);
DC=500;
dutytime=datatime/DC;
Tmax=floor(max(dutytime));
min_d=0;
max_d=1;
ddss=linspace(0,1,s+1);
S=zeros(size(norm_compl));

%Transform damage matrix into state matrix
for i=1:size(norm_compl,1)
    for j=1:size(norm_compl,2)
        if norm_compl(i,j)==min_d
            S(i,j)=1;
        elseif norm_compl(i,j)≥max_d
            S(i,j)=s;
        else
         for n=1:numel(ddss)−1
         if norm_compl(i,j)>ddss(n) && norm_compl(i,j)≤ddss(n+1)
                    S(i,j)=n;
                end
            end
        end
    end
end

%F transition at each discrete−time
Fnstt={};
Fil_1=[sum(S(:,1)),zeros(1,s−1)];
Init=[Fil_1;zeros(s−1,s)];
Fnstt{1,1}=Init;

for n=2:size(S,2)
```

```
    Sr=S(:,(n-1:n));
    Fn=zeros(s);
    for i=1:s
        for j=i:s
            for k=1:size(Sr,1)
                for l=2:size(Sr,2)
                    if Sr(k,l-1)==i && Sr(k,l)==j
                        Fn(i,j)=Fn(i,j)+1;
                    end
                end
            end
        end
    end
    Fnstt{n,1}=Fn;
end

%F transition matrix%
F=zeros(s);
for i=1:s
    for j=i:s
        for k=1:size(S,1)
            for l=2:size(S,2)
                if S(k,l-1)==i && S(k,l)==j
                    F(i,j)=F(i,j)+1;
                end
            end
        end
    end
end

%MLE for PTM
P_mle=zeros(s);
for i=1:s
    for j=1:s
```

```
        P_mle(i,j)=F(i,j)/sum(F(i,1:end));
    end
end
```

```
%%%%%%%%%%%%%%%%%%%%log-evidence.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%Codes for Model Class Selection
%%%Evidence computing.
%%%Is needed the sample of the likelihood function
%M. Chiachio. June 2011
%% Setting initial data (Wei'10)
format compact;
clear all;
load newdata; %load  Young mod. (N/m2)
load datatime; %load the time-cycle
norm_compl=newdata';
datatime;
global PMFe
global T
global D_e
global mdl
global R

%*******************input parameters**********************
DC=500; %number of cycles in a DC (≤500)
nx=2^7; %number of experimental points
tol=15; % percentual range (100*1/tol) tolerance of data

%**********************************************************
nzro_compl=[]; %time zero is avoided (trivial case)
for i=2:size(norm_compl,2)
    nzro_compl=cat(2,nzro_compl,norm_compl(:,i));

end
```

```
abs_st=1; %absorbing state
nzro_compl=absrvnt(nzro_compl,abs_st);

nzro_compl=treatdata(nzro_compl);
norm_compl=nzro_compl;

dutytime=datatime/DC; %display time data
Tmax=floor(max(dutytime));
T_e=dutytime; %experimental time

T=T_e(2:end);
[D_e,PMFe]=non_smoothing(norm_compl,T,nx);

D_e=treatdata(D_e);
PMFe=treatdata(PMFe);
[PMFe]=adjs_zero(PMFe);

mu_samples=mean(norm_compl,1);
desv_samples=sqrt(var(norm_compl,1,1));
median_samples=median(norm_compl,1);

%%
load RESULTS_I
load Expect_target
Nsim=numel(RESULTS_I(:,1));%Total nº of simulations

burn=115; %%See in Convergence_control folder

%Prior definition.Only for Gaussian prior.
sigma_prior=[0.4,0.2,0.1];
mu=[0.75 0.75 0.75 0.75 0.75];

G={}; %It initialize the matrix storage
H={}; %Idem
```

```

tic
for j=1:3 %3 types of priors
%Prior i. By only changing the stdv.
sigma=sigma_prior(j);
covm=sigma^2*eye(5);
G_accum=[];
H_accum=[];
for i=burn:Nsim %%Is discount the burn-in period.

    x=RESULTS_I(i,:);
    g_value=exp(-0.5*(x-mu)*inv(covm)*(x-mu)'); %Prior
    likeli=Expect_target(i); %%Likelihood value for x
    h_value=g_value*log(likeli); %% And integration constant
    G_accum=[G_accum;g_value];
    H_accum=[H_accum;h_value];
   x=[];% It avoid possible errors
end
G{j,1}=G_accum; %for each prior the simulation result is stored
H{j,1}=H_accum; %The function h is also stored

end
toc
 save Gcomput G;
 save Hcomput H;
 %%
%%%Evidence: Expectation of Prior (G) over M-H sample
%%%%See Ka-Ven Yueng, Wiley 2010.
Evdnc=[];LogEvdnc=[];RelEntrop=[];
for k=1:3 %For each Prior
Evdnc(k)=(1/Nsim)*(sum(G{k,1}));
%%%Log-Evidence: Log of Evidence
LogEvdnc(k)=log(Evdnc(k));
%%%Log Goodness of fit
```

