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Consensus and Multi-Criteria Three-Way Decision Models and Applications in Group Decision-Making

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Consensus and Multi-Criteria Three-Way Decision Models and Applications in Group Decision-Making

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Abstract

Group Decision-Making (GDM) aims to integrate the diverse judgments and preferences of multiple Decision-Makers (DMs) to reach collective agreements, often supported by Multi-Criteria Decision-Making (MCDM) methods and facilitated through the Consensus-Reaching Process (CRP). However, conventional GDM approaches face significant challenges in handling the complexity and uncertainty of real-world decision problems. Existing consensus models often overlook the dynamic nature of opinion reliability in social networks, leading to potentially biased or unstable consensus outcomes. At the same time, traditional MCDM methods typically rely on Two-Way Decision (2WD) logic, which struggles to capture hesitation and uncertainty in complex decision contexts. Although the Three-Way Decision (3WD) theory introduces a more flexible framework by adding a delay option, it is rarely integrated with consensus processes and often ignores subjective preferences, multi-source information, and psychological factors like regret. Furthermore, the integration of 3WD and consensus models remains underexplored, despite their potential to address the limitations of conventional decision-making frameworks. Additionally, the practical application of these integrated models to real-world GDM scenarios, such as Energy Internet (EI) project evaluation, has not been fully developed, limiting their impact on real-world decision-making.

To address these interconnected challenges, this thesis aims to construct a comprehensive framework for integrating consensus processes and multi-criteria 3WD models, with the goal of enhancing the efficiency and robustness of GDM in uncertain and complex environments. Specifically, this thesis is structured around four interconnected research topics:

(1) The group consensus evolution process in social trust networks. Traditional GDM models often overlook the dynamic nature of opinion reliability and trust evolution in social networks. This thesis introduces novel methods for evaluating opinion reliability, adjusting trust propagation, and modeling realistic opinion dynamics, aiming to capture the influence of social trust on the consensus-reaching process. This forms the foundational layer of the proposed framework, emphasizing the critical role of trust in consensus dynamics.

(2) The construction of a preference-based regret multi-criteria 3WD model. Recognizing that DMs often express subjective preferences and experience psychological reactions like regret or hesitation, this thesis develops a 3WD model based on regret theory within a Linguistic Z-Number (LZN) framework. This model accounts for both classification and ranking, addressing the limitations of conventional 3WD approaches in capturing the uncertainty and subjectivity of expert judgments. This component complements the consensus process by providing a refined approach for decision evaluation under uncertainty, building on the strengths of traditional MCDM models while addressing their limitations in handling complex, uncertain information.

(3) The integration of group consensus and multi-criteria 3WD models. Building on Set Pair Analysis (SPA) theory, this thesis extends the conventional two-state decision framework to a three-state model, capturing consensus, conflict, and uncertainty. This approach aims to enhance

the granularity of expert behavior modeling, providing a more realistic representation of complex decision environments. It acts as the critical link between consensus formation and decision evaluation, integrating the benefits of both perspectives and bridging the gap between GDM and 3WD.

(4) The application research on GDM in EI project evaluation. In the context of accelerating the transformation towards a sustainable circular economy, this objective aims to bridge the gap between theoretical advancements in GDM and practical applications in EI project evaluation. To achieve this, a multi-criteria decision-making framework based on Flexible Linguistic Expressions (FLEs) and Multi-Granularity Cloud-Rough Set (MGCRS) will be developed. This framework will address several key challenges, including the quantitative transformation of discrete linguistic terms into continuous cloud information, the integration of probabilistic linguistic modeling, and the processing of highly uncertain data. Additionally, a comprehensive index system covering technical, environmental, and economic dimensions of EI projects will be established to enhance decision accuracy and robustness. The proposed models will be validated using real-world data, such as the Beijing-Tianjin-Hebei region, to demonstrate their effectiveness in supporting complex decision-making processes.

Overall, this research presents a comprehensive framework that not only extends the theoretical foundations of GDM and 3WD but also bridges the gap between abstract decision theory and real-world applications. By integrating the strengths of consensus modeling, 3WD theory, MCDM, and practical applications, it provides systematic solutions for enhancing decision accuracy, stability, and effectiveness in complex decision-making scenarios.

Resumen

La toma de decisiones en grupo (GDM) tiene como objetivo integrar las valoraciones y preferencias diversas de múltiples tomadores de decisiones (DMs) para alcanzar acuerdos colectivos, a menudo apoyados por métodos de toma de decisiones multicriterio (MCDM) y facilitados a través del proceso de alcance de consenso (CRP). Sin embargo, los enfoques convencionales de GDM se enfrentan a desafíos significativos al manejar la complejidad e incertidumbre de los problemas de decisión del mundo real. Los modelos de consenso existentes, a menudo pasan por alto la naturaleza dinámica de la confiabilidad de las opiniones en las redes sociales, lo que puede llevar a resultados de consenso potencialmente sesgados o inestables. Al mismo tiempo, los métodos tradicionales de MCDM típicamente se basan en la lógica de decisiones de dos vías (2WD), que tiene dificultades para capturar la vacilación e incertidumbre en contextos de decisión complejos. Aunque la teoría de las decisiones de tres vías (3WD) introduce un marco más flexible al agregar una opción de demora, rara vez se integra con los procesos de consenso y a menudo ignora las preferencias subjetivas, la información de múltiples fuentes y factores psicológicos como el arrepentimiento. Además, la integración de modelos de 3WD y consenso sigue sin estar suficientemente explorada, a pesar de su potencial para abordar las limitaciones de los marcos de toma de decisiones convencionales. Además, la aplicación práctica de estos modelos integrados a escenarios reales de GDM, como la evaluación de proyectos de Internet de la Energía (EI), no se ha desarrollado completamente, limitando su impacto en la toma de decisiones del mundo real.

Para abordar estos desafíos interconectados, esta tesis tiene como objetivo construir un marco integral para integrar los procesos de consenso y modelos de 3WD multicriterio, con el objetivo de mejorar la eficiencia y robustez del GDM en entornos inciertos y complejos. Específicamente, esta tesis se estructura en torno a cuatro temas de investigación interconectados:

(1) El proceso de evolución del consenso grupal en redes de confianza social. Los modelos tradicionales de GDM a menudo pasan por alto la naturaleza dinámica de la confiabilidad de las opiniones y la evolución de la confianza en las redes sociales. Esta tesis introduce métodos novedosos para evaluar la confiabilidad de las opiniones, ajustar la propagación de la confianza y modelar dinámicas de opinión realistas, con el objetivo de capturar la influencia de la confianza social en el proceso de alcance de consenso. Esto forma la capa fundamental del marco propuesto, enfatizando el papel crítico de la confianza en la dinámica del consenso.

(2) La construcción de un modelo de 3WD multicriterio basado en preferencias y teoría del arrepentimiento. Reconociendo que los DMs a menudo expresan preferencias subjetivas y experimentan reacciones psicológicas como el arrepentimiento o la vacilación, esta tesis desarrolla un modelo de 3WD basado en la teoría del arrepentimiento dentro de un marco de números Z lingüísticos (LZN). Este modelo tiene en cuenta tanto la clasificación como la jerarquización, abordando las limitaciones de los enfoques convencionales de 3WD para capturar la incertidumbre y subjetividad en los juicios de expertos. Este componente complementa el proceso de consenso al proporcionar

un enfoque refinado para la evaluación de decisiones bajo incertidumbre, construyendo sobre las fortalezas de los modelos tradicionales de MCDM mientras aborda sus limitaciones para manejar información compleja e incierta.

(3) La integración de modelos de consenso grupal y 3WD multicriterio. Basándose en la teoría del análisis de pares conjuntos (SPA), esta tesis extiende el marco convencional de decisiones de dos estados a un modelo de tres estados, capturando consenso, conflicto e incertidumbre. Este enfoque busca mejorar la granularidad del modelado del comportamiento de los expertos, proporcionando una representación más realista de entornos de decisión complejos. Actúa como el vínculo crítico entre la formación de consenso y la evaluación de decisiones, integrando los beneficios de ambas perspectivas y cerrando la brecha entre GDM y 3WD.

(4) La investigación aplicada sobre GDM en la evaluación de proyectos de EI. En el contexto de la aceleración hacia una economía circular sostenible, este objetivo busca cerrar la brecha entre los avances teóricos en GDM y las aplicaciones prácticas en la evaluación de proyectos de EI. Para lograr esto, se desarrollará un marco de toma de decisiones multicriterio basado en expresiones lingüísticas flexibles (FLE) y conjuntos rugosos de multi-granularidad en nube (MGCRS). Este marco abordará varios desafíos clave, incluyendo la transformación cuantitativa de términos lingüísticos discretos en información continua de nube, la integración de modelos lingüísticos probabilísticos y el procesamiento de datos altamente inciertos. Además, se establecerá un sistema de índices comprensivo que cubra dimensiones técnicas, ambientales y económicas de los proyectos de EI para mejorar la precisión y robustez de las decisiones. Los modelos propuestos han sido validados utilizando datos del mundo real, como la región de Beijing-Tianjin-Hebei, para demostrar su efectividad en el apoyo a procesos de toma de decisiones complejos.

En general, esta investigación presenta un marco integral que no solo extiende los fundamentos teóricos de GDM y 3WD, sino que también cierra la brecha entre la teoría abstracta de decisiones y las aplicaciones del mundo real. Al integrar las fortalezas del modelado de consenso, la teoría de 3WD, MCDM y aplicaciones prácticas, proporciona soluciones sistemáticas para mejorar la precisión, estabilidad y efectividad de las decisiones en contextos de toma de decisiones complejos.

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Chapter I

PhD dissertation

1 Introduction

The rapid development of digital transformation has significantly increased the complexity of decision-making problems in the real world [HS24]. With the continuous emergence of emerging technologies such as artificial intelligence, big data, the Internet of Things and blockchain, various decision-making problems have evolved into complex, high-dimensional problems characterized by large-scale, heterogeneous, uncertain, and dynamic information environments. Traditional decision-making methods, which often rely on simplified models and fixed assumptions, struggle to cope with these new complexities. In this context, DMs not only need to deal with the sharp increase in information volume, but also handle the uncertainty, fuzziness and dynamics of the information. These challenges make traditional decision-making methods often difficult to meet the demands of the modern decision-making environment [MZG^{+}24]. To overcome these challenges, GDM emerges as a practical and effective approach [LKP21]. By integrating the professional knowledge and experience of multiple DMs, GDM helps to reduce cognitive biases and improve the quality of decision-making outcomes through collective wisdom. Typically, the GDM process consists of two key stages: (1) the CRP, which promotes mutual understanding and integration of expert opinions; (2) the selection stage, at which the optimal solution is selected [LKLP22]. In this process, CRP is particularly crucial because it not only determines the efficiency of decision-making but also directly affects the quality of decision-making. Especially in complex environments involving multiple criteria, i.e., MCDM, the advantages of GDM are more significant because it can effectively integrate the viewpoints and preferences of different experts, reducing information asymmetry and decision-making biases[Tha21]. Furthermore, Multi-Criteria Group Decision-Making (MCGDM) extends this framework by aiming to reach the best collective decision that balances the viewpoints and preferences of all DMs, ensuring that the final choice is both comprehensive and representative of the group's collective expertise [BCPBP23].

