Area-stationary surfaces in contact sub-Riemannian manifolds

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Introducción, resumen y conclusiones

A lo largo de la historia del cálculo de variaciones, se le ha dedicado un interés especial a los problemas variacionales relacionados con el área. Entre estos, destacan por su importancia el problema isoperimétrico y el problema de Plateau. El problema isoperimétrico consiste en encontrar las regiones con un volumen dado v con el menor perímetro posible. En el problema de Plateau, buscamos las superficies (o hipersuperficies) que minimizan el área con borde prefijado. Estas cuestiones están muy relacionadas con el estudio de las superficies minimales y con curvatura media constante. Para una introducción a la geometría de superficies recomendamos las monografías de do Carmo [**35**] y de Montiel y Ros [**66**].

Estos problemas, en el espacio euclídeo y, posteriormente, en variedades riemannianas, han sido estudiados con técnicas de teoría geométrica de la medida, ecuaciones en derivadas parciales, geometría diferencial y transporte óptimo, entre muchas otras.

En las últimas decadas, los problemas variacionales relacionados con el funcional área en variedades sub-riemannianas han sido objeto de un estudio intensivo desde distintos puntos de vista. Se han estudiado desigualdades isoperimétricas, existencia de regiones isoperimétricas, teoría geométrica de la medida, la estructura del conjunto singular de hipersuperficies de clase C^1, regularidad de los mínimos, grafos minimales, el problema de Bernstein, estabilidad de superficies con curvatura media constante, y muchos otros.

En [**71**], la siguiente desigualdad isoperimétrica:

$$|\partial\Omega| \geqslant C|\Omega|^{4/3} \tag{0.1}$$

fue demostrada en el grupo de Heisenberg $\mathbb{H}^1$ por Pansu. Aunque el exponente es óptimo, la constante $C > 0$ no lo es. En [**71**] se conjeturó que las regiones isoperimétricas en $\mathbb{H}^1$ forman una familia de bolas topológicas que no son bolas métricas. El borde de estas regiones, que llamaremos esferas de Pansu y denotaremos por S_λ, $\lambda > 0$, está caracterizado por la propiedad que cada S_λ está foliada por geodésicas sub-riemannianas de curvatura constante λ, [**20**, § 2.3]. Recientemente, Chanillo y Yang [**21**] han generalizado la desigualdad (0.1) a variedades pseudo-hermíticas de dimensión tres, con torsión pseudo-hermítica nula. Para un resumen bastante detallado de los últimos avances sobre la desigualdad isoperimétrica óptima en $\mathbb{H}^1$, el lector interesado puede consultar la monografía de Capogna, Danielli, Pauls y Tyson [**20**].

En geometría sub-riemanniana, fuera del caso compacto, el único resultado conocido de existencia de regiones isoperimétricas fue dado por Leodardi y Rigot en grupos de Carnot [**58**]. En su trabajo utilizaron de manera intensiva las propiedades del perfil isoperimétrico en un grupo de Carnot $\mathbb{G}$. Como las regiones isoperimétricas en un grupo de Carnot $\mathbb{G}$ son invariantes por una familia uniparamétrica de dilataciones intrínsecas, el perfil isoperimétrico $I_\mathbb{G}$ de $\mathbb{G}$ está dado por $I_\mathbb{G}(v) = Cv^q$, donde C es una constante positiva y $q \in (0, 1)$. En particular, la función $I_\mathbb{G}$ es estrictamente cóncava, una propiedad que desempeña un papel fundamental en su demostración.

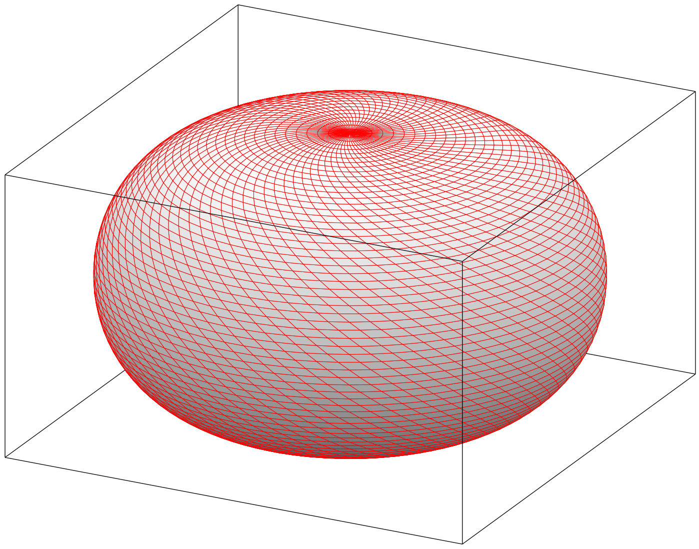

Figure 1. Una esfera de Pansu S_λ en $\mathbb{H}^1$.

Leonardi y Rigot además han probado que los conjunto isoperimétricos son dominios de isoperimetría en el caso particular del grupo de Heisenberg. Sin embargo, su resultado no se puede aplicar a algunos grupos sub-riemannianos que no son de Carnot. Un ejemplo es el grupo roto-traslacional, descrito en [**20**]. Algunos puntos cruciales de la prueba de Leonardi y Rigot son analizados en [**20**, § 8.2].

Observamos que en variedades no compactas, la existencia de regiones isoperimétricas no es trivial. Ritoré [**74**, Thm. 2.16] observó la existencia de variedades riemannianas completas no compactas en las cuales no existen regiones isoperimétricas para ningún valor del volumen, como los planos de revolución con curvatura de Gauss estrictamente creciente.

La conjetura de Pansu no ha presentado avances hasta los años '00, cuando fueron probados dos casos particulares. En [**81**] Ritoré y Rosales demuestran que las regiones encerradas por las esferas de Pansu son conjuntos isoperimétricos en $\mathbb{H}^1$, entre todas las regiones encerradas por superficies C^2. En su artículo han obtenido una caracterización de las superficies estacionarias para el área (con o sin restricciones sobre el volumen). Concretamente, las líneas características deben cortar a las líneas singulares de forma ortogonal. Esta propiedad fue observada también por Cheng, Hwang y Yang [**26**] en su trabajo sobre grafos estacionarios para el área, y probada por medio de la primera fórmula de variación del área sub-riemanniana y utilizando la estructura local del conjunto singular, estudiada por Cheng, Hwang, Malchiodi y Yang [**25**]. Esta condición, junto con el análisis de los campos de Jacobi sobre la superficie, fue utilizada en [**81**] para clasificar las superficies orientadas, inmersas, compactas, conexas y completas de clase C^2 estacionarias para el área, que encierran un volumen prefijado.

En [**64**], Monti y Rickly han demostrado que el borde de una región isoperimétrica convexa, en $\mathbb{H}^1$, está foliado por geodésicas sub-riemannianas. Esto implica que el borde es una esfera de Pansu. Su prueba se basa en un resultado de regularidad de tipo Sobolev para una clase de campos vectoriales con variación acotada.

Otra línea de investigación emergente en espacios sub-riemannianos está relacionada con problemas de teoría geométrica de la medida, como el estudio de conjuntos de perímetro finito y superficies regulares desde un punto de vista intrínseco. Estas temáticas han sido investigadas de modo particular en los grupos de Heisenberg $\mathbb{H}^n$ y de Carnot $\mathbb{G}$. En [**41**, Theorem 7.1], Franchi, Serapioni y Serra Cassano han probado un teorema de estructura para conjuntos de perímetro localmente finito, análogo al resultado euclídeo:

THEOREM 0.1 (Estructura de conjuntos de Caccioppoli intrínsecos). *Si $E \subset \mathbb{H}^n$ es un conjunto con $\mathbb{H}$-perímetro localmente finito, entonces el borde reducido $\partial^*_{\mathbb{H}} E$ es $\mathbb{H}$-rectificable, es decir:*

$$\partial^*_{\mathbb{H}} E = N \cup \bigcup_{i=1}^{\infty} N_i,$$

donde $\mathcal{H}^{2n+1}_{\mathbb{H}}(N) = 0$ y N_i es un subconjunto compacto de una hipersuperficie $\mathbb{H}$-regular S_i.

En [**41**, Definition 6.1] los autores definen S una superficie $\mathbb{H}$-regular si, para todo $p \in S$, existen una bola métrica $B(p,r)$ y una función $f \in C^1_{\mathbb{H}}(B)(p,r)$, tales que:

$$S \cap B(p,r) = \{q \in \mathbb{H}^n : f(q) = 0, \nabla_{\mathbb{H}} f(q) \neq 0\}.$$

Posteriormente, esta definición ha sido generalizada a estructuras más generales por Franchi, Serapioni y Serra Cassano [**42, 43**] y Citti y Manfredini [**29**]. Observamos que una superficie $\mathbb{H}$-regular puede ser muy irregular desde el punto de vista euclídeo. Kirchheim y Serra Cassano han construido ejemplos de superficies $\mathbb{H}$-regulares que son fractales, [**57**].

Franchi, Serapioni y Serra Cassano han mostrado que una superficie $\mathbb{H}$-regular puede ser descrita de forma equivalente como un grafo intrínseco, [**43**, Theorem 3.27], véase también [**44**]. Una región S en $\mathbb{H}^n$ es un grafo intrínseco, sobre un dominio Ω dentro de un plano vertical, si las geodésicas sub-riemannianas de curvatura cero, que salen desde Ω de forma ortogonal, cortan a S en sólo un punto, [**43**, Definition 3.6].

Una caracterización alternativa de grafos intrínsecos es estudiada por Ambrosio, Serra Cassano y Vittone [**4**], Bigolin y Serra Cassano [**12, 13**] y Bigolin, Caravenna y Serra Cassano [**11**]. Una función continua ϕ es una solución de $\nabla^{\phi}\phi = w$, donde $w \in C(\Omega, \mathbb{R}^{2n-1})$ y Ω es un dominio de un plano vertical, si y solo si ϕ induce un grafo $\mathbb{H}$-regular. ∇^{ϕ} denota al gradiente intrínseco en S y $\nabla^{\phi}\phi = w$ es una EDP no lineal de primer orden, relacionada con el operador de Burgers. Monti y Vittone han mostrado que un conjunto de perímetro finito, con el normal horizontal continuo, es una superficie $C^1_{\mathbb{H}}$, [**65**].

Considerando a una superficie regular euclídea en $\mathbb{H}^n$, definimos el *conjunto singular* como el conjunto de los puntos en los cuales el plano tangente coincide con la distribución horizontal, es decir:

$$\Sigma_0 := \{p \in \Sigma : T_p\Sigma \equiv \mathcal{H}_p\}.$$

En un entorno de un punto singular, una superficie puede ser vista como un grafo euclídeo sobre el plano xy. Observamos que una superficie de clase $C^{1,1}$ con conjunto singular no vacío, no es $\mathbb{H}$-regular. Por otra parte, esto puede no ser verdad para superficies de clase $C^{1,\alpha}$. Se puede comprobar que el grafo intrínseco de $f(y,t) := |t|^{\alpha}$

es $\mathbb{H}$-regular por $\alpha \in]1/2, 1[$, [**4**, Corollary 5.11], aunque el plano tangente euclídeo en el origen coincide con la distribución horizontal. Esta superficie es de clase $C^{1,2\alpha-1}$.

Las dimensiónes de Hausdorff euclídea e intrínseca de Σ_0 fueron estudiadas por Balogh [**5**], y también por Derridj [**34**] y Cheng, Hwang y Yang [**26**]. Resumimos el siguiente resultado de Balogh:

THEOREM 0.2.
(i) *Si* Σ *es una hipersuperficie regular de clase* C^1 *en* $\mathbb{R}^{2n+1}$*, entonces* $\mathcal{H}_{\mathbb{H}}^{2n+1}(\Sigma_0) = 0$*;*
(ii) *Si* Σ *es una hipersuperficie regular de clase* $C^{1,1}$ *en* $\mathbb{R}^{2n+1}$*, entonces* $dim_E(\Sigma_0) < 2n$ *y* $dim_{\mathbb{H}}(\Sigma_0) < 2n$*;*
(iii) *Si* Σ *es una hipersuperficie regular de clase* C^2*, entonces* $dim_E(\Sigma_0) < n$ *y* $dim_{\mathbb{H}}(\Sigma_0) < n$*;*
(iv) *Para todos* $\alpha > 0$ *existe* Σ_α*, hipersuperficie regular de clase* $C^{1,1}$*, tal que* $dim_E((\Sigma_\alpha)_0) \geqslant 2n - \alpha$*;*
(v) *Existe* Σ *de clase* $\bigcap_{0<\alpha<1} C^{1,\alpha}$ *tal que* $\mathcal{H}_E^{2n}(\Sigma) < +\infty$ *y* $\mathcal{H}_E^{2n}(\Sigma_0) > 0$.

Cheng, Hwang, Malchiodi e Yang han encontrado una caracterización mas geométrica de la estructura del conjunto singular de superficies de clase C^2 en $\mathbb{H}^1$, [**25**]:

PROPOSITION 0.3. *Sea* Σ *una superficie de clase* C^2 *y sea* p *un punto singular. Entonces*
(i) *o* p *es un punto singular aislado;*
(ii) *o* p *está contenido en una curva singular de clase* C^1*. En este caso las curvas características encuentran las singulares con el mismo ángulo de incidencia y reflexión, cuando* Σ *tenga curvatura media acotada.*

Recientemente, en [**24, 27**], han probado además:

PROPOSITION 0.4. *Sea* Σ *una superficie de clase* C^1 *en* $\mathbb{H}^1$*. Asumimos que* Σ *es una solución débil de la ecuación de las superficies minimales. Entonces, tenemos que*
(i) *para todos* $p \in \Sigma - \Sigma_0$*, existe una única curva característica y una única curva "transversa". Además sus proyecciones sobre el plano* xy *son curvas de clase* C^2*;*
(ii) *el conjunto singular está formado por curvas y el conjunto de los puntos singulares no degenerado por curvas de clase* C^1*;*
(iii) *dos curvas características que se encuentran en un punto singular no degenerado* p *forman el mismo ángulo con la tangente de la curva singular en* p.

Observamos que han generalizado la Proposition 0.4 a 3-variedades pseudo-hermitianas, [**24**, § 8].

Bigolin y Vittone han mostrado que, en general, no existe unicidad de las curvas características que pasan por un punto de una superficie $\mathbb{H}$-regular, [**14**].

Los resultados sobre la estructura de una superficie cerca de Σ_0 implican consecuencias muy fuertes en la clasificación de las superficies estacionarias para el área, puesto que el comportamiento de Σ en un entorno de Σ_0 es bastante rígido. Por ejemplo, este tipo de resultados fueron utilizados de manera muy fuerte por Ritoré y Rosales en [**81**], y se utilizarán en la Sección 2.9 de esta tesis, donde clasificamos las superficies estacionarias con conjunto singular vacío en el grupo roto-traslacional $\mathcal{RT}$, Lema 2.54 y Lema 2.55:

Sea Σ una superficie completa y estacionaria de clase C^2, con conjunto singular no vacío. Entonces Σ es el helicoide que gira a la derecha o un plano de la forma $\{(x, y, \theta) \in \mathcal{RT} : ax + by + c = 0, a, b \in \mathbb{R}, c \in \mathbb{S}^1\}$.

Sobre la regularidad de los minimizantes, la situación es diferente con respecto al caso riemanniano. Más en detalle, Cheng, Hwang e Yang [**26**] y Ritoré [**76**] han construido ejemplos de grafos Lipschitz sobre el plano xy, que son minimizantes para el área. Por otra parte, estos ejemplos no son grafos $\mathbb{H}$-regulares. El único resultado de regularidad para grafos intrínsecos es debido a Capogna, Citti y Manfredini, [**17, 18**]:

THEOREM 0.5. *Sea Σ un grafo intrínseco Lipschitz sobre un plano vertical en $\mathbb{H}^1$. Sea además Σ límite de grafos minimales riemannianos. Entonces Σ es de clase $C^{1,\alpha}$, $0 < \alpha < 1$, y las curvas características son de clase C^∞.*

THEOREM 0.6. *Sea Σ un grafo intrínseco Lipschitz sobre un plano vertical en $\mathbb{H}^n$. Sea además Σ límite de grafos minimales riemannianos. Entonces Σ es de clase C^∞.*

Observamos que el Teorema 0.5 es óptimo y ha sido recientemente generalizado a grupos de Lie tridimensionales por Barbieri y Citti, [**8**].

Por lo que respecta a la existencia de grafos minimizantes sobre un dominio Ω, con borde prefijado, Serra Cassano y Vittone han probado existencia para grafos sobre el plano xy, con datos en el borde en $L^1(\partial\Omega)$ y $\partial\Omega$ Lipschitz, [**85**]. Además probaron que estos minimizantes son localmente acotados y que un grafo sobre el plano xy es siempre $\mathcal{H}^{2n}_E$-rectificable en $\mathbb{H}^n$. Resultados análogos pueden encontrarse en Cheng, Hwang e Yang [**26**] y Cheng y Hwang [**23**]. En [**85**] se demuestra además la existencia de grafos intrínsecos sobre un dominio Ω de un plano vertical en $\mathbb{H}^n$, con dato al borde fijado. En este caso, si suponemos que la función que define el grafo está en $L^{2n+1}_{loc}(\Omega)$, demuestran que es también localmente acotada.

Otro problema muy conocido en análisis geométrico es el problema de Bernstein. En $\mathbb{R}^n$, consiste en clasificar los grafos minimales enteros. En $\mathbb{R}^n$ las soluciones son minimizantes para el área, por un argumento estándar de calibración. Por otra parte, en $\mathbb{H}^n$, Danielli, Garofalo, Nhieu y Pauls han probado que existen grafos enteros, horizontales y minimales que no son estables, [**30, 31, 32**]. De cualquier modo, las únicas superficies orientadas, inmersas, completas y estables en $\mathbb{H}^1$ son los planos verticales. Casos particulares han sido demostrados por Barone-Adesi, Serra Cassano y Vittone [**9**] y Danielli, Garofalo, Nhieu and Pauls [**33**].El caso general ha sido probado en [**55**], por Hurtado, Ritoré y Rosales. En $\mathbb{H}^n$, para $n \geqslant 5$, Barone Adesi, Serra Cassano y Vittone [**9**] han dado ejemplos de grafos $\mathbb{H}$-regulares minimizantes, que no son planos horizontales. El problema general en la clase de las superficies $\mathbb{H}$-regulares está todavía abierto.

Notamos que en [**55**] Hurtado, Ritoré y Rosales clasifican también las superficies inmersas, orientadas, completas y estables de clase C^2 en $\mathbb{H}^1$, con conjunto singular no vacío: la única es el paraboloide hiperbólico.

La herramienta fundamental es la segunda fórmula de variación del área sub-riemanniana para superficies de clase C^2 con conjunto singular no vacío. Mientras que es sencillo mover el conjunto singular en la primera fórmula de variación del área, mover Σ_0 en la segunda variación es mucho más complicado. Esencialmente, no podemos derivar bajo el signo de integral, puesto que la segunda derivada de $|N_H|$ es no acotada cerca de Σ_0. La única posibilidad es que nuestra variación está constituida por superficies minimales en un entorno de Σ_0, por variaciones generales en $\Sigma - \Sigma_0$, y, por medio de una partición de la unidad sobre las curvas características, unir las dos variaciones.

El problema de tipo Bernstein ha sido generalizado por Rosales en [**82**], para superficies C^2 completas, orientadas, horizontales, estables con curvatura media constante inmersas en los espacios forma sasakianos $\mathbb{S}^1$ y $\widetilde{SL}(2,\mathbb{R})$.

En esta tesis estudiamos problemas relacionados con superficies estacionarias para el área en variedades sub-riemannianas de contacto. Siguiendo la monografía de Blair [**15**], una *variedad de contacto* es una variedad M^{2n+1} de clase C^∞ y dimensión impar, tal que existe una 1-forma ω con $d\omega$ no degenerada restringida a $\mathcal{H} := \ker(\omega)$. Como:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

la *distribución horizontal* $\mathcal{H} := \ker(\omega)$ es completamente no integrable. Un ejemplo muy conocido de variedad de contacto es el espacio euclídeo $\mathbb{R}^{2n+1}$ con la forma de contacto estándar:

$$\omega_0 := dt + \sum_{i=1}^{n}(x_i dy_i - y_i dx_i). \tag{0.2}$$

Una *variedad sub-riemanniana de contacto* es una variedad de contacto con una métrica definida positiva $g_{\mathcal{H}}$ sobre $\mathcal{H}$.

Una clase muy estudiada en geometría de contacto es la de las variedades *riemannianas de contacto*, véanse la monografía de Blair [**15**] y Tanno [**90**]. Dada una variedad de contacto, se puede demostrar la existencia de una métrica riemanniana g y de un tensor J de tipo $(1,1)$ tales que

$$g(T,X) = \omega(X), \quad 2g(X,J(Y)) = d\omega(X,Y), \quad J^2(X) = -X + \omega(X)\,T. \tag{0.3}$$

La estructura compuesta por (M,ω,g,J) se llama variedad riemanniana de contacto. La clase de las variedades sub-riemannianas de contacto es diferente de esta última. Recordamos que, en nuestra definición, la métrica $g_{\mathcal{H}}$ está dada, y puede ser extendida a una métrica riemanniana g en TM. por otra parte, en general no existe un tensor J de tipo $(1,1)$ que satisfaga todas las condiciones en (0.3). Obsérvese que la segunda condición en (0.3) define J en $\mathcal{H}$ de manera unívoca, pero este J no satisface en general la tercera condición en (0.3), como se puede fácilmente ver tomando $(\mathbb{R}^3,\omega_0)$ con una adecuada métrica definida positiva sobre $\ker(\omega_0)$.

En el Capítulo 1 probamos un resultado de existencia de regiones isoperimétricas en ciertas variedades sub-riemaniannas de contacto, donde el cociente por el grupo de las isométrias de contacto, los difeomorfismos que preservan la estructura de contacto y la métrica sub-riemanniana, es compacto, Teorema 1.26. Este es el análogo al resultado en el caso riemanniano probado por Morgan [**67, 68**].

En la demostración del Teorema 1.26 seguimos la idea de Morgan: consideramos una succesión minimizante de conjuntos de volumen v cuyos perímetros aproximan el ínfimo de los perímetros de todos los conjuntos de volumen v. Si esta succesión sub-converge, sin perder una parte del volumen inicial, la semicontinuidad inferior del perímetro implica que el conjunto límite es una región isoperimétrica con volumen v. Si una porción del volumen se ha perdido, la Proposición 1.24 implica que la succesión minimizante puede ser dividida en una parte convergente y otra divergente, esta última compuesta por conjuntos de volumen uniformemente positivo, véanse Ritoré [**74, 75**] y Ritoré y Rosales [**79**], para el caso riemanniano. La parte convergente tiene como límite una región isoperimétrica para su volumen, y es acotada por el Lema 1.23. Ahora podemos trasladar de forma conveniente la parte divergente para recobrar una porción del volumen perdido. Un punto importante es que siempre podemos recobrar una fracción fija del volumen, debido al Lema 1.27, véase también Leonardi y Rigot [**58**, Lemma 4.1].

A lo largo de la demostración del Teorema 1.26 surgen dos problemas importantes. Hemos mostrado en el Lema 1.23 la acotación de las regiones isoperimétricas, y un resultado de estructura para succesiones minimizantes en la Proposición 1.24. Un punto clave para demostrar la acotación es el Lema de Deformación 1.22, donde agrandamos un conjunto dado de perímetro finito proporcionando una variación de perímetro controlada por un múltiplo del volumen añadido. Esta es una observación muy útil de Almgren [**1**, V1.2(3)], [**67**, Lemma 13.5]. El Lema de Deformación es el único punto donde utilizamos de manera determinante que nuestra variedad sub-riemanniana es de contacto, para construir una foliación por medio de hiperfuperficies con curvatura media uniformemente acotada. Nuestra prueba del Lema de Deformación no parece generalizarse de forma sencilla a variedades sub-riemannianas más generales. El resultado de estructura de sucesiones minimizantes aparece por primera vez, aunque era conocido por expertos en teoría geométrica de la medida, en Ritoré [**74**] para superficies riemannianas, y en Ritoré y Rosales [**79**] para variedades riemannianas de dimensión arbitraria. En ciertos casos, la Proposición 1.24 asegura una demostración directa de la existencia de regiones isoperimétricas.

En el Capítulo 2, probamos una condición necesaria para que una superficie minimal C^2, con conjunto singular vacío, sea estable, en una clase muy amplía de variedades pseudo-hermitianas, que incluyen los grupos de Lie unimodulares, Proposición 2.48:

> *Sea Σ una superficie C^2 completa y orientable, con conjunto singular vacío, inmersa en una variedad pseudo-hermitiana $(M, \mathcal{H}, g_{\mathcal{H}}, J)$. Supongamos que $g(R(T,Z)\nu_h, Z) - Z(g(\tau(Z), \nu_h)) = 0$ en Σ y que la cantidad $W - c_1 g(\tau(Z), \nu_h)$ es constante a lo largo de las curvas características. Asumamos además que las curvas características en Σ son todas cerradas o no cerradas. Si Σ es una superficie minimal estable, entonces $W - c_1 g(\tau(Z), \nu_h) \leqslant 0$ sobre Σ. Además, si $W - c_1 g(\tau(Z), \nu_h) = 0$ entonces Σ es una superficie estable y vertical.*

Esta clase es importante, porque Perrone [**73**] ha demostrado que las variedades riemannianas de contacto simplemente conexas, de dimensión tres, homogéneas según Bootby y Wang, [**16**] (existe un grupo de Lie simplemente conexo que actúa de manera transitiva como grupo de los difeomorfismos de contacto), son grupos de Lie. La condición que encontramos involucra la curvatura escalar de Webster W y la torsión pseudo-hermítica τ de la variedad, que son invariantes pseudo-hermíticos.

Esta caracterización ha sido obtenida por medio del estudio de un operador de estabilidad, que ha sido construido a partir de la segunda fórmula de variación del área sub-riemanniana. En la Sección 2.7 construimos otro operador de estabilidad que tiene en cuenta la parte singular de la superficie. Con estas dos herramientas, podemos clasificar las superficies completas y estables en el grupo de los movimientos rígidos del plano euclídeo $\mathcal{RT}$, Teorema 2.61:

> *Sea Σ una superficie C^2 orientada, completa y estable inmersa en $\mathcal{RT}$. Entonces*
> 1. *si Σ es una superficie no singular, entonces es un plano vertical;*
> 2. *si Σ es una superficie con conjunto singular no vacío, entonces es el helicoide que gira a la derecha.*

El grupo $\mathcal{RT}$ es interesante por dos razones. Desde el punto de vista geométrico es una de las variedades pseudo-hermíticas con torsión no nula más simples. Además es un modelo para la corteza visual del ojo humano que desempeña un papel importante en la teoría de la reconstrucción de imágenes, como ha sido observado por Citti y Sarti [**28**] y Citti, Sarti y Petitot [**84**]. Fijando una curva Γ, podemos reconstruir una imagen resolviendo un problema de Plateau. Esto es equivalente a encontrar una superficie minimal estable Σ cuyo borde sea Γ, es decir Σ tal que $A'(\Sigma)(0) = 0$ y $A''(\Sigma)(0) \geqslant 0$, para todas las variaciones que fijan $\partial\Sigma = \Gamma$.

El objetivo principal del Capítulo 3 es generalizar las fórmulas de variación del área sub-riemanniana a variedades sub-riemannianas de contacto de dimensión arbitraria.

Decimos que una hipersuperficie C^1 es de clase C^2_h si Σ es el conjunto de nivel de una función que admite dos derivadas horizontales continuas. Notamos que $C^2_{\mathbb{H}} \subset C^2_h$, porque no pedimos que Σ_0 sea vacío. Probamos que, en una hipersuperficie C^2_h, la medida de Hausdorff euclídea $(n+3)$-dimensional de Σ_0 es nula, Teorema 3.3. Esta estimación es bastante sorprendente, si se compara con el Teorema 0.2. Además, como nuestra estimación no es muy sofisticada, creemos que se verifica $\mathcal{H}^{n+1}_E(\Sigma_0) = 0$ bajo las hipótesis del Teorema 3.3. En nuestra prueba simplemente observamos que Σ_0 está contenido en una superficie $\mathbb{H}$-regular de codimensión n y estimamos $\mathcal{H}^{n+3}_E(S)$.

Utilizando el Teorema 3.3, para $n \geqslant 4$, obtenemos una primera variación general del área, que mueve el conjunto singular. Como Σ_0 está contenida en una unión finita de bolas euclídeas $(2n-1)$-dimensionales, podemos construir una familia oportuna de funciones "cut-off" ϕ_ε, $\varepsilon > 0$. Como $|\nabla\phi_\varepsilon|$ está controlado por ε y las funciones ϕ_ε se anulan en el conjunto singular, tomando límite cuando $\varepsilon \to 0$ obtenemos la fórmula general.

Concluimos probando la segunda fórmula de variación para hipersuperficies C^2 con conjunto singular vacío. Movemos la hipersuperficie por medio de un grupo uniparamétrico de difeomorfismos inducido por un campo vectorial arbitrario U. Mostramos que, despejando $U = U_{ht} + U^{\perp}_{ht}$, donde $U_{ht} \in T\Sigma \cap \mathcal{H}$ y $U^{\perp}_{ht} \in (T\Sigma \cap \mathcal{H})^{\perp}$, sólo la parte ortogonal intrínseca aparece en la fórmula. El resultado análogo riemanniano es muy conocido, aunque, una demostración rigurosa se conoce sólo para hipersuperficies de clase C^3.

Por último, observamos que, para una mayor comodidad del lector, los tres capítulos son autocontenidos e independientes. Notamos además que corresponden a las referencias bibliográficas [**47, 45, 46**].

Introduction, abstract and conclusions

In the history of the Calculus of Variations, a special interest has been devoted to variational problems related to the area functional. Among these ones, the Isoperimetric problem and Plateau's problem stand out and have received a special attention. The isoperimetric problem consists on finding the regions of volume v with the smallest perimeter. In Plateau's problem, we try to understand which surfaces (or hypersurfaces) with a given fixed boundary minimize area. These two questions are strongly related to the study of minimal surfaces and constant mean curvature surfaces. For a beginners' introduction to the geometry of surfaces we suggest the monographs by do Carmo [**35**] and Montiel and Ros [**66**].

These topics, in the Euclidean space, and consequently in Riemannian manifolds, have been studied with techniques of Geometric Measure Theory, Partial Differential Equations, Differential Geometry, Optimal Transport, etc...

In the last decades, an intensive work has been done on variational problems related to the area functional in sub-Riemannian manifolds, covering many aspects such as isoperimetric inequalities, existence of isoperimetric regions, geometric measure theory, structure of the singular set of C^1 hypersurfaces, regularity of minimizers, minimizing graphs, the Bernstein problem, stability of constant mean curvature surfaces, and many others.

In [**71**], the following isoperimetric inequality

$$|\partial\Omega| \geqslant C|\Omega|^{4/3} \tag{0.4}$$

was proven in the Heisenberg group $\mathbb{H}^1$ by Pansu, while the exponent is sharp, the constant $C > 0$ is not. Also in [**71**], it was conjectured the the isoperimetric regions in $\mathbb{H}^1$ were a family of topological balls that are not metric balls. The boundary of these regions, that will be called Pansu's spheres and denoted by S_λ, $\lambda > 0$, is characterized by the property that each S_λ is ruled by sub-Riemannian geodesics of constant curvature λ, [**20**, § 2.3]. Recently, Chanillo and Yang, [**21**], have generalized the inequality (0.4) to pseudo-hermitian three-manifolds with vanishing pseudo-hermitian torsion. For a quite complete account of recent development concerning the optimal isoperimetric inequality in $\mathbb{H}^1$, the interest reader should consult the monograph by Capogna, Danielli, Pauls and Tyson [**20**].

In sub-Riemannian Geometry, apart from the compact case, the only known existence result of isoperimetric regions, has been given by Leonardi and Rigot for Carnot groups [**58**]. In their paper they made an extensive use of the properties of the isoperimetric profile in a Carnot group $\mathbb{G}$. Since isoperimetric regions in $\mathbb{G}$ are invariant by intrinsic dilations, the isoperimetric profile $I_{\mathbb{G}}$ of $\mathbb{G}$ is given by $I_{\mathbb{G}}(v) = Cv^q$, where C is a positive constant and $q \in (0, 1)$. In particular, the function $I_{\mathbb{G}}$ is strictly concave, a property that plays a fundamental role in their proof. Leonardi and Rigot also proved that isoperimetric sets are domains of isoperimetry in the particular case of the Heisenberg group. However, their result cannot be applied to some interesting sub-Riemannian groups which are not of Carnot type. An example is the

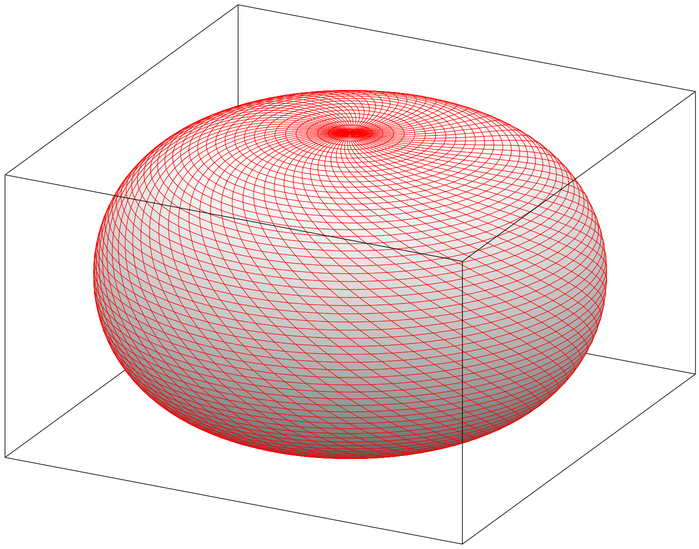

FIGURE 2. A Pansu's sphere S_λ in $\mathbb{H}^1$.

roto-translational group, described in [**20**]. Some of the crucial points of the proof of Leonardi and Rigot are discussed in [**20**, § 8.2].

We remark that in non-compact manifolds, the existence of isoperimetric regions is not trivial. Ritoré [**74**, Thm. 2.16] observed the existence of complete non-compact Riemannian manifolds for which isoperimetric regions do not exist for any value of the volume, such as planes of revolution with strictly increasing Gauss curvature.

Pansu's conjecture has presented no advances up to the late '00, when two special case were proven. In [**81**], Ritoré and Rosales showed that the regions enclosed by Pansu's spheres are isoperimetric sets in $\mathbb{H}^1$, among all regions bounded by a C^2 surface. In their paper a characterization of an area-stationary surface (with or without volume constraint), is obtained, namely, that the characteristic lines must meet the singular ones in an orthogonal way. This property was also observed by Cheng, Hwang and Yang [**26**] in their study of area-stationary graphs, and proven by computing a first variation formula of the sub-Riemannin area and using the local structure of the singular set, studied by Cheng, Hwang, Malchiodi and Yang [**25**]. Finally, this condition, together with an analysis of the Jacobi fields on the surface, was used to classify the oriented, immersed, complete and connected C^2 area-stationary surfaces enclosing a fixed volume.

In [**64**], Monti and Rickly showed that the boundary of an Euclidean convex isoperimetric region, in $\mathbb{H}^1$, has to be ruled by sub-Riemannian geodesics. This implies that the boundary is a Pansu's sphere. Their proof is based on a Sobolev-type regularity result for a class of vector fields of bounded variation.

Another growing research line in sub-Riemannian spaces, is related to geometric measure theory problems, such as the study of finite perimeter sets and regular surfaces from an intrinsic viewpoint. These topics have been investigated in particular in the Heisenberg groups $\mathbb{H}^n$ and in Carnot groups $\mathbb{G}$. In [**41**, Theorem 7.1], Franchi, Serapioni and Serra Cassano proved a structure theorem for a locally finite perimeter set, analogous to the Euclidean one,

THEOREM 0.7 (Structure of $\mathbb{H}$-Caccioppoli sets). *If $E \subset \mathbb{H}^n$ is a locally finite $\mathbb{H}$-perimeter set, then the reduced boundary $\partial^*_{\mathbb{H}} E$ is $\mathbb{H}$-rectifiable, i.e.*

$$\partial^*_{\mathbb{H}} E = N \cup \bigcup_{i=1}^{\infty} N_i,$$

where $\mathcal{H}^{2n+1}_{\mathbb{H}}(N) = 0$ e N_i is a compact subset of an $\mathbb{H}$-regular hypersurface S_i.

In [**41**, Definition 6.1] the authors define an $\mathbb{H}$-regular surface S, if for any $p \in S$ exists a metric ball $B(p,r)$ and a function $f \in C^1_{\mathbb{H}}(B)(p,r)$, such that

$$S \cap B(p,r) = \{q \in \mathbb{H}^n : f(q) = 0, \nabla_{\mathbb{H}} f(q) \neq 0\}.$$

Later, this definition was generalized to more general ambients by Franchi, Serapioni and Serra Cassano [**42, 43**] and Citti and Manfredini [**29**]. We remark that $\mathbb{H}$-regular surfaces can be very irregular, from the Euclidean point of view. Kirchheim and Serra Cassano have constructed examples of $\mathbb{H}$-regular surfaces that are fractal sets, [**57**].

Franchi, Serapioni and Serra Cassano have also pointed out that an $\mathbb{H}$-regular surface can be equivalently described as an intrinsic graph, [**43**, Theorem 3.27], see also [**44**]. A region S in $\mathbb{H}^n$ is an intrinsic graph, over a domain Ω inside a vertical hyperplane, if the sub-Riemannian geodesics of curvature zero, leaving Ω orthogonally, meet S in just one point, [**43**, Definition 3.6].

An alternative characterization of intrinsic graphs is studied by Ambrosio, Serra Cassano and Vittone [**4**], Bigolin and Serra Cassano [**12, 13**] and Bigolin, Caravenna and Serra Cassano [**11**]. A continuous function ϕ is a distributional solution of $\nabla^\phi \phi = w$, where $w \in C(\Omega, \mathbb{R}^{2n-1})$ and Ω is a domain in a vertical hyperplane, if and only if ϕ induces a $\mathbb{H}$-regular graph. Here ∇^ϕ denotes the intrinsic gradient in S and $\nabla^\phi \phi = w$ is a first-order non-linear PDE, related to Burgers' operator. Monti and Vittone [**65**] have shown that a finite perimeter set with a continuous horizontal normal is a $C^1_{\mathbb{H}}$ surface.

Considering a Euclidean regular surface in $\mathbb{H}^n$, we define the *singular set* as the set of points in which the tangent plane coincides with the horizontal distribution, i.e.,

$$\Sigma_0 := \{p \in \Sigma : T_p\Sigma \equiv \mathcal{H}_p\}.$$

In a neighborhood of a singular point, a surface can be viewed as an Euclidean graph over the xy-plane. We observe that $C^{1,1}$ Euclidean regular surfaces with non-empty singular set are not $\mathbb{H}$-regular surfaces. On the other hand, this may be false for Euclidean $C^{1,\alpha}$ surfaces. Indeed, one may check that the intrinsic graph of $f(y,t) := |t|^\alpha$ is $\mathbb{H}$-regular for $\alpha \in]1/2, 1[$, [**4**, Corollary 5.11], even though the Euclidean tangent plane at the origin coincides with the horizontal one. This surface is of class $C^{1,2\alpha-1}$.

The Euclidean and intrinsic Hausdorff dimension of Σ_0, are discussed by Balogh [**5**], but also by Derridj [**34**] and Cheng, Hwang and Yang [**26**]. We summarize the following result by Balogh

THEOREM 0.8. (i) *If Σ is a C^1 smooth, regular hypersurface in $\mathbb{R}^{2n+1}$, then $\mathcal{H}^{2n+1}_{\mathbb{H}}(\Sigma_0) = 0$;*

(ii) *if Σ is a $C^{1,1}$ smooth, regular hypersurface in $\mathbb{R}^{2n+1}$, then $dim_E(\Sigma_0) < 2n$ and $dim_{\mathbb{H}}(\Sigma_0) < 2n$;*

(iii) *if Σ is C^2 smooth, then $dim_E(\Sigma_0) < n$ and $dim_{\mathbb{H}}(\Sigma_0) < n$;*

(iv) *for any* $\alpha > 0$ *there exists a* $C^{1,1}$ *smooth, regular hypersurface* Σ_α *such that* $dim_E((\Sigma_\alpha)_0) \geqslant 2n - \alpha$;
(v) *there exists a regular hypersurface* Σ *of smoothness* $\bigcap_{0<\alpha<1} C^{1,\alpha}$ *such that* $\mathcal{H}_E^{2n}(\Sigma) < +\infty$ *and* $\mathcal{H}_E^{2n}(\Sigma_0) > 0$.

Cheng, Hwang, Malchiodi and Yang have found a more geometric characterization of the structure of the singular set, [**25**], for C^2 surfaces in $\mathbb{H}^1$. They showed

PROPOSITION 0.9. *Let* Σ *be a surface of class* C^2 *and let* p *a singular point. Then*
(i) *or* p *is an isolated singular point;*
(ii) *or* p *is contained in a* C^1 *smooth singular curve. In this case the characteristic curves meet the singular one with the same incident and reflected angle, when* Σ *has bounded mean curvature.*

Recently, in [**24, 27**], they also proved

PROPOSITION 0.10. *Let* Σ *be a* C^1 *smooth surface* Σ *in* $\mathbb{H}^1$. *Assume that* Σ *is a weak solution of the minimal surface equation. Then, we have*
(i) *for each* $p \in \Sigma - \Sigma_0$, *there exists a unique characteristic and a unique "seed" curve. Furthermore the projections of the characteristic and seed curves, on the xy-plane, are* C^2 *smooth;*
(ii) *the singular set is path-connected and the set of non-degenerate singular points consists of* C^1 *smooth curves;*
(iii) *two characteristic curves issuing from a non-degenerate singular point* p *have the same angle with the tangent line of the singular curve through* p.

We remark that they have generalized Proposition 0.10 to 3-dimensional pseudo-hermitian manifolds, [**24**, § 8].

Bigolin and Vittone have remarked that, in general, we have no uniqueness for the characteristic curves passing through a point in a $\mathbb{H}$-regular surface, [**14**].

Results about the structure of a surface near Σ_0 imply strong consequences related to the classification of area-stationary surfaces, since in general the behavior of Σ in a neighborhood of Σ_0 is very rigid. For example, these type of results were heavily used by Ritoré and Rosales in [**81**] and in Section 2.9 of this thesis, where we classify area-stationary surfaces with non-empty singular set in the roto-translation group $\mathcal{RT}$, Lemma 2.54 and Lemma 2.55

> *Let* Σ *be a complete area-stationary surface of class* C^2 *with non-empty singular set. Then* Σ *is a right-handed helicoid or a plane* $\{(x, y, \theta) \in \mathcal{RT} : ax + by + c = 0, a, b \in \mathbb{R}, c \in \mathbb{S}^1\}$.

About regularity of the minimizers, the situation is different with respect to the Riemannian case. In fact Cheng, Hwang and Yang [**26**] and Ritoré [**76**] have constructed examples of graphs over the xy-plane, the so called t-graphs, which are area-minimizing and merely Euclidean Lipschitz. On the other hand, these examples are not $\mathbb{H}$-regular graphs. The only regularity result for intrinsic graphs, at our knowledge, is due to Capogna, Citti and Manfredini, [**17, 18**]

THEOREM 0.11. *Let* Σ *be an Lipschitz intrinsic graph over a vertical plane in* $\mathbb{H}^1$. *Let also* Σ *be the limit of Riemannian minimal graphs. Then* Σ *is of class* $C^{1,\alpha}$, $0 < \alpha < 1$, *and the characteristic curves are* C^∞.

THEOREM 0.12. *Let Σ be an Lipschitz intrinsic graph over a vertical plane in $\mathbb{H}^n$. Let also Σ be the limit of Riemannian minimal graphs. Then Σ is C^∞smooth.*

We remark that Theorem 0.11 is sharp and has been recently generalized to tridimensional Lie groups by Barbieri and Citti, [**8**].

Concerning the existence of minimizing graphs over a domain Ω, with a fixed boundary value, Serra Cassano and Vittone have proved existence for t-graphs, with boundary value in $L^1(\partial\Omega)$ and $\partial\Omega$ Lipschitz, [**85**]. They proved also that such minimizers are locally bounded and that a t-graph is always $\mathcal{H}^{2n}_E$-rectifiable in $\mathbb{H}^n$. Analogous results are present also in Cheng, Hwang and Yang [**26**], Cheng and Hwang [**23**] and Pauls [**72**]. Furthermore, in [**85**], it is also shown the existence of minimizing intrinsic graphs over a domain Ω of a vertical hyperplane in $\mathbb{H}^n$, with fixed boundary data. Moreover, if we suppose that the function which defines the graph is in $L^{2n+1}_{loc}(\Omega)$, it is also locally bounded.

Another well-known problem in Geometric Analysis is the Bernstein problem. In $\mathbb{R}^n$, it consists of classifying the entire minimal graphs. In $\mathbb{R}^n$ the solutions are area-minimizing, by a standard calibration argument. In $\mathbb{H}^n$, however, Danielli, Garofalo, Nhieu and Pauls have shown that there exist entire, horizontal, minimal graphs that are not stable, [**30, 31, 32**]. On the other hand the only complete, oriented, immersed and stable C^2 surfaces in $\mathbb{H}^1$ are vertical planes. Particular cases are provided by Barone-Adesi, Serra Cassano and Vittone [**9**] and Danielli, Garofalo, Nhieu and Pauls [**33**]. The general case is proved in [**55**], by Hurtado, Ritoré and Rosales. In $\mathbb{H}^n$, for $n \geqslant 5$, Barone Adesi, Serra Cassano and Vittone [**9**] have shown the existence of minimizing, intrinsic $\mathbb{H}$-regular graphs, that are not horizontal planes. The general problem in the class of $\mathbb{H}$-regular surfaces is still open.

We remark that in [**55**] Hurtado, Ritoré and Rosales also classify the C^2 immersed, oriented, complete, stable surfaces in $\mathbb{H}^1$, with non-empty singular set: the only one is the hyperbolic paraboloid.

Their key tool is a second variation formula of the sub-Riemannian area for C^2 surfaces with non-empty singular set. While in the first variation of the area is easy to move the singular set, moving Σ_0 in the second variation is much more involved. Essentially we cannot differentiate under the integral sign, since the second derivative of $|N_h|$ is not bounded near Σ_0. The only possibility is to perform variations consisting on minimal surfaces near Σ_0, to perform a general variation in $\Sigma - \Sigma_0$, and, by means of a partition of unity on characteristic curves, to fit together both variations.

The Bernstein-type problem is generalized by Rosales in [**82**], for C^2 complete, oriented, horizontal, stable, constant mean curvature surfaces without singular points immersed in the Sasakian space forms $\mathbb{S}^1$ and $\widetilde{SL}(2,\mathbb{R})$.

In this thesis we study topics related to area-stationary surfaces in contact sub-Riemannian manifolds. Following the monograph by Blair [**15**], a *contact manifold* is a C^∞ manifold M^{2n+1} of odd dimension so that there is a one-form ω such that $d\omega$ is non-degenerate when restricted to $\mathcal{H} := \ker(\omega)$. Since

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

the *horizontal distribution* $\mathcal{H} := \ker(\omega)$ is completely non-integrable. A well-known example of a contact manifold is the Euclidean space $\mathbb{R}^{2n+1}$ with the standard contact one-form

$$\omega_0 := dt + \sum_{i=1}^{n}(x_i dy_i - y_i dx_i). \tag{0.5}$$

A *contact sub-Riemannian manifold* is a contact manifold equipped with a positive defined metric $g_{\mathcal{H}}$ on $\mathcal{H}$.

A usual class defined in contact geometry is the one of contact Riemannian manifolds, see the Blair's monograph [**15**] and Tanno [**90**]. Given a contact manifold, one can ensure the existence of a Riemannian metric g and an $(1,1)$-tensor field J so that

$$g(T,X) = \omega(X), \quad 2g(X, J(Y)) = d\omega(X,Y), \quad J^2(X) = -X + \omega(X)\,T. \tag{0.6}$$

The structure given by (M,ω,g,J) is called a contact Riemannian manifold. The class of contact sub-Riemannian manifolds is different from this one. Recall that, in our definition, the metric $g_{\mathcal{H}}$ is given, and it can be extended to a Riemannian metric g in TM. However, there is not in general an $(1,1)$-tensor field J satisfying all conditions in (0.6). Observe that the second condition in (0.6) uniquely defines J on $\mathcal{H}$, but this J does not satisfy in general the third condition in (0.6), as it is easily seen in $(\mathbb{R}^3,\omega_0)$ choosing an appropriate positive definite metric in $\ker(\omega_0)$.

In Chapter 1 we prove an existence result for isoperimetric regions in contact sub-Riemannian manifolds such that the quotient by the group of contact isometries, the diffeomorphisms that preserve the contact structure and the sub-Riemannian metric, is compact, Theorem 1.26. This is the analogous result to of Morgan's Riemannian one, [**67, 68**].

In the proof of Theorem 1.26 we follow closely Morgan's scheme: we pick a minimizing sequence of sets of volume v whose perimeters approach the infimum of the perimeter of sets of volume v. If the sequence subconverges without losing any fraction of the original volume, the lower semicontinuity of the perimeter implies that the limit set is an isoperimetric region of volume v. If some fraction of the volume is missing then Proposition 1.24 implies that the minimizing sequence can be broken into a converging part and a diverging one, the latter being composed of sets of uniformly positive volume, see Ritoré [**74, 75**] and Ritoré and Rosales [**79**], for the Riemannian case. The converging part has a limit, which is an isoperimetric region for its volume, and is bounded by Lemma 1.23. Hence we can suitably translate the diverging part to recover some of the lost volume. An important point here is that we always recover a fixed fraction of the volume because of Lemma 1.27, see Leonardi and Rigot [**58**, Lemma 4.1].

Along the proof of Theorem 1.26 two important technical points have to be solved, as mentioned in the previous paragraph. We prove in Lemma 1.23 boundedness of the isoperimetric regions, and a structure result for minimizing sequences in Proposition 1.24 . The key point to prove boundedness is the Deformation Lemma 1.22, where we slightly enlarge a given finite perimeter set producing a variation of perimeter which can be controlled by a multiple of the increase of volume. This is a extremely useful observation of Almgren [**1**, V1.2(3)], [**67**, Lemma 13.5]. The Deformation Lemma is the only point where we strongly use the fact that our underlying sub-Riemannian manifold is of contact type, to construct a foliation by hypersurfaces with controlled mean curvature. Our proof of the Deformation Lemma 1.22 does not seem to generalize easily to more general sub-Riemannian manifolds. The structure result for minimizing sequences appeared for the first time, although it was known to experts in Geometric Measure Theory, in Ritoré [**75**] for Riemannian surfaces, and in Ritoré and Rosales [**79**] for Riemannian manifolds of any dimension. In some cases, Proposition 1.24 provides direct proofs of existence of isoperimetric regions.

In Chapter 2, we prove a necessary condition for C^2 stable minimal surfaces with empty singular set in a large class of pseudo-hermitian manifolds, that includes the uni-modular Lie groups, Proposition 2.48

> *Let Σ be a C^2 complete orientable surface with empty singular set immersed in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. We suppose that $g(R(T,Z)\nu_h, Z) - Z(g(\tau(Z), \nu_h)) = 0$ on Σ and the quantity $W - c_1 g(\tau(Z), \nu_h)$ is constant along characteristic curves. We also assume that all characteristic curves in Σ are either closed or non-closed. If Σ is a stable minimal surface, then $W - c_1 g(\tau(Z), \nu_h) \leqslant 0$ on Σ. Moreover, if $W - c_1 g(\tau(Z), \nu_h) = 0$ then Σ is a stable vertical surface.*

This is an important class since Perrone [**73**] has shown that simply connected contact Riemannian 3-manifolds, homogeneous in the sense of Bootby and Wang, [**16**] (there exists a connected Lie group acting transitively as a group of contact diffeomorphisms), are Lie groups. The condition we found involves the Webster scalar curvature W and the pseudo-hermitian torsion τ of the manifold, that are pseudo-hermitian invariants.

This characterization is obtained by the study of a stability operator, which is constructed from the second variation formula of the sub-Riemannian area. In Section 2.7 we construct another stability operator that takes account the singular set. With these two tools, we give a classification of complete stable surfaces in the group of the rigid motions of the Euclidean plane $\mathcal{RT}$, Theorem 2.61

> *Let Σ be a C^2 stable, immersed, oriented and complete surface in $\mathcal{RT}$. Then*
> 1. *if Σ is a non-singular surface, then it is a vertical plane;*
> 2. *if Σ is a surface with singular set, then it is a right-handed helicoid.*

The $\mathcal{RT}$ group is interesting for two reasons. From the geometric point of view it is one of the simplest pseudo-hermitian manifolds which has non-vanishing torsion. Moreover it is a model of the visual cortex of the human eye which plays an important role in the theory of image reconstruction, as observed by Citti and Sarti [**28**] and Citti, Sarti and Petitot [**84**]. Given a boundary curve Γ, we can reconstruct an image by solving a Plateau's problem. This is equivalent to find a stable minimal surface Σ with boundary Γ, i.e. to find Σ such that $A'(\Sigma)(0) = 0$ and $A''(\Sigma)(0) \geqslant 0$, for variations that fix $\partial\Sigma = \Gamma$.

The main purpose of Chapter 3 is to generalize variation formulas for the sub-Riemannian area in contact sub-Riemannian manifolds of arbitrary dimensions.

We call a C^1 hypersurface Σ of class C^2_h if Σ is a level set of a function with two continuous horizontal derivatives. We remark that $C^2_{\mathbb{H}} \subset C^2_h$, since we do not require that Σ_0 is empty. We prove that, in a C^2_h hypersurface, the $(n+3)$-Euclidean Hausdorff measure of Σ_0 vanishes, Theorem 3.3. This estimate is quite surprising, compared with Theorem 0.8. In particular, since our estimation is unsophisticated and we believe that $\mathcal{H}^{n+1}_E(\Sigma_0) = 0$ holds under the hypothesis of Theorem 3.3. In the proof we simply observe that Σ_0 is contained in a $\mathbb{H}$-regular surface S of codimension n and we estimate $\mathcal{H}^{n+3}_E(S)$.

By Theorem 3.3, for $n \geqslant 4$, we can produce a general first variation formula for C^2_h hypersurfaces moving the singular set. Since Σ_0 is contained in a finite union of $(2n-1)$-dimensional Euclidean balls, we construct a suitable family of cut-off functions ϕ_ε, $\varepsilon > 0$. As $|\nabla \phi_\varepsilon|$ is controlled by ε and ϕ_ε vanishes on the singular set, we can take limits when $\varepsilon \to 0$ to get a general formula.

Finally we prove a second variation formula for C^2 hypersurfaces with empty singular set. We move the surface by a diffeomorphism induced by an arbitrary vector field U. We show that, splitting $U = U_{ht} + U_{ht}^{\perp}$, where $U_{ht} \in T\Sigma \cap \mathcal{H}$ and $U_{ht}^{\perp} \in (T\Sigma \cap \mathcal{H})^{\perp}$, only the intrinsic orthogonal part $U_{ht}^{\perp}$ appears in the formula. The analoguos Riemannian's results are well-known. However, at our knowledge, a rigorous proof was known only for C^3 surfaces.

Finally we remark that, for the reader convenience, the three chapters are self-contained and independent. We also point out that they correspond to the following references [**47, 45, 46**].

CHAPTER 1

Existence of isoperimetric regions in contact sub-Riemannian manifolds

1.1. Preliminaries

A *contact manifold* [**15**] is a C^∞ manifold M^{2n+1} of odd dimension so that there is a one-form ω such that $d\omega$ is non-degenerate when restricted to $\mathcal{H} := \ker(\omega)$. Since

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

the *horizontal distribution* $\mathcal{H} := \ker(\omega)$ is completely non-integrable. One can easily prove the existence of a unique vector field T in M so that

$$\omega(T) = 1, \qquad (\mathcal{L}_T\omega)(X) = 0, \tag{1.1}$$

where $\mathcal{L}$ is the Lie derivative and X is any smooth vector field on M. The vector field T is usually called the *Reeb vector field* of the contact manifold M. It is a direct consequence that $\omega \wedge (d\omega)^n$ is an orientation form in M.

A well-known example of a contact manifold is the Euclidean space $\mathbb{R}^{2n+1}$ with the standard contact one-form

$$\omega_0 := dt + \sum_{i=1}^{n}(x_i dy_i - y_i dx_i). \tag{1.2}$$

A *contact transformation* between contact manifolds is a diffeomorphism preserving the horizontal distributions. A *strict contact transformation* is a diffeomorphism preserving the contact one-forms. A strict contact transformation preserves the Reeb vector fields. Darboux's Theorem [**15**, Thm. 3.1] shows that, given a contact manifold and some point $p \in M$, there is an open neighborhood U of p and a strict contact transformation f from U into an open set of $\mathbb{R}^{2n+1}$ with its standard contact structure induced by ω_0. Such a local chart will be called a *Darboux chart*.

The length of a piecewise horizontal curve $\gamma : I \to M$ is defined by

$$L(\gamma) := \int_I |\gamma'(t)|\, dt,$$

where the modulus is computed with respect to the metric $g_{\mathcal{H}}$. The Carnot-Carathéodory distance $d(p,q)$ between p, $q \in M$ is defined as the infimum of the lengths of piecewise smooth horizontal curves joining p and q. A minimizing geodesic is any curve $\gamma : I \to M$ such that $d(\gamma(t),\gamma(t')) = |t-t'|$ for each t, $t' \in I$. We shall say that the sub-Riemannian manifold $(M, g_{\mathcal{H}}, \omega)$ is complete if (M,d) is a complete metric space. By Hopf-Rinow's Theorem [**51**, p. 9] bounded closed sets are compact and each pair of points can be joined by a minimizing geodesic. From [**62**, Chap. 5] a minimizing geodesic in a contact sub-Riemannian manifold is a smooth curve that satisfies the geodesic equations, i.e., it is normal.

The metric $g_{\mathcal{H}}$ can be extended to a Riemannian metric g on M by requiring that T be a unit vector orthogonal to $\mathcal{H}$. The scalar product of two vector fields X and Y with respect to the metric g will be often denoted by $\langle X, Y\rangle$. The Levi-Civita connection induced by g will be denoted by D. An important property of the metric g is that the integral curves of the Reeb vector field T defined in (3.1) are *geodesics*, see [**15**, Thm. 4.5]. To check this property we observe that condition $(\mathcal{L}_T\omega)(X)$ in (3.1) applied to a horizontal vector field X yields $\omega([T,X])=0$ so that $[T,X]$ is horizontal. Hence, for any horizontal vector field X, we have

$$\big\langle X, D_T T\big\rangle = -\big\langle D_T X, T\big\rangle = -\big\langle D_X T, T\big\rangle = 0,$$

where in the last equality we have used $|T|=1$. Since we trivially have $\big\langle T, D_T T\big\rangle = 0$, we get $D_T T = 0$, as we claimed.

A usual class defined in contact geometry is the one of contact Riemannian manifolds, see [**15**], [**90**]. Given a contact manifold, one can ensure the existence of a Riemannian metric g and a $(1,1)$-tensor field J so that

$$g(T,X)=\omega(X),\quad 2g(X,J(Y))=d\omega(X,Y),\quad J^2(X)=-X+\omega(X)\,T. \tag{1.3}$$

The structure given by (M,ω,g,J) is called a contact Riemannian manifold. The class of contact sub-Riemannian manifolds is different from this one. Recall that, in our definition, the metric $g_{\mathcal{H}}$ is given, and it is extended to a Riemannian metric g in TM. However, there is not in general an $(1,1)$-tensor field J satisfying all conditions in (3.3). Observe that the second condition in (3.3) uniquely defines J on $\mathcal{H}$, but this J does not satisfy in general the third condition in (3.3), as it is easily seen in $(\mathbb{R}^3,\omega_0)$ choosing an appropriate positive definite metric in $\ker(\omega_0)$.

The Riemannian volume form dv_g in (M,g) coincides with Popp's measure [**62**, § 10.6]. The volume of a set $E\subset M$ with respect to the Riemannian metric g will be denoted by $|E|$.

A *contact isometry* in $(M,g_{\mathcal{H}},\omega)$ is a strict contact transformation that preserves $g_{\mathcal{H}}$. Contact isometries preserve the Reeb vector fields and they are isometries of the Riemannian manifold (M,g). The group of contact isometries of $(M,g_{\mathcal{H}},\omega)$ will be denoted by $\mathrm{Isom}_\omega(M,g)$.

It follows from [**70**, Thm. 1] that, given a compact set $K\subset M$ there are positive constants ℓ, L, r_0, such that M is *Ahlfors-regular*

$$\ell r^Q \leqslant |B(x,r)| \leqslant L r^Q, \tag{1.4}$$

for all $x\in K$, $0<r<r_0$. Here Q is the *homogeneous dimension* of M, defined as

$$Q := 2n+2. \tag{1.5}$$

Related to the homogeneous dimension we shall also consider the isoperimetric exponent

$$q := (Q-1)/Q. \tag{1.6}$$

In the case of contact sub-Riemannian manifolds this result also follows taking Darboux charts. Inequalities (1.4) immediately imply the *doubling property*: given a compact set $K\subset M$, there are positive constants C, r_0 such that

$$|B(x,2r)| \leqslant C|B(x,r)|, \tag{1.7}$$

for all $x\in K$, $0<r<r_0$. Moreover, (1.4) also implies that, given a compact subset $K\subset M$, there are positive constants C, r_0, such that

$$\frac{|B(x_0,r)|}{|B(x,s)|} \geqslant C\bigg(\frac{r}{s}\bigg)^Q, \tag{1.8}$$

for any $x_0\in K$, $x\in B(x_0,r)$, $0<r\leqslant s<r_0$.

Given a Borel set $E \subset M$ and an open set $\Omega \subset M$, the *perimeter* of E in Ω can be defined, following the Euclidean definition by De Giorgi, by

$$P(E,\Omega) := \sup\left\{ \int_E \operatorname{div} X \, dv_g : X \in \mathfrak{X}_0^1(\Omega), X \text{ horizontal}, |X| \leqslant 1 \right\},$$

where $\mathfrak{X}_0^1(\Omega)$ is the space of vector fields of class C^1 and compact support in Ω and div is the divergence in the Riemannian manifold (M, g). When $\Omega = M$ we define $P(E) := P(E, M)$. A set E is called of *finite perimeter* if $P(E) < +\infty$, and of *locally finite perimeter* if $P(E,\Omega) < +\infty$ for any bounded open subset $\Omega \subset M$. See [**41**] and [**48**] for similar definitions.

A function $u \in L^1(M)$ is of *bounded variation* in an open set Ω if

$$|Du|(\Omega) := \sup\left\{ \int_\Omega u \operatorname{div} X \, dv_g : X \in C_0^1(M), X \text{ horizontal}, |X| \leqslant 1, \operatorname{supp} X \subset \Omega \right\}$$

is finite. We shall say that $|Du|(\Omega)$ is the *total variation* of u in Ω. The space of functions with bounded variation in M will be denoted by $BV(M)$. If u is a smooth function then

$$|Du|(\Omega) = \int_\Omega |\nabla_h u| \, dv_g,$$

where $\nabla_h u$ is the orthogonal projection to $\mathcal{H}$ of the gradient ∇u of u in (M, g).

It follows easily that $P(E,\Omega)$ is the total variation of the characteristic function $\mathbf{1}_E$ of E. A sequence of finite perimeter sets $\{E_i\}_{i\in\mathbb{N}}$ converges to a finite perimeter set E if $\mathbf{1}_{E_i}$ converges to $\mathbf{1}_E$ in $L^1_{loc}(M)$.

Finite perimeter sets are defined up to a set of measure zero. We can always choose a representative so that all density one points are included in the set and all density zero points are excluded [**49**, Chap. 3]. We shall always take such a representative without an explicit mention.

There is a more general definition of functions of bounded variation and of sets of finite perimeter in metric measure spaces, using a relaxation procedure, using as energy functional the L^1 norm of the minimal upper gradient, [**59**], [**3**]. If $(M, g_{\mathcal{H}}, \omega)$ is a contact sub-Riemannian manifold then the definition of perimeter given above coincides with the one in [**59**], [**3**]. See [**59**, § 5.3], [**3**, Ex. 3.2].

In case E has C^1 boundary Σ, it follows from the Divergence Theorem in the Riemannian manifold (M, g) that the perimeter $P(E)$ coincides with the sub-Riemannian area of Σ defined by

$$A(\Sigma) := \int_\Sigma |N_h| \, d\Sigma, \tag{1.9}$$

where N is a unit vector field normal to Σ, N_h the orthogonal projection of N to the horizontal distribution, and $d\Sigma$ is the Riemannian measure of Σ.

The following usual properties for finite perimeter sets E, $F \subset M$ in an open set $\Omega \subset M$ are proven in [**59**]

1. $P(E,\Omega) = P(F,\Omega)$ when the symmetric difference $E \triangle F$ satisfies $|(E \triangle F) \cap \Omega)| = 0$.
2. $P(E \cup F,\Omega) + P(E \cap F,\Omega) \leqslant P(E,\Omega) + P(F,\Omega)$.
3. $P(E,\Omega) = P(M \setminus E,\Omega)$.

The set function $\Omega \mapsto P(E,\Omega)$ is the restriction to the open subsets of the finite Borel measure $P(E, \cdot\,)$ defined by

$$P(E,B) := \inf\{P(E,A) : B \subset A, A \text{ open}\}, \tag{1.10}$$

where B is a any Borel set.

We fix a point $p \in M$ and we consider the open balls $B_r := B(p,r)$, $r > 0$. Then the following property holds from the definitions

$$P(E \cap B_r) \leqslant P(E, B_r) + P(E \setminus B_r, \partial B_r), \tag{1.11}$$

where $P(E \setminus B_r, \partial B_r)$ is defined from (1.10).

The following results are proved in general metric measure spaces

PROPOSITION 1.1 (Lower semicontinuity [**3**],[**59**]). *The function $E \to P(E, \Omega)$ is lower semicontinuous with respect to the $L^1(\Omega)$ topology.*

PROPOSITION 1.2 (Compactness [**59**]). *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of finite perimeter sets such that $\{\mathbf{1}_{E_i}\}_{i\in\mathbb{N}}$ is bounded in $L^1_{loc}(M)$ norm and satisfying $\sup_i P(E_i, \Omega) < +\infty$ for any relatively compact open set $\Omega \subset M$. Then there exists a finite perimeter set E in M and a subsequence $\{\mathbf{1}_{E_{n_i}}\}_{i\in\mathbb{N}}$ converging to $\mathbf{1}_E$ in $L^1_{loc}(M)$.*

THEOREM 1.3 (Gauss-Green for finite perimeter sets). *Let $E \subset M$ be a set of finite perimeter. Then there exists a $P(E)$-measurable vector field $\nu_E \in TM$ such that*

$$-\int_E \operatorname{div} X \, dv_g = \int_M g_{\mathcal{H}}(\nu_E, X) \, dP(E),$$

for all $X \in \mathcal{H}$ and $|\nu_E| = 1$ for $P(E)$-a.e. $x \in M$.

The proof consists essentially in taking local coordinates and applying Riesz Representation Theorem [**38**, § 1.8] to the linear functional $f \mapsto -\int f \operatorname{div}_{\mathcal{H}} X dv_g$, where f is any function with compact support in M. This result was proven in the Heisenberg group $\mathbb{H}^n$ in [**41**].

DEFINITION . Let E be a finite perimeter set. The *reduced boundary* $\partial^* E$ is composed of the points $x \in \partial E$ which satisfy

(i) $P(E, B_r(x)) > 0$, for all $r > 0$;
(ii) exists $\lim_{r\to 0} \fint_{B_r(x)} \nu_E dP(E)$ and its modulus is one.

The following approximation result, whose proof is a straightforward adaptation of the Euclidean one, [**49**, Chap. 1], holds.

PROPOSITION 1.4. *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold, and let $u \in BV(\Omega)$. Then there exists a sequence $\{u_i\}_{i\in\mathbb{N}}$ of smooth functions such that $u_i \to u$ in $L^1(\Omega)$ and $\lim_{i\to+\infty} |\nabla_h u_i|(\Omega) = |\nabla_h u|(\Omega)$.*

The localization lemma [**3**, Lemma 3.5], see also [**59**], allows us to prove

PROPOSITION 1.5. *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold, $E \subset M$ a finite perimeter set, $p \in M$, and $B_r := B(p,r)$. Then, for almost all $r > 0$, the set $E \setminus B_r$ has finite perimeter, and*

$$P(E \setminus B_r, \partial B_r) \leqslant \frac{d}{dr} |E \cap B_r|.$$

The *isoperimetric profile* of M is the function $I_M : (0, |M|) \to \mathbb{R}^+ \cup \{0\}$ given by

$$I_M(v) := \inf\{P(E) : E \subset M, |E| = v\}.$$

A set $E \subset M$ is an *isoperimetric region* if $P(E) = I_M(|E|)$. The isoperimetric profile must be seen as an optimal isoperimetric inequality in the manifold M, since for any set $E \subset M$ we have

$$P(E) \geqslant I_M(|E|),$$

with equality if and only if E is an isoperimetric region.

1.2. A relative isoperimetric inequality and an isoperimetric inequality for small volumes

In this section we consider a contact sub-Riemannian manifold $(M, g_{\mathcal{H}}, \omega)$. We shall say that M supports a 1-*Poincaré inequality* if there are constants C_P, $r_0 > 0$ such that

$$\int_{B(p,r)} |u - u_{p,r}|\, dv_g \leqslant C_P r \int_{B(p,r)} |\nabla_h u|\, dv_g$$

holds for every $p \in M$, $0 < r < r_0$, and $u \in C^\infty(M)$. Here $u_{p,r}$ is the average value of the function u in the ball $B(p,r)$ with respect to the measure dv_g

$$u_{p,r} := \frac{1}{|B(p,r)|} \int_{B(p,r)} u\, dv_g,$$

that will also be denoted by

$$\fint_{B(x,r)} u\, dv_g.$$

We shall prove that a 1-Poincaré inequality holds in M provided $M/\operatorname{Isom}_\omega(M,g)$ is compact, using the following result by Jerison

THEOREM 1.6 ([**56**, Thm. 2.1]). *Let $X_1, \ldots, X_m$ be C^∞ vector fields satisfying Hörmander's condition defined on a neighborhood Ω of the closure $\overline{E}_1$ of the Euclidean unit ball $E_1 \subset \mathbb{R}^d$.*

Then there exists constants $C > 0$, $r_0 > 0$ such that, for any $x \in E_1$ and every $0 < r < r_0$ such that $B(x, 2r) \subset \Omega$,

$$\int_{B(x,r)} |f - \tilde{f}_{x,r}|\, d\mathcal{L} \leqslant Cr \int_{B(x,r)} \left(\sum_{i=1}^m X_i(f)^2 \right)^{1/2} d\mathcal{L}, \tag{1.12}$$

for any $f \in C^\infty(\overline{B}(x,r))$, where the integration is taken with respect to the Lebesgue measure $\mathcal{L}$, the balls are computed with respect to the Carnot-Carathéodory distance associated to $X_1, \ldots, X_m$, and $\tilde{f}_{x,r}$ is the mean value with respect to the Lebesgue measure.

REMARK 1.7. Jerison really proved the 2-Poincaré inequality

$$\int_{B(x,r)} |f - \tilde{f}_{x,r}|^2\, d\mathcal{L} \leqslant Cr^2 \int_{B(x,r)} \left(\sum_{i=1}^m X_i(f)^2 \right) d\mathcal{L}.$$

However, as stated by Hajłasz and Koskela [**52**, Thm. 11.20], his proof also works for the L^1 norm in both sides of the inequality.

REMARK 1.8. The dependence of the constants C, r_0 is described in [**56**, p. 505].

Using Jerison's result we can easily prove

LEMMA 1.9 (Poincaré's inequality). *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold, and $K \subset M$ a compact subset. Then there exist constants C_K, $r_0 > 0$, only depending on K, such that*

$$\int_{B(p,r)} |u - u_{p,r}|\, dv_g \leqslant C_K r \int_{B(p,r)} |\nabla_h u|\, dv_g, \tag{1.13}$$

for all $p \in K$, $0 < r < r_0$, $u \in C^\infty(M)$.

PROOF. For every $p \in K$ we consider a Darboux chart centered at p, i.e., an open neighborhood U_p of p together with a diffeomorphism $\phi_p : U_p \to \mathbb{R}^{2n+1}$ with $\phi_p(p) = 0$ and $\phi_p^* \omega_0 = \omega$, where ω_0 is the standard contact form in Euclidean space.

We denote by $h_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, for $\lambda > 0$, the intrinsic dilation of ratio λ, defined by $h_\lambda(z,t) := (\lambda z, \lambda^2 t)$, for $(z,t) \in \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$. For every $p \in K$ we choose $\lambda(p) > 0$ so that the image of U_p by $\varphi_p := h_{\lambda(p)} \circ \phi_p$ contains the closure $\overline{E}_1$ of the unit ball $E_1 \subset \mathbb{R}^{2n+1}$. From the open covering $\{\varphi_p^{-1}(E_1)\}_{p\in K}$ of K we extract a finite subcovering $\varphi_{p_1}^{-1}(E_1), \ldots, \varphi_{p_r}^{-1}(E_1)$. From now on we fix some p_i, $i = 1, \ldots r$, and we take $p = p_i$, $\varphi = \varphi_i$, $U = U_{p_i}$.

We consider the scalar product $h := (\varphi^{-1})^* g_{\mathcal{H}}$ in the contact distribution $\mathcal{H}_0 := \ker(\omega_0)$. Let $X_1, \ldots, X_{2n}$ be an orthonormal basis of $\mathcal{H}_0$ with respect to h in Ω. Observe that φ is a contact transformation from (U, ω) to (Ω, ω_0) that preserves the sub-Riemannian metrics. Hence φ is an isometry between metric spaces when we consider on (U, ω) its associated Carnot-Carathéodory distance d and on (Ω, ω_0) the distance induced by the family of vector fields $X_1, \ldots, X_{2n}$. Moreover, if $u \in C^\infty(M)$ then, for every $p \in U$, we have

$$|(\nabla_h u)_p| = \left(\sum_{i=1}^{2n} (X_i)_p(\varphi \circ u) \right)^{1/2}. \tag{1.14}$$

Let $\mu := \varphi^{-1}(dv_g)$. Since φ is a diffeomorphism, μ and $d\mathcal{L}$ satisfy

$$\ell\, \mathcal{L}(E) \leqslant \mu(E) \leqslant L\, \mathcal{L}(E), \tag{1.15}$$

for some constants ℓ, $L > 0$, and E contained in a compact neighborhood of E_1 in $\varphi(U)$.

By Jerison's result, there are C, $r_0 > 0$ so that

$$\int_{B(x,r)} |f - \tilde{f}_{x,r}|\, d\mathcal{L} \leqslant Cr \int_{B(x,r)} \left(\sum_{i=1}^{2n} (X_i(f))^2 \right)^{1/2} d\mathcal{L}$$

for all $f \in C^\infty(\Omega)$, $z \in E_1$. Since

$$\int_{B(x,r)} |f - \tilde{f}_{x,r}|\, d\mathcal{L} = \frac{1}{\mathcal{L}(B(x,r))} \int_{B(x,r)} \int_{B(x,r)} |f(y) - f(z)|\, d\mathcal{L}(y)\, d\mathcal{L}(z),$$

we can use inequalities (1.15) to prove that there is $C' > 0$ such that

$$\int_{B(x,r)} |f - \tilde{f}_{x,r}|\, d\mathcal{L} \geqslant C' \int_{B(x,r)} |f - f_{x,r}|\, d\mu,$$

where $f_{x,r}$ is the mean of f in the ball $B(x,r)$. So we obtain from (1.12) and again from (1.15) that there are C, $r_0 > 0$ so that

$$\int_{B(x,r)} |f - f_{x,r}|\, d\mu \leqslant Cr \int_{B(x,r)} \left(\sum_{i=1}^{2n} X_i(f)^2 \right)^{1/2} d\mu.$$

From the definition of μ, the fact that φ is an isometry, and (1.14) we obtain (1.13) for $p \in \varphi^{-1}(E_1)$. We repeat this process for every open set $\varphi_{p_i}^{-1}(E_1)$, $i = 1, \ldots, r$. Taking the maximum of the constants C so obtained and the minimum of the radii r_0 it follows that (1.13) holds for all $p \in K$. □

Using this result we get

LEMMA 1.10. *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold such that the quotient $M/\operatorname{Isom}_\omega(M,g)$ is compact. Then there exist constants C_P, $r_0 > 0$, only depending on M, such that*

$$\int_{B(p,r)} |u - u_{p,r}|\, dv_g \leqslant C_P r \int_{B(p,r)} |\nabla_h u|\, dv_g, \tag{1.16}$$

for all $p \in M$, $0 < r < r_0$, $u \in C^\infty(M)$.

REMARK 1.11. Poincaré's inequality also holds for functions of bounded variation by an approximation argument, see [**49**].

From the 1-Poincaré inequality (1.13) and inequality (1.8) we can prove, using Theorem 5.1 and Corollary 9.8 in [**52**] (see also Remark 3 after the statement of Theorem 5.1 in [**52**]), that, given a compact set $K \subset M$, there are positive constants C, r_0 so that

$$\bigg(\fint_{B(x,r)} |u - u_{x,r}|^{Q/(Q-1)} \bigg)^{(Q-1)/Q} \leqslant Cr \bigg(\fint_{B(x,r)} |\nabla_h u| \bigg), \tag{1.17}$$

for all $u \in C^\infty(M)$, $x \in K$, $0 < r < r_0$. Furthermore, it is well-known that the q-Poincaré's inequality (1.17) implies the following relative isoperimetric inequality, [**38**] and [**49**]

LEMMA 1.12 (Relative isoperimetric inequality). *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold, and $K \subset M$ a compact subset. There exists constants $C_I > 0$, $r_0 > 0$, only depending on K, so that, for any set $E \subset M$ with locally finite perimeter, we have*

$$C_I \min \big\{ |\, E \cap B(x,r)\,|, |\, E^c \cap B(x,r)\,| \big\}^{(Q-1)/Q} \leqslant P(E, B(x,r)), \tag{1.18}$$

for any $x \in K$.

REMARK 1.13. A relative isoperimetric inequality in compact subsets of $\mathbb{R}^n$ for sets E with $\mathcal{C}^1$ boundary was proven in [**19**] for the sub-Riemannian structure given by a family of Hörmander vector fields. As the authors remark their result holds for any family of vector fields on a connected manifold.

REMARK 1.14. As for Poincaré's inequality, the relative isoperimetric inequality (1.18) holds in the whole of M provided $M/\operatorname{Isom}_\omega(M,g)$ is compact.

LEMMA 1.15 (Isoperimetric inequality for small volumes). *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold so that the quotient $M/\operatorname{Isom}_\omega(M)$ is compact. Then there exists $v_0 > 0$ and $C_I > 0$ such that*

$$P(E) \geqslant C_I |E|^{(Q-1)/Q}, \tag{1.19}$$

for any finite perimeter set $E \subset M$ with $|E| < v_0$.

PROOF. This is a classical argument [**58**, Lemma 4.1]. We fix $\delta > 0$ small enough so that Poincaré's inequality holds for balls of radius smaller than or equal to δ. Since $M/\operatorname{Isom}_\omega(M)$ is compact, there exists $v_0 > 0$ so that $|B(x,\delta)| \geqslant 2v_0$ holds for all $x \in M$. Let $E \subset M$ be a set of finite perimeter with $|E| < v_0$. We fix a maximal family of points $\{x_i\}_{i\in\mathbb{N}}$ with the properties

$$d(x_i, x_j) \geqslant \frac{\delta}{2} \quad \text{for } i \neq j, \qquad E \subseteq \bigcup_{i\in\mathbb{N}} B(x_i, \delta). \tag{1.20}$$

Letting $q := (Q-1)/Q$ we have

$$|E|^q \leqslant \left(\sum_{i\in\mathbb{N}} |B(x_i,\delta)\cap E|\right)^q \leqslant \sum_{i\in\mathbb{N}} |B(x_i,\delta)\cap E|^q \leqslant C_1 \sum_{i\in\mathbb{N}} P(E, B(x_i,\delta)) \tag{1.21}$$

from (1.20), the concavity of the function $x \mapsto x^q$, and the relative isoperimetric inequality in Lemma 1.12. For $z \in M$, we define $A(z) := \{x_i : z \in B(x_i,\delta)\}$, so that $B(x,\delta/4) \subset B(z,2\delta)$ and $B(z,\delta/4) \subset B(x,2\delta)$ for $x \in A(z)$. Since the balls $B(x_i,\delta/4)$ are disjoint by (1.20), we get

$$\#A(z) \min_{x\in A(z)} |B(x,\delta/4)| \leqslant \left|\bigcup_{x\in A(z)} B(x,\delta/4)\right| \leqslant |B(z,2\delta)|. \tag{1.22}$$

On the other hand, since $B(z,\delta/4) \subset B(x,2\delta)$ we have

$$|B(x,\delta/4)| \geqslant C_D^{-3}\,|B(z,\delta/4)|, \tag{1.23}$$

where $C_D > 0$ is the doubling constant. We conclude from (1.22) and (1.23) that

$$\#A(z) \leqslant C_D^6,$$

and so

$$\sum_{i\in\mathbb{N}} P(E, B(x_i,\delta)) \leqslant C\, P(E).$$

This inequality and (1.21) yields (1.19). □

Remark 1.16. Another approach to isoperimetric inequalities in Carnot-Carathéodory spaces is provided by Gromov [**50**, § 2.3].

Remark 1.17. An isoperimetric inequality for small volumes in compact Riemannian manifolds was proven by Berard and Meyer [**10**].

1.3. The Deformation Lemma. Boundedness of isoperimetric regions

In order to prove Theorem 1.26, we need to construct a foliation of a punctured neighborhood of any point in M by smooth hypersurfaces with bounded mean curvature. We briefly recall this definition. Let $\Sigma \subset M$ be a C^2 hypersurface in M. The *singular set* Σ_0 of Σ is the set of points in Σ where the tangent hyperplane to Σ coincides with the horizontal distribution. If Σ is orientable then there exists a globally defined unit normal vector field N to Σ in (M,g), from which a horizontal unit normal ν_h can be defined on $\Sigma \setminus \Sigma_0$ by

$$\nu_h := \frac{N_h}{|N_h|}, \tag{1.24}$$

where N_h is the orthogonal projection of N to the horizontal distribution. The sub-Riemannian *mean curvature* of Σ is the function, defined in $\Sigma \setminus \Sigma_0$, by

$$H := -\sum_{i=1}^{2n-1} \langle D_{e_i}\nu_h, e_i\rangle, \tag{1.25}$$

where D is the Levi-Civita connection in (M,g), and $\{e_1,\dots,e_{2n-1}\}$ is an orthonormal basis of $T\Sigma \cap \mathcal{H}$. We recall that, given a vector field X defined on Σ, the divergence of X in Σ, $\operatorname{div}_\Sigma X$, is defined by

$$\operatorname{div}_\Sigma X := \sum_{i=1}^{2n} \langle D_{e_i}\nu_h, e_i\rangle, \tag{1.26}$$

where $\{e_1,...,e_{2n}\}$ is an orthonormal basis of $T\Sigma$.

We define the tensor

$$\Sigma(X,Y) := \big\langle D_X T, Y\big\rangle, \tag{1.27}$$

where X and Y are vector fields on M. In the case of the Heisenberg group we have $D_X T = J(X)$, so that $\Sigma(X,Y) = \big\langle J(X), Y\big\rangle$.

At every point of $\Sigma\setminus\Sigma_0$, we may choose an orthonormal basis of $T\Sigma$ consisting on an orthonormal basis $\{e_1,\ldots,e_{2n-1}\}$ of $T\Sigma\cap\mathcal{H}$ together with the vector

$$S := \big\langle N,T\big\rangle\,\nu_h - |N_h|\,T, \tag{1.28}$$

which is orthogonal to N and of modulus 1. Hence we obtain in $\Sigma\setminus\Sigma_0$

$$\operatorname{div}_\Sigma \nu_h = -H + \big\langle D_S\nu_h, S\big\rangle. \tag{1.29}$$

From (1.28) and equality $|\nu_h| = 1$ we immediately get $\big\langle D_S\nu_h, S\big\rangle = -|N_h|\,\big\langle D_S\nu_h, T\big\rangle$, which is equal to $|N_h|\,\Sigma(\nu_h, S)$. Since the vector field S can be rewritten in the form $S = |N_h|^{-1}\big(\big\langle N,T\big\rangle\,N - T\big)$, and $D_T T = 0$, we finally get

$$\big\langle D_S\nu_h, S\big\rangle = \big\langle N,T\big\rangle\,\Sigma(\nu_h, N),$$

and so

$$\operatorname{div}_\Sigma \nu_h = -H + \big\langle N,T\big\rangle\,\Sigma(\nu_h, N). \tag{1.30}$$

The mean curvature (1.25) appears in the expression of the first derivative of the sub-Riemannian area functional (1.9).

Lemma 1.18. *Let $\Sigma\subset M$ be an orientable hypersurface of class C^2 in a contact sub-Riemannian manifold $(M, g_{\mathcal{H}}, \omega)$, and let U be a vector field with compact support in $M\setminus\Sigma_0$ and associated one-parameter family of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$. Then*

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(\Sigma)) = -\int_\Sigma H\,\big\langle U,N\big\rangle\,d\Sigma. \tag{1.31}$$

Proof. Let $u := \big\langle U,N\big\rangle$. Following the proof of [**80**, Lemma 3.2] we obtain

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(\Sigma)) = \int_\Sigma \{U^\perp(|N_h|) + |N_h|\operatorname{div}_\Sigma U^\perp\}\,d\Sigma.$$

For the first summand in the integrand we obtain

$$\begin{aligned} U^\perp(|N_h|) = U^\perp\big(\big\langle N,\nu_h\big\rangle\big) &= \big\langle D_{U^\perp}N, \nu_h\big\rangle + \big\langle N, D_{U^\perp}\nu_h\big\rangle \\ &= -\big\langle \nabla_\Sigma u, \nu_h\big\rangle - \big\langle N,T\big\rangle\,\Sigma(\nu_h, U^\perp) \\ &= -(\nu_h)^\top(u) - \big\langle N,T\big\rangle\,\Sigma(\nu_h, U^\perp), \end{aligned}$$

since $D_{U^\perp}N = -\nabla_\Sigma u$. So we get from the previous formula

$$\begin{aligned} &U^\perp(|N_h|) + |N_h|\operatorname{div}_\Sigma U^\perp = \\ &\quad= -(\nu_h)^\top(u) - \big\langle N,T\big\rangle\,\Sigma(\nu_h, U^\perp) + u\operatorname{div}_\Sigma(|N_h|N) \\ &\quad= -\operatorname{div}_\Sigma(u(\nu_h)^\top) + u\operatorname{div}_\Sigma(\nu_h)^\top - \big\langle N,T\big\rangle\,\Sigma(\nu_h, U^\perp) + u\operatorname{div}_\Sigma(|N_h|N) \\ &\quad= -\operatorname{div}_\Sigma(u(\nu_h)^\top) + u\operatorname{div}_\Sigma(\nu_h) - u\,\big\langle N,T\big\rangle\,\Sigma(\nu_h, N), \end{aligned}$$

where we have used $\nu_h = (\nu_h)^\top + |N_h|N$ in the final step. Since U has compact support out of Σ_0, where ν_h is well defined, we conclude from the Riemannian Divergence Theorem and (1.30)

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(\Sigma)) = \int_\Sigma u\,\{\operatorname{div}_\Sigma(\nu_h) - \big\langle N,T\big\rangle\,\Sigma(\nu_h,N)\}\,d\Sigma = -\int_\Sigma H\,\big\langle U,N\big\rangle\,d\Sigma,$$

which completes the proof of the Lemma. □

The local model of a sub-Riemannian manifold is the contact manifold $(\mathbb{R}^{2n+1},\omega_0)$, where $\omega_0 := dt+\sum_{i=1}^n(x_i dy_i - y_i dx_i)$ is the standard contact form in $\mathbb{R}^{2n+1}$, together with an arbitrary positive definite metric $g_{\mathcal{H}_0}$ in $\mathcal{H}_0$. A basis of the horizontal distribution is given by

$$X_i := \frac{\partial}{\partial x_i} + y_i\,\frac{\partial}{\partial t}, \qquad Y_i := \frac{\partial}{\partial y_i} + x_i\,\frac{\partial}{\partial t}, \qquad i=1,\ldots,n,$$

and the Reeb vector field is

$$T := \frac{\partial}{\partial t}.$$

The metric $g_{\mathcal{H}_0}$ will be extended to a Riemannian metric on $\mathbb{R}^{2n+1}$ so that the Reeb vector field is unitary and orthogonal to $\mathcal{H}_0$. We shall usually denote the set of vector fields $\{X_1,Y_1,\ldots,X_n,Y_n\}$ by $\{Z_1,\ldots,Z_{2n}\}$. The coordinates of $\mathbb{R}^{2n+1}$ will be denoted by $(x_1,y_1,\ldots,x_n,y_n,t)$, and the first $2n$ coordinates will be abbreviated by z or (x,y). We shall consider the map $F:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ defined by

$$F(x_1,y_1,\ldots,x_n,y_n) := (-y_1,x_1,\ldots,-y_n,x_n).$$

Given a C^2 function $u:\Omega\subset\mathbb{R}^{2n}\to\mathbb{R}$ defined on an open subset Ω, we define the graph $G_u := \{(z,t) : z\in\Omega, t=u(z)\}$. By (1.9), the sub-Riemannian area of the graph is given by

$$A(G_u) = \int_{G_u} |N_h|\,dG_u,$$

where dG_u is the Riemannian metric of the graph and $|N_h|$ is the modulus of the horizontal projection of a unit normal to G_u. We consider on Ω the basis of vector fields $\big\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_n}\big\}$.

By the Riemannian area formula

$$dG_u = \operatorname{Jac} d\mathcal{L}^{2n}, \tag{1.32}$$

where $d\mathcal{L}^{2n}$ is Lebesgue measure in $\mathbb{R}^{2n}$ and Jac is the Jacobian of the canonical map $\Omega\to G_u$ given by

$$\operatorname{Jac} = \det(g_{ij} + (\nabla u+F)_i(\nabla u+F)_j)^{1/2}_{i,j=1,\ldots,2n}, \tag{1.33}$$

where $g_{ij} := g(Z_i,Z_j)$, ∇ is the Euclidean gradient of $\mathbb{R}^{2n}$ and $(\nabla u+F)_i$ is the i-th Euclidean coordinate of the vector field $\nabla u+F$ in Ω. We have

$$(\nabla u+F)_i = \begin{cases} u_{x_{(i+1)/2}} - y_{(i+1)/2}, & i \text{ odd},\\ u_{y_{i/2}} + x_{i/2}, & i \text{ even}.\end{cases}$$

Let us compute the composition of $|N_h|$ with the map $\Omega\to G_u$. The tangent space TG_u is spanned by

$$Z_i + (\nabla u+F)_i\,T, \quad i=1,\ldots,2n. \tag{1.34}$$

So the projection to Ω of the singular set $(G_u)_0$ is the set $\Omega_0\subset\Omega$ defined by $\Omega_0 := \{z\in\Omega : (\nabla u+F)(z)=0\}$. Let us compute a *downward pointing* normal vector $\tilde N$ to G_u writing

$$\tilde N = \sum_{i=1}^{2n}(a_i Z_i) - T. \tag{1.35}$$

The horizontal component of $\tilde N$ is $\tilde N_h = \sum_{i=1}^{2n} a_i Z_i$. We have

$$\sum_{i=1}^{2n} a_i g_{ij} = g(\tilde N_h, Z_j) = g(\tilde N, Z_j) = -(\nabla u+F)_j\big\langle \tilde N, T\big\rangle = (\nabla u+F)_j,$$

since Z_j is horizontal, $\tilde{N}$ is orthogonal to Z_j defined by (1.34), and (1.35). Hence

$$(a_1, \ldots, a_{2n}) = b(\nabla u + F),$$

where b is the inverse of the matrix $\{g_{ij}\}_{i,j=1,\ldots,2n}$. So we get

$$|\tilde{N}| = (1 + \langle \nabla u + F, b(\nabla u + F) \rangle)^{1/2}, \tag{1.36}$$

and

$$|\tilde{N}_h| = \langle \nabla u + F, b(\nabla u + F) \rangle^{1/2},$$

where $\langle , \rangle$ is the Euclidean Riemannian metric in $\mathbb{R}^{2n}$, and so

$$|N_h| = \frac{|\tilde{N}_h|}{|\tilde{N}|} = \frac{\langle \nabla u + F, b(\nabla u + F) \rangle^{1/2}}{\big(1 + \langle \nabla u + F, b(\nabla u + F) \rangle\big)^{1/2}}. \tag{1.37}$$

Observe that, from (1.35) and (1.36) we also get that the scalar product of the unit normal N with the Reeb vector field T is given by

$$g(N,T) = -\frac{1}{\big(1 + \langle \nabla u + F, b(\nabla u + F) \rangle\big)^{1/2}}. \tag{1.38}$$

Hence we obtain from (1.9), (1.32), (1.33) and (1.37)

$$A(G_u) = \int_\Omega \langle \nabla u + F, b(\nabla u + F) \rangle^{1/2} \, \frac{\det(g_{ij} + (\nabla u + F)_i (\nabla u + F)_j)^{1/2}}{\big(1 + \langle \nabla u + F, b(\nabla u + F) \rangle\big)^{1/2}} \, d\mathcal{L}^{2n}. \tag{1.39}$$

Now we use formula (1.39) to compute the mean curvature of a graph.

Lemma 1.19. *Let us consider the contact sub-Riemannian manifold $(\mathbb{R}^{2n+1}, g_{\mathcal{H}_0}, \omega_0)$, where ω_0 is the standard contact form in $\mathbb{R}^{2n+1}$ and $g_{\mathcal{H}_0}$ is a positive definite metric in the horizontal distribution $\mathcal{H}_0$. Let $u : \Omega \subset \mathbb{R}^{2n} \to \mathbb{R}$ be a C^2 function. We denote by $g = (g_{ij})_{i,j=1,\ldots,2n}$ the metric matrix and by $b = g^{-1} = (g^{ij})_{i,j=1,\ldots,2n}$ the inverse metric matrix. Then the mean curvature of the graph G_u, computed with respect to the downward pointing normal, is given by*

$$-\operatorname{div}\left(\frac{b(\nabla u + F)}{\langle \nabla u + F, b(\nabla u + F) \rangle^{1/2}}\right) + \mu, \tag{1.40}$$

where μ is a bounded function in $\Omega \setminus \Omega_0$, and div *is the usual Euclidean divergence in Ω.*

Proof. Given a smooth function v with compact support in Ω, we shall compute the first derivative of the function $s \mapsto A(G_{u+sv})$ and we shall compare it with the general first variation of the sub-Riemannian area (1.31). Let us fix some compact set $K \subset \Omega$.

We use the usual notation in Calculus of Variations. Let us denote by

$$G(z,u,p) := \frac{\det(g_{ij} + (p+F)_i (p+F)_j)^{1/2}_{i,j=1,\ldots,2n}}{\big(1 + \langle p + F, b(p+F) \rangle\big)^{1/2}}, \tag{1.41}$$

where $p \in \mathbb{R}^{2n}$. Observe that G is a C^∞ function well defined in Ω. From (1.33) and (1.38) we obtain

$$G(z,u,\nabla u) := -\operatorname{Jac} g(T,N). \tag{1.42}$$

Recall that $g_{ij} = g_{ij}(z,u)$, $F = F(z)$. Let us denote also

$$F(z,u,p) := \langle p + F, b(p+F) \rangle^{1/2} \, G(z,u,p). \tag{1.43}$$

Then we can write

$$A(G_u) := \int_\Omega F(z,u,\nabla u)\, d\mathcal{L}^{2n}.$$

So we have

$$\frac{d}{ds}\Big|_{s=0} A(G_{u+sv}) = \int_\Omega (F_u\, v + \big\langle F_p, \nabla v\big\rangle\, d\mathcal{L}^{2n},$$

where $\big\langle F_p, X\big\rangle(z,u,p) = \frac{d}{ds}\big|_{s=0}(z,u,p+sX)$ is the gradient of $p \mapsto F(z,u,p)$. Applying the Divergence Theorem

$$\frac{d}{ds}\Big|_{s=0} A(G_{u+sv}) = \int_\Omega v\,(F_u - \operatorname{div} F_p)\, d\mathcal{L}^{2n}. \tag{1.44}$$

Observe that, from (1.43)

$$F_u = \frac{\big\langle p+F, \frac{\partial b}{\partial u}(p+F)\big\rangle}{2\big\langle p+F, b(p+F)\big\rangle^{1/2}}\, G + \big\langle p+F, b(p+F)\big\rangle^{1/2} G_u,$$

which is bounded from above since b is a symmetric positive definite matrix, and so there is $C>0$ depending on K so that $\big\langle \nabla u + F, b(\nabla u + F)\big\rangle \geqslant C\,|\nabla u + F|^2$, and the numerator satisfies $\big\langle \nabla u + F, \frac{\partial b}{\partial t}(\nabla u + F)\big\rangle \leqslant C'|\nabla u + F|^2$. On the other hand

$$F_p = G\,\frac{b(p+F)}{\big\langle p+F, b(p+F)\big\rangle^{1/2}} + \big\langle p+F, b(p+F)\big\rangle^{1/2} G_p,$$

so that

$$\begin{aligned}\operatorname{div} F_p = G \operatorname{div}\left(\frac{b(p+F)}{\big\langle p+F, b(p+F)\big\rangle^{1/2}}\right) + \Big\langle \nabla G, \frac{b(p+F)}{\big\langle p+F, b(p+F)\big\rangle^{1/2}}\Big\rangle \\ + \operatorname{div}\big(\big\langle p+F, b(p+F)\big\rangle^{1/2} G_p\big).\end{aligned}$$

Observe that the last two terms are bounded and that G_p is bounded, so that we get from (1.44) and the previous discussion

$$\frac{d}{ds}\Big|_{s=0} A(G_{u+sv}) = \int_\Omega v \left\{ G \operatorname{div}\left(\frac{b(\nabla u + F)}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}\right) + \mu' \right\} d\mathcal{L}^{2n},$$

where G and μ' are bounded functions in K.

Taking into account that the variation $s \mapsto u + sv$ is the one obtained by moving the graph G_u by the one-parameter group of diffeomorphisms associated to the vector field $U := vT$, which has normal component $g(U,N) = v\,g(T,N)$, that $dG_u = \operatorname{Jac} d\mathcal{L}^{2n}$, and equation (1.42), we conclude

$$\frac{d}{ds}\Big|_{s=0} A(G_{u+sv}) = \int_\Omega g(U,N)\left\{ -\operatorname{div}\left(\frac{b(\nabla u + F)}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}\right) + \mu \right\} dG_u,$$

where $\mu := \mu'(g(N,T)\operatorname{Jac})^{-1}$ is a bounded function. Comparing this formula with the general first variation one (1.31), and taking into account that $g(U,N)$ is arbitrary we get (1.40). □

REMARK 1.20. If $g = g_0$ is the standard Riemannian metric in the Heisenberg group so that $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ is orthonormal then $(g_{ij})_{i,j=1,\ldots,2n}$ is the identity matrix, $b = \mathrm{Id}$, $\mu = 0$, and we have the usual mean curvature equation, see [**26**].

LEMMA 1.21. *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold. Given $p \in M$, there exists a neighborhood U of p so that $U \setminus \{p\}$ is foliated by surfaces with mean curvature uniformly bounded outside any neighborhood $V \subset U$ of p.*

PROOF. Since the result is local, we may assume, using a Darboux's chart, that our contact sub-Riemannian manifold is $(\mathbb{R}^{2n+1}, g, \omega_0)$, where ω_0 is the standard contact form in (1.2) and g is an arbitrary positive definite metric in the horizontal distribution $\mathcal{H}_0$. We also assume $p = 0$.

For each $\lambda > 0$, we consider the hypersurface $\mathbb{S}_\lambda$ given by the graph of the function

$$u_\lambda(z) = \frac{1}{2\lambda^2}\{\lambda|z|(1-\lambda^2|z|^2)^{1/2} + \arccos(\lambda|z|)\}, \quad |z| \leqslant \frac{1}{\lambda}, \tag{1.45}$$

and its reflection with respect to the hyperplane $t = 0$, see [**78**]. Each $\mathbb{S}_\lambda$ is a topological sphere of class C^2 with constant mean curvature λ in the Heisenberg group $\mathbb{H}^{2n+1}$ and two singular points $\pm(0, \pi/(4\lambda^2))$. The family $\{\mathbb{S}_\lambda\}_{\lambda>0}$ is a foliation of $\mathbb{R}^{2n+1} \setminus \{0\}$. From now on we fix some $\lambda > 0$ and let $u := u_\lambda$.

From Lemma 1.19 it is sufficient show that

$$\operatorname{div}\left(\frac{b(\nabla u + F)}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}\right) \tag{1.46}$$

is bounded near the singular points, In fact the mean curvature is continuous away from the singular set by the regularity of $\mathbb{S}_\lambda$.

Let $g^i := (g^{i1}, \ldots, g^{i(2n)})$ be the vector in $\mathbb{R}^{2n}$ corresponding to the i-th row of the matrix b. We have

$$\operatorname{div}\left(\frac{b(\nabla u + F)}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}\right) = \sum_{i=1}^{2n} \partial_i \left(\frac{\big\langle g^i, \nabla u + F\big\rangle}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}\right),$$

where ∂_i is the partial derivative with respect the i-th variable, i.e., $x_{(i+1)/2}$ when i is odd and $y_{i/2}$ when i is even. Taking derivatives we get

$$\begin{aligned}\sum_{i=1}^{2n} \partial_i \left(\frac{\big\langle g^i, \nabla u + F\big\rangle}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}\right) &= \sum_{i=1}^{2n} \frac{\big\langle \partial_i g^i, \nabla u + F\big\rangle + \big\langle g^i, \partial_i(\nabla u + F)\big\rangle}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}} \\ &- \big\langle g^i, \nabla u + F\big\rangle \frac{\frac{1}{2}\big\langle \nabla u + F, (\partial_i b)(\nabla u + F)\big\rangle + \big\langle \partial_i(\nabla u + F), b(\nabla u + F)\big\rangle}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{3/2}}.\end{aligned}$$

It is clear that the first and the third summands are bounded. So we only have to prove that

$$\begin{aligned}&\sum_{i=1}^{2n} \big\langle g^i, \partial_i(\nabla u + F)\big\rangle \big\langle \nabla u + F, b(\nabla u + F)\big\rangle \\ &\qquad - \sum_{i,j,k,\ell=1}^{2n} \big\langle g^i, \nabla u + F\big\rangle \big\langle \partial_i(\nabla u + F), b(\nabla u + F)\big\rangle \leqslant C\,|\nabla u + F|^{3/2}\end{aligned} \tag{1.47}$$

for some positive constant C. We easily see that the left side of (1.47) is equal to

$$\begin{aligned}&\sum_{i,j,k,\ell=1}^{2n} g^{ij} g^{k\ell} \partial_i(\nabla u + F)_j (\nabla u + F)_k (\nabla u + F)_\ell \\ &\qquad - \sum_{i,j,k,\ell=1}^{2n} g^{ij} g^{k\ell} \partial_i(\nabla u + F)_k (\nabla u + F)_j (\nabla u + F)_\ell.\end{aligned} \tag{1.48}$$

Taking into account that

$$\partial_i F_j + \partial_j F_i = 0, \quad \text{for all } i, j = 1, \ldots, 2n,$$

and the symmetries of (1.48), we get that (1.48) is equal to

$$\sum_{i,j,k,\ell=1}^{2n} g^{ij}g^{k\ell}u_{ij}(\nabla u+F)_k(\nabla u+F)_\ell - \sum_{i,j,k,\ell=1}^{2n} g^{ij}g^{k\ell}u_{ik}(\nabla u+F)_j(\nabla u+F)_\ell,$$

so we only need to show that each term

$$\frac{(\nabla u+F)_k(\nabla u+F)_\ell\, u_{ij}}{\langle \nabla u+F, b(\nabla u+F)\rangle^{3/2}}$$

is bounded to complete the proof. Since

$$|(\nabla u+F)_k| \leqslant |\nabla u+F|, \text{ and } \big\langle \nabla u+F, b(\nabla u+F\big\rangle^{1/2} \geqslant C|\nabla u+F|,$$

for some positive constant $C>0$, it is enough to show that

$$\frac{u_{ij}}{|\nabla u+F|} \tag{1.49}$$

is bounded.

A direct computation yields

$$\frac{\partial u}{\partial x_i} = -\frac{\lambda|z|x_i}{(1-\lambda^2|z|^2)^{1/2}}, \qquad \frac{\partial u}{\partial y_i} = -\frac{\lambda|z|y_i}{(1-\lambda^2|z|^2)^{1/2}},$$

and so

$$|\nabla u+F|^2 = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}-y_i\right)^2 + \left(\frac{\partial u}{\partial y_i}+x_i\right)^2 = |z|^2\left(1+\frac{\lambda^2|z|^2}{1-\lambda^2|z|^2}\right).$$

Hence

$$C_1|z| \leqslant |\nabla u+F| \leqslant C_2|z|, \tag{1.50}$$

for some constants C_1, $C_2>0$ near $z=0$.

On the other hand

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i\partial x_j} &= -\delta_{ij}\,\frac{\lambda|z|}{(1-\lambda^2|z|^2)^{1/2}} - \frac{\lambda x_i x_j}{|z|(1-\lambda^2|z|^2)^{3/2}},\\ \frac{\partial^2 u}{\partial y_i\partial y_j} &= -\delta_{ij}\,\frac{\lambda|z|}{(1-\lambda^2|z|^2)^{1/2}} - \frac{\lambda y_i y_j}{|z|(1-\lambda^2|z|^2)^{3/2}},\\ \frac{\partial^2 u}{\partial x_i y_j} &= -\frac{\lambda x_i y_j}{|z|(1-\lambda^2|z|^2)^{3/2}},\end{aligned}$$

and so

$$|u_{ij}| \leqslant C|z|,$$

for some constant $C>0$. This inequality, together with (1.50), shows that (1.49) is bounded.

□

Lemma 1.22 (Deformation Lemma). *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold and $\Omega\subset M$ a finite perimeter set. Then there exists a small deformation $\widetilde{\Omega}_r \supset \Omega$, $0<r\leqslant r_0$, such that*

$$P(\partial(\widetilde{\Omega}_r-\Omega)) \leqslant C\,|\widetilde{\Omega}_r-\Omega|,$$

where C is a positive constant.

PROOF. For $p \in \mathrm{int}(\Omega)$ sufficiently close to $\partial\Omega$, there exists by Lemma 1.21 a local foliation by hypersurfaces F_r, $0 < r \leqslant r_0$, with mean curvature uniformly bounded outside a small neighborhood of p. Let U_r be the regions bounded by F_r and let $\nu_h(q)$ the horizontal unit normal at $q \in F_r$ of the surface F_r, for $r \in [d(p, \partial\Omega), r_0]$. Letting $\Omega_r := \Omega^c \cap U_r$, $\widetilde{\Omega}_r := \Omega_r \cup \Omega$, we have that there exists $C > 0$ so that $\operatorname{div} \nu_h \leqslant C$ by (1.30) and the boundedness of the mean curvature. So we have

$$\begin{aligned} C|\Omega_r| &\geqslant \int_{\Omega_r} \operatorname{div}(\nu_h) dv_g = \int_M g_{\mathcal{H}}(\nu_h, \nu) dP(F_r \cap \Omega^c) + \int_M g_{\mathcal{H}}(\nu_h, \nu) dP(\partial^* \Omega_r \cap U_r) \\ &\geqslant P(\partial F_r \cap \Omega^c) - P(\partial^* \Omega_r \cap U_r), \end{aligned}$$

where ν is defined in the Gauss-Green formula. We have used $g_{\mathcal{H}}(\nu_h, \nu) \equiv 1$ in the first integral and $g_{\mathcal{H}}(\nu_h, \nu) \geqslant -1$ in the second one. But also that from the definition of $dP(.)$ it follows

$$\int_\Omega dP(E) = P(E, \Omega),$$

see [**40**, p. 879–880] and [**41**, p. 491–494]. □

LEMMA 1.23. *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold, and $E \subset M$ be a set minimizing perimeter under a volume constraint. Then E is bounded.*

PROOF. We fix $p \in M$ and denote the ball $B(p, r)$ by B_r. We let $V(r) := |E \cap (M \setminus B_r)|$, so that $V(r) \to 0$ when $r \to \infty$ since E has finite volume. Let us assume that $V(r) > 0$ for all $r > 0$. Applying the isoperimetric inequality for small volumes when r is large enough to the set $E \cap (M \setminus B_r)$ we get, taking q as in (1.6),

$$\begin{aligned} C_I\, V(r)^q &\leqslant P(E \cap (M \setminus B_r)) \\ &\leqslant P(E, M \setminus \overline{B}_r) + P(E \cap B_r, \partial B_r) \\ &\leqslant P(E, M \setminus \overline{B}_r) + |V'(r)| \\ &\leqslant P(E) - P(E, B_r) + |V'(r)|. \end{aligned} \tag{1.51}$$

We now fix some $r_0 > 0$. For $r > r_0$, the Deformation Lemma shows the existence of a set E_r so that E_r is a small deformation of $E \cap B_r$, $E_r \setminus (E \cap B_r)$ is properly contained in B_{r_0}, $|E_r| = |E|$ (which implies $|E \setminus E_r| = V(r)$), and $P(E_r, B_r) \leqslant P(E, B_r) + C\, V(r)$. So we have

$$\begin{aligned} P(E_r) &\leqslant P(E_r, B_r) + P(E_r \cap B_r, \partial B_r) \\ &= P(E_r, B_r) + P(E \cap B_r, \partial B_r) \\ &\leqslant P(E_r, B_r) + |V'(r)| \end{aligned} \tag{1.52}$$

By the isoperimetric property of E we also have

$$P(E) \leqslant P(E_r) \leqslant P(E_r, B_r) + |V'(r)|, \tag{1.53}$$

for all $r \geqslant r_0$.

From (1.51), (1.52) and (1.53) we finally get

$$C_I V(r)^q \leqslant C\, V(r) + 2\, |V'(r)|. \tag{1.54}$$

Since $V(r) = V(r)^{1-q} V(r)^q \leqslant (C_I/2)\, V(r)^q$ for r large enough, we get

$$-\frac{C_I}{2}\, V(r)^q \geqslant 2\, V'(r),$$

or, equivalently,

$$(V^{1/Q})' \leqslant -\frac{C_I Q}{2} < 0,$$

which forces $V(r)$ to be negative for r large enough. This contradiction proves the result. □

1.4. Structure of minimizing sequences

In this section we will prove an structure result for minimizing sequences in a non-compact contact sub-Riemannian manifold. Partial versions of this result were obtained for Riemannian surfaces, [**74**], [**75**], and for Riemannian manifolds [**79**].

PROPOSITION 1.24. *Let $(M, g_{\mathcal{H}}, \omega)$ be a non-compact contact sub-Riemannian manifold. Consider a minimizing sequence $\{E_k\}_{k\in\mathbb{N}}$ of sets of volume v converging in $L^1_{loc}(M)$ to a finite perimeter set $E \subset M$, eventually empty. Then there exist sequences of finite perimeter sets $\{E_k^c\}_{k\in\mathbb{N}}$, $\{E_k^d\}_{k\in\mathbb{N}}$ such that*

1. *$\{E_k^c\}_{k\in\mathbb{N}}$ converges to E in $L^1(M)$, $\{E_k^d\}_{k\in\mathbb{N}}$ diverges, and $|E_k^c| + |E_k^d| = v$.*
2. *$\lim_{k\to\infty} P(E_k^c) + P(E_k^d) = I_M(v)$.*
3. *$\lim_{k\to\infty} P(E_k^c) = P(E)$.*
4. *If $|E| \neq 0$, then E is an isoperimetric region of volume $|E|$.*
5. *Moreover, if $M/\operatorname{Isom}_\omega(M, g)$ is compact then $\lim_{k\to\infty} P(E_k^d) = I_M(v - |E|)$. In particular, $I_M(v) = I_M(|E|) + I_M(v - |E|)$.*

PROOF. We fix a point $p \in M$ and we consider the balls $B(r) := B(p, r)$. Let $m(r) := |E \cap B(p, r)|$, $m_k(r) := |E_k \cap B(r)|$.

We can choose a sequence of diverging radii $r_k > 0$ so that, considering a subsequence of $\{E_k\}_{k\in\mathbb{N}}$, we would had

$$\int_{B(r_k)} |\mathbf{1}_E - \mathbf{1}_{E_k}| \leqslant \frac{1}{k}, \tag{1.55}$$

$$P(E_k \setminus B(r_k), \partial B(r_k)) \leqslant \frac{v}{k}. \tag{1.56}$$

In order to prove (1.55) and (1.56) we consider a sequence of radii $\{s_k\}_{k\in\mathbb{N}}$ so that $s_{k+1} - s_k \geqslant k$ for all $k \in \mathbb{N}$. Taking a subsequence of $\{E_k\}_{k\in\mathbb{N}}$, we may assume that

$$\int_{B(s_{k+1})} |\mathbf{1}_E - \mathbf{1}_{E_k}| \leqslant \frac{1}{k},$$

so that (1.55) holds for all $r \in (0, s_{k+1})$. To prove (1.56) we observe that $m_k(r)$ is an increasing function. By Lebesgue's Theorem

$$\int_{s_k}^{s_{k+1}} m'(r)\, dr \leqslant m(s_{k+1}) - m(s_k) \leqslant v,$$

which implies that there is a set of positive measure in $[s_k, s_{k+1}]$ so that $m'(r) \leqslant \frac{v}{k}$. By Ambrosio's localization Lemma [**2**, Lemma 3.5] we have, for almost everywhere r,

$$P(E_k \setminus B(r)), \partial B(r)) \leqslant m_k'(r).$$

This implies that there is $r_k \in [s_k, s_{k+1}]$ so that (1.56) holds.

Now we define

$$E_k^c := E \cap B(r_k), \qquad E_k^d := E \setminus B(r_{k+1}).$$

Now we prove 1. Since E has finite volume and (1.55) holds we conclude that $\{E_k^c\}_{k\in\mathbb{N}}$ converges in $L^1(M)$ to E. The divergence of the sequence $\{E_k^d\}_{k\in\mathbb{N}}$ and equality $|E_k^c|+|E_k^d|=v$ follow from the definitions of E_k^c and E_k^d.

In order to prove 2 we take into account that

$$P(E_k^c)\leqslant P(E_k,B(r_k))+P(E_k\cap\partial B(r_k),\partial B(r_k)),$$
$$P(E_k^d)\leqslant P(E_k,M\setminus\overline{B}(r_k))+P(E_k\cap\partial B(r_k),\partial B(r_k)).$$

By (1.56) we have

$$P(E_k)\leqslant P(E_k^c)+P(E_k^d)\leqslant P(E_k)+\frac{2v}{k}.$$

Taking limits when $k\to\infty$ we get 2.

To prove 3 we shall first show that

$$P(E)=\liminf_{k\to\infty}P(E_k^c) \tag{1.57}$$

reasoning by contradiction. Since E_k^c converges in $L^1(M)$ to E, we may assume that the strict inequality $P(E)<\liminf_{k\to\infty}P(E_k)$ holds. Reasoning as above we obtain an non-decreasing and diverging sequence of radii $\{\rho_k\}_{k\in\mathbb{N}}$ so that $\rho_k<r_k$ and

$$P(E\cap\partial B(\rho_k),\partial B(\rho_k))\leqslant\frac{v}{k},$$

for all $k\in\mathbb{N}$. Let $E_k':=E\cap B(\rho_k)$. The perimeter of E_k' satisfies

$$P(E_k')\leqslant P(E,B(\rho_k))+P(E\cap\partial B(\rho_k),\partial B(\rho_k))\leqslant P(E)+\frac{v}{k},$$

and for the volume $|E_k'|$ we have

$$\lim_{k\to\infty}|E_k'|=|E|=v-\lim_{k\to\infty}|E_k^d|.$$

We fix two points $p_1\in\operatorname{int}(E)$, $p_2\in\operatorname{int}(M\setminus E)$, close enough to the boundary of E, so that we can apply the Deformation Lemma in a neighborhood of each point. This allows us to make small corrections of the volume and obtain, for $k\in\mathbb{N}$ large enough, a set E_k'' of finite perimeter so that

$$|E_k''|+|E_k^d|=v,$$

and

$$P(E_k'')\leqslant P(E_k')+C\,\big||E_k'|-|E_k^d|\big|\leqslant P(E)+\frac{v}{k}+C\,\big||E_k'|-|E_k^d|\big|,$$

so that

$$\liminf_{k\to\infty}P(E_k'')\leqslant P(E).$$

Then $F_k:=E_k''\cup E_k^d$ is sequence of sets of volume v with

$$\liminf_{k\to\infty}P(F_k)\leqslant P(E)+\liminf_{k\to\infty}P(E_k^d)<\liminf_{k\to\infty}(P(E_k^c)+P(E_k^d))=I_M(v),$$

which clearly gives us a contradiction and proves (1.57). To complete the proof of 3 we observe that we can replace the inferior limit in (1.57) by the true limit of the sequence since every subsequence of a minimizing sequence is also minimizing.

To prove 4 we consider a finite perimeter set F with $|F|=|E|$ and $P(F)<P(E)$ and we reason as in the proof of 3 with F instead of E.

Let us finally see that 5 holds. From 2 and 3 we see that $\lim_{k\to\infty}P(E_k^d)$ exists and it is equal to $I_M(v)-P(E)$. If this limit were smaller than $I_M(v-|E|)$ then we could slightly modify the sequence $\{E_k^d\}_{k\in\mathbb{N}}$ to produce another one $\{F_k\}_{k\in\mathbb{N}}$ with $|F_k|=v-|E|$ and $\lim_{k\to\infty}P(F_k)=\lim_{k\to\infty}P(E_k^d)<I_M(v-|E|)$, which gives a

contradiction. If $\lim_{k\to\infty} P(E_k^d)$ were larger than $I_M(v-|E|)$ then we could find a set F with $|F| = v - |E|$ so that

$$I_M(v-|E|) < P(F) < \lim_{k\to\infty} P(E_k^d).$$

Modifying again slightly the volume of F we produce a sequence $\{F_k\}_{k\in\mathbb{N}}$ so that $|E| + |F_k| = v$ and $\lim_{k\to\infty} P(F_k) = P(F)$. Since E is bounded, we can translate the sets F_k so that they are at positive distance from E. Hence

$$\lim_{k\to\infty} P(E\cup F_k) = \lim_{k\to\infty} P(E) + P(F_k) = P(E) + P(F) < I_M(v),$$

a contradiction that proves 5. □

REMARK 1.25. The proof of the first three items in the statement of Proposition 1.24 works in quite general metric measure spaces. The proof of the last two ones needs the compactness of the isoperimetric regions.

1.5. Proof of the main result

We shall prove in this section our main result

THEOREM 1.26. *Let $(M, g_{\mathcal{H}}, \omega)$ be a contact sub-Riemannian manifold such that the quotient $M/\operatorname{Isom}_\omega(M,g)$ is compact. Then, for any $0 < v < |M|$, there exists on M an isoperimetric region of volume v.*

First we need the following result [**58**, Lemma 4.1]

LEMMA 1.27. *Let $E \subset M$ be a set with positive and finite perimeter and measure. Assume that $m \in (0, \inf_{x\in M} |B(x,r_0)|/2)$, where $r_0 > 0$ is the radius for which the relative isoperimetric inequality holds, is such that $|E\cap B(x,r_0)| < m$ for all $x \in M$. Then we have*

$$C\,|E|^Q \leqslant m\,P(E)^Q, \tag{1.58}$$

for some constant $C > 0$ that only depends on Q.

PROOF. We closely follow the proof of [**58**, Lemma 4.1]. We consider a maximal family of points $\mathcal{A}$ in M so that $d(x,x') \geqslant r_0/2$ for all x, $x' \in \mathcal{A}$, $x \neq x'$, and $|E\cap B(x,r_0/2)| > 0$ for all $x \in \mathcal{A}$. Then $\bigcup_{x\in\mathcal{A}} B(x,r_0)$ cover almost all of E. We have

$$\begin{aligned}|E| &\leqslant \sum_{x\in\mathcal{A}} |E\cap B(x,r_0)| \leqslant m^{1/Q} \sum_{x\in\mathcal{A}} |E\cap B(x,r_0)|^q \\ &\leqslant m^{1/Q} C_I \sum_{x\in\mathcal{A}} P(E, B(x,r_0)),\end{aligned}$$

since $(1/Q) + q = 1$ and $|E\cap B(x,r_0)| < m$. The last inequality follows from the relative isoperimetric inequality since $|E\cap B(x,r_0)| < m \leqslant |B(x,r_0)|/2$ and so $\min\{|E\cap B(x,r_0)|, |E^c\cap B(x,r_0)|\} = |E\cap B(x,r_0)|$. The overlapping is controlled in the same way as in [**58**] to conclude the proof. □

Using the following result we can prove Proposition 1.29

LEMMA 1.28 ([**3**, Thm. 4.3]). *The measure $P(E,\cdot)$ satisfies*

$$\tau < \liminf_{\delta\to 0} \frac{P(E, B(x,\delta))}{\delta^{Q-1}} \leqslant \limsup_{\delta\to 0} \frac{P(E, B(x,\delta))}{\delta^{Q-1}} < +\infty,$$

for $P(E,\cdot)$-a.e. $x \in M$, with $\tau > 0$.

PROPOSITION 1.29. *Given $v_0 > 0$, there exists a constant $C(v_0) > 0$ so that*

$$I_M(v) \leqslant C(v_0)\, v^{(Q-1)/Q}, \tag{1.59}$$

for all $v \in (0, v_0]$.

PROOF. For any $x \in M$ we have

$$I_M(|B(x,r)|) \leqslant P(B(x,r)) \leqslant cr^{Q-1} \leqslant \frac{c}{C^{(Q-1)/Q}} |B(x,r)|^{(Q-1)/Q},$$

where we have used $|B(x,r)| \geqslant Cr^Q$ to get $r^Q \leqslant C^{-1/Q}|B(x,r)|^{1/Q}$ and Lemma 1.28 with $E = B(x,r)$ and $\delta = 2r$.

□

PROOF OF THEOREM 1.26. We fix a volume $0 < v < |M|$, and we consider a minimizing sequence $\{E_k\}_{k\in\mathbb{N}}$ of sets of volume v whose perimeters approach $I_M(v)$. In case M is compact, we can extract a convergent subsequence to a finite perimeter set E with $|E| = v$ and $P(E) = I_M(v)$.

We assume from now on that M is not compact. By Lemma 1.27, for any $m > 0$ such that $mv < \inf_{x\in M} |B(x, r_0)|/2$, there is a a constant $C > 0$, only depending on Q, so that, for any finite perimeter set $E \subset M$ satisfying $|E \cap B(x, r_0)| < m\,|E|$ for all $x \in M$, we have

$$C|E|^Q \leqslant (m|E|)\, P(E)^Q,$$

and so

$$P(E) \geqslant \left(\frac{C}{m}\right)^{1/Q} |E|^{(Q-1)/Q}. \tag{1.60}$$

From Proposition 1.29 we deduce that, given $v > 0$, there is a constant $C(v) > 0$ so that $I_M(w) \leqslant C(v)\, w^{(Q-1)/Q}$ for all $w \in (0, v]$. Taking $m_0 > 0$ small enough so that

$$\left(\frac{C}{m_0}\right)^{1/Q} |E|^{(Q-1)/Q} > 2C(v)\, |E|^{(Q-1)/Q} \tag{1.61}$$

we conclude, using (1.60), (1.61) and (1.59)

$$P(E) \geqslant 2\, I_M(|E|). \tag{1.62}$$

We conclude from (1.62) that, for k large enough, the sets in the minimizing sequence $\{E_k\}_{k\in\mathbb{N}}$ cannot satisfy the property $|E \cap B(x, r_0)| < m|E|$ for all $x \in M$. So we can take points $x_k \in M$ such that

$$|E_k \cap B(x_k, r_0)| \geqslant m_0|E_k| = m_0 v,$$

for k large enough. Since $M/\mathrm{Isom}_\omega(M, g)$ is compact, we translate the whole minimizing sequence (and still denote it in the same way), so that $\{x_k\}_{k\in\mathbb{N}}$ is bounded. By passing to a subsequence, denoted in the same way, we assume that $\{x_k\}_{k\in\mathbb{N}}$ converges to some point $x_0 \in M$. By the compactness Lemma there is a convergent subsequence, still denoted by $\{E_k\}_{k\in\mathbb{N}}$ that converges to some finite perimeter set E, and

$$m_0 v \leqslant \liminf_{k\to\infty} |E_k \cap B(x_0, r_0)| = |E \cap B(x_0, r_0)|,$$

and

$$|E| \leqslant \liminf_{k\to\infty} |E_k| = v.$$

So we have proven the following fact: from every minimizing sequence of sets of volume $v > 0$, one can produce, suitably applying isometries of M to each member of the sequence, a new minimizing sequence $\{E_k\}_{k\in\mathbb{N}}$ that converges to some finite perimeter set E with $m_0 v \leqslant |E| \leqslant v$, where $m_0 > 0$ is a universal constant that only depends on v. Hence a fraction of the total volume is captured by the minimizing sequence.

Now take a minimizing sequence $\{E_k\}_{k\in\mathbb{N}}$ that converges to some finite perimeter set E of volume $m_0 v \leqslant |E| < v$. The set E is isoperimetric for volume $|E|$ and hence bounded by Lemma 1.23. By Proposition 1.24, the sequence $\{E_k\}_{k\in\mathbb{N}}$ can be replaced by another minimizing sequence $\{E_k^c \cup E_k^d\}_{k\in\mathbb{N}}$ so that $E_k^c \to E$ and E_k^d diverges. Moreover, $\{E_k^d\}_{k\in\mathbb{N}}$ is minimizing for volume $v - |E|$. Hence one obtains

$$I_M(|E|) + I_M(v - |E|) = I_M(v).$$

If $|E| = v$ we are done since $P(E) \leqslant \liminf_{k\to\infty} P(E_k) = I_M(|E|)$ and hence E is an isoperimetric region. So assume that $|E| < v$ and observe that $|E| \geqslant m_0\, v$. It is clear that E is an isoperimetric region of volume $|E|$. The minimizing sequence can be broken into two pieces: one of them converging to E and the other one diverging. The diverging part is a minimizing sequence for volume $v - |E|$. We let $F_0 := E$.

Now we apply again the previous arguments to the diverging part of the sequence, which is minimizing for volume $v - |E|$. We translate the sets to capture part of the volume and we get a new isoperimetric region F_1 with volume

$$v - |F_0| \geqslant |F_1| \geqslant m_0(v - |F_0|),$$

and a new diverging minimizing sequence for volume $v - |F_0| - |F_1|$. By induction we get a sequence of isoperimetric regions $\{F_k\}_{k\in\mathbb{N}}$ so that the volume of F_k satisfies

$$|F_k| \geqslant m_0\bigg(v - \sum_{i=0}^{k-1} |F_i|\bigg).$$

Hence we have

$$\sum_{i=0}^{k} |F_i| \geqslant (k+1)m_0 v - k m_0 \sum_{i=0}^{k-1} |F_i| \geqslant (k+1)m_0 v - k m_0 \sum_{i=0}^{k} |F_i|,$$

and so

$$\sum_{i=0}^{k} |F_i| \geqslant \frac{(k+1)m_0 v}{1 + k m_0}.$$

Taking limits when $k \to \infty$ we get

$$\lim_{k\to\infty} \sum_{i=0}^{k} |F_i| = v.$$

Moreover,

$$\sum_{i=0}^{\infty} P(F_i) = I_M(v).$$

Each region F_i is bounded, so that we can place them in M using the isometry group so that they are at positive distance (each one contained in an annulus centered at some given point). Hence $F := \bigcup_{i=0}^{\infty} F_i$ is an isoperimetric region of volume v. In fact, F must be bounded by Lemma 1.23, so we only need a finite number of steps to recover all the volume.

□

CHAPTER 2

First and second variation formulae in three-dimensional pseudo-hermitian manifolds

2.1. Preliminaries

A three-dimensional contact manifold [**15**] is a three-dimensional smooth manifold M so that there exists a one-form ω such that $d\omega$ is non-degenerate when restricted to $\mathcal{H} := ker(\omega)$. Since

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

the *horizontal distribution* $\mathcal{H}$ is completely non-integrable. It is well known the existence of a unique *Reeb vector field* T in M so that

$$\omega(T) = 1, \qquad (\mathcal{L}_T\omega)(X) = 0, \tag{2.1}$$

where $\mathcal{L}$ is the Lie derivative and X any smooth vector field on M. It is a direct consequence that $\omega \wedge d\omega$ is an orientation form on M. A well-known example of contact manifold is the Euclidean space $\mathbb{R}^3$ with the contact one-form

$$\omega_0 := dt + xdy - ydx. \tag{2.2}$$

A *contact transformation* between contact manifolds is a diffeomorphism preserving the horizontal distribution. A *strict contact transformation* is a diffeomorphism preserving the contact one-form. A strict contact transformation preserves the Reeb vector field. Darboux's Theorem [**15**, Theorem 3.1] shows that, given a contact manifold M and some point $p \in M$, there exists an open neighborhood U of p and a strict contact transformation f from U into a open set of $\mathbb{R}^3$ with its standard contact structure induced by ω_0. Such a local chart will be called a *Darboux chart*.

A positive definite metric $g_{\mathcal{H}}$ on $\mathcal{H}$ defines a *contact sub-Riemannian* manifold $(M, g_{\mathcal{H}}, \omega)$ on M [**63**]. The first Heisenberg group is the contact sub-Riemannian manifold $\mathbb{H}^1 \equiv (\mathbb{R}^3, g_0, \omega_0)$, where g_0 is the Riemannian metric on $\mathcal{H}$ defined requiring that

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}$$

form an orthonormal basis at each point.

The length of a piecewise horizontal curve $\gamma : I \to M$ is defined by

$$L(\gamma) := \int_I |\gamma'(t)|\, dt,$$

where the modulus is computed with respect to the metric $g_{\mathcal{H}}$. The Carnot-Carathéodory distance $d(p, q)$ between p, $q \in M$ is defined as the infimum of the lengths of piecewise smooth horizontal curves joining p and q. A minimizing geodesic is any curve $\gamma : I \to M$ such that $d(\gamma(t), \gamma(t')) = |t - t'|$ for each t, $t' \in I$. From [**63**, Chap. 5] a minimizing geodesic in a contact sub-Riemannian manifold is a smooth curve that satisfies the geodesic equations, i.e., it is normal.

The metric $g_{\mathcal{H}}$ can be extended to a Riemannian metric g on M by requiring that T be a unit vector orthogonal to $\mathcal{H}$. The scalar product of two vector fields X and Y with respect to the metric g will be often denoted by $\langle X, Y\rangle$ instead of $g(X,Y)$. The Levi-Civita connection induced by g will be denoted by D. An important property of the metric g is that the integral curves of the Reeb vector field T are *geodesics* [**15**, Theorem 4.5].

A usual class defined in contact geometry is the one of contact Riemannian manifolds, see [**15**], [**90**]. Given a contact manifold, one can assure the existence of a Riemannian metric g and an $(1,1)$-tensor field J so that

$$g(T,X) = \omega(X), \quad 2g(X, J(Y)) = d\omega(X,Y), \quad J^2(X) = -X + \omega(X)\,T. \tag{2.3}$$

The structure given by (M, ω, g, J) is called a contact Riemannian manifold. The class of contact sub-Riemannian manifolds is different from this one. Recall that, in our definition, the metric $g_{\mathcal{H}}$ is given, and it is extended to a Riemannian metric g in TM. However, there is not in general an $(1,1)$-tensor field J satisfying all conditions in (2.3). Observe that the second condition in (2.3) uniquely defines J on $\mathcal{H}$, but this J does not satisfy in general the third condition in (2.3), as it is easily seen in $(\mathbb{R}^3, \omega_0)$ choosing an appropriate positive definite metric in $\ker(\omega_0)$. When M is three-dimensional the structure (M, ω, g, J) is equivalent to a strongly pseudo-convex pseudo-hermitian structure [**15**, Corollary 6.4] and we will call briefly $(M, g_{\mathcal{H}}, \omega, J)$ a *pseudo-hermitian manifold*.

The Riemannian volume form dv_g in (M,g) is Popp's measure [**63**, § 10.6]. The volume of a set $E \subset M$ with respect to the Riemannian metric g will be denoted by $|E|$.

A *contact isometry* in $(M, g_{\mathcal{H}}, \omega)$ is a strict contact transformation that preserves $g_{\mathcal{H}}$. Contact isometries preserve the Reeb vector fields and they are isometries of the Riemannian manifold (M,g).

In a contact sub-Riemannian 3-manifold $(M, g_{\mathcal{H}}, \omega)$, we define a linear operator $J : \mathcal{H} \to \mathcal{H}$ on an orthonormal basis $\{X, Y\}$ of $\mathcal{H}$ with respect to the metric $g_{\mathcal{H}}$ by

$$g(J(X), Y) = -g(J(Y), X) = sgn(c_1), \quad g(J(X), X) = g(J(Y), Y) = 0, \tag{2.4}$$

where we have denoted $c_1(p) = -g([X,Y](p), T_p)$. We remark that $c_1(p)$ never vanish since $span\{X, Y\} = TM$ and $sgn(c_1)$ equals 1 or -1 in the whole manifold. Furthermore J can be extended to the whole tangent space by requiring $J(T) = 0$. Now we define a connection ∇, that we will call the *(contact) sub-Riemannian connection*, as the unique connection having non-vanishing torsion defined by

$$\operatorname{Tor}(X,Y) = g(X,T)\tau(Y) - g(Y,T)\tau(X) + c_1 g(J(X),Y)T, \tag{2.5}$$

where $\tau : TM \to \mathcal{H}$ is defined by

$$\tau(V) = -\frac{1}{2} J(\mathcal{L}_T J)(V)$$

for all $V \in TM$. Clearly τ vanishes outside $\mathcal{H}$. Alternatively if we consider the endomorphism

$$\sigma(V) := D_V T : TM \to \mathcal{H}.$$

we have that

$$g(\sigma(V), Z) = g(\tau(V), Z) + \frac{c_1}{2} g(J(V), Z). \tag{2.6}$$

Equation (2.6) can be viewed as an alternative definition of J and τ, where J and τ are antisymmetric and symmetric respectively. We shall call τ the *(contact) sub-Riemannian torsion.* We remark that ∇ and τ are generalizations of the well-known pseudo-hermitian connection and pseudo-hermitian torsion in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$,[**25**, Appendix] and [**37**]. From the above definitions it follows easily

$$\nabla_V T = 0, \tag{2.7}$$

$$(\nabla_V J)Z = 0 \tag{2.8}$$

and

$$g(J(V), V) = 0, \tag{2.9}$$

for all $V, Z \in TM$. Here $J^2 = -Id$ on $\mathcal{H}$ but satisfies the second equation in (2.3) if and only if (M, g, J) is a pseudo-hermitian manifold. It implies the normalization $c_1 = 2$ and at our knowledge all definitions of pseudo-hermitian manifolds and Riemannian contact manifolds require it implicitly. But there exist interesting examples that do not satisfy $c_1 = 2$ as the roto-translation group RT that we will study in the last section. The difference between the Levi-Civita and the pseudo-hermitian connections can be computed using Koszul's formulas as in [**37**, p.38]

$$2g(D_X Y - \nabla_X Y, Z) = g(Tor(X, Z), Y) + g(Tor(Y, Z), X) - g(Tor(X, Y), Z). \tag{2.10}$$

In a contact sub-Riemannian 3-manifold $(M, g_{\mathcal{H}}, \omega)$ we can generalize the definition of the *Webster scalar curvature* given in pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$ by

$$W := -g(R(X, Y)Y, X), \tag{2.11}$$

where $\{X, Y\}$ is an orthonormal basis of $\mathcal{H}$ and R is the pseudo-hermitian curvature tensor defined by

$$R(Z, W)V = \nabla_W \nabla_Z V - \nabla_Z \nabla_W V + \nabla_{[Z,W]} V, \tag{2.12}$$

for all $Z, W, V \in TM$.

In the following we restrict ourselves to the case in which c_1 is a constant. We briefly call such a manifold a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$, since it has analogous properties respect to a pseudo-hermitian manifold defined in [**25**, Appendix] and [**37**].

2.2. The first variation formula for C^2 surfaces.

We define the *area* of a $\mathcal{C}^1$ surface Σ immersed in M by

$$A(\Sigma) = \int_\Sigma |N_h| d\Sigma, \tag{2.13}$$

where N is the unit normal vector with respect to the metric g, N_h is the orthogonal projection of N to $\mathcal{H}$ and $d\Sigma$ is the Riemannian area element of Σ. The singular set Σ_0 consists of those points p where $\mathcal{H}_p$ coincides with the tangent plane $T_p\Sigma$ of M. We define the *horizontal unit normal vector* $\nu_h(p)$ and the *characteristic vector field* $Z(p)$ by

$$\nu_h(p) := \frac{N_h(p)}{|N_h(p)|}, \quad Z(p) := J(\nu_h)(p) \tag{2.14}$$

for all $p \in \Sigma - \Sigma_0$. Since Z_p is orthogonal to ν_h and horizontal, we get that Z_p is tangent to Σ and generates $T_p\Sigma \cap \mathcal{H}_p$. We call *characteristic curves* of Σ the integral curves of Z in $\Sigma - \Sigma_0$. Now setting

$$S := g(N, T)\nu_h - |N_h|T \tag{2.15}$$

we get that $\{Z_p, S_p\}$ is an orthonormal basis of $T_p\Sigma$ for $p \in \Sigma - \Sigma_0$.

Now we consider a $\mathcal{C}^1$ vector field U with compact support on Σ and denote by Σ_t the variation of Σ induced by U, i.e., $\Sigma_t = \{exp_p(tU_p) | p \in \Sigma\}$, where exp is the exponential map of M with respect to g. Furthermore we denote by B the Riemannian shape operator and by θ the 1-form associated to the connection ∇ and ν_h

$$\theta(v) := g(\nabla_v \nu_h, Z), \tag{2.16}$$

for all $v \in T_pM$.

Lemma 2.1. *Let Σ be an oriented immersed $\mathcal{C}^2$ surface in a contact sub-Riemannian three-dimensional manifold $(M, g_{\mathcal{H}}, \omega)$. Consider a point $p \in \Sigma - \Sigma_0$, the horizontal Gauss map ν_h and the basis $\{Z, S\}$ of T_pM already defined. For any $v \in T_pM$ we have*

(i) $|N_h| Z(|N_h|) = -g(N,T) Z(g(N,T))$;
(ii) $|N_h|^{-1} Z(g(N,T)) = |N_h| Z(g(N,T)) - g(N,T) Z(|N_h|)$;
(iii) $g(B(Z), S) = \frac{c_1}{2} - g(\tau(Z), \nu_h) + |N_h|^{-1} Z(g(N,T))$
$= -g(\sigma(Z), \nu_h) + |N_h|^{-1} Z(g(N,T))$;
(iv) $g(B(S), Z) = -g(N,T)^2 g(\tau(\nu_h), Z) + \frac{c_1}{2}(|N_h|^2 - g(N,T)^2) - |N_h|\theta(S)$;
(v) $|N_h|^{-1} Z(g(N,T)) = -c_1 g(N,T)^2 + |N_h|^2 g(\tau(Z), \nu_h) - |N_h|\theta(S)$.

Proof. From $Z(|N_h|^2) = Z(1 - g(N,T)^2)$ we immediately obtain (i). Using (i) and $|N| = 1$ we get

$$\begin{aligned} |N_h| Z(g(N,T)) - g(N,T) Z(|N_h|) &= (|N_h| + |N_h|^{-1} g(N,T)) Z(g(N,T)) \\ &= |N_h|^{-1} Z(g(N,T)) \end{aligned}$$

which proves (ii). From $N = g(N,T)T + |N_h|\nu_h$, (2.6) and (2.15) we have

$$\begin{aligned} g(D_Z N, S) &= Z(g(N,T)) g(T,S) + g(N,T) g(\sigma(Z), S) + Z(|N_h|) g(\nu_h, S) + |N_h| g(D_Z \nu_h, S) \\ &= g((c_1/2) J(Z) + \tau(Z), \nu_h) + |N_h| Z(g(N,T)) - g(N,T) Z(|N_h|), \end{aligned}$$

where we have used

$$g(D_Z \nu_h, S) = -|N_h| g(D_Z \nu_h, T) = |N_h| g(\sigma(Z), \nu_h).$$

Now from (ii) we get (iii). On the other hand

$$g(D_S N, Z) = g(N,T) g(\sigma(S), Z) + |N_h| g(D_S \nu_h, Z),$$

by (2.10) and (2.15) we obtain (iv). Finally we get (v) subtracting (iii) and (iv) since $g(B(Z), S) - g(B(S), Z) = 0$. $\square$

The next lemma is proved in [**77**] for the Heisenberg group $\mathbb{H}^n$, but it holds in a general contact sub-Riemannian 3-manifold $(M, g_{\mathcal{H}}, \omega)$ with the same proof.

Lemma 2.2. *Let Σ be a C^1 surface in M, $p \in \Sigma$ and $\{E_1, E_2\}$ any basis of $T_p\Sigma$. Then we have*

$$|N_h|(p) = \frac{|v_p|}{G(E_1, E_2)^{1/2}},$$

where $v_p := g(T_p, E_1) E_2 - g(T_p, E_2) E_1$ and $G(E_1, E_2)$ is the Gram determinant of $\{E_1, E_2\}$.

Proof. We consider

$$N_h = \lambda E_1 + \mu E_2 + |N_h|^2 N \tag{2.17}$$

so that $|N_h|^2 g(N,T) = -(\lambda g(E_1,T) + \mu g(E_2,T))$. From $N_h = N - g(N,T)T$ we have

$$g(N_h, E_i) = -g(N,T)g(T,E_i), i = 1,2. \tag{2.18}$$

Now compute λ and μ taking scalar product in (2.17) with E_1 and E_2 in (2.17) and using (2.18) we have

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \frac{-g(N,T)}{G(E_1,E_2)} \begin{pmatrix} g(E_2,E_2) & -g(E_1,E_2) \\ -g(E_1,E_2) & g(E_1,E_1) \end{pmatrix} \begin{pmatrix} g(T,E_1) \\ g(T,E_2) \end{pmatrix}.$$

Hence we have obtained

$$|N_h|^2 g(N,T) = \frac{g(N,T)}{G(E_1,E_2)} |v|^2$$

which prove the statement in the case $g(N,T) \neq 0$. If $g(N,T) = 0$ we simply check that $|v|^2 = G(E_1,E_2)^2$, writing E_1 and E_2 in term of an orthonormal basis $\{w,T\}$ of $T_p\Sigma$. □

Now we introduce the notion of intrinsic regularity, [**41**], [**42**] and [**43**]. Let Ω be an open subset of M, we say $f : \Omega \to \mathbb{R}$ of class $\mathcal{C}^1_{\mathcal{H}}$ in Ω when Xf exists and it is continuous for any $X \in \mathcal{H}$. We define $f \in \mathcal{C}^k_{\mathcal{H}}(\Omega)$ when $Xf \in \mathcal{C}^{k-1}_{\mathcal{H}}(\Omega)$ for all $X \in \mathcal{H}$. Since c_1 is a real constant immediately we obtain that $f \in \mathcal{C}^{2k}_{\mathcal{H}}(\Omega)$ implies $f \in \mathcal{C}^k(\Omega)$. We define a surface Σ a *$\mathcal{H}$-regular surface* of class $C^k_{\mathcal{H}}$ if for any $p \in \Sigma$ exist $B_r(p)$, a metric ball of radius r centered in p, and a function $f \in C^k_{\mathcal{H}}$ such that

$$\Sigma \cap B_r(p) = \{p \in B_r(p) : f(p) = 0, \nabla_{\mathcal{H}} f(p) \neq 0\},$$

see [**41**] for the definition in the Heisenberg group.

LEMMA 2.3. *Let Σ be an oriented immersed $\mathcal{C}^2$ surface in a contact sub-Riemannian three-dimensional manifold $(M, g_{\mathcal{H}}, \omega)$ and let $f \in \mathcal{C}^1(M)$. Then we have*

$$div_\Sigma(fS) = S(f) + fg(N,T)\theta(Z) - f|N_h|g(\tau(Z),Z),$$

and

$$\begin{aligned} div_\Sigma(fZ) =& Z(f) - fg(N,T)\theta(S) + fg(N,T)|N_h|g(\tau(\nu_h),Z) \\ &+ c_1 fg(N,T)|N_h|g(J(\nu_h),Z), \end{aligned}$$

where div_Σ is the Riemannian divergence with respect to an orthonormal basis of $T\Sigma$.

PROOF. We have

$$div_\Sigma(fZ) = Z(f) + fg(D_S Z, S)$$

and by (2.15)

$$g(D_S Z, S) = g(N,T)g(D_S Z, \nu_h) - g(N,T)|N_h|g(D_{\nu_h} Z, T)$$

and using (2.10) we prove the second equation. For the first one we note that

$$div_\Sigma(fS) = S(f) + fg(D_Z S, Z)$$

and we can conclude using

$$g(D_Z S, Z) = g(N,T)g(D_Z \nu_h, Z) - |N_h|g(D_Z T, Z)$$

together with (2.10). □

Now we can present the key Lemma to obtain the first variation.

LEMMA 2.4. *Let Σ be an oriented immersed C^1 surface in a contact sub-Riemannian three-dimensional manifold $(M, g_{\mathcal{H}}, \omega)$. Then the first variation of the sub-Riemannian area induced by the vector field U, that is $\mathcal{C}_0^1(\Sigma)$ along the variation, is given by*

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) = \int_{\Sigma-\Sigma_0} \{ -S(g(U,T)) + c_1 g(N,T) g(J(\nu_h), U_h) \\ + |N_h| g(\nabla_Z U_h, Z) + |N_h| g(U,T) g(\tau(Z), Z)\} d\Sigma.\end{aligned} \tag{2.19}$$

PROOF. For every $p \in \Sigma$ and the orthonormal basis $\{Z, S\}$ of $T_p\Sigma$, we consider extensions $E_1(s), E_2(s)$ of Z, S along the curve $s \mapsto \varphi_s(p)$ so that $[E_i, U] = 0$, i.e., the vector fields E_i are invariant under the flow generated by U. By Lemma 2.2 the Jacobian of the map φ_s at p is given by $G(E_1(s), E_2(s))^{1/2}$. We get

$$A(\varphi_s(\Sigma)) = \int_{\Sigma} |V(s)| d\Sigma,$$

where $V(s) := g(T, E_1(s))E_2(s) - g(T, E_2(s))E_1(s)$. We can express the first derivative of the area as

$$\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma - \Sigma_0)) = \int_{\Sigma} \frac{g(\nabla_U V, V(0))}{|V(0)|} d\Sigma.$$

Now $g(T, E_1(0)) = 0$ and $g(T, E_2(s)) = -|N_h|$ imply $V(0) = |N_h|Z$. Since $[E_i, U] = 0$ and (2.7) we have

$$\frac{g(D_U V, V(0))}{|V(0)|} = -U(g(E_2, T)) + |N_h| g(\nabla_U E_1, Z) = -(g(\nabla_U E_2, T)) + |N_h| g(\nabla_U E_1, Z)$$

and substituting we obtain

$$\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) = \int_{\Sigma-\Sigma_0} \{-g(\nabla_U E_2, T) + |N_h| g(\nabla_U E_1, Z)\} d\Sigma.$$

Finally since

$$\begin{aligned} g(\nabla_U E_2, T)) &= g(\nabla_S U, T) - c_1 g(N,T) g(J(\nu_h), U) \\ &= S(g(U,T)) - c_1 g(N,T) g(J(\nu_h), U) \end{aligned}$$

and from

$$g(\nabla_U E_1, E_1) = g(\nabla_Z U_h, Z) + g(U,T) g(\tau(Z), Z)$$

we get (2.19).

□

Now we are able to get variation formulas in generic directions.

COROLLARY 2.5. *Let Σ be an oriented immersed $\mathcal{C}^2$ surface in a contact sub-Riemannian three-dimensional manifold $(M, g_{\mathcal{H}}, \omega)$. Then the first variation of the area induced by the tangent vector field $U = lZ + hS$, with $l, h \in \mathcal{C}_0^1(\Sigma - \Sigma_0)$, is*

$$\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) = \int_{\Sigma} \mathrm{div}_{\Sigma}(|N_h| U) d\Sigma.$$

Furthermore when $\partial\Sigma = \emptyset$ the above term vanishes.

PROOF. By (2.19) we get

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) &= \int_{\Sigma-\Sigma_0} \{c_1 l g(N,T) g(J(\nu_h), Z) + |N_h| Z(l)\} d\Sigma \\ &+ \int_{\Sigma-\Sigma_0} \{S(|N_h| h) + h g(N,T) |N_h| g(\nabla_Z \nu_h, Z) - h |N_h|^2 g(\tau(Z), Z)\} d\Sigma \\ &= \int_\Sigma \mathrm{div}_\Sigma(l |N_h| Z) d\Sigma + \int_\Sigma \mathrm{div}_\Sigma(h |N_h| S) d\Sigma,\end{aligned}$$

where we have used $|N_h| Z(l) = Z(|N_h| l) - l Z(|N_h|)$, Lemma 2.1, formula (2.19) and Lemma 2.3. When $\partial\Sigma = \emptyset$ we can use the Riemannian divergence theorem to prove that the variation vanishes. □

COROLLARY 2.6. *Let Σ be an oriented immersed $\mathcal{C}^2$ surface in a contact sub-Riemannian 3-manifold $(M, g_{\mathcal{H}}, \omega)$. Then the first variation of the area induced by a normal vector field $U = uN$, with $u \in \mathcal{C}^1_0(\Sigma)$, is*

$$\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) = \int_\Sigma u\, g(\nabla_Z \nu_h, Z) d\Sigma - \int_\Sigma \mathrm{div}_\Sigma(u g(N,T) S) d\Sigma.$$

Furthermore if $u \in C^1_0(\Sigma - \Sigma_0)$ we get

$$\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) = \int_\Sigma u\, g(\nabla_Z \nu_h, Z) d\Sigma.$$

PROOF. By (2.19) and Lemma 2.3 we get

$$\begin{aligned}\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) &= \int_\Sigma \{-S(g(N,T)u) + u |N_h|^2 g(\nabla_Z \nu_h, Z) \\ &\qquad + u g(N,T) |N_h| g(\tau(Z), Z)\} d\Sigma \\ &= \int_\Sigma u\, g(\nabla_Z \nu_h, Z) d\Sigma - \int_\Sigma \mathrm{div}_\Sigma(u g(N,T) S) d\Sigma.\end{aligned}$$

When $u \in C^1_0(\Sigma - \Sigma_0)$, we can use the Riemannian divergence theorem to conclude

$$\int_\Sigma \mathrm{div}_\Sigma(u g(N,T) S) d\Sigma = 0.$$

□

REMARK 2.7. When $(M, g_{\mathcal{H}}, \omega)$ is the Heisenberg group $\mathbb{H}^1$ we have that Corollary 2.6 coincides with [**81**, Lemma 4.3]. Furthermore Corollary 2.6 coincides with [**25**, eq. (2.8')] where the authors considered non-singular variations in a pseudo-hermitian 3-manifolds $(M, g_{\mathcal{H}}, \omega, J)$. Different versions of Corollary 2.6 can be found also in [**60**], [**61**] and [**53**], [**54**], for Carnot groups and vertically rigid manifolds, respectively.

DEFINITION. Let Σ be a surface of class $\mathcal{C}^2_{\mathcal{H}}$. Corollary 2.6 allows us to define the *mean curvature* of Σ in a point $p \in \Sigma - \Sigma_0$ as

$$H := -g(\nabla_Z \nu_h, Z). \tag{2.20}$$

A *minimal surface* is a surface of class $\mathcal{C}^2_{\mathcal{H}}$ whose mean curvature H vanishes.

We observe that (2.20) extends the definition of mean curvature from C^2 surfaces to C_h^2 surfaces. We note that our definition of mean curvature coincides with [**4**], [**25**], [**53**] and [**81**] among others, for surfaces of class $\mathcal{C}^2$ and it is motivated by Proposition 2.22 in Section 2.5. In [**77**] the author also defined the mean curvature for surfaces of class $C^2_{\mathcal{H}}$.

2.3. Characteristic Curves and Jacobi-like vector fields.

In this section we give a characterization of characteristic curves in a constant mean curvature surface and we define special vector fields along characteristic curves that are the natural generalization of Jacobi vector fields along geodesics in Sasakian sub-Riemannian manifolds.

PROPOSITION 2.8. *Let Σ be an oriented immersed $\mathcal{C}^2_{\mathcal{H}}$ surface of constant mean curvature $H = c_1\lambda$ in a contact sub-Riemannian three-dimensional manifold $(M, g_{\mathcal{H}}, \omega)$. Then, outside the singular set, the equation of characteristic curves is*

$$\nabla_Z Z + c_1\lambda\, J(Z) = 0, \tag{2.21}$$

where ∇ denote the pseudo-hermitian connection. We will call λ the curvature of the characteristic curve.

PROOF. It is an immediate consequence of (2.7), (2.20) and $|Z| = 1$. □

REMARK 2.9. Let γ be a Carnot-Caratheodory geodesic in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. Then the unit tangent vector field $\dot\gamma$ to γ satisfies [**83**, Proposition 15]

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma + c_1\lambda J(\dot\gamma) = 0 \\ \dot\gamma(\lambda) = -\frac{1}{c_1} g(\tau(\dot\gamma), \dot\gamma), \end{cases} \tag{2.22}$$

where ∇ is the pseudo-hermitian connection. This implies that characteristic curves in a constant mean curvature surface are sub-Riemannian geodesics if and only if $g(\tau(\dot\gamma), \dot\gamma) = 0$. For instance, this is satisfied in manifolds with vanishing torsion.

PROPOSITION 2.10. *We consider a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$, a curve $\alpha : I \to M$ of class C^1 defined on some interval $I \subset M$ and a C^1 unit horizontal vector field U along α. For fixed $\lambda \in \mathbb{R}$, suppose we have a well-defined map $F : I \times I' \to M$ given by $F(\varepsilon, s) = \gamma_\varepsilon(s)$, where I' is a open interval containing the origin, and $\gamma_\varepsilon(s)$ is a characteristic curve of curvature λ with initial conditions $\gamma_\varepsilon(0) = \alpha(\varepsilon)$ and $\dot\gamma_\varepsilon(0) = U(\varepsilon)$. Then the vector field $V_\varepsilon(s) := (\partial F/\partial\varepsilon)(\varepsilon, s)$ satisfies the following properties:*

(i) *V_ε is a C^∞ vector field along γ_ε and satisfies $[\dot\gamma_\varepsilon, V_\varepsilon] = 0$;*
(ii) *along γ_ε we have*

$$\dot\gamma_\varepsilon(\lambda g(V_\varepsilon, T) + g(V_\varepsilon, \dot\gamma_\varepsilon)) = -g(V_\varepsilon, T) g(\tau(\dot\gamma_\varepsilon), \dot\gamma_\varepsilon),$$

in particular $\lambda g(V_\varepsilon, T) + g(V_\varepsilon, \dot\gamma_\varepsilon)$ is constant along sub-Riemannian geodesics;
(iii) *V_ε satisfies the equation*

$$V_\varepsilon'' + R(\dot\gamma_\varepsilon, V_\varepsilon)\dot\gamma_\varepsilon + c_1\lambda\{J(V_\varepsilon') + g(V_\varepsilon, T) J(\tau(\dot\gamma_\varepsilon))\} + \nabla_{\dot\gamma_\varepsilon} Tor(V_\varepsilon, \dot\gamma_\varepsilon) = 0, \tag{2.23}$$

$$\nabla_{\dot\gamma_\varepsilon} Tor(V_\varepsilon, \dot\gamma_\varepsilon) = -g(V_\varepsilon, T)'' T + g(V_\varepsilon, T)'\tau(\dot\gamma_\varepsilon) + g(V_\varepsilon, T)\nabla_{\dot\gamma_\varepsilon}\tau(\dot\gamma_\varepsilon)$$

where V' denotes the covariant derivative along γ_ε and R the curvature tensor with respect to the pseudo-hermitian connection.

(iv) *the vertical component of* V_ε*,* $g(V_\varepsilon, T)$*, satisfies the differential equation*

$$g(V_\varepsilon, T)''' + \beta_1(s) g(V_\varepsilon, T)' + c_1\beta_2(s) g(V_\varepsilon, T) = 0,$$

with

$$\beta_1(s) = W - c_1 g(\tau(\dot\gamma_\varepsilon), J(\dot\gamma_\varepsilon)) + c_1^2\lambda^2,$$

$$\beta_2(s) = c_1\lambda g(\tau(\dot\gamma_\varepsilon), \dot\gamma_\varepsilon) + g(R(\dot\gamma_\varepsilon, T)\dot\gamma_\varepsilon, J(\dot\gamma_\varepsilon)) - \dot\gamma_\varepsilon(g(\tau(\dot\gamma_\varepsilon), J(\dot\gamma_\varepsilon))),$$

where W *is the pseudo-hermitian scalar curvature and* $'$ *is the derivative respect to* s*.*

PROOF. For simplicity we avoid the subscript ε in the computation. The proof of (i) is analogous to the one of [**82**, Lemma 3.3 (i)]. From $[\dot\gamma, V] = 0$ and (2.5) we have

$$\begin{aligned} g(V,T)' &= \dot\gamma(g(V,T)) = g(\nabla_{\dot\gamma} V, T) = g(\nabla_V \dot\gamma + Tor(\dot\gamma, V), T) \\ &= g(Tor(\dot\gamma, V), T) = c_1 g(J(\dot\gamma), V). \end{aligned} \tag{2.24}$$

Now (2.24) together with

$$g(V,\dot\gamma)' = g(Tor(\dot\gamma, V), \dot\gamma) - c_1\lambda g(V, J(\dot\gamma)) = -g(V,T) g(\tau(\dot\gamma), \dot\gamma) - c_1\lambda g(V, J(\dot\gamma))$$

proves (ii). Now using (2.8) we get

$$\nabla_V J(\dot\gamma) = J(\nabla_V \dot\gamma) = J(V') + g(V,T) J(\tau(\dot\gamma)),$$

that permits us to compute $\nabla_V(\nabla_{\dot\gamma}\dot\gamma + c_1\lambda J(\dot\gamma))$ to obtain the first equation in (iii). The second one is simply obtained using (2.5) and (2.24).

To prove (iv) we have, differentiating (2.24)

$$\frac{1}{c_1} g(V,T)'' = c_1\lambda g(\dot\gamma, V) + g(J(\dot\gamma), V') \tag{2.25}$$

and consequently

$$\begin{aligned} \frac{1}{c_1} g(V,T)''' &= \dot\gamma(c_1\lambda g(\dot\gamma, V)) + c_1\lambda g(\dot\gamma, V') + g(V'', J(\dot\gamma)) \\ &= 2c_1\lambda g(V,\dot\gamma)' + c_1\lambda^2 g(V,T)' + g(V'', J(\dot\gamma)). \end{aligned}$$

Taking into account (ii) we get

$$\frac{1}{c_1} g(V,T)''' = -c_1\lambda^2 g(V,T)' - 2c_1\lambda g(V,T) g(\tau(\dot\gamma), \dot\gamma) + g(V'', J(\dot\gamma)). \tag{2.26}$$

The only term we have to deal with is $g(R(\dot\gamma, V)\dot\gamma + c_1\lambda J(V'), J(V'))$. Now by point (iii) we have

$$\begin{aligned} g(V'', J(\dot\gamma)) = &- g(R(\dot\gamma, V)\dot\gamma, J(\dot\gamma)) - c_1\lambda\{g(V,T) g(\tau(\dot\gamma), \dot\gamma) + g(V', \dot\gamma)\} \\ &- g(V,T)' g(\tau(\dot\gamma), J(\dot\gamma)) - g(V,T) g(\nabla_{\dot\gamma}\tau(\dot\gamma), J(\dot\gamma)). \end{aligned}$$

From $V = g(V,T)T + g(V,\dot\gamma)\dot\gamma + g(V, J(\dot\gamma))J(\dot\gamma)$ we obtain

$$-g(R(\dot\gamma, V)\dot\gamma, J(\dot\gamma)) = -g(V, J(\dot\gamma))W - g(V,T) g(R(\dot\gamma, T)\dot\gamma, J(\dot\gamma)).$$

Furthermore since

$$-g(\nabla_{\dot\gamma}\tau(\dot\gamma), J(\dot\gamma)) = -\dot\gamma(\tau(\dot\gamma), J(\dot\gamma)) + c_1\lambda g(\tau(\dot\gamma), \dot\gamma)$$

and

$$g(V', \dot\gamma) = g(V, \dot\gamma)' - g(\nabla_{\dot\gamma}\dot\gamma, V) = -g(V,T) g(\tau(\dot\gamma), \dot\gamma)$$

we finally get

$$
\begin{aligned}
g(V'', J(\dot\gamma)) = & -\frac{1}{c_1} g(V,T)'W + c_1\lambda g(V,T) g(\tau(\dot\gamma),\dot\gamma) - g(\tau(\dot\gamma), J(\dot\gamma)) g(V,T)' \\
& - g(V,T)\{g(R(\dot\gamma,T)\dot\gamma, J(\dot\gamma)) - \dot\gamma(g(\tau(\dot\gamma)), J(\dot\gamma))\}.
\end{aligned} \tag{2.27}
$$

We conclude summing (2.26) and (2.27) and simplifying. □

DEFINITION. Let $\gamma : I \to \Sigma$ be a characteristic curve, where I is a real interval and Σ is a surface. A vector field V along γ is called a *Jacobi-like field* if it satisfies (2.23) for all $s \in I$.

REMARK 2.11. Special cases of Proposition 2.10 can be found in [**21**], [**81**] and [**82**].

2.4. The structure of the singular set.

The local model of a three-dimensional contact sub-Riemannian manifold is the contact manifold $(\mathbb{R}^3, \omega_0)$, where ω_0 defined in (2.2) is the standard contact form in $\mathbb{R}^3$, together with an arbitrary positive definite metric $g_{\mathcal{H}_0}$ in $\mathcal{H}_0$. A basis of the horizontal distribution is given by

$$
X := \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial t}, \qquad Y := \frac{\partial}{\partial y} + x\,\frac{\partial}{\partial t},
$$

and the Reeb vector field is

$$
T := \frac{\partial}{\partial t}.
$$

The metric $g_{\mathcal{H}_0}$ will be extended to a Riemannian metric on $\mathbb{R}^3$ so that the Reeb vector field is unitary and orthogonal to $\mathcal{H}_0$. We shall usually denote the set of vector fields $\{X, Y\}$ by $\{Z_1, Z_2\}$. The coordinates of $\mathbb{R}^3$ will be denoted by (x, y, t), and the first 2 coordinates will be abbreviated by z. We shall consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$
F(x, y) := (-y, x).
$$

Given a C^2 function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ defined on an open subset Ω, we define the graph $G_u := \{(z, t) : z \in \Omega, t = u(z)\}$. By (2.13), the sub-Riemannian area of the graph is given by

$$
A(G_u) = \int_{G_u} |N_h|\, dG_u,
$$

where dG_u is the Riemannian metric of the graph and $|N_h|$ is the modulus of the horizontal projection of a unit normal to G_u. We consider on Ω the basis of vector fields $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$.

By the Riemannian area formula

$$
dG_u = \operatorname{Jac} d\mathcal{L}^2, \tag{2.28}
$$

where $d\mathcal{L}^2$ is Lebesgue measure in $\mathbb{R}^2$ and Jac is the Jacobian of the canonical map $\Omega \to G_u$ given by

$$
\operatorname{Jac} = \{\det(g) + g_{11}(u_y + x)^2 + g_{22}(u_x - y)^2 - 2g_{12}(u_x - y)(u_y + x)\}^{1/2} \tag{2.29}
$$

where g is the matrix of the metric, with elements $g_{ij} := g(Z_i, Z_j)$.

Let us compute the composition of $|N_h|$ with the map $\Omega \to G_u$. The tangent space TG_u is spanned by

$$
X + (u_x - y)\,T, \quad Y + (u_y + x)\,T. \tag{2.30}
$$

So the projection to Ω of the singular set $(G_u)_0$ is the set $\Omega_0 \subset \Omega$ defined by $\Omega_0 := \{z \in \Omega : (\nabla u + F) = 0\}$, where ∇ is the Euclidean gradient in $\mathbb{R}^2$. Let us compute a *downward pointing* normal vector $\tilde{N}$ to G_u writing

$$\tilde{N} = \sum_{i=1}^{2} (a_i Z_i) - T. \tag{2.31}$$

The horizontal component of $\tilde{N}$ is $\tilde{N}_h = \sum_{i=1}^{2} a_i Z_i$. We have

$$\sum_{i=1}^{2} a_i g_{ij} = g(\tilde{N}_h, Z_j) = g(\tilde{N}, Z_j) = -(\nabla u + F)_j \langle \tilde{N}, T \rangle = (\nabla u + F)_j,$$

since Z_j is horizontal, $\tilde{N}$ is orthogonal to Z_j defined by (2.30), and (2.31). Hence

$$(a_1, a_2) = b(\nabla u + F),$$

where b is the inverse of the matrix $\{g_{ij}\}_{i,j=1,2}$. So we get

$$|\tilde{N}| = (1 + \langle \nabla u + F, b(\nabla u + F) \rangle)^{1/2}, \tag{2.32}$$

and

$$|\tilde{N}_h| = \langle \nabla u + F, b(\nabla u + F) \rangle^{1/2},$$

where $\langle , \rangle$ is the Euclidean Riemannian metric in $\mathbb{R}^2$, and so

$$|N_h| = \frac{|\tilde{N}_h|}{|\tilde{N}|} = \frac{\langle \nabla u + F, b(\nabla u + F) \rangle^{1/2}}{\left(1 + \langle \nabla u + F, b(\nabla u + F) \rangle\right)^{1/2}}. \tag{2.33}$$

Observe that, from (2.31) and (2.32) we also get

$$g(N, T) = -\frac{1}{\left(1 + \langle \nabla u + F, b(\nabla u + F) \rangle\right)^{1/2}}. \tag{2.34}$$

Hence we obtain from (2.13), (2.28), (2.29) and (2.33)

$$A(G_u) = \int_\Omega \langle \nabla u + F, b(\nabla u + F) \rangle^{1/2} \frac{\det(g_{ij} + (\nabla u + F)_i (\nabla u + F)_j)^{1/2}}{\left(1 + \langle \nabla u + F, b(\nabla u + F) \rangle\right)^{1/2}} \, d\mathcal{L}^2. \tag{2.35}$$

We can compute the mean curvature of a graph G_u [**47**, Lemma 4.2]

LEMMA 2.12. *Let us consider the contact sub-Riemannian manifold* $(\mathbb{R}^3, g_{\mathcal{H}_0}, \omega_0)$, *where* ω_0 *is the standard contact form in* $\mathbb{R}^3$ *and* $g_{\mathcal{H}_0}$ *is a positive definite metric in the horizontal distribution* $\mathcal{H}_0$. *Let* $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ *be a* C^2 *function. We denote by* $g = (g_{ij})_{i,j=1,2}$ *the metric matrix and by* $b = g^{-1} = (g^{ij})_{i,j=1,2}$ *the inverse metric matrix. Then the mean curvature of the graph* G_u, *computed with respect to the downward pointing normal, is given by*

$$-\operatorname{div}\left(\frac{b(\nabla u + F)}{\langle \nabla u + F, b(\nabla u + F) \rangle^{1/2}}\right) + \mu, \tag{2.36}$$

where μ *is a bounded function in* $\Omega \setminus \Omega_0$, *and* div *is the usual Euclidean divergence in* Ω.

Furthermore in dimension three we get

LEMMA 2.13. *Let us consider the contact sub-Riemannian manifold* $(\mathbb{R}^3, g_{\mathcal{H}_0}, \omega_0)$, *where* ω_0 *is the standard contact form in* $\mathbb{R}^3$ *and* $g_{\mathcal{H}_0}$ *is a positive definite metric in the horizontal distribution* $\mathcal{H}_0$. *Let* $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ *be a* C^2 *function. Then*

$$\operatorname{div}\left(\frac{b(\nabla u + F)}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}\right) = det(g)\operatorname{div}\left(\frac{\nabla u + F}{\big\langle \nabla u + F, \nabla u + F\big\rangle^{1/2}}\right) + \rho,$$

where ρ *is a bounded function in* $\Omega \setminus \Omega_0$.

PROOF. The proof is a standard computation. We only note that ρ is of the form

$$\frac{\rho_1(b)(u_x - y)^3 + \rho_2(b)(u_y + x)^3 + \rho_3(b)(u_x - y)^2(u_y + x) + \rho_4(b)(u_x - y)(u_y + x)^2}{(g^{11}(u_x - y)^2 + 2g^{12}(u_x - y)(u_y + x) + g^{22}(u_y + x)^2)^{3/2}},$$

where $\rho_i(b)$ are sums and products of the coefficients g^{ij}. □

Let $\Sigma \subset M$ a $\mathcal{C}^2$ be a surface and let $p \in \Sigma_0$. Then there exists a neighborhood U of p that is a Darboux chart and Σ can be viewed as a graph G_u in $(\mathbb{R}^3, g_{\mathcal{H}_0}, \omega_0)$ above defined. The projection Ω_0 of the singular set in $(G_u)_0$ does not depend on the metric $g_{\mathcal{H}_0}$. The characteristic curves in G_u with respect to $g_{\mathcal{H}_0}$ and the standard Heisenberg metric g_0 coincide, as they are determined by $TG_u \cap \mathcal{H}$. This implies

THEOREM 2.14. *Let* Σ *be a* $\mathcal{C}^2$ *oriented immersed surface with constant mean curvature* H *in* $(M, g_{\mathcal{H}}, \omega)$. *Then the singular set* Σ_0 *consists of isolated points and* $\mathcal{C}^1$ *curves with non-vanishing tangent vector. Moreover, we have*

(i) if $p \in \Sigma_0$ *is isolated then there is* $r > 0$ *and* $\lambda \in \mathbb{R}$ *with* $|2\lambda| = |H|$ *such that the set described as*

$$D_r(p) = \{\gamma^{\lambda}_{p,v}(s) | v \in T_p\Sigma, |v| = 1, s \in [0, r)\},$$

is an open neighborhood of p *in* Σ, *where* $\gamma^{\lambda}_{p,v}$ *denote the characteristic curve starting from* p *in the direction* v *with curvature* λ (2.21);

(ii) if p *is not isolated, it is contained in a* $\mathcal{C}^1$ *curve* $\Gamma \subset \Sigma_0$. *Furthermore there is a neighborhood* B *of* p *in* Σ *such that* $B - \Gamma$ *is the union of two disjoint connected open sets* B_+ *and* B_- *contained in* $\Sigma - \Sigma_0$, *and* ν_h *extends continuously to* Γ *from both sides of* $B - \Gamma$, *i.e., the limits*

$$\nu_h^+(q) = \lim_{x \to q, x \in B_+} \nu_h(x), \qquad \nu_h^-(q) = \lim_{x \to q, x \in B_-} \nu_h(x)$$

exist for any $q \in \Gamma \cap B$. *These extensions satisfy* $\nu_h^+(q) = -\nu_h^-(q)$. *Moreover, there are exactly two characteristic curves* $\gamma_1^{\lambda} \subset B_+$ *and* $\gamma_2^{\lambda} \subset B_-$ *starting from* q *and meeting transversally* Γ *at* q *with initial velocities* $(\gamma_1^{\lambda})'(0) = -(\gamma_2^{\lambda})'(0)$. *The curvature* λ *does not depend on* q *and satisfies* $|\lambda| = |H|$.

PROOF. By [**25**, Theorem B], Lemma 2.12 and Lemma 2.13, Σ_0 consists of isolated points and $\mathcal{C}^1$ curves with non-vanishing tangent vector. Also (i) follows easily.

Writing

$$\nu_h = \frac{b(\nabla u + F)}{\big\langle \nabla u + F, b(\nabla u + F)\big\rangle^{1/2}}$$

because of [**25**, Theorem 3.10, Corollary 3.6], we get (ii). □

COROLLARY 2.15. *Let* Σ *be a* $\mathcal{C}^2$ *minimal surface with singular set* Σ_0. *Then* Σ *is area stationary if and only if the characteristic curves meet the singular curves orthogonally with respect the metric* g.

The proof is a straightforward adaptation of the Heisenberg one, [**81**, Theorem 4.16]. Another version of the last corollary is presented in [**26**, Proposition 6.2] and [**23**, p. 20].

REMARK 2.16. [**81**, Proposition 4.19] implies that, for Σ a $\mathcal{C}^2$ oriented immersed area-stationary surface (with or without a volume constraint), any singular curve of Σ is a $\mathcal{C}^2$ smooth curve.

REMARK 2.17. Another approach to characterize the local behavior of the singular set is provided in [**86**], where the author constructs a circle bundle over the surface and studies the projection of the singular set.

Now we are able to generalize [**25**, Theorem E] to general three-dimensional contact sub-Riemannian manifolds.

THEOREM 2.18. *Let Σ be a $\mathcal{C}^2$ closed, connected surface immersed in a three-dimensional contact sub-Riemannian manifold M, with bounded mean curvature. Then $g(\Sigma) \leqslant 1$, where $g(\Sigma)$ denote the genus of Σ.*

PROOF. By Theorem 2.14 the singular set Σ_0 consists of singular curves and isolated singular points. The line field associated to the characteristic foliation, extended to the singular curves, has a contribution to the index only due to the isolated singular points, Theorem 2.14. Now consider a partition of unity $\{\eta_i\}_{i\in I}$ subordinate to a covering of Σ with Darboux's charts $\{U_i\}_{i\in I}$. By [**25**, Lemma 3.8] the index associated to the characteristic line field with respect to the Heisenberg metric in the Darboux coordinates is 1 and follows that the index of the vector field

$$\sum_{i\in I} \eta_i(\varphi_i^{-1}(Z_0))$$

is 1 in each singular point of Σ, since the Darboux's diffeomorphism preserve the index, [**88**, Lemma 27, p. 446]. Here φ_i denotes the Darboux's diffeomorphism in each chart U_i and Z_0 denote the characteristic vector field associate to the Heisenberg metric in $\varphi_i(U_i)$. We get $\chi(\Sigma) \geqslant 0$, by the Hopf index Theorem, [**88**].

On the other hand for a closed surface $\chi(\Sigma) = 2 - 2g(\Sigma)$, which implies $g(\Sigma) \leqslant 1$. □

REMARK 2.19. When Σ is a compact $\mathcal{C}^2$ surface without boundary in a three-dimensional pseudo-hermitian sub-Riemannian manifold, Theorem I in [**24**] implies immediately $\chi(\Sigma) \geqslant 0$. Then $g(\Sigma) \leqslant 1$.

2.5. The first variation formula for $C^2_{\mathcal{H}}$ surfaces

Now we present a first variation formula for surfaces of class $\mathcal{C}^2_{\mathcal{H}}$ using variations supported in the non-singular set. Given a surface Σ of class $\mathcal{C}^2_{\mathcal{H}}$, we can express Σ as the zero level set of a function $f \in C^2_{\mathcal{H}}$ with non-vanishing horizontal gradient.

REMARK 2.20. In $\mathbb{H}^1$, by [**39**, Proposition 1.20], see also [**9**, Lemma 2.4] and the proof of Theorem 6.5, step 1, in [**41**], the family of smooth surfaces $\{\Sigma_j\}_{j\in\mathbb{N}} = \{p \in M : f_j(p) = 0\}$, where $f_j := \rho_j * f$ and ρ_j are the standard Friedrichs' mollifiers, converge to Σ on compact subsets of $\Sigma - \Sigma_0$. Furthermore also the second order derivatives of f_j with respect to horizontal vectors fields converge to the second derivatives of f. We denote by $Z_j, (\nu_h)_j$ and N_j, respectively, the characteristic vector field, the horizontal unit normal and the unit normal of Σ_j. Furthermore let

$(Z_j(p), S_j(p))$ be an orthonormal basis of $T_p\Sigma_j$. We have that

$$\nu_h = \frac{(Xf)X + (Yf)Y}{\sqrt{(Xf)^2 + (Yf)^2}}, \quad Z = \frac{-(Yf)X + (Xf)Y}{\sqrt{(Xf)^2 + (Yf)^2}}$$

and

$$(\nu_h)_j = \frac{(Xf_j)X + (Yf_j)Y}{\sqrt{(Xf_j)^2 + (Yf_j)^2}}, \quad Z_j = \frac{-(Yf_j)X + (Xf_j)Y}{\sqrt{(Xf_j)^2 + (Yf_j)^2}}$$

so Z_j (resp. (ν_j)) converges to Z (resp. ν_h) with their horizontal derivatives. On the other hand

$$N = \frac{(Xf)X + (Yf)Y + (Tf)T}{\sqrt{(Xf)^2 + (Yf)^2 + (Tf)^2}}$$

and

$$N_j = \frac{(Xf_j)X + (Yf_j)Y + (Tf_j)T}{\sqrt{(Xf_j)^2 + (Yf_j)^2 + (Tf_j)^2}},$$

which implies that N_j (resp. S_j) converges to N (resp. S) but there are not convergence of their derivatives.

LEMMA 2.21. *Let Σ be a C^1 surface immersed in M, such that the derivative in the Z-direction of ν_h exists and is continuous. Assume $\Sigma_0 = \emptyset$. Then exists a family of smooth surfaces $\{\Sigma_j\}_{i\in\mathbb{N}}$ such that*

$$\lim_{j\to+\infty} g(\nabla_{Z_j}(\nu_h)_j, Z_j) = g(\nabla_Z \nu_h, Z)$$

uniformly on compact subsets of Σ.

PROOF. It is sufficient to prove the result locally in a Darboux's chart. So we consider Σ in $(\mathbb{R}^3, \mathcal{H}_0, g_{\mathcal{H}_0})$, where $g_{\mathcal{H}_0}$ is an arbitrary positive definite smooth metric in $\mathcal{H}_0$. We denote by $(\nu_h)_0$ the horizontal unit normal with respect to the Heisenberg metric g_0. By Remark 2.20 the statement holds in the Heisenberg metric. As in (2.31) and (2.32) we have

$$(\nu_h)_j = \frac{(g^{11}X(f_j) + g^{12}Y(f_j))X + (g^{12}X(f_j) + g^{22}Y(f_j))Y}{\sqrt{\langle (X(f_j), Y(f_j)), b(X(f_j), Y(f_j))\rangle}}$$

and

$$((\nu_h)_j)_0 = \frac{X(f_j)\,X + Y(f_j)\,Y}{\sqrt{\langle (X(f_j), Y(f_j)), b(X(f_j), Y(f_j))\rangle}}.$$

Similar expressions hold for ν_h and $(\nu_h)_0$. The Z-direction does not depend on the metric, since it is determined by $T\Sigma \cap \mathcal{H}$. Furthermore, since the coefficients g^{il} are smooth, the convergence also holds in the arbitrary metric.

□

Now we are able to prove

PROPOSITION 2.22. *Let Σ be an oriented immersed $\mathcal{C}^2_{\mathcal{H}}$ surface in a contact sub-Riemannian three-dimensional manifold $(M, g_{\mathcal{H}}, \omega)$. Then the first variation of the area induced by the vector field $U = f\nu_h + lZ + hT$, with $f, l, h \in \mathcal{C}^1_0(\Sigma - \Sigma_0)$, is*

$$\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) = -\int_\Sigma g(U, N)\, H d\Sigma.$$

PROOF. Due to the linearity of (2.19) respect to U we can compute separately the variations in the direction of Z, ν_h and T. By (2.19) the variation in the direction of ν_h becomes

$$\int_\Sigma f|N_h|g(\nabla_Z \nu_h, Z)d\Sigma.$$

The variation produced by T is (2.19)

$$\int_\Sigma \{-S(h) + hg(N,T)|N_h|g(\tau(Z),Z)\}d\Sigma = \int_\Sigma hg(N,T)g(\nabla_Z \nu_h, Z)d\Sigma,$$

because of

$$\begin{aligned}
&\int_\Sigma \{-S(h) + hg(N,T)|N_h|g(\tau(Z),Z)\}d\Sigma = \\
&\qquad = \lim_{j\to+\infty} \int_{\Sigma_j} \{-S_j(h) + hg(N_j,T)|(N_h)_j|g(\tau(Z_j),Z_j)\}d\Sigma_j \\
&\qquad = \lim_{j\to+\infty} \Big\{ \int_{\Sigma_j} hg(N_j,T)\, g(\nabla_{Z_j}\nu_h^j, Z_j)d\Sigma_j - \int_{\Sigma_j} \operatorname{div}_{\Sigma_j}(hg(N_j,T)^2 S_j)d\Sigma_j \Big\} \\
&\qquad = \int_\Sigma hg(N,T)\, g(\nabla_Z \nu_h, Z)d\Sigma,
\end{aligned}$$

where we have used the Riemannian divergence theorem in the last equality. In an analogous way

$$\begin{aligned}
&\int_\Sigma \{c_1 g(N,T)g(J(\nu_h),Z) + |N_h|Z(l)\}d\Sigma = \\
&\qquad = \lim_{j\to+\infty} \int_{\Sigma_j} \{c_1 g(N_j,T)g(J((\nu_h))_j, Z_j) + |(N_h)_j|Z_j(l)\}d\Sigma_j.
\end{aligned}$$

Now since $|N_h|Z(l) = Z(|N_h|l) - lZ(|N_h|)$, by Lemma 2.1 and Lemma 2.3 we get

$$\int_\Sigma \{c_1 g(N,T)g(J(\nu_h),Z) + |N_h|Z(l)\}d\Sigma = \lim_{j\to+\infty} \int_{\Sigma_j} \operatorname{div}_{\Sigma_j}(l|(N_h)_j|Z_j)d\Sigma_j = 0,$$

so we have proved that the variation produced by Z vanishes. Since $g(U,N) = f|N_h| + hg(N,T)$ we finally get

$$\frac{d}{ds}\Big|_{s=0} A(\varphi_s(\Sigma)) = -\int_\Sigma g(U,N)\, H d\Sigma.$$

□

REMARK 2.23. Proposition 2.22 holds also for a C^1 surface Σ in which ν_h (or equivalently Z) is C^1 in the Z-direction. It does not imply $C^2_{\mathcal{H}}$ regularity when $T\Sigma\cap\mathcal{H}$ has dimension one. We thank F. Serra Cassano for pointing out this fact.

2.6. Second variation formulas

In this section we will compute a second variation formula for a minimal surface considering variations in the direction of N and T in the regular part and variation induced by the Reeb vector field supported near the singular set of the surface. We restrict ourselves to the case of pseudo-hermitian manifolds. Consider the orthonormal basis $\{Z, \nu_h, T\}$. We can compute

$$\begin{aligned} &[Z, \nu_h] = c_1 T + \theta(Z)Z + \theta(\nu_h)\nu_h \\ &[Z, T] = g(\tau(Z), Z)Z + \{g(\tau(Z), \nu_h) + \theta(T)\}\nu_h \\ &[\nu_h, T] = \{g(\tau(Z), \nu_h) - \theta(T)\}Z + g(\tau(\nu_h), \nu_h)\nu_h \end{aligned} \tag{2.37}$$

where θ is defined in (2.16) and we have computed $\theta(T) = -1 + g(D_T \nu_h, Z)$ using (2.10).

LEMMA 2.24. *Let Σ be a C^2 immersed oriented surface with constant mean curvature H in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. We consider a point $p \in \Sigma - \Sigma_0$ and we denote by $\alpha : I \to \Sigma - \Sigma_0$ the integral curve of S_p. Then the results in Proposition 2.10 hold with $U_{\alpha(\varepsilon)} = Z_{\alpha(\varepsilon)}$. Furthermore in $\Sigma - \Sigma_0$, the normal vector N is $\mathcal{C}^\infty$ in the direction of the characteristic field Z.*

PROOF. From (i) in Proposition 2.10 and from (2.21) follows that V_ε and $\dot\gamma_\varepsilon$ are C^∞ along characteristic curves and we express the unit normal to Σ along γ_ε by

$$N = \pm \frac{\dot\gamma_\varepsilon \times V_\varepsilon}{|\dot\gamma_\varepsilon \times V_\varepsilon|},$$

where $\times$ denotes the cross product in (M, g). We conclude that N is C^∞ along γ_ε. □

2.6.1. Second variation in the regular set. Now we present a variation formula in the regular part of the surface induced by a vector field of the form $vN + wT$.

LEMMA 2.25. *Let Σ be a $\mathcal{C}^2$ surface of constant mean curvature H in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. Then we have*

$$g(R(T, Z)\nu_h, Z) = -\nu_h(g(\tau(Z), Z)) + Z(g(\tau(Z), \nu_h)) - 2\omega(\nu_h)g(\tau(Z), \nu_h) + 2Hg(\tau(Z), Z). \tag{2.38}$$

PROOF. By (2.10) it is not difficult to show (see [**37**, Theorem 1.6] for the case in which $c_1 = 2$)

$$g(R(T, Z)\nu_h, Z) = g(R^{LC}(\nu_h, Z)T, Z), \tag{2.39}$$

where R^{LC} is the curvature tensor with respect to the Levi-Civita connection, that can be easily computed as

$$\begin{aligned} g(R^{LC}(\nu_h, Z)T, Z) = &-\nu_h(g(\tau(Z), Z)) + Z(g(\tau(Z), \nu_h)) \\ &- 2\omega(\nu_h)g(\tau(Z), \nu_h) + 2Hg(\tau(Z), Z), \end{aligned}$$

take in account (2.10) and (2.6). □

THEOREM 2.26. *Let Σ be a $\mathcal{C}^2$ minimal surface in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$, with singular set Σ_0. We consider a $\mathcal{C}^1$ vector field $U = vN + wT$, where N is the unit normal vector to Σ and $w, v \in \mathcal{C}^1_0(\Sigma - \Sigma_0)$. Then the second derivative of the area for the variation induced by U is given by*

$$A''(0) = \int_\Sigma \{|N_h|^{-1} Z(u)^2 + u^2 q\} d\Sigma + \int_\Sigma div_\Sigma(\xi Z + \zeta Z + \eta S) d\Sigma, \tag{2.40}$$

with

$$\xi = g(N,T)\{|N_h|\theta(S) + c_1 g(N,T)^2 + (1+g(N,T)^2)g(\tau(Z),\nu_h)\}u^2,$$
$$\zeta = |N_h|^2|\{g(N,T)(|N_h|\theta(S) + c_1 g(N,T)^2 + (1+g(N,T)^2)g(\tau(Z),\nu_h))w^2 - 2g(B(Z),S)vw\},$$
$$\eta = (|N_h|^2 v^2 - (g(N,T)v + w)^2)g(\tau(Z),Z)$$

and

$$q = |N_h|\{-W + c_1^2 + c_1 g(\tau(Z),\nu_h)\} - |N_h|(|N_h|(c_1 + g(\tau(Z),\nu_h)) - \theta(S))^2 + g(N,T)g(R(Z,T)\nu_h, Z) - g(N,T)Z(g(\tau(Z),\nu_h)),$$

where $u = g(U,N)$, R is the pseudo-hermitian curvature tensor and B is the Riemannian shape operator.

Remark 2.27. The second variation formula (2.40) involves many geometrical quantities, depending on the ambient manifold and the surface. We stress that the divergence terms vanish when the surface has empty singular set. The only term that is not adapted to the sub-Riemannian setting is $g(B(Z),S)$, which we have kept for simplicity. Its expression is given in Lemma 2.1. We also remark that (2.40) will be useful to obtain the stability operators in Section 2.7.

Remark 2.28. If Σ is area stationary without boundary, then

$$\int_\Sigma div_\Sigma(fS) = 0,$$

for every $f \in \mathcal{C}_0^1(\Sigma)$, by Corollary 2.15.

Proof. (of Theorem 2.26) We can reason as in the proof of the first variation formula. We have

$$U(|V|) = \frac{1}{|V|}g(\nabla_U V, V)$$

and

$$U(U(|V|)) = -\frac{1}{|V|^3}g(\nabla_U V, V)^2 + \frac{1}{|V|}\left(g(\nabla_U\nabla_U V, V) + |\nabla_U V|^2\right).$$

We fix

$$\lambda := g(\nabla_U V, V),$$

so we get

$$U(U(|V|)) = \frac{1}{|V|}\left\{\underbrace{g(\nabla_U\nabla_U V, V)}_{II} + \underbrace{g(\nabla_U V, \nabla_U V - \frac{\lambda}{|V|^2}V)}_{I}\right\}. \tag{2.41}$$

As $V = g(E_1,T)E_2 - g(E_2,T)E_1$ we compute

$$\nabla_U V(0) = U(g(E_1,T))S - U(g(T,E_2))Z + |N_h|\nabla_U E_1. \tag{2.42}$$

Observing $g(E_1,T) = 0$ and

$$-\frac{\lambda V(0)}{|V(0)|^2} = -g(\nabla_U V, Z)Z,$$

we have

$$\nabla_U V - \frac{\lambda V(0)}{|V(0)|^2} = U(g(E_1,T))S + |N_h|g(\nabla_U E_1, S)S + |N_h|g(\nabla_U E_1, N)N. \tag{2.43}$$

By $|N_h|S = g(N,T)N - T$ and $|N_h|N + g(N,T)S = \nu_h$ we obtain

$$I = g(\nabla_U E_1, N)^2.$$

Using (i) in Lemma 2.1, $T = g(N,T)N - |N_h|S$ and $\nu_h = g(N,T)S + |N_h|N$ we have

$$I = \{Z(g(U,N) + g(N,T)|N_h|g(U,N)(c_1 + g(\tau(Z),\nu_h)) - g(U,S)|N_h|\theta(S)\}^2 \tag{2.44}$$

and

$$\begin{aligned}|N_h|^{-1}I = {} & |N_h|^{-1}Z(u)^2 + 2g(N,T)uZ(u)(c_1 + g(\tau(Z),\nu_h)) \\ & + g(N,T)^2|N_h|u^2(c_1 + g(\tau(Z),\nu_h))^2 + 2Z(u)|N_h|w\theta(S) \\ & + 2g(N,T)|N_h|^2uw\theta(S)(c_1 + g(\tau(Z),\nu_h)) + w^2|N_h|^3\theta(S)^2,\end{aligned} \tag{I}$$

where $u = g(U,N)$.

Now we consider

$$\begin{aligned}|N_h|^{-1}II = {} & \underbrace{g(R(V,U)U,Z)}_{A} + \underbrace{g(\nabla_V\nabla_U U,Z)}_{B} + \underbrace{g(\nabla_{[U,V]}U,Z) + g(\nabla_U[U,V],Z)}_{C} \\ & + g(\underbrace{\nabla_U Tor_\nabla(U,V),Z)}_{D},\end{aligned}$$

as

$$g(\nabla_U\nabla_V U,V) = g(R(V,U)U,V) + g(\nabla_V\nabla_U U,V) + g(\nabla_{[U,V]}U,V).$$

By equation (2.10) we obtain

$$\nabla_U U = -g(U,\nu_h)^2 g(\tau(Z),Z)T - g(U,T)g(U,\nu_h)(c_1 Z + \tau(\nu_h)), \tag{2.45}$$

furthermore by (i) in Lemma 2.1 and $D_U U = 0$ we have

$$\begin{aligned}B = & - Z(v(g(N,T)v + w)|N_h|^2(c_1 g(\tau(Z),\nu_h))) \\ & - g(N,T)Z(g(N,T))v(g(N,T)v + w)(c_1 g(\tau(Z),\nu_h))\end{aligned}$$

and for the linearity of R we get

$$A = |N_h|^2 v(R(Z,U)\nu_h, Z).$$

On the other hand as $[U,V] = U(g(E_1,T))E_2 - U(g(E_2,T))E_1$ and Σ stationary we have

$$\begin{aligned}C = {} & (Z(g(U,T)) + c_1 g(U,\nu_h))(2g(U,\nu_h)\theta(S) \\ & - (g(N,T)g(U,T) - |N_h|g(U,\nu_h))g(\tau(Z),\nu_h)) \\ & - S(g(U,T))g(U,T)g(\tau(Z),Z) - U(U(g(T,E_2)))\end{aligned}$$

and writing $-U(U(g(T,E_2))$ as

$$-S(g(\nabla_U U,T)) + c_1 g(N,T)g(\nabla_U U,Z) - c_1 g(\nabla_S U, J(U)) - c_1 g(Tor_\nabla(U,S),J(U))$$

that is

$$\begin{aligned}& - S(g(U,\nu_h)^2 g(\tau(Z),Z)) - c_1 g(N,T)g(U,T)g(U,\nu_h)(2g(\tau(Z),\nu_h)) \\ & + c_1 g(U,\nu_h)(g(N,T)g(U,T) - |N_h|g(U,\nu_h))g(\tau(Z),\nu_h) - c_1 g(U,\nu_h)^2\theta(S),\end{aligned}$$

we get that C equals

$$\begin{aligned}& Z(g(N,T)v + w)ug(\tau(Z),\nu_h) + c_1|N_h|^2 v^2\theta(S) \\ & - c_1 g(N,T)|N_h|(g(N,T)v + w)v(2g(\tau(Z),\nu_h)) \\ & + 2Z(g(N,T)v + w)|N_h|v\theta(S) + S(g(U,\nu_h)^2 g(\tau(Z),Z)) \\ & - S(g(U,T))g(U,T)g(\tau(Z),Z).\end{aligned}$$

Now $V \in \mathcal{H}$ implies $D = -|N_h| g(U,\nu_h)^2 (g(\tau(Z),Z))^2 + g(U,T) g(\nabla_U \tau(V), Z)$. On the other hand

$$g(\nabla_U \tau(V), Z) = g(\nabla_U \tau(V) - \nabla_V U, Z) + g(\nabla_V \tau(U), Z),$$

where $g(\nabla_V \tau(U), Z) = Z(|N_h| g(U,\nu_h) g(\tau(Z),\nu_h)) - Z(|N_h|) g(U,\nu_h) g(\tau(Z),\nu_h)$ and

$$g(\nabla_U \tau(V) - \nabla_V \tau(U), Z) = g((\nabla_U \tau)V - (\nabla_V \tau)U, Z) + g(\tau(Z), [U,V] + Tor(U,V)).$$

We have that $g((\nabla_U \tau)V - (\nabla_V \tau)U, Z)$ is equal to

$$|N_h| g(U,\nu_h) g((\nabla_{\nu_h}\tau)Z - (\nabla_Z \tau)\nu_h, Z) + g(U,T)|N_h| g((\nabla_T \tau)Z, Z)$$

and by Theorem 1.6 in [**37**] and the fact that $(\nabla_X \tau)Y$ is a tensor we obtain

$$\begin{aligned} g((\nabla_U \tau)V - (\nabla_V \tau)U, Z) = &-|N_h| g(U,\nu_h) g(R(T,Z)\nu_h, Z) \\ &+ |N_h| g(U,T)(T(g(\tau(Z),Z)) + 2g(\tau(Z),\nu_h)\omega(T)). \end{aligned}$$

Finally since $g(Tor(U,V),\tau(Z)) = g(U,T)|N_h|(g(\tau(Z),\nu_h)^2 + (g(\tau(Z),Z))^2)$ and writing $g(\tau(Z),[U,V])$ as

$$- S(g(U,T)) g(U,T) g(\tau(Z),Z) + g(N,T) g(\tau(Z),\nu_h)(Z(g(U,T)) + c_1 g(U,\nu_h))$$

together with (i) in Lemma 2.1 we obtain that D equals

$$\begin{aligned} &- |N_h| g(U,\nu_h)^2 (g(\tau(Z),Z))^2 + |N_h|(g(N,T)v + w)^2 (T(g(\tau(Z),Z)) + g(\tau(Z),\nu_h)^2 \\ &+ (g(\tau(Z),Z))^2) - |N_h|^2 (g(N,T)v + w) v g(R(T,Z)\nu_h, Z) + 2\omega(T) g(\tau(Z),\nu_h) \\ &+ |N_h|(g(N,T)v + w) Z(|N_h| v g(\tau(Z),\nu_h)) + c_1 g(N,T)|N_h| v (g(N,T)v + w) g(\tau(Z),\nu_h) \\ &- S(g(U,T)) g(U,T) g(\tau(Z),Z) + g(N,T) Z(g(N,T)v + w)(g(N,T)v + w) g(\tau(Z),\nu_h). \end{aligned}$$

The sum of all terms that contain $g(\tau(Z),Z)$, after have used lemma 2.3 is

$$div_\Sigma((|N_h|^2 v^2 - (g(N,T)v+w)^2) g(\tau(Z),Z) S) + g(N,T)(g(N,T)v+w)^2 \nu_h(g(\tau(Z),Z)).$$

By the definition of θ we have

$$\begin{aligned} Z(\theta(S)) &= g(R(S,Z)\nu_h, Z) + \theta([Z,S]) \\ &= g(R(S,Z)\nu_h, Z) + g(N,T)\theta(S)^2 - g(N,T)|N_h|\theta(S)(c_1 + g(\tau(Z),\nu_h)), \end{aligned} \tag{2.46}$$

where we used that $[Z,S]$ is tangent to Σ. We note that since $\nabla_Z \nu_h \equiv 0$ (2.46) make sense when Σ is of class C^2. Furthermore by Lemma 2.3 and equation (2.46) we have that $2Z(u)|N_h| w\theta(S) + 2Z(g(N,T)v + w)|N_h| v\theta(S)$ is equal to

$$\begin{aligned} &div_\Sigma(g(N,T)|N_h|(v^2 + w^2)\theta(S)Z) + \frac{Z(g(N,T))}{|N_h|}\theta(S)(v^2 + w^2) \\ &+2div_\Sigma(|N_h| vw\theta(S)Z) - 2vwZ(|N_h|)\theta(S) - 2vw|N_h| g(R(S,Z)\nu_h, Z) \\ &- g(N,T)|N_h|(v^2 + w^2) g(R(S,Z)\nu_h, Z) \end{aligned}$$

and similary B equals

$$\begin{aligned} &c_1 g(N,T)|N_h| v (g(N,T)v + w)(c_1 g(\tau(Z),\nu_h)) \\ &-div_\Sigma(|N_h|^2 v (g(N,T)v + w)(c_1 g(\tau(Z),\nu_h))Z). \end{aligned}$$

In the same way $g(N,T)Z(u^2)(g(\tau(Z),\nu_h)) + g(N,T)^2 |N_h| u^2 (c_1 + g(\tau(Z),\nu_h))^2$ can be expressed as

$$\begin{aligned} &div_\Sigma(g(N,T) u^2 (g(\tau(Z),\nu_h))Z) - g(N,T) u^2 Z(g(\tau(Z),\nu_h)) + u^2 |N_h| c_1 (c_1 + g(\tau(Z),\nu_h)) \\ &+ u^2 (g(\tau(Z),\nu_h))\theta(S) - u^2 |N_h|^3 (g(\tau(Z),\nu_h))^2. \end{aligned}$$

Furthermore $-Z(g(N,T)v+w)(u+g(N,T)^2v-g(N,T)w)+|N_h|(g(N,T)v+w)Z(|N_h|vg(\tau(Z),\nu_h))$ become

$$\begin{aligned}&div_\Sigma(g(\tau(Z),\nu_h)(g(N,T)v+w)(v+g(N,T)w)Z)\\&+g(N,T)(-\theta(S)-|N_h|(2g(\tau(Z),\nu_h)))g(\tau(Z),\nu_h)(g(N,T)v+w)(v+g(N,T)w)\\&-g(N,T)(g(N,T)v+w)^2Z(g(\tau(Z),\nu_h))-Z(g(N,T))g(\tau(Z),\nu_h)w(g(N,T)v+w)\end{aligned}$$

and all the other terms not considered are

$$|N_h|^{-1}Z(u)^2+|N_h|^3v^2g(R(Z,\nu_h)\nu_h,Z)-2|N_h|^2v(g(N,T)v+w)g(R(T,Z)\nu_h,Z)$$

and

$$\begin{aligned}&2g(N,T)|N_h|^2uw\theta(S)(2g(\tau(Z),\nu_h))+|N_h|^3\theta(S)^2w^2+2|N_h|^2v^2\theta(S)\\&-4g(N,T)|N_h|v(g(N,T)v+w)-|N_h|(g(N,T)v+w)^2(g(\tau(Z),\nu_h)^2\\&-2\omega(T)g(\tau(Z),\nu_h)).\end{aligned}$$

Since $g(R(Z,\nu_h)\nu_h,Z)=-W$ we have that the terms in which the curvature tensor appears are equal to

$$-|N_h|Wu^2+g(N,T)|N_h|^2(w^2-v^2)g(R(T,Z)\nu_h,Z) \tag{2.47}$$

and by equation (2.38) we have (2.47) sums with $g(N,T)(g(N,T)v+w)^2(\nu_h(g(\tau(Z),Z))+2\omega(\nu_h)g(\tau(Z),\nu_h)-Z(g(\tau(Z),\nu_h)))$ is

$$-|N_h|Wu^2-g(N,T)u^2g(R(T,Z)\nu_h,Z).$$

Finally a long but standard computation shows that the remaining terms add up to

$$\begin{aligned}&c_1u^2|N_h|(c_1+g(\tau(Z),\nu_h))-|N_h|(|N_h|(c_1+g(\tau(Z),\nu_h))\\&-\theta(S))^2u^2-g(N,T)u^2Z(g(\tau(Z),\nu_h))\\&+g(N,T)(g(N,T)v+w)^2(\nu_h(g(\tau(Z),Z))+2\omega(\nu_h)g(\tau(Z),\nu_h)-Z(g(\tau(Z),\nu_h))).\end{aligned}$$

Since that $u^2=v^2+2g(N,T)vw+g(N,T)^2w^2$, we get the statement.

□

REMARK 2.29. If we suppose that our variation is not produced from Riemannian geodesics, i.e., we remove the hypothesis $D_UU=0$, we get the additional term

$$-\int_\Sigma div_\Sigma(|N_h|g(D_UU,Z)Z)d\Sigma+\int_\Sigma div_\Sigma(g(D_UU,T)S)d\Sigma. \tag{2.48}$$

It is worth pointing out that (2.48) vanishes when the variation functions w,v have support in the regular set.

PROOF. From the term $-U(U(g(E_2,T)))$ we have $S(g(D_UU,T))-c_1g(N,T)g(D_UU,Z)$. Furthermore

$$|N_h|g(\nabla_ZD_UU,Z)=-Z(|N_h|g(D_UU,Z))+g(D_UU,Z)Z(|N_h|)$$

and $|N_h|^{-1}g(D_UU,T)g(\tau(V),V)=g(D_UU,T)|N_h|g(\tau(Z),Z)$. By Lemma 2.1 we have

$$Z(|N_h|)=g(N,T)(|N_h|\theta(S)+c_1-|N_h|^2(2g(\tau(Z),\nu_h)))$$

and, as no others terms are involved, we conclude applying Lemma 2.3. □

2.6.2. Second variation moving the singular set. By Theorem 2.14 the singular set of a C^2 surface is composed of singular curves and isolated singular points without accumulation points. First we present a second variation formula induced by a vertical variation near a singular curve, i.e., in a tubular neighborhood of radius $\varepsilon > 0$ of the singular curve that is the union of all the characteristic curves centered at $(\Sigma_0)_c$ defined in the interval $[-\varepsilon, \varepsilon]$.

LEMMA 2.30. *Let Σ be a complete $\mathcal{C}^2$ area-stationary surface immersed in M, with a singular curve Γ of class $\mathcal{C}^3$. Let $w \in \mathcal{C}_0^2(\Sigma)$. We consider the variation of Σ given by $p \to \exp_p(rw(p)T_p)$. Let U be a tubular neighborhood of $supp(w) \cap \Gamma$, and assume that w is constant along the characteristic curves in U. Then there is a tubular neighborhood $U' \subset U$ of $supp(w) \cap \Gamma$ so that*

$$\begin{aligned} A''(\varphi_r(U')) = \frac{d^2}{dr^2}A(\varphi_r(U')) &= \int_\Sigma \{2w^2|N_h|(g(\tau(Z),\nu_h)^2 + g(\tau(Z),Z)^2)\}d\Sigma \\ &+ \int_\Sigma div_\Sigma(w^2 g(\tau(Z),Z)S)d\Sigma + \int_\Gamma S(w)^2 d\Gamma. \end{aligned} \tag{2.49}$$

PROOF. We consider the singular curve Γ parametrized by arc-length with variable ε. By Theorem 2.15 we can parametrize Σ in a neighborhood of $supp(w) \cap \Gamma$ by (s, ε), so that the curves with ε constant are the characteristic curves of Σ. As E_i are Jacobi-like vector fields it is easy to prove that $g(E_i, T)'' = 0$, so that $g(E_i, T) = g(E_i, T)'(0)r + g(E_i, T)(0)$ and, in particular, we have $g(E_1, T) = 0$. This means that $|V(r)| = |g(E_2, T)||E_1| = |F(p, s, r)||E_1|$ which vanishes if and only if $F(p, s, r) = 0$. As

$$\frac{\partial F(p,0,0)}{\partial s} = -Z(|N_h|) = \frac{g(N,T)}{N_h} Z(g(N,T)) = -c_1,$$

we can apply the implicit function theorem i.e., there exists $s(\varepsilon, r)$ such that the curve $F(p, s(\varepsilon, r), r) = 0$ is a graph on $\Gamma(\varepsilon)$. We have obtained

$$\begin{aligned} A(\varphi_r(U')) &= \int_{-s_0}^{s_0}\int_{-\varepsilon_0}^{\varepsilon_0} |F(p,s,r)||E_1|ds d\varepsilon \\ &= \int_{-\varepsilon_0}^{\varepsilon_0} \left\{ \int_{-s_0}^{s(\varepsilon,r)} F(p,s,r)|E_1|ds - \int_{s(\varepsilon,r)}^{s_0} F(p,s,r)|E_1|ds \right\} d\varepsilon = \int_{-\varepsilon_0}^{\varepsilon_0} f_\varepsilon(r)d\varepsilon \end{aligned}$$

and

$$\begin{aligned} f''_\varepsilon(r) = 2\frac{\partial F(p,s(\varepsilon,r),r)}{\partial r}|E_1|\frac{\partial s(\varepsilon,r)}{\partial r} + \int_{-s_0}^{s(\varepsilon,r)} (2\frac{\partial F(p,s,r)}{\partial r}\frac{\partial |E_1|}{\partial r} + F(p.s,r)\frac{\partial^2 |E_1|}{\partial r^2})ds \\ - \int_{s(\varepsilon,r)}^{s_0} \Big(2\frac{\partial F(p,s,r)}{\partial r}\frac{\partial |E_1|}{\partial r} + F(p,s,r)\frac{\partial^2 |E_1|}{\partial r^2}\Big)ds \\ = \dot{w}(\varepsilon)^2 + \int_{-s_0}^{s_0} 2S(w)U(|E_1|) + |N_h|U(U(|E_1|))ds \end{aligned}$$

we have that the second variation formula becomes

$$\begin{aligned} A''(\varphi_r(U')) &= \int_{(\Sigma_0)_c} S(w)^2 + \int_{\Sigma} \{2S(w)U(|E_1|) + |N_h|U(U(|E_1|))\} d\Sigma \\ &= \int_{\Sigma} \{2wS(w)g(\tau(Z),Z) + w^2|N_h|(2g(\tau(Z),\nu_h)^2 + g(\tau(Z),Z)^2)\} d\Sigma \\ &+ \int_{(\Sigma_0)_c} S(w)^2 \\ &= \int_{\Sigma} \{2w^2|N_h|(g(\tau(Z),\nu_h)^2 + g(\tau(Z),Z)^2)\} d\Sigma \\ &+ \int_{\Sigma} div_{\Sigma}(w^2 g(\tau(Z),Z)S) d\Sigma \\ &+ \int_{(\Sigma_0)_c} S(w)^2, \end{aligned}$$

where we have used lemma 2.3 and

$$\begin{aligned} U(U(|E_1|)) &= g(\nabla_U\nabla_U E_1, E_1) + g(\nabla_U E_1, \nabla_U E_1 - (g(\nabla_U E_1, E_1)/|E_1|^2)E_1) \\ &= wg(\nabla_U \tau(E_1), E_1) + g(\nabla_U E_1, N)^2 + g(\nabla_U E_1, E_2)^2 \\ &= Z(w)^2 + w^2\{2g(\tau(Z),\nu_h)^2 + g(\tau(Z),Z)^2\} \end{aligned}$$

as

$$g(\nabla_U \tau(E_1), E_1) = w(g(\tau(Z),Z)^2 + g(\tau(Z),\nu_h)^2)$$

because of [**37**, equation 1.77].

□

REMARK 2.31. Formula (2.49) looks different from (2.40). For example τ is involved, but not W. This depends on the fact that the surface is moved only in the direction of the Reeb vector field. The curvature tensor that appears is $g(R(T,e_1)T,e_1)$, which vanishes.

REMARK 2.32. The hypothesis $\Gamma \in C^3$ is purely technical. We only need it when we apply the implicit function theorem and it can be weakened. On the other hand in all examples in our knowledge singular curves in area-stationary surfaces are C^∞.

Finally we consider variations which are constant in a neighborhood of the singular set. This hypothesis is reasonable when we move the surface close to isolated singular points, otherwise the second variation blows-up. By a tubular neighborhood of a singular point q we mean the union of all the characteristic segments of length ε going into q or coming out from q.

LEMMA 2.33. *Let Σ be a complete $\mathcal{C}^2$ area-stationary surface immersed in M with an isolated singular point p_0. Let $w \in \mathcal{C}^2_0(\Sigma)$. We consider the variation of Σ given by $p \to \exp_p(rw(p)T_p)$. Let U be a tubular neighborhood of p_0 and assume that w is constant near p_0. Then there is a tubular neighborhood $U' \subset U$ of p_0 so that*

$$A''(\varphi_r(U')) = \int_{\Sigma} \{2w^2|N_h|(g(\tau(Z),\nu_h)^2 + g(\tau(Z),Z)^2)\} d\Sigma.$$

The proof is analogous to the previous one using variations moving singular curves. We note that in this case $F(p,s,r) = 0$ if and only if p is equal to the original singular point p_0 and the statement follows.

REMARK 2.34. We note that Theorem 2.26 coincides with [**55**, Theorem 3.7] in the spacial case of the Heisenberg group $\mathbb{H}^1$ and with [**82**, Theorem 5.2] for three-dimensional Sasakian sub-Riemannian manifolds. It can be easily seen by (iv) in Lemma 2.1. Furthermore we will see in the next section that (2.40) generalizes the second variation formula in [**25**].

2.7. Two stability operators.

The first stability operator which we present gives a criterion for instability in the regular set of a surface. It is the counterpart of the Riemannian one.

PROPOSITION 2.35. *Let Σ be a $\mathcal{C}^2$ immersed surface with unit normal vector N and singular set Σ_0 in a pseudo-hermitian manifold $(M, g_{\mathcal{H}}, \omega, J)$. Consider two functions $u \in \mathcal{C}_0(\Sigma - \Sigma_0)$ and $v \in \mathcal{C}_0(\Sigma - \Sigma_0)$ which are $\mathcal{C}^1$ and $\mathcal{C}^2$ in the Z-direction respectively. If Σ is stable then the index form*

$$\mathcal{I}(u,v) := \int_\Sigma \{|N_h|^{-1} Z(u)Z(v) + quv\} d\Sigma = -\int_\Sigma u\mathcal{L}(v) d\Sigma \geqslant 0$$

where $\mathcal{L}$ is the following second order differential operator

$$\begin{aligned}\mathcal{L}(v) :=& |N_h|^{-1}\{Z(Z(v)) + |N_h|^{-1} g(N,T)(-2|N_h|\theta(S) - c_1 \\ &+ 2|N_h|^2(c_1 + g(\tau(Z),\nu_h)))Z(v) - q|N_h|v\},\end{aligned} \tag{2.50}$$

with q defined in Theorem 2.26.

PROOF. Following [**55**, Proposition 3.14] we prove that $\mathcal{L}(v) = div_\Sigma(|N_h|^{-1}Z(v)Z) + |N_h|^{-1}q$. In fact

$$div_\Sigma(|N_h|^{-1}Z(v)Z) = Z(|N_h|^{-1}Z(v)) + |N_h|^{-1}Z(v) div_\Sigma Z.$$

So by Lemma 2.1 we have

$$\begin{aligned}Z(|N_h|^{-1}Z(v)) =& |N_h|^{-1}Z(Z(v)) \\ &+ |N_h|^{-2} g(N,T)(-|N_h|\theta(S) - c_1 + |N_h|^2(c_1 + g(\tau(Z),\nu_h)))Z(v)\end{aligned}$$

and $div_\Sigma(Z) = -g(N,T)\theta(S) + g(N,T)|N_h|(c_1 + g(\tau(Z),\nu_h))$.

Finally it is sufficient to observe

$$\begin{aligned}0 =& \int_\Sigma div_\Sigma(|N_h|^{-1}Z(v)uZ) d\Sigma = \int_\Sigma u\, div_\Sigma(|N_h|^{-1}Z(v)Z) d\Sigma + \int_\Sigma |N_h|^{-1}Z(v)Z(u) d\Sigma \\ =& \int_\Sigma u\mathcal{L}(v) d\Sigma + \mathcal{I}(u,v).\end{aligned}$$

Really we need $u, v \in \mathcal{C}_0^1(\Sigma - \Sigma_0)$, but this condition can be weakened with an approximation argument, as in [**55**, Proposition 3.2].

□

Now we present the analogues of [**55**, Lemma 3.17] and [**55**, Lemma 4.1], which are a sort of integration by parts and a useful stability operator for non-singular surfaces respectively. The proofs of the following Lemmas are straightforward generalizations of the Heisenberg case.

LEMMA 2.36. *Let Σ be a $\mathcal{C}^2$ immersed surface in a pseudo-hermitian 3-manifolds $(M, g_{\mathcal{H}}, \omega, J)$, with unit normal vector N and singular set Σ_0. Consider two functions $u \in \mathcal{C}_0(\Sigma - \Sigma_0)$ and $v \in \mathcal{C}(\Sigma - \Sigma_0)$ which are $\mathcal{C}^1$ and $\mathcal{C}^2$ in the Z-direction respectively. Then we have*

$$\int_\Sigma |N_h|\{Z(u)Z(v) + uZ(Z(v)) + c_1|N_h|^{-1}g(N,T)uZ(v)\}d\Sigma = 0.$$

LEMMA 2.37. *Let Σ be a $\mathcal{C}^2$ immersed minimal surface in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$, with unit normal vector N and singular set Σ_0. For any function $u \in \mathcal{C}_0(\Sigma - \Sigma_0)$ which is also $\mathcal{C}^1$ in the Z-direction we have*

$$\mathcal{I}(u|N_h|, u|N_h|) = \int_\Sigma |N_h|\{Z(u)^2 - \mathcal{L}(|N_h|)u^2\}d\Sigma.$$

Now it is interesting to compute $\mathcal{L}(|N_h|)$.

LEMMA 2.38. *Let Σ be a $\mathcal{C}^2$ immersed minimal surface in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. Then*

$$\begin{aligned}\mathcal{L}(|N_h|) = W &+ c_1 g(\tau(Z), \nu_h) - 2c_1|N_h|^{-2}(|N_h|\theta(S) \\ &- |N_h|^2 g(\tau(Z), \nu_h)) - c_1^2|N_h|^{-2}g(N,T)^2.\end{aligned}$$

PROOF. (of Lemma 2.38) By (v) in Lemma 2.1 we have that

$$Z(|N_h|) = g(N,T)(|N_h|\theta(S) + c_1 - |N_h|^2(c_1 + g(\tau(Z), \nu_h)))$$

and so

$$\begin{aligned}Z(Z(|N_h|)) = &- |N_h|(|N_h|\theta(S) + c_1 - |N_h|^2(c_1 + g(\tau(Z), \nu_h)))^2 \\ &+ Z(|N_h|)(g(N,T)\theta(S) - 2g(N,T)|N_h|(c_1 + g(\tau(Z), \nu_h))) \\ &+ g(N,T)|N_h|Z(\theta(S)) - g(N,T)|N_h|^2 Z(g(\tau(Z), \nu_h)).\end{aligned}$$

Now we observe that

$$\begin{aligned}-g(N,T)(c_1|N_h|^{-1} + \theta(S))Z(|N_h|) = Z(|N_h|)&(g(N,T)\theta(S) \\ &- g(N,T)|N_h|(c_1 + g(\tau(Z), \nu_h))) \\ &+ \frac{g(N,T)}{|N_h|}(-2|N_h|\theta(S) - c_1 \\ &+ |N_h|^2(c_1 + g(\tau(Z), \nu_h)))Z(|N_h|)\end{aligned}$$

and

$$\begin{aligned}-|N_h|(|N_h|\theta(S) &+ c_1 - |N_h|^2(c_1 + g(\tau(Z), \nu_h)))^2 = \\ &= -|N_h|^3(\theta(S) - |N_h|(c_1 + g(\tau(Z), \nu_h)))^2 \\ &- c_1^2|N_h| - 2c_1|N_h|^2(\theta(S) - |N_h|(c_1 + g(\tau(Z), \nu_h))).\end{aligned}$$

Now it is sufficient to substitute, use Lemma 2.1 (v), and (2.46) to obtain the required formula.

□

REMARK 2.39. By Lemma 2.38 it is easy to prove that our formula coincides with the one in [**25**] in the special case of $\mathcal{C}^3$ surfaces, as the authors obtained that formula by deriving the mean curvature. By

$$Z\left(\frac{g(N,T)}{|N_h|}\right) = |N_h|^{-3}Z(g(N,T))$$

and (v) in Lemma 2.1 we have

$$\mathcal{L}(|N_h|) = W - c_1 g(\tau(Z),\nu_h) + 2c_1 Z\Big(\frac{g(N,T)}{|N_h|}\Big) + c_1^2 \frac{g(N,T)^2}{|N_h|^2}. \tag{2.51}$$

Equation (2.51) gives an easy criterion for the stability of *vertical surfaces*, which are the surfaces for which $g(N,T)\equiv 0$ holds. In the Heisenberg group these surfaces are vertical planes and their stability was first proved in [**32**].

We conclude this section by pasting the variations in the regular and in the singular set, to obtain a stability operator in the spirit of [**55**, Proposition 4.11]. By a tubular neighborhood of $(\Sigma_0)_c \cap supp(u)$ we mean the union of the tubular neighborhood of each singular curve and each singular point intersected with $supp(u)$. We are interested in a finite number of singular curves and singular points, since u is compactly supported, Theorem 2.14.

THEOREM 2.40. *Let Σ be a $\mathcal{C}^2$ oriented minimal surface immersed in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$, with singular set Σ_0 and $\partial\Sigma = \emptyset$. If Σ is stable then, for any function $u \in \mathcal{C}_0^1(\Sigma)$ such that $Z(u) = 0$ in a tubular neighborhood of a singular curve and constant in a tubular neighborhood of an isolated singular point, we have $\mathcal{Q}(u) \geqslant 0$, where*

$$\mathcal{Q}(u) := \int_{\Sigma} \{|N_h|^{-1} Z(u)^2 + qu^2\} d\Sigma + 2\int_{(\Sigma_0)_c} (\xi+\zeta) g(Z,\nu) u^2 d(\Sigma_0)_c + \int_{(\Sigma_0)_c} S(u)^2 d(\Sigma_0)_c.$$

Here $d(\Sigma_0)_c$ is the Riemannian length measure on $(\Sigma_0)_c$, ν is the external unit normal to $(\Sigma_0)_c$ and q, ξ, ζ are defined in Theorem 2.26.

PROOF. First we observe that $\mathcal{Q}(u)$ is well defined for any $u \in \mathcal{C}_0(\Sigma)$, which is piecewise $\mathcal{C}^1$ in the Z-direction and $\mathcal{C}^1$ when restricted to Σ_0. First we prove

$$\begin{aligned} &\mathcal{Q}(v) \geqslant 0, \text{ for any } v \in \mathcal{C}_0^1(\Sigma) \text{ such that } Z(v/g(N,T)) = 0 \text{ in a small tubular} \\ &\qquad\qquad \text{neighborhood } E \text{ of } (\Sigma_0)_c. \end{aligned} \tag{2.52}$$

Here we denote $\Sigma_0 \cap supp(u)$ by $(\Sigma_0)_c$. Clearly the last hypothesis implies $|N_h|^{-1}Z(u)^2 \in \mathcal{L}^1(\Sigma)$. Denoting by σ_0 the radius of E and by K the support of v, respectively, we let E_σ be the tubular neighborhood of $(\Sigma_0)_c$ of radius $\sigma \in (0, \sigma_0/2)$ and let h_σ, g_σ be $\mathcal{C}_0^\infty(\Sigma)$ functions such that $g_\sigma = 1$ on $K \cap \overline{E_\sigma}$, $supp(g_\sigma) \subset E_{2\sigma}$ and $h_\sigma + g_\sigma = 1$ on K. Finally we define

$$U_\sigma = (h_\sigma v) N + g_\sigma \frac{v}{g(N,T)} T. \tag{2.53}$$

Observe that $supp(U_\sigma) \subset K$ and $g(U_\sigma, N) = v$ on K. Now we define a variation $\varphi_r^\sigma(p) = exp_p(r(U_\sigma)_p)$ and the area functional $A_\sigma(r) = A(\varphi_r^\sigma(\Sigma))$. As this variation is vertical when restricted to E_σ we have that $A''(\varphi_r^\sigma(E_\sigma))$ is given by

$$A''(\varphi_0^\sigma(E_\sigma)) = \int_{E_\sigma} \{2v^2 |N_h| (g(\tau(Z),\nu_h)^2 + g(\tau(Z),Z)^2)\} d\Sigma + \int_{(\Sigma_0)_c} S(v)^2 d\Sigma_0$$

and by Theorem 2.26 we have

$$A''(\varphi_0^\sigma(\Sigma - E_\sigma)) = \int_{\Sigma - E_\sigma} \{|N_h|^{-1} Z(u)^2 + u^2 q\} d\Sigma + \int_{\Sigma - E_\sigma} div_\Sigma(\xi Z + \zeta Z + \eta S) d\Sigma.$$

If Σ is stable then $A''(0) \geqslant 0$, so using the Riemannian divergence theorem we have

$$\int_{E_\sigma} \{2v^2|N_h|(g(\tau(Z),\nu_h)^2 + (g(\tau(Z),Z))^2)\}d\Sigma + \int_{(\Sigma_0)_c} S(v)^2 d\Sigma_0$$
$$+ \int_{\Sigma - E_\sigma} \{|N_h|^{-1}Z(u)^2 + u^2 q\}d\Sigma + 2\int_{\partial E_\sigma} (\xi Z + \zeta Z)g(Z,\nu)dl \geqslant 0,$$

where ν is the unit normal pointing into E_σ and dl denote the Riemannian length element. Letting $\sigma \to 0$, by the dominated convergence theorem we have proved condition (2.52).

Now we suppose $u \in \mathcal{C}_0^1(\Sigma)$ with $Z(u) = 0$ in a tubular neighborhood E of $(\Sigma_0)_c$. Then for any $\sigma \in (0,1)$ let D_σ be the open neighborhood of $(\Sigma_0)_c$ such that $|g(N,T)| = 1-\sigma$ on ∂D_σ. Exists $\sigma_0 > 0$ such that $D_\sigma \subset E$ for $\sigma \in (0,\sigma_0)$. Now we define the function $\phi_\sigma : \Sigma \to [0,1]$ given by

$$\phi_\sigma = \begin{cases} |g(N,T)|, & \text{in } \overline{D_\sigma}, \\ 1-\sigma, & \text{in } \Sigma - D_\sigma. \end{cases}$$

We note that ϕ_σ is continuous, piecewise $\mathcal{C}^1$ in the Z-direction and the sequence $\{\phi_\sigma\}_{\sigma\in(0,\sigma_0)}$ pointwise converge to 1 when $\sigma \to 0$. Using Lemma 2.1 we have that $|N_h|^{-1}Z(g(N,T))^2$ extends to a continuous function on Σ, so

$$\lim_{\sigma\to\sigma_0} \int_\Sigma |N_h|^{-1}Z(\phi_\sigma)^2 d\Sigma = 0.$$

Now slightly modifying ϕ_σ around ∂D_σ we can consider a sequence of $\mathcal{C}^1$ functions $\{\psi_\sigma\}_{\sigma\in(0,\sigma_0)}$ with the same properties. Defining $v_\sigma = \psi_\sigma u$ we have $\mathcal{Q}(v_\sigma) \geqslant 0$ for any $\sigma \in (0,\sigma_0)$ by condition 2.52. Now is enough to use the dominated convergence theorem and the Cauchy-Schwartz inequality in $L^2(\Sigma)$ to show $\mathcal{Q}(v_\sigma) \to \mathcal{Q}(u)$ for $\sigma \to 0$ and prove the statement.

□

2.8. Stable minimal surfaces inside a three-dimensional pseudo-hermitian sub-Riemannian manifolds.

We present a generalization of [**82**, Proposition 6.2] in the case of a minimal vertical surface of class $\mathcal{C}^2$ inside a three dimensional pseudo-hermitian manifold. A surface Σ with unit normal vector N is a *vertical surface* if $g(N,T) \equiv 0$. Obviously a vertical surface has empty singular set.

PROPOSITION 2.41. *Let Σ be a $\mathcal{C}^2$ vertical minimal surface inside a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$.*

(i) *If $W - c_1 g(\tau(Z),\nu_h) > 0$ on Σ, then Σ is unstable.*
(ii) *If $W - c_1 g(\tau(Z),\nu_h) \leqslant 0$ on Σ, then Σ is stable.*

PROOF. For vertical surfaces (2.51) becomes

$$\mathcal{I}(u|N_h|, u|N_h|) = \int_\Sigma |N_h| \Big\{ Z(u)^2 - (W - c_1 g(\tau(Z),\nu_h))u^2 \Big\} d\Sigma.$$

When $W - c_1 g(\tau(Z),\nu_h) > 0$ and Σ is compact we can use the function $u \equiv 1$ to get the instability. In the non-compact case we can prove (i) with a suitable cut off of the constant function 1. Point (ii) is immediate. □

It is remarkable that the sign of the quantities $W - c_1 g(\tau(Z), \nu_h)$ can be studied at least for three-dimensional Lie groups carrying out a pseudo-hermitian structure. We have the following classification result [**73**, Theorem 3.1]

PROPOSITION 2.42. *Let M be a simply connected contact 3-manifolds, homogeneous in the sense of Boothby and Wang,* [**16**]. *Then M is one of the following Lie group:*

(1) *if M is unimodular*
   - *the first Heisenberg group $\mathbb{H}^1$ when $W = |\tau| = 0$;*
   - *the three-sphere group $SU(2)$ when $W > 2|\tau|$;*
   - *the group $\widetilde{SL(2,\mathbb{R})}$ when $-2|\tau| \neq W < 2|\tau|$;*
   - *the group $\widetilde{E(2)}$, universal cover of the group of rigid motions of the Euclidean plane, when $W = 2|\tau| > 0$;*
   - *the group $E(1,1)$ of rigid motion of Minkowski 2-space, when $W = -2|\tau| < 0$;*

(2) *if M is non-unimodular, the Lie algebra is given by*

$$[X,Y] = \alpha Y + 2T, \quad [X,T] = \gamma Y, \quad [Y,T] = 0, \quad \alpha \neq 0,$$

where $\{X, Y\}$ is an orthonormal basis of $\mathcal{H}$, $J(X) = Y$ and T is the Reeb vector field. In this case $W < 2|\tau|$ and when $\gamma = 0$ the structure is Sasakian and $W = -\alpha^2$.

Here $|\tau|$ denote the norm of the matrix of the pseudo-hermitian torsion with respect to an orthonormal basis.

A Lie Group is *unimodular* when his left invariant Haar measure is also right invariant [**73**, p. 248].

We remark that in [**73**] the author gave the classification in terms of the equivalent invariant $W_1 = W/4$ and $|\tau_1| = 2\sqrt{2}|\tau|$. It is easy to show that if M is unimodular then

$$W = \frac{c_1(c_3 - c_2)}{2} \quad \text{and} \quad |\tau| = \frac{|c_2 + c_3|}{2}, \tag{2.54}$$

where the Lie algebra of M is defined by

$$[X,Y] = c_1 T, \quad [X,T] = c_2 Y, \quad [Y,T] = c_3 X,$$

with $\{X, Y\}$ orthonormal basis of $\mathcal{H}$, $J(X) = Y$, T the Reeb vector field and the normalization $c_1 = -2$. In the non-unimodular case we have

$$W = -\alpha^2 - \gamma \quad \text{and} \quad |\tau| = |\gamma|. \tag{2.55}$$

Furthermore in a unimodular sub-Riemannian Lie group G the matrix of τ in the X, Y, T basis is

$$\begin{pmatrix} 0 & \frac{c_2+c_3}{2} & 0 \\ \frac{c_2+c_3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and by [**37**, p. 38] we can compute the following derivatives

$$\begin{aligned} \nabla_X X &= 0, \quad \nabla_Y X = 0, \quad \nabla_T X = \frac{c_3 - c_2}{2} Y, \\ \nabla_X Y &= 0, \quad \nabla_Y Y = 0, \quad \nabla_T Y = \frac{c_2 - c_3}{2} X. \end{aligned} \tag{2.56}$$

If we consider another orthonormal basis $\{X_1, Y_1, T\}$ where $J(X_1) = Y_1, X_1 = a_1X + a_2Y, Y_1 = -a_2X + a_1Y$ the new torsion matrix becomes

$$\begin{pmatrix} (c_2+c_3)a_1a_2 & \frac{c_2+c_3}{2}(a_1^2-a_2^2) & 0 \\ \frac{c_2+c_3}{2}(a_1^2-a_2^2) & (c_2+c_3)a_1a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.57}$$

LEMMA 2.43. *Let Σ a surface of constant mean curvature H immersed in a unimodular Lie group G . Then*

$$g(R(T,Z)\nu_h, Z) - Z(g(\tau(Z), \nu_h)) = 2Hg(\tau(Z), Z),$$

which vanishes when Σ is minimal.

PROOF. By (2.57) we can express $g(\tau(Z), \nu_h) = (c_2+c_3)(1-2g(Z,X)^2)/2$ and

$$Z(g(\tau(Z),\nu_h)) = -2(c_2+c_3)g(Z,X)(g(\nabla_Z Z, X) + g(\nabla_Z X, Z)).$$

Taking into account (2.21) and (2.56) we obtain

$$Z(g(\tau(Z),\nu_h)) = 2Hg(\tau(\nu_h),\nu_h). \tag{2.58}$$

On the other hand $\nu_h(g(\tau(Z), Z)) = (c_2+c_3)\nu_h(g(\nu_h, X)g(\nu_h, Y))$, calculating

$$\begin{aligned} \nu_h(g(\tau(Z),Z)) =& (c_2+c_3)g(\nu_h,Y)(g(\nabla_{\nu_h}\nu_h, X) + g(\nabla_{\nu_h}X,\nu_h)) \\ &+ (c_2+c_3)g(\nu_h,X)(g(\nabla_{\nu_h}\nu_h,Y) + g(\nabla_{\nu_h}Y,\nu_h)) \\ &= -2\theta(\nu_h)g(\tau(Z),\nu_h), \end{aligned} \tag{2.59}$$

where we have used (2.56). Finally taking into account (2.39), (2.38), (2.58) and (2.59) we get the claim. □

LEMMA 2.44. *Let Σ be a $\mathcal{C}^2$ immersed minimal surface in M. Consider two functions $u \in \mathcal{C}(\Sigma - \Sigma_0)$ and $v \in \mathcal{C}(\Sigma - \Sigma_0)$ which are $\mathcal{C}^1$ and C^2 in the Z-direction, respectively. If v never vanishes, then*

$$\begin{aligned} \mathcal{I}(uv^{-1}|N_h|, uv_{-1}|N_h|) =& \int_\Sigma |N_h| v^{-2} Z(u)^2 d\Sigma \\ &+ \int_\Sigma |N_h| u^2 \{Z(v^{-1})^2 - \frac{1}{2} Z(Z(v^{-4})) - \frac{c_1}{2}\frac{g(N,T)}{|N_h|} Z(v^{-4})\} d\Sigma \\ &- \int_\Sigma |N_h| \mathcal{L}(|N_h|)(uv^{-1})^2 d\Sigma. \end{aligned} \tag{2.60}$$

The proof is the same as of [**55**, Lemma 4.3] except that Lemma 2.36 is used instead of [**55**, Lemma 3.17].

PROPOSITION 2.45. *Let Σ be a complete orientable $\mathcal{C}^2$ minimal surface with empty singular set immersed in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. We suppose that $g(R(T,Z)\nu_h, Z) - Z(g(\tau(Z),\nu_h)) = 0$ on Σ . If*

$$(W - c_1 g(\tau(Z), \nu_h))(p_0) \geqslant 0$$

for some $p_0 \in \Sigma$, then the operator $\mathcal{L}$ satisfies $\mathcal{L}(|N_h|) \geqslant 0$ on the characteristic curve γ_0 passing through p_0. Moreover, $\mathcal{L}(|N_h|) = 0$ over Σ if and only if $g(N,T) = 0$ and $W - c_1 g(\tau(Z),\nu_h) = 0$ on γ_0.

Proof. We consider a point $p \in \Sigma$. Let I be an open interval containing the origin and $\alpha : I \to \Sigma$ a piece of the integral curve of S passing through p. Consider the characteristic curve $\gamma_\varepsilon(s)$ of Σ with $\gamma_\varepsilon(0) = \alpha(\varepsilon)$. We define the map $F : I \times \mathbb{R} \to \Sigma$ given by $F(\varepsilon, s) = \gamma_\varepsilon(s)$ and denote $V(s) := (\partial F/\partial\varepsilon)(0, s)$ which is a Jacobi-like vector field along γ_0, Proposition 2.10. Clearly $V(0) = (S)_p$. We denote by $'$ the derivatives of functions depending on s, and the covariant derivative along γ_0 respect to ∇ and $\dot\gamma_0$ by Z. By (2.24) and (2.25)

$$g(V, T)' = -c_1 g(V, \nu_h), \tag{2.61}$$

$$\frac{1}{c_1} g(V, T)'' = c_1 \lambda g(V, Z) - Z(g(V, \nu_h)) \tag{2.62}$$

since

$$g(V', \nu_h) = Z(g(V, \nu_h)) + g(V, J(\nabla_Z Z)) = Z(g(V, \nu_h)).$$

Now we show that $\{V, Z\}$ is a basis of $T\Sigma$ along γ_0. It is sufficient to show that $g(V, T)$ and $g(V, \nu_h)$ do not vanish simultaneously. Suppose there exists s_0 such that $g(V, T)(s_0) = g(V, \nu_h)(s_0) = 0$. This means that $V(s_0)$ is co-linear with $(Z)_{s_0}$ and

$$g(V, T)'(s_0) = g(V, T)''(s_0) = 0$$

by (2.61) and (2.62). As $g(V, T)$ satisfies the differential equation in Proposition 2.10 (iv) we deduce $g(V, T) = 0$ along γ_0 which is impossible as $g(V, T)(0) = -|N_h| < 0$. We have proved that $g(V, T)$ never vanishes along γ_0 as Σ_0 is empty.

By (2.58) we have $W - c_1 g(\tau(Z), \nu_h) = k^2$, with $k \geqslant 0$. If $k = 0$ then solving the ordinary differential equation in Proposition 2.10 (iv) we have

$$g(V, T)(s) = as^2 + bs + c,$$

where a, b, c are given by

$$\begin{aligned} a &= g(V, T)''(0)/2 = -c_1 Z(g(N, T))/2, \\ b &= g(V, T)'(0) = -c_1 g(N, T), \\ c &= g(V, T)(0) = -|N_h|. \end{aligned}$$

Now $g(V, T) \neq 0$ implies $b^2 - 4ac < 0$ or $a = b = 0$. In the first case we get

$$b^2 - 4ac = \{c_1^2 g(N, T)^2 - 2c_1 |N_h| Z(g(N, T))\} \geqslant -\{c_1^2 g(N, T)^2 + 2c_1 |N_h| Z(g(N, T))\}$$

and the right term is equal to

$$-|N_h|^2 \left\{ 2c_1 Z\left(\frac{g(N, T)}{|N_h|}\right) + c_1^2 \frac{g(N, T)^2}{|N_h|^2} \right\},$$

which implies $\mathcal{L}(|N_h|) \geqslant 0$. On the other hand $a = b = 0$ implies that Σ is a vertical surface and $\mathcal{L}(|N_h|) = 0$. We note that in any vertical surface $b^2 - 4ac = 0$ so that $\mathcal{L}(|N_h|) = 0$.

Now we suppose $k \neq 0$. Then by Proposition 2.10 (iv) we get

$$g(V, T)(s) = \frac{1}{k}(a \sin(ks) - b \cos(ks)) + c,$$

with a, b, c given by

$$\begin{aligned} a &= g(V, T)'(0) = -c_1 g(N, T), \\ b &= \frac{1}{k} g(V, T)''(0) = -\frac{c_1}{k} Z(g(N, T)), \\ c &= \frac{1}{k^2} g(V, T)''(0) + g(V, T)(0) = \frac{b}{k} - |N_h|. \end{aligned}$$

As in [**82**, Proof of Proposition 6.6] we have $g(V,T)(s) \neq 0$ for all s if and only if

$$0 < k^2|N_h|^2 - 2k|N_h|b - a^2 = |N_h|^2\mathcal{L}(|N_h|),$$

which implies $\mathcal{L}(|N_h|) > 0$.

□

LEMMA 2.46. *Let Σ be a C^2 complete, oriented, immersed, CMC surface with empty singular set in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. Then any characteristic curve of Σ is an injective curve or a closed curve.*

PROOF. Since characteristic curves are the trajectories of the vector field Z, they are injective or closed curves.

□

REMARK 2.47. It is remarkable that Lemma 2.46, together with [**82**, Remark 6.8], implies that [**82**, Theorem 6.7] holds for all homogeneous Sasakian sub-Riemannian 3-manifolds. We only have to reason as in the last part of the proof of Proposition 2.48 below.

PROPOSITION 2.48. *Let Σ be a $\mathcal{C}^2$ complete orientable surface with empty singular set immersed in a pseudo-hermitian 3-manifold $(M, g_{\mathcal{H}}, \omega, J)$. We suppose that $g(R(T,Z)\nu_h, Z) - Z(g(\tau(Z),\nu_h)) = 0$ on Σ and the quantity $W - c_1 g(\tau(Z),\nu_h)$ is constant along characteristic curves. We also assume that all characteristic curves in Σ are either closed or non-closed. If Σ is a stable minimal surface, then $W - c_1 g(\tau(Z),\nu_h) \leqslant 0$ on Σ. Moreover, if $W - c_1 g(\tau(Z),\nu_h) = 0$ then Σ is a stable vertical surface.*

PROOF. We need to prove that when exists $p \in \Sigma$ such that $W - c_1 g(\tau(Z),\nu_h) > 0$ in p and $\mathcal{L}(|N_h|) \neq 0$ over the characteristic curve passing through p in Σ, then Σ is unstable, in virtue of Proposition 2.45. We consider $p \in \Sigma$ such that $\mathcal{L}(|N_h|)(p) > 0$. We denote by $\gamma_0(s)$ the characteristic curve passing through p and we denote by $\alpha(\varepsilon)$ the integral curve of S passing through p, parametrized by arc-length. As the surface is not singular Σ is foliated by characteristic curves, we denote by $\gamma_\varepsilon(s)$ the characteristic curve passing through $\alpha(\varepsilon)$ parametrized by arc-length. We obtain a C^1 map $F : I \times I' \to \Sigma$ given by $F(\varepsilon, s) = \gamma_\varepsilon(s)$ which parametrizes a neighborhood of the characteristic curve γ_0 on Σ, where I' is an interval, compact or not, where live the parameter s and $I = [-\varepsilon_0, \varepsilon_0]$ with $\varepsilon_0 \in \mathbb{R}$ eventually small. By Proposition 2.10 $V_\varepsilon(s) := (\partial F/\partial\varepsilon)(\varepsilon, s)$ is a Jacobi-like vector field along γ_ε and the function $g(V_\varepsilon, T)$ never vanishes since $\Sigma_0 = \emptyset$. Furthermore $V_\varepsilon(0) = (S)_{\alpha(\varepsilon)}$ implies that $g(V_\varepsilon, T) < 0$. We define the function $f_\varepsilon := g(V_\varepsilon, S)$ and it is immediate that $g(V_\varepsilon, T) = -f_\varepsilon|N_h|$ and $g(V_\varepsilon, \nu_h) = f_\varepsilon g(N,T)$ where $|N_h|$ and $g(N,T)$ are evaluated along γ_ε. The Riemannian area element of Σ with respect to the coordinates (ε, s) is given by

$$d\Sigma = (|V_\varepsilon|^2 - g(V_\varepsilon, \dot{\gamma_\varepsilon}))^{1/2} = f_\varepsilon\, ds\, d\varepsilon.$$

We define the function

$$v(\varepsilon, s) := |g(V_\varepsilon, T)(s)|^{1/2} = (f_\varepsilon|N_h|)^{1/2}, \tag{2.63}$$

which is positive, continuous on $I \times I'$ and C^∞ along characteristic curves, by Proposition 2.10. Denoting $v_\varepsilon(s) = v(\varepsilon, s)$ and denoting by $'$ the derivatives with respect to s, by (2.61) and (2.62) we get

$$(v_\varepsilon^{-2})' = g(V_\varepsilon, T)^{-2} g(V_\varepsilon, T)' = -c_1 \frac{g(N,T)}{f_\varepsilon|N_h|^2},$$

$$(v_\varepsilon^{-4})' = -2g(V_\varepsilon, T)^{-3} g(V_\varepsilon, T)' = -2c_1 \frac{g(N,T)}{f_\varepsilon^2 |N_h|^3},$$

$$\begin{aligned}(v_\varepsilon^{-4})'' &= 6g(V_\varepsilon, T)^{-4}(g(V_\varepsilon, T)')^2 - 2g(V_\varepsilon, T)^{-3} g(V_\varepsilon, T)'' \\ &= 4c_1^2 \frac{g(N,T)^2}{f_\varepsilon^2 |N_h|^4} - 2c_1 \frac{Z(|N_h|^{-1} g(N,T))}{f_\varepsilon^2 |N_h|^2},\end{aligned}$$

where we have used $g(V_\varepsilon, \nu_h) = -g(V_\varepsilon, T)|N_h|^{-1} g(N,T)$, and consequently

$$\begin{aligned}((v_\varepsilon^{-2})')^2 - \frac{1}{2}(v_\varepsilon^{-4})'' - \frac{c_1}{2}\frac{g(N,T)}{|N_h|}(v_\varepsilon^{-4})' &= c_1 \frac{Z(|N_h|^{-1} g(N,T))}{f_\varepsilon^2 |N_h|^2} \\ &= \frac{\mathcal{L}(|N_h|)}{2f_\varepsilon^2 |N_h|^2} - \frac{W - c_1 g(\tau(\nu_h), Z) + c_1^2 |N_h|^{-2} g(N,T)^2}{2f_\varepsilon^2 |N_h|^2}.\end{aligned} \tag{2.64}$$

Now we consider a function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi \in C_0^\infty(I)$ and $\phi(0) > 0$. Let ρ a positive constant such that $|\phi'(\varepsilon)| \leqslant \rho$ for any $\varepsilon \in \mathbb{R}$. We distinguish two cases. First we suppose that the family of curves γ_ε is defined in the whole real line for ε small enough. For any $n \in \mathbb{N}$ we consider the function $u_n : I \times I' \to \mathbb{R}$ defined by $u_n(\varepsilon, s) := \phi(\varepsilon)\phi(s/n)$, with $I' = \mathbb{R}$. At this point we can conclude as in [**82**, proof of Theorem 6.7].

In the second case we consider a family of closed curves γ_ε with eventually different length l_ε. We can parametrize all the curves as $\gamma_\varepsilon(t) : I' \to \Sigma$, with $t = s\, l_0/l_\varepsilon$ and $I' = [0, l_0]$. In this case we get

$$\begin{aligned}((v_\varepsilon^{-2})')^2 - \frac{1}{2}(v_\varepsilon^{-4})'' - \frac{c_1}{2}\frac{g(N,T)}{|N_h|}(v_\varepsilon^{-4})' &= \frac{l_0}{l_\varepsilon} c_1 \frac{Z(|N_h|^{-1} g(N,T))}{f_\varepsilon^2 |N_h|^2} \\ &= \frac{l_o}{l_\varepsilon}\frac{\mathcal{L}(|N_h|)}{2f_\varepsilon^2 |N_h|^2} - \frac{l_0}{l_\varepsilon}\frac{W - c_1 g(\tau(\nu_h), Z) + c_1^2 |N_h|^{-2} g(N,T)^2}{2f_\varepsilon^2 |N_h|^2}.\end{aligned}$$

Now it is sufficient reasoning as above changing the definition of the function $\phi_n(t) := \phi(0)$ to conclude as in [**82**, proof of Theorem 6.7].

We observe that, chosen a point $p \in \Sigma$, the curve γ_0 passing through p can be closed (resp. non-closed) but the other characteristic curves γ_ε can be non-closed (reps. closed) even for ε_0 small. In this case we can choose our initial point in another non-closed (reps. closed) curves.

□

REMARK 2.49. The proof of Proposition 2.48 works under weaker assumptions, i.e., when the closed and non-closed characteristic curves of Σ are not dense ones into others.

COROLLARY 2.50. *There are not complete stable minimal surfaces with empty singular set in the three-sphere group $SU(2)$.*

PROOF. By Proposition 2.42 in $SU(2)$ we have $W - 2g(\tau(Z), \nu_h) > 0$ and we get the statement using Theorem 2.48. □

REMARK 2.51. In [**82**, Corollary 6.9(ii)] the author shows that complete stable minimal surfaces with empty singular set do not exist in the pseudo-hermitian 3-sphere, which is the only Sasakian structure of $SU(2)$.