```
LogGdoft(k)=(1/Evdnc(k))*(1/Nsim)*(sum(H{k,1}));
%%%Relative Entropy btwn prior & posterior
RelEntrop(k)=LogGdoft(k)−LogEvdnc(k);
end
ModClSl=[Evdnc;LogEvdnc;LogGdoft;RelEntrop]';
%%
%%%Evidence with improper prior. Integration constant
M=10000;
%Prior definition.Gaussian for simplicity
sigma_prior=[0.4];
mu=[0.2 0.2 0.2 0.2 0.8];
K_accum=[];
for i=1:M %%Is discount the burn−in period.
    x=rand(1,5);
    k_value=ftarget(x,dutytime); %Prior evaluated in M−H draws
    K_accum=[K_accum;k_value];
end
Evdnc=(1/M)*(sum(K_accum));
```

```

%%%%%%%%%%%%%%%%%%%%%MtrplsHstngs.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%METROPOLIS HASTING ALGORITHM
%by Manuel Chiachio−−May 2011 (Granada)



%%%TARGET DISTRIBUTION%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Defined as a m−function called ftarget.m
%Input: a vector x. Output: a function value (escalar)
%Number of parameters or size of x:
N=3;
%%%PROPOSAL DISTRIBUTION%%%%%%%%%%%%%%
%%%Multivariate Gaussian uncorrelated function
```

```
%Defined as a m-function called fnormal.m
%Input: xvalue(point evaluation),mu(mean),covm (cov. matrix)
%(point evaluation)and mu(mean) are changing inside the algorithm.
sig=0.01; %Too sensible for convergency
covm=sig^2*eye(N); %We assume no correlation for the prop.
R = chol(covm); %Cholesky decomposition of cov. matrix
%Output: a function value (escalar)
n=15000; %Nº of trials. Length of the chain
RESULTS_II=[];
tic


    x=zeros(n,N); %Initializing the chain for computational tasks
    %x(1,:)=rand(1,N); % Start point of the M. Chain
    x(1,:)=[0.5,0.45,0.85];
    %%%%
    for i=2:n
        %Gaussian multivariate pdf for new steps

        y_aux=x(i-1,:)+randn(1,N)*R;% a new set of parameters
        ind1=find(y_aux≥1,1,'first');%looks for not allowed values
        ind2=find(y_aux≤0.05,1,'first');%idem
        %It restrict the parameter to be (0,1)
        while isempty(ind1)==false || isempty(ind2)==false,...
        ||(y_aux(2)/y_aux(1))>4.5
            y_aux=x(i-1,:)+randn(1,N)*R;
            ind1=find(y_aux≥1,1,'first'); %It recount new  1
            ind2=find(y_aux≤0.05,1,'first'); %It recount new 0
        end
        y=y_aux;
        %generate a uniform for comparison
        u=rand(1);
        alpha=min([1,ftarget(y,dutytime)*fnormal(x(i-1,:),y,covm)/...
        (ftarget(x(i-1,:),dutytime)*fnormal(y,x(i-1,:),covm))]);
```

```
        if u≤alpha
            x(i,:)=y;
        else
            x(i,:)=x(i−1,:);
        end
        end
    RESULTS_II=[RESULTS_II,x];

toc
```

```
%Function ftarget called by the evidence computing
function fvalue=ftarget(x,dutytime)

%%%%%%%%%%%%PTM for non stationary model
T=dutytime;
b=25; %Number of states
%%%%Model Parameters%%%%%%%%%%%%%%%%

alfa1=x(1);
beta1=x(2);
alfa2=x(3);
beta2=x(4);
p=x(5);

%%%%%%%%Model evaluation%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[accPTMy,accQy,real_rt]=PTM_nstt(b,p,alfa1,beta1,alfa2,beta2,T);
%%%%%%%%%%%Likelihood function%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[Fnstt,F,P_mle]=Ftrnscnt_nnst(b);%%%DATA
L_t=[];
for tim=1:numel(real_rt)

    %N_xy=combint_nstt(Fnstt{tim,1}); %The factorial for each time
```

```
    pval_Matrix=[];
    for i=1:b
        j_pval_Matrix=[];
        for j=i:b %Max size of any PTM is bxb. Only upper triang matrix
            if accQy{tim,1}(i,j)==0
            j=j+1; %It discardes the zero elements for each row
            else
                j_pval_stat=(accQy{tim,1}(i,j))^É
%Check the difference with
 %the other nonzero elemnts
               (Fnstt{tim,1}(i,j));
                j_pval_Matrix=[j_pval_Matrix,j_pval_stat];
                end
                end
        pval_stat=prod(j_pval_Matrix);
        pval_Matrix=[pval_Matrix,pval_stat];

    end
    L_aux=prod(pval_Matrix);
    %L_aux=log10(prod(pval_Matrix));

    L_t=[L_t;L_aux];
end
L_nstt=prod(L_t);
%L_nstt=sum(L_t);%%%Likelihood value
fvalue=L_nstt;
```

```
%%Auxiliary script for evaluating the convergence of target
%distribution over N simulation.
%Manuel Chiachio. June 2011

