This is the accepted version of the article with DOI 10.1016/j.fss.2019.06.001

Specificity Measures Based on Fuzzy Set Similarity

Nicolás Marín[a], Gustavo Rivas-Gervilla[a], Daniel Sánchez[a], and Ronald R. Yager[b]

[a]Department of Computer Science and Artificial Intelligence, University of Granada, Spain
[b]Iona College, New York, USA

Abstract: In this work we propose to define specificity measures on the basis of measures of similarity for fuzzy sets. Specifically, we calculate specificity of a fuzzy set as the similarity between the set and its closest singleton defined on the same domain. Our contribution is completed by the study of the axiomatic relationship between specificity and similarity measures. We also provide several illustrative examples of specificity measures obtained using our approach on the basis of different similarity measures.

Keywords: specificity, similarity of fuzzy sets

1 Introduction

Given any fuzzy set A, its specificity can be defined intuitively as the degree to which A is a crisp singleton, measured as a real value in $[0, 1]$. Specificity can be employed for many purposes. For instance, in the setting of decision making (where A is an assignment of degrees to alternatives, indicating the degree of satisfaction of the decision maker with each alternative), specificity can be employed to measure the tranquility (lack of anxiety) in making a decision, that is, "the emotional ease with which a decision maker can select one alternative from a group of alternatives based on a decision function" [Yag82]. In the particular case of A being a possibility distribution representing the available information about the value of a variable V (a case in which, when there is no contradictory information, the fuzzy set is assumed to be normalized), its specificity gives us information about the degree of uncertainty about the actual value of V following A [Yag98], 0 and 1 meaning full uncertainty and no uncertainty at all, respectively [DP87; Yag82]. Hence, specificity is a very important measure of uncertainty in possibility theory, just like entropy is a key uncertainty measure in probability theory [Yag08].

Specificity also plays an important role in measurement of performance of decision support systems [Yag84], deductive reasoning systems [DP99], default reasoning [Yag92a], and measuring referential success in referring expression generation [Mar+18], among other potential applications.

In view of the definition of specificity we have just mentioned, it is obvious that the maximum specificity degree 1 is achieved when A is a crisp singleton since, in that case, the single object in A is the actual value of V, without any uncertainty. In this paper we use this idea to show that the similarity between A and its closest singleton, obtained by means of a similarity measure between fuzzy sets, is a specificity measure. This idea goes along the line of [Yag98], where a measure of specificity is proposed on the basis of the complement of a normalized distance between the set A and its closest singleton. However, the new proposal here is more general, since the complement of a normalized distance can be seen as a particular case of similarity measure. It is important to remark that our approach has nothing to do with those in [Gar+06b; Yag91; Yag08], where similarity and t-indistinguishability relations are defined on the reference domain of the possibility distribution, being employed as a way to compute specificity of fuzzy sets defined on continuous domains.

There are many definitions of families of specificity measures in the literature, as well as individual measures not belonging to any of such families, as we shall see later. Similarly, different definitions of similarity measures for fuzzy sets, based on different sets of axioms, have been provided [CGS13]. In both cases, the different definitions differ in the axioms required for the measures. In this paper we also study sets of axioms for measures of similarity that guarantee, as sufficient conditions, that specificity measures are obtained. We also illustrate our results by showing how different specificity measures, some belonging to existing families and some others new, can be obtained from different similarity measures for fuzzy sets by using our approach.

The paper is organized as follows: we recall different formal definitions of specificity measures in Section 2. Section 3 briefly summarizes the state of the art about axioms and formal definitions for similarity measures provided by

Couso et al. in [CGS13]. Section 4 introduces our proposal for definition of specificity based on similarity, and study some relations between axioms of both kinds of measures. In Section 5 we show different specificity measures obtained from measures of similarity using our approach. Finally, Section 6 contains our conclusions and future research perspectives.

2 Specificity measures

There exist many different proposals of specificity measures according to several axiomatic definitions [Gar+03; Sá+17; Yag90; Yag82; Yag92b; Yag12]. Let $\mathcal{O}$ be a finite set and let $\mathcal{F}(\mathcal{O})$ be the set of all fuzzy sets defined on $\mathcal{O}$. A first axiomatic definition is the following:

Definition 2.1 [Yag82; Yag92b]
A *specificity measure* is a mapping $Sp : \mathcal{F}(\mathcal{O}) \to [0, 1]$ that fulfils the following axioms:

$\mathbb{A}$ $Sp(A) = 1 \iff A$ is a crisp singleton.

$\mathbb{B}$ $Sp(\emptyset) = 0$.

$\mathbb{c}$ If $A, B \in \mathcal{F}(\mathcal{O})$ are normal fuzzy sets such that $A \subseteq B$, then $Sp(A) \geq Sp(B)$.

Axiom $\mathbb{A}$ says that the specificity of a set reaches its maximum value, 1, iff the set is a crisp singleton, i.e., iff there is no doubt about the value of the variable represented by the set. On its turn, $\mathbb{B}$ states that the specificity of a set is minimal (equal to 0) when all the alternative values are not possible [Yag82]. As we shall see below, these two axioms are shared by all axiomatic definitions in the literature.

Definition 2.1 considers as third axiom the monotonicity condition $\mathbb{c}$ that tells us that, when there are two normal fuzzy sets, i.e., at least one (and the same) value is fully possible for each of the variables represented by the fuzzy sets, then the lower the possibility of the rest of values, the higher the specificity of the fuzzy set.

A second definition is the following:

Definition 2.2 [Yag12]
A *specificity measure* is a mapping $Sp : \mathcal{F}(\mathcal{O}) \to [0, 1]$ that fulfils the following axioms:

$\mathbb{A}$ $Sp(A) = 1 \iff A$ is a crisp singleton.

$\mathbb{B}$ $Sp(\emptyset) = 0$.

$\mathbb{C}$ Let $A, B \in \mathcal{F}(\mathcal{O})$ such that $a_1 \geq b_1$ and $a_i \leq b_i$ $\forall i \geq 2$, where a_i and b_i are the membership degrees of A and B, arranged in decreasing order. Then $Sp(A) \geq Sp(B)$.

Definition 2.2 replaces condition $\mathbb{c}$ by $\mathbb{C}$. This condition does not require the fuzzy sets to be normalized. In this case, $\mathbb{C}$ states that when the maximum possibility is larger for A than for B, and the rest of possibility degrees are smaller for A than for B, then the specificity of the fuzzy set increases or remains equal. That is, specificity of A is increasing in a_1 and decreasing in a_i for each $i \geq 2$.