However, traditional GDM models often treat expert opinions as independent and equally reliable, assuming that each DM's input is of similar credibility and influence.This assumption overlooks the fact that real-world decision-making often occurs in socially interconnected environments, where individual opinions are influenced by social interactions and trust dynamics. With the rise of social networks and online platforms like Facebook, WeChat, and Twitter, interpersonal communication has become more frequent and complex, fundamentally changing the nature of decision-making [GWL^{+}24]. In these environments, DMs are no longer isolated but embedded in trust-based social networks, where their views are shaped through ongoing consultations with neighbors, friends, or colleagues. To capture this phenomenon, the field of Social Network Group Decision-Making (SNGDM) has emerged [GZ21]. SNGDM models often utilize opinion dynamics to describe how individual opinions evolve over time, considering factors such as trust, credibility, and social influence [BAPV24]. However, most studies in this field assume that experts are completely reliable and that the trust value among them is given or fixed. In practice, the credibility of expert opinions may vary greatly, and subjective biases may affect the way trust is distributed [GMLG24]. Although system reliability has long been studied in the field of engineering, the concept of human reliability – especially in the context of social decision-making – has still not been fully explored. This brings new challenges to the modeling of consensus formation in real-world social networks.

Meanwhile, MCDM has long served as the cornerstone of decision-making theory, providing systematic methods for evaluating and ranking alternatives based on multiple criteria. Traditional MCDM models typically rely on binary decision logic, often referred to as 2WD, which forces DMs to either accept or reject an alternative without considering the inherent uncertainties and ambiguities present in real-world data [PHWM22]. To overcome this limitation, the 3WD theory is proposed [Yao09], which is rooted in decision-theoretic rough sets and Bayesian theory, introducing a third option – delay – to provide a buffer for indecision. This method has been proven to be able to effectively reduce decision-making risks and adapt to uncertainties. Although the 3WD model has many advantages, most existing models only focus on classification and are specifically designed for static decision information systems (DIS) [XZZL24]. However, DMs usually not only need to classify the altenatives but also rank them in practice, especially when dealing with emergency situations such as multiple data sources or medical triage. Furthermore, decisions in the real world are rarely made in isolation; Experts may have subjective preferences, historical experiences, and psychological reactions such as regret or hesitation [PHW24]. These factors are usually expressed in vague or linguistic terms, but the current 3WD model has limited ability to handle such soft information, especially when there are differences in the reliability evaluated by experts.

Recent studies have attempted to apply the 3WD theory to the scenario of group consensus [LFX22, WLL22]. However, most models adopt a two-state perspective on the consensus process (i.e., consensus or conflict) and use the degree of consensus to evaluate progress. This simplification ignores a common situation in complex GDM – uncertainty – where DMs often hesitate and may need further information or careful consideration. To make up for this defect, the Sequential Three-Way Decision (S3WD) model is introduced as an extension of granular computing [Yao13]. It operates at multiple granularity levels through three modules (trisecting, acting, and outcome). With the emergence of new information, the elements in the delay area can be iteratively re-evaluated. This mechanism is particularly applicable to GDM environments where delays or non-commitments pose risks but require a more flexible and continuous evaluation process.

Although these theories – GDM, MCDM, 3WD and S3WD – have all made substantial progress, current research still tends to view them in isolation. At present, there is a lack of an integrated model that can integrate the consensus reaching process, multi-criteria 3WD decision-making logic and practical applications into a coherent and practical framework. Against this backdrop, our research aims to extend the GDM framework from both theoretical and practical perspectives by focusing on how group consensus evolves and how complex multi-criteria 3WD models can be better supported and applied. In particular, we center our investigation around consensus models, multi-criteria 3WD models, and the application in real-world GDM scenarios. These directions are closely connected and together form a coherent path—from understanding how individuals reach consensus in social networks, to developing more flexible and psychologically grounded decision models, and finally to applying these models in practical domains. Our goal is to provide a systematic response to the limitations of existing GDM approaches while strengthening the bridge between theory and practice.

To this end, the research will proceed along four interrelated lines:

(1) Exploring the evolution of group consensus in social trust networks via opinion reliability. We will incorporate opinion reliability into opinion dynamics to better reflect how trust, credibility, and individual influence shape the CRP in social networks. This involves developing new methods for evaluating reliability, adjusting trust propagation mechanisms, and modeling realistic opinion evolution paths.

(2) Constructing a preference-based regret multi-criteria 3WD model. Recognizing that expert evaluations are often subjective and influenced by preferences and psychological reactions, we propose a 3WD model based on regret theory under the LZNs framework. This model will enable both classification and ranking of alternatives while accounting for preference heterogeneity and the uncertainty of expert judgments.

(3) Extending a sequential three-state consensus model for multi-criteria 3WD problems. Drawing on SPA, we aim to build a three-state framework—capturing consensus, conflict, and uncertainty—that enhances the granularity of expert behavior modeling in the CRP. The model will include iterative adjustment and feedback strategies to support gradual opinion alignment in complex group settings.

(4) Applying the proposed models to real-world decision-making problems. To ensure the practicality of our approach, we will implement and validate the developed models in application areas such as energy planning, recommendation systems, and environmental evaluation. This step will translate theoretical contributions into tangible tools that help address complex decisions involving multiple criteria and stakeholders.

This thesis mainly consists of two parts: the first one illustrates the existing problems, the basic concepts, and models, and the results obtained from the proposed models. The second part is a compilation of the main publications that are associated with this thesis. To improve the readability of subsequent content, we explain the three essential abbreviations that often appear in this thesis.

(1) GDM refers to the decision-making activities carried out by more than two people, including alternative evaluation, CRP, and the selection process.

(2) CRP is a key stage in GDM where DMs negotiate, adjust, or revise their opinions through iterative interaction to reach an acceptable level of agreement before finalizing the decision.

(3) MCDM refers to a decision-making process in which multiple criteria are considered simultaneously when evaluating a set of alternatives, which helps DMs to rank or select alternatives under complex conditions.

(4) 3WD is a decision-making theory based on decision-theoretic rough sets and Bayesian decision rules. It classifies objects into three disjoint regions—acceptance, rejection, and non-commitment (or delay)—to reduce decision risk and better handle uncertainty.

The rest of the thesis is organized as follows: Section 2 introduces some related preliminaries. Section 3 justifies the development of the thesis through discussing the basic ideas and challenges of current researches. Section 4 presents the objectives of the thesis. Section 5 introduces the methodologies used in the thesis. Section 6 discusses the summary of the proposals in the thesis. Section 7 presents a discussion of the results obtained in the thesis. Section 8 gives the conclusions of the thesis. Finally, some future works are discussed in Section 9.

2 Preliminaries

In this section, we provide an overview of the basic concepts for the GDM framework and the 3WD model. Specifically, Section 2.1 recalls the two important components of GDM including the CRP and selection process. Section 2.2 reviews the traditional 3WD method and decision rules.

2.1 Group decision-making

GDM usually involves handling uncertain and subjective information provided by multiple DMs. Due to cognitive limitations, subjective judgments, and incomplete knowledge, input information in real-world settings is often expressed in vague or linguistic terms. Therefore, appropriate methods are required to represent and process such uncertainty effectively. The GDM process typically consists of two main stages: the CRP and the selection process. The goal of CRP is to encourage DMs to adjust their opinions to improve the level of group consensus. During this process, DMs interact and negotiate with each other to achieve a reliable group solution. Opinion dynamics is a typical approach used to model CRP, as it simulates how individual opinions evolve and influence each other within a group, ultimately leading to consensus formation. Once the group consensus is reached, the selection stage is implemented to help DMs rank all alternatives and identify the most desirable one.

2.1.1 Several Uncertain Information Representations

This subsection introduces several uncertain information representations including LZNs, SPA, cloud model, and Pawlak rough set.

Definition 1. *[WCZ17] Let X be a universe discourse, $A = \{A_0, A_1, ..., A_{2g_1}\}$ and $B = \{B_0, B_1, ..., B_{2g_2}\}$ be two finite and totally ordered linguistic term sets, where g_1 and g_2 are non-negative integers. A LZN for $x \in X$ is defined as $Z = (A_{\varphi(x)}, B_{\phi(x)})$, where $A_{\varphi(x)} \in A$ is a restriction of the value that uncertain variables allow to take and $B_{\phi(x)} \in B$ is the measure of reliability of the first component $A_{\varphi(x)}$. $\varphi(x)$ and $\phi(x)$ are the subscripts of linguistic terms A and B, respectively.*

Definition 2. *[Zha89] For a problem W with N features, a set pair including two related set A and B is denoted as $H(A, B)$, in which S features are mutual, P features are opposite and $F(F = N - S - P)$ features are neither mutual nor opposite between A and B. The Connection Number (CN) $u(A, B)$ of the set pair $H(A, B)$ is defined as:*

$$u(A, B) = \frac{S}{N} + \frac{F}{N}\delta_1 + \frac{P}{N}\delta_2, \tag{I.1}$$

where $\frac{S}{N}$, $\frac{F}{N}$ and $\frac{P}{N}$ denote the identity degree, discrepancy degree and contrary degree of the set pair $H(A, B)$, respectively. N is the total number of features. $\delta_1 \in [-1, 1]$ is the discrepancy coefficient of the set pair $H(A, B)$. $\delta_2 = -1$ is the contrary coefficient of the set pair $H(A, B)$. Suppose $\frac{S}{N} = a$, $\frac{F}{N} = b$, and $\frac{P}{N} = c$, then the CN can be rewritten as:

$$u(A, B) = a + b\delta_1 + c\delta_2, \tag{I.2}$$

where $0 \le a, b, c \le 1$ and $a + b + c = 1$.

Definition 3. *[LLG09] Let U be the universe of discourse and $\tilde{A}$ be a qualitative concept in U. If $x \in U$ is a random instantiation of the qualitative concept $\tilde{A}$ that satisfies $x \sim N(Ex, En'^2)$ and $En' \sim N(En, He^2)$, and the certainty degree y of x belonging to concept $\tilde{A}$ is a probability distribution, which satisfies*

$$y = e^{-\frac{(x-Ex)^2}{2En'^2}}, \tag{I.3}$$

then the distribution of x in the universe U is called a normal cloud, and the cloud drop can be denoted as (x, y). The overall quantitative properties of concept $\tilde{A}$ can be perfectly depicted in cloud C with three numerical features: expectation Ex, entropy En, and hyper entropy He. Cloud C can be described as $C = (Ex, En, He)$.

Definition 4. *[Paw82] Let U be a non-empty finite universe and $R \in U \times U$ be a binary equivalence relation over universe U, then (U, R) is Pawlak approximation space. The lower and upper approximations for $X \in U$ are defined as follows:*

$$\begin{aligned} &\underline{R}(X) = \cup\{[x]_R | [x]_R \subseteq X, X \in U\}, \\ &\overline{R}(X) = \cup\{[x]_R | [x]_R \cap X \neq \emptyset, X \in U\}, \end{aligned} \tag{I.4}$$

where $[x]_R$ is the equivalence class of x under the binary equivalence relation R. $X(X \in U)$ is called Pawlak rough set if $\overline{R}(X) \neq \underline{R}(X)$.

2.1.2 Opinion dynamics

Opinion dynamics are a fusion process of individual opinions through interactions among a group of individuals, in which they continuously update and fuse their opinions through the fusion rules and reach a consensus, polarization, or fragmentation in the final stage. One of the most representative models in opinion dynamics is the bounded confidence model, which includes the Deffuant-Weisbuch (DW) [DNAW00] and the Hegselmann-Krause (HK) [RK02]. These models simulate how individuals only interact with others whose opinions are within a certain confidence bound, thereby capturing the gradual evolution of consensus or polarization in GDM.

In the DW model, two experts whose opinion distance is smaller than a threshold τ_1 will be chosen randomly to communicate and update new opinions in each round. The updated rules for opinions x_i^{t+1} and x_j^{t+1} of a pair of experts v_i and v_j are defined as follows:

$$\begin{cases} x_i^{t+1} = \alpha x_i^t + (1-\alpha) x_j^t \\ x_j^{t+1} = \alpha x_j^t + (1-\alpha) x_i^t \end{cases}, i, j = 1, 2, ..., k, i \neq j; t = 0, 1, 2, ... \tag{I.5}$$

where α is a predetermined convergence parameter. The bounded confidence threshold τ_1 shows that experts will exchange their opinions effectively with other experts who share similar opinions, and the convergence parameter α reflects the degree to which experts reserve their opinions.

In the HK model, the updated opinion of expert v_i is obtained by averaging opinions of neighbors whose opinion distance with expert v_i is smaller than a threshold τ_2. The updated opinion x_i^{t+1} is denoted as follows:

$$x_i^{t+1} = \frac{1}{\|N_i^t\|} \sum_{j \in N_i^t} x_j^t, i = 1, 2, ..., k; t = 0, 1, 2, ... \tag{I.6}$$

where $N_i^t = \{j | \left| x_i^t - x_j^t \right| \leq \tau_2\}$ is the confidence set and $\left\| N_i^t \right\|$ is the cardinality of the set N_i^t.

The main difference between the DW model and the HK model is the ways of opinion fusion. The former claims that whether a pair of individuals compromise with each other, while the latter holds that individual opinions move towards the average opinions of their neighbors.

2.1.3 Selection process

After the CRP in GDM, the selection process is carried out to determine the most suitable alternative from a set of feasible options. This stage primarily relies on ranking methods, which help prioritize alternatives based on aggregated evaluations from multiple DMs. In complex decision environments involving multiple conflicting criteria, ranking methods offer a clear and systematic solution for making final decisions.

Most of these ranking techniques are developed within the framework of MCDM, which focuses on evaluating alternatives across several criteria. Among the widely adopted methods are the Technique for Order Preference by Similarity to an Ideal Solution (TOPSIS) [BOYI12], Grey Relational Analysis (GRA) [KYH08], VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR)[OT07], and the Linear Programming Technique for Multidimensional Analysis of Preference (LINMAP) [SS73]. These methods typically construct the ranking process by measuring how close each alternative is to a predefined ideal solution or by evaluating dominance relationships among alternatives. For instance, TOPSIS selects the option closest to the ideal and farthest from the negative-ideal solution. VIKOR emphasizes a compromise solution by balancing collective satisfaction and individual regret. GRA assesses similarity to a reference alternative and is particularly effective when dealing with incomplete or uncertain data. LINMAP applies linear programming to jointly derive ideal points and criteria weights from preference data.

Ranking-based methods are essential to the selection process because they bridge the gap between consensus and actionable decisions. They convert qualitative judgments and numerical evaluations into a prioritized list of alternatives, enabling DMs to make rational and justifiable choices. In this way, ranking techniques serve as the final and critical step in transforming collective opinion into group decision outcomes.

2.2 Three-way decision theory

The three-way decision (3WD) theory proposed by Yao [Yao09, Yao10] based on the Bayesian theory aims to divide the universal set into three disjoint regions: the positive region, the boundary region and the negative region. The three regions correspond to acceptance action, delay action and rejection action, respectively. Generally, the 3WD model includes two states $\Omega = \{X, \neg X\}$ and three actions $\mathrm{A} = \{a_P, a_B, a_N\}$. X and $\neg X$ represent that an object x belongs to X and does not belong to X. a_P, a_B and a_N stand for acceptance, delay and rejection actions, which are denoted as $x \in POS(X)$, $x \in BND(X)$ and $x \in NEG(X)$, respectively. R is an equivalence relation on X and $[x]_R$ is the equivalence class of R including the object x. $\Pr(X|[x]_R)$ and $\Pr(\neg X|[x]_R)$ are the conditional probabilities of the object x belonging to and not belonging to X. In particular, $\lambda_{\bullet\circ}(\bullet = P, B, N; \circ = P, N)$ are used to measure the losses of taking three actions, which are shown in Table I.1. λ_{PP}, λ_{BP} and λ_{NP} represent the losses taking actions a_P, a_B and a_N when $x \in X$. Similarly, λ_{PN}, λ_{BN} and λ_{NN} denote the losses taking actions a_P, a_B and a_N when $x \in \neg X$. A reasonable assumption is considered in the 3WD model: $0 \leq \lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}$ and $0 \leq \lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$.

The expected loss function $R(a_\bullet|[x]_R)$ of taking three actions can be calculated as follows:

(1) $R(a_P|[x]_R) = \lambda_{PP} \Pr(X|[x]_R) + \lambda_{PN} \Pr(\neg X|[x]_R)$;

(2) $R(a_B|[x]_R) = \lambda_{BP} \Pr(X|[x]_R) + \lambda_{BN} \Pr(\neg X|[x]_R)$;

(3) $R(a_N|[x]_R) = \lambda_{NP} \Pr(X|[x]_R) + \lambda_{NN} \Pr(\neg X|[x]_R)$.

According to the Bayesian theory, the following decision rules of minimum losses can be

Table I.1: The loss functions in 3WD theory.

Actions	$X(P)$	$\neg X(N)$
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

demonstrated as follows:

(P) If $R(a_P|[x]_R) \leq R(a_B|[x]_R)$ and $R(a_P|[x]_R) \leq R(a_N|[x]_R)$, then $x \in POS(X)$;

(B) If $R(a_B|[x]_R) \leq R(a_P|[x]_R)$ and $R(a_B|[x]_R) \leq R(a_N|[x]_R)$, then $x \in BND(X)$;

(N) If $R(a_N|[x]_R) \leq R(a_P|[x]_R)$ and $R(a_N|[x]_R) \leq R(a_B|[x]_R)$, then $x \in NEG(X)$.

The above rules (P)-(N) relate to the loss function $\lambda_{\bullet\circ}$ and the conditional probability $\Pr(X|[x]_R)$, then the decision rules can be rewritten as:

(P1) If $\Pr(X|[x]_R) \geq \alpha$ and $\Pr(X|[x]_R) \geq \gamma$, then $x \in POS(X)$;

(B1) If $\Pr(X|[x]_R) \leq \alpha$ and $\Pr(X|[x]_R) \geq \beta$, then $x \in BND(X)$;

(N1) If $\Pr(X|[x]_R) \leq \gamma$ and $\Pr(X|[x]_R) \leq \beta$, then $x \in NEG(X)$;

where $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$, $\gamma = \frac{\lambda_{PN}-\lambda_{NN}}{(\lambda_{PN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{PP})}$ and $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$.

When the condition $(\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN}) \leq (\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN})$ holds, we can obtain $\alpha \geq \beta$. In this case, the simplified decision rules (P1)-(N1) are listed as follows:

(P2) If $\Pr(X|[x]_R) \geq \alpha$, then $x \in POS(X)$;

(B2) If $\beta < \Pr(X|[x]_R) < \alpha$, then $x \in BND(X)$;

(N2) If $\Pr(X|[x]_R) \leq \beta$, then $x \in NEG(X)$.

Based on the granular computing theory, Yao and Deng [YD11] proposed the S3WD model, extending the classic 3WD model from coarse granularity to fine granularity and from one step to multiple steps. In cases of coarse granularity, the precise classification of objects cannot be obtained when information is insufficient. S3WD is used to gradually classify the delay objects into the positive and negative regions through granularity refinement until the decision problem transforms into a 2WD problem, thereby minimizing the additional cost of misclassification. In the S3WD process, the boundary region gradually shrinks until the definitive decision results are obtained as information accumulates and granularity refines, which conforms to the basic principles of human cognition and decision-making.

3 Justification

GDM plays a vital role in a wide range of real-world settings, including enterprise strategy formulation, environmental policy evaluation, investment analysis, and public project prioritization. In such contexts, decisions are often made collectively by multiple experts or stakeholders with diverse preferences, knowledge backgrounds, and evaluation behaviors. To facilitate effective and coordinated group decisions, two important tools have been developed in the field: group consensus models, which aim to align conflicting opinions, and 3WD theory, which introduces a more flexible decision paradigm by allowing a delay option in addition to acceptance and rejection.

On one hand, group consensus is a core step in GDM, ensuring collective agreement through iterative opinion adjustments. In particular, the social trust networks among experts significantly influence how consensus is formed. However, traditional consensus models typically ignore opinion reliability and trust dynamics, resulting in potentially biased or unrealistic consensus results. On the other hand, 3WD originally developed as an extension to overcome the limitations of traditional 2WD models in GDM, introduces an intermediate "delay" option. This addition enables decision-makers to defer judgment when information is insufficient or uncertainty is high, thus better reflecting real human decision-making behavior. However, most existing 3WD models fail to incorporate experts' preferences, regret psychology, and multi-source decision information, limiting their applicability in complex decision scenarios. Furthermore, in many practical GDM situations, consensus and decision-making are intertwined—the decision process depends on achieving consensus, and consensus, in turn, is influenced by how decisions are framed and evaluated. However, there is a lack of integrated models that combine group consensus mechanisms with 3WD models, especially under multi-criteria and uncertain conditions. Finally, as GDM is increasingly applied to real-world evaluation problems (e.g., EI project assessments), it faces new challenges: experts often express evaluations in discrete probabilistic linguistic terms, which are difficult to quantify, compare, and integrate. This complexity calls for new methods that bridge the gap between symbolic and numerical representations of evaluation information.

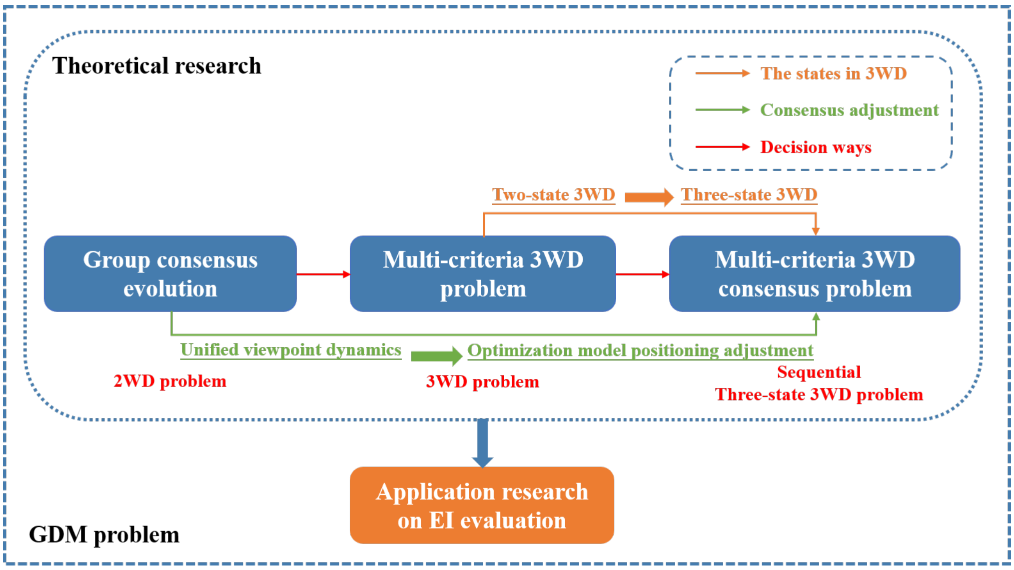

Fig. 1: The logical relationship between several researches.

To address the above issues, this thesis focuses on four interconnected research topics: (1) The group consensus evolution process in social trust network via opinion reliability. (2) The construction of a preference-based multi-criteria 3WD model. (3) The combination of group consensus and multi-criteria 3WD models. (4) The application research on EI project evaluation with FLEs. The main procedure of our research is shown in Fig.1. The justifications of each topic are described as follows:

(1) **The group consensus evolution process in social trust network via opinion reliability.** Upon conducting the state-of-art review of the relevant studies on the existing group consensus models and SNGDM studies, we find that:

- Reliability is an important characteristic of experts in SNGDM problems. However, most previous researchers rarely have considered the reliability of experts, or simply assumed that all experts are completely reliable. In reality, reliability of experts may significantly affect the opinion evolution process, which needs to be further explored in SNGDM problems.

- Most SNGDM studies have assumed that trust values between experts in social networks are provided by experts. However, unreliable experts may not accurately quantify the trust values due to the impacts of subjective and objective factors, which may lead to unreasonable predefined trust values given by experts. Therefore, it deserves to explore how to obtain the trust values and derive the complete trust matrix and weights of experts in social trust network accurately when considering the opinion reliability.

- In CRP, experts are more willing to accept opinions of the experts they trust, which can be affected by opinion dynamics. The existing models paid little attention to the evolution of social networks and ignored the characteristic of experts like opinion reliability, which needs to be further extended using opinion dynamics.

(2) **The construction of a preference-based multi-criteria 3WD model.** Through a literature review regarding the 3WD models with the extant contributions, we realize that:

- Most 3WD models have been applied to classify data of DISs in databases where preference information has not been involved. However, preference information generated from the historical data of different DISs may affect the decision results. Meanwhile, different experts recorded in databases may have the subjective preferences for different DISs. Therefore, the preference information is necessary to be considered and solved in 3WD model.

- Experts often express evaluation values using natural language in practical decision-making problems. Some existing MCDM or 3WD methods have used fuzzy information to measure the uncertainty of evaluation information which might be assumed to be totally reliable. Besides, experts feel regret or rejoice due to psychological factors when comparing three actions in 3WD models, which has not been well studied.

- The existing 3WD research has only focused on how to determine the conditional probabilities or loss functions of single DIS. In complex decision-making scenarios, multi-source DISs may be collected to make integrated decisions to obtain more reasonable results. Therefore, the classification and ranking methods for MDISs are worth exploring.

(3) **The combination of group consensus and multi-criteria 3WD models.** Through a literature review, we acknowledge that the existing studies greatly contributed to the development of the MCDM and 3WD theory, but there still exist some research gaps that need to be investigated:

- In traditional MCDM problems, experts are typically classified as either consensus experts or conflict experts, the latter of whom adjust their opinions to facilitate the group consensus. However, experts may encounter a state of uncertainty between consensus and conflict in the complex decision-making environment, that is, they are unsure about whether they should revise their opinions and need to make further judgements. For the two types of non-consensus experts—uncertainty experts and conflict experts, the consensus reaching strategies are worth studying.

- The 3WD theory extends two actions of 2WD theory to three actions by introducing the delay action. However, the classic 3WD theory includes only two states of consensus and conflict in MCGDM problems. By introducing the discrepancy degree, SPA theory provides a three-state perspective, which opens up a new view to expand the two states of classic 3WD theory.

- The idea of S3WD can be leveraged to guide the CRP in MCDM problems. However, the challenge of granularity being too coarse for further classification in the S3WD process remains unresolved. Meanwhile, the methods for reclassifying the delay region and adjusting the opinions of conflict experts need further exploration. Designing the adjustment and feedback strategies for uncertainty experts and conflict experts based on S3WD theory has a significant effect on the CRP in MCDM problems.

(4) **The application research on EI project evaluation with flexible linguistic expressions.** Upon reviewing relevant studies in multi-criteria evaluation and uncertain linguistic environments, we identify several limitations in existing GDM applications to EI project evaluation:

- Current EI evaluation methods often rely on discrete linguistic terms to express complex environmental indicators (e.g., sustainability, policy adaptation, ecological risk). However, such linguistic expressions, while intuitive, lack continuity and are difficult to compare and integrate. This limits the effective aggregation and interpretation of evaluation information.

- Most existing research on FLEs focuses on symbolic normalization or qualitative reasoning. Few studies have addressed how to quantitatively transform discrete FLEs into continuous information to better support decision-making.

- Although the MGRS theory is widely used for handling uncertainty in evaluation, it is primarily designed for discrete datasets. It lacks the ability to flexibly process continuous or highly fuzzy information, which is increasingly common in practical EI assessment scenarios.

- The integration between probabilistic linguistic modeling (e.g., cloud models) and rough sets is still underdeveloped. Without this integration, it is hard to ensure both uncertainty retention and decision robustness in the evaluation process.

Upon identifying the aforementioned gaps and assumptions, our research addresses four interrelated topics: (1) the consensus evolution process in social trust networks; (2) the construction of a preference-based multi-criteria 3WD model; (3) the integration of group consensus and 3WD models; and (4) the application of GDM models to real-world EI project evaluation. These four topics together aim to extend GDM theory and bridge it with practical decision-making. From the perspective of consensus reaching, the first and third topics focus on automatic consensus evolution in social networks and consensus optimization in the 3WD environment, respectively. The consensus process is thus studied across a spectrum—from unified opinion dynamics to optimization-based position adjustment. From the perspective of 3WD modeling, the second and third topics deal with multi-criteria 3WD under subjective preferences and regret information, and extend it from traditional two-state to three-state logic. Moreover, the third topic explores sequential adjustment mechanisms, thereby expanding the 3WD framework from single-step to process-based decision modeling. Beyond theoretical expansion, the fourth topic focuses on application-oriented research, applying the developed models in the evaluation of EI projects. This not only verifies the feasibility of the proposed methods in real-world decision scenarios but also strengthens the link between abstract GDM theory and practical sustainability challenges, such as circular economy planning. Overall, the four topics form a coherent research path—from theoretical advancement in consensus and 3WD models to integrated modeling and practical application—offering systematic solutions to existing limitations in GDM.

4 Objectives

GDM is prevalent in various organizations and fields, such as in enterprises, government, communities, and families. Consensus reaching in GDM is a necessary process for promoting communication and understanding among group members. Meanwhile, the 3WD theory addresses the limitations of binary classification in multi-criteria GDM, providing an effective framework for handling complexity and uncertainty. Therefore, studying consensus and multi-criteria 3WD models in GDM and their applications is crucial. The specific objectives are summarized as follows:

(1) **To explore the evolution of group consensus in social trust network via opinion reliability.** Distinguished from the traditional consensus models, this research will focus on the evolution of group consensus in social network GDM by incorporating opinion reliability into the opinion dynamics framework. Unlike existing models that often overlook the reliability of experts' opinions, this study will define opinion reliability based on social trust network structure and individual characteristics, establishing a comprehensive trust degree to reflect the credibility of experts. Key components include trust propagation and aggregation mechanisms for constructing the social trust matrix, as well as the design of opinion evolution rules based on extended HK models. The effectiveness and feasibility of this approach will be validated through numerical experiments such as supplier performance evaluation and the widely studied Zachary's karate club network.

(2) **To construct a preference-based regret multi-criteria 3WD model.** This study aims to extend the conventional 3WD model by incorporating preference information, which is often neglected in multi-criteria GDM contexts. Using linguistic Z-numbers, which improve the precision of expert evaluations by integrating reliability, this model will focus on two core aspects: criteria weight determination based on preference information and the derivation of conditional probabilities and loss functions using regret theory. The study will also introduce a Z-LINMAP method adapted for LZN environments to derive preference coefficients, consistency-order, and inconsistency-order, forming the foundation for defining equivalence classes. Finally, the proposed model will be validated through real-world applications, such as image recognition cases involving human-computer interaction, to demonstrate its practical effectiveness.

(3) **To extend a multi-criteria sequential three-state three-way consensus model.** Building on the SPA theory, this study aims to expand the classic two-state 3WD model to a three-state structure, incorporating an additional uncertainty state. This approach will introduce a consensus set pair probability space, categorizing objects into consensus, uncertainty, and conflict states. The model will include the development of decision rules, simplified decision processes, and feedback mechanisms to adjust conflicting and uncertain opinions dynamically. The theoretical contributions will be supported by experimental analyses, including case studies on ranking alternatives based on the connection number (CN) and sequential adjustment strategies for enhanced consensus.

(4) **To enhance the application research on GDM in EI project evaluation.** This objective focuses on bridging the gap between theoretical GDM advancements and real-world EI project evaluation. The study will develop a multi-criteria decision-making framework based on FLEs and MGCRS, addressing critical challenges such as transforming discrete linguistic terms into continuous cloud information and managing highly uncertain data. This framework will include comprehensive index systems covering technical, environmental, and

economic dimensions, validated using real-world cases, such as the Beijing-Tianjin-Hebei region, to enhance decision accuracy and robustness in EI project selection.

5 Methodology

This section introduces the methodology used in the doctoral thesis proposal. Considering the main idea of this study to explore the consensus and multi-criteria 3WD model and applications within GDM, the related methods are provided as follows:

(1) **Hypothesis formulation.** The hypothesis provides a good and useful tool to guide the decision-making process, which should be reasonable and suitable for the discussed GDM problems. For instance, the assumption that non-cooperative behavior is absent for all decision-makers should be satisfied when constructing the consensus reaching process, as this is a critical condition for effective consensus adjustment and feedback mechanism. In addition, when exploring the 3WD model, we should reasonably assume that the loss functions corresponding to the acceptance, delay and rejection actions gradually increase when in the belonging state, while they gradually decrease in the non-belonging state.

(2) **Model construction.** To explore the consensus and multi-criteria 3WD model, we will construct the consensus reaching model and optimization model to guide the measure of consensus and reach the group consensus for decision-makers. In addition, different multi-criteria 3WD models will be constructed to characterize the practical GDM problems to classify all alternatives or experts.

(3) **Simulation analysis.** The simulation experiment is used to validate the proposal and demonstrate the effectiveness of the proposed methods, involving the numerical study and sensitive analysis. With the simulation experiment, the feasibility and validity of the proposed model for GDM problems are discussed. For example, the case of GDM problems in practical fields will be conducted to reflect the process of the proposed models, and the sensitive experiment will be performed by varying different parameters to explore the impact on the decision-making results.

(4) **Comparative study.** The comparative analysis is used to reflect and highlight the advantages of the proposed methods and models by comparing two or more studies with the proposal. In this study, we will summarize and analyze the existing studies regarding the topic and compare the decision-making results between those methods, thus further highlighting the characteristics of the proposed model.

6 Summary

In this section, we summarize the main proposals in this thesis and explain the main contents and the obtained results associated with the journal publications. The published, submitted, or ongoing works are listed as follows:

(1) H. Wang, Y. Ju, E. Herrera-Viedma, P. Dong, Y. Liang, A social network group decision making framework with opinion dynamics considering opinion reliability, Computers & Industrial Engineering 183 (2023) 109523. DOI: https://doi.org/10.1016/j.cie.2023.109523.

- Status: **Published**.
- Impact Factor (JCR 2023): 6.7
- Subject Category: Computer Science, Interdisciplinary Applications, Ranking 21/170 (Q1).
- Subject Category: Engineering, Industrial, Ranking 11/69 (Q1).

(2) H. Wang, Y. Ju, P. Dong, A. Wang, F. J. Cabrerizo, Preference-based regret three-way decision method on multiple decision information systems with linguistic z-numbers, Information Sciences 654 (2024) 119861. DOI: https://doi.org/10.1016/j.ins.2023.119861.

- Status: **Published**.
- Impact Factor (JCR 2022): 8.1
- Subject Category: Computer Science, Information Systems, Ranking 13/158 (Q1).

(3) H. Wang, Y. Ju, P. Dong, P. Maresova, T. Ju, E. Herrera-Viedma, Multi-criteria sequential three-state three-way decision consensus model based on set pair analysis theory, Information Sciences 661 (2024) 120199. DOI: https://doi.org/10.1016/j.ins.2024.120199.

- Status: **Published**.
- Impact Factor (JCR 2022): 8.1
- Subject Category: Computer Science, Information Systems, Ranking 13/158 (Q1).

(4) H. Wang, Y. Ju, C. Porcel, Energy internet project evaluation in circular economy practices: A novel multi-criteria decision-making framework with flexible linguistic expressions based on multi-granularity cloud-rough set, Computers & Industrial Engineering 201 (2025) 110890. DOI: https://doi.org/10.1016/j.cie.2025.110890.

- Status: **Published**.
- Impact Factor (JCR 2023): 6.7
- Subject Category: Computer Science, Interdisciplinary Applications, Ranking 21/170 (Q1).
- Subject Category: Engineering, Industrial, Ranking 11/69 (Q1).

The remainder of this section is organized into the four objectives defined in Section 4. Subsection 6.1 introduces a social network group consensus model based on opinion reliability. Subsection 6.2 proposes a preference-based multi-criteria 3WD model for multiple information systems. Subsection 6.3 deeply combines the consensus model and the 3WD model for MCDM problems. Finally, Subsection 6.4 presents a multiple-criteria evaluation framework based on multi-granularity cloud-rough set and extends the applications of GDM methods for energy internet projects.

6.1 The research on the group consensus evolution model in social trust network

The research on the group consensus evolution model in social trust network mainly covers the social network evolution, trust propagation mechanism, and opinion evolution process.

6.1.1 The social network evolution

Social network evolution is influenced primarily by two factors: random connections [GZ06] and opinion similarity [DZM19]. However, experts are more inclined to interact with those who not only share similar opinions but are also considered reliable. Therefore, opinion reliability is a critical factor in shaping social network evolution in GDM. Therefore, we design the social network evolution rules considering two factors: (i) opinion similarity Sm_{ij}^t $(i, j = 1, 2, ..., k, i \neq j; t = 0, 1, 2, ...)$ between opinions of experts v_i and v_j, and (ii) reliability R_j^t $(j = 1, 2, ..., k; t = 0, 1, 2, ...)$ of expert v_j. We set the opinion similarity threshold $Sm*$ and the reliability threshold $R*$ as the judgment rules. Without loss of bias, 0.5 is considered to be the fairest threshold for $Sm*$ and $R*$. $A^t = (a_{ij}^t)_{k\times k}$ is the adjacency matrix at time t for the social network. Given a predefined probability γ_{ij} for establishing a connection from expert v_i to v_j, the social network evolution rules are divided into two cases:

Case 1: when $a_{ij}^{t-1} < 0.5$,

$$a_{ij}^t = \begin{cases} 1, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t \geq R^* \\ \gamma_{ij}, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t < R^* \\ \gamma_{ij}, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t \geq R^* \\ 0, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t < R^* \end{cases}; \tag{I.7}$$

Case 2: when $a_{ij}^{t-1} \geq 0.5$,

$$a_{ij}^t = \begin{cases} 1, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t \geq R^* \\ 1-\gamma_{ij}, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t < R^* \\ 1-\gamma_{ij}, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t \geq R^* \\ 0, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t < R^* \end{cases}. \tag{I.8}$$

6.1.2 The trust propagation mechanism

In a social trust network, the comprehensive trust degree t_{ij}^t of expert v_i on expert v_j at time t depends on the leadership of v_j, the reliability of v_j and the opinion similarity between them, which are represented as the centrality degree CD_i^t of expert v_i, the opinion reliability R_j^t of expert v_j, and the opinion similarity Sm_{ij}^t between two experts v_i and v_j, respectively. Notably, the opinion reliability R_i^t of an expert at time $(t+1)$ in a group is considered as a combination of the stability similarity and the weighted similarity. The stability similarity is the difference between the experts' opinions at time $(t+1)$ and at time t. Meanwhile, the weighted similarity is the distance between the expert's opinion at time $(t+1)$ and opinions of other reliable experts at time t.

Definition 5. *In a social trust network, the comprehensive trust degree t_{ij}^t of expert v_i on expert v_j at time t depends on the leadership of v_j, the reliability of v_j and the opinion similarity between* *them, and is defined as*

$$t_{ij}^{t} = \alpha Sm_{ij}^{t} + \beta R_{j}^{t} + (1-\alpha-\beta)CD_{j}^{t}, i,j = 1,2,...,k, i \neq j, t = 0,1,2,... \tag{I.9}$$

where $\alpha \in [0,1]$, $\beta \in [0,1]$ *and* $(1-\alpha-\beta) \in [0,1]$ *are the similarity coefficient, reliability coefficient and centrality coefficient, respectively.* R_j^t *and* CD_j^t *reflect the opinion reliability and centrality degree of expert* v_j*.*

The existing studies only roughly multiply propagation efficiency by Einstein product operator, which has limited influence on the final trust information and cannot truly reflect the actual propagation process. Therefore, we define $\bar{t}_{ij}^{t}$ $(i,j = 1,2,...,k, i \neq j, t = 0,1,2,...)$ as the actual trust value propagated from v_i to v_j at time t as follows: $\bar{t}_{ij}^{t} = \tilde{t}_{ij}^{t} \cdot \sin(\frac{\pi \times Sm_{ij}^{t}}{2})$. Suppose that $\{(v_i, v_{\sigma(1)}), (v_{\sigma(1)}, v_{\sigma(2)}), (v_{\sigma(2)}, v_{\sigma(3)}), ..., (v_{\sigma(d)}, v_j)\}$ is a directed path from v_i to v_j, then t-norm-based trust propagation value $\hat{t}_{ij}^{t}$ at time t considering the propagation efficiency can be written as:

$$\begin{aligned} \hat{t}_{ij}^{t} &= E_{\otimes}(\bar{t}_{i\sigma(1)}^{t}, \bar{t}_{\sigma(1)\sigma(2)}^{t}, ..., \bar{t}_{\sigma(d)j}^{t}) \\ &= \frac{2 \cdot \bar{t}_{i\sigma(1)}^{t} \cdot \bar{t}_{\sigma(d)j}^{t} \prod_{z=1}^{d} \bar{t}_{\sigma(z-1)\sigma(z)}^{t}}{(2-\bar{t}_{i\sigma(1)}^{t}) \cdot (2-\bar{t}_{\sigma(d)j}^{t}) \prod_{z=1}^{d} \bar{t}_{\sigma(z-1)\sigma(z)}^{t} + \bar{t}_{i\sigma(1)}^{t} \cdot \bar{t}_{\sigma(d)j}^{t} \prod_{z=1}^{d} \bar{t}_{\sigma(z-1)\sigma(z)}^{t}}. \end{aligned} \tag{I.10}$$

In a social network, there may be several paths between experts, which means that we can get several different trust propagation values. Therefore, we need to aggregate these values to obtain the final trust value from v_i to v_j. The Order Weighted Averaging (OWA) operator is often used to compute the trust aggregation value. The larger the trust propagation value obtained by a path is, the higher the weight assigned to the path is. However, the number of mediators in a path may impact the trust aggregation process: the smaller the number of mediators is, the more accurate the obtained trust value is. Therefore, the path with less mediators should be assigned larger weight. The following optimization model is used to obtain the trust aggregation value:

$$\begin{aligned} &\max t_{ij}^{t'} \\ &s.t. \begin{cases} t_{ij}^{t'} = \sum_{k=1}^{h} w_{ij}^{kt} \times \hat{t}_{ij}^{kt} \\ \sum_{k=1}^{h} w_{ij}^{kt} = 1 \\ w_{ij}^{1t} \geq w_{ij}^{2t} \geq \cdots \geq w_{ij}^{ht} \\ 0 \leq w_{ij}^{kt} \leq 1 (k = 1,2,...,h) \\ i,j = 1,2,3,...,k, i \neq j, t = 0,1,2,... \end{cases}, \end{aligned} \tag{I.11}$$

where w_{ij}^{kt} and $\hat{t}_{ij}^{kt}$ is the weight and trust aggregation value of the path with the k^{th} least number of mediators at time t. $t_{ij}^{t'}$ is the final trust value aggregating all paths from v_i to v_j. Then we can get the trust matrix $T^{t'} = (t_{ij}^{t'})_{k \times k}$ and the weight matrix $W^t = (w_{ij}^{t})_{k \times k}$ after normalization procedure.

6.1.3 The opinion evolution process

In the opinion evolution process, the HK model is preferred over the DW model due to its synchronous updating mechanism. However, the traditional HK model only considers opinion similarity within a bounded range. To address its limitations, an extended HK model is proposed that incorporates both individuals' own opinions and those of more reliable experts. The opinion evolution rule is as follows:

$$x_i^{t+1} = \rho \frac{\sum_{j_1 \in \{N_1^t \cup i\}} w_{ij_1}^{t} x_{j_1}^{t}}{\sum_{j_1 \in \{N_1^t \cup i\}} w_{ij_1}^{t}} + (1-\rho) \frac{\sum_{j_2 \in \{N_2^t \cup i\}} w_{ij_2}^{t} x_{j_2}^{t}}{\sum_{j_2 \in \{N_2^t \cup i\}} w_{ij_2}^{t}}, i = 1,2,...,k; t = 1,2,3,... \tag{I.12}$$

where $\rho \in [0,1]$ is an adaptive coefficient. For convenience, this paper assumes all adaptive coefficients of experts are equal in each round. $w_{ij_1}^t$ and $w_{ij_2}^t$ is the expert v_{j_1}'s weight and expert v_{j_2}'s weight assigned by expert v_i at time t, respectively. N_1^t is the similarity set of experts pairs who meet the condition that the opinion distance with expert v_i is less than $(1-Sm^*)$ i.e., $N_1^t = \{(i,j_1) \middle| |x_i^t - x_{j_1}^t| \leq (1-Sm^*), i, j_1 = 1,2,...,k; t = 1,2,3,...\}$. N_2^t is the reliability set of expert whose reliability is more than $R*$, i.e., $N_2^t = \{j_2 | R_{j_2}^t \geq R*, j_2 = 1,2,...,k; t = 1,2,3,...\}$. The opinion of expert depends on his/her own opinion and opinions of other experts belonging to N_1^t or N_2^t. The extended HK model will be degenerated into the traditional HK model when $\rho = 1$ and all $w_{ij_1}^t$ are equal.

The above three parts operate iteratively, beginning with the initialization of opinions, reliability, and network parameters. At each iteration, the social network evolves based on updated opinion similarity and expert reliability, followed by trust propagation and aggregation to refine the trust matrix and expert weights. Then, expert opinions are updated synchronously through the extended HK model that incorporates both opinion similarity and reliability. These steps are repeated, with parameters recalculated in each round, until the opinions converge and group consensus is reached.

The journal paper concerning this part is:

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6.2 The research on the construction of a preference-based multi-criteria 3WD model

The research on the preference-based multi-criteria 3WD model mainly includes the Z-LINMAP method, the determination of conditional probability, and the loss functions.

Given the LZNs evaluation information and preference set of experts, we construct a Z-LINMAP method to derive the criteria weights, positive and negative ideal Z-numbers based on experts' preferences as follows:

$$
\begin{aligned}
&\min \sum_{(k,l)\in Q} \theta\mu_{kl} + (1-\theta)\mu'_{kl} \\
&s.t. \begin{cases}
\sum_{(k,l)\in Q} (S_l^+ - S_k^+) = \varepsilon, \\
\mu_{kl} + (S_l^+ - S_k^+) \geq 0, \forall (k,l) \in Q, \\
\sum_{(k,l)\in Q} (S_k^- - S_l^-) = \varepsilon', \\
\mu'_{kl} + (S_k^- - S_l^-) \geq 0, \forall (k,l) \in Q, \\
S_l^+ - S_k^+ = \sum_{j=1}^{m} w_j(\rho_1(f(A_{lj}) - f(A_j^+))^2 + \rho_2(g(B_{lj}) - B(b_j^+))^2 - \\
\qquad \rho_1(f(A_{kj}) - f(A_j^+))^2 - \rho_2(g(B_{kj}) - g(B_j^+))^2), \forall (k,l) \in Q, \\
S_k^- - S_l^- = \sum_{j=1}^{m} w_j(\rho_1(f(A_{lj}) - f(A_j^-))^2 + \rho_2(g(B_{lj}) - g(B_j^-))^2 - \\
\qquad \rho_1(f(A_{kj}) - f(A_j^-))^2 - \rho_2(g(B_{kj}) - g(B_j^-))^2), \forall (k,l) \in Q, \\
w_j, f(A_j^+), f(A_j^-), g(B_j^+), g(B_j^-) \in [0,1], j = 1,2,...,m, \\
\mu_{kl} \geq 0, \mu'_{kl} \geq 0, \forall (k,l) \in Q.
\end{cases}
\end{aligned}
\tag{I.13}
$$

where $S_i^+ = \sum_{j=1}^{m} w_j d(Z_{ij}, Z_j^+)^2$ and $S_i^- = \sum_{j=1}^{m} w_j d(Z_j, Z_j^-)^2$ $(i = l, k)$ are the positive weighted distance and negative weighted distance of alternative x_i. $\mu_{kl} = \max\{0, (S_k^+ - S_l^+)\}$ and $\mu'_{kl} = \max\{0, (S_l^- - S_k^-)\}$ are the positive and negative preference coefficients.

The consistency-order and inconsistency-order are defined to derive two equivalence classes ($[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$) and corresponding conditional probabilities, as follows:

$$\begin{array}{l} [x_k]_{P_{con}} = \{x_l | (x_k P_{con} x_l) \vee (x_l P_{con} x_k) \vee x_k, x_l \in X)\} \\ [x_k]_{P_{incon}} = \{x_l | (x_k P_{incon} x_l) \vee (x_l P_{incon} x_k) \vee x_k, x_l \in X)\} \end{array}, \tag{I.14}$$

$$\begin{array}{l} \Pr(X|[x_k]_{P_{con}}) = \dfrac{\left(\sum_{x_a \in [x_k]_{P_{con}}} Z_a\right)/\#[x_k]_{P_{con}}}{\left(\sum_{x_a \in [x_k]_{P_{con}}} Z_a\right)/\#[x_k]_{P_{con}} + \left(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b\right)/\#[x_k]_{P_{incon}}} \\ \Pr(X|[x_k]_{P_{incon}}) = \dfrac{\left(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b\right)/\#[x_k]_{P_{incon}}}{\left(\sum_{x_a \in [x_k]_{P_{con}}} Z_a\right)/\#[x_k]_{P_{con}} + \left(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b\right)/\#[x_k]_{P_{incon}}} \end{array}. \tag{I.15}$$

According to Table I.2 and regret theory, we derive nine extended loss functions as follows:

$\overline{\lambda}_{PP}^{hk} = \lambda_{PP}^{hk} + r(\lambda_{PP}^{hk} - \lambda_{BP}^{hk}) + \lambda_{PP}^{hk} + r(\lambda_{PP}^{hk} - \lambda_{NP}^{hk}) = 0;$

$\overline{\lambda}_{BP}^{hk} = \lambda_{BP}^{hk} + r(\lambda_{BP}^{hk} - \lambda_{PP}^{hk}) + \lambda_{BP}^{hk} + r(\lambda_{BP}^{hk} - \lambda_{NP}^{hk}) = 2\sigma_h \overline{S}_k^+ + 1 - e^{-\delta_h \sigma_h \overline{S}_k^+};$

$\overline{\lambda}_{NP}^{hk} = \lambda_{NP}^{hk} + r(\lambda_{NP}^{hk} - \lambda_{PP}^{hk}) + \lambda_{NP}^{hk} + r(\lambda_{NP}^{hk} - \lambda_{BP}^{hk}) = 2\overline{S}_k^+ + 1 - e^{-\delta_h \overline{S}_k^+} + 1 - e^{-\delta_h (1-\sigma_h) \overline{S}_k^+};$

$\overline{\lambda}_{PN}^{hk} = \lambda_{PN}^{hk} + r(\lambda_{PN}^{hk} - \lambda_{BN}^{hk}) + \lambda_{PN}^{hk} + r(\lambda_{PN}^{hk} - \lambda_{NN}^{hk}) = 2\overline{S}_k^- + 1 - e^{-\delta_h (1-\sigma_h) \overline{S}_k^-} + 1 - e^{-\delta_h \overline{S}_k^-};$

$\overline{\lambda}_{BN}^{hk} = \lambda_{BN}^{hk} + r(\lambda_{BN}^{hk} - \lambda_{PN}^{hk}) + \lambda_{BN}^{hk} + r(\lambda_{BN}^{hk} - \lambda_{NN}^{hk}) = 2\sigma_h \overline{S}_k^- + 1 - e^{-\delta_h \sigma_h \overline{S}_k^-};$

$\overline{\lambda}_{NN}^{hk} = \lambda_{NN}^{hk} + r(\lambda_{NN}^{hk} - \lambda_{PN}^{hk}) + \lambda_{NN}^{hk} + r(\lambda_{NN}^{hk} - \lambda_{BN}^{hk}) = 0.$

Table I.2: The relative loss functions provided by e_h.

Actions	$X(P)$	$\neg X(N)$
a_P	$\lambda_{PP}^{hk} = 0$	$\lambda_{PN}^{hk} = \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z_j^-)^2$
a_B	$\lambda_{BP}^{hk} = \sigma_h \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z_j^+)^2$	$\lambda_{BN}^{hk} = \sigma_h \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z_j^-)^2$
a_N	$\lambda_{NP}^{hk} = \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z_j^+)^2$	$\lambda_{NN}^{hk} = 0$

Furthermore, the new regret simplified decision rules are obtained as follows:

(P3) If $\Pr_h(X|[x_k]_{P_{con}}) \geq \alpha^{hk}$ and $\Pr_h(X|[x_k]_{P_{con}}) \geq \gamma^{hk}$, then $x_k \in POS^h(X)$;

(B3) If $\Pr_h(X|[x_k]_{P_{con}}) < \alpha^{hk}$ and $\Pr_h(X|[x_k]_{P_{con}}) > \beta^{hk}$, then $x_k \in BND^h(X)$;

(N3) If $\Pr_h(X|[x_k]_{P_{con}}) < \gamma^{hk}$ and $\Pr_h(X|[x_k]_{P_{con}}) \leq \beta^{hk}$, then $x_k \in NEG^h(X)$;

where $\alpha^{hk} = \frac{\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{BN}^{hk}}{(\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{BN}^{hk}) + (\overline{\lambda}_{BP}^{hk} - \overline{\lambda}_{PP}^{hk})}$, $\gamma^{hk} = \frac{\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{NN}^{hk}}{(\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{NN}^{hk}) + (\overline{\lambda}_{NP}^{hk} - \overline{\lambda}_{PP}^{hk})}$; $\beta^{hk} = \frac{\overline{\lambda}_{BN}^{hk} - \overline{\lambda}_{NN}^{hk}}{(\overline{\lambda}_{BN}^{hk} - \overline{\lambda}_{NN}^{hk}) + (\overline{\lambda}_{NP}^{hk} - \overline{\lambda}_{BP}^{hk})}$.

The journal article associated to this part is:

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6.3 The research on the combination of group consensus and multi-criteria 3WD models

The research on the combination of group consensus and multi-criteria 3WD models mainly includes the Three-State 3WD (TS3WD) model and the consensus feedback mechanism.

The TS3WD model includes three states $\Omega = \left\{C, \widetilde{C}, \overline{C}\right\}$ and three actions $\mathrm{A} = \{a_P, a_B, a_N\}$. C, $\widetilde{C}$ and $\overline{C}$ represent that an object x_i is the consensus state, the uncertainty state and the conflict state, respectively. a_P, a_B and a_N stand for acceptance, delay and rejection actions, which are denoted as $x_i \in POS(C)$, $x_i \in BND(C)$ and $x_i \in NEG(C)$, respectively. For an alternative-expert pair (x_i, e_k) consisting of alternative x_i and expert e_k, $Pr_k(C|[x_i])$, $Pr_k(\widetilde{C}|[x_i])$ and $Pr_k(\overline{C}|[x_i])$ are the conditional probabilities of the three states of the alternative-expert pair (x_i, e_k), which can be obtained that $Pr_k(C|[x_i]) = S_k^i$, $Pr_k(\widetilde{C}|[x_i]) = F_k^i$ and $Pr_k(\overline{C}|[x_i]) = 1 - S_k^i - F_k^i$ based on the consensus set pair probability space. The extended loss functions $\lambda_{\bullet\circ}(\bullet = P, B, N; \circ = P, B, N)$ are denoted to measure the losses of taking three actions. λ_{PP}, λ_{BP} and λ_{NP} represent the losses taking actions a_P, a_B and a_N when $x \in C$. λ_{PB}, λ_{BB} and λ_{NB} denote the losses taking actions a_P, a_B and a_N when $x \in \widetilde{C}$. λ_{PN}, λ_{BN} and λ_{NN} denote the losses taking actions a_P, a_B and a_N when $x \in \overline{C}$. A reasonable assumption is considered in the TS3WD model: $0 \le \lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $0 \le \lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$. When determining the optimistic extended loss functions, the loss of taking action a_P is less than the loss of taking action a_N if $x \in \widetilde{C}$, then $0 \le \lambda_{BB} \le \lambda_{PB} < \lambda_{NB}$ is assumed. Similarly, $0 \le \lambda_{BB} \le \lambda_{NB} \le \lambda_{PB}$ is considered when determining the pessimistic extended loss functions.

The expected extended loss functions $R_k(a_\bullet|[x_i])$ of taking three actions for expert e_k under alternative x_i can be calculated as:

(1) $R_k(a_P|[x_i]) = \lambda_{PP}\mathrm{Pr}_k(C|[x_i]) + \lambda_{PB}\mathrm{Pr}_k(\widetilde{C}|[x_i]) + \lambda_{PN}\mathrm{Pr}_k(\overline{C}|[x_i])$;

(2) $R_k(a_B|[x_i]) = \lambda_{BP}\mathrm{Pr}_k(C|[x_i]) + \lambda_{BB}\mathrm{Pr}_k(\widetilde{C}|[x_i]) + \lambda_{BN}\mathrm{Pr}_k(\overline{C}|[x_i])$;

(3) $R_k(a_N|[x_i]) = \lambda_{NP}\mathrm{Pr}_k(C|[x_i]) + \lambda_{NB}\mathrm{Pr}_k(\widetilde{C}|[x_i]) + \lambda_{NN}\mathrm{Pr}_k(\overline{C}|[x_i])$.

Table I.3: The classification results for $POS(x_i^t)$, $BND(x_i^t)$ and $NEG(x_i^t)$.

Cases	*Nonempty number*	$POS(x_i^t)$	$BND(x_i^t)$	$NEG(x_i^t)$
Case 1	3	*nonempty*	*nonempty*	*nonempty*
Case 2	2	*nonempty*	*nonempty*	$\emptyset$
Case 3	2	*nonempty*	$\emptyset$	*nonempty*
Case 4	2	$\emptyset$	*nonempty*	*nonempty*
Case 5	1	*nonempty*	$\emptyset$	$\emptyset$
Case 6	1	$\emptyset$	$\emptyset$	*nonempty*
Case 7	1	$\emptyset$	*nonempty*	$\emptyset$

Given the the identity degree$S_i^k \in [0,1]$, the discrepancy degree $F_i^k \in [0,1]$, and the contrary degree$P_i^k \in [0,1]$ of expert e_k under the alternative x_i, the TS3WD decision rules (P4)-(N4) are listed as follows:

(P4) If $S_k^i \ge \frac{(\lambda_{PB}-\lambda_{BB}-\lambda_{PN}+\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})}F_k^i + \alpha$ *and* $S_k^i \ge \frac{(\lambda_{PB}-\lambda_{NB}+\lambda_{NN}-\lambda_{PN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}F_k^i + \gamma$, *then* $(x_i, e_k) \in POS(C)$;

(B4) If $S_k^i \le \frac{(\lambda_{PB}-\lambda_{BB}-\lambda_{PN}+\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})}F_k^i + \alpha$ *and* $S_k^i \ge \frac{(\lambda_{BB}-\lambda_{NB}+\lambda_{NN}-\lambda_{BN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}F_k^i + \beta$, *then* $(x_i, e_k) \in$

BND(C);
(N4) If $S_k^i \leq \frac{(\lambda_{PB}-\lambda_{NB}+\lambda_{NN}-\lambda_{PN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}F_k^i + \gamma$ *and* $S_k^i \leq \frac{(\lambda_{BB}-\lambda_{NB}+\lambda_{NN}-\lambda_{BN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}F_k^i + \beta$, *then* $(x_i, e_k) \in NEG(C)$.

Based on the principle of minimum opinion adjustment, a consensus adjustment model for all experts belonging to $NEG(x_i^t)$ in the tth iteration is proposed to obtain the opinion modifications under all criteria. Considering that the minimum opinion modifications, the main idea of the model is to modify opinions of experts in the conflict state under alternative x_i according to the opinions of experts in the consensus state, in which the experts modifying the opinions would be divided into the consensus state in the next iteration. The model is as follows:

$$\min \sum_{j=1}^{n} w_j (m_{ij}^k - \bar{m}_{ij}^k)^2$$
$$s.t. \begin{cases} u_{ij}(e_k, e_h) = \begin{cases} 1 + 0\delta_1 + 0\delta_2, \left| m_{ij}^k - m_{ij}^h \right| \leq \tau_1 \\ 0 + 1\delta_1 + 0\delta_2, \tau_1 < \left| m_{ij}^k - m_{ij}^h \right| < \tau_2, j = 1, 2, ..., n \\ 0 + 0\delta_1 + 1\delta_2, \left| m_{ij}^k - m_{ij}^h \right| \geq \tau_2 \end{cases} \\ u_{ij}(\bar{e}_k, e_h) = \begin{cases} 1 + 0\delta_1 + 0\delta_2, \left| \bar{m}_{ij}^k - m_{ij}^h \right| \leq \tau_1 \\ 0 + 1\delta_1 + 0\delta_2, \tau_1 < \left| \bar{m}_{ij}^k - m_{ij}^h \right| < \tau_2, j = 1, 2, ..., n \\ 0 + 0\delta_1 + 1\delta_2, \left| \bar{m}_{ij}^k - m_{ij}^h \right| \geq \tau_2 \end{cases} \\ u_i(e_k, e_h) = f\left(u_{i1}(e_k, e_h), u_{i2}(e_k, e_h), ..., u_{in}(e_k, e_h)\right), \\ u_i(\bar{e}_k, e_h) = f\left(u_{i1}(\bar{e}_k, e_h), u_{i2}(\bar{e}_k, e_h), ..., u_{in}(\bar{e}_k, e_h)\right), \\ u_i(e_k) = f\left(u_i(e_k, e_{o(1)}), u_i(e_k, e_{o(2)}), ..., u_i(e_k, e_{o(\varepsilon_i^t)})\right), \\ u_i(\bar{e}_k) = f\left(u_i(\bar{e}_k, e_{o(1)}), u_i(\bar{e}_k, e_{o(2)}), ..., u_i(\bar{e}_k, e_{o(\varepsilon_i^t)})\right), \\ \bar{e}_k \in POS(x_i^{t+1}), \\ \forall e_k \in NEG(x_i^t); \forall e_{o(\cdot)} \in POS(x_i^t). \end{cases} \tag{I.16}$$

According to the number of nonempty regions, all possible classification results for three regions $POS(x_i^t)$, $BND(x_i^t)$ and $NEG(x_i^t)$ are listed in Table I.3. The seven cases are assigned seven different consensus feedback rules for two regions $BND(x_i^t)$ and $NEG(x_i^t)$, as shown in Table I.4.

Table I.4: The consensus feedback rules for $BND(x_i^t)$ and $NEG(x_i^t)$.

Cases	The adjustment strategy for $BND(x_i^t)$	The adjustment strategy for $NEG(x_i^t)$
Case 1	Classify again in next iteration	Adjust opinions toward $POS(x_i^t)$
Case 2	Classify again in next iteration	No adjustment
Case 3	END	Adjust opinions toward $POS(x_i^t)$
Case 4	Classify again in next iteration	Adjust opinions toward $POS(x_i^{t-1})$
Case 5	END	END
Case 6	END	Adjust opinions toward $POS(x_i^{t-1})$
Case 7	Update the coefficients and classify again in next iteration	No adjustment

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Sciences 661 (2024) 120199. DOI: https://doi.org/10.1016/j.ins.2024.120199.

6.4 The application of GDM methods in energy internet project evaluation

To study the application of GDM methods, we mainly focus on the energy internet project evaluation with FLEs based on MGCRS.

For the EI project evaluation problem, $E = \{e_1, e_2, ..., e_K\}$ is the set of experts, and $V = \{v_1, v_2, ..., v_K\}$ is the set of experts' weights. $B = \{b_1, b_2, ..., b_m\}$ is the criteria set of EI projects, and the criteria weight set is $W = \{W_1, W_2, ..., W_m\}$. There are n EI projects $X = \{x_1, x_2, ..., x_n\}$ to be evaluated. Experts express their preference through providing linguistic terms set $L = \{l_0, l_1, ..., l_g\}$ with symbolic proportions, i.e., FLEs. The evaluation matrix provided by expert e_k using FLEs is $M^k = (m_{ij}^k)_{n\times m}$, where m_{ij}^k is an FLE provided by expert e_k over the EI project x_i under criterion b_j. The transformation between FLE and cloud model is shown as follows:

$$\min \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{T} d(\tilde{C}_{ij}^t, C^t) * \tilde{p}_{ij}^t + \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{T}\left|\tilde{p}_{ij}^t - p_{ij}^t\right|}{mn}$$
$$s.t. \begin{cases} \sum_{t=1}^{T}\tilde{p}_{ij}^t = 1 \\ 0 \le \tilde{p}_{ij}^t \le 1 \\ d(\tilde{C}_{ij}^t, C_{ij}^t) = \sqrt{\frac{1}{2}((Ex_{ij}^t - \widetilde{Ex}_{ij}^t)^2 + (En_{ij}^t - \widetilde{En}_{ij}^t)^2 + (He_{ij}^t - \widetilde{He}_{ij}^t)^2)} \\ \widetilde{Ex}_{ij}^t - 3\widetilde{En}_{ij}^t - 9\widetilde{He}_{ij}^t > 0 \\ \widetilde{En}_{ij}^t \ge 3\widetilde{He}_{ij}^t \\ \widetilde{Ex}_{ij}^t - 3\widetilde{En}_{ij}^t \ge U^L \\ \widetilde{Ex}_{ij}^t + 3\widetilde{En}_{ij}^t \le U^U \\ \widetilde{En}_{ij}^t \ge 0, \widetilde{En}_{ij}^t \ge 0, \widetilde{He}_{ij}^t \ge 0 \\ \widetilde{Ex}_{ij} = \sum_{t=1}^{T}\widetilde{Ex}_{ij}^t * \tilde{p}_{ij}^t \\ \widetilde{En}_{ij} = \sqrt{\sum_{t=1}^{T}(\widetilde{En}_{ij}^t)^2 * \tilde{p}_{ij}^t} \\ \widetilde{He}_{ij} = \sqrt{\sum_{t=1}^{T}(\widetilde{He}_{ij}^t)^2 * \tilde{p}_{ij}^t} \\ i = 1, 2, ..., n; j = 1, 2, ..., m. \end{cases} \quad . \tag{I.17}$$

To determine weights of criteria using Shannon entropy method, we design the following steps:

Step 1. Normalize the collective interval-cloud matrix by calculating $\underline{h}_{ij}$ and $\overline{h}_{ij}$.

$$\underline{h}_{ij} = \frac{\underline{C}_{ij}}{\sum_{i=1}^{n}\underline{C}_{ij}}, \overline{h}_{ij} = \frac{\overline{C}_{ij}}{\sum_{i=1}^{n}\overline{C}_{ij}},\ i = 1, 2, ..., n; j = 1, 2, ..., m. \tag{I.18}$$

Step 2. Calculate the lower entropy $\underline{g}_j$ of $\underline{h}_{ij}$ and the upper entropy $\overline{g}_j$ of $\overline{h}_{ij}$.

$$\begin{aligned} \underline{g}_j &= -\frac{1}{\ln n}\sum_{i=1}^{n}\underline{h}_{ij}\ln\underline{h}_{ij}, j = 1, 2, ..., m, \\ \overline{g}_j &= -\frac{1}{\ln n}\sum_{i=1}^{n}\overline{h}_{ij}\ln\overline{h}_{ij}, j = 1, 2, ..., m. \end{aligned} \tag{I.19}$$

Step 3. Obtain the downward limit $\underline{W}_j$ and upward limit $\overline{W}_j$ of criteria weights.

$$\underline{W}_j = \frac{1-\underline{g}_j}{\sum_{j=1}^{m}(1-\underline{g}_j)}, \overline{W}_j = \frac{1-\bar{g}_j}{\sum_{j=1}^{m}(1-\bar{g}_j)}, j=1,2,...,m. \tag{I.20}$$

Step 4. Calculate the average weight W_j of criterion b_j.

$$W_j = \frac{\overline{W}_j + \underline{W}_j}{\sum_{j=1}^{m}(\overline{W}_j + \underline{W}_j)}, j=1,2,...,m. \tag{I.21}$$

Based on the above methods, the weight set $W = \{W_1, W_2, ..., W_m\}$ can be obtained by the Shannon entropy method based on all evaluation matrixes using FLEs. After this, we define the MCGRS:

Definition 6. *Let (X,E,F,R,B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j=1,2,...,m)$ is the binary cloud relation between universe X and E. For any $A \in F(E)$, $e \in E$ and $x \in X$, the comprehensive multi-granularity lower approximation $\underline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i)$ and upper approximation $\overline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i)$ of A with respect to (X,E,F,R,B) are as follows:*

$$\begin{aligned} \underline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i) &= \sum_{j=1}^{m} W_j \underset{e\in E}{\wedge} \max(N(R^j(x_i,e_k)), A(e_k)), x_i \in X, \\ \overline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i) &= \sum_{j=1}^{m} W_j \underset{e\in E}{\vee} \min(R^j(x_i,e_k), A(e_k)), x_i \in X. \end{aligned} \tag{I.22}$$

Furthermore, the approximation evaluation value $R_{\sum_{j=1}^{m} R^j}(A)(x_i)$ of A for x_i using MGCRS over two universes is as follows:

$$R_{\sum_{j=1}^{m} R^j}(A)(x_i) = \theta \overline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i) + (1-\theta)\underline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i), \tag{I.23}$$

where θ is the preference coefficient and $\theta \in [0,1]$.

Finally, the reference cloud $A = (Ex_A(E), En_A(E), He_A(E))$ is obtained by $A = \sum_{j=1}^{m} W_j C^j$, which is as follows:

$$\begin{cases} Ex_A(E) = \sum_{j=1}^{m} W_j Ex^j \\ En_A(E) = \sqrt{\sum_{j=1}^{m} W_j (En^j)^2} \\ He_A(E) = \sqrt{\sum_{j=1}^{m} W_j (He^j)^2} \end{cases}, \tag{I.24}$$

where $C^j = \gamma C^{j+} + (1-\gamma)C^{j-}, j=1,2,...,m$ is the jth reference cloud $C^j = \{C_1^j, C_2^j, ..., C_K^j\}$ under criterion $b_j (j=1,2,...,m)$.

The journal article associated to this part is:

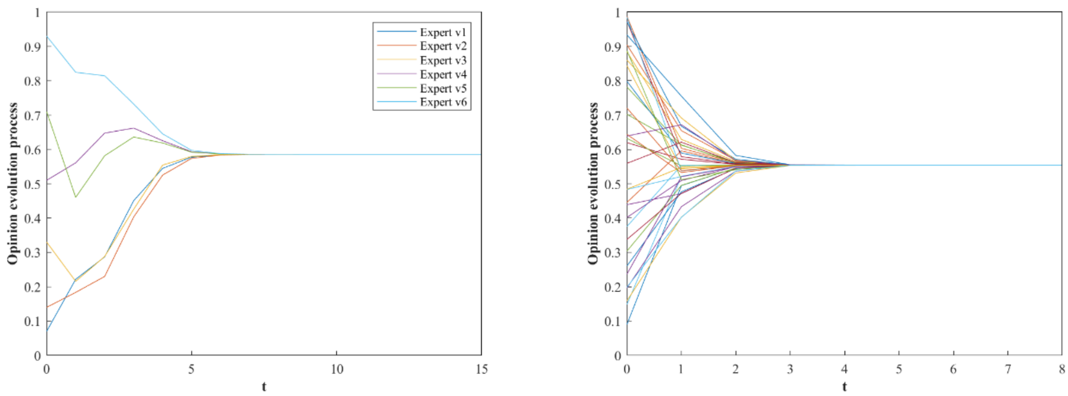
H. Wang, Y. Ju, C. Porcel, Energy internet project evaluation in circular economy practices: A novel multi-criteria decision-making framework with flexible linguistic expressions based on multi-granularity cloud-rough set, Computers & Industrial Engineering 201 (2025) 110890. DOI: `https://doi.org/10.1016/j.cie.2025.110890`.

7 Discussion of results

This section mainly makes several discussions about the results obtained in all the mentioned stages of this thesis.

7.1 The research on the group consensus evolution model in social trust network

To describe the steps of our proposal in detail and illustrate the usefulness of our proposal, we have applied the proposed model to two numerical experiments about supplier performance evaluation and Zachary's karate club, as shown in Fig. 2. The two case studies demonstrate the effectiveness of incorporating opinion reliability into the opinion evolution process. In the supplier evaluation case, reliable experts influenced the convergence path, helping to avoid blind adjustments and encouraging selective opinion revisions. Expert reliability showed a consistent upward trend and eventually stabilized as a consensus was reached. In the Zachary's karate club case, the group achieved consensus rapidly, driven by the reinforcement of opinion similarity and trust through the evolving network. The initially subjective and divergent reliabilities were effectively corrected by the structure of the social network, which enhanced both opinion alignment and convergence speed. These results confirm that integrating opinion reliability into social network dynamics not only improves decision quality but also accelerates the CRP.



(a) The opinion evolution process about supplier performance evaluation.

(b) The opinion evolution process about Zachary's karate club.

Fig. 2: The numerical experiments.

In addition to the numerical experiments, we have conducted comparative simulations on Erdos–Renyi (ER) [Gil59] and Watts–Strogatz (WS) [WS98] random networks and parameter sensitivity analyses (including ρ, $R*$, and Sm^*) to further verify the applicability and robustness of the proposed model. The results demonstrate that the proposed model can consistently achieve group consensus across different network structures, confirming its generalizability. Compared to traditional HK models, the extended HK model produces compromise solutions that balance opinion similarity and reliability, avoiding over-reliance on a few experts. Moreover, simulation experiments reveal that while increasing the opinion similarity threshold contributes to convergence, the impact of the reliability threshold is more significant and consistent. A higher reliability threshold restricts excessive opinion modification, improving the stability of the convergence process. These findings highlight the importance of carefully selecting model parameters to ensure both convergence efficiency and realistic consensus outcomes. The conducted comparative analysis confirms the advantages and innovation of our model in the context of SNGDM. Unlike previous methods that overlook opinion reliability, our approach incorporates it alongside similarity and centrality in trust estimation, while clearly distinguishing between social and trust networks. By integrating a dynamic feedback mechanism with an extended HK model, our framework enables more realistic and flexible consensus building. These improvements promote robust convergence and reduce the impact of unreliable or overly dominant experts, addressing key gaps in existing trust-based consensus models.

7.2 The research on the construction of a preference-based multi-criteria 3WD model

In this study, we have conducted a comprehensive sensitivity analysis to investigate the impact of risk avoidance and regret aversion coefficients on the outcomes of the 3WD model. The results reveal that both parameters significantly influence the classification outcomes. From Fig. 3, as the risk avoidance coefficient increases, experts tend to make more decisive judgments, resulting in a larger number of tasks being assigned to the positive (POS) and negative (NEG) regions, and a smaller number falling into the boundary (BND) region. This indicates that a higher level of risk aversion encourages experts to reduce ambiguity. Conversely, when the risk avoidance coefficient is low, more tasks are classified into the BND region, reflecting a more cautious or uncertain stance. In terms of regret aversion, we observed that, under the 3WD setting, an increase in the regret aversion coefficient leads to a higher average $\bar{\alpha}$ and a lower $\bar{\beta}$, thereby expanding the BND region. In the 2WD scenario, however, the regret aversion coefficient primarily affects the single threshold $\bar{\gamma}$, exerting only a limited influence on the classification. These findings highlight the critical role of parameter tuning in ensuring reasonable and interpretable decision outcomes.

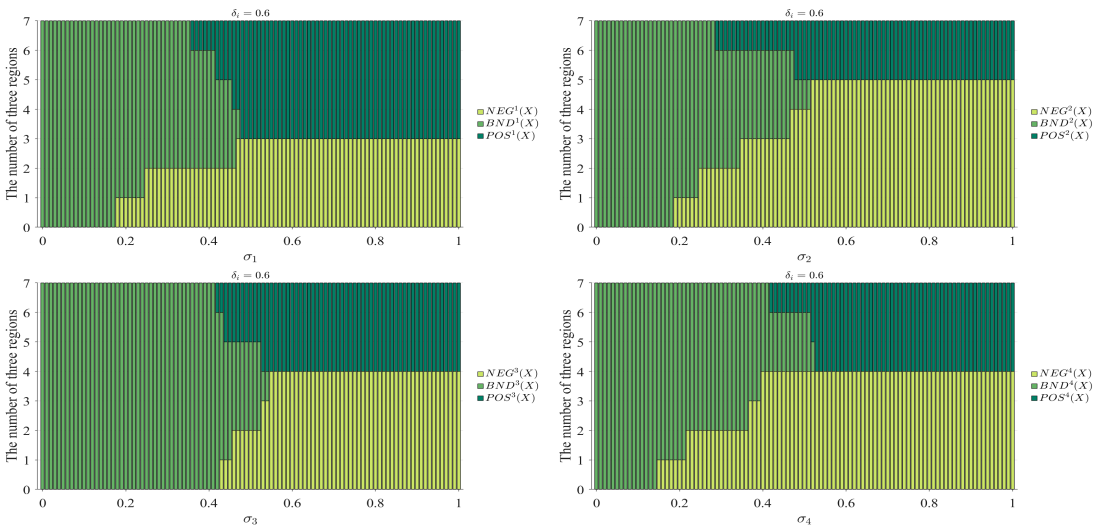


Fig. 3: The results under different risk avoidance coefficients σ_h.

In addition, we have also carried out a comparative analysis to assess the influence of expert preferences on decision results. By keeping the evaluation matrix constant and altering only the preference matrices, we observed significant differences in classification. For example, expert e_1, under a revised preference set, categorized all tasks as unsuitable for computer-only recognition, whereas expert e_2 assigned a greater number of tasks to human-computer collaboration. These results illustrate the sensitivity of the proposed regret-based 3WD model to preference information, confirming its ability to effectively capture the subjective tendencies of different experts in multi-expert decision-making scenarios. To further validate the robustness of our method, we have compared its performance with five classical MCDM methods, including variants based on TOPSIS, GRA, VIKOR, and Weighted Averaging Aggregation (WAA) methods. Despite methodological differences, the ranking outcomes showed a high degree of consistency. Notably, all six methods identified alternative x_4 as the best option. The Spearman's correlation coefficients between our model and the others all exceeded 0.82, indicating a strong agreement in ranking results. This confirms the reliability and stability of the proposed approach.

In summary, the proposed preference-based regret 3WD model demonstrates several theoretical and practical advantages. First, it directly utilizes LZNs to represent expert evaluations, avoiding information loss caused by conversion to other fuzzy number types. Second, the integration of the Z-LINMAP method allows for the coherent fusion of preference and evaluation information, improving the realism of weight and probability calculations. Third, the model incorporates regret theory to construct regret-based relative loss functions, thus capturing the psychological behaviors of experts more accurately. Finally, the model supports both classification and ranking under Multiple Decision Information Systems (MDISs), making it a versatile and effective tool for complex decision-making problems.

7.3 The research on the combination of group consensus and multi-criteria 3WD models

To validate the effectiveness of the proposed model, extensive numerical experiments have been conducted using the Ecoli and Iris datasets from the UCI repository. The results demonstrate the model's ability to simulate expert opinion adjustments and capture consensus dynamics under various decision conditions. In the Ecoli dataset, expert opinions were frequently adjusted for all thirteen alternatives, with a clear distinction between positive and negative directions. This illustrates that the model can effectively guide experts toward consensus. For alternatives such as x_3, x_6, x_{10}, and x_{13}, more experts adjusted their opinions positively, indicating alignment in evaluation, while other alternatives showed dominant negative adjustments, reflecting divergence. The Iris dataset further confirms the model's flexibility through sensitivity analysis. As the consensus coefficient τ_1 increases, the number of alternatives classified into the positive region increases, while those in the boundary region decrease, highlighting that a looser consensus condition facilitates agreement. Meanwhile, the conflict coefficient τ_2 plays a crucial role in distinguishing experts with strong disagreements, as it influences the size of the negative region. The interaction between τ_1 and τ_2 offers decision-makers a flexible tool to balance between coarse- and fine-grained consensus, depending on expert willingness and confidence levels.

A comparative analysis with three existing 3WD methods has been applied to the Breast Cancer dataset shows that our model achieves superior performance, as shown in Table I.5. Our method requires fewer iterations to reach consensus and yields a lower deferment rate and higher comprehensive score, indicating faster convergence and better decision quality. Furthermore, theoretical comparison reveals that our approach uniquely integrates MCDM and S3WD theory while classifying expert–alternative pairs instead of single entities. This finer granularity enhances the model's ability to localize disagreement and guide targeted adjustments.

In summary, both the numerical and theoretical results validate the robustness, adaptability,

Table I.5: Performance comparisons of our proposal and the existing 3WD methods.

Methods	Iterations	Average deferment rate	Average comprehensive score
Classic 3WD method [Yao10]	–	0.8558	0.7853
S3WD-HK method	5	0.1547	0 .8928
Wang et al.'s [WLL22] method	5	0.1473	0.8952
Our proposal	3	0.1450	0.8971

and superiority of the proposed model in handling multi-expert decision-making scenarios with uncertain and dynamic consensus behavior.

7.4 The application of GDM methods in energy internet project evaluation

The proposed MCDM framework provides a substantial theoretical advance in the circular economy practice. First, a new method for transforming discrete FLEs into continuous cloud information enhances the management of uncertainty by transforming discrete data into a continuous format, allowing for a more nuanced and precise integration of assessment criteria. This theoretical advance contributes to a deeper understanding of how various EI projects impact sustainability, leading to a more accurate representation of their contribution to circular economy. Secondly, with the support of the Shannon entropy method, a tailored evaluation index system is developed to expand the theoretical framework for evaluating the different impacts of EI projects. By using this approach to assign appropriate weight to assessment criteria, the framework ensures that assessments reflect the multifaceted nature of green innovation and sustainability in the circular economy. This theoretical refinement improves the ability to capture the effectiveness of projects in improving energy efficiency and achieving sustainable development goals. Finally, the application of MGCRS and integrated multi-granularity approximations has introduced significant theoretical advances to the decision-making process. This approach provides a powerful mechanism for ranking and optimizing EI projects by managing ongoing cloud information. The use of optimistic and pessimistic MGCRSs in both areas can improve the accuracy and reliability of project evaluations. In theory, this innovation provides a more precise and practical basis for optimizing energy grid connections, thereby supporting the circular economy and advancing energy integration. Totally, these theoretical contributions collectively strengthen the understanding and evaluation of green innovation and circular economy by improving uncertainty management, refining evaluation frameworks, and advancing decision-making methