ON THE DEVELOPMENT OF EARLY ALGEBRAIC THINKING

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This article deals with the question of the development of algebraic thinking in young students. In contrast to mental approaches to cognition, we argue that thinking is made up of material and ideational components such as (inner and outer) speech, forms of sensuous imagination, gestures, tactility, and actual actions with signs and cultural artifacts. Drawing on data from a longitudinal classroom-based research program where 8-year old students were followed as they moved from Grade 2 to Grade 3 to Grade 4, our developmental research question is investigated in terms of the manner in which new relationships between embodiment, perception, and symbol-use emerge and evolve as students engage in patterning activities.

Keywords: Algebraic thinking; Cognition; Development; Knowledge objectification

Sobre el desarrollo de pensamiento algebraico temprano

Este artículo aborda la cuestión del desarrollo del pensamiento algebraico en estudiantes jóvenes. En contraste con los enfoques mentales de la cognición, sostenemos que el pensamiento está compuesto por componentes materiales y del mundo de las ideas tales como el discurso (interior y exterior), formas de imaginación sensitiva, gestos, tacto y acciones reales con signos y artefactos culturales. Con base en datos obtenidos de un programa de investigación longitudinal basado en el aula en el que se siguió el paso de estudiantes de 8 años de segundo grado a tercero y a cuarto, nuestra pregunta de investigación acerca del desarrollo es investigada en términos de la forma en que surgen y evolucionan nuevas relaciones entre el cuerpo, la percepción y el inicio del uso de símbolos a medida que los estudiantes participan en actividades sobre patrones.

Términos clave: Cognición; Desarrollo; Objetivación del conocimiento; Pensamiento algebraico

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As a subject of scrutiny, development is a relatively recent phenomenon. It appeared in the 18th century as a central concept in the new understanding of nature and the individual as a natural being. One of the first ideas of development was articulated by what came to be known as pre-formation theory. Preformationist theoreticians, like Charles Bonnet (1769), argued that development is the unfolding or growing of preformed structures: The process of bringing out the latent possibilities already possessed by the individual. Within this line of thought, preformationists often portrayed the child as a miniature of the developed adult. Other schools adopted a more dynamic stance, arguing that development is led by final causes (Gould, 1977). From the mid-19th century onward, the causes were seen in the context of the theory of evolution. It is in this context that we find Ernst Haeckel suggesting that heredity and adaptation are the two constitutive physiological functions of living things (Haeckel, 1912, p. 6). He went on to claim that

> *The series of forms through which the individual organism passes during its development from the ovum to the complete bodily structure is a brief, condensed repetition of the long series of forms which the animal ancestors of the said organism, or the ancestral forms of the species, have passed through from the earliest period of organic life down to the present day.* (pp. 2-3)

And it was the same Haeckel who transposed the previous law—that he termed the fundamental law of biogeny—to the development of the mind, asserting a kind of parallelism between historic (or phylogenetic) and life-term (or ontogenetic) developments. He said: “the psychic development of the child is but a brief repetition of the phylogenetic [(i.e., historical)] evolution” (Haeckel quoted by Mengal, 1993, p. 94).

In his genetic epistemology, Piaget was led to revisit Haeckel’s law of parallelism. He concluded that “We mustn’t exaggerate the parallel between history and the individual development, but in broad outline there certainly are stages that are the same” (Bringuier, 1980, p. 48). Piaget’s theory rests indeed on the existence of omnipresent universal mechanisms (e.g., assimilation and accommodation) that explain intellectual development. These mechanisms are biological, not contextual, and account for the broad similarities between phylogenetic and ontogenetic developments. Yet, early cross-cultural psychologists like Werner (1957) and Elias (1991) insisted that the investigation of human development must take into account the contextual, historical, and cultural factors where development occurs. Werner argued that it is impossible to equate a given intellectual stage of a child in a modern society to the stage an adult could have reached in an ancient society because the respective environments, as well as the genetic processes involved in them, are completely different. German sociologist Norbert Elias also mentioned the differences that necessarily result as a consequence of variations in cultural settings. Whereas in traditional societies children participate directly in the life of adults earlier and their learning is done in situ (as apprentices), modern children are instructed indirectly in mediating institutions, or schools (Elias, 1991, pp. 66-67).[1]

As we can see from the previous brief overview, theories of development are in agreement in considering development as related to change. Yet, developmental questions remain difficult to investigate. The reason is very simple: Some theories ascribe different factors or circumstances to the changes that define development (e.g., preformed structures, final causes, universal mechanisms of knowledge formation, culture, etc.). In fact, things are even more complex. Theories of development do not necessarily assume the same ideas about what has been developed. Thus, thinking, as an object of developmental scrutiny, may mean something very different across two separate theories. In traditional cognitive psychology, for instance, thinking is generally understood as something purely mental. However, thinking can also be understood as something that includes a material dimension—not only our body (as contemporary theories of embodiment claim), but something beyond our skin as well, encompassing the materiality of the cultural artifacts around us—. The result is that there are not only differences from one theory to another concerning the factors or circumstances that are considered to be responsible for development to occur; there are also differences about the conceptualization of the entity that is in the process of development.

In this article, I deal with the question of the development of algebraic thinking in young students. In the next section, I sketch the concepts of thinking and development that have oriented the longitudinal classroom research presented here. Then, I comment on some research on early algebra and discuss our findings.

THEORETICAL FRAMEWORK

The research reported here is framed by a theoretical perspective—the theory of knowledge objectification (Radford, 2008a)—grounded in a dialectical philosophy and psychology developed after the works of Hegel, Marx, and Vygotsky. Within the theory of knowledge objectification, thinking is considered a relationship between the thinking subject and the cultural forms of thought in which the subject finds itself immersed. More precisely, thinking is a unity of a sensing subject and a historically and culturally constituted conceptual realm where things appear already bestowed with meaning and objectivity. What this objectivity means is not something transcendentally true. Objectivity in the Hegelian sense adopted here means that things have significance for the thinking subject and for others as well (Hegel, 1978, p. 291). It is in this sense that "Thinking constitutes the unity of subjectivity and objectivity" (p. 289). This is why think-

[1] For a more detailed account see Furinghetti and Radford (2008).

ing is not something produced by an isolated or solipsistic mind, but rather involves Otherness. Indeed, thinking necessarily involves something that is not of our own doing—e.g., language as overt or inner speech or the shapes and other aspects of things in the world to which we attend through perception, tactility hearing, action, etc.—.

Hence, thinking, as we understand it here, is not about representing knowledge; it is the activity of bringing together, in the Hegelian dialectical sense, the thinking subject and cultural forms of thought through language, body, artifacts, and semiotic activity more generally. In other terms, thinking is a tangible social practice materialized in the body (e.g., through kinaesthetic actions, gestures, perception, visualization), in the use of signs (e.g., mathematical symbols, graphs, written and spoken words), and artifacts of different sorts (rulers, calculators, and so on).

To put the previous ideas in psychological terms, thinking is not something that solely happens in the head. Thinking is rather considered as made up of material and ideational components including (inner and outer) speech, objectified forms of sensuous imagination, gestures, tactility, and our actual actions with cultural artifacts. Now, conceiving of thinking as a sensuous, material process that resorts to the body and material culture does not mean that thinking is a collection of items. Thinking is rather a dynamic unity of material and ideal components.

How does thinking develop? The idea of thinking as a unity of subjective-objective material and ideal components paves the way to the dialectical account of its development. And it is in fact this idea that Vygotsky articulated in his late work, in particular in his investigation of the relationship between thinking and speech. For Vygotsky, thinking is not merely embodied in speech, nor is thinking outer speech without sound. In a truly Hegelian spirit, Vygotsky articulated the relationship between thought and speech not as a thing, but as a process of contradictory units that become organized into a dialectical unity or system. The dialectical unit of thinking and speech leads to a new psychic entity that might be termed *speech thinking* (Rieber & Carton, 1987, p. 387) and that is more than each one of its parts. Within the new unity that arises from the dialectical fusion of its units, "Thought is restructured as it is transformed into speech. It is not expressed but completed in the word" (Vygotsky, 1987, p. 251).

The drive of development is to be found in the overcoming of the contradiction of the units. The contradiction appears in the manner in which each of them signifies. Vygotsky provides the example of thinking and speech in the child. In mastering the external aspect of speech, Vygotsky noted that

> *The child begins with the initial single word utterance and moves to the coupling of two or three words, then to the simple phrase and the coupling of phrases, and still later to the complex sentence and connected speech composed of a series of complex sentences*. (p. 250)

The child hence moves from the part to the whole. In the semantic mastering of speech, the movement goes in the opposite direction:

> *The child's first word is not a one-word sentence but a whole phrase. Thus, in the development of the semantic aspect of speech, the child begins with the whole—with the sentence—and only later moves to the mastery of particular units of meaning, to the mastery of the meanings of separate words.* (p. 250)

Meaning in outer speech moves in the opposite direction of meaning in the semantic field. When the contradiction (i.e., the oppositional movement of meaning) is overcome, the units are transformed and subsumed into a new dialectical unity. In this new *speech thinking unity,* speech or word meaning changes. "The discovery that word meaning changes and develops" Vygotsky asserted, "is our new and fundamental contribution to the theory of thinking and speech. It is our major discovery" (Vygotsky, 1987, p. 245).

Naturally, there is much more to thinking than speech, as Vygotsky himself recognized:

> *It has been proven that in a large number of cases thinking takes place without any evidence of the presence of even internal speech. One of the German schools of psychology, the Wurzburg school, has shown that intense mental work may proceed not only without words, but also without any images at all... Processes such as the act of pondering a chess board may also take place without internal speech, solely by means of a combination of visual images.* (Luria & Vygotsky, 1998, p. 140)

To tackle the question of the development of thinking it is thus necessary to take into account the various components that intervene in it (e.g., perception, gestures, speech, artifacts, and symbols). It is also imperative to investigate the manner in which each one of these components signify and become transformed as new complexes of meaning arise and evolve. Thus, to ask the question of the development of algebraic thinking is to ask about the appearance of new structuring relationships between the material-ideational components of thinking (e.g., gesture, inner and outer speech) and the manner in which these relationships are organized and reorganized.

Now, since in the approach sketched here development is not considered to follow an innate path, it is also necessary to consider the contextual conditions that make new forms of thinking possible in the first place. Algebraic thinking—in all its intricacies and subtleties—is not something that will appear spontaneously in ontogeny. Algebraic thinking, as we know it today, has been a lengthy process of conceptualizations and reconceptualizations that, historically speaking, led sometimes to dead ends (Høyrup, 2008). The historical evolution of algebraic thinking required translations into other languages and new cultural interpretations. Algebraic thinking as we know it now has been the result of the interpretations of Babylonian mathematics by the Greeks, that in turn were reinterpreted and developed further by the Arabs in the 9th century and then by the Renaissance mathematicians in the 16th century (Radford, 2001). Embedded in the 16th century forms of labor and production, and new forms of social relations, new algebraic ideas, syntheses, and generalizations about numbers, problems, and patterns became possible (Radford, 2006). The investigation of the development of thinking in students can only be carried out against the background of the historically constituted cultural forms of mathematical thinking that our classroom settings target through the pedagogical choice of the problems and activities.

It is against this theoretical framework that the question of the development of young students' algebraic thinking is investigated in the following sections. In the next section I provide a short overview of current research on early algebra; then, I describe the methodology followed in the research presented here.

Research on Early Algebra

The idea that young students—even with limited knowledge of arithmetic—can start learning some algebraic concepts has received increasing experimental support in the past few years. It has been found that young students can start developing an understanding of key algebraic concepts, such as algebraic aspects of equations and problem solving (Brizuela & Schliemann, 2004), and pattern generalization—e.g., to describe the terms of a sequence according to the position they occupy therein (Becker & Rivera, 2008; Moss & Beatty, 2006; Radford, 2010a; Warren & Cooper, 2008)—.

The recent book edited by Cai and Knuth (2011) presents a state of the art research on early algebra; the significant number of contributions to the book attests to the international interest in rethinking the teaching and learning of algebra in young students. In her commentary chapter on the Cai and Knuth book, Carolyn Kieran summarizes the research trends as pivoting around the following focal themes:

- thinking about the general in the particular;
- thinking rule-wise about patterns;
- thinking relationally about quantity, number, and numerical operations;
- thinking representationally about the relations in problem situations;
- thinking conceptually about the procedural;
- anticipating, conjecturing, and justifying; and
- gesturing, visualizing, and languaging (Kieran, 2011, p. 581).

Kieran goes on to argue that "An additional, but non-negligible, thread running through almost all the chapters is that algebraic thinking does not develop unaided in students. The role of the teacher is crucial" (p. 592). Kieran's remark is consonant with Ferdinand Rivera's findings described in another recent book. Rivera notes that "In my Grade 2 class, when they [the students] were presented with two numerical and figural pattern tasks prior to formal instruction, none of them exhibited functional thinking relative to the numerical tasks" (Rivera, 2011, p. 195). Rivera observed that the students

> *were primarily engaged in empirical counting (e.g., counting all at each stage; counting on from one stage to the next; skip counting by 2s or by 4s; combinations of count-all and count-on) with very little indication of a concern toward structural understanding.* (p. 195)

The awareness of a structural understanding, as we shall see in the remainder of this article, is indeed a crucial aspect of the emergence of algebraic thinking. As the previous brief account intimates, early algebra has become a very active research field within mathematics education. However, many research questions remain open (Carraher & Schliemann, 2007; Carraher, Schliemann, Brizuela, & Earnest, 2006). For instance, little is known about how algebraic thinking develops in young students. This article seeks to contribute to this research question.

Methodology: Data Collection and Analysis

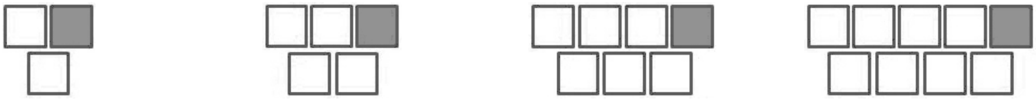
Our data comes from a 3-year longitudinal research program conducted in an urban primary school in which a class of 25 8-year old students was followed as the students moved from Grade 2, to Grade 3, and to Grade 4. The data was collected during regular mathematics lessons designed by the teacher and our research team. To collect data, we used four or five video cameras, each filming one small group of students (groups of 2 or 3). The data that is presented here comes from episodes of what happened when the students were dealing with questions about pattern generalization. We focus in particular on one student, Carlos, whose developmental path is representative of our findings. In tune with our theoretical framework, to investigate the development of early algebraic thinking we conducted a multi-semiotic data analysis. Once the videotapes were fully transcribed, we identified salient episodes within the activities. Focusing on the selected episodes, we carried out a low-motion and a frame-by-frame fine-grained video microanalysis to study the role of and the relationship between gestures, language, and mathematical signs.

Results and Discussions

Findings of the three episodes—Grade 2, Grade 3, and Grade 4—are presented in this section.

First Episode: Grade 2

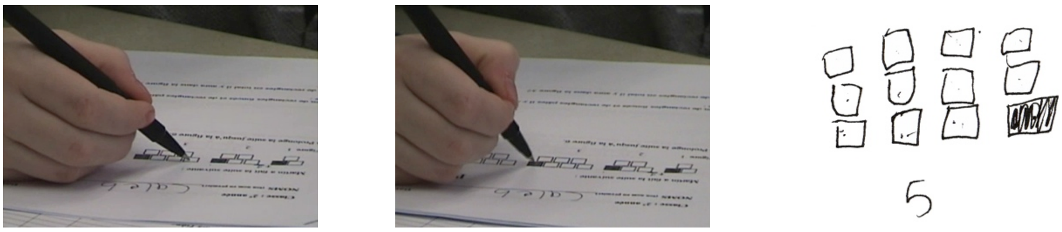
The first algebra activity that the students tackled in Grade 2 revolved around the sequence shown in Figure 1.



Term 1 Term 2 Term 3 Term 4

Figure 1. The first four pictures of a sequence given to the students

In the first part of the activity, the students were asked to extend the sequence up to Term 6. Carlos, one of the students, started counting the squares aloud, accompanying the counting process with a rhythmic upper body movement and pen-pointing gestures. He counted all the squares in an orderly way, beginning with the squares in the top row, from left to right, then addressing those in the bottom row (see Figure 2, Pictures 1 and 2). Then he drew Term 5 in an orderly manner, starting from the bottom row, left to right. Although Term 5 contains almost the right number of squares, it certainly does not conform to the two-row arrangement of the given terms of the sequence (see Figure 2, Picture 3).



Picture 1 Picture 2 Picture 3

Figure 2. Carlos's drawing of Term 5

To come up with an interpretation of Carlos's actions, let us note that, generally speaking, to extend a pattern sequence, the students need to grasp a regularity that involves the linkage of two different structures: one spatial and the other numerical. From the spatial structure emerges a sense of the squares' spatial position, whereas their numerosity emerges from a numerical structure. While Carlos attends to the numerical structure in the generalizing activity, the spatial structure is not coherently emphasized. This does not mean that Carlos does not see the terms as composed of two horizontal rows. As with other students, Carlos's emphasis on the numerical structure somehow leaves in the background the geometric structure. This emphasis reappeared when he finished drawing Term 5: Since the shape of the term did not provide him with a clue about its numerosity, he might have felt the need to count the squares again. We could say that the shape of the terms of the sequence is used to facilitate the counting process—as he always counted the squares in a term in a spatial orderly way—, but that the geometric structure does not come to be related to the numerical one in a meaningful and efficient way. Carlos's process can be contrasted with Kyle's, where shape is emphasized but numerosity is not well-attended. Kyle drew Term 5 as having 2 rows but drew 4 squares on the bottom and 4 squares on the top row. These examples—as well as those reported by Rivera (2010) with other Grade 2 students—suggest that the linkage of spatial and numerical structures constitutes an important aspect of the development of algebraic thinking. Each structure offers a different or opposing—or contradicting, in the dialectic-Vygotskian terminology—form of signifying, and its linkage results in a complex meaning.

That the aforementioned linkage of geometric and numerical structures is less natural than it may appear at first sight can be made evident if we resort to studies in special education. It is well known that children with Down Syndrome tend to reproduce terms such as Term 5 considering their shape without paying much attention to numerical details; by contrast, children with Williams Syndrome tend to present more analytical thinking, and focus on the numerical in detriment to the spatial (Brigaglia, 2010). Or as Bellugi, Lai, and Wand (1997) note, William Syndrome subjects are typically impaired at reproducing global forms, while Down Syndrome subjects tend to produce global forms without local information. Coming back to our Grade 2 students, it is interesting to note that in extending the sequences, the students did not use deictic spatial terms, like bottom or top. (There was one exception: Kyle, who talked once about the top row, without then using it in a systematic manner.) In the cases in which the students did succeed in linking the spatial and numerical structures, the spatial structure appeared ostensibly only, i.e., in the embodied realm of action and perception (Radford, 2011).

The geometric structure reached the realm of language the next day, when the teacher discussed the sequence with the students. During the debriefing of the first day, it was agreed with the teacher that it would be important to bring to the students' attention the linkage of the numerical and spatial structures. To do so, the teacher drew the first five terms of the sequence on the blackboard and refereed to an imaginary student who counted by rows: "This student," she said to the class, "noticed that in Term 1 [she pointed to the name of the term] there is one rectangle on the bottom [and she pointed to the rectangle on the bottom], one on the top [pointing to the rectangle], plus one dark rectangle [pointing to the dark rectangle]." Next, she moved to Term 2 and repeated in a rhythmic manner the same counting process coordinating the spatial deictics, bottom and top, the corresponding spatial rows of the term, and the number of rectangles therein. To make sure that everyone was following, she started again from Term 1 and, at Term 3, she invited the students to join her in the counting process, going together up to Term 5 (see Figure 3).

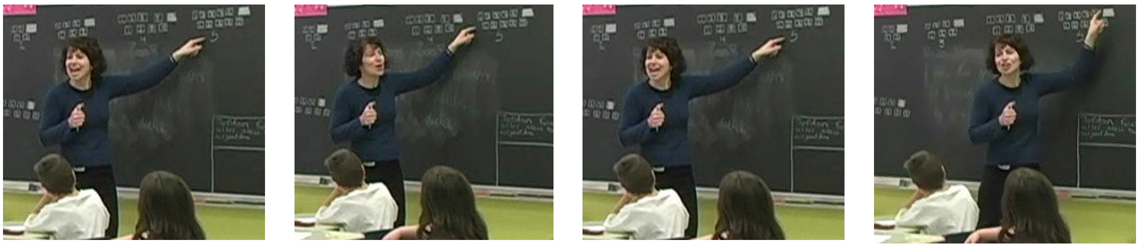


Figure 3. The teacher draws the first five terms of the sequence on the blackboard

Then, the teacher asked the class about the number of squares in Term 25. Mary raised her hand and answered: “25 on the bottom, 25 on top, plus 1.” The class spent some time dealing with remote terms, such as Term 50, and 100. Schematically speaking, the students’ answers were “x on the bottom, x on the top, plus 1” where x was always a specific number. Since at the time the students were able to make systematic additions up to 25, the teacher made calculators available to them and asked the students to explain the steps to calculate the total number of squares in specific terms. Schematically speaking, the students’ answer was “$x + x + 1$” (where x was always a specific number).

The students came back to small-group work and continued their tasks. In one of the questions, they had to explain how Pierre should proceed to build a big term of the sequence. The goal of this and other similar questions was to give the students the opportunity to objectify a numerical-spatial regularity of the given terms of the sequence and to use it to imagine and deal with remote (or even unspecified) terms. Carlos wrote: “Pierre wants to build Term 10,000. Pierre has to put 10,000 on the bottom[;] on the top he has to put 10,001.”

In our PME 34 paper (Radford, 2010b) we dealt with the nature of the students’ emergent algebraic thinking. What we want to discus here is the question of development. As stated in our theoretical framework, conceptual development is marked by the appearance of new relationships between the material-ideational components of thinking; it brings forward new forms of psychic functioning. If during the first day Carlos and other students were emphasizing the analytic process of counting squares one by one, from the second day on, their perception of the terms and the counting processes changed. The link between the spatial and geometric structures was achieved and, as illustrated by Mary’s and Carlos’s answers, spatial deictics became part of their linguistic repertoire. These changes bear witness to the appearance of new relationships between gesture, speech, perception, imagination, and counting. A new unity of material and ideational components of thinking was forged. Thus, the students were able not only to imagine remote terms (e.g., Term 100)—which would be difficult to imagine within the relationships of ideal and material components of thinking underpinning pure analytic, one-by-one counting procedures—but also to devise formulas to calculate the number of squares in terms beyond perception (e.g., “$100 + 100 + 1$”).

The joint counting process in which the teacher and the students engaged during the second day is, of course, an instance of a zone of proximal development. The explicit use of rhythm, gestures, and linguistic deictics by the teacher, followed later by the students, opened up new possibilities for the student to use efficient and evolved cultural forms of mathematical generalization that they successfully applied to other sequences with different shapes. The joint counting process made it possible for the students to notice and articulate new forms of mathematical generalization. In particular, they became aware of the fact that the counting process can be based on a relational idea: to link the number of the term to relevant parts of it (e.g., the squares on the bottom row). This requires an altogether new perception of the number of the term and of the terms themselves. The term appears now not as a mere bunch of ordered squares but as something susceptible to being decomposed, the decomposed parts bearing potential clues for algebraic relationships to occur. But it is not only perception that is developmentally modified. In the same way as perception develops, so do speech (e.g., through spatial deictics) and gesture (through rhythm and precision). Indeed, perception, speech, gesture, and imagination develop in an interrelated manner. They come to form a new unity of the material-ideational components of thinking, where words, gestures, and signs more generally, are used as means of objectification, or as Vygotsky (1987) put it, "as means of voluntary directing attention, as means of abstracting and isolating features, and as a means of […] synthesizing and symbolising" (p. 164).

Second Episode: Grade 3

As usual, in Grade 3 the students were presented with generalizing tasks to be tackled in small groups. The first task featured a figural sequence, *Sn*, having n circles horizontally and $n-1$ vertically, of which the first four terms were given. Contrary to what he did first in Grade 2, from the outset, Carlos perceived the sequence, and took advantage of the spatial configuration of its terms. Talking to his teammates about Term 4 he said: "here [pointing to the vertical part] there are four. Like you take all this [i.e., the vertical part] together [he draws a line around], and you take all this [i.e., the horizontal part] together [he draws a line around; see Figure 4, Picture 1]. So, we should draw 5 like that [through a vertical gesture he indicates the place where the vertical part should be drawn] and [making a horizontal gesture] 5 like that" (see Figure 4, Pictures 2 and 3).

Picture 1 Picture 2 Picture 3 Picture 4 Picture 5

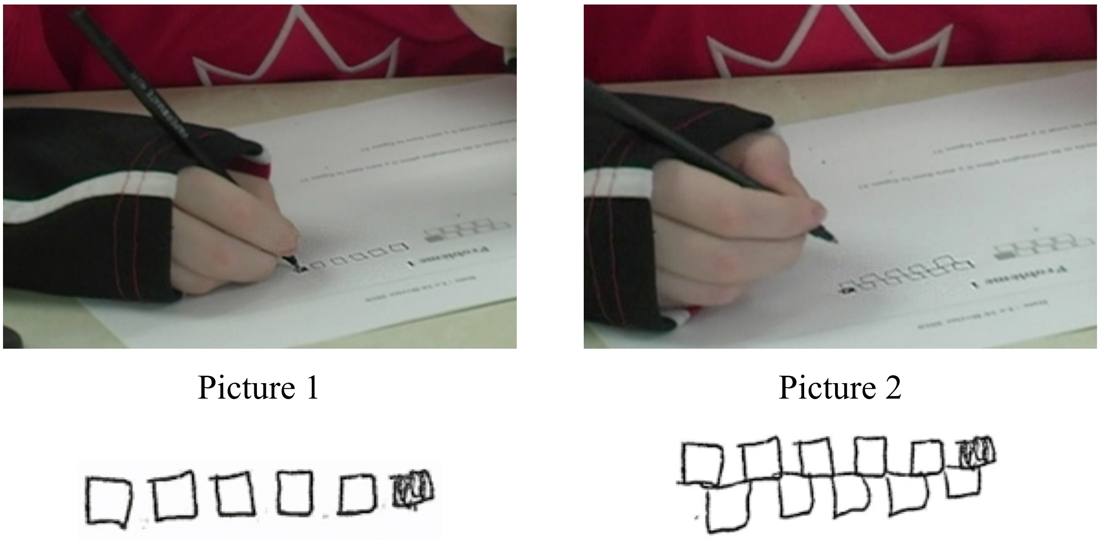
Figure 4. Carlos talking about the spatial configuration of the sequence terms

When the teacher came to see the group, she asked Carlos to sketch Term 10 for her, then Term 50. The first answer was given using unspecified deictics and gestures. He quickly said: "10 like this (vertical gesture) and 10 like that (horizontal gesture)". The specific deictic term "vertical" was used in answering the question about Term 50. He said: "50 on the vertical… and 49…" When the teacher left, the students kept discussing how to write the answer to the question about Term 6. Carlos wrote: "6 vertical and 5 horizontal".

In developmental terms, we see the evolution of the unity of ideational-material components of algebraic thinking. Now, Carlos by himself and with great ease coordinates gestures, perception, and speech. The coordination of these outer components of thinking is much more refined compared to what we observed in Grade 2. This refinement is what we have called a semiotic contraction (Radford, 2008b) and it is a symptom of learning and conceptual development.

Third Episode: Grade 4

To check developmental questions, in Grade 4 we gave the students the sequence with which they started in Grade 2 (see Figure 1). This time, from the outset, Carlos perceived the terms as being divided into two rows. Talking to his teammates and referring to the top row of Term 5, he said as if talking about something banal: "5 white squares, 'cause in Term 1, there is one white square (making a quick pointing gesture)… Term 2, two [squares] (making another quick pointing gesture); 3, (another quick pointing gesture) three." He drew the five white squares on the top row of Term 5 and added: "after that you add a dark square" (see Figure 5, Picture 1). Then, referring to the bottom row of Term 4: "there are four; there [Term 5] are five." He drew the term as shown in Figure 5, Picture 2.



Picture 1 Picture 2

Picture 3 Picture 4

Figure 5. Carlos draws Term 5

When the teacher came to see their work, Carlos and his teammates explained "We looked at Term 2, it's the same thing [i.e., two white squares on top]… Term 6 will have six white squares." In his answer to the question about explaining what Pierre has to do to build a big term of the sequence, Carlos wrote: "He needs [to put as many white squares as] the number of the term on top and on the bottom, plus a dark square on top." The algebraic formula that he provided is shown in Figure 6.

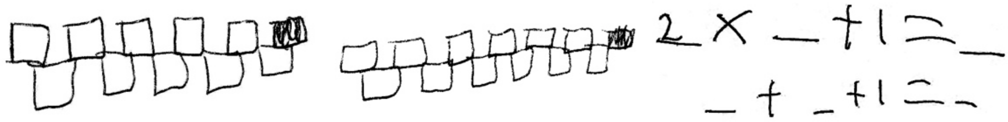


Figure 6. Carlos's drawings of Terms 5 and 6 and corresponding formulas

From a developmental perspective, we see how Carlos's use of language has been refined. In Grade 2 he was resorting to particular terms (Term 1,000) to answer the same question. Here he deals with indeterminacy in an easy way, through the expression "the number of the term". He even goes further and produces two symbolic expressions to calculate the total of squares in the unspecified term.

SYNTHESIS AND CONCLUDING REMARKS

This article seeks to contribute to the question of the development of young students' algebraic thinking. Framed by the theory of objectification, it was suggested that thinking is a unity of material and ideal components—inner and outer speech, forms of sensuous visualization and imagination, gestures and tactility, etc. —. Our developmental question did not consist in investigating how particular psychic functions or components of algebraic thinking evolve, but rather in studying the new links that arise between these functions or components as children engage in classroom activities. As Vygotsky (1999) put it, "higher mental functions arise as a specific neoformation, as a new structural whole that is characterized by the new functional relations that are being established within it" (p. 45). In harmony with this view, we consider development to consist of the refinement of previous, and the appearance of new, structuring relationships between the material-ideational components of thinking.

Within this framework, early algebraic thinking is based on the student's possibilities to grasp patterns in culturally evolved co-variation ways and use them to deal with questions of remote and unspecified terms. The culturally evolved co-variation ways that experienced individuals with algebra exhibit constitute the ideal form to which development is teleologically attuned through the design of our classroom activities. This ideal form is what serves as a cultural reference point in instruction—and which constitutes the objective dimension of thinking as the unity of subjectivity and objectivity, in Hegel's terms—. Development is indeed the result of the interaction of the real forms (as shown in concrete activity) and the ideal form in culture.

Cognitively speaking, for development to occur, the students have to resort to a coordination of numeric and spatial structures. The awareness of these structures and their coordination entail a complex relationship between (inner or outer) speech, forms of visualization and imagination, gesture, and activity on signs (e.g., numbers and proto-algebraic notations). Our data offer a glimpse of the evolution of algebraic thinking. It shows how in Grade 2 spontaneous perception was successfully transformed through the joint work of the teacher and the students (see also Radford, 2010c). This joint work, we suggested, might be conceptualized as occurring in a zone of proximal development out of which the students created new psychological functions. As Vygotsky (1999) notes, the new intellectual operations "are not simply invented by children or acquired from adults but arise… only after a series of qualitative transformations of which each promotes the next step, being itself promoted by the preceding step, and connects them as a stage of a single process historical in nature" (p. 49). A substantial refinement of the new relationships between the material-ideational components of algebraic thinking was accomplished in Grade 3. In Grade 4 we witnessed the marked evolution of language allowing the students to deal with indeterminate terms of sequences (e.g., through the insertion of general expressions such as "the number of the term") and the appearance of a new component—symbolic activity (see Figure 6). At this point we are investigating how the new symbolic activity gives rise to abstract algebraic notations—.

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