2.9. Classification of complete, stable, minimal surfaces in the roto-translation group $\mathcal{RT}$.

We consider the group of rigid motions of the Euclidean plane. The underlying manifold is $\mathbb{R}^2 \times \mathbb{S}^1$ where the horizontal distribution $\mathcal{H}$ is generated by the vector fields

$$X = \frac{\partial}{\partial \alpha} \quad \text{and} \quad Y = cos(\alpha)\,\frac{\partial}{\partial x} + sin(\alpha)\,\frac{\partial}{\partial y},$$

the Reeb vector field is

$$T = sin(\alpha)\,\frac{\partial}{\partial x} - cos(\alpha)\,\frac{\partial}{\partial y}$$

and the contact form is $\omega = sin(\alpha)\,dx - cos(\alpha)\,dy$, [**20**]. Furthermore we have the following expressions Lie brackets

$$[X, Y] = -T,\ [X, T] = Y,\ [Y, T] = 0$$

which imply $W = 1/2$ and that the matrix of the pseudo-hermitian torsion with respect to the basis $\{X, Y, T\}$ is

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

By [**54**, Theorem 1.2] a characteristic curve $\gamma(t) = (x(t), y(t), \alpha(t))$ of curvature $\lambda = 0$ with initial conditions $\gamma(0) = (x_0, y_0, \alpha_0)$ and $\dot{\gamma}(0) = (\dot{x_0}, \dot{y_0}, \dot{\alpha_0})$ in $\mathcal{RT}$ is of the form

$$\gamma(t) = (x_0 + R_0 \cos(\alpha_0)t,\ y_0 + R_0 \sin(\alpha_0)t,\ \alpha_0) \tag{2.65}$$

when $\theta_0 = 0$ or

$$\gamma(t) = (x_0 + (R_0/\dot{\alpha}_0)(\sin(\alpha(t)) - \sin(\alpha_0)),\ y_0 + (R_0/\dot{\alpha}_0)(\cos(\alpha_0) - \cos(\alpha(t))),\ \alpha_0 + \dot{\alpha}_0 t) \tag{2.66}$$

otherwise, where $R_0 = \sqrt{\dot{x_0}^2 + \dot{y_0}^2}$. We underline that the first family of curves is composed by sub-Riemannian geodesic but the second one only when $R_0 = 0$.

We investigate the equation of a minimal surface Σ defined as the zero level set of a function $u(\alpha, x, y)$. We consider the horizontal unit normal and the characteristic field

$$\nu_H = \frac{(u_\alpha X + (\cos(\alpha)u_x + \sin(\alpha)u_y)Y)}{(u_\alpha^2 + \cos^2(\alpha)u_x^2 + \sin^2(\alpha)u_y^2)^{1/2}},\ Z = \frac{(\cos(\alpha)u_x + \sin(\alpha)u_y)X - u_\alpha Y}{(u_\alpha^2 + \cos^2(\alpha)u_x^2 + \sin^2(\alpha)u_y^2)^{1/2}}$$

respectively. By a direct computation we get the minimal surface equation

$$\begin{aligned} &u_\alpha^2(\cos^2(\alpha)u_{xx} + 2\cos(\alpha)\sin(\alpha)u_{xy} + \sin^2(\alpha)u_{yy}) + (\cos(\alpha)u_x + \sin(\alpha)u_y)^2 u_{\alpha\alpha} \\ &- u_\alpha(\cos(\alpha)u_x + \sin(\alpha)u_y)(2\cos(\alpha)u_{\alpha x} + 2\sin(\alpha)u_{\alpha y} - \sin(\alpha)u_x + \cos(\alpha)u_y) = 0. \end{aligned} \tag{2.67}$$

Remark 2.52. In $\mathcal{RT}$ we can express

$$g(\tau(Z), Z) = g(Z, X)g(Z, Y) = -g(\nu_h, X)g(\nu_h, Y)$$

and

$$g(\tau(Z), \nu_h) = 1/2 - g(\nu_h, Y)^2$$

which imply $W - g(\tau(Z), \nu_h) = g(\nu_h, Y)^2 = g(Z, X)^2$.

Corollary 2.53. *Let Σ be a C^2 stable, oriented, complete, immersed minimal surface in $\mathcal{RT}$ with empty singular set. Then Σ is a vertical plane of the form $\Sigma_a = \{(x, y, \alpha) \in \mathcal{RT} : \alpha = a \in \mathcal{S}^1\}$.*

We note that there exists another family of vertical surfaces composed of the left-handed helicoids $\Sigma_b = \{(x, y, \alpha) \in \mathcal{RT} : \cos(b\,\alpha)x + \sin(b\,\alpha)y = 0, b \in \mathcal{S}^1\}$, that are unstable minimal surfaces. In fact the horizontal normal of Σ_b is

$$\nu_h = \frac{(-\sin(\alpha)x + \cos(\alpha)y)X + Y}{(1 + (-\sin(\alpha)x + \cos(\alpha)y)^2)^{1/2}}$$

which implies $W - g(\tau(Z), \nu_h) > 0$ outside the line $\{x = y = 0\}$.

LEMMA 2.54. *In $\mathcal{RT}$ there do not exist minimal surfaces with isolated singular points.*

PROOF. We can suppose that the singular point is the origin. Then $T_0\Sigma = span\{\partial_x, \partial_\alpha\}$. The unique way to construct a minimal surface is to to put together all characteristic curves starting from 0, in the directions of $T_0\Sigma$ with curvature $\lambda = 0$, Theorem 2.14. But in this way we construct a right-handed helicoid denoted Σ_c below, which contains a singular line. □

LEMMA 2.55. *Let Σ be a complete area-stationary surface of class C^2 in $\mathcal{RT}$ which contains a singular curve Γ. Then Σ is a right-handed helicoid Σ_c or a plane $\Sigma_{a,b,c}$ defined below.*

PROOF. We consider a singular curve $\Gamma(\varepsilon)$ in Σ. Then as Σ is foliated by characteristic curves we can parametrize it by the map $F(\varepsilon, s) = \gamma_\varepsilon(s)$, where $\gamma_\varepsilon(s)$ is the characteristic curves with initial data $\gamma_\varepsilon(0) = \Gamma(\varepsilon)$ and $\dot\gamma_\varepsilon(0) = J(\dot\Gamma(\varepsilon))$. We define the function $V_\varepsilon(s) := (\partial F/\partial\varepsilon)(s, \varepsilon)$ that is a smooth Jacobi-like vector field along $\gamma_\varepsilon(s)$. The vertical component of V_ε satisfies the ordinary differential equation

$$g(V_\varepsilon, T)''' + k_\varepsilon\, g(V_\varepsilon, T)' = 0,$$

with $k_\varepsilon = g(\dot\gamma_\varepsilon(s), X)^2$ that is constant along $\gamma_\varepsilon(s)$. We suppose that a characteristic curve $\gamma_{\bar\varepsilon}(s)$ is not a sub-Riemannian geodesic, it means that $0 < k_{\bar\varepsilon} < 1$. As $g(V_\varepsilon, T)'(0) = 0$ and $g(V_\varepsilon, T)''(0) = 0$ by (2.24), (2.25) and the fact that Γ is a singular curve, we get

$$g(V_\varepsilon, T)(s) = -\frac{1}{\sqrt{k_\varepsilon}} \sin(\sqrt{k_\varepsilon}\, s) \tag{2.68}$$

and we find another singular point at distance $\pi/\sqrt{k_\varepsilon}$. The singular point is contained in a singular curve Γ_1 composed of points of the type $\gamma_\varepsilon(s_\varepsilon)$, with $s_\varepsilon = \pi/\sqrt{k_\varepsilon}$. Σ area-stationary implies $g(\dot\Gamma_1(\varepsilon), J(\dot\gamma_\varepsilon(s_\varepsilon))) = 0$. Now we prove that $g(V_\varepsilon, \dot\gamma_\varepsilon)(s)$ is constant along γ_ε. It is zero in the initial point and we suppose it is increasing or decreasing. By point (ii) in Proposition 2.10 we get that it has a maximum or a minimum in s_ε and so $V_\varepsilon(s_\varepsilon)$ and $\dot\gamma_\varepsilon(s_\varepsilon)$ are co-linear. This is impossible and we have proved $V_\varepsilon(s_\varepsilon) = \dot\Gamma_1(\varepsilon)$. Finally integrating $g(V_\varepsilon, \dot\gamma_\varepsilon)(s)$ along γ_ε by point (ii) in Proposition 2.10 we get

$$0 = \int_0^{s_\varepsilon} g(V_\varepsilon, \dot\gamma_\varepsilon)'(s)ds = -\int_0^{s_\varepsilon} g(V_\varepsilon, T)(s) g(\tau(\dot\gamma_\varepsilon), \dot\gamma_\varepsilon)(s)ds,$$

that is impossible since $g(V_\varepsilon, T) > 0$ on $(0, s_\varepsilon)$ and $g(\tau(\dot\gamma_\varepsilon), \dot\gamma_\varepsilon) = g(\dot\gamma_\varepsilon, X)\sqrt{1 - g(\dot\gamma_\varepsilon, X)^2}$ is a constant different from zero. We have proved that each γ_ε is a sub-Riemannian geodesic and $k = k_\varepsilon$ is equal to 0 or 1. When $k = 0$ we get the surface a right-hand helicoid and when $k = 1$ we get a plane.

□

REMARK 2.56. In [**86**, Example 2.1] the author gives examples of minimal surfaces of equations $ax + b\sin(\alpha) + c = 0$ and $x - y + c(\sin(\alpha + \cos(\alpha))) + d = 0$. Also the surfaces $ay - b\cos(\alpha) + c = 0$ and $x + y + c(\sin(\alpha + \cos(\alpha))) + d = 0$ are minimal surfaces with a similar property, in fact they satisfy $g(\tau(Z), \nu_h) = 0$. We remark that all these examples are not area-stationary.

For example in the surface described by $x+\sin(\alpha) = 0$ we have $Z = (-\cos(\alpha)X + \cos(\alpha)Y/(2|\cos(\alpha)|)$ that is not orthogonal to the singular curves $\Gamma_1 = \{(-1, y, \pi/2) \in \mathcal{RT} : y \in \mathbb{R}\}$ and $\Gamma_2 = \{(1, y, 3\pi/2) \in \mathcal{RT} : y \in \mathbb{R}\}$.

LEMMA 2.57. *Let Σ be a surface defined by a function $u(x, y) = 0$, with $u : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^2$ and $(u_x, u_y) \neq (0, 0)$. Then Σ is a minimal surface that is area-stationary if and only if it is a plane $\Sigma_{a,b,c} = \{(x, y, \alpha) \in \mathcal{RT} : ax+by+c = 0,\ a, b \in \mathbb{R},\ c \in \mathcal{S}^1\}$.*

PROOF. It is sufficient observe that u_α or $u_{\alpha\alpha}$ multiply each term of equation (2.67). Furthermore it is clear that a surface Σ of the type $u = u(x, y)$ contains two singular curves whose union is $\Sigma_0 = \{(x, y, \alpha)\mathcal{RT} : \cos(\alpha)u_x + \sin(\alpha)u_y = 0\}$; by Lemma 2.55 the surface is a plane $\Sigma_{a,b,c} = \{(x, y, \alpha) \in \mathcal{RT} : ax + by + c = 0,\ a, b \in \mathbb{R},\ c \in \mathcal{S}^1\}$. □

In the sequel we investigate the stability of the two families of area-stationary surfaces that contains singular curves.

PROPOSITION 2.58. *All planes $\Sigma_{a,b,c} = \{(x, y, \alpha) \in \mathcal{RT} : ax + by + c = 0,\ a, b \in \mathbb{R},\ c \in \mathcal{S}^1\}$ are unstable area-stationary surfaces.*

PROOF. We take for simplicity a plane of equation $y = 0$. Then we have

$$\nu_h = \frac{\sin(\alpha)}{|\sin(\alpha)|}Y \quad Z = \frac{\sin(\alpha)}{|\sin(\alpha)|}X.$$

Then we get $g(\tau(Z), \nu_h) = -1/2$ and $W - g(\tau(Z), \nu_h) = 1$ by Remark 2.52. Furthermore using (2.56) we can compute $\theta(S) = -|N_h|$ and putting a function $u = u(x)$, with $u \in C_0^\infty([-x_0, x_0])$ and $x_0 > 0$, in the stability operator in Theorem 2.40 we get

$$\mathcal{Q}(u) = \left(\int\limits_{[-x_0,x_0]} u(x)^2\, dx\right)\left(-\int\limits_{[0,2\pi]} \frac{1}{4}|\sin(\alpha)|^3|\cos(\alpha)|\, d\alpha\right) + 2\int\limits_{[-x_0,x_0]} u'(x)^2\, dx$$

and as

$$\inf\left\{\left(\int\limits_{\mathbb{R}} u'(x)\, dx\right)\left(\int\limits_{\mathbb{R}} u(x)^2\, dx\right)^{-1} : u \in C_0^\infty(\mathbb{R})\right\} = 0,$$

there exists a function $u \in C_0^\infty([-x_0, x_0])$ such that $\mathcal{Q}(u) < 0$.

□

REMARK 2.59. A plane characterized by equation $ax+by+c\alpha = d$ is not minimal if $a, b, c \neq 0$.

PROOF. That plane is minimal if and only if the following equation holds:

$$c\{ab(\cos^2\alpha - \sin^2\alpha) + \cos\alpha\sin\alpha(b^2 - a^2)\} = 0,$$

that implies $c = 0$ or $a = b = 0$. □

PROPOSITION 2.60. *Let $\Sigma_c = \{(x, y, \alpha) \in \mathcal{RT} : x\sin(c\,\alpha) - y\cos(c\,\alpha) = 0, c \in \mathcal{S}^1\}$. Then Σ_c is a stable, area-stationary surface.*

PROOF. By a direct substitution in (2.67) Σ_c is minimal. Now we suppose $c = 1$ for simplicity and we have

$$\nu_h = \frac{x\cos(\alpha) + y\sin(\alpha)}{|x\cos(\alpha) + y\sin(\alpha)|}X, \quad Z = \frac{x\cos(\alpha) + y\sin(\alpha)}{|x\cos(\alpha) + y\sin(\alpha)|}Y$$

outside the only singular curve $\Gamma_0 = \{(x, y, \alpha) \in \mathcal{S}^1 : x = y = 0\}$, so the characteristic curves meet orthogonally the singular one.

Now by (2.56) we have $\theta(S) = |N_h|$ and by Remark 2.52 we get $-W+g(\tau(Z), \nu_h)) = 0$ and $g(\tau(Z), \nu_h) = 1/2$. Then the stability operator for non-singular surfaces in Theorem 2.40 become

$$\begin{aligned} \mathcal{Q}(u) = &\int_\Sigma \left\{ |N_h|^{-1} Z(u)^2 + |N_h| \left(1 - \frac{1}{4}|N_h|^2\right) u^2 \right\} d\Sigma \\ &+ 4 \int_{\Gamma_0} (u\big|_{\Gamma_0})^2 d\Gamma_0 + \int_{\Gamma_0} S(u\big|_{\Gamma_0})^2 d\Gamma_0, \end{aligned} \tag{2.69}$$

which is non-negative for all functions $u \in C_0^1(\Sigma_c)$. □

THEOREM 2.61. *Let Σ be a stable, area-stationary, immersed, oriented and complete surface of class C^2 in $\mathcal{RT}$. Then we distinguish two cases:*

(i) *if Σ is a non-singular surface, then it is a vertical plane Σ_a;*

(ii) *if Σ is a surface with non-empty singular set, then it is the right-handed helicoid Σ_c.*

Finally we would remark that the family of planes Σ_a are area-minimizing by a standard calibration argument, in fact they form a family of area-stationary surfaces who foliate RT.

CHAPTER 3

Variational formulas in contact sub-Riemannian manifolds

3.1. Preliminaries

3.1.1. Contact sub-Riemannian manifolds. A *contact manifold* is a C^∞ manifold M of odd dimension $2n+1$ together with a one-form ω such that $d\omega$ is non-degenerate when restricted to the *horizontal distribution* $\mathcal{H} := \ker(\omega)$, [**15**]. The $(2n+1)$-form $\omega \wedge (d\omega)^n$ is an orientation form in M. Since

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]),$$

$\mathcal{H}$ is completely non-integrable. The *Reeb vector field* T in M is the unique one that satisfies

$$\omega(T) = 1, \qquad \mathcal{L}_T\omega = 0, \tag{3.1}$$

where $\mathcal{L}$ is the Lie derivative in M.

A well-known example of a contact manifold is the Euclidean space $\mathbb{R}^{2n+1}$ with the contact one-form $\omega_0 := dt + \sum_{i=1}^n (x_i dy_i - y_i dx_i)$. A *contact transformation* between contact manifolds is a diffeomorphism preserving the horizontal distributions. An *strict contact transformation* is a diffeomorphism preserving the contact one-forms. An strict contact transformation preserves the Reeb vector fields. Darboux's Theorem [**15**, Thm. 3.1] implies that, given a contact manifold and some point $p \in M$, exists an open neighborhood U of p and a strict contact transformation f from U into an open set of $\mathbb{R}^{2n+1}$ with its standard contact structure. Such a local chart will be called a *Darboux chart*.

A *contact sub-Riemannian* structure $(M, g_{\mathcal{H}}, \omega)$ is given by a positive definite metric $g_{\mathcal{H}}$ on $\mathcal{H}$, [**62**]. The metric $g_{\mathcal{H}}$ can be extended to a Riemannian metric g on M by requiring that T be a unit vector orthogonal to $\mathcal{H}$. The scalar product of two vector fields X and Y with respect to the metric g will be often denoted by $\langle X, Y\rangle$. The Levi-Civita connection associated to g will be denoted by D. The integral curves of the Reeb vector field T are *geodesics* of the metric g. To check this property we observe that condition $\mathcal{L}_T\omega = 0$ in (3.1) implies

$$\omega([T,X]) = 0 \quad \text{for any } X \in \mathcal{H}. \tag{3.2}$$

Hence, for any horizontal vector field X, we have

$$\langle X, D_T T\rangle = -\langle D_T X, T\rangle = -\langle D_X T, T\rangle = 0.$$

We trivially have $\langle T, D_T T\rangle = 0$, and so we get $D_T T = 0$, as claimed.

A usual class defined in contact geometry is the one of *contact Riemannian manifolds*, see [**15**], [**90**]. Given a contact manifold, one can ensure the existence of a Riemannian metric g and an $(1,1)$-tensor field J, so that

$$g(T,X) = \omega(X), \quad 2g(X, J(Y)) = d\omega(X,Y), \quad J^2(X) = -X + \omega(X)\,T. \tag{3.3}$$

The structure given by (M, ω, g, J) is called a contact Riemannian manifold. The class of contact sub-Riemannian manifolds is different from this one. Recall that, in our definition, the metric $g_{\mathcal{H}}$ is given, and it is extended to a Riemannian metric g in TM. For such extended metric g, there is not in general an $(1,1)$-tensor field J satisfying all conditions in (3.3). Observe that the second condition in (3.3) uniquely defines J on $\mathcal{H}$, but this J does not satisfy in general the third condition in (3.3), as it is easily seen in $(\mathbb{R}^3, \omega_0)$ choosing an appropriate positive definite metric in $\ker(\omega_0)$.

We shall denote by Ω the volume form in (M, g) inducing the same orientation as $\omega \wedge (d\omega)^n$. A wedge product is defined in M for vectors and vector fields by

$$\big\langle e_1 \wedge \ldots \wedge e_{2n}, v\big\rangle = \Omega(e_1, \ldots, e_{2n}, v). \tag{3.4}$$

The Riemannian volume element in (M, g) will be denoted by dv_g. It coincides with Popp's measure [**62**, § 10.6]. The volume of a set $E \subset M$ with respect to the Riemannian metric g will be denoted by $V(E)$.

The *length* of a piecewise horizontal curve $\gamma : I \to M$ is defined by

$$L(\gamma) := \int_I |\gamma'(t)|\, dt,$$

where the modulus is computed with respect to the metric $g_{\mathcal{H}}$. The Carnot-Carathéodory distance $d(p, q)$ between p, $q \in M$ is defined as the infimum of the lengths of piecewise smooth horizontal curves joining p and q. A minimizing geodesic is any curve $\gamma : I \to M$ such that $d(\gamma(t), \gamma(t')) = |t - t'|$ for each t, $t' \in I$. We shall say that the sub-Riemannian manifold $(M, G_{\mathcal{H}}, \omega)$ is complete if (M, d) is a complete metric space. By Hopf-Rinow's Theorem [**51**, p. 9] bounded closed sets are compact and each pair of points can be joined by a minimizing geodesic. From [**62**, Chap. 5] a minimizing geodesic in a contact sub-Riemannian manifold is a smooth curve that satisfies the geodesic equations, i.e., it is normal.

3.1.2. The sub-Riemannian connection. In a contact sub-Riemannian manifold, we can decompose the endomorphism $X \in TM \to D_X T$ into its antisymmetric and symmetric parts, which we will denoted by J and τ, respectively. Explicitly we have

$$\begin{aligned} 2\big\langle J(X), Y\big\rangle &= \big\langle D_X T, Y\big\rangle - \big\langle D_Y T, X\big\rangle, \\ 2\big\langle \tau(X), Y\big\rangle &= \big\langle D_X T, Y\big\rangle + \big\langle D_Y T, X\big\rangle. \end{aligned} \tag{3.5}$$

Observe that $J(X), \tau(X) \in \mathcal{H}$ when $X \in \mathcal{H}$, and that $J(T) = \tau(T) = 0$. Also, note that

$$2\big\langle J(X), Y\big\rangle = -\big\langle [X, Y], T\big\rangle, \qquad X, Y \in \mathcal{H}. \tag{3.6}$$

We will call τ the (*contact*) *sub-Riemannian torsion*.

Now we define the (*contact*) *sub-Riemannian connection* ∇ as the unique metric connection, [**22**, eq. (I.5.3)], with torsion tensor $\operatorname{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ given by

$$\operatorname{Tor}(X, Y) := \big\langle X, T\big\rangle \tau(Y) - \big\langle Y, T\big\rangle \tau(X) + 2\big\langle J(X), Y\big\rangle T. \tag{3.7}$$

We arrive at definition (3.7) by requiring $\nabla T \equiv 0$. We recall Koszul's formula (compare with [**22**, p. 17])

$$\begin{aligned} 2\big\langle \nabla_X T, Y\big\rangle &= X(\big\langle T, Y\big\rangle) + T(\big\langle X, Y\big\rangle) - Y(\big\langle X, T\big\rangle) \\ &\quad + \big\langle [X, T], Y\big\rangle - \big\langle [X, Y], T\big\rangle + \big\langle [Y, T], X\big\rangle \\ &\quad + \big\langle \operatorname{Tor}(X, T), Y\big\rangle - \big\langle \operatorname{Tor}(X, Y), T\big\rangle + \big\langle \operatorname{Tor}(Y, T), X\big\rangle. \end{aligned} \tag{3.8}$$

$\nabla T \equiv 0$ implies $\langle \nabla_X T, Y\rangle - \langle \nabla_Y T, X\rangle = 0$ for all $X, Y \in TM$, and so we get from (3.8)

$$X(\langle T, Y\rangle) - Y(\langle T, X\rangle) - \langle [X, Y], T\rangle - \langle \operatorname{Tor}(X, Y), T\rangle = 0,$$

which implies

$$\langle \operatorname{Tor}(X, Y), T\rangle = 2\langle J(X), Y\rangle, \qquad \text{for all } X, Y \in \mathfrak{X}(M). \tag{3.9}$$

Again by Koszul's formulas for D and ∇ we obtain

$$2\langle D_X T - \nabla_X T, Y\rangle = -\langle \operatorname{Tor}(X, T), Y\rangle + \langle \operatorname{Tor}(X, Y), T\rangle - \langle \operatorname{Tor}(Y, T), X\rangle. \tag{3.10}$$

Using (3.5) and (3.10), still assuming $\nabla T \equiv 0$, we have

$$\begin{aligned} 2\langle \tau(X), Y\rangle &= \langle D_X T, Y\rangle + \langle D_Y T, X\rangle - \langle \nabla_X T, Y\rangle - \langle \nabla_Y T, X\rangle \\ &= \langle \operatorname{Tor}(T, Y), X\rangle + \langle \operatorname{Tor}(T, X), Y\rangle. \end{aligned}$$

Hence, assuming $(X, Y) \mapsto \langle \operatorname{Tor}(T, Y), X\rangle$ is symmetric, we have

$$\operatorname{Tor}(T, X) = \tau(X), \qquad \text{for all } X \in \mathfrak{X}(M). \tag{3.11}$$

Finally, we make the assumption

$$\langle \operatorname{Tor}(X, Y), Z\rangle = 0, \qquad \text{for all } X, Y, Z \in \mathcal{H}. \tag{3.12}$$

Definition (3.7) is then equivalent to (3.9), (3.11) and (3.12).

Let us check now that $\nabla T \equiv 0$ if (3.7) holds. From (3.8), (3.7) and (3.5) we get

$$\langle \nabla_X T, Y\rangle = 0, \qquad \text{for all } X, Y \in \mathcal{H}. \tag{3.13}$$

On the other hand,

$$\langle \nabla_T T, X\rangle = -\langle T, \nabla_X T\rangle - \langle T, \operatorname{Tor}(T, X)\rangle - \langle T, [T, X]\rangle = 0, \qquad \text{for any } X \in \mathcal{H}, \tag{3.14}$$

since $|T| = 1$, and $\operatorname{Tor}(T, X)$, $[T, X]$ are horizontal by (3.11) and (3.2). As $\langle \nabla_X T, T\rangle = 0$ for any $X \in TM$, we get from (3.13) and (3.14) that $\nabla T \equiv 0$.

The difference $D_X Y - \nabla_X Y$ can be computed for any pair of vector fields X, $Y \in \mathfrak{X}(M)$ using Koszul's formulas for D and ∇ to get

$$D_X Y - \nabla_X Y = \langle Y, T\rangle \big(\tau(X) + J(X)\big) + \langle X, T\rangle J(Y) - \langle \tau(X) + J(X), Y\rangle T. \tag{3.15}$$

The *sub-Riemannian curvature operator* R associated to the sub-Riemannian connection is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad X, Y, Z \in \mathfrak{X}(M). \tag{3.16}$$

The *sub-Riemannian curvature tensor* is defined by

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle, \qquad X, Y, Z, W \in \mathfrak{X}(M). \tag{3.17}$$

Since $\nabla T = 0$ we have

$$R(X, Y)T = 0, \qquad \text{for any } X, Y \in TM. \tag{3.18}$$

Moreover, the curvature tensor has the symmetries

$$\begin{aligned} R(X, Y, Z, W) &= -R(Y, X, Z, W), \\ R(X, Y, Z, W) &= -R(X, Y, W, Z), \end{aligned} \tag{3.19}$$

for all $X, Y, Z, W \in TM$.

We shall denote the Riemannian curvature operator and the Riemannian curvature tensor by $R^g(X,Y)Z$ and $R^g(X,Y,Z,W)$, respectively. In the sequel, we shall need the following relation

$$\begin{aligned} R^g(T,X)Y = R(T,X)Y &+ (\nabla_X J)(Y) \\ &+ \big(\langle J(\tau(X)),Y\rangle + \langle \tau(\tau(X)),Y\rangle\big)\,T \\ &- \big(\langle (\nabla_T J)(X),Y\rangle + \langle (\nabla_T \tau)(X),Y\rangle\big)\,T, \end{aligned} \tag{3.20}$$

for $X,Y\in\mathcal{H}$. To obtain (3.20) we have used (3.15), that $[X,T]$, $\nabla_X Y$ and $\nabla_T Y$ are horizontal, and $D_T T=0$. Observe that, if $E\in\mathfrak{X}(M)$ then $\langle J(E),E\rangle=0$. Then, for any vector field $W\in\mathfrak{X}(M)$, we have

$$\langle (\nabla_W J)(E),E\rangle = 0, \tag{3.21}$$

and so one of the terms in (3.20) vanishes when $X=Y$.

Our definition of ∇ coincides with previous ones in pseudo-hermitian manifolds [**83**] and special cases of contact sub-Riemannian manifolds which can be obtained as limits of contact Riemannian manifolds, where ∇ is the generalized pseudo-hermitian connection and τ the generalized pseudo-hermitian torsion, [**90**].

From the definition of the volume form Ω in (3.4) it follows that $\nabla_U\Omega=0$ for any vector field U in M and so

$$\nabla_U(E_1\wedge\ldots\wedge E_{2n}) = \nabla_U E_1\wedge\ldots\wedge E_{2n}+\cdots+E_1\wedge\ldots\wedge\nabla_U E_{2n}, \tag{3.22}$$

for any family $E_1,\ldots,E_{2n}\in\mathfrak{X}(M)$.

3.1.3. Hypersurfaces of class C_h^2.

Let $\Sigma\subset M$ a hypersurface of class C^1: we define the *sub-Riemannian area* of Σ by

$$A(\Sigma) := \int_\Sigma |N_h|\,d\Sigma, \tag{3.23}$$

where N is a unit vector field normal to Σ, N_h the orthogonal projection of N to the horizontal distribution, and $d\Sigma$ is the Riemannian measure of Σ. The *singular set* Σ_0 is

$$\Sigma_0 := \{p\in S : T_p\Sigma = \mathcal{H}_p\}, \tag{3.24}$$

of points where the tangent space to Σ is the horizontal distribution $\mathcal{H}_p$. We will always assume the existence of a unit vector field N normal to Σ globally defined. We define, in the *regular set* $\Sigma\setminus\Sigma_0$, the *horizontal unit normal* given by

$$\nu_h := \frac{N_h}{|N_h|}, \tag{3.25}$$

and we will take on $\Sigma\setminus\Sigma_0$ the vector field

$$S := \langle N,T\rangle\,\nu_h - |N_h|\,T, \tag{3.26}$$

which is tangent to Σ and orthogonal to $T\Sigma\cap\mathcal{H}$. We will say that a local basis $E_1,\ldots,E_{2n-1}$ of $T\Sigma\cap\mathcal{H}$ is positively oriented if $E_1,\ldots,E_{2n-1},S,N$ is positively oriented, which is equivalent to the positive orientation of the local basis $E_1,\ldots,E_{2n-1},\nu_h,T$.

In this paper we will consider hypersurfaces defined locally as the level set of a function having two continuous horizontal derivatives, i.e., such that locally $\Sigma\equiv\{f=0\}$ and Xf, $X(Yf)$ exist and are continuous for any $X,Y\in\mathcal{H}$. We will further assume that either there is $X\in\mathcal{H}$ such that $Xf\neq 0$, or there is a pair of horizontal vector fields $X,Y\in(f)$ such that $[X,Y]f\neq 0$. These conditions are equivalent to Σ being of class C^1 with two continuous horizontal derivatives. Under these conditions, we shall say that $f \in C_h^2$ and that Σ is of class C_h^2. We remark that the class of $C_{\mathbb{H}}^2$-regular hypersurfaces defined by Franchi, Serapioni and Serra-Cassano, [**41**], is contained C_h^2, and that $C_h^2 \subset C^1$.

Hypersurfaces of class C_h^2 can be approximated, in the regular set, by hypersurfaces of class C^∞. This is a local argument that can be described in the Heisenberg group $\mathbb{H}^n$. We consider on $\mathbb{H}^n$ a smooth function ρ such that $0 \leqslant \rho \leqslant 1$, $\int_{\mathbb{H}^n} \rho = 1$ and $\rho(p) = \rho(p^{-1}) = \rho(-p)$ for any $p \in \mathbb{H}^n$, where $^{-1}$ denotes the inverse with respect to the group multiplication. We define the *intrinsic mollifiers*

$$\rho_\varepsilon(x) := \frac{\rho\left((\delta_{1/\varepsilon}(p)\right)}{\varepsilon^{2n+2}}, \tag{3.27}$$

where δ_s is the one-parameter family of intrinsic dilations. For any $f \in L^1_{loc}(\mathbb{H}^n)$, we define the *intrinsic convolution*, [**9**, Lemma 2.4] and [**39**, Proposition 1.20], as

$$(\rho_\varepsilon * f)(p) := \int_{\mathbb{H}^n} \rho_\varepsilon(p') f((p')^{-1} \cdot p)\, dp' = \int_{\mathbb{H}^n} \rho_\varepsilon(p \cdot (p')^{-1}) f(p')\, dp'. \tag{3.28}$$

LEMMA 3.1. *We denote by $B_\varepsilon(p)$ the intrinsic ball of radius ε centered in p in $\mathbb{H}^n$ and by $\cdot$ the group multiplication in $\mathbb{H}^n$. Then in $\mathbb{H}^n$ the following properties hold*

(i) *supp $(\rho_\varepsilon * f) \subset B_\varepsilon(0)\cdot$ supp (f);*
(ii) *if $f \in L^\infty(\overline{\Omega}) \cap C(\Omega)$, then $(\rho_\varepsilon * f) \to f$ uniformly in compact subsets of $\Omega \subset \mathbb{H}^n$ when $\varepsilon \to 0^+$;*
(iii) *$X(\rho_\varepsilon * f)) = \rho_\varepsilon * (Xf))$, for any X such that Xf exists and is continuous.*

PROOF. Property (i) is straightforward. Property (ii) follows by applying the dominate convergence theorem to

$$\begin{aligned} |(\rho_\varepsilon * f)(p) - f(p)| &\leqslant \int_{\mathbb{R}^{2n+1}} \rho_\varepsilon(p') |f((p')^{-1} \cdot p) - f(p)|\, dp' \\ &\leqslant \int_{\mathbb{R}^{2n+1}} \rho(q) |f((\delta_\varepsilon(q))^{-1} \cdot p) - f(p)|\, dq, \end{aligned}$$

where we have used the change of variables $\delta_{1/\varepsilon}(p') = q$, since $|f((\delta_\varepsilon(q))^{-1} \cdot p) - f(p)| \leqslant 2|f|_{L^\infty}(\overline{\Omega})$ and $f((\delta_\varepsilon(q))^{-1} \cdot p) \to f(p)$, assuming that $p \in \overline{\Omega'} \subset \Omega$.

Property (iii) follows from definition 3.28 and from the fact that the Jacobian determinant of an intrinsic left translation is equal to 1. □

REMARK 3.2. We consider a point $p \in \Sigma$ and a neighborhood V of p in M. Then Darboux's diffeomorphism, [**15**, Thm. 3.1], maps $(V, g_{\mathcal{H}}, \omega)$ onto $(U \subset \mathbb{R}^{2n+1}, \tilde{g}_{\mathcal{H}}, \omega_0)$, where ω_0 is the Heisenberg contact form and $\tilde{g}_{\mathcal{H}}$ an arbitrary positive definite metric in the horizontal distribution.

Given a surface $\Sigma \subset U$, expressed as the level set of a function $f \in C^2_{\mathcal{H}}$, we can approximate Σ with a family of smooth surfaces $\{\Sigma_k\}_{k \in \mathbb{N}}$, defined as the zero level set of the functions $f_k := f * \rho_{1/k}$. From Lemma 3.1, f_k converges to f in compact subsets of Σ and also the horizontal derivatives up to the second order in compact subsets of Σ converge.

This implies C^1 convergence, so N_k converge to N, $(\nu_h)_k$ converge to ν_h and $(e_i)_k$ converge to e_i, for all $e_i \in T\Sigma$, in any compact subset. Here the subscript k denotes the vectors in the surface Σ_k. Furthermore we can express

$$(\nu_h)_k = \frac{X_1(f_k)X_1 + Y_1(f_k)Y_1 + \cdots + X_n(f_k)X_n + Y_n(f_k)Y_n}{\sqrt{X_1(f_k)^2 + Y_1(f_k)^2 + \cdots + X_n(f_k)^2 + Y_n(f_k)^2}},$$

where $\{X_i, Y_i\}_{i=1,\ldots,n}$ is an orthonormal basis of $\mathcal{H}_0 = \ker \omega_0$, and we have also the convergence of $\nabla_{(e_i)_k}(\nu_h)_k$ to $\nabla_{e_i}\nu_h$ for any $e_i \in T\Sigma \cap \mathcal{H}$ on compact subsets of Σ.

All the convergences still hold when we come back to $(V, g_{\mathcal{H}}, \omega)$ as the Darboux's diffeomorphism preserve the directions in $T\Sigma \cap \mathcal{H}$.

3.2. Size of the singular set of a C_h^2 hypersurface

We prove the following estimate on the size of the singular set of a C_h^2 hypersurface

THEOREM 3.3. *Let $\Sigma \subset M^{2n+1}$ be a hypersurface of class C_h^2 in a contact sub-Riemannian manifold. Then $\mathcal{H}_E^{n+3}(\Sigma_0) = 0$.*

PROOF. Since Darboux's Theorem preserves $T\Sigma$ and $\mathcal{H}$, we prove the result in $\mathbb{H}^n$.

We can suppose that Σ is locally the zero level set of a C_h^2 function. In Σ_0 we have $\nabla_{\mathbb{H}} f = 0$ and $(X_i(Y_i f), Y_i(X_i f)) \neq (0,0)$, for all $i \in \{1, \ldots, n\}$ (otherwise $T(f) = 0$ should be 0). We conclude that $\Sigma_0 \subset (\Sigma \cap S)$, where S is the $\mathbb{H}$-regular n-codimensional submanifold defined as the intersection of the zero level sets of the $C_{\mathbb{H}}^1$-function $Z_i(f)$, where $Z_i = X_i$ or Y_i. Then, from [**7**, Theorem 2.4], we get

$$\mathcal{H}_E^{n+3}(\Sigma_0) \leqslant \mathcal{H}_E^{n+3}(S) \leqslant \mathcal{H}_{\mathbb{H}}^{n+3}(S) = 0.$$

□

REMARK 3.4. We conjecture that, under the hypothesis of Theorem 3.3, $\mathcal{H}_E^{n+1}(\Sigma_0)$ should be 0. Our estimation is probably not sharp. although the estimate $\mathcal{H}_E^{n+3}(S) \leqslant \mathcal{H}_{\mathbb{H}}^{n+3}(S)$ is optimal for arbitrary sets in $\mathbb{H}^n$, $n > 1$, [**6, 7**], but at our knowledge it is not known if it is optimal for regular graphs. Moreover, in the proof of Theorem 3.3, we only estimate $\mathcal{H}_E(S)$ and not $\mathcal{H}_E(S \cap \Sigma)$.

3.3. The first variation formula

We consider a hypersurface $\Sigma \subset M$ of class C_h^2. At a given point $p \in \Sigma \setminus \Sigma_0$, the *mean curvature* of Σ at p is defined by

$$H(p) = -\sum_{i=1}^{2n-1} \langle \nabla_{e_i}\nu_h, e_i \rangle, \tag{3.29}$$

where $e_1, \ldots, e_{2n-1}$ is any orthonormal basis of $T_p\Sigma \cap \mathcal{H}_p$.

Let U be a vector field with compact support in M, and $\{\varphi_s\}_{s\in\mathbb{R}}$ its associated one-parameter group of diffeomorphisms. Since Σ is of class C^1, the sub-Riemannian area of $\Sigma_s := \varphi_s(\Sigma)$ is defined for all $s \in \mathbb{R}$. By the Riemannian area formula we have

$$A(\Sigma_s) = \int_\Sigma (|(N_s)_h| \circ \varphi_s) \operatorname{Jac}(\varphi_s)\, d\Sigma, \tag{3.30}$$

where N_s is a unit normal to $\varphi_s(\Sigma)$ chosen so that N_s is continuous in s, and $\operatorname{Jac}(\varphi_s)$ is the Riemannian Jacobian of the map φ_s, defined as the squared root of the determinant of the symmetric matrix $\big(\langle (d\varphi_t)_p(e_s), (d\varphi_s)_p(e_j)\rangle\big)_{i,j=1,\ldots,2n}$, where e_i is any orthonormal basis of $T_p\Sigma$.

Fix $p \in \Sigma \setminus \Sigma_0$ and let $e_1, \ldots, e_{2n-1}$ be a positively oriented orthonormal basis of $T_p\Sigma \cap \mathcal{H}_p$. We consider extensions $E_1, \ldots, E_{2n}$ of $e_1, \ldots, e_{2n}$ along the integral curve $\alpha(s)$ of U passing through p, such that each E_i is invariant under the flow of U. We can always extend E_i in a neigborhood of the curve so that $[E_i, U] = 0$. Observe that

$\nabla_{U_p} E_i$ is really the covariant derivative DE_i/ds of E_i along the integral curve of U through p, [**36**, p. 50]. Let

$$V(s,p) := \left(\sum_{i=1}^{2n} \langle E_i, T \rangle E_1 \wedge \cdots \wedge \overset{(i)}{T} \wedge \cdots \wedge E_{2n} \right) (\alpha(s)). \tag{3.31}$$

As shown in [**77**] in the Heisenberg group $\mathbb{H}^n$, we have

$$\big((|(N_s)_h| \circ \varphi_s) \ \mathrm{Jac}(\varphi_s) \big)(p) = |V(s,p)|, \tag{3.32}$$

and we call the expression in (3.32) the *horizontal Jacobian*. From (3.30) and (3.32) we get

$$A(\Sigma_s) = \int_\Sigma |V(s,p)| \, d\Sigma(p). \tag{3.33}$$

We can take one derivative under the integral sign in (3.30) since the singular set in a C^1 surface has $\mathcal{H}_E^{2n-1}$-measure 0, and

$$\frac{d}{ds}\bigg|_{s=0} |V(s,p)| = U_p(|V|) = \frac{\langle \nabla_{U_p} V, V_p \rangle}{|V_p|}, \tag{3.34}$$

which is a uniformly bounded function of s and p near $s=0$. We have used the notation $V_p := V(0,p)$ in (3.34). For future reference, we calculate

LEMMA 3.5. *Let Σ be a hypersurface of class C_h^2 in a contact sub-Riemannian manifold M^{2n+1}. Let U be a vector field in M, $p \in \Sigma \setminus \Sigma_0$, $\{e_1, \ldots, e_{2n-1}\}$ an orthonormal basis of $T_p\Sigma \cap \mathcal{H}_p$, $e_{2n} := S_p = \langle N_p, T_p \rangle (\nu_h)_p - |(N_h)_p| \, T_p$, and $E_1, \ldots, E_{2n}$ extensions of $e_1, \ldots, e_n$ along the integral curve of U through p invariant under the flow of U. Then we have*

$$\begin{aligned} \nabla_{U_p} V = &- \sum_{i=1}^{2n-1} \langle \nabla_{U_p} E_i, N_p \rangle \, e_i \\ &+ \Big\{ - \langle \nabla_{U_p} E_i, T_p \rangle + |(N_h)_p| \sum_{i=1}^{2n-1} \langle \nabla_{U_p} E_i, e_i \rangle \Big\} \, (\nu_h)_p \end{aligned} \tag{3.35}$$

where V is defined by (3.31)*, and N is the unit normal to Σ.*

PROOF. From (3.31) we get

$$\begin{aligned} \nabla_{U_p} V = \sum_{i=1}^{2n-1} &\langle \nabla_{U_p} E_i, T_p \rangle e_1 \wedge \ldots \wedge \overset{(i)}{T_p} \wedge \ldots \wedge e_{2n} \\ &+ \sum_{i=1}^{2n} \langle e_i, T_p \rangle \nabla_{U_p} \big(E_1 \wedge \ldots \wedge \overset{(i)}{T_p} \wedge \ldots E_{2n} \big). \end{aligned} \tag{3.36}$$

Since we have

$$e_1 \wedge \ldots \wedge \overset{(i)}{T_p} \wedge \ldots \wedge e_{2n} = \begin{cases} -\langle N_p, T_p \rangle \, e_i, & i = 1, \ldots, 2n-1, \\ -(\nu_h)_p, & i = 2n, \end{cases} \tag{3.37}$$

and

$$\langle e_i, T_p \rangle = \begin{cases} 0, & i = 1, \ldots, 2n-1, \\ -|(N_h)_p|, & i = 2n, \end{cases} \tag{3.38}$$

we obtain

$$\begin{aligned}(3.39)\quad \nabla_{U_p}V = -\sum_{i=1}^{2n-1}\big\langle N_p,T_p\big\rangle\big\langle\nabla_{U_p}E_i,T_p\big\rangle\, e_i - \big\langle\nabla_{U_p}E_{2n},T_p\big\rangle\,(\nu_h)_p \\ - |(N_h)_p|\,\nabla_{U_p}\big(E_1\wedge\ldots\wedge E_{2n-1}\wedge T\big).\end{aligned}$$

Writing $\nabla_{U_p}E_i = \big\langle\nabla_{U_p}E_i,e_i\big\rangle\, e_i + \big\langle\nabla_{U_p}E_i,(\nu_h)_p\big\rangle\,(\nu_h)_p + \cdots$, we get

$$(3.40)\quad \nabla_{U_p}\big(E_1\wedge\ldots\wedge E_{2n-1}\wedge T\big) = -\sum_{i=1}^{2n-1}\big\langle\nabla_{U_p}E_i,e_i\big\rangle\,(\nu_h)_p + \sum_{i=1}^{2n-1}\big\langle\nabla_{U_p}E_i,(\nu_h)_p\big\rangle\, e_i,$$

and so we finally obtain

$$\begin{aligned}(3.41)\quad \nabla_{U_p}V = \sum_{i=1}^{2n-1}\big\{ -\big\langle N_p,T_p\big\rangle\big\langle\nabla_{U_p}E_i,T_p\big\rangle - |(N_h)_p|\big\langle\nabla_{U_p}E_i,(\nu_h)_p\big\rangle\big\}\, e_i \\ + \big\{ -\big\langle\nabla_{U_p}E_{2n},T_p\big\rangle + \sum_{i=1}^{2n-1}|(N_h)_p|\big\langle\nabla_{U_p}E_i,e_i\big\rangle\big\}\,(\nu_h)_p,\end{aligned}$$

from which the result follows. □

PROPOSITION 3.6. *Let $\Sigma\subset M$ be a hypersurface of class C_h^2 in a contact sub-Riemannian manifold. Let N be a unit normal to Σ. Consider a C^∞ vector field U with compact support in M, and let $\{\varphi_s\}_{s\in\mathbb{R}}$ be the associated one-parameter group of diffeomorphisms. Then we have*

$$\begin{aligned}(3.42)\quad \frac{d}{ds}\bigg|_{s=0} A(\varphi_s(\Sigma)) = \int_\Sigma \big\{ -e_{2n}(\langle U,T\rangle) + \big\langle U,T\big\rangle|N_h|\sum_{i=1}^{2n-1}\big\langle\tau(e_i),e_i\big\rangle \\ - 2\big\langle J(U),e_{2n}\big\rangle + |N_h|\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}U_{ht},e_i\big\rangle - \big\langle U,N_h\big\rangle H\big\}\, d\Sigma(p)\end{aligned}$$

where the function between brackets is evaluated at $p\in\Sigma$, and $\{e_1,\ldots,e_{2n-1}\}$ is a positively oriented orthonormal basis of $T_p\Sigma\cap\mathcal{H}_p$.

PROOF. Equation (3.34) implies that the integrand in the first variation formula is given by $\big\langle\nabla_{U_p}V,(\nu_h)_p\big\rangle$. From (3.35), using equality $\nabla_{U_p}E_i = \nabla_{e_i}U + \operatorname{Tor}(U_p,e_i)$, the expression of the torsion (3.7), the decomposition $U = U_{ht} + \big\langle U,\nu_h\big\rangle\,\nu_h + \big\langle U,T\big\rangle\, T$, and the definition of H given in (3.29), we get (3.42). □

We wish to simplify formula (3.42) when the support of U is contained in the regular set of Σ. For C^2 hypersurfaces, this is a consequence of the Divergence Theorem applied to certain tangent vector fields. For C_h^2 hypersurfaces, we will approximate Σ by C^∞ surfaces out of the singular set, and pass to the limit to obtain the same result.

LEMMA 3.7. *Let $\Sigma\subset M$ be a hypersurface of class C^2 in a contact sub-Riemannian manifold, and let N be a unit normal to Σ. Then we have*

(i) $\operatorname{div}_\Sigma(fS) = e_{2n}(f) - f|N_h|\sum_{i=1}^{2n-1}\big\langle\tau(e_i),e_i\big\rangle - f\big\langle N,T\big\rangle H$, *for any $f\in C^1(\Sigma\setminus\Sigma_0)$.*

(ii) $\operatorname{div}_\Sigma(|N_h|\,U) = -2\big\langle J(U),e_{2n}\big\rangle + |N_h|\,\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}U,e_i\big\rangle$, *for any tangent and horizontal C^1 vector field U with support in $\Sigma\setminus\Sigma_0$.*

The functions in these formulas are evaluated at a point $p\in\Sigma\setminus\Sigma_0$, and $e_1,\ldots,e_{2n-1}$ is any orthonormal basis of $T_p\Sigma\cap\mathcal{H}_p$. Recall that $e_{2n}=S_p$.

PROOF. We note that

$$\operatorname{div}_\Sigma(fS)(p) = e_{2n}(f) + f(p) \sum_{i=1}^{2n-1} \big\langle D_{e_i} S, e_i \big\rangle,$$

since $|S| = 1$. Using (3.15) we express $\big\langle D_{e_i} S, e_i\big\rangle$ in terms of $\big\langle \nabla_{e_i} S, e_i\big\rangle$, which is computed from the expression of S given by (3.26), and the definition of H in equation (3.29). In this way we get (i).

In order to prove (ii) we write, for given $p \in \Sigma$,

$$\operatorname{div}_\Sigma(|N_h|\,U)(p) = U_p(|N_h|) + |(N_h)_p| \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} U, e_i\big\rangle + |(N_h)_p| \big\langle D_{e_{2n}} U, e_{2n}\big\rangle, \tag{3.43}$$

since $\big\langle D_{e_i} U, e_i\big\rangle = \big\langle \nabla_{e_i} U, e_i\big\rangle$ for $= 1, \ldots, 2n-1$ by (3.15), and U is orthogonal to e_i when $i = 1, \ldots, 2n-1$. We have

$$\begin{aligned} U_p(|N_h|) &= \big\langle D_{U_p} N_h, (\nu_h)_p\big\rangle = \big\langle D_{U_p}(N - \big\langle N, T\big\rangle T), (\nu_h)_p\big\rangle \\ &= \big\langle N_p, T_p\big\rangle \big\langle D_{U_p} N, e_{2n}\big\rangle - \big\langle D_{U_p} T, e_{2n}\big\rangle \end{aligned} \tag{3.44}$$

as $\nu_h = \big\langle N, T\big\rangle S + |N_h|\, N$ (which follows from (3.26)), and (3.26). On the other hand

$$|(N_h)_p| \big\langle D_{e_{2n}} U, e_{2n}\big\rangle = \big\langle N_p, T_p\big\rangle \big\langle D_{e_{2n}} U, N_p\big\rangle + \big\langle D_{e_{2n}} T, U_p\big\rangle, \tag{3.45}$$

since $|N_h|\, S = \big\langle N, T\big\rangle N - T$ by (3.26) and U is horizontal. Adding (3.44) and (3.45) we get

$$U_p(|N_h|) + |(N_h)_p| \big\langle D_{e_{2n}} U, e_{2n}\big\rangle = -2 \big\langle J(U_p), e_{2n}\big\rangle, \tag{3.46}$$

using the symmetries of τ and the second fundamental form of Σ. This equation, together with (3.43), implies (ii). □

LEMMA 3.8. *Let $\Sigma \subset M$ be a hypersurface of class C_h^2 in a contact sub-Riemannian manifold and let N be a unit normal to Σ. Then we have*

$$\int_\Sigma \Big\{ - e_{2n}(f) + f |N_h| \sum_{i=1}^{2n-1} \big\langle \tau(e_i), e_i\big\rangle \Big\}\, d\Sigma = \int_\Sigma f \big\langle N, T\big\rangle H \, d\Sigma, \tag{3.47}$$

for any $f \in C^1(\Sigma \setminus \Sigma_0)$, and

$$\int_\Sigma \Big\{ - 2 \big\langle J(U), e_{2n}\big\rangle + |N_h| \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} U, e_i\big\rangle \Big\}\, d\Sigma = 0, \tag{3.48}$$

for any horizontal C^1 vector field U, tangent to Σ, and with compact support in $\Sigma \setminus \Sigma_0$.

PROOF. We prove (3.48). In the same way (3.47) can be proved. We approximate Σ with a family of C^∞ hypersurfaces $\{\Sigma_k\}_{k\in\mathbb{N}}$, as in Remark 3.2. On each Σ_k there holds

$$\int_{\Sigma_k} \Big\{ - 2 \big\langle J(U), (e_{2n})_k\big\rangle + |(N_h)_k| \sum_{i=1}^{2n-1} \big\langle \nabla_{(e_i)_k} U, (e_i)_k\big\rangle \Big\}\, d\Sigma = 0, \tag{3.49}$$

by Lemma 3.7. Finally the right side of (3.49) converges to the right-hand side of (3.48) by Remark 3.2. □

THEOREM 3.9 (First variation of the sub-Riemannian area). *Let $\Sigma \subset M^{2n+1}$ be a hypersurface of class C^2_h in a contact sub-Riemannian manifold. Consider a C^∞ vector field U with compact support in M, and let $\{\varphi_s\}_{s\in\mathbb{R}}$ be the associated one-parameter group of diffeomorphisms. Then, for $n \geqslant 4$, we have*

$$\frac{d}{ds}\bigg|_{s=0} A(\varphi_s(\Sigma)) = -\int_\Sigma H\langle U, N\rangle\, d\Sigma, \tag{3.50}$$

where H is the sub-Riemannian mean curvature of Σ and $d\Sigma$ is the Riemannian measure of Σ. On the other hand, (3.50) *holds for variations supported in $\Sigma \setminus \Sigma_0$ in any dimension.*

PROOF. The proof follows from Proposition 3.6, Lemma 3.8 and Lemma 3.10. □

Following [**89**, Lemma 2.4], see also [**69**, Lemma 3.1] for details, we can prove

LEMMA 3.10. *Let $\Sigma \subset M$ be a (compact) C^1 hypersurface. Let $\Sigma_0 \subset \Sigma$ be such that $\mathcal{H}^{2n-1}_E(\Sigma_0) = 0$, where $\mathcal{H}^{2n-1}_E$ denotes the $(2n-1)$-dimensional Hausdorff measure associated to the Riemannian metric. Given $\varepsilon > 0$, there exist a smooth function $\varphi_\varepsilon : \Sigma \to [0,1]$ with compact support and such that*

(i) $\mathcal{H}^{2n}(\{\varphi_\varepsilon \neq 1\}) \leqslant \varepsilon$;
(ii) $\int_\Sigma |\nabla\varphi_\varepsilon| d\mathcal{H}^{2n} \leqslant \varepsilon$;
(iii) $\varphi_\varepsilon \equiv 0$ *in a neighborhood of* Σ_0 .

PROOF. Since $\mathcal{H}^{2n-1}(\Sigma_0) = 0$ we may cover Σ_0 with Riemannian balls $B(z_k, r_k)$ where $z_k \in \Sigma$ and

$$\sum_k r_k^{2n-1} < \varepsilon. \tag{3.51}$$

Furthermore we require $r_k < \varepsilon/2$. We may also assume that the collection of balls is finite since Σ is compact. Now for each k let ϕ_k be a smooth function such that $0 \leqslant \phi_k \leqslant 1$ with

$$\phi_k(x) = \begin{cases} 0 \text{ , for } x \in B(z_k, r_k) \\ 1 \text{ , for } x \in B(z_k, 2r_k) \end{cases} \tag{3.52}$$

and

$$|\nabla\phi_k| \leqslant \frac{2}{r_k} \tag{3.53}$$

for all x. We now define $\psi(x) = \phi_1(x)\cdots\phi_k(x)$, so we have $0 \leqslant \psi \leqslant 1$ and $\psi = 0$ on $U' = \bigcup_k B(z_k, r_k)$ which contain Σ_0 and $\psi = 1$ on $\mathbb{R}^{2n+1} \setminus U$, where $U = \bigcup_k B(z_k, 2r_k)$.

Furthermore ψ is smooth and we have

$$\begin{aligned} \int_\Sigma |\nabla_\Sigma\psi|^2 d\mathcal{H}^{2n} &\leqslant \int_\Sigma |\nabla\psi|^2 d\mathcal{H}^{2n} \\ &\leqslant \int_\Sigma \sum_k |\nabla\phi_k|^2 d\mathcal{H}^{2n} \\ &\leqslant \sum_k \frac{4}{r_k^2}\mathcal{H}^{2n}(B(z_k, 2r_k)\cap\Sigma) \\ &\leqslant 2^{2n+2}C_0\sum_k r_k^{2n-1} < C\varepsilon, \end{aligned} \tag{3.54}$$

where we have used $\mathcal{H}^{2n}_E(B(z_k, 2r_k)\cap\Sigma) \leqslant C_0(2r_k)^{2n}$. □

In case $\Sigma = \partial\Omega$, where Ω is a bounded open subset of M, the first derivative of the volume of Ω can be computed in the usual way and we get

$$\frac{d}{ds}\Big|_{s=0} V(\varphi_s(\Omega)) = \int_\Sigma \langle U, N\rangle \, d\Sigma,$$

assuming that N is the *outer* unit normal to Σ. As usual a hypersurface Σ is a critical point of the area if and only if $H = 0$ and it is a critical point of the area under a volume constraint if and only if H is constant. Analogous results are no longer valid in a contact sub-Riemannian manifold of dimension 3, see [**81**].

3.4. The second variation formula

First of all, we present two Lemmas.

LEMMA 3.11. *Let Σ be an oriented C^2 surface immersed inside a contact sub-Riemannian manifold $(M, g_{\mathcal{H}}, \omega)$, $\{e_1, \ldots, e_{2n-1}\}$ an orthogonal basis of $T\Sigma \cap \mathcal{H}$, and B the Riemannian shape operator. Then we have*

(i) $|N_h| e_i(|N_h|) = -\langle N, T\rangle e_i(\langle N, T\rangle)$;

(ii) $|N_h|^{-1} e_i(\langle N, T\rangle) = |N_h| e_i(\langle N, T\rangle) - \langle N, T\rangle e_i(|N_h|)$;

(iii) $\langle \nabla_{e_i} N, S\rangle = -|N_h|^{-1} e_i(\langle N, T\rangle)$;

(iv) $\langle \nabla_S N, e_i\rangle = |N_h| \langle \nabla_S \nu_h, e_i\rangle$;

(v) $\langle B(e_i), S\rangle = |N_h|^{-1} e_i(\langle N, T\rangle) + \langle J(\nu_h), e_i\rangle - \langle \tau(e_i), \nu_h\rangle$;

(vi) $\langle B(S), e_i\rangle = -|N_h| \langle \nabla_S \nu_h, e_i\rangle + |N_h|^2 \langle J(\nu_h), e_i\rangle$ $-\langle N, T\rangle^2 (\langle J(\nu_h), e_i\rangle + \langle \tau(e_i), \nu_h\rangle)$;

(vii) $|N_h|^{-1} e_i(\langle N, T\rangle) = -|N_h| \langle \nabla_S \nu_h, e_i\rangle - 2\langle N, T\rangle^2 \langle J(\nu_h), e_i\rangle + |N_h|^2 \langle \tau(e_i), \nu_h\rangle$;

(viii) $|N_H|^{-3} e_i(\langle N, T\rangle) = e_i\left(\frac{\langle N,T\rangle}{|N_H|}\right)$.

The proof is the analogous of [**45**, Lemma 3.1].

LEMMA 3.12. *We get*

$$\sum_{i=1}^{2n-1} \langle (\nabla_{e_i}\tau)\nu_h - (\nabla_{\nu_h}\tau)e_i, e_i\rangle = \sum_{i=1}^{2n-1} \{\langle \nabla_{e_i}\tau(\nu_h), e_i\rangle - \nu_h(\tau_{ii}) + 2\langle \nabla_{\nu_h} e_i, \tau(e_i)\rangle - \langle \nabla_{e_i}\nu_h, \tau(e_i)\rangle\}.$$

and

$$\langle (\nabla_{e_i}\tau)\nu_h - (\nabla_{\nu_h}\tau)e_i, e_i\rangle = R(e_i, T, \nu_h, e_i).$$

PROOF. By (3.10) we get

$$\langle R(e_i, T)\nu_h, e_i\rangle = \langle R^{LC}(\nu_h, e_i)e_i, T\rangle - \langle D_{e_i} J(\nu_h), e_i\rangle - \langle J(e_i), \nabla_{e_i}\nu_h\rangle$$

where R^{LC} denote the curvature tensor respect to the Levi-Civita connection. Using $D_X T = J(X) + \tau(X)$, we have

$$\langle D_{e_i} D_{\nu_h} T, e_i\rangle = \langle D_{e_i}\tau(\nu_h), e_i\rangle + \langle D_{e_i} J(\nu_h), e_i\rangle,$$
$$\langle D_{\nu_h} D_{e_i} T, e_i\rangle = \nu_h(\tau_{ii}) - \langle J(e_i) + \tau(e_i), D_{\nu_h} e_i\rangle,$$
$$\langle \nabla_{[\nu_h, e_i]} T, e_i\rangle = \langle J([\nu_h, e_i]) + \tau([\nu_h, e_i]), e_i\rangle$$

and from (3.10) we can easily obtain the second equation, while the first one is a standard computation.

□

Now we compute the second variation of the volume

LEMMA 3.13 (Second variation of the volume). *Let $\Sigma\subset M^{2n+1}$ be a hypersurface of class C^2 in a contact sub-Riemannian manifold. Consider a C^∞ vector field U with compact support in $M\smallsetminus\Sigma_0$, and let $\{\varphi_s\}_{s\in\mathbb{R}}$ be the associated one-parameter group of diffeomorphisms. Then we have*

$$\frac{d^2}{ds^2}\bigg|_{s=0} V(\varphi_s(\Omega)) = -\int_\Sigma \Big\{ \big\langle \nabla_{U^\perp}U^\perp, N\big\rangle) + \big\langle U,T\big\rangle e_{2n}(\big\langle U,\nu_h\big\rangle) - \big\langle U,\nu_h\big\rangle e_{2n}(\big\langle U,T\big\rangle) + \big\langle U,N\big\rangle\big\langle U,T\big\rangle\big\langle \tau(\nu_h),\nu_h\big\rangle + \big\langle U,N\big\rangle \sum_{i=1}^{2n-1}\big\langle \nabla_{e_i}U^\perp + Tor(U^\perp,e_i),e_i\big\rangle\Big\} d\Sigma.$$

PROOF. From Lemma A.3, we have

$$V''(0) = -\int_\Sigma \{U^\perp(\big\langle U^\perp,N\big\rangle) + \big\langle U,N\big\rangle \operatorname{div}_\Sigma U^\perp\} d\Sigma$$

and since $\big\langle N,E_{2n}\big\rangle = 0$, it is equal to

$$V''(0) = -\int_\Sigma \{\big\langle \nabla_{U^\perp}U^\perp, N\big\rangle) - \big\langle U,e_{2n}\big\rangle\big\langle \nabla_{U^\perp}E_{2n},U\big\rangle + \big\langle U,N\big\rangle \sum_{i=1}^{2n}\big\langle \nabla_{e_i}U^\perp + Tor(U^\perp,e_i),e_i\big\rangle\} d\Sigma,$$

as $\operatorname{div}_\Sigma U^\perp = \sum_{i=1}^{2n}\big\langle \nabla_{e_i}U^\perp + Tor(U^\perp,e_i),e_i\big\rangle$, by (3.10). On the other hand

$$\begin{aligned}\big\langle \nabla_{e_{2n}}U^\perp + Tor(U^\perp,e_{2n}),e_{2n}\big\rangle =&\big\langle N,T\big\rangle e_{2n}(\big\langle U,\nu_h\big\rangle) - |N_H| e_{2n}(\big\langle U,T\big\rangle)\\ &+ \big\langle N,T\big\rangle\big\langle U,N\big\rangle\big\langle \tau(\nu_h),\nu_h\big\rangle\end{aligned}$$

and

$$\begin{aligned}\big\langle \nabla_{U^\perp}E_{2n},N\big\rangle =& e_{2n}(\big\langle U,\nu_h\big\rangle|N_H| + \big\langle N,T\big\rangle e_{2n}(\big\langle U,T\big\rangle)\\ &+ |N_H|\big\langle U,N\big\rangle\big\langle \tau(\nu_h),\nu_h\big\rangle,\end{aligned}$$

which, together with $-|N_H|e_{2n} + \big\langle N,T\big\rangle N = T$, implies

$$V''(0) = -\int_\Sigma \{\big\langle \nabla_{U^\perp}U^\perp, N\big\rangle) + \big\langle U,T\big\rangle e_{2n}(\big\langle U,\nu_h\big\rangle) - \big\langle U,\nu_h\big\rangle e_{2n}(\big\langle U,T\big\rangle) + \big\langle U,N\big\rangle\big\langle U,T\big\rangle\big\langle \tau(\nu_h),\nu_h\big\rangle + \big\langle U,N\big\rangle \sum_{i=1}^{2n-1}\big\langle \nabla_{e_i}U^\perp + Tor(U^\perp,e_i),e_i\big\rangle\} d\Sigma.$$

□

THEOREM 3.14 (Second variation of $A-HV$). *Let $\Sigma\subset M^{2n+1}$ be a CMC hypersurface of class C^2 in a contact sub-Riemannian manifold. Consider a C^∞ vector field U with compact support in $M\smallsetminus\Sigma_0$, and let $\{\varphi_s\}_{s\in\mathbb{R}}$ be the associated one-parameter group of diffeomorphisms. Then we have*

(3.55)

$$\frac{d^2}{ds^2}\bigg|_{s=0}\{A(\varphi_s(\Sigma)) - HV(\varphi_s(\Omega))\} = \int_\Sigma \Big\{\Big|\nabla^h_\Sigma\Big(\frac{\big\langle U,N\big\rangle}{|N_H|}\Big)\Big|^2 - \frac{\big\langle U,N\big\rangle^2}{|N_H|^2}\, q\Big\}\,(|N_H| d\Sigma),$$

where q is defined by

$$
\begin{aligned}
q :=& W - 2\langle J(\nu_h), \tau(\nu_h)\rangle + |\sigma|^2 + 4\Big\langle \nabla_\Sigma^h\left(\frac{\langle N,T\rangle}{|N_H|}\right), J(\nu_h)\Big\rangle \\
&+ 4\frac{\langle N,T\rangle^2}{|N_H|^2}|J(\nu_h)|^2 + 2\frac{\langle N,T\rangle}{|N_H|}\operatorname{div}_\Sigma^h(J(\nu_h)) - \frac{\langle N,T\rangle^2}{|N_H|^2}\sum_{i=1}^{2n-1}|J(e_i)_{ht}|^2,
\end{aligned}
\tag{3.56}
$$

and σ denotes the intrinsic second fundamental form, as defined in Lemma B.1.

PROOF. From (3.33), in order to compute the second variation of the sub-Riemannian area, it is enough to calculate

$$
\frac{d^2}{ds^2}\bigg|_{s=0}|V(s,p)| = U_p(U(|V|)) = \frac{1}{|V_p|}\Big\{\langle\nabla_{U_p}\nabla_U V, V_p\rangle + |(\nabla_{U_p}V)_{ht}|^2\Big\},
$$

where W_{ht} denotes the orthogonal projection of the vector field W over $T\Sigma\cap\mathcal{H}$. Given a vector field W in M, we shall denote by $W_{ht}^\perp$ the only vector field so that $W = W_{ht} + W_{ht}^\perp$, and W_{ht}, $W_{ht}^\perp$ are orthogonal. We trivially have $W_{ht}^\perp = \langle W,\nu_h\rangle\,\nu_h + \langle W,T\rangle\,T$.

We define the horizontal gradient on Σ by

$$
\nabla_\Sigma^h f = \sum_{i=1}^{2n-1} e_i(f)\,e_i,
$$

the horizontal divergence on Σ by

$$
\operatorname{div}_\Sigma^h U = \sum_{i=1}^{2n-1}\langle\nabla_{e_i}U, e_i\rangle,
$$

and the horizontal Laplacian on Σ by

$$
\Delta_\Sigma^h f = \operatorname{div}_\Sigma^h(\nabla_\Sigma^h f).
$$

We get

$$
\begin{aligned}
\langle\nabla_{U_p}\nabla_U V, (\nu_h)_p\rangle =& \sum_{i=1}^{2n}\langle\nabla_{U_p}\nabla_U E_i, T_p\rangle\,\langle e_1\wedge\ldots\wedge \overset{(i)}{T_p}\wedge\ldots e_{2n}, (\nu_h)_p\rangle \\
&+\sum_{i=1}^{2n} 2\,\langle\nabla_{U_p}E_i, T_p\rangle\langle\nabla_{U_p}\big(E_1\wedge\ldots\wedge\overset{(i)}{T}\wedge\ldots\wedge E_{2n}\big), (\nu_h)_p\rangle \\
&+\sum_{i=1}^{2n}\langle e_i, T_p\rangle\langle\nabla_{U_p}\nabla_U\big(E_1\wedge\ldots\wedge\overset{(i)}{T}\wedge\ldots\wedge E_{2n}\big), (\nu_h)_p\rangle.
\end{aligned}
$$

From (3.22) and (3.26) we have

$$
\langle\nabla_{U_p}(E_1\wedge\ldots\wedge\overset{(i)}{T}\wedge\ldots\wedge E_{2n}), (\nu_h)_p\rangle = \langle\nabla_{U_p}E_{2n}, e_i\rangle, \quad i=1,\ldots,2n-1.
$$

Also we get

$$
\langle\nabla_{U_p}\big(E_1\wedge\ldots\wedge E_{2n-1}\wedge T\big), (\nu_h)_p\rangle = -\sum_{i=1}^{2n-1}\langle\nabla_{U_p}E_i, e_i\rangle.
$$

Hence from (3.37) and (3.38) we obtain

$$\begin{aligned}\big\langle \nabla_{U_p}\nabla_U V,(\nu_h)_p\big\rangle &= -\big\langle \nabla_{U_p}\nabla_U E_{2n},T_p\big\rangle\\ &+2\sum_{i=1}^{2n-1}\big\langle \nabla_{U_p}E_i,T_p\big\rangle\big\langle \nabla_{U_p}E_{2n},e_i\big\rangle-2\sum_{i=1}^{2n-1}\big\langle \nabla_{U_p}E_{2n},T_p\big\rangle\big\langle \nabla_{U_p}E_i,e_i\big\rangle\\ &-|(N_h)_p|\big\langle \nabla_{U_p}\nabla_U\big(E_1\wedge\ldots\wedge E_{2n-1}\wedge T\big),(\nu_h)_p\big\rangle.\end{aligned}$$

For the first summand in the above formula we get

$$\begin{aligned}\big\langle \nabla_{U_p}\nabla_U E_{2n},T_p\big\rangle &= \big\langle \nabla_{U_p}\big(\nabla_{E_{2n}}U+\mathrm{Tor}(U,E_{2n})\big),T_p\big\rangle\\ &= R(U_p,e_{2n},U_p,T_p)+\big\langle \nabla_{e_{2n}}(\nabla_U U),T\big\rangle+\big\langle \nabla_{U_p}\mathrm{Tor}(U,E_{2n}),T_p\big\rangle\\ &= e_{2n}\big\langle \nabla_U U,T\big\rangle+\big\langle \nabla_{U_p}\mathrm{Tor}(U,E_{2n}),T_p\big\rangle,\end{aligned}$$

since $R(U_p,e_{2n},U_p,T_p)=-R(U_p,e_{2n},T_p,U_p)=0$. For the last one we get

$$\begin{aligned}\big\langle \nabla_{U_p}\nabla_U\big(E_1\wedge\ldots\wedge E_{2n-1}\wedge T\big),(\nu_h)_p\big\rangle &= -\sum_{i=1}^{2n-1}\big\langle \nabla_{U_p}\nabla_U E_i,e_i\big\rangle\\ &+\sum_{i,j=1,i\neq j}^{2n-1}-\big\langle \nabla_{U_p}E_i,e_i\big\rangle\big\langle \nabla_{U_p}E_j,e_j\big\rangle+\big\langle \nabla_{U_p}E_i,e_j\big\rangle\big\langle \nabla_{U_p}E_j,e_i\big\rangle,\end{aligned}$$

and

$$\begin{aligned}\big\langle \nabla_{U_p}\nabla_U E_i,e_i\big\rangle &= \big\langle \nabla_{U_p}\big(\nabla_{E_i}U+\mathrm{Tor}(U,E_i)\big),e_i\big\rangle\\ &= R(U_p,e_i,U_p,e_i)+\big\langle \nabla_{e_i}(\nabla_U U)+\nabla_{U_p}\mathrm{Tor}(U,E_i),e_i\big\rangle.\end{aligned}$$

So we finally have

$$\begin{aligned}\big\langle \nabla_{U_p}\nabla_U V,(\nu_h)_p\big\rangle &= -e_{2n}\big\langle \nabla_U U,T\big\rangle-\big\langle \nabla_{U_p}\mathrm{Tor}(U,E_{2n}),T_p\big\rangle\\ &+2\sum_{i=1}^{2n-1}\big\langle \nabla_{U_p}E_i,T_p\big\rangle\big\langle \nabla_{U_p}E_{2n},e_i\big\rangle-2\sum_{i=1}^{2n-1}\big\langle \nabla_{U_p}E_{2n},T_p\big\rangle\big\langle \nabla_{U_p}E_i,e_i\big\rangle\\ &+|(N_h)_p|\sum_{i=1}^{2n-1}\Big\{-R(e_i,U_p,U_p,e_i)+\big\langle \nabla_{e_i}(\nabla_U U)+\nabla_{U_p}\mathrm{Tor}(U,e_i),e_i\big\rangle\Big\}\\ &+|(N_h)_p|\sum_{i,j=1,i\neq j}^{2n-1}\Big\{\big\langle \nabla_{U_p}E_i,e_i\big\rangle\big\langle \nabla_{U_p}E_j,e_j\big\rangle-\big\langle \nabla_{U_p}E_i,e_j\big\rangle\big\langle \nabla_{U_p}E_j,e_i\big\rangle\Big\},\end{aligned}\tag{3.57}$$

from which it follows

(3.58)

$$\begin{aligned}\big\langle\nabla_{U_p}\nabla_U V,(\nu_h)_p\big\rangle &= -e_{2n}\big\langle\nabla_U U,T\big\rangle - \big\langle\nabla_{U_p}\operatorname{Tor}(U,E_{2n}),T_p\big\rangle\\ &+2\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}U+\operatorname{Tor}(U_p,e_i),T_p\big\rangle\big\langle\nabla_{e_{2n}}U+\operatorname{Tor}(U_p,e_{2n}),e_i\big\rangle\\ &-2\sum_{i=1}^{2n-1}\big\langle\nabla_{e_{2n}}U+\operatorname{Tor}(U_p,e_{2n}),T_p\big\rangle\big\langle\nabla_{e_i}U+\operatorname{Tor}(U_p,e_i),e_i\big\rangle\\ &+|(N_h)_p|\sum_{i=1}^{2n-1}\Big\{-R(e_i,U_p,U_p,e_i)+\big\langle\nabla_{e_i}(\nabla_U U)+\nabla_{U_p}\operatorname{Tor}(U,e_i),e_i\big\rangle\Big\}\\ &+|(N_h)_p|\sum_{i,j=1,i\neq j}^{2n-1}\Big\{\big\langle\nabla_{e_i}U+\operatorname{Tor}(U_p,e_i),e_i\big\rangle\big\langle\nabla_{e_j}U+\operatorname{Tor}(U_p,e_j),e_j\big\rangle\\ &\quad-\big\langle\nabla_{e_i}U+\operatorname{Tor}(U_p,e_i),e_j\big\rangle\big\langle\nabla_{e_j}U+\operatorname{Tor}(U_p,e_j),e_i\big\rangle\Big\}.\end{aligned}$$

On the other hand, Lemma 3.5 implies

$$\frac{1}{|V_p|}\,|(\nabla_{U_p}V)_{ht}|^2=\frac{1}{|(N_h)_p|}\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}U+\operatorname{Tor}(U_p,e_i),N_p\big\rangle^2. \tag{3.59}$$

We have obtained that the second variation of the area equals the integral over Σ of (3.58)+(3.59), which coincides with the bilinear form $B(U,U)$ defined in (A.1). Splitting $B(U,U)=B(U,U_{ht})+B(U_{ht},U_{ht}^\perp)+B(U_{ht}^\perp,U_{ht}^\perp)$, from Lemma A.1 and Lemma A.2, we get that $A''(0)=B(U_{ht}^\perp,U_{ht}^\perp)$.

Since

$$\begin{aligned}-2\sum_{i=1}^{2n-1}&\big\langle\nabla_{e_{2n}}U_{ht}^\perp+\operatorname{Tor}(U_{ht}^\perp,e_{2n}),T_p\big\rangle\big\langle\nabla_{e_i}U_{ht}^\perp+\operatorname{Tor}(U_{ht}^\perp,e_i),e_i\big\rangle=\\ &=2\big\langle U,\nu_h\big\rangle e_{2n}(\big\langle U,T\big\rangle)H-e_{2n}(\big\langle U,T\big\rangle^2)\sum_{i=1}^{2n-1}\big\langle\tau(e_i),e_i\big\rangle,\end{aligned}$$

using Lemma 3.7 and Lemma 3.13, we obtain

(3.60)

$$\begin{aligned}&-HV''(0)-2\sum_{i=1}^{2n-1}\int_\Sigma\{\big\langle\nabla_{e_{2n}}U_{ht}^\perp+\operatorname{Tor}(U_{ht}^\perp,e_{2n}),T_p\big\rangle\big\langle\nabla_{e_i}U_{ht}^\perp+\operatorname{Tor}(U_{ht}^\perp,e_i),e_i\big\rangle\}d\Sigma=\\ &=\int_\Sigma\Big\{H\big\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,N\big\rangle+H\big\langle U,N\big\rangle\big\langle U,T\big\rangle\big\langle\tau(\nu_h),\nu_h\big\rangle-\big\langle U,T\big\rangle^2|N_H|\Big(\sum_{i=1}^{2n-1}\big\langle\tau(e_i),e_i\big\rangle\Big)^2\\ &+\operatorname{div}_\Sigma(H\big\langle U,T\big\rangle\big\langle U,\nu_h\big\rangle e_{2n})-\operatorname{div}_\Sigma(\big\langle U,T\big\rangle^2\sum_{i=1}^{2n-1}\big\langle\tau(e_i),e_i\big\rangle e_{2n})-|N_H|\big\langle U,\nu_h\big\rangle^2H^2\\ &+2H|N_H|\big\langle U,T\big\rangle\big\langle U,\nu_h\big\rangle\sum_{i=1}^{2n-1}\big\langle\tau(e_i),e_i\big\rangle+\big\langle U,T\big\rangle^2\sum_{i=1}^{2n-1}e_{2n}(\big\langle\tau(e_i),e_i\big\rangle)\Big\}\,d\Sigma.\end{aligned}$$

On the other hand

$$\big\langle\nabla_{e_i}U_{ht}^\perp+\operatorname{Tor}(U_{ht}^\perp,e_i),e_i\big\rangle=\big\langle U,\nu_h\big\rangle\big\langle\nabla_{e_i}\nu_h,e_i\big\rangle+\big\langle U,T\big\rangle\big\langle\tau(e_i),e_i\big\rangle,$$

and, by Lemma 3.7,

(3.61)

$$-e_{2n}(\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle) = -\operatorname{div}_\Sigma(\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle e_{2n}) - |N_H|\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle\sum_{i=1}^{2n-1}\langle\tau(e_i),e_i\rangle$$
$$-\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle\langle N,T\rangle H.$$

Furthermore $\langle\nabla_{U_p}\operatorname{Tor}(U,e_i),e_i\rangle$ equals

(3.62)

$$\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle\langle\tau(e_i),e_i\rangle - (e_i(\langle U,T\rangle)+2\langle U,\nu_h\rangle\langle J(\nu_h),e_i\rangle)\langle U,\nu_h\rangle\langle\tau(\nu_h),e_i\rangle$$
$$+\langle U,T\rangle\langle(\nabla_{U_{ht}^\perp}\tau)(e_i),e_i\rangle + \langle U,T\rangle\langle U,\nu_h\rangle\langle\nabla_{e_i}\nu_h,\tau(e_i)\rangle$$
$$+\langle U,T\rangle e_i(\langle U,\nu_h\rangle)\langle\tau(\nu_h),e_i\rangle + \langle U,T\rangle^2|\tau(e_i)|^2$$

and consequently also

(3.63)

$$\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle\langle\tau(e_i),e_i\rangle - (e_i(\langle U,T\rangle)+2\langle U,\nu_h\rangle\langle J(\nu_h),e_i\rangle)\langle U,\nu_h\rangle\langle\tau(\nu_h),e_i\rangle$$
$$+\langle U,T\rangle\langle U,\nu_h\rangle(-R(e_i,T,\nu_h,e_i)+\operatorname{div}_\Sigma^h(\tau(\nu_h)_{ht})+\langle\nabla_{e_i}\nu_h,e_i\rangle\langle\tau(\nu_h),\nu_h\rangle)$$
$$+\langle U,T\rangle^2(T(\langle\tau(e_i),e_i\rangle)-2\langle\nabla_T e_i,\tau(e_i)\rangle)+\langle U,T\rangle e_i(\langle U,\nu_h\rangle)\langle\tau(\nu_h),e_i\rangle$$
$$+\langle U,T\rangle^2|\tau(e_i)|^2,$$

where we have use Lemma 3.12. Finally we get that

(3.64)

$$-HV''(0) - 2\sum_{i=1}^{2n-1}\int_\Sigma\{\langle\nabla_{e_{2n}}U_{ht}^\perp+\operatorname{Tor}(U_{ht}^\perp,e_{2n}),T_p\rangle\langle\nabla_{e_i}U_{ht}^\perp+\operatorname{Tor}(U_{ht}^\perp,e_i),e_i\rangle\}\,d\Sigma$$
$$+\int_\Sigma|(N_h)_p|\sum_{i,j=1,i\neq j}^{2n-1}(\langle\nabla_{e_i}U+\operatorname{Tor}(U_p,e_i),e_i\rangle\langle\nabla_{e_j}U+\operatorname{Tor}(U_p,e_j),e_j\rangle)\,d\Sigma$$
$$-\int_\Sigma e_{2n}(\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle)\,d\Sigma + \int_\Sigma|N_H|\sum_{i=1}^{2n-1}\langle\nabla_{U_p}\operatorname{Tor}(U,e_i),e_i\rangle\,d\Sigma$$

equals

$$\begin{aligned}&\int_\Sigma\Big\{H\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,N_h\rangle-\operatorname{div}_\Sigma(\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\rangle e_{2n})\\&+\operatorname{div}_\Sigma(H\langle U,T\rangle\langle U,\nu_h\rangle e_{2n})-\operatorname{div}_\Sigma(\langle U,T\rangle^2\sum_{i=1}^{2n-1}\langle\tau(e_i),e_i\rangle e_{2n})\\&+\langle N,T\rangle\langle U,T\rangle^2\sum_{i=1}^{2n-1}\nu_h(\langle\tau(e_i),e_i\rangle)+\langle N,T\rangle\langle U,T\rangle^2H\langle\tau(\nu_h),\nu_h\rangle\Big\}\,d\Sigma\\&+\int_\Sigma|N_H|\sum_{i=1}^{2n-1}\Big\{-(e_i(\langle U,T\rangle)+2\langle U,\nu_h\rangle\langle J(\nu_h),e_i\rangle)\langle U,\nu_h\rangle\langle\tau(\nu_h),e_i\rangle\\&+\langle U,T\rangle^2|\tau(e_i)|^2+\langle U,T\rangle\langle U,\nu_h\rangle(-R(e_i,T,\nu_h,e_i)+\operatorname{div}_\Sigma^h(\tau(\nu_h)_{ht}))\\&-2\langle\nabla_T e_i,\tau(e_i)\rangle)+\langle U,T\rangle e_i(\langle U,\nu_h\rangle)\langle\tau(\nu_h),e_i\rangle\Big\}\,d\Sigma.\end{aligned}\tag{3.65}$$

Now, by Lemma 3.7, we have

(3.66)
$$|N_H| \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i}(\nabla_{U_{ht}^\perp} U_{ht}^\perp), e_i \big\rangle = -H \big\langle \nabla_{U_{ht}^\perp} U_{ht}^\perp, N_h \big\rangle - 2\big\langle N, T\big\rangle \big\langle \nabla_{U_{ht}^\perp} U_{ht}^\perp, J(\nu_h)\big\rangle + \mathrm{div}_\Sigma(|N_H|(\nabla_{U_{ht}^\perp} U_{ht}^\perp)_{ht})$$

and

(3.67)
$$-\big\langle \nabla_{U_{ht}^\perp} \mathrm{Tor}(U_{ht}^\perp, E_{2n})\big\rangle = 2\{ -\big\langle (\nabla_{U_{ht}^\perp} J)(U_{ht}^\perp), e_{2n}\big\rangle + \big\langle N, T\big\rangle \big\langle \nabla_{U_{ht}^\perp} U_{ht}^\perp, J(\nu_h)\big\rangle - \big\langle U, \nu_h\big\rangle \big\langle \nabla_{e_{2n}} U_{ht}^\perp + \mathrm{Tor}(U_{ht}^\perp, e_{2n}), J(\nu_h)\big\rangle\}.$$

We also note

$$-\int_\Sigma |N_H| \sum_{i=1}^{2n-1} R(e_i, U_{ht}^\perp, U_{ht}^\perp, e_i)\, d\Sigma = \tag{3.68}$$
$$= \int_\Sigma |N_H| \{-\big\langle U, \nu_h\big\rangle^2 W - \big\langle U, \nu_h\big\rangle \big\langle U, T\big\rangle s(T, \nu_h)\}\, d\Sigma,$$

where $s(T,\nu_h) := \sum_{i=1}^{2n-1} R(e_i, T, \nu_h, e_i)$ and $W := \sum_{i=1}^{2n-1} R(e_i, \nu_h, \nu_h, e_i)$. So adding the right side of (3.66), (3.67), (3.68) and (3.65), we obtain

(3.69)
$$\int_\Sigma \Big\{ -\mathrm{div}_\Sigma(\big\langle \nabla_{U_{ht}^\perp} U_{ht}^\perp, T\big\rangle e_{2n})$$
$$+ \mathrm{div}_\Sigma(H\big\langle U, T\big\rangle \big\langle U, \nu_h\big\rangle e_{2n}) - \mathrm{div}_\Sigma(\big\langle U, T\big\rangle^2 \sum_{i=1}^{2n-1} \big\langle \tau(e_i), e_i\big\rangle e_{2n})$$
$$+ \big\langle U, T\big\rangle^2 \big\langle N, T\big\rangle \sum_{i=1}^{2n-1} \nu_h(\big\langle \tau(e_i), e_i\big\rangle) + \mathrm{div}_\Sigma(|N_H|(\nabla_{U_{ht}^\perp} U_{ht}^\perp)_{ht})\Big\}\, d\Sigma$$
$$+ \int_\Sigma |N_H| \Big\{ (\big\langle \big\langle U, T\big\rangle \nabla_\Sigma^h(\big\langle U, \nu_h\big\rangle) - \big\langle U, \nu_h\big\rangle \nabla_\Sigma^h(\big\langle U, T\big\rangle), \tau(\nu_h)\big\rangle$$
$$- 2\big\langle U, \nu_h\big\rangle^2 \big\langle J(\nu_h), \tau(\nu_h)\big\rangle) - \big\langle U, \nu_h\big\rangle^2 W$$
$$+ \sum_{i=1}^{2n-1} (\big\langle U, T\big\rangle^2 |\tau(e_i)|^2 + \big\langle U, T\big\rangle \big\langle U, \nu_h\big\rangle (\big\langle \nabla_{e_i} \tau(\nu_h), e_i\big\rangle - 2s(\nu_h, T))$$
$$- 2\big\langle U, T\big\rangle^2 \big\langle \nabla_T e_i, \tau(e_i)\big\rangle)\Big\}\, d\Sigma - 2\int_\Sigma \{\big\langle (\nabla_{U_{ht}^\perp} J)(U_{ht}^\perp), e_{2n}\big\rangle$$
$$+ \big\langle U, \nu_h\big\rangle^2 \big\langle \nabla_{e_{2n}} \nu_h, J(\nu_h)\big\rangle + \big\langle U, \nu_h\big\rangle \big\langle U, N\big\rangle \big\langle \tau(\nu_h), J(\nu_h)\big\rangle\}\, d\Sigma.$$

On the other hand, since

$$2\sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht}^\perp + \mathrm{Tor}(U_{ht}^\perp, e_i), T\big\rangle \big\langle \nabla_{e_{2n}} U_{ht}^\perp + \mathrm{Tor}(U_{ht}^\perp, e_{2n}), e_i\big\rangle = \tag{3.70}$$
$$= 2\big\langle \nabla_\Sigma^h(\big\langle U, T\big\rangle) + 2\big\langle U, \nu_h\big\rangle J(\nu_h), \big\langle U, \nu_h\big\rangle \nabla_{e_{2n}} \nu_h + \big\langle U, N\big\rangle \tau(\nu_h)\big\rangle$$

and

$$\begin{aligned}\big\langle(\nabla_{U_{ht}^{\perp}}J)(U_{ht}^{\perp}),e_{2n}\big\rangle &= \big\langle N,T\big\rangle\big\langle(\nabla_{U_{ht}^{\perp}}J)(\nu_h),\nu_h\big\rangle\\ &= \big\langle N,T\big\rangle(\big\langle\nabla_{U_{ht}^{\perp}}J(\nu_h),\nu_h\big\rangle+\big\langle\nabla_{U_{ht}^{\perp}}\nu_h,J(\nu_h)\big\rangle)=0,\end{aligned}$$

we obtain that (3.69) added with the left term in (3.70) is equal to

$$\begin{aligned}&\int_\Sigma\Big\{\operatorname{div}_\Sigma(|N_H|(\nabla_{U_{ht}^{\perp}}U_{ht}^{\perp})_{ht})-\operatorname{div}_\Sigma(\big\langle\nabla_{U_{ht}^{\perp}}U_{ht}^{\perp},T\big\rangle e_{2n})\\
&+\operatorname{div}_\Sigma(H\big\langle U,T\big\rangle\big\langle U,\nu_h\big\rangle e_{2n})-\operatorname{div}_\Sigma(\big\langle U,T\big\rangle^2\sum_{i=1}^{2n-1}\big\langle\tau(e_i),e_i\big\rangle e_{2n})\\
&+\big\langle U,T\big\rangle^2\big\langle N,T\big\rangle\sum_{i=1}^{2n-1}\nu_h(\big\langle\tau(e_i),e_i\big\rangle)\Big\}\,d\Sigma\\
&+\int_\Sigma|N_H|\Big\{\big\langle\nabla_\Sigma^h(\big\langle U,\nu_h\big\rangle\big\langle U,T\big\rangle),\tau(\nu_h)\big\rangle\\
&+2\big\langle U,\nu_h\big\rangle|N_H|^{-1}\big\langle\nabla_\Sigma^h(\big\langle U,T\big\rangle),\nabla_{e_{2n}}\nu_h\big\rangle\\
&+\big\langle N,T\big\rangle|N_H|^{-1}\big\langle\nabla_\Sigma^h(\big\langle U,T\big\rangle^2),\tau(\nu_h)\big\rangle)-\big\langle U,\nu_h\big\rangle^2W\\
&+\big\langle U,T\big\rangle^2\sum_{i=1}^{2n-1}(|\tau(e_i)|^2-2\big\langle\nabla_Te_i,\tau(e_i)\big\rangle)\\
&+(\big\langle U,T\big\rangle\big\langle U,\nu_h\big\rangle\operatorname{div}_\Sigma^h(\tau(\nu_h)_{ht})-2s(T,\nu_h))\Big\}\,d\Sigma\\
&+2\int_\Sigma\{\big\langle U,\nu_h\big\rangle^2\big\langle\nabla_{e_{2n}}\nu_h,J(\nu_h)\big\rangle+\big\langle U,\nu_h\big\rangle\big\langle U,T\big\rangle\big\langle N,T\big\rangle\big\langle\tau(\nu_h),J(\nu_h)\big\rangle\}\,d\Sigma.\end{aligned}\tag{3.71}$$

Furthermore, the term

$$-|N_H|\sum_{i,j=1}^{2n-1}\big\langle\nabla_{e_i}U+\operatorname{Tor}(U_p,e_i),e_j\big\rangle\big\langle\nabla_{e_j}U+\operatorname{Tor}(U_p,e_j),e_i\big\rangle$$

equals

$$\begin{aligned}&-|N_H|\sum_{i=1}^{2n-1}\{\big\langle U,T\big\rangle^2|\tau(e_i)|^2+2\big\langle U,T\big\rangle\big\langle U,\nu_h\big\rangle\big\langle\nabla_{e_i}\nu_h,\tau(e_i)\big\rangle\\
&+\big\langle U,\nu_h\big\rangle^2(|\nabla_{e_i}\nu_h|^2+2\big\langle N,T\big\rangle|N_H|^{-1}\big\langle J(e_i),\nabla_{e_i}\nu_h\big\rangle)\},\end{aligned}\tag{3.72}$$

where we can express

$$|\nabla_\Sigma^h\nu_h|^2+2\frac{\big\langle N,T\big\rangle}{|N_H|}\sum_{i=1}^{2n-1}\big\langle J(e_i),\nabla_{e_i}\nu_h\big\rangle=|\sigma|^2-\sum_{i=1}^{2n-1}\frac{\big\langle N,T\big\rangle^2}{|N_H|^2}|J(e_i)_{ht}|^2.\tag{3.73}$$

Finally we need to compute $\big\langle\nabla_{e_i}U_{ht}^{\perp}+\operatorname{Tor}(U_{ht}^{\perp},e_i),\nu_h+\big\langle N,T\big\rangle|N_H|^{-1}T\big\rangle$. In fact

$$|N_H|\big\langle\nabla_{e_i}U_{ht}^{\perp}+\operatorname{Tor}(U_{ht}^{\perp},e_i),\nu_h+\big\langle N,T\big\rangle|N_H|^{-1}T\big\rangle^2=|N_H|^{-1}\big\langle\nabla_{e_i}U_{ht}^{\perp}+\operatorname{Tor}(U_{ht}^{\perp},e_i),N\big\rangle^2.\tag{3.74}$$

From Lemma 3.11 we simply have that

$$\begin{aligned}
&\big\langle \nabla_{e_i} U_{ht}^{\perp} + \mathrm{Tor}(U_{ht}^{\perp}, e_i), \nu_h + \big\langle N, T\big\rangle |N_H|^{-1} T\big\rangle = \\
&e_i \left(\frac{\langle U, N\rangle}{|N_H|} \right) + \big\langle U, T\big\rangle \big\langle |N_H|^{-1} \nabla_{e_{2n}} \nu_h, e_i\big\rangle + 2 \frac{\langle N, T\rangle}{|N_H|} \left(\frac{\langle U, N\rangle}{|N_H|} \right) \big\langle J(\nu_h), e_i\big\rangle.
\end{aligned} \tag{3.75}$$

and consequently the right and left sides in (3.75), summed for $i = 1, \ldots, 2n-1$, can be expressed as

$$\begin{aligned}
&|N_H| \Big\{ \Big| \nabla_\Sigma^h \left(\frac{\langle U, N\rangle}{|N_H|} \right) \Big|^2 + 4 \frac{\langle N, T\rangle^2}{|N_H|^2} \left(\frac{\langle U, N\rangle}{|N_H|} \right)^2 |J(\nu_h)|^2 + \big\langle U, T\big\rangle^2 ||N_H|^{-1} \nabla_{e_{2n}} \nu_h|^2 \\
&+ 2 \frac{\langle N, T\rangle}{|N_h|} \big\langle \nabla_\Sigma^h \left(\left(\frac{\langle U, N\rangle}{|N_H|} \right)^2 \right), J(\nu_h)\big\rangle + 2 \big\langle U, T\big\rangle \big\langle \nabla_\Sigma^h \left(\frac{\langle U, N\rangle}{|N_H|} \right), |N_H|^{-1} \nabla_{e_{2n}} \nu_h\big\rangle \\
&+ 4 \frac{\langle N, T\rangle}{|N_h|} \big\langle U, T\big\rangle \left(\frac{\langle U, N\rangle}{|N_H|} \right) \big\langle |N_H|^{-1} \nabla_{e_{2n}} \nu_h, J(\nu_h)\big\rangle \Big\}.
\end{aligned} \tag{3.76}$$

Then we have obtained that $A''(0) + HV''(0) = (3.71) + (3.73) + (3.76)$. Now we want to compute the terms involving horizontal and tangent derivatives of U in (3.71) and (3.76). These have the following expressions

$$\begin{aligned}
&|N_H| \big\langle \nabla_\Sigma^h (\big\langle U, T\big\rangle \big\langle U, \nu_h\big\rangle), 2|N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)\big\rangle = \\
&\mathrm{div}_\Sigma (\big\langle U, T\big\rangle \big\langle U, \nu_h\big\rangle (2 \nabla_{e_{2n}} \nu_h + |N_H| \tau(\nu_h)_{ht})) \\
&- |N_H| \big\langle U, T\big\rangle \big\langle U, \nu_h\big\rangle \Big\{ \mathrm{div}_\Sigma^h (2|N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)_{ht}) \\
&+ 2 \frac{\langle N, T\rangle}{|N_H|} \big\langle J(\nu_h), 2|N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)\big\rangle \Big\},
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
&\big\langle N, T\big\rangle \big\langle \nabla_\Sigma^h (\big\langle U, T\big\rangle^2), |N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)\big\rangle \\
&+ 2|N_H| \big\langle U, T\big\rangle^2 \big\langle \nabla_\Sigma^h \left(\frac{\langle N, T\rangle}{|N_h|} \right), |N_H|^{-1} \nabla_{e_{2n}} \nu_h\big\rangle = \\
&= \mathrm{div}_\Sigma \left(\big\langle N, T\big\rangle \big\langle U, T\big\rangle^2 (|N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)_{ht}) \right) \\
&- |N_H| \big\langle U, T\big\rangle^2 \Big\{ \big\langle \nabla_\Sigma^h \left(\frac{\langle N, T\rangle}{|N_h|} \right), -|N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)\big\rangle \\
&\qquad + \frac{\langle N, T\rangle}{|N_h|} \mathrm{div}_\Sigma^h (\big\langle N, T\big\rangle \big\langle U, T\big\rangle^2 (|N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)_{ht})) \\
&\qquad + 2 \left(\frac{\langle N, T\rangle}{|N_h|} \right)^2 \big\langle J(\nu_h), |N_H|^{-1} \nabla_{e_{2n}} \nu_h + \tau(\nu_h)\big\rangle, \Big\}
\end{aligned} \tag{3.78}$$

and

$$\begin{aligned}
&2\big\langle N,T\big\rangle\big\langle\nabla^h_\Sigma\left(\left(\frac{\langle U,N\rangle}{|N_H|}\right)^2\right),J(\nu_h)\big\rangle = 2\operatorname{div}_\Sigma\left(\big\langle N,T\big\rangle\left(\frac{\langle U,N\rangle}{|N_H|}\right)^2 J(\nu_h)\right)\\
&\quad - 2|N_H|\left(\frac{\langle U,N\rangle}{|N_H|}\right)^2\Big\{2\frac{\langle N,T\rangle^2}{|N_H|^2}\big\langle U,\nu_h\big\rangle^2|J(\nu_h)|^2+\operatorname{div}^h_\Sigma\left(\frac{\langle N,T\rangle}{|N_h|}J(\nu_h)\right)\Big\}.
\end{aligned}\tag{3.79}$$

On the other hand,

$$\begin{aligned}
&\operatorname{div}^h_\Sigma(|N_H|^{-1}\nabla_{e_{2n}}\nu_h) =\\
&\quad = \big\langle N,T\big\rangle\big\langle\nabla^h_\Sigma\left(\frac{\langle N,T\rangle}{|N_H|}\right),\nabla_{e_{2n}}\nu_h\big\rangle + |N_H|^{-1}\sum_{i=1}^{2n-1}R(e_i,e_{2n},\nu_h,e_i)\\
&\quad + |N_H|^{-1}\sum_{i=1}^{2n-1}\{-\big\langle\nabla_{e_{2n}}e_i,\nabla_{e_i}\nu_h\big\rangle+\big\langle\nabla_{[e_i,e_{2n}]}\nu_h,e_i\big\rangle\}\\
&\quad = \frac{\langle N,T\rangle}{|N_h|}\Big\{W-2\big\langle J(\nu_h),|N_H|^{-1}\nabla_{e_{2n}}\nu_h\big\rangle+|\sigma|^2\\
&\quad - \left(\frac{\langle N,T\rangle}{|N_H|}\right)^2\sum_{i=1}^{2n-1}|J(e_i)_{ht}|^2\Big\} - s(T,\nu_h) - \sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}\nu_h,\tau(e_i)\big\rangle,
\end{aligned}\tag{3.80}$$

and

$$\operatorname{div}^h_\Sigma\left(\frac{\langle N,T\rangle}{|N_H|}J(\nu_h)\right) = \big\langle\nabla^h_\Sigma\left(\frac{\langle N,T\rangle}{|N_H|}\right),J(\nu_h)\big\rangle+\frac{\langle N,T\rangle}{|N_H|}\operatorname{div}^h_\Sigma(J(\nu_h)),\tag{3.81}$$

where we express

$$\big\langle[e_i,e_{2n}],e_j\big\rangle = \big\langle N,T\big\rangle\big\langle\nabla_{e_i}\nu_h,e_j\big\rangle+\big\langle\nabla_{e_{2n}}e_j,e_i\big\rangle-|N_H|\big\langle\tau(e_i),e_j\big\rangle$$

and

$$\big\langle[e_i,e_{2n}],e_{2n}\big\rangle = \big\langle N,T\big\rangle\big\langle\nabla_{e_{2n}}\nu_h,e_i\big\rangle-2\big\langle N,T\big\rangle|N_H|\big\langle J(\nu_h),e_i\big\rangle-\big\langle N,T\big\rangle|N_H|\big\langle\tau(\nu_h),e_j\big\rangle$$

to obtain that

$$|N_H|^{-1}\sum_{i=1}^{2n-1}\{-\big\langle\nabla_{e_{2n}}e_i,\nabla_{e_i}\nu_h\big\rangle+\big\langle\nabla_{[e_i,e_{2n}]}\nu_h,e_i\big\rangle\}$$

equals

$$\begin{aligned}
&\frac{\langle N,T\rangle}{|N_H|}\Big\{\big\langle\nabla_{e_{2n}}\nu_h,\nabla_{e_{2n}}\nu_h-2|N_H|J(\nu_h)-|N_H|\tau(\nu_h)\big\rangle+|\sigma|^2\\
&\qquad - \frac{\langle N,T\rangle^2}{|N_H|^2}\sum_{i=1}^{2n-1}|J(e_i)_{ht}|^2\Big\} - \sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}\nu_h,\tau(e_i)\big\rangle.
\end{aligned}\tag{3.82}$$

Now we can decompose the second variation formula in few parts. In particular we can compute easily the tangential divergence part,

(3.83)

$$
\begin{aligned}
D := & 2\operatorname{div}_\Sigma\left(\big\langle N,T\big\rangle\left(\frac{\big\langle U,N\big\rangle}{|N_H|}\right)^2 J(\nu_h)\right)-\operatorname{div}_\Sigma\left(\big\langle U,T\big\rangle^2\sum_{i=1}^{2n-1}\langle\tau(e_i),e_i\rangle e_{2n}\right)\\
&+\operatorname{div}_\Sigma\left(\big\langle N,T\big\rangle\big\langle U,T\big\rangle^2(|N_H|^{-1}\nabla_{e_{2n}}\nu_h+\tau(\nu_h)_{ht})\right)-\operatorname{div}_\Sigma(\big\langle\nabla_{U_{ht}^\perp}U_{ht}^\perp,T\big\rangle e_{2n})\\
&+\operatorname{div}_\Sigma(H\big\langle U,T\big\rangle\big\langle U,\nu_h\big\rangle e_{2n})+\operatorname{div}_\Sigma(\big\langle U,T\big\rangle\big\langle U,\nu_h\big\rangle(2\nabla_{e_{2n}}\nu_h+|N_H|\tau(\nu_h)_{ht}))\\
&+\operatorname{div}_\Sigma(|N_H|(\nabla_{U_{ht}^\perp}U_{ht}^\perp)_{ht}).
\end{aligned}
$$

Finally, by Lemma 3.11 and Lemma 3.12, simplifying, we get that $-|N_H|\big\langle U,\nu_h\big\rangle^2$, $-2\big\langle N,T\big\rangle|N_H|^{-1}\big\langle U,\nu_h\big\rangle\big\langle U,T\big\rangle$, and $-\big\langle N,T\big\rangle|N_H|^{-2}\big\langle U,T\big\rangle^2$, multiply the same quantity q, defined in (3.56). □

REMARK 3.15. Theorem 3.14 coincides, in 3-dimensional pseudo-hermitian manifolds, with the second variation formula for C^3 minimal surfaces in [**25**, Proposition 6.1] and the stability operator for C^2 minimal surfaces in [**45**, Lemma 8.3].

REMARK 3.16. In Appendix B, we prove a second variation formula of the functional $A - HV$, (B.15), for a C^3 smooth surfaces Σ in a contact sub-Riemannian manifold. The proof of (B.15) is easy, since we can differentiate the mean curvature of Σ. Furthermore we remark that (B.15) coincides with the expression in Theorem 3.14.

DEFINITION. An area-stationary hypersurface Σ is *stable* if $A''(0) \geqslant 0$ for any variation induced by a vector field $U \in C_0^\infty(M \setminus \Sigma_0)$. A volume-preserving area-stationary hypersurface is *stable under a volume constrain* if $(A - HV)''(0) \geqslant 0$ for any variation induced by a vector field $U \in C_0^\infty(M \setminus \Sigma_0)$, with $\int_\Omega \operatorname{div} U \, dv_g = 0$.

APPENDIX A

Tangential variations

We collect in this Appendix some lemmas concerning the second variation formulas of area and volume with respect to variations induced by horizontal and tangent vector fields. We define the bilinear operator

$$B(U,W) := \int_\Sigma C(U,W)\,d\Sigma, \tag{A.1}$$

where

(A.2)

$$\begin{aligned}
C(U,W) :=& - e_{2n}\big\langle \nabla_U W, T\big\rangle - \big\langle \nabla_{U_p} \operatorname{Tor}(W,E_{2n}), T_p\big\rangle \\
&+ \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} U + \operatorname{Tor}(U_p,e_i), T_p\big\rangle\big\langle \nabla_{e_{2n}} W + \operatorname{Tor}(W_p,e_{2n}), e_i\big\rangle \\
&+ \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} W + \operatorname{Tor}(W_p,e_i), T_p\big\rangle\big\langle \nabla_{e_{2n}} U + \operatorname{Tor}(U_p,e_{2n}), e_i\big\rangle \\
&- \sum_{i=1}^{2n-1} \big\langle \nabla_{e_{2n}} U + \operatorname{Tor}(U_p,e_{2n}), T_p\big\rangle\big\langle \nabla_{e_i} W + \operatorname{Tor}(W_p,e_i), e_i\big\rangle \\
&- \sum_{i=1}^{2n-1} \big\langle \nabla_{e_{2n}} W + \operatorname{Tor}(W_p,e_{2n}), T_p\big\rangle\big\langle \nabla_{e_i} U + \operatorname{Tor}(U_p,e_i), e_i\big\rangle \\
&+ |(N_h)_p| \sum_{i=1}^{2n-1} \Big\{ - R(e_i,U_p,W_p,e_i) + \big\langle \nabla_{e_i}(\nabla_U W), e_i\big\rangle \\
&\qquad + \big\langle \nabla_{U_p} \operatorname{Tor}(W,e_i), e_i\big\rangle \Big\} \\
&+ |(N_h)_p| \sum_{i,j=1, i\neq j}^{2n-1} \Big\{ \big\langle \nabla_{e_i} U + \operatorname{Tor}(U_p,e_i), e_i\big\rangle\big\langle \nabla_{e_j} W + \operatorname{Tor}(W_p,e_j), e_j\big\rangle \\
&\qquad - \big\langle \nabla_{e_i} U + \operatorname{Tor}(U_p,e_i), e_j\big\rangle\big\langle \nabla_{e_j} W + \operatorname{Tor}(W_p,e_j), e_i\big\rangle \Big\} \\
&+ \frac{1}{|(N_h)_p|} \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} U + \operatorname{Tor}(U_p,e_i), N_p\big\rangle\big\langle \nabla_{e_i} W + \operatorname{Tor}(W_p,e_i), N_p\big\rangle,
\end{aligned}$$

where we have to understand

$$\begin{aligned}
-\big\langle \nabla_{U_p} \operatorname{Tor}(W,E_{2n}), T_p\big\rangle = -2\{&\big\langle (\nabla_U J)(W), e_{2n}\big\rangle + \big\langle J(\nabla_U W), e_{2n}\big\rangle \\
&+ \big\langle J(W), \nabla_{e_{2n}} U + \operatorname{Tor}(U,e_{2n})\big\rangle\}
\end{aligned}$$

and

(A.3)

$$\big\langle \nabla_{U_p} \operatorname{Tor}(W,e_i),e_i\big\rangle = \big\langle \nabla_U W,T\big\rangle\big\langle \tau(e_i),e_i\big\rangle - \big\langle \nabla_{e_i}U+\operatorname{Tor}(U,e_i),T\big\rangle\big\langle \tau(W),e_i\big\rangle + \big\langle W,T\big\rangle\big\langle (\nabla_U\tau)(e_i),e_i\big\rangle + \big\langle W,T\big\rangle\big\langle \nabla_{e_i}U+\operatorname{Tor}(U,e_i),\tau(e_i)\big\rangle.$$

Then we have

LEMMA A.1. *Let Σ be a C^2 hypersurface in M, then $B(U,U_{ht}) = 0$, for all $U \in C_0^\infty(\Sigma)$.*

PROOF. We will prove that $B(U,U_{ht})$ equals

$$\frac{d}{ds}\Big|_{s=0}\int_{\Sigma_s} \operatorname{div}_{\Sigma_s}(|N_h|U_{ht})d\Sigma_s \tag{A.4}$$

which vanishes since

$$\int_{\Sigma_s} \operatorname{div}_{\Sigma_s}(|N_h|U_{ht})d\Sigma_s \tag{A.5}$$

vanishes for all s. We stress that (A.5) is the first variation of the area of Σ_s with respect to the horizontal tangent vector U_{ht}. From the area formula it follows that (A.4) can be rewritten as

(A.6)

$$\int_\Sigma \{ - U(\big\langle \nabla_{E_{2n}}U_{ht}+\operatorname{Tor}(U_{ht},E_{2n})),T\big\rangle + |N_h|\sum_{i=1}^{2n-1}\big\langle \nabla_{E_i}U_{ht}+\operatorname{Tor}(U_{ht},E_i),E_i\big\rangle)$$
$$+ (\big\langle \nabla_{E_{2n}}U_{ht}+\operatorname{Tor}(U_{ht},E_{2n})),T\big\rangle$$
$$+ |N_h|\sum_{i=1}^{2n-1}\big\langle \nabla_{E_i}U_{ht}+\operatorname{Tor}(U_{ht},E_i),E_i\big\rangle)\operatorname{div}_\Sigma U\}d\Sigma,$$

where $E_1,\dots,E_{2n-1}$ is an orthonormal basis of $T\Sigma_s\cap\mathcal{H}$ and E_{2n} is the unit vector generating $T\Sigma_s\cap\operatorname{span}\{T\}$. We remark that this is different from the one used in Chapter 3.

Now, by (3.18), we have

$$-U(\big\langle \nabla_{E_{2n}}U_{ht},T\big\rangle) = -e_{2n}(\big\langle \nabla_U U_{ht},T\big\rangle).$$

On the other hand

$$-\big\langle \nabla_U \operatorname{Tor}(U_{ht},E_{2n}),T\big\rangle = -\big\langle (\nabla_U \operatorname{Tor})(U_{ht},e_{2n}) - \operatorname{Tor}(\nabla_U U_{ht},e_{2n}) - \operatorname{Tor}(U_{ht},\nabla_U E_{2n}),T\big\rangle,$$

that equals the term $-\big\langle \nabla_{U_p}\operatorname{Tor}(U_{ht},E_{2n}),T_p\big\rangle$ in A.2, plus the factor

$$-\big\langle \operatorname{Tor}(U_{ht},[U,E_{2n}]),T\big\rangle =$$
$$-\sum_{i=1}^{2n-1}\big\langle \operatorname{Tor}(U_{ht},e_i),T\big\rangle\big\langle \nabla_U E_{2n}-\nabla_{e_{2n}}U-\operatorname{Tor}(U,e_{2n}),e_i\big\rangle + \big\langle \operatorname{Tor}(U_{ht},e_{2n}),T\big\rangle\big\langle \nabla_{e_{2n}}U+\operatorname{Tor}(U,e_{2n}),e_{2n}\big\rangle. \tag{A.7}$$

To expand (A.7) we observe that $[U,e_{2n}]$ is tangent to Σ. This follows since

$$\big\langle [U,e_{2n}],N\big\rangle = -\big\langle e_{2n},A(U^\top)\big\rangle + \big\langle A(e_{2n}),U^\top\big\rangle = 0,$$

where we have used

$$D_U N = -\nabla_\Sigma(\big\langle U,N\big\rangle) - A(U^\top).$$

Here ∇_Σ and A denote the tangential gradient and the Riemannian Weingarten operator of Σ, respectively.

Now remarking that $U(|N_h|) + |N_H| \operatorname{div}_\Sigma U$ is the first variation formula, we get

$$\begin{aligned}(U(|N_h|) + |N_H| \operatorname{div}_\Sigma U) \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), e_i \big\rangle = \\ = - \sum_{i=1}^{2n-1} \big\langle \nabla_{e_{2n}} U + \operatorname{Tor}(U, e_{2n}), T \big\rangle \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), e_i \big\rangle \\ + |N_H| \sum_{i,j=1}^{2n-1} \big\langle \nabla_{e_j} U + \operatorname{Tor}(U, e_j), e_j \big\rangle \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), e_i \big\rangle.\end{aligned} \tag{A.8}$$

Then

$$\begin{aligned}-\big\langle \nabla_{E_{2n}} U_{ht} + \operatorname{Tor}(U_{ht}, E_{2n})), T \big\rangle \operatorname{div}_\Sigma U = \\ = -\big\langle \operatorname{Tor}(U_{ht}, E_{2n})), T \big\rangle \big\langle \nabla_{E_{2n}} U + \operatorname{Tor}(U, E_{2n})), e_{2n} \big\rangle \\ - \sum_{i=1}^{2n-1} \big\langle \nabla_{E_{2n}} U_{ht} + \operatorname{Tor}(U_{ht}, E_{2n})), T \big\rangle \big\langle \nabla_{e_i} U + \operatorname{Tor}(U, e_i), e_i \big\rangle,\end{aligned} \tag{A.9}$$

where we have computed $\operatorname{div}_\Sigma U$ using (3.10). The last term to be treated is

$$|N_h| \sum_{i=1}^{2n-1} \{ \big\langle \nabla_U \nabla_{E_i} U_{ht} + \nabla_U \operatorname{Tor}(U_{ht}, E_i), E_i \big\rangle + \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), \nabla_U E_i \big\rangle \}. \tag{A.10}$$

By (3.16)

$$\big\langle \nabla_U \nabla_{E_i} U_{ht}, e_i \big\rangle = -R(e_i, U, U_{ht}, e_i) + \big\langle \nabla_{e_i} \nabla_U U_{ht}, e_i \big\rangle + \big\langle \nabla_{[U,E_i]} U_{ht}, e_i \big\rangle,$$

with

$$\begin{aligned}\big\langle \nabla_{[U,E_i]} U_{ht}, e_i \big\rangle = \sum_{j=1}^{2n-1} \big\langle [U, E_i], e_j \big\rangle \big\langle \nabla_{e_j} U_{ht}, e_i \big\rangle \\ + \big\langle [U, E_i], e_{2n} \big\rangle \big\langle \nabla_{e_{2n}} U_{ht}, e_i \big\rangle,\end{aligned}$$

since $[U, E_i]$ is tangent to Σ. The second summand in (A.10) equals the analogous one in (A.2) plus

$$\big\langle \operatorname{Tor}(U_{ht}, [U, E_i]), e_i \big\rangle = \big\langle [U, E_i], T \big\rangle \big\langle \operatorname{Tor}(U_{ht}, T), e_i \big\rangle. \tag{A.11}$$

Finally, since $E_1, \ldots, E_{2n-1}, \nu_h, T$ is an orthonormal basis,

$$\begin{aligned}\big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), \nabla_U E_i \big\rangle = - \sum_{j=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), e_j \big\rangle \big\langle e_i, \nabla_U E_j \big\rangle \\ + \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), \nu_h \big\rangle \big\langle \nu_h, \nabla_U E_i \big\rangle \\ = - \sum_{j=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), e_j \big\rangle \big\langle e_i, \nabla_{e_j} U + \operatorname{Tor}(U, e_j) \big\rangle \\ - \sum_{j=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht}, e_j \big\rangle \big\langle e_i, [U, e_j] \big\rangle \\ + |N_H|^{-2} \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), N \big\rangle \big\langle N, \nabla_{e_i} U + \operatorname{Tor}(U, e_i) \big\rangle \\ - \big\langle N, T \big\rangle |N_H|^{-2} \big\langle \operatorname{Tor}(U_{ht}, e_i), T \big\rangle \big\langle N, \nabla_U E_i \big\rangle,\end{aligned}$$

where we have used $\big\langle [U,E_i],N\big\rangle=0$ and $N=|N_H|N_h+\big\langle N,T\big\rangle T$. Since $|N_H|e_{2n}=-T+\big\langle N,T\big\rangle N$ we get

$$|N_H|\sum_{i=1}^{2n-1}\big\langle [U,E_i],e_{2n}\big\rangle\big\langle \nabla_{e_{2n}}U_{ht},e_i\big\rangle+|N_H|\sum_{i=1}^{2n-1}\big\langle [U,E_i],T\big\rangle\big\langle\,\mathrm{Tor}(U_{ht},T),e_i\big\rangle=$$
$$=\sum_{i=1}^{2n-1}\big\langle \nabla_{e_i}U+\mathrm{Tor}(U,e_i),T\big\rangle\big\langle\nabla_{e_{2n}}U_{ht}+\mathrm{Tor}(U_{ht},e_{2n}),e_i\big\rangle$$

and

$$-\big\langle N,T\big\rangle|N_H|^{-1}\sum_{i=1}^{2n-1}\big\langle\,\mathrm{Tor}(U_{ht},e_i),T\big\rangle\big\langle N,\nabla_U E_i\big\rangle=\sum_{i=1}^{2n-1}\big\langle\,\mathrm{Tor}(U_{ht},e_i),T\big\rangle\big\langle\nabla_U E_{2n},e_i\big\rangle.$$

After simplifying we get that (A.4) equals $B(U,U_{ht})$.

□

LEMMA A.2. *Let Σ a C^2 hyperfurface in M. Then $B(U_{ht},U_{ht}^{\perp})=0$, for all $U\in C_0^{\infty}(\Sigma)$.*

PROOF. We will prove that $B(U_{ht},U_{ht}^{\perp})$ equals

$$\int_{\Sigma}\mathrm{div}_{\Sigma}(\mathrm{div}_{\Sigma}(|N_h|U_{ht}^{\perp})U_{ht})d\Sigma$$

which vanishes, where $(\mathrm{div}_{\Sigma}(|N_h|U_{ht}^{\perp})$ is the first variation of the area of Σ induced by $U_{ht}^{\perp}$. We have

$$\mathrm{div}_{\Sigma}(\mathrm{div}_{\Sigma}(|N_h|U_{ht}^{\perp})U_{ht})=\mathrm{div}_{\Sigma}(|N_h|U_{ht}^{\perp})\,\mathrm{div}_{\Sigma}U_{ht}+U_{ht}(\mathrm{div}_{\Sigma}(|N_h|U_{ht}^{\perp})).$$

Now we compute

$$\begin{aligned}&\mathrm{div}_{\Sigma}(|N_h|U_{ht}^{\perp})\,\mathrm{div}_{\Sigma}U_{ht}=\\&\quad-\sum_{i=1}^{2n-1}\big\langle\nabla_{e_{2n}}U_{ht}^{\perp}+\mathrm{Tor}(U_{ht}^{\perp},e_{2n}),T\big\rangle\big\langle\nabla_{e_i}U_{ht}+\mathrm{Tor}(U_{ht},e_i),e_i\big\rangle\\&\quad-\big\langle\nabla_{e_{2n}}U_{ht}^{\perp}+\mathrm{Tor}(U_{ht}^{\perp},e_{2n}),T\big\rangle\big\langle\nabla_{e_{2n}}U_{ht}+\mathrm{Tor}(U_{ht},e_{2n}),e_{2n}\big\rangle\\&\quad+|N_H|\,\mathrm{div}_{\Sigma}U_{ht}\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}U_{ht}^{\perp}+\mathrm{Tor}(U_{ht}^{\perp},e_i),e_i\big\rangle.\end{aligned}$$

On the other hand

$$\begin{aligned}&U_{ht}(\mathrm{div}_{\Sigma}(|N_h|U_{ht}^{\perp}))=\\&\quad-\big\langle\nabla_{U_{ht}}\nabla_{E_{2n}}U_{ht}^{\perp}+\nabla_{U_{ht}}\mathrm{Tor}(U_{ht}^{\perp},E_{2n}),T\big\rangle\\&\quad+U_{ht}(|N_H|)\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}U_{ht}^{\perp}+\mathrm{Tor}(U_{ht}^{\perp},e_i),e_i\big\rangle\\&\quad+|N_H|\sum_{i=1}^{2n-1}\{\big\langle\nabla_{U_{ht}}\nabla_{E_i}U_{ht}^{\perp}+\nabla_{U_{ht}}\mathrm{Tor}(U_{ht}^{\perp},E_i),e_i\big\rangle\\&\qquad+\big\langle\nabla_{e_i}U_{ht}^{\perp}+\mathrm{Tor}(U_{ht}^{\perp},e_i),\nabla_{U_{ht}}E_i\big\rangle\}.\end{aligned}$$

Since $U_{ht}(|N_H|) + |N_H| \operatorname{div}_\Sigma U_{ht}$ is the first variation of the area of Σ respect to U_{ht}, we get

$$\begin{aligned}(U_{ht}(|N_H|) + |N_H| \operatorname{div}_\Sigma U_{ht}) \sum_{i=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_i), e_i \big\rangle = \\ = - \sum_{i=1}^{2n-1} \big\langle \nabla_{e_{2n}} U_{ht} + \operatorname{Tor}(U_{ht}, e_{2n}), T \big\rangle \big\langle \nabla_{e_i} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_i), e_i \big\rangle \\ + |N_H| \sum_{i,j=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i), e_i \big\rangle \big\langle \nabla_{e_j} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_j), e_j \big\rangle.\end{aligned}$$

By (3.16) and (3.18)

$$-\big\langle \nabla_{U_{ht}} \nabla_{E_{2n}} U_{ht}^\perp, T \big\rangle = -e_{2n}(\big\langle \nabla_{U_{ht}} U_{ht}^\perp, T \big\rangle) - \big\langle \nabla_{[U_{ht}, E_{2n}]} U_{ht}^\perp, T \big\rangle,$$

where

$$\begin{aligned}-\big\langle \nabla_{[U_{ht}, E_{2n}]} U_{ht}^\perp, T \big\rangle = - \sum_{i=1}^{2n-1} \big\langle \nabla_{U_{ht}} E_{2n} - \nabla_{e_{2n}} U_{ht} - \operatorname{Tor}(U_{ht}, e_{2n}), e_i \big\rangle \big\langle \nabla_{e_i} U_{ht}^\perp, T \big\rangle \\ + \big\langle \nabla_{e_{2n}} U_{ht} + \operatorname{Tor}(U_{ht}, e_{2n}), e_{2n} \big\rangle \big\langle \nabla_{e_{2n}} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_{2n}), T \big\rangle.\end{aligned}$$

since $[U_{ht}, E_{2n}]$ is tangent to Σ and $\big\langle \operatorname{Tor}(U_{ht}^\perp, e_{2n}), T \big\rangle = 0$. Now $-\big\langle \nabla_{U_{ht}} \operatorname{Tor}(U_{ht}^\perp, E_{2n}), T \big\rangle$ differs from the analogous term in $B(U_{ht}, U_{ht}^\perp)$ only for

$$\begin{aligned}-\big\langle \operatorname{Tor}(U_{ht}^\perp, [U_{ht}, E_{2n}]), T \big\rangle = \\ - \big\langle \nabla_{U_{ht}} E_{2n} - \nabla_{e_{2n}} U_{ht} - \operatorname{Tor}(U_{ht}, e_{2n}), e_i \big\rangle \big\langle \operatorname{Tor}(U_{ht}^\perp, e_i), T \big\rangle,\end{aligned}$$

where we have used $\big\langle [U_{ht}, E_{2n}], N \big\rangle = 0$ and $\big\langle \operatorname{Tor}(U_{ht}^\perp, e_{2n}), T \big\rangle = 0$. Analogously in $\big\langle \nabla_{U_{ht}} \operatorname{Tor}(U_{ht}^\perp, E_i), e_i \big\rangle$ we only need to consider

$$\begin{aligned}|N_H| \sum_{i=1}^{2n-1} \big\langle \operatorname{Tor}(U_{ht}^\perp, [U_{ht}, E_i]), e_i \big\rangle = |N_H| \sum_{i,j=1}^{2n-1} \big\langle [U_{ht}, E_i]), e_j \big\rangle \big\langle \operatorname{Tor}(U_{ht}^\perp, e_j), e_i \big\rangle \\ + |N_H| \sum_{i=1}^{2n-1} \big\langle [U_{ht}, E_i]), e_{2n} \big\rangle \big\langle \operatorname{Tor}(U_{ht}^\perp, e_{2n}), e_i \big\rangle.\end{aligned}$$

By (3.16)

$$\big\langle \nabla_{U_{ht}} \nabla_{E_i} U_{ht}^\perp, e_i \big\rangle = -R(e_i, U_{ht}, U_{ht}^\perp, e_i) + \big\langle \nabla_{e_i} \nabla_{U_{ht}} U_{ht}^\perp, e_i \big\rangle + \big\langle \nabla_{[U_{ht}, E_i]} U_{ht}^\perp, e_i \big\rangle,$$

where

$$\big\langle \nabla_{[U_{ht}, E_i]} U_{ht}^\perp, e_i \big\rangle = \sum_{j=1}^{2n-1} \big\langle [U_{ht}, E_i], e_j \big\rangle \big\langle \nabla_{e_j} U_{ht}^\perp, e_i \big\rangle + \big\langle [U_{ht}, E_i], e_{2n} \big\rangle \big\langle \nabla_{e_{2n}} U_{ht}^\perp, e_i \big\rangle.$$

The last term to be treated is

$$\begin{aligned}\big\langle \nabla_{e_i} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_i), \nabla_{U_{ht}} E_i \big\rangle = \\ = - \sum_{j=1}^{2n-1} \big\langle \nabla_{e_i} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_i), e_j \big\rangle \big\langle e_i, \nabla_{e_j} U_{ht} + \operatorname{Tor}(U_{ht}, e_j) + [U_{ht}, E_j] \big\rangle \\ + |N_H|^{-2} \big\langle \nabla_{e_i} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_i), N \big\rangle \big\langle N, \nabla_{e_i} U_{ht} + \operatorname{Tor}(U_{ht}, e_i) \big\rangle \\ - \big\langle N, T \big\rangle |N_H|^{-2} \big\langle \nabla_{e_i} U_{ht}^\perp + \operatorname{Tor}(U_{ht}^\perp, e_i), T \big\rangle \big\langle N, \nabla_{U_{ht}} E_i \big\rangle.\end{aligned}$$

Finally since $|N_H|e_{2n} = -T + \langle N, T\rangle N$ and $\langle [U_{ht}, E_i], N\rangle = 0$, we have

$$|N_H|\langle [U_{ht}, E_i], e_{2n}\rangle = \langle \nabla_{e_i} U_{ht} + \mathrm{Tor}(U_{ht}, e_i), T\rangle$$

and $-\langle N, T\rangle |N_H|^{-2}\langle \nabla_{U_{ht}} E_i, N\rangle = -\langle \nabla_{U_{ht}} E_i, e_{2n}\rangle$. Now, simplifying, the statement follows.

□

LEMMA A.3. *Let $\Sigma \subset M$ be a C^2 surface. Then $U_{ht}(V'(s))$ vanishes.*

PROOF. From

$$V'(s) = -\int_{\Sigma_s} \langle U_{ht}^{\perp}, N\rangle \, d\Sigma_s = -\int_{\Sigma} \langle U_{ht}^{\perp}, N\rangle |\operatorname{Jac}\varphi(s)| \, d\Sigma,$$

we get

$$\begin{aligned} U_{ht}(V'(s)) &= -\int_{\Sigma} \{U_{ht}(\langle U_{ht}^{\perp}, N\rangle) + \langle U_{ht}^{\perp}, N\rangle \operatorname{div}_{\Sigma}(U_{ht})\} \, d\Sigma \\ &= -\int_{\Sigma} \operatorname{div}_{\Sigma}(\langle U_{ht}^{\perp}, N\rangle U_{ht}) \, d\Sigma, \end{aligned}$$

where we used $U_{ht}(|\operatorname{Jac}\varphi(s)|) = \operatorname{div}_{\Sigma}(U_{ht})$, [**87**, § 9]. Now the statement follows by the Riemannian divergence theorem.

□

APPENDIX B

A second variation formula for hypersurfaces with higher regularity

We consider a hypersurface $\Sigma \subset M$ of enough regularity to ensure the validity of the following computations, C^3 will suffice. Let $\Omega = \partial\Sigma$ be the region enclosed by Σ, and assume that Σ has constant mean curvature H. We consider a smooth vector field U with compact support and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$. According to formula (3.50), the first variation of the sub-Riemannian area along the deformation is given by

$$-\int_{\Sigma_s} H_s \left\langle U, N_s \right\rangle d\Sigma_s.$$

So the second derivative of the sub-Riemannian area is given by

$$\frac{d^2}{ds^2}\bigg|_{s=0} A(\Sigma_s) = -\int_{\Sigma} \left(\frac{d}{ds}\bigg|_{s=0} H_s \right) \left\langle U, N \right\rangle d\Sigma + H \frac{d^2}{ds^2}\bigg|_{s=0} V(\Omega_s).$$

Hence, to calculate $A''(0) - HV''(0)$, we only need to compute the derivative of the mean curvature H along the deformation. Since H is constant, it is enough to compute $U(H)$ assuming that U is normal to Σ, i.e., $U = \left\langle U, \nu_h \right\rangle \nu_h + \left\langle U, T \right\rangle T$. So we have

(B.1)
$$\begin{aligned} -\frac{d}{ds}\bigg|_{s=0} H_s = -U(H) &= \sum_{i=1}^{2n-1} \left\langle \nabla_U \nabla_{E_i} \nu_h, E_i \right\rangle + \left\langle \nabla_{E_i} \nu_h, \nabla_U E_i \right\rangle \\ &= \sum_{i=1}^{2n-1} \bigg\{ R(U, E_i, \nu_h, E_i) + \left\langle \nabla_{E_i} \nabla_U \nu_h, E_i \right\rangle + \left\langle \nabla_{[U,E_i]} \nu_h, E_i \right\rangle \\ &\qquad + \left\langle \nabla_{E_i} \nu_h, \nabla_{E_i} U + [U, E_i] + \operatorname{Tor}(U, E_i) \right\rangle \bigg\}. \end{aligned}$$

We compute the terms appearing in the last summand. We first observe

Lemma B.1. *Consider the endomorphism of $T\Sigma \cap \mathcal{H}$ given by*

$$-A(e) = \left(\nabla_e \nu_h + \frac{\left\langle N, T \right\rangle}{|N_h|} J(e) \right)_{ht}. \tag{B.2}$$

Then A is self-adjoint,

$$\left\langle A(e), v \right\rangle = \left\langle e, A(v) \right\rangle, \qquad e, v \in T\Sigma \cap \mathcal{H},$$

and the trace of A is equal to the mean curvature of Σ.

A will be called the *intrinsic Weingarten operator* in Σ. We also define the *intrinsic second fundamental form* σ as $\sigma(v) = \left\langle A(v), v \right\rangle$, for any $v \in T\Sigma \cap \mathcal{H}$. Finally by $|\sigma|^2$ we mean

$$|\sigma|^2 = \sum_{i=1}^{2n-1} \big\langle A(e_i), A(e_i)\big\rangle,$$

where $\{e_1, \ldots, e_{2n-1}\}$ is an orthonormal basis of $T\Sigma \cap \mathcal{H}$.

LEMMA B.2. *In the above conditions,*

$$\begin{aligned}(\text{B.3}) \quad \nabla_U \nu_h = \nabla_{U_{ht}} \nu_h - \frac{\big\langle N, T\big\rangle}{|N_h|}\, \nabla^h_\Sigma \big\langle U, T\big\rangle - \nabla^h_\Sigma \big\langle U, \nu_h\big\rangle \\ - \big\langle U, T\big\rangle\, (\tau(\nu_h))_{ht} - 2\, \frac{\big\langle N, T\big\rangle}{|N_h|}\, \big\langle U, \nu_h\big\rangle\, J(\nu_h).\end{aligned}$$

Alternatively, we can write (B.3) *as*

$$\begin{aligned}(\text{B.4}) \quad \nabla_U \nu_h = \nabla_{U_{ht}} \nu_h - \nabla^h_\Sigma \bigg(\frac{\big\langle U, N\big\rangle}{|N_h|} \bigg) + \big\langle U, T\big\rangle\, \nabla^h_\Sigma \bigg(\frac{\big\langle N, T\big\rangle}{|N_h|} \bigg) \\ - \big\langle U, T\big\rangle\, (\tau(\nu_h))_{ht} - 2\, \frac{\big\langle N, T\big\rangle}{|N_h|}\, \big\langle U, \nu_h\big\rangle\, J(\nu_h).\end{aligned}$$

PROOF. We denote by E_i the vector fields invariant by the flow generated by U. Then, from $\big\langle [U_{ht}, e_i], N\big\rangle = 0$, we deduce $\big\langle [U_{ht}^\perp, e_i], N\big\rangle = 0$. Consequently

$$\begin{aligned}\big\langle \nabla_{U_{ht}^\perp} \nu_h, e_i\big\rangle &= \big\langle \nabla_{U_{ht}^\perp} (|N_H|^{-1} N), e_i\big\rangle \\ &= -\big\langle \nabla_{U_{ht}^\perp} E_i, |N_H|^{-1} N\big\rangle \\ &= -\big\langle \nabla_{e_i} U_{ht}^\perp, |N_H|^{-1} N\big\rangle - \big\langle\, \mathrm{Tor}(U_{ht}^\perp, e_i), |N_H|^{-1} N\big\rangle,\end{aligned}$$

for $i \in \{1, \ldots, 2n-1\}$. Now from 3.7 and

$$\big\langle \nabla_{e_i} U_{ht}^\perp, |N_H|^{-1} N\big\rangle = e_i(|N_H|^{-1} \big\langle U, N\big\rangle) + \big\langle U, T\big\rangle e_i(|N_H|^{-1} \big\langle N, T\big\rangle)$$

we get (B.4). On the other hand we can express

$$\big\langle \nabla_{e_i} U_{ht}^\perp, |N_H|^{-1} N\big\rangle = e_i(\big\langle U, \nu_h\big\rangle) + \frac{\big\langle N, T\big\rangle}{|N_h|} e_i(\big\langle U, T\big\rangle)$$

to obtain (B.3). □

We have

LEMMA B.3. *Under the above conditions on* Σ, *if* U *is horizontal and tangent we have*

$$(\text{B.5}) \qquad \int_\Sigma \bigg\{ \big\langle \nabla^h_\Sigma f, U\big\rangle + f\, \mathrm{div}^h_\Sigma\, U + 2\, \frac{\big\langle N, T\big\rangle}{|N_h|} f \big\langle U, J(\nu_h)\big\rangle \bigg\}\, |N_h|\, d\Sigma = 0.$$

In case $U = \nabla^h_\Sigma g$, *we get*

$$(\text{B.6}) \qquad \int_\Sigma \bigg\{ \big\langle \nabla^h_\Sigma f, \nabla^h_\Sigma g\big\rangle + f\, \Delta^h_\Sigma g + 2\, \frac{\big\langle N, T\big\rangle}{|N_h|} f \big\langle \nabla^h_\Sigma g, J(\nu_h)\big\rangle \bigg\}\, |N_h|\, d\Sigma = 0.$$

PROOF. We simply compute $\mathrm{div}_\Sigma(|N_h| f U)$. □

Coming back to (B.1) we have

$$\begin{aligned}\sum_{i=1}^{2n-1} \big\langle \nabla_{E_i}\nu_h, \nabla_{E_i}U \big\rangle &= \sum_{i=1}^{2n-1} \big\langle U, \nu_h \big\rangle \, |\nabla_{E_i}\nu_h|^2 \\ &= \big\langle U, \nu_h \big\rangle \bigg(|\sigma|^2 + \sum_{i=1}^{2n-1} \left(\frac{\langle N, T \rangle}{|N_h|} \right)^2 \, |(J(E_i)_{th}|^2 \bigg),\end{aligned} \tag{B.7}$$

and

(B.8)

$$\begin{aligned}\big\langle \nabla_{E_i}\nu_h, \mathrm{Tor}(U, E_i) \big\rangle &= \big\langle U, T \big\rangle \, \big\langle \nabla_{E_i}\nu_h, \tau(E_i) \big\rangle \\ &= \big\langle U, T \big\rangle \bigg(- \big\langle A(E_i), \tau(E_i) \big\rangle - \frac{\langle N, T \rangle}{|N_h|} \, \big\langle J(E_i)_{th}, \tau(E_i) \big\rangle \bigg).\end{aligned}$$

It remains to compute

$$\big\langle \nabla_{[U,E_i]}\nu_h, E_i \big\rangle + \big\langle \nabla_{E_i}\nu_h, [U, E_i] \big\rangle.$$

We decompose $[U, E_i] = [U, E_i]_{ht}^{\perp} + [U, E_i]_{ht}$ and so the above expression is equal to

$$\big\langle \nabla_{[U,E_i]_{ht}^{\perp}}\nu_h, E_i \big\rangle + 2 \left\langle \left(\nabla_{E_i}\nu_h + \frac{\langle N, T \rangle}{|N_h|} \, J(E_i) \right)_{ht}, [U, E_i]_{ht} \right\rangle.$$

We observe

$$\begin{aligned}\big\langle [U, E_i], T \big\rangle &= -E_i \big\langle U, T \big\rangle - 2 \big\langle U, \nu_h \big\rangle \big\langle J(\nu_h), E_i \big\rangle. \\ \big\langle [U, E_i], \nu_h \big\rangle &= \frac{\langle N, T \rangle}{|N_h|} \, \big(E_i \big\langle U, T \big\rangle + 2 \big\langle U, \nu_h \big\rangle \big\langle J(\nu_h), E_i \big\rangle \big).\end{aligned}$$

Since $[U, E_i]$ is tangent to Σ we get that $[U, E_i]_{ht}^{\perp}$ is also and so

$$\begin{aligned}\nabla_{[U,E_i]_{th}^{\perp}}\nu_h = \big(E_i \big\langle U, T \big\rangle &+ 2 \big\langle U, \nu_h \big\rangle \big\langle J(\nu_h), E_i \big\rangle \big) \times \\ &\times \left(\nabla_{\Sigma}^h \left(\frac{\langle N, T \rangle}{|N_h|} \right) - \tau(\nu_h)_{th} + 2 \left(\frac{\langle N, T \rangle}{|N_h|} \right)^2 J(\nu_h) \right).\end{aligned}$$

and so

$$\sum_{i=1}^{2n-1} \big\langle \nabla_{[U,E_i]_{th}^{\perp}}\nu_h, E_i \big\rangle$$

is equal to

$$\Big\langle \nabla_{\Sigma}^h \big\langle U, T \big\rangle + 2 \big\langle U, \nu_h \big\rangle \, J(\nu_h), + \nabla_{\Sigma}^h \left(\frac{\langle N, T \rangle}{|N_h|} \right) - \tau(\nu_h)_{ht} + 2 \left(\frac{\langle N, T \rangle}{|N_h|} \right)^2 J(\nu_h) \Big\rangle. \tag{B.9}$$

On the other hand, the expression

$$2 \left\langle \left(\nabla_{E_i}\nu_h + \frac{\langle N, T \rangle}{|N_h|} \, J(E_i) \right)_{ht}, [U, E_i]_{ht} \right\rangle = -2 \big\langle A(E_i), [U, E_i]_{ht} \big\rangle$$

is equal to

$$2 \, \big\langle A(E_i), -\nabla_U E_i + \nabla_{E_i} U + \mathrm{Tor}(U, E_i) \big\rangle.$$

If we take the basis E_i composed of eigenvectors, then the product of the left side with $\nabla_U E_i$ vanishes, and so we get that the above expression is equal to

$$2 \big\langle U, \nu_h \big\rangle \big\langle A(E_i), \nabla_{E_i}\nu_h \big\rangle + 2 \big\langle U, T \big\rangle \big\langle A(E_i), \tau(E_i) \big\rangle.$$

Summing up we get

$$-\sum_{i=1}^{2n-1} 2\left\langle A(E_i),[U,E_i]_{ht}\right\rangle = -2\big\langle U,\nu_h\big\rangle|\sigma|^2+2\big\langle U,T\big\rangle\sum_{i=1}^{2n-1}\big\langle A(E_i),\tau(E_i)\big\rangle. \tag{B.10}$$

Now we collect all the previous computations to obtain the following expression for the second derivative of the functional $A-HV$. We remark that, from

$$-|N_H|^{-1}\nabla_{e_{2n}}\nu_h = \nabla_\Sigma^h\Big(\frac{\langle N,T\rangle}{|N_h|}\Big)-\tau(\nu_h)_{ht}+2\Big(\frac{\langle N,T\rangle}{|N_h|}\Big)^2 J(\nu_h), \tag{B.11}$$

the variation of the mean curvature is given by

(B.12)

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} H_s = {}&\operatorname{div}_\Sigma^h\left(\nabla_\Sigma^h\left(\frac{\langle U,N\rangle}{|N_H|}\right)+\big\langle U,T\big\rangle\left(\tau(\nu_h)_{ht}-\nabla_\Sigma^h\left(\frac{\langle N,T\rangle}{|N_h|}\right)\right)\right.\\
&\left.+2\frac{\langle N,T\rangle}{|N_h|}\big\langle U,\nu_h\big\rangle J(\nu_h)\right)\\
&+\big\langle U,\nu_h\big\rangle\Big(|\sigma|^2-\left(\frac{\langle N,T\rangle}{|N_h|}\right)^2\sum_{i=1}^{2n-1}|J(e_i)_{ht}|^2-2\big\langle J(\nu_h),|N_H|^{-1}\nabla_{e_{2n}}\nu_h\big\rangle\Big)\\
&-\big\langle\nabla_\Sigma^h\langle U,T\rangle,|N_H|^{-1}\nabla_{e_{2n}}\nu_h\big\rangle\\
&-\big\langle U,T\big\rangle\Big(\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}\nu_h,\tau(e_i)\big\rangle+2\sum_{i=1}^{2n-1}\big\langle A(e_i),\tau(e_i)\big\rangle\Big)\\
&+\sum_{i=1}^{2n-1}R(e_i,U,\nu_h,e_i).
\end{aligned}$$

And so the second variation $(A''-HV'')(0)$ is given by

$$\begin{aligned}
&-\int_\Sigma\Bigg\{\operatorname{div}_\Sigma^h\left(\nabla_\Sigma^h\left(\frac{\langle U,N\rangle}{|N_H|}\right)\right.\\
&\left.+\big\langle U,T\big\rangle\left(\tau(\nu_h)_{ht}-\nabla_\Sigma^h\left(\frac{\langle N,T\rangle}{|N_h|}\right)\right)+2\frac{\langle N,T\rangle}{|N_h|}\big\langle U,\nu_h\big\rangle J(\nu_h)\right)\\
&-\big\langle\nabla_\Sigma^h\langle U,T\rangle,|N_H|^{-1}\nabla_{e_{2n}}\nu_h\big\rangle\\
&+\big\langle U,\nu_h\big\rangle\Big(|\sigma|^2-\left(\frac{\langle N,T\rangle}{|N_h|}\right)^2\sum_{i=1}^{2n-1}|J(e_i)_{ht}|^2-2\big\langle J(\nu_h),|N_H|^{-1}\nabla_{e_{2n}}\nu_h\big\rangle\Big)\\
&-\big\langle U,T\big\rangle\Big(\sum_{i=1}^{2n-1}\big\langle\nabla_{e_i}\nu_h,\tau(e_i)\big\rangle+2\sum_{i=1}^{2n-1}\big\langle A(e_i),\tau(e_i)\big\rangle\Big)\\
&+\sum_{i=1}^{2n-1}R(e_i,U,\nu_h,e_i)\Bigg\}\frac{\langle U,N\rangle}{|N_H|}\big(|N_H|d\Sigma\big).
\end{aligned} \tag{B.13}$$

Since

$$\begin{aligned}
\big\langle U,T\big\rangle\left(\tau(\nu_h)_{ht}-\nabla_\Sigma^h\left(\frac{\langle N,T\rangle}{|N_h|}\right)\right)+2\frac{\langle N,T\rangle}{|N_h|}\big\langle U,\nu_h\big\rangle J(\nu_h) =\\
=\big\langle U,T\big\rangle|N_H|^{-1}\nabla_{e_{2n}}\nu_h+2\frac{\langle N,T\rangle}{|N_h|}\frac{\langle U,N\rangle}{|N_H|}J(\nu_h),
\end{aligned}$$

we can apply Lemma B.3 to obtain the following expression for the second derivative of the functional $A - HV$,

(B.14)

$$\begin{aligned}
&\int_\Sigma \Bigg\{ \left| \nabla_\Sigma^h \left(\frac{\langle U, N\rangle}{|N_H|} \right) \right|^2 \\
&- 2 \left(\frac{\langle U, N\rangle}{|N_H|} \right)^2 \left(\left\langle J(\nu_h), \nabla_\Sigma^h \left(\frac{\langle N, T\rangle}{|N_H|} \right) + \frac{\langle N, T\rangle}{|N_h|} \operatorname{div}_\Sigma^h (J(\nu_h)) \right\rangle \right\} (|N_H| d\Sigma) \\
&+ \int_\Sigma \Bigg\{ \left\langle \nabla_\Sigma^h \langle U, T\rangle, |N_H|^{-1} \nabla_{e_{2n}} \nu_h \right\rangle + \langle U, \nu_h\rangle \Bigg(W + |\sigma|^2 - \left(\frac{\langle N, T\rangle}{|N_h|} \right)^2 \sum_{i=1}^{2n-1} |J(e_i)_{ht}|^2 \\
&- 2 \left\langle J(\nu_h), |N_H|^{-1} \nabla_{e_{2n}} \nu_h \right\rangle \Bigg) \\
&- \langle U, T\rangle \Bigg(\sum_{i=1}^{2n-1} \left\langle \nabla_{e_i} \nu_h, \tau(e_i) \right\rangle + 2 \sum_{i=1}^{2n-1} \left\langle A(e_i), \tau(e_i) \right\rangle - s(T, \nu_h) \Bigg) \Bigg\} \frac{\langle U, N\rangle}{|N_H|} \big(|N_H| d\Sigma\big) \\
&- \int_\Sigma \operatorname{div}_\Sigma^h (\langle U, T\rangle |N_H|^{-1} \nabla_{e_{2n}} \nu_h) \frac{\langle U, N\rangle}{|N_H|} (|N_H| d\Sigma).
\end{aligned}$$

Finally, from (3.80), we get that the second derivative of $(A'' - HV'')(0)$ equals

$$\int_\Sigma \left\{ \left| \nabla_\Sigma^h \left(\frac{\langle U, N\rangle}{|N_H|} \right) \right|^2 - \left(\frac{\langle U, N\rangle}{|N_H|} \right)^2 q \right\} (|N_H| d\Sigma), \tag{B.15}$$

where

$$\begin{aligned}
q = {} & W + |\sigma|^2 + 2\left(\left\langle J(\nu_h), \nabla_\Sigma^h \left(\frac{\langle N, T\rangle}{|N_H|} \right) \right\rangle \right) - 2 \left\langle J(\nu_h), |N_H|^{-1} \nabla_{e_{2n}} \nu_h \right\rangle \\
& + 2 \frac{\langle N, T\rangle}{|N_h|} \operatorname{div}_\Sigma^h (J(\nu_h)) - \left(\frac{\langle N, T\rangle}{|N_h|} \right)^2 \sum_{i=1}^{2n-1} |J(e_i)_{ht}|^2.
\end{aligned} \tag{B.16}$$

We remark that, because of (B.11), (B.15) coincides with the formula in Theorem 3.14.

REMARK B.4. In the Heisenberg group $\mathbb{H}^n$, denoting $Z = J(\nu_h)$, we have

$$q = |\sigma|^2 + 4Z \left(\frac{\langle N, T\rangle}{|N_h|} \right) + (2n+2) \left(\frac{\langle N, T\rangle}{|N_h|} \right)^2.$$

In fact, computing the divergence with respect to a basis $\{e_1, \ldots, e_{2n-1}\}$ of eigenvectors of A, we get

$$\operatorname{div}_\Sigma^h (J(\nu_h)) = \frac{\langle N, T\rangle}{|N_h|} \sum_{i=1}^{2n-1} |J(e_i)_{ht}|^2.$$

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