%%%Model 1%%%%%%
nzro_compl=[]; %time zero is avoided (trivial case)
```

```
for i=2:size(norm_compl,2)
nzro_compl=cat(2,nzro_compl,norm_compl(:,i));
end
abs_st=1; %absorbing state
nzro_compl=absrvnt(nzro_compl,abs_st);

nzro_compl=treatdata(nzro_compl);
norm_compl=nzro_compl;

dutytime=datatime/DC; %display time data
Tmax=floor(max(dutytime));
T_e=dutytime; %experimental time

T=T_e(2:end); %vector of time where model has to be evaluated.
[D_e,PMFe]=non_smoothing(norm_compl,T,nx);

D_e=treatdata(D_e);
PMFe=treatdata(PMFe);
[PMFe]=adjs_zero(PMFe);

mu_samples=mean(norm_compl,1);
desv_samples=sqrt(var(norm_compl,1,1));
median_samples=median(norm_compl,1);
%%
tic
load RESULTS_I;
Expect_target=[];
for i=1:numel(RESULTS_I(:,1))
 aux_target=ftarget(RESULTS_I(i,:),dutytime);
 Expect_target=[Expect_target;aux_target];
end
toc
```

```
%%Script for evaluating the convergence of target
%distribution over N simulation.
%Manuel Chiachio. June 2011
%%Model 2 %%%%%%%%%%%

nzro_compl=[]; %time zero is avoided (trivial case)
for i=2:size(norm_compl,2)
    nzro_compl=cat(2,nzro_compl,norm_compl(:,i));

end
abs_st=1; %absorbing state
nzro_compl=absrvnt(nzro_compl,abs_st);

nzro_compl=treatdata(nzro_compl);
norm_compl=nzro_compl;

dutytime=datatime/DC; %display time data
Tmax=floor(max(dutytime));
T_e=dutytime; %experimental time

T=T_e(2:end); %vector of time where model has to be evaluated.
%[D_e,PMFe,bndwth,dens]=smoothing(norm_compl,T,nx,tol);
[D_e,PMFe]=non_smoothing(norm_compl,T,nx);

D_e=treatdata(D_e);
PMFe=treatdata(PMFe);
[PMFe]=adjs_zero(PMFe);

mu_samples=mean(norm_compl,1);
desv_samples=sqrt(var(norm_compl,1,1));
median_samples=median(norm_compl,1);
%%
tic
load RESULTS_II;
```

```
Expect_target=[];
for i=1:numel(RESULTS_II(:,1))
 aux_target=ftarget(RESULTS_II(i,:),dutytime);
 Expect_target=[Expect_target;aux_target];
end
toc
```

```
%%%%%%%%%%%%%%%%%%%%%%AUXILIARY FUNCTIONS%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%binprod.m%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%J-MCHR 03/2011
%Binary product of matrices for large exponents

function [PTM]=binprod(P,X)

%%test
% P=rand(3);
% X=6;
% PTM1=P^X;

if X<0;
disp('MCHR_Warning: negative time in binprod!!. Converted to positive');
end
X=abs(round(X)); %this algorithm only allow integers positive exponents

if X==1
    PTM=P;
elseif X==0
    PTM=eye(size(P));
else
    PTM=eye(size(P)); %initialize
ex=0; %initialize
    while X-ex>1
        n=floor(log2(X-ex));
```

```
        M=P;
        for i=1:n
            M=M*M;
        end
        PTM=PTM*M;
        ex=ex+2^(n); %exponent of PTM in each iteration
        if ex == X;
            break
            PTM;
        elseif X-ex==1
            PTM=PTM*P;
            break
        end
        end
end
```

```
function [D_e,PMFe]=non_smoothing(norm_compl,T,nx)

D_e=[];
PMFe=[];
D_ac=[];

for n=1:numel(T) %using not-measured data is not allowed

        D_ac=[D_ac,norm_compl(:,n)];
     [stairs_ecdf,Dmg] = ecdf(D_ac(:,n));
     for j=1:numel(Dmg)-1
        if Dmg(j)==Dmg(j+1) || Dmg(j+1)<=Dmg(j)
            Dmg(j+1)=Dmg(j)+1e-100;
        end
        end

    PMFe=[PMFe;linspace(0,1,nx)];
```

```
    D_e=[D_e;interp1(stairs_ecdf,Dmg,linspace(0,1,nx))];
    end
```

```
function [PMFe]=adjs_zero(PMFe)
%J—MCHR. March 2011
%WITH THIS FUNCTION THE FIRST COLUMN OF THE EXPERIMENTAL MATRIX (CDF) IS
%OBLIGATED TO BE ZEROS. THIS IS TO AVOID COMPUTATIONAL ERRORS.


for i=1:size(PMFe,1)

    if PMFe(i,1)>0 && PMFe(i,1)<1
        PMFe(i,1)=0;
    end

end
```

```
%%Function to homogenize the absorvent state
function M=absrvnt(M,abs_st)

for i=1:size(M,1)
    for j=1:size(M,2)

        if M(i,j)>abs_st
            M(i,j)=abs_st;
        end

    end
end
```

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