Finally, the last definition slightly modifies Definition 2.2 so that the specificity strictly increases with respect to a_1. In this way, each minimal change in a_1 is reflected in the specificity measure of the modified fuzzy set.

Definition 2.3 [Sá+17; Yag98]
A *specificity measure* is a mapping $Sp : \mathcal{F}(\mathcal{O}) \to [0, 1]$ that fulfils the following axioms:

$\mathbb{A}$ $Sp(A) = 1 \iff A$ is a crisp singleton.

$\mathbb{B}$ $Sp(\emptyset) = 0$.

$\mathbb{C}^{\uparrow}$ $Sp(A)$ is strictly increasing with respect to a_1.

$\mathbb{C}^{\downarrow}$ $Sp(A)$ is decreasing with respect to a_i for all $i \geq 2$.

It can be shown that each definition is more restrictive than the previous ones. This is proved in the following proposition:

Proposition 1
The following implications hold:

1. $\mathbb{C} \implies \mathbb{c}$.
2. $\mathbb{C}^{\uparrow} \wedge \mathbb{C}^{\downarrow} \implies \mathbb{C}$.

Proof:

1. Let Sp a specificity measure according to Definition 2.2, so it fulfils $\mathbb{C}$. Let $A, B \in \mathcal{F}(\mathcal{O})$ two normal fuzzy sets such that $A \subseteq B$. Then, because A and B are normal, it is $a_1 = b_1 = 1$ and, since $A \subseteq B$, it is $a_i \le b_i$ for all $i \ge 2$. Hence, since Sp fulfils $\mathbb{C}$, it is $Sp(A) \ge Sp(B)$.
2. Let Sp be a specificity measure according to Definition 2.3, so it fulfils $\mathbb{C}^{\uparrow}$ and $\mathbb{C}^{\downarrow}$. Let $A, B \in \mathcal{F}(\mathcal{O})$ such that $a_1 \ge b_1$ and $a_i \le b_i \ \forall i \ge 2$. Let $n = |\mathcal{O}| > 0$. For every $1 \le i \le n+1$ it is always possible to define at least one fuzzy set $B^i \in \mathcal{F}(\mathcal{O})$ satisfying:
$$b^i_j = \begin{cases} a_j & j \ge i \\ b_j & j < i \end{cases}$$
Then it is $a_j = a^1_j$ and $b_j = b^{n+1}_j \ \forall 1 \le j \le n$ and hence $Sp(A) = Sp(B^1)$ and $Sp(B^{n+1}) = Sp(B)$, since specificity depends only on the multiset of membership degrees in the fuzzy set, and it is independent of the assignment of such degrees to objects. It is also $b^1_1 = a_1 \ge b_1 = b^2_1$ and $b^1_j = b^2_j = a_j \ \forall j \ge 2$ so, as Sp fulfils $\mathbb{C}^{\uparrow}$, it is $Sp(A) = Sp(B^1) \ge Sp(B^2)$. Furthermore, for every $2 \le i \le n$ it is $b^i_j = b^{i+1}_j = b_j \ \forall j < i$, $b^i_j = b^{i+1}_j = a_j \ \forall j \ge i+1$, and $b^i_i = a_i \le b_i = b^{i+1}_i$ and, as Sp fulfils $\mathbb{C}^{\downarrow}$, it is $Sp(B^i) \ge Sp(B^{i+1})$. Hence it is $Sp(A) = Sp(B^1) \ge Sp(B^2) \ge \cdots \ge Sp(B^n) \ge Sp(B^{n+1}) = Sp(B)$, and Sp satisfies $\mathbb{C}$.

■

Many specificity measures have been proposed in the literature, some of them through parametric families like the linear family [Yag90], the product family [Yag98], measures based on combinations of fuzzy set operators [Gar+03] and fuzzy integrals [Gar+06a], among others [Mar+18]. Each measure has different properties and characteristics that make them useful for different purposes [Mar+17]. We shall see some of them in Section 4.1 and Section 5.

3 Similarity measures

3.1 Axioms for similarity measures

Similarity measures are a kind of comparison measures that compute a degree of equality between two fuzzy sets. There are several axiomatic definitions of similarity measures, for which we take as reference the review in [CGS13]. Table 1 shows the set of axioms compiled in [CGS13] that could be required for any comparison measure m (either similarity or dissimilarity) between fuzzy sets. For example, axiom G2 says that the comparison measure must be symmetrical, while G4 tells that the comparison between two fuzzy sets can be computed by means of their differences and intersection. Axioms with an asterisk are a stronger versions of their homonyms (for example, axiom G1* clearly implies axiom G1).

Table 2 shows an additional list of axioms, also compiled in [CGS13], that have been considered in some definitions of similarity measures. For example, if a similarity measure fulfils axiom S4, this measure reaches its maximum value when the two compared sets are the same.

3.2 Some axiomatic definitions of similarity for fuzzy sets

On the basis of the previous axioms, several axiomatic definitions of similarity measures can be found in the literature. In this section we overview some of them.

In [LS03] an axiomatic definition is proposed for similarity measures between intuitionistic fuzzy sets. We show the corresponding axioms for the particular case of fuzzy sets. In the following definitions the axioms of Tables 1 and 2 are indicated where applicable (note that sometimes, the axioms proposed correspond to the conjunction of some axioms in Tables 1 and 2).

