

Dynamic Magnification Factor of SDOF Oscillators under Harmonic Loading

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Abstract

The magnification factor for the steady-state response of a SDOF system under harmonic loading is described in many structural dynamics textbooks; the well known analytical solution is easily obtained from the solution to the damped equation of motion for harmonic loading. The complete and steady state solutions can differ significantly. An analytical expression for the maximum response to the complete solution (steady state plus transient) remains elusive; however, a simple analytical expression is identified herein for the undamped case. Differences in the magnification factors obtained for both solutions are discussed.

Keywords: Dynamic magnification factor, steady state, transient state.

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1. Introduction

One of the fundamental topics in most if not all structural dynamics books is the well studied equation of motion for a damped harmonic oscillator [1-5], which for free vibration is given as:

$$m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = 0 \quad (1)$$

where m , k , and c are the mass, the stiffness and the coefficient of viscous damping, respectively. When subjected to an external force $p(t)$, as presented in Figure 1, the equation of motion is:

$$m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = p(t) \quad (2)$$

where $p(t)$ is the force applied to the mass.

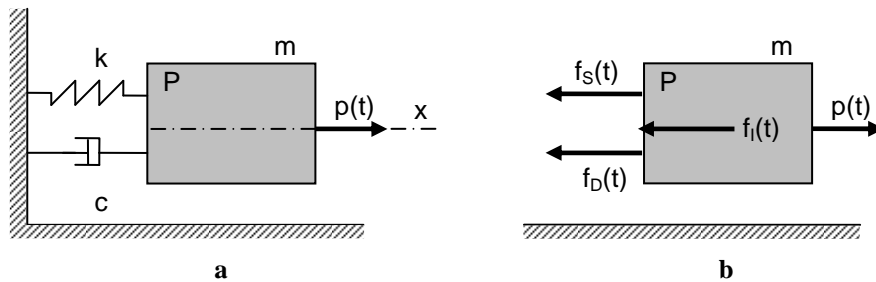


Figure 1. Oscillator: (a) idealized physical configuration and (b) forces acting on the mass for acceleration $\ddot{x} > 0$.

Customarily, two new parameters are defined as functions of m , c , and k . The undamped natural frequency, ω_0 , is given by:

$$\omega_0 = \sqrt{\frac{k}{m}} \quad (3)$$

and the damping ratio, ξ , is given by:

$$\xi = \frac{c}{2 \cdot m \cdot \omega_0} \quad (4)$$

Introducing these two parameters allows (2) to be rewritten as:

$$\ddot{x}(t) + 2 \cdot \xi \cdot \omega_0 \cdot \dot{x}(t) + \omega_0^2 \cdot x(t) = \frac{p(t)}{m} \quad (5)$$

If the excitation is sinusoidal ($p(t) = p_0 \cdot \sin(\Omega \cdot t)$), where p_0 and Ω are the amplitude and frequency of the applied force, respectively, the equation of motion can be solved analytically. The well-known solution is:

$$x(t) = \underbrace{e^{-\xi \omega_0 t} (c_1 e^{-i \omega_1 t} + c_2 e^{i \omega_1 t})}_{\text{Transient}} + \underbrace{\frac{p_0 [-2\xi \omega_0 \Omega \cos(\Omega t) + (\omega_0^2 - \Omega^2) \sin(\Omega t)]}{m [\omega_0^4 + 2(2\xi^2 - 1)\omega_0^2 \Omega^2 + \Omega^4]}}_{\text{Steady-state}} \quad (6)$$

where

$$\omega_1 = \omega_0 \sqrt{1 - \xi^2} \quad (7)$$

is the damped natural frequency of the system and c_1 and c_2 are two complex conjugated constants that can be evaluated once the boundary conditions have been specified. As indicated in (6), the response is composed of a *transient* term that vanishes with time and a *steady-state* term that is a harmonic function of time.

This paper addresses the solution to (5) given by (6). The peak response (maximum of absolute value) is sought and is compared with the amplitude of the steady state portion of the response in (6). Results are considered as a function of a frequency ratio, β , where β the ratio of the frequency of the excitation (Ω) and the *undamped natural frequency* of the system (ω_0). As will be seen, particularly for $\beta > 1$, the peak response is significantly greater than the *steady-state* peak response.

2. Total response versus steady-state response

At rest boundary conditions (i.e. $x(0)=0$ and $\dot{x}(0)=0$) are considered because those are the actual conditions under any earthquake motion or any other dynamic motion in structural engineering. For these boundary conditions, (6) becomes:

$$x(t) = \frac{\left(\frac{p_0}{k}\right)}{\underbrace{\sqrt{1-2(1-2\xi^2)\beta^2+\beta^4}}_a} \left[\underbrace{\frac{-e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}} \beta \cos(\omega_1 t - \phi_1)}_I + \underbrace{\cos(\Omega t - \phi_2)}_{II} \right]$$

$$\phi_1 = \arctan\left(\frac{-\beta^2 + 1 - 2\xi^2}{-2\xi\sqrt{1-\xi^2}}\right); \quad (8)$$

$$\phi_2 = \arctan\left(\frac{-2\xi\beta}{1-\beta^2}\right)$$

The *transient* and a *steady-state* components of (8) are identified by $a \cdot I$ (meaning a times I) and $a \cdot II$ respectively. The maximum of the *steady-state* component is easily computed since this part is a constant (a) multiplied by a function (II) whose peak value is 1. Therefore, the peak *steady-state response* is given by a :

$$\max x_{steady} = a = \frac{1}{\sqrt{1-2(1-2\xi^2)\beta^2+\beta^4}} \frac{p_0}{k} \quad (9)$$

The term of Eq. 9 that multiplies p_0/k has been called the *steady-state dynamic magnification factor* in numerous textbooks. Figure 2 shows the *steady-state dynamic magnification factor* as function of β for different values of the *damping ratio* ξ .

Resonance is easily appreciated in Figure 2. For the *undamped* case ($\xi=0$) the resonant frequency is equal to the *undamped natural frequency* (ω_0). It will be apparent that the resonant frequencies are maintained even for the complete solution to (8), that is, where the transient portion is included in determining the peak response amplitude.

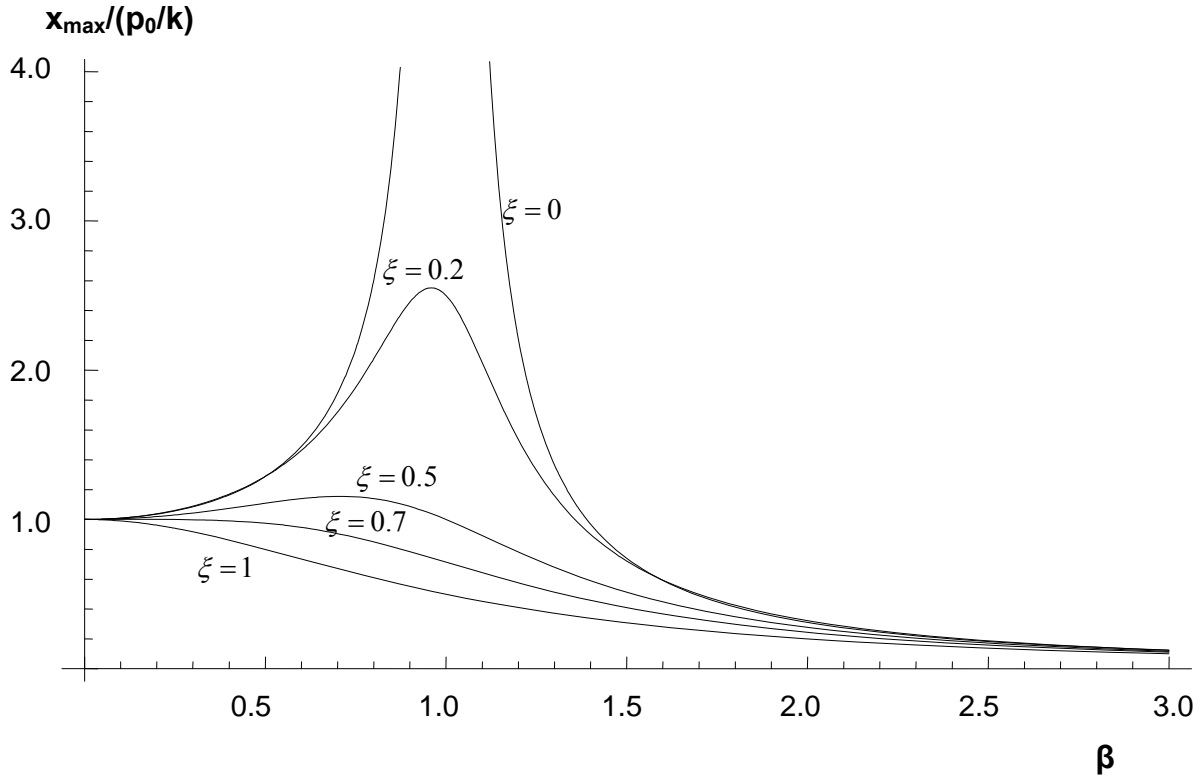


Figure 2. Variation of the steady-state dynamic magnification factor with ξ and β .

Although a simple, explicit, function describing the peak response to (8) cannot be obtained easily, the value of the peak response does not depend on the value of ω_0 , as should be apparent upon closer inspection of Eq. (8). Consider that the responses of two systems having the same β and ξ but different *undamped natural frequencies* ω_{0a} and ω_{0b} would be described by:

$$x_a(t) = \frac{\left(\frac{p_0}{k}\right)}{\sqrt{[1-2(1-2\xi^2)\beta^2 + \beta^4]}} \left[\frac{-e^{-\xi\omega_{0a}t}}{\sqrt{1-\xi^2}} \beta \cos(\omega_{0a}\sqrt{1-\xi^2}t - \phi_1) + \cos(\beta\omega_{0a}t - \phi_2) \right] \quad (10)$$

$$x_b(t) = \frac{\left(\frac{p_0}{k}\right)}{\sqrt{[1-2(1-2\xi^2)\beta^2 + \beta^4]}} \left[\frac{-e^{-\xi\omega_{0b}t}}{\sqrt{1-\xi^2}} \beta \cos(\omega_{0b}\sqrt{1-\xi^2}t - \phi_1) + \cos(\beta\omega_{0b}t - \phi_2) \right] \quad (11)$$

Let us substitute for time t in Eq. (11) the modified time $\frac{\omega_{0a}}{\omega_{0b}}t$:

$$x_b\left(\frac{\omega_{0a}}{\omega_{0b}} t\right) = \frac{\left(\frac{p_0}{k}\right)}{\sqrt{[1 - 2(1 - 2\xi^2)\beta^2 + \beta^4]}} \left[\frac{-e^{-\xi\omega_{0a}t}}{\sqrt{1 - \xi^2}} \beta \cos(\omega_{0a}\sqrt{1 - \xi^2}t - \phi_1) + \cos(\beta\omega_{0a}t - \phi_2) \right]$$

$$\Rightarrow x_b\left(\frac{\omega_{0a}}{\omega_{0b}} t\right) = x_a(\omega_{0a}t) \quad (12)$$

Therefore, the peak response (over all time) is independent of ω_0 , and can be obtained using Eq. (8) for an arbitrary value of the *undamped natural frequency* (e.g. $\omega_0 = 1.0\text{Hz}$). Systems having other values of ω_0 will have the same peak, but occurring at a different time (see Figure 3). For a system having undamped natural frequency $\omega_{0,s}$ the time of the peak is delayed an amount

$$\Delta t = t_0 \left(1 - \frac{1}{\omega_{0,s}}\right) \quad (13)$$

where t_0 is the time corresponding to the maximum response for $\omega_0 = 1.0\text{Hz}$, and $\omega_{0,s}$ is the *undamped natural frequency* of the system being analyzed.

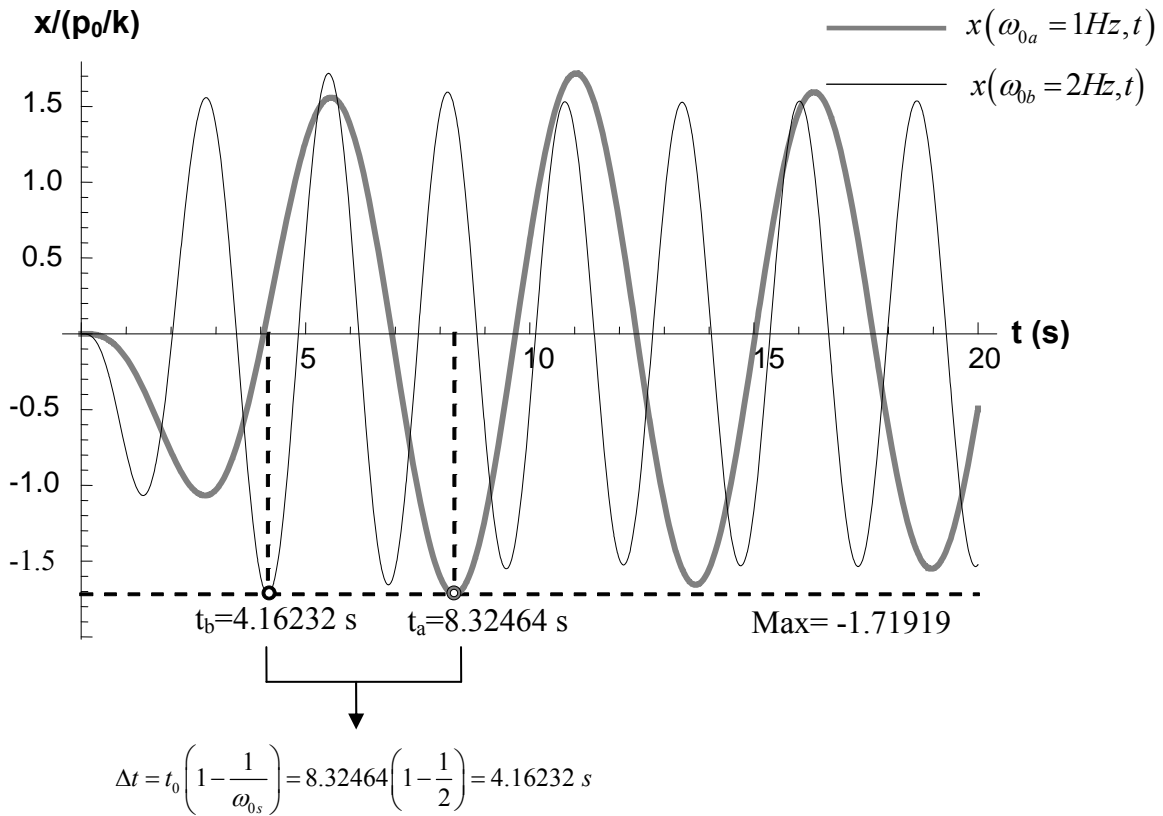


Figure 3. Responses corresponding to $\xi = 0.2$, $\beta = 1.2$ for different natural frequencies. The same maximum occurs, but at different times as shown.

Peak values of the response expression can be determined using various mathematical software programs. In the present case, the nonlinear constrained global optimization package function FindMaximum was used within the Mathematica® program.

Plots of both *steady-state* and *complete magnification factor* (including both transient and steady-state responses) for different values of *damping ratio* are presented in Figure 4. Results were obtained using the *undamped natural frequency* set to 1 Hz ($\omega_0 = 1.0\text{Hz}$).

Figure 4 shows that the *complete magnification factor* is always greater than or equal to the *steady-state magnification factor*. The closer β is to ω_1/ω_0 the smaller the difference is between both magnification factors, except for the case $\xi = 1$. Also, the lower the value of the

damping ratio (ξ) the greater is the difference between the *complete* and *steady-state maximum responses*. For instance, in case of $\xi = 0$, with a value of $\beta = 0.7$, the *complete magnification factor* is 3.32 while the *steady-state magnification factor* is 1.96. In this case the *steady-state magnification factor* represents only 60% of the *complete maximum response*. For $\xi = 0.5$ and $\beta = 3.0$, Figure 4 shows that the *steady-state magnification factor* is around the 45% of the *complete magnification factor*.

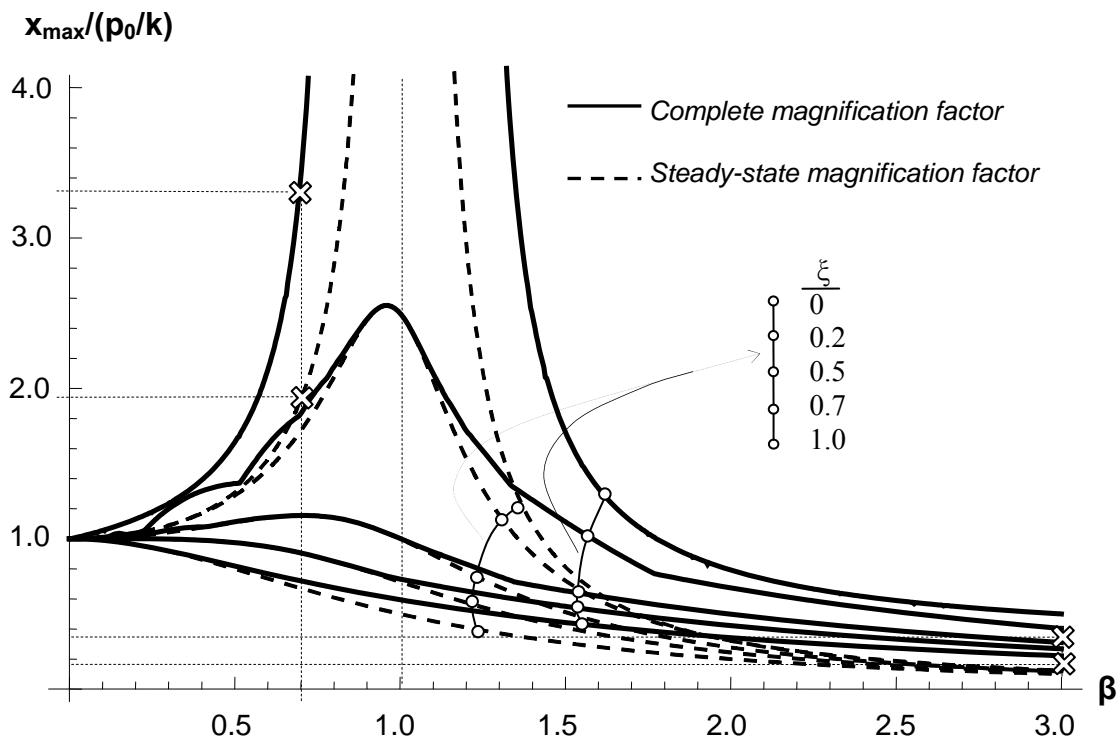


Figure 4. Complete and steady-state magnification factors for $\omega_0 = 1$ Hz.

To further illustrate differences in the magnification factors obtained for the complete and steady-state solutions, the ratio of these magnification factors is plotted in Figure 5 as a function of β and ξ . Dashed lines in Figure 5 correspond to the ratio between peak response given by the complete solution (obtained from Eq. (8)) and the peak response given by the steady-state magnification factors (obtained from Eq. (9)) for different values of the ξ and β . Values of β vary from 0 to 3.0 in steps of 0.05. In case that the damping ratio is equal to zero

($\xi = 0$) and the frequency ratio is equal to one ($\beta = 1$, i.e. resonance response) a discontinuity appear in Figure 5.

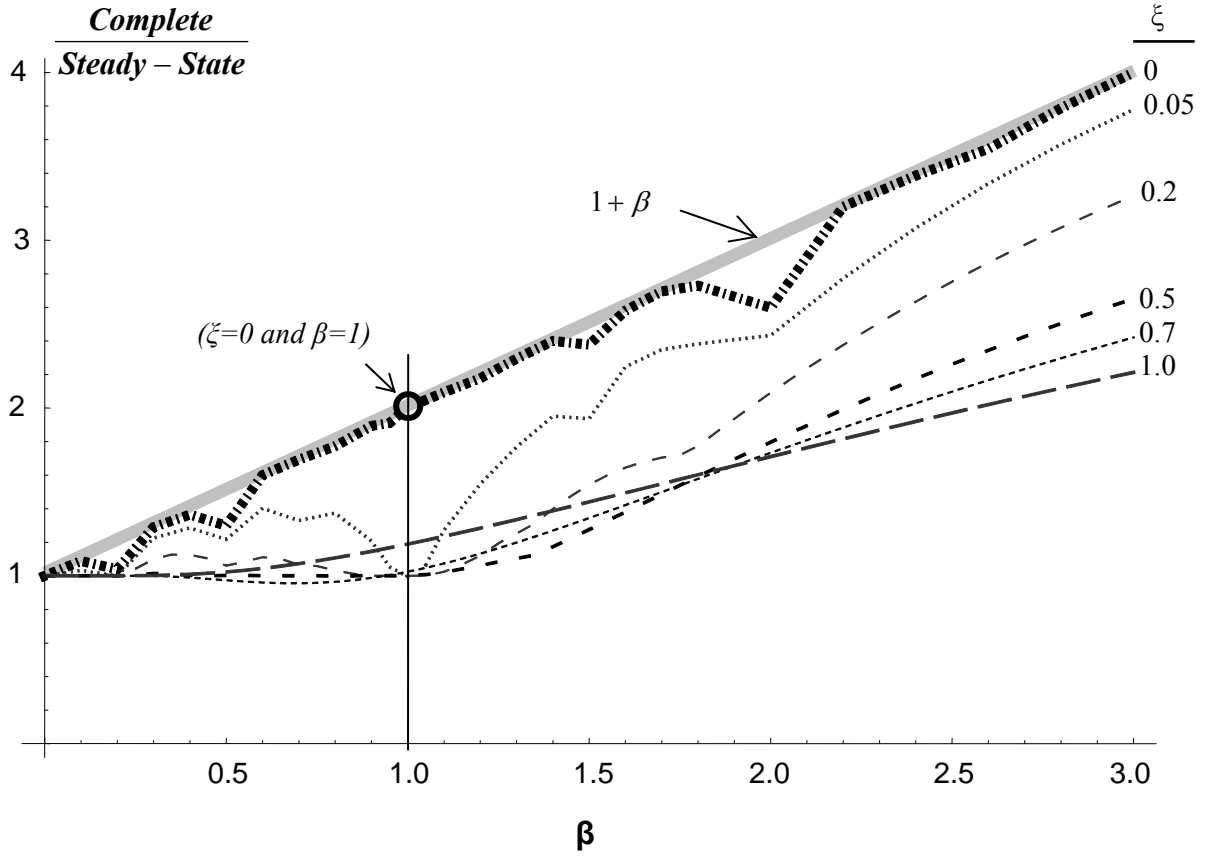


Figure 5. Complete to steady-state magnification factors ratio.

In case that $\beta = 1$ and $\xi \in (0,1)$ the ratio of peaks response (complete over steady-state) is equal to one. This coincides with the observation given by Clough and Penzien (1993) in §3.3 of their book. For the case of $\beta = 1$ and $\xi=0$, the ratio of peaks response presents a discontinuity and two values are possible, as can be seen in Figure 5:

$$\frac{\max x_{complete}(\xi = 0, \beta = 1)}{\max x_{steady-state}(\xi = 0, \beta = 1)} = \begin{cases} \lim_{\xi \rightarrow 0} \frac{\max x_{complete}(\xi, \beta = 1)}{\max x_{steady-state}(\xi, \beta = 1)} = 1 \\ \lim_{\beta \rightarrow 1} \frac{\max x_{complete}(\xi = 0, \beta)}{\max x_{steady-state}(\xi = 0, \beta)} = 2 \end{cases} \quad (14)$$

Clough and Penzien (1993) approached the solution imposing $\beta = 1$, so they got the first solution given by expression (14).

It is apparent that the maximum ratios are obtained for the highest value of β and the lowest damping ratio, ξ . Inspection of the plots indicates that for the undamped case, the ratio of the magnification factors for the complete and steady-state solutions is almost linear. Irregularities in the ratio of magnification factors are apparent for $\xi \leq 0.2$.

In the undamped case, a function that expresses the ratio of the complete and the steady-state magnification factor as a function of β can be simply adjusted. This function is given by (continuous thick grey line in Figure 5):

$$\frac{\max x_{complete}(\xi = 0, \beta)}{\max x_{steady}(\xi = 0, \beta)} = \beta + 1 \quad (15)$$

Thus, considering Eq. (9), an explicit expression for the complete maximum response for the undamped case is given by:

$$\max x_{complete}(\xi = 0, \beta) = (\beta + 1) \cdot \frac{\left(\frac{p_0}{k}\right)}{\sqrt{[1 - 2\beta^2 + \beta^4]}} = \frac{\left(\frac{p_0}{k}\right)}{|(1 - \beta)|} \quad (16)$$

with the exception of $\beta = 1$ and $\xi = 0$, as it was shown in Eq. (14).

Furthermore, Figure 5 demonstrates that Eq. (15) provides an upper bound to the ratio of complete and steady-state magnification factors. Thus, for any damping level ($\xi \leq 1$), the product of Eqs. (9) and (15) is an upper bound to the complete solution.

4. Conclusions

The preceding demonstrates the significance of the *transient part* of the response of a SDOF system subjected to a harmonically varying load of sine-wave form. Near resonance, differences between peak transient and peak full responses are negligible; these differences become significant as the forcing frequency begins to deviate significantly from the undamped frequency of vibration of the oscillator.

Traditionally, structural dynamics books define the maximum of the *steady-state response* of SDOF systems as the *dynamic magnification factor*. A better term for the conventional magnification factor is the *steady-state magnification factor*, since it represents an important component of the complete magnification factor.

A new explicit expression for the maximum response for the complete solution for the undamped case (i.e. for the most unfavourable situation) was developed, and is given by Eq. (16), with the exception of the case of $\beta = 1$ and $\xi=0$ where two values are possible. An upper bound on the complete solution for cases with damping was identified.

6. References.

1. Chopra, A.K. *Dynamics of Structures-Theory and Applications to Earthquake Engineering*, 3rd Edition. Prentice Hall.
2. Clough, R.W, and Penzien, J. *Dynamics of Structures*, 2nd Edition. McGraw-Hill. 1993.
3. Villaverde, R. *Fundamental Concepts of Earthquake Engineering*. CRC Press. 2009.
4. R. R. Craig Jr. and A. J. Kurdila. *Fundamentals of Structural Dynamics*. Wiley, New York, 2006.
5. García-Reyes L.E. *Dinámica Estructural Aplicada al Diseño Sísmico*. Uniandes. Bogotá. 1997.