The proof of a proposition

Antonio Francisco Roldán López de Hierro

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Proposition

Let $a, b, c \in \mathbb{R}$ be three real numbers such that $a < 0 \leq b < c$, and let $f, g, k, h : [0,1] \to \mathbb{R}$ be the functions defined, for each $s \in [0,1]$, by:

$$f(s) = a + (b-a)s, \quad g(s) = c - (c-b)s,$$
$$k = \min\{\, f^2,\, fg,\, g^2 \,\} \quad \text{and} \quad h = \max\{\, f^2,\, fg,\, g^2 \,\}.$$

Let denote $s_1 = \dfrac{-a}{b-a}$. Then, the following properties are fulfilled.

1. $s_1 \in (0,1]$ is the unique solution of the equation $f(s) = 0$. Furthermore, $b = 0$ (respectively, $b > 0$) if, and only if, $s_1 = 1$ (respectively, $s_1 < 1$).

2. $f(s)g(s) \leq g(s)^2$ for each $s \in [0,1]$, so

$$k = \min\{\, f^2,\, fg \,\} \quad \text{and} \quad h = \max\{\, f^2,\, g^2 \,\}.$$

3. The function $k = \min\{\, f^2,\, fg \,\} : [0,1] \to [ac, b^2]$ is a continuous and strictly increasing bijection from $[0,1]$ onto $[ac, b^2]$ (where $ac < 0 \leq b^2$). In fact, for each $s \in [0,1]$:

$$k(s) = \min\{\, f^2(s),\, f(s)g(s) \,\} = \begin{cases} f(s)g(s), & \text{if } s \in [0, s_1], \\ f^2(s), & \text{if } s \in [s_1, 1], \end{cases}$$

and the restricted functions:

$$k|_{[0,s_1]} = fg|_{[0,s_1]} : [0, s_1] \to [ac, 0] \quad \text{and} \quad k|_{[s_1,1]} = f^2|_{[s_1,1]} : [s_1, 1] \to [0, b^2]$$

are also continuous and strictly increasing bijections between the indicated closed intervals.

4. If $a^2 > c^2$, then

$$s_2 = \frac{-a-c}{2b-a-c} \quad \text{and} \quad t_3 = \frac{b(c-a)}{2b-a-c}$$

are well-defined. Furthermore, if $b = 0$, then $s_1 = s_2 = 1$, and if $b \neq 0$, then $s_2 \in (0, s_1)$. In fact, the unique solutions of the equation $f^2(s) = g^2(s)$ in $[0,1]$ are $s_2 \in (0, s_1)$ and $s = 1$. Furthermore, $f^2(s_2) = g^2(s_2) = t_3^2$.

Proposition

5. The function $h=\max\{f^2,g^2\}:[0,1]\to[\,b^2,\max\{a^2,c^2\}\,]$ is a continuous and strictly decreasing bijection from $[0,1]$ onto $[\,b^2,\max\{a^2,c^2\}\,]$. In fact, for each $s\in[0,1]$:

$$h(s)=\begin{cases} \textit{[Case } a^2\le c^2 \textit{]} & g^2(s), \qquad \text{if } s\in[0,1], \\ \textit{[Case } a^2> c^2 \textit{]} & \begin{cases} f^2(s), & \text{if } s\in[0,s_2], \\ g^2(s), & \text{if } s\in[s_2,1]. \end{cases} \end{cases}$$

Furthermore,

- if $a^2\le c^2$, the function $h=g^2:[0,1]\to[\,b^2,\max\{a^2,c^2\}\,]$ is a continuous and strictly decreasing bijection from $[0,1]$ onto $[\,b^2,\max\{a^2,c^2\}\,]$,
- if $a^2>c^2$, the restricted functions

$$h|_{[0,s_2]}=f^2|_{[0,s_2]}:[0,s_2]\to[\,t_3^2,\max\{a^2,c^2\}\,] \quad \text{and} \quad h|_{[s_2,1]}=g^2|_{[s_2,1]}:[s_2,1]\to[b^2,t_3^2]$$

  are also continuous and strictly decreasing bijections between the indicated closed intervals.

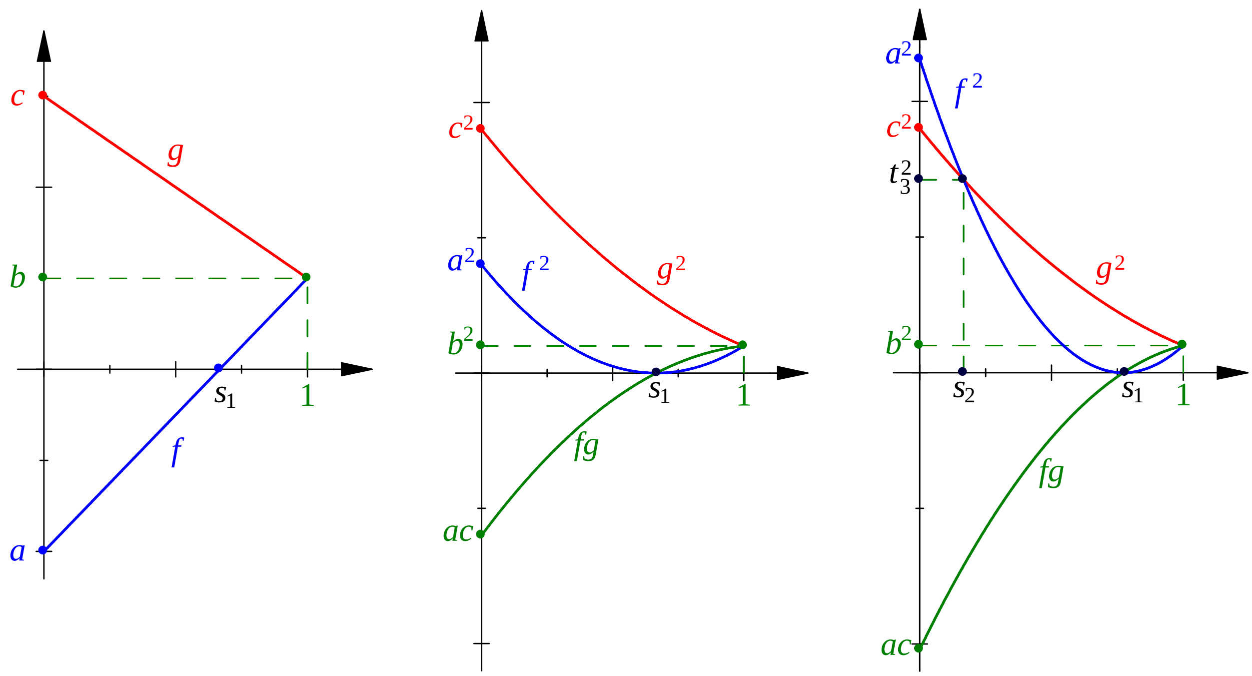
6. The function $h=\max\{f^2,g^2\}$ is convex on $[0,1]$, and the function $k=\min\{f^2,fg\}$ is concave in $[0,s_1]$ and convex in $[s_1,1]$ (in particular, k is concave in a neighborhood of $s=0$ and, when $b>0$, it is convex in a neighborhood of $s=1$).

7. The function $h=\max\{f^2,g^2\}$ is differentiable C^∞ on $[0,1]\diagdown\{s_2\}$ and the function $k=\min\{f^2,fg\}$ is differentiable C^∞ on $[0,1]\diagdown\{s_1\}$.

Proof.

By hypothesis, $b-a>0$ and $c-b>0$. Clearly, f and g are affine functions (that is, polynomials of degree 1), f is strictly increasing on $[0,1]$ ($f'(s)=b-a>0$), g is strictly decreasing on $[0,1]$ ($g'(s)=-(c-b)<0$), they are continuous and C^∞ on $[0,1]$ and $f''(s)=g''(s)=0$ for all $s\in[0,1]$. As a result, the functions f, g, f^2, fg, and g^2 are continuous on $[0,1]$, so $k=\min\{f^2,fg,g^2\}$ and $h=\max\{f^2,fg,g^2\}$ are also continuous on $[0,1]$. These functions satisfy:

$$f(0)=a,\quad f(1)=g(1)=b,\quad g(0)=c,\quad f^2(0)=a^2,\quad f^2(1)=g^2(1)=b^2 \quad \text{and} \quad g^2(0)=c^2.$$



(a) Functions f and g. (b) Case $a^2 \leq c^2$. (c) Case $a^2 > c^2$.

Figure 1: Indicative plots: two cases for f^2, g^2 and fg depending on whether $a^2 \leq c^2$ or $a^2 > c^2$.

Notice that:

$$\begin{aligned} [\, f(0) = a,\ f(1) = b,\ f \text{ strictly increasing}\,] \quad &\Rightarrow \quad a \leq f(s) \leq b \quad \text{for all } s \in [0,1]; \\ [\, g(0) = c,\ g(1) = b,\ g \text{ strictly decreasing}\,] \quad &\Rightarrow \quad 0 \leq b \leq g(s) \leq c \quad \text{for all } s \in [0,1]. \end{aligned}$$

In particular, as $b \geq 0$,

$$f(s) \leq b \leq g(s) \quad \text{for all } s \in [0,1], \tag{1}$$

$$g(s) \geq 0 \quad \text{for all } s \in [0,1]. \tag{2}$$

Notice that, for all $s \in [0,1]$,

$$g(s) - f(s) = [\, c - (c-b)s\,] - [\, a + (b-a)s\,] = (c-a) + (-c+a)s = (c-a)(1-s).$$

Hence, $g - f$ is an affine function such that

$$\begin{cases} \bullet \;\; g(s) - f(s) > 0 \quad \text{for all } s \in [0,1), \\ \bullet \;\; g(1) - f(1) = 0, \\ \bullet \;\; g - f \;\text{ is strictly decreasing on } [0,1]. \end{cases} \tag{3}$$

Similarly, for all $s \in [0,1]$,

$$\begin{aligned} & g(s) + f(s) = [\, c - (c-b)s \,] + [\, a + (b-a)s \,] = (a+c) + (2b-a-c)s; \\ & g(0) + f(0) = c + a; \quad g(1) + f(1) = b + b = 2b \geq 0. \end{aligned} \tag{4}$$

Hence, $g + f$ is also an affine function whose monotonicity depends on the sign of the number $2b - a - c$. In particular, for all $s \in [0,1]$,

$$g(s)^2 - f(s)^2 = (g(s) + f(s))(g(s) - f(s)) = (c-a)(1-s)\,[\,(a+c) + (2b-a-c)s\,]. \tag{5}$$

Item

Item 1. $s_1 \in (0,1]$ is the unique solution of the equation $f(s) = 0$. Furthermore, $b = 0$ (respectively, $b > 0$) if, and only if, $s_1 = 1$ (respectively, $s_1 < 1$).

Since $b - a > 0$, then $s_1 = \dfrac{-a}{b-a} = \frac{|a|}{b+|a|}$ is well-defined and, as $b \geq 0$ and $|a| > 0$, it satisfies $s_1 \in (0,1]$. Clearly, it is the unique solution of the equation $f(s) = 0$.

Since the function f is strictly increasing in $[0,1]$ and $f(s_1) = 0$, then

$$\begin{cases} \bullet \;\; f(s) < 0 \quad \text{for all } s \in [0, s_1), \\ \bullet \;\; f(s_1) = 0, \\ \bullet \;\; f(s) > 0 \quad \text{for all } s \in (s_1, 1]. \end{cases} \tag{6}$$

Furthermore, $f(s_1) = f^2(s_1) = f(s_1)g(s_1) = 0$. Notice that $s_1 = 1 \Leftrightarrow b = 0$, and also $s_1 < 1 \Leftrightarrow b > 0$. Then, the interval $(s_1, 1]$ is non-empty if, and only if, $b > 0$ (if $b = 0$, then $s_1 = 1$ and the third condition of (6) is empty).

Item

Item 2. $f(s)g(s) \leq g(s)^2$ for each $s \in [0,1]$, so

$$k = \min\{ f^2, fg \} \quad \text{and} \quad h = \max\{ f^2, g^2 \}.$$

By (1) and (2), $f(s) \leq g(s)$ and $g(s) \geq 0$ for all $s \in [0,1]$. Then $f(s)g(s) \leq g(s)^2$ for all $s \in [0,1]$. In particular:

$$k = \min\{ f^2, fg, g^2 \} = \min\{ f^2, fg \} \quad \text{and} \quad h = \max\{ f^2, fg, g^2 \} = \max\{ f^2, g^2 \}.$$

Item

Item 3. The function $k = \min\{ f^2, fg \} : [0,1] \to [ac, b^2]$ is a continuous and strictly increasing bijection from $[0,1]$ onto $[ac, b^2]$ (where $ac < 0 \leq b^2$). In fact, for each $s \in [0,1]$:

$$k(s) = \min\{ f^2(s), f(s)g(s) \} = \begin{cases} f(s)g(s), & \text{if } s \in [0, s_1], \\ f^2(s), & \text{if } s \in [s_1, 1], \end{cases}$$

and the restricted functions:

$$k|_{[0,s_1]} = fg|_{[0,s_1]} : [0, s_1] \to [ac, 0] \quad \text{and} \quad k|_{[s_1,1]} = f^2|_{[s_1,1]} : [s_1, 1] \to [0, b^2]$$

are also continuous and strictly increasing bijections between the indicated closed intervals.

We know that the function $k = \min\{ f^2, fg \}$ is continuous on $[0,1]$, and, since $f(s_1) = 0$, it verifies

$$\begin{cases} \bullet\ k(0) = \min\{ f^2(0), f(0)g(0) \} = \min\{ a^2, ac \} = ac < 0, \\ \bullet\ k(s_1) = \min\{ f^2(s_1), f(s_1)g(s_1) \} = \min\{ 0, 0 \} = 0, \\ \bullet\ k(1) = \min\{ f^2(1), f(1)g(1) \} = \min\{ b^2, b^2 \} = b^2. \end{cases}$$

Taking into account that $g(s) \geq 0$ for all $s \in [0,1]$ and (6), then

$$\begin{cases} \bullet\ f(s)g(s) \leq 0 \leq f^2(s) \quad \text{for all } s \in [0, s_1], \\ \bullet\ f^2(s) \leq f(s)g(s) \quad \text{for all } s \in [s_1, 1]. \end{cases}$$

Hence, for all $s \in [0,1]$,

$$k(s) = \min\{\, f^2(s),\, f(s)g(s) \,\} = \begin{cases} f(s)g(s), & \text{if } s \in [0, s_1], \\ f^2(s), & \text{if } s \in [s_1, 1]. \end{cases}$$

Notice that for all $s \in [0, s_1)$, since $f(s) < 0$,

$$\begin{aligned} k'(s) &= f'(s)g(s) + f(s)g'(s) = (b-a)g(s) + f(s)\left(-\left(c-b\right)\right) \\ &= (b-a)g(s) - f(s)\left(c-b\right) = (b-a)g(s) + |f(s)|\left(c-b\right) \geq |f(s)|\left(c-b\right) > 0, \end{aligned} \tag{7}$$

and if $s \in (s_1, 1]$, then $f(s) > 0$, so

$$k'(s) = 2f(s)f'(s) = 2(b-a)f(s) > 0. \tag{8}$$

In any case, $k'(s) > 0$ for all $s \in [0,1] \diagdown \{s_1\}$. As k is continuous, then it is strictly increasing on $[0,1]$, and as $k(0) = ac < 0$ and $k(1) = b^2 \geq 0$, then the function $k = \min\{\, f^2,\, fg \,\} : [0,1] \to [ac, b^2]$ is a continuous and strictly increasing bijection from $[0,1]$ onto $[ac, b^2]$ (where $ac < 0 \leq b^2$). In fact, as $k(s_1) = 0$, then the restricted functions:

$$k|_{[0,s_1]} = fg|_{[0,s_1]} : [0, s_1] \to [ac, 0] \quad \text{and} \quad k|_{[s_1,1]} = f^2|_{[s_1,1]} : [s_1, 1] \to [0, b^2]$$

are also continuous and strictly increasing bijections between the indicated closed intervals.

Notice that the unique point where k can be not differentiable is at $s = s_1$, but we know that k is differentiable in $[0,1] \diagdown \{s_1\}$.

Item

Item 4. If $a^2 > c^2$, then

$$s_2 = \frac{-a-c}{2b-a-c} \quad \text{and} \quad t_3 = \frac{b(c-a)}{2b-a-c}$$

are well-defined. Furthermore, if $b = 0$, then $s_1 = s_2 = 1$, and if $b \neq 0$, then $s_2 \in (0, s_1)$. In fact, the unique solutions of the equation $f^2(s) = g^2(s)$ in $[0,1]$ are $s_2 \in (0, s_1)$ and $s = 1$. Furthermore, $f^2(s_2) = g^2(s_2) = t_3^2$.

Suppose that $a^2 > c^2$. In this case, $|a| > |c|$, that is, $-a > c$, so $a + c < 0$ and $|a + c| > 0$. Hence

$$2b - a - c = 2b - (a + c) = 2b + |a + c| \geq |a + c| > 0.$$

Therefore, the denominators of

$$s_2 = \frac{-a-c}{2b-a-c} = \frac{|a+c|}{2b+|a+c|} \quad \text{and} \quad t_3 = \frac{b(c-a)}{2b-a-c} = \frac{b(c-a)}{2b+|a+c|}$$

are non null, so s_2 and t_3 are well-defined in this case. Clearly, $s_2 > 0$ and $t_3 \geq 0$.

If $b = 0$, it is apparent that $s_1 = s_2 = 1$.

Next, suppose that $b > 0$. In this case:

$$\begin{aligned} s_2 < s_1 \;&\Leftrightarrow\; \frac{-a-c}{2b-a-c} < \frac{-a}{b-a} \;\Leftrightarrow\; (-a-c)\,(b-a) < -a\,(2b-a-c) \\ &\Leftrightarrow\; -ab + a^2 - bc + ac < -2ab + a^2 + ac \;\Leftrightarrow\; -ab - bc < -2ab \\ &\Leftrightarrow\; ab < bc \;\Leftrightarrow\; 0 < b(c-a), \text{ True.} \end{aligned}$$

Hence, $s_2 \in (0, s_1)$. Notice that $f(1)^2 = b^2 = g(1)^2$, so $s = 1$ is a solution of the equation $f^2(s) = g^2(s)$. Furthermore, notice that, in the case $a^2 > c^2$, using (5), the unique solutions of the equation $f^2(s) = g^2(s)$ in $[0, 1]$ are $s = 1$ and

$$s = \frac{-(a+c)}{2b-a-c} = \frac{|a+c|}{2b+|a+c|} = s_2 \in (0, s_1).$$

In fact,

$$f^2(s_2) = \left(a + (b-a)\cdot\frac{-a-c}{2b-a-c}\right)^2 = \left(\frac{a(2b-a-c)+(b-a)(-a-c)}{2b-a-c}\right)^2 = \left(\frac{-b(c-a)}{2b-a-c}\right)^2 = t_3^2, \quad (9)$$

$$g^2(s_2) = \left(c - (c-b)\cdot\frac{-a-c}{2b-a-c}\right)^2 = \left(\frac{c(2b-a-c)-(c-b)(-a-c)}{2b-a-c}\right)^2 = \left(\frac{b(c-a)}{2b-a-c}\right)^2 = t_3^2. \quad (10)$$

Item

Item 5. The function $h = \max\{f^2, g^2\} : [0,1] \to [\,b^2, \max\{a^2, c^2\}\,]$ is a continuous and strictly decreasing bijection from $[0,1]$ onto $[\,b^2, \max\{a^2, c^2\}\,]$. In fact, for each $s \in [0,1]$:

$$h(s) = \begin{cases} [\,\textit{Case } a^2 \le c^2\,] & g^2(s), \qquad \text{if } s \in [0,1], \\ [\,\textit{Case } a^2 > c^2\,] & \begin{cases} f^2(s), & \text{if } s \in [0, s_2], \\ g^2(s), & \text{if } s \in [s_2, 1]. \end{cases} \end{cases}$$

Furthermore,

- if $a^2 \le c^2$, the function $h = g^2 : [0,1] \to [\,b^2, \max\{a^2, c^2\}\,]$ is a continuous and strictly decreasing bijection from $[0,1]$ onto $[\,b^2, \max\{a^2, c^2\}\,]$,
- if $a^2 > c^2$, the restricted functions

$$h|_{[0,s_2]} = f^2|_{[0,s_2]} : [0, s_2] \to [\,t_3^2, \max\{a^2, c^2\}\,] \quad \text{and} \quad h|_{[s_2,1]} = g^2|_{[s_2,1]} : [s_2, 1] \to [b^2, t_3^2]$$

are also continuous and strictly decreasing bijections between the indicated closed intervals.

Notice that

$$h(0) = \max\{f^2(0), g^2(0)\} = \max\{a^2, c^2\},$$
$$h(1) = \max\{f^2(1), g^2(1)\} = \max\{b^2, b^2\} = b^2.$$

(Case 1) First, suppose that $a^2 \le c^2$, that is, $0 < -a \le c$ and $c + a \ge 0$. By (4), $g(0) + f(0) = c + a \ge 0$ and $g(1) + f(1) = 2b \ge 0$. As $g + f$ is an affine function, we can deduce that:

$$g(s) + f(s) \ge 0 \quad \text{for all } s \in [0,1].$$

Combining it with (3), as $g(s) - f(s) \ge 0$, then

$$g(s)^2 - f(s)^2 = (g(s) + f(s))(g(s) - f(s)) \ge 0,$$

so $g(s)^2 \geq f(s)^2$ for all $s \in [0,1]$, which means that:

$$\text{when } a^2 \leq c^2, \quad h = \max\{\, f^2,\, g^2\} = g^2.$$

As g is strictly decreasing on $[0,1]$, then

$$\begin{cases} g(s) > 0, & \text{if } s \in [0,1), \\ g(1) = b \geq 0, & \text{if } s = 1. \end{cases}$$

As g is differentiable, then for each $s \in [0,1)$,

$$h'(s) = 2g(s)g'(s) = 2g(s)\left(-\left(c-b\right)\right) = -2(c-b)g(s) < 0.$$

Therefore, h is strictly decreasing on $[0,1]$. In particular, as $h(0) = \max\{\, a^2,\, c^2\} = c^2$ and $h(1) = b^2$, then the function $h = g^2 : [0,1] \to [\, b^2,\, c^2\,] = [\, b^2,\, \max\{a^2, c^2\}\,]$ is a continuous and strictly decreasing bijection from $[0,1]$ onto $[\, b^2,\, \max\{a^2, c^2\}\,]$ (notice that $b^2 < c^2 = \max\{a^2, c^2\}$).

(Case 2) Now suppose that $a^2 > c^2$. On item 4 we proved that, in this case, $a + c < 0$, $|a+c| > 0$ and

$$s_2 = \frac{-a-c}{2b-a-c} = \frac{|a+c|}{2b+|a+c|} \quad \text{and} \quad t_3 = \frac{b(c-a)}{2b-a-c} = \frac{b(c-a)}{2b+|a+c|}$$

are well-defined. Furthermore, if $b = 0$, then $s_1 = s_2 = 1$, and if $b \neq 0$, then $s_2 \in (0, s_1)$. In fact, we checked that, when $b > 0$, $s_2 \in (0, s_1)$ and $s = 1$ are the unique solutions of the equation $f^2(s) = g^2(s)$ in $[0,1]$. Such numbers divide the interval $[0,1]$ into the intervals $[0, s_2)$ and $(s_2, 1]$, where $0 < s_2 < s_1 \leq 1$. By (5),

$$g(0)^2 - f(0)^2 = c^2 - a^2 < 0,$$
$$g\left(\frac{s_2+s_1}{2}\right)^2 - f\left(\frac{s_2+s_1}{2}\right)^2 = (c-a)\left(1 - \frac{s_2+s_1}{2}\right)\left[(a+c) + (2b-a-c)\cdot\frac{s_2+s_1}{2}\right] > 0.$$

Notice that we have here used that:

$$(a+c) + (2b-a-c)\cdot\frac{s_2+s_1}{2} = (a+c) + (2b-a-c)\cdot\frac{\dfrac{-a}{b-a} + \dfrac{-a-c}{2b-a-c}}{2} = \frac{b(c-a)}{2(b-a)} > 0.$$

As the function $g^2 - f^2$ is continuous on $[0,1]$, and its unique zeros are s_2 and 1, we deduce that

$$\begin{cases} \bullet\ \ g(s)^2 - f(s)^2 < 0 \quad \text{for all } s \in [0, s_2), \\ \bullet\ \ g(s_2)^2 - f(s_2)^2 = 0, \\ \bullet\ \ g(s)^2 - f(s)^2 > 0 \quad \text{for all } s \in (s_2, 1), \\ \bullet\ \ g(1)^2 - f(1)^2 = 0. \end{cases}$$

As a consequence, in the case $a^2 > c^2$,

$$h(s) = \max\{ f(s)^2, g(s)^2 \} = \begin{cases} f(s)^2, & \text{if } s \in [0, s_2], \\ g(s)^2, & \text{if } s \in [s_2, 1]. \end{cases}$$

Therefore, for all $s \in [0, s_2)$, since $s < s_2 < s_1$, $f(s) < 0$ by (6), so

$$h'(s) = 2f(s)f'(s) = 2f(s)(b-a) < 0,$$

and for all $s \in (s_2, 1)$, as $g(s) > g(1) = b \geq 0$, then $g(s) > 0$, so

$$h'(s) = 2g(s)g'(s) = 2g(s)(-(c-a)) = -2(c-a)g(s) < 0.$$

In any case, $h'(s) < 0$ for all $s \in [0,1) \setminus \{s_2\}$. Notice that, in fact, h is differentiable in $[0,1] \setminus \{s_2\}$, so s_2 is the unique point where h can be not differentiable.

As h is continuous on $[0,1]$, we deduce that h is strictly decreasing on $[0,1]$. By (9)-(10), $f(s_2)^2 = g(s_2)^2 = t_3^2$. As a result, in this case ($a^2 > c^2$), the restricted functions

$$\begin{aligned} h|_{[0,s_2]} &= f^2|_{[0,s_2]} : [0, s_2] \to [\, t_3^2,\, a^2 \,] = [\, t_3^2,\, \max\{a^2, c^2\} \,] \quad \text{and} \\ h|_{[s_2,1]} &= g^2|_{[s_2,1]} : [s_2, 1] \to [b^2, t_3^2] \end{aligned}$$

are also continuous and strictly decreasing bijections between the indicated closed intervals.

Item

Item 6. The function $h = \max\{f^2, g^2\}$ is convex on $[0,1]$, and the function $k = \min\{f^2, fg\}$ is concave in $[0,s_1]$ and convex in $[s_1,1]$ (in particular, k is concave in a neighborhood of $s = 0$ and, when $b > 0$, it is convex in a neighborhood of $s = 1$).

The functions f^2 and g^2 are pieces of convex parabolas because, for each $s \in [0,1]$,

$$f^2(s) = [a + (b-a)s]^2 = (b-a)^2 \left[s - \left(\frac{-a}{b-a} \right) \right]^2 = (b-a)^2 [s - s_1]^2 \quad \text{and}$$
$$g^2(s) = [c - (c-b)s]^2 = (c-b)^2 \left[s - \frac{c}{c-b} \right]^2 .$$

As f^2 and g^2 are convex in $[0,1]$, then the function $h = \max\{f^2, g^2\}$ also is (the maximum of convex function also is). To study the function $k = \min\{f^2, fg\}$, recall that, by (7)-(8),

$$k'(s) = \begin{cases} (b-a)g(s) - f(s)(c-b), & \text{if } s \in [0,s_1), \\ 2(b-a)f(s), & \text{if } s \in (s_1,1]. \end{cases}$$

Therefore, for all $s \in [0,s_1)$,

$$k''(s) = (b-a)g'(s) - f'(s)(c-b) = -(b-a)(c-b) - (b-a)(c-b) = -2(b-a)(c-b) < 0,$$

and if $s \in (s_1,1]$, then

$$k''(s) = 2(b-a)f'(s) = 2(b-a)^2 > 0.$$

Therefore, k is concave in $[0,s_1]$ and convex in $[s_1,1]$. Since $s_1 > 0$, then $[0,s_1]$ is a neighborhood of $s = 0$. The interval $[s_1,1]$ is a neighborhood of $s = 1$ only if $s_1 < 1$, and this is equivalent to suppose that $b > 0$. In particular, k is concave in a neighborhood of $s = 0$ and, when $b > 0$, it is convex in a neighborhood of $s = 1$.

Item

Item 7. The function $h = \max\{f^2, g^2\}$ is differentiable C^∞ on $[0,1] \setminus \{s_2\}$ and the function $k = \min\{f^2, fg\}$ is differentiable C^∞ on $[0,1] \setminus \{s_1\}$.

The functions f and g are differentiable C^∞ on $[0,1]$ (in fact, $f''(s) = g''(s) = 0$ for all $s \in [0,1]$).

Therefore, the expression:

$$k(s)=\begin{cases} f(s)g(s), & \text{if } s\in[0,s_1],\\ f^2(s), & \text{if } s\in[s_1,1],\end{cases}$$

means that k is differentiable C^∞ on $[0,1]\diagdown\{s_1\}$, and the expression:

$$h(s)=\begin{cases} \textit{[Case } a^2\le c^2 \textit{]} & g^2(s), \qquad \text{if } s\in[0,1],\\ \textit{[Case } a^2> c^2 \textit{]} & \begin{cases} f^2(s), & \text{if } s\in[0,s_2],\\ g^2(s), & \text{if } s\in[s_2,1],\end{cases}\end{cases}$$

means that h is differentiable C^∞ on $[0,1]\diagdown\{s_2\}$. ■