G1	$0 \leq m(A,B) \leq 1\ \forall A,B \in \mathcal{F}(\mathcal{O})$.
G1*	$0 \leq m(A,B) \leq 1\ \forall A,B \in \mathcal{F}(\mathcal{O})$ and there exists a pair of fuzzy sets $C,D \in \mathcal{F}(\mathcal{O})$ such that $m(C,D) = 1$.
G2	$m(A,B) = m(B,A)\ \forall A,B \in \mathcal{F}(\mathcal{O})$.
G3	Let $\mathcal{O}$ a finite set and $\rho : \mathcal{O} \to \mathcal{O}$ be a permutation. For each $A \in \mathcal{F}(\mathcal{O})$, A^ρ is a fuzzy set such that $A^\rho(o) = A(\rho(o))$. Then $m(A,B) = m(A^\rho, B^\rho)$.
G3*	$\mathcal{O}$ is a finite set and there exists $h : [0,1]^2 \to \mathbb{R}$ such that $m(A,B) = \sum_{o\in\mathcal{O}} h[A(o), B(o)]$.
G4	There exists a mapping $f : \mathcal{F}(\mathcal{O})^3 \to \mathbb{R}$ such that $m(A,B) = f(A \cap B, A \setminus B, B \setminus A)$.
G4*	There exists a mapping $F_m : \mathbb{R}^3 \to \mathbb{R}$ and a fuzzy measure $M : \mathcal{F}(\mathcal{O}) \to \mathbb{R}$ such that, for all $A,B \in \mathcal{F}(\mathcal{O})$, $m(A,B) = F_m(M(A\cap B), M(A \setminus B), M(B \setminus A))$.
G5	If $A \cap B = \emptyset$, $A' \cap B' = \emptyset$, $m(A,\emptyset) \leq m(A',\emptyset)$ and $m(B,\emptyset) \leq m(B',\emptyset)$, then $m(A,B) \leq m(A',B')$.

Table 1. Axioms for general comparison measures between fuzzy sets [CGS13].

S1	$\forall A,B,C \in \mathcal{F}(\mathcal{O})$, if $A \subseteq B \subseteq C$ and $max_{o\in\mathcal{O}} A(o) = max_{o\in\mathcal{O}} B(o)$ then $\mathfrak{P}(A,C) \leq \mathfrak{P}(A,B)$.
S1*	$\forall A,B,C \in \mathcal{F}(\mathcal{O})$, if $A \subseteq B \subseteq C$ then $\mathfrak{P}(A,C) \leq \mathfrak{P}(A,B)$.
S1*-var	$\forall A,B,C \in \mathcal{F}(\mathcal{O})$, if $A \subsetneq B \subsetneq C$ then $\mathfrak{P}(A,C) < \mathfrak{P}(A,B)$.
S2	$\forall A,B,C \in \mathcal{F}(\mathcal{O})$, if $A \subseteq B \subseteq C$ then $\mathfrak{P}(A,C) \leq \mathfrak{P}(B,C)$.
S2*	If $A,B,C \in \mathcal{F}(\mathcal{O})$ satisfy · $A(o_0) < B(o_0) \leq C(o_0)$, for some $o_0 \in \mathcal{O}$, and · $B(o) = A(o), \forall o \in \mathcal{O}, o \neq o_0$, then $\mathfrak{P}(A,C) < \mathfrak{P}(B,C)$.
S3	$\forall A \in \mathcal{P}(\mathcal{O}), \mathfrak{P}(A,\overline{A}) = 0$.
S3*	$\mathfrak{P}(A,\overline{A}) = 0 \iff A \in \mathcal{P}(\mathcal{O})$.
S4	$\mathfrak{P}(C,C) = max_{A,B\in\mathcal{F}(\mathcal{O})}\mathfrak{P}(A,B)\ \forall C \in \mathcal{F}(\mathcal{O})$.
S4*	$C = D \iff \mathfrak{P}(C,D) = max_{A,B\in\mathcal{F}(\mathcal{O})}\mathfrak{P}(A,B)$.
S5	$\mathfrak{P}(A,B) = 0 \implies A \cap B = \emptyset$.
S5*	Consider $A,B \in \mathcal{F}(\mathcal{O})$ and $o_0 \in \mathcal{O}$. Define C and D such that: · $C(o) = A(o)$ y $D(o) = B(o), \forall o \neq o_0$, · $C(o_0) = A(o_0) + \alpha$ and $D(o_0) = B(o_0) + \alpha$, where $0 \leq \alpha \leq 1 - max(A(o_0), B(o_0))$ Then: · If $max_{o\in\mathcal{O}}(C \cap D)(o) = max_{o\in\mathcal{O}}(A \cap B)(o)$ then $\mathfrak{P}(C,D) = \mathfrak{P}(A,B)$. · If $max_{o\in\mathcal{O}}(C \cap D)(o) > max_{o\in\mathcal{O}}(A \cap B)(o)$ then $\mathfrak{P}(C,D) > \mathfrak{P}(A,B)$.
S6	Consider $A,B \in \mathcal{F}(\mathcal{O})$ and $o_0 \in \mathcal{O}$. Define C and D such that: · $C(o) = A(o)$ y $D(o) = B(o), \forall o \neq o_0$, · $C(o_0) = A(o_0) + \alpha$ and $D(o_0) = B(o_0) + \alpha$, where $0 \leq \alpha \leq 1 - max(A(o_0), B(o_0))$ Then, $\mathfrak{P}(A,C) = \mathfrak{P}(B,D)$.
S7	If $A,B,C,D \in \mathcal{F}(\mathcal{O})$ satisfy $A \cap B \subseteq C \cap D$, $A \setminus B \supseteq C \setminus D$, and $B \setminus A \supseteq D \setminus C$ then $\mathfrak{P}(A,B) \leq \mathfrak{P}(C,D)$.

Table 2. Axioms for similarity measures between fuzzy sets [CGS13].

Definition 3.1 [LS03]
A mapping $\mathfrak{P} : \mathcal{F}(\mathcal{O})^2 \to [0, 1]$ is a *similarity measure*, if $\mathfrak{P}$ satisfies the following axioms:

G1 $0 \leq \mathfrak{P}(A, B) \leq 1, \ \forall A, B \in \mathcal{F}(\mathcal{O})$.

G2 $\mathfrak{P}(A, B) = \mathfrak{P}(B, A)$.

G1* and S4 If $A = B$, then $\mathfrak{P}(A, B) = 1$.

S1* and S2 If $A \subseteq B \subseteq C$, $A, B, C \in \mathcal{F}(\mathcal{O})$, then $\mathfrak{P}(A, C) \leq \mathfrak{P}(A, B)$ and $\mathfrak{P}(A, C) \leq \mathfrak{P}(B, C)$.

This definition can be seen somehow as comprised of the most basic axioms for similarity. Note that axiom G1 is already part of the definition of function $\mathfrak{P}$. This is also the case in the rest of axiomatic definitions in this section, so we shall explicitly include G1 in all definitions for a better comparison between them.

The following axiomatic definition is a particularization to the case of fuzzy sets of the proposal in [Bus00] for interval-valued fuzzy sets:

Definition 3.2
A mapping $\mathfrak{P} : \mathcal{F}(\mathcal{O})^2 \to [0, 1]$ is a *similarity measure* if $\mathfrak{P}$ satisfies the following axioms:

G1 $0 \leq \mathfrak{P}(A, B) \leq 1, \ \forall A, B \in \mathcal{F}(\mathcal{O})$.

G2 $\mathfrak{P}(A, B) = \mathfrak{P}(B, A)$.

S3 $\mathfrak{P}(A, \bar{A}) = 0, \ \forall A \in \mathcal{P}(\mathcal{O})$.

G1* and S4 $\mathfrak{P}(A, A) = 1$.

S1* and S2 If $A \subseteq B \subseteq C$, then $\mathfrak{P}(A, B) \geq \mathfrak{P}(A, C)$ and $\mathfrak{P}(B, C) \geq \mathfrak{P}(A, C)$.

Definition 3.2 is more restrictive that Definition 3.1 since it adds to the latter axiom S3.

A third definition can be found in [MMR06], where similarity measures are defined as follows:

Definition 3.3 [MMR06]
A mapping $\mathfrak{P} : \mathcal{F}(\mathcal{O})^2 \to [0, 1]$ is a *similarity measure* if $\mathfrak{P}$ satisfies the following axioms:

G1 $0 \leq \mathfrak{P}(A, B) \leq 1, \ \forall A, B \in \mathcal{F}(\mathcal{O})$.

G2 $\mathfrak{P}(A, B) = \mathfrak{P}(B, A), \ \forall A, B \in \mathcal{F}(\mathcal{O})$.

S5 Let $A \neq \emptyset \neq B$, if $\mathfrak{P}(A, B) = 0 \implies A \cap B = \emptyset$.

G1* and S4* $\mathfrak{P}(A, B) = 1 \iff A = B$.

S1*, S2 and G2 If either $A \supseteq B \supseteq C$ or $A \subseteq B \subseteq C$ then $\mathfrak{P}(A, C) \leq min(\mathfrak{P}(A, B), \mathfrak{P}(B, C))$.

Definition 3.3 is also more restrictive that Definition 3.1, in this case by replacing axiom S4 by S4*, and adding axiom S5. It differs from Definition 3.2 in these two axioms and also in that axiom S3 is not necessarily satisfied by Definition 3.3.

Finally, another proposal can be found in [ZL06], where a similarity measure is defined for interval-valued fuzzy sets, that can be particularized to the case of fuzzy sets as follows:

Definition 3.4 [ZL06; ZZM09]
A mapping $\mathfrak{P} : \mathcal{F}(\mathcal{O})^2 \to [0, 1]$ is a *similarity measure*, if $\mathfrak{P}$ satisfies the following axioms:

G1 $0 \leq \mathfrak{P}(A, B) \leq 1, \ \forall A, B \in \mathcal{F}(\mathcal{O})$.

G2 $\mathfrak{P}(A, B) = \mathfrak{P}(B, A)$.

S3 $\mathfrak{P}(A, \bar{A}) = 0, \ \forall A \in \mathcal{P}(\mathcal{O})$.

G1* and S4* $\mathfrak{P}(A, B) = 1 \iff A = B$.

S1* and S2 For any $A, B, C \in \mathcal{F}(\mathcal{O})$, if $A \subseteq B \subseteq C$, then $\mathfrak{P}(A, C) \leq \mathfrak{P}(A, B)$ and $\mathfrak{P}(A, C) \leq \mathfrak{P}(B, C)$.

Definition 3.4 is similar to Definition 3.2 except for the inclusion of axiom S4* instead of S4. With respect to Definition 3.3, it differs in that it requires axiom S3 instead of S5.

In the Sections 4.1 and 5 we shall see examples of measures satisfying these axioms, as well as other measures based on entropy, fuzzy implications, and set operations, respectively.

4 Defining specificity in terms of similarity

As we have mentioned before, given a fuzzy set representing the information available about the value of a variable, its specificity indicates to which extent there is not uncertainty about the actual value of the variable. Specificity measures can be seen as the degree to which the fuzzy set is a crisp singleton, since crisp singletons are the possibility distributions that guarantee no uncertainty. This idea is reinforced by the axioms defining specificity measures, particularly by axiom $\mathbb{A}$, which specifies that the maximum specificity is achieved by crisp singletons only, but also by the different monotonicity conditions imposed in the three definitions described in Section 2: specificity is strictly increasing or increasing with respect to a_1, and it is decreasing with respect to a_i, $i \geq 2$, which is equivalent to say that specificity increases as A *becomes more similar to a singleton.*

On this basis, the main idea in this work is to compute the specificity of a fuzzy set A by means of its similarity to the closest crisp singleton. Let us consider the following definition:

Definition 4.1
Let $\mathcal{O} \neq \emptyset$ be a finite set with $|\mathcal{O}| = n$. For every $A \in \mathcal{F}(\mathcal{O})$, let $A^{\downarrow} \in \mathcal{F}(\mathcal{O})$ be a fuzzy set defined such that $A^{\downarrow}(o_i) = a_i$, where $a_1 \geq a_2 \geq \cdots \geq a_n$ are the membership degrees of set A arranged in decreasing order. Let $S = \{o_1\} \subseteq \mathcal{O}$. Let $\mathfrak{P}$ be a similarity measure between fuzzy sets. Then we define $Sp_{\mathfrak{P}} : \mathcal{F}(\mathcal{O}) \to [0,1]$ as:

$$Sp_{\mathfrak{P}}(A) = \begin{cases} 0 & A = \emptyset \\ \mathfrak{P}(A^{\downarrow}, S) = \mathfrak{P}(A^{\downarrow}, \{o_1\}) & \text{otherwise} \end{cases} \tag{1}$$

Note that, as in the case of specificity measures, $Sp_{\mathfrak{P}}(A)$ solely depends on the membership degrees of this fuzzy set, and is independent of which object each membership degree is assigned to, so there is no problem in considering the similarity between $A^{\downarrow}$ and S, instead of the similarity between A and S. This allows us to align membership degrees so that the maximum degree of A is assigned to o_1 when measuring similarity to $S = \{o_1\}$, which is an intuitive way to define specificity in terms of similarity to the closest singleton, as shown for the case of distance in [Yag98].

However, $Sp_{\mathfrak{P}}$ is not always a specificity measure, depending on the properties of $\mathfrak{P}$. In the following, sufficient conditions for $\mathfrak{P}$ under which $Sp_{\mathfrak{P}}$ in Equation (1) defines an specificity measure over $\mathcal{F}(\mathcal{O})$ are shown.

Proposition 2
If $\mathfrak{P}$ fulfils G1* and S4*, then $Sp_{\mathfrak{P}}$ satisfies $\mathbb{A}$.

Proof:

$\Rightarrow$) $A = \{o\} \subseteq \mathcal{O} \Longrightarrow a_1 = 1,\ a_i = 0\ \forall i \geq 2 \underset{Def.4.1}{\Longrightarrow} A^{\downarrow} = \{o_1\} = S \underset{Eq.(1)}{\Longrightarrow} Sp_{\mathfrak{P}}(A) = \mathfrak{P}(S,S) \underset{S4*}{=} max_{A,B\in\mathcal{F}(\mathcal{O})}\mathfrak{P}(A,B) \underset{G1*}{=} 1.$

$\Leftarrow$) $Sp_{\mathfrak{P}}(A) = 1 \underset{Eq.(1)}{\Longrightarrow} \mathfrak{P}(A^{\downarrow}, S) = 1 \underset{G1*}{=} max_{A,B\in\mathcal{F}(\mathcal{O})}\mathfrak{P}(A,B) \underset{S4*}{\Longrightarrow} A^{\downarrow} = S \underset{Def.4.1}{\Longrightarrow} A = \{o\} \subseteq \mathcal{O}.$

■

Proposition 3
$Sp_{\mathfrak{P}}$ satisfies $\mathbb{B}$ for any similarity measure $\mathfrak{P}$.

Proof: Immediate by Equation (1). ■

Proposition 4
If $\mathfrak{P}$ fulfils G2 and S1 then $Sp_{\mathfrak{P}}$ satisfies $\mathbb{C}$.

Proof: If $A, B \in \mathcal{F}(\mathcal{O})$ are two normal fuzzy sets such that $A \subseteq B$, then it is $a_i \leq b_i\ \forall i$, with $a_1 = b_1 = 1$. Hence it is $S = \{o_1\} \subseteq A^{\downarrow} \subseteq B^{\downarrow}$. Furthermore, $max_{o\in O}S(o) = max_{o\in\mathcal{O}}A^{\downarrow}(o) = 1 \underset{S1}{\Longrightarrow} \mathfrak{P}(S, B^{\downarrow}) \leq \mathfrak{P}(S, A^{\downarrow}) \underset{G2}{\Longrightarrow} \mathfrak{P}(B^{\downarrow}, S) \leq \mathfrak{P}(A^{\downarrow}, S) \underset{Eq.(1)}{\Longrightarrow} Sp_{\mathfrak{P}}(B) \leq Sp_{\mathfrak{P}}(A).$ ■

In the proofs of the following propositions, the fuzzy set A_i^{ε} [BM+10], obtained from a fuzzy set A with $\varepsilon > 0$, is used. This set is defined in [BM+10] to be:

$$A_i^\varepsilon(o_j) = \begin{cases} A(o_j) & j \neq i \\ min(1, A(o_j) + \varepsilon) & j = i \end{cases} \quad (2)$$

Proposition 5
If $\mathfrak{P}$ fulfils S2* then $Sp_{\mathfrak{P}}$ satisfies $\mathbb{C}^{\uparrow}$.

Proof: Let us consider A defined on $\mathcal{O}$, and let $A(o_i) = a_1 < 1$ and A_i^ε for some $\varepsilon > 0$. Then it is $a_j = A^{\downarrow}(o_j) = A_i^{\varepsilon\downarrow}(o_j) = a_j^\varepsilon \; \forall j \geq 2$ and $a_1 = A^{\downarrow}(o_1) < A_i^{\varepsilon\downarrow}(o_1) = a_1^\varepsilon = min(1, a_1 + \varepsilon) \leq 1 = S(o_1) \underset{S2*}{\Longrightarrow} \mathfrak{P}(A^{\downarrow}, S) < \mathfrak{P}(A_i^{\varepsilon\downarrow}, S) \underset{Eq.(1)}{\Longrightarrow} Sp_{\mathfrak{P}}(A) < Sp_{\mathfrak{P}}(A_i^\varepsilon)$, so $Sp_{\mathfrak{P}}$ is strictly increasing with respect to a_1. ■

Proposition 6
If $\mathfrak{P}$ fulfils S7 then $Sp_{\mathfrak{P}}$ satisfies $\mathbb{C}^{\downarrow}$.

Proof: Let us consider $A \neq \emptyset$ defined on $\mathcal{O}$ with $|\mathcal{O}| > 1$, with $A(o_i) = a_j < a_1$ for some $j \geq 2$, and let A_i^ε for some $0 < \varepsilon \leq a_1 - a_j$, that is, $a_j < a_j + \varepsilon \leq a_1$ (in order to fulfil the preconditions of axiom $\mathbb{C}^{\downarrow}$). Then it is $A_i^{\varepsilon\downarrow}(o_k) = A^{\downarrow}(o_k) \; \forall k > j$ and also for $k = 1$, and $A_i^{\varepsilon\downarrow}(o_k) \geq A^{\downarrow}(o_k) \; \forall 1 < k \leq j$. Hence, for any intersection and difference of fuzzy sets, it is $A_i^{\varepsilon\downarrow} \cap \{o_1\} = A^{\downarrow} \cap \{o_1\}$, $A_i^{\varepsilon\downarrow} \setminus \{o_1\} \supseteq A^{\downarrow} \setminus \{o_1\}$ and $\{o_1\} \setminus A_i^{\varepsilon\downarrow} = \{o_1\} \setminus A^{\downarrow} \underset{S7}{\Longrightarrow} \mathfrak{P}(A_i^{\varepsilon\downarrow}, \{o_1\}) \leq \mathfrak{P}(A^{\downarrow}, \{o_1\}) \underset{Eq.(1)}{\Longrightarrow} Sp_{\mathfrak{P}}(A_i^\varepsilon) \leq Sp_{\mathfrak{P}}(A)$, so $Sp_{\mathfrak{P}}$ is decreasing with respect to $a_j = A(o_i)$. ■

According to the previous propositions, the following theorems stablish sets of axioms for $\mathfrak{P}$ as sufficient conditions under which Equation (1) defines a specificity measure according to Definitions 2.1, 2.2 and 2.3:

Theorem 1
If $\mathfrak{P}$ satifies G1*, G2, S4* and S1, then $Sp_{\mathfrak{P}}$ is a specificity measure over $\mathcal{F}(\mathcal{O})$ according to Definition 2.1.

As a consequence of this theorem, the following corollary follows:

Corollary 1
Similarity measures fulfiling either Definition 3.3 or Definition 3.4 yield specificity measures following Definition 2.1 using our approach as introduced in Definition 4.1.

Theorem 2
If $\mathfrak{P}$ fulfils G1*, S4*, S2* and S7, then $Sp_{\mathfrak{P}}$ is a specificity measure over $\mathcal{F}(\mathcal{O})$ according to Definition 2.3 and, as a consequence of Proposition 1, also according to Definition 2.2 and Definition 2.1.

In the following section, we present some illustrative examples of these results.

4.1 Some examples

As we have seen, similarity measures following Definitions 3.3 and 3.4 allow us, through Theorem 1, to obtain specificity measures using Definition 4.1. Let us introduce a couple of concrete examples.

The following measure is the analogous of Gregson's model [BA09]:

$$\mathfrak{P}_1(A, B) = \begin{cases} 1 & A = B = \emptyset \\ \frac{|A \cap B|}{|A \cup B|} & \text{otherwise} \end{cases} \quad (3)$$

It can be shown that $\mathfrak{P}_1$ fulfils the axioms in Definition 3.3, using the minimum for the intersection and the sigma-count for cardinality computation:

Proof:

G1) It is clear by definition, since $|A \cap B| \leq |A \cup B|$.

G2) It is clear, since intersection and union are symmetrical set operations.

G1* ∧ S4*) ⇒) $\mathfrak{P}_1(A,B) = 1 \implies A = B = \emptyset$ or $|A \cap B| = |A \cup B|$. In the second case, since $A \cup B \neq \emptyset$, we have that $min(A(o), B(o)) = max(A(o), B(o)),\ \forall o \in \mathcal{O} \implies A(o) = B(o),\ \forall o \in \mathcal{O} \implies A = B$.

⇐) $A = B \implies A \cap B = A \cup B \implies |A \cap B| = |A \cup B| \implies \mathfrak{P}_1(A,B) = 1$.

S5) $A \neq \emptyset \neq B \implies A \cup B \neq \emptyset$. Since $\mathfrak{P}_1(A,B) = 0 \implies |A \cap B| = 0 \implies A \cap B = \emptyset$.

S1*,S2 and G2) Consider the case when $A \supseteq B \supseteq C$ (the other case is analogous). In this case, $A \cup B = A \cup C = A$, $A \cap C = B \cap C = C$, $A \cap B = B$ and $B \cup C = B$. In addition, $|C| \leq |B| \leq |A|$. Then $\mathfrak{P}_1(A,C) = \frac{|C|}{|A|} \leq min(\mathfrak{P}_1(A,B) = \frac{|B|}{|A|}, \mathfrak{P}_1(B,C) = \frac{|C|}{|B|})$.

■

Taking the above analyzed measure of Equation (3) we obtain through Definition 4.1:

$$Sp_{\mathfrak{P}_1}(A) = \frac{a_1}{1 + \sum_{i=2}^{n} a_i} \tag{4}$$

which is a specificity measure according to Definitions 2.1, 2.2, and 2.3. This measure was introduced in [Mar+18] as a new measure on the basis of the proposal in that paper.

As an additional example, in [ZL06] several similarity measures are defined throught entropy measures, fulfiling Definition 3.4. Particularly, they use the following entropy measures:

- $E_1(A) = 1 - \frac{1}{n}\sum_{o \in \mathcal{O}} |2A(o) - 1|$
- $E_2(A) = 1 - \sqrt{\frac{1}{n}\sum_{o \in \mathcal{O}} (2A(o) - 1)^2}$

as well as the following fuzzy sets:

- $f(A,B)(o) = \frac{1 + |A(o) - B(o)|}{2}$
- $g(A,B)(o) = \frac{1 + |A(o) - B(o)|^p}{2}, p > 0$

Then, the following are similarity measures according to Definition 3.4 [ZL06]:

1. $\mathfrak{P}_E^f(A,B) = E(f(A,B))$
2. $\bar{\mathfrak{P}}_E^f(A,B) = E\left(\overline{f(A,B)}\right)$
3. $\mathfrak{P}_E^g(A,B) = E(g(A,B))$
4. $\bar{\mathfrak{P}}_E^g(A,B) = E\left(\overline{g(A,B)}\right)$

The following specificity measures in the sense of Definition 2.1 can be obtained from the similarity measures $\mathfrak{P}_{E_1}^f$ and $\bar{\mathfrak{P}}_{E_2}^g$ according to Definition 4.1:

$$Sp_{\mathfrak{P}_{E_1}^f}(A) = \mathfrak{P}_{E_1}^f(A_{\mathcal{X}_\mathcal{O}}, s) = 1 - \frac{1}{n}\sum_{x \in \mathcal{X}_\mathcal{O}} |A_{\mathcal{X}_\mathcal{O}}(x) - s(x)| = \frac{a_1 + \sum_{i \geq 2}(1 - a_i)}{n} \tag{5}$$

$$Sp_{\bar{\mathfrak{P}}_{E_2}^g}(A) = \bar{\mathfrak{P}}_{E_2}^g(A_{\mathcal{X}_\mathcal{O}}, s) = 1 - \frac{1}{\sqrt{n}}\sqrt{\sum_{x \in \mathcal{X}_\mathcal{O}} |A_{\mathcal{X}_\mathcal{O}}(x) - s(x)|^{2p}} =$$

$$= 1 - \frac{1}{\sqrt{n}}\sqrt{(1 - a_1)^{2p} + \sum_{i \geq 2} a_i^{2p}} \tag{6}$$

To the best of our knowledge, these measures have not been defined previously in the literature. Other measures can be derived by different combinations of entropy measures and fuzzy sets.

5 Discussion

In the previous section we have proposed a way of constructing specificity measures from measures of similarity: Equation (1) defines a measure of specificity when the conditions analyzed by Theorems 1 and 2 are accomplished. In this section we first study the relationship between different axiomatic definitions of similarity in the literature and our proposal. Afterwards, specificity measures derived from some widely employed measures of similarity, outside the axiomatic definitions, are proposed.

5.1 Axiomatic definitions of similarity measures and $Sp_{\mathfrak{P}}$

In contrast to Corollary 1, it can be proved that the fulfilment of axioms in Definition 3.2 is not a sufficient condition for a similarity measure to define a specificity measure following our approach. As a consequence, the same can be said of Definition 3.1.

To show that, let us consider the following similarity measure between fuzzy sets in $\mathcal{F}(\mathcal{O})$:

$$\mathfrak{P}(A,B) = \begin{cases} 1 & A \cap B \neq \emptyset \text{ or } A = B \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where the intersection is computed via the minimum. Then, we are going to prove that, although it is a similarity measure according to Definition 3.2 (Proposition 7), it cannot define a specificity measure (Proposition 8).

Proposition 7
$\mathfrak{P}$ in Equation (7) is a similarity measure following Definition 3.2.

Proof: Axioms G1, G2, S3, and G1* and S4 are immediate. To prove axioms S1* and S2 it is enough to consider that $\emptyset \neq F \subseteq G$ implies $\mathfrak{P}(F,G) = 1$ and $\mathfrak{P}(\emptyset, F) = \mathfrak{P}(\emptyset, G) = 0$, and also $\mathfrak{P}(\emptyset,\emptyset) = 1$. Then, under these conditions:

- When $\emptyset \neq A \subseteq B \subseteq C$ it is $\mathfrak{P}(A,B) = \mathfrak{P}(A,C) = \mathfrak{P}(B,C) = \mathfrak{P}(A,C) = 1$.
- When $\emptyset = A \subset B \subseteq C$ it is $\mathfrak{P}(A,B) = \mathfrak{P}(A,C) = 0$ and $1 = \mathfrak{P}(B,C) > \mathfrak{P}(A,C) = 0$.
- When $\emptyset = A = B \subset C$ it is $1 = \mathfrak{P}(A,B) > \mathfrak{P}(A,C) = 0$ and $\mathfrak{P}(B,C) = \mathfrak{P}(A,C) = 0$.
- When $\emptyset = A = B = C$ it is $\mathfrak{P}(A,B) = \mathfrak{P}(A,C) = \mathfrak{P}(B,C) = \mathfrak{P}(A,C) = 1$.

∎

On the basis of the similarity measure in Equation (7), through Definition 4.1 we obtain:

$$Sp_{\mathfrak{P}(A)} = \begin{cases} 0 & A = \emptyset \\ 1 & \text{otherwise} \end{cases} \tag{8}$$

The following proposition holds:

Proposition 8
$Sp_{\mathfrak{P}}$ in Equation (8) is not a specificity measure.

Proof: It is enough to consider that given $A = \mathcal{O}$ with $|\mathcal{O}| > 1$, $Sp_{\mathfrak{P}}(A) = 1$. Then, $Sp_{\mathfrak{P}}$ does not satisfy axiom $\mathbb{A}$. ∎

Hence, neither the axioms in Definition 3.2 nor those in 3.1 define sufficient conditions for measures of similarity in order to define specificity measures following Definition 4.1.

5.2 Other measures

Let us remark that the value of Definition 4.1 goes beyond Theorems 1 and 2, since they are not defining necessary conditions, but sufficient ones. Hence, it is possible to obtain specificity measures from similarity measures not satisfying the conditions in Theorems 1 and 2. In this section we illustrate this claim with some widely used similarity measures.

- In [BA09] the following similarity measure is defined where T, $\otimes$ are t-norms and I is a fuzzy implication operator:

$$\mathfrak{P}_{I,\otimes}(A,B) = T_{o\in\mathcal{O}} \otimes (I(A(o),B(o)), I(B(o),A(o))) \tag{9}$$

It can be shown that this measure does not fulfil G1* and S4*, for example, when $I(a,b) = max(b, 1-a)$.

Proof: Let $A = \sum_{o\in\mathcal{O}} {}^{0.5}/_{o}$, i.e., the fuzzy set to which all elements belong with grade 0.5. Since $I(0.5,0.5) = max(0.5,0.5) = 0.5$, and $min(0.5,0.5) = 0.5$. Then $\mathfrak{P}_{I,T,\otimes}(A,A) \leq 0.5$ as the minimum is the greatest t-norm. ∎

Thus, Theorems 1 and 2 cannot be applied. However, Definition 4.1 can be used to derive a specificity measure from the measure in Equation (9) in this case.

Again, let us consider $I(a,b) = max(b, 1-a)$ as implicator. If $T = min$, we have that

$$Sp_{\mathfrak{P}_{I,min,\otimes}} = min(a_1, 1-a_2) \tag{10}$$

for any t-norm $\otimes$, which is a specificity measure according to Definitions 2.1 and 2.2. On the other hand, if we consider the same implicator and the product t-norm instead of the minimum, then

$$Sp_{\mathfrak{P}_{I,*,\otimes}} = a_1 \Pi_{i\geq 2}(1-a_i) \tag{11}$$

for any t-norm $\otimes$, which is a specificity measure from the product family with parameter $k = 0$ [Yag90], satisfying Definitions 2.1, 2.2, and 2.3.

- The following measure of similarity can be derived from the proposal in [Tve77]:

$$\mathfrak{P}_{Tversky}(A,B) = \theta|A\cap B| - \alpha|A\backslash B| - \beta|B\backslash A|; \alpha,\beta,\theta \geq 0 \tag{12}$$

This measure does not meet axiom S4*, whatever the values of the parameters. This is easy to see only by considering the case when $A = B = \emptyset$.

Let us consider $\theta = 1$, $\alpha = 1/(n-1)$, and $\beta = 0$. Let us use the minimum and sigma-count for computing intersection and cardinality, respectively. Then, for any fuzzy set difference, we obtain

$$Sp_{\mathfrak{P}_{Tversky}}(A) = a_1 - \frac{1}{n-1}\sum_{i\geq 2} a_i \tag{13}$$

which is a measure of specificity of the linear family [Yag90] according to Definitions 2.1, 2.2, and 2.3.

6 Conclusions and future work

We have shown that it is possible to obtain specificity measures from measures of similarity between fuzzy sets. Our proposal obtains specificity measures by computing the similarity between a fuzzy set and its closest singleton. Though, with this approach, not every similarity leads to a specificity measure, we have proved that certain sets of axioms for similarity measures are sufficient conditions for defining specificity measures. We also show in the paper that the power of our approach is not limited to the mentioned sets of axioms. We have illustrated these facts using some well known similarity measures. This work opens a very promising research line in the study of specificity measures with semantics grounded on the semantics of similarity.

Acknowledgements

This work has been partially supported by the Spanish Ministry of Economy and Competitiveness and the European Regional Development Fund (Fondo Europeo de Desarrollo Regional - FEDER) under project TIN2014-58227-P. In addition, this work has been partially supported by the Spanish Ministry of Science, Innovation and Universities and the European Regional Development Fund - ERDF (Fondo Europeo de Desarrollo Regional - FEDER) under project PGC2018-096156-B-I00 *Recuperación y Descripción de Imágenes mediante Lenguaje Natural usando Técnicas de Aprendizaje Profundo y Computación Flexible.* This work has been also partially supported by the Spanish Ministry of Education, Culture and Sport grant FPU16/05199 and by the Universidad de Granada grant "Becas de Iniciación para estudiantes de Máster del Plan Propio de Investigación (2017)".

References

[BA09] Ismat Beg and Samina Ashraf. "Similarity Measures for Fuzzy Sets". In: *Applied and Computational Mathematics* 8.2 (2009), pp. 192–202.

[BM+10] Bernadette Bouchon-Meunier, Giulianella Coletti, Marie-Jeanne Lesot, and Maria Rifqi. "Towards a Conscious Choice of a Fuzzy Similarity Measure: A Qualitative Point of View". In: *Computational Intelligence for Knowledge-Based Systems Design.* Ed. by David Hutchison et al. Vol. 6178. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 1–10.

[Bus00] Humberto Bustince. "Indicator of inclusion grade for interval-valued fuzzy sets. Application to approximate reasoning based on interval-valued fuzzy sets". In: *International Journal of Approximate Reasoning* 23.3 (2000), pp. 137–209.

[CGS13] Inés Couso, Laura Garrido, and Luciano Sánchez. "Similarity and dissimilarity measures between fuzzy sets: A formal relational study". In: *Information Sciences* 229 (2013), pp. 122–141.

[DP87] Didier Dubois and Henri Prade. "Properties of Measures of Information in Evidence and Possibility Theories". In: *Fuzzy Sets and Systems.* Measures of Uncertainty 24.2 (1987), pp. 161–182.

[DP99] Didier Dubois and Henri Prade. "Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions". In: *Fuzzy Sets and Systems* 100 (1999), pp. 73 –132.

[Gar+06a] L. Garmendia, R. R. Yager, E. Trillas, and A. Salvador. "A t-norm based specificity for fuzzy sets on compact domains". In: *International Journal of General Systems* 35.6 (2006), pp. 687–698.

[Gar+03] Luis Garmendia, Ronald R. Yager, Enric Trillas, and Adela Salvador. "On t-norms based measures of specificity". In: *Fuzzy Sets and Systems* 133.2 (2003), pp. 237–248.

[Gar+06b] Luis Garmendia, Ronald R. Yager, Enric Trillas, and Adela Salvador. "Measures of specificity of fuzzy sets under T-indistinguishabilities". In: *IEEE Transactions on Fuzzy Systems* 14.4 (2006), pp. 568–572.

[LS03] Zhizhen Liang and Pengfei Shi. "Similarity measures on intuitionistic fuzzy sets". In: *Pattern Recognition Letters* 24.15 (2003), pp. 2687–2693.

[Mar+17] N. Marín, G. Rivas-Gervilla, D. Sanchez, and R. R. Yager. "On families of bounded specificity measures". In: *2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE).* 2017, pp. 1–6.

[Mar+18] N. Marín, G. Rivas-Gervilla, D. Sánchez, and R. R. Yager. "Specificity Measures and Referential Success". In: *IEEE Transactions on Fuzzy Systems* 26.2 (2018), pp. 859–868.

[MMR06] Banibrata Mondal, Dipankar Mazumdar, and Swapan Raha. "Similarity in Approximate Reasoning". In: *International Journal of Computational Cognition* 4.3 (2006), pp. 46–56.

[Sá+17] J. L. G. Sánchez, R. G. del Campo, L. Garmendia, and R. R. Yager. "Comparing families of measures of k-specificity. Measure of crispness". In: *2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE).* 2017, pp. 1–6.

[Tve77] Amos Tversky. "Features of similarity". In: *Psychological Review* 84.4 (1977), pp. 327–352.

[Yag90] R. R. Yager. "Ordinal Measures of Specificity". In: *International Journal of General Systems* 17.1 (1990), pp. 57–72.

[Yag82] Ronald R. Yager. “Measuring Tranquility and Anxiety in Decision Making: An Application of Fuzzy Sets”. In: *International Journal of General Systems* 8.3 (1982), pp. 139–146.

[Yag84] Ronald R. Yager. “Measuring the Quality of Linguistic Forecasts”. In: *International Journal of Man-Machine Studies* 21.3 (1984), pp. 253–257.

[Yag91] Ronald R. Yager. “Similarity based specificity measures”. In: *International Journal of General Systems* 19.2 (1991), pp. 91–105.

[Yag92a] Ronald R. Yager. “Default knowledge and measures of specificity”. In: *Information Sciences* 61.1-2 (1992), pp. 1–44.

[Yag92b] Ronald R Yager. “On the specificity of a possibility distribution”. In: *Fuzzy Sets and Systems* 50.3 (1992), pp. 279–292.

[Yag98] Ronald R. Yager. “Measures of Specificity”. In: *Computational Intelligence: Soft Computing and Fuzzy-Neuro Integration with Applications*. Ed. by Okyay Kaynak, Lotfi A. Zadeh, Burhan Türksen, and Imre J. Rudas. Springer Berlin Heidelberg, 1998, pp. 94–113.

[Yag08] Ronald R. Yager. “Measures of specificity over continuous spaces under similarity relations”. In: *Fuzzy Sets and Systems* 159.17 (2008), pp. 2193–2210.

[Yag12] Ronald R Yager. “Expansible measures of specificity”. In: *International Journal of General Systems* 41.3 (2012), pp. 247–263.

[ZL06] Wenyi Zeng and Hongxing Li. “Relationship between similarity measure and entropy of interval valued fuzzy sets”. In: *Fuzzy Sets and Systems* 157.11 (2006), pp. 1477–1484.

[ZZM09] Hongying Zhang, Wenxiu Zhang, and Changlin Mei. “Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure”. In: *Knowledge-Based Systems* 22.6 (2009), pp. 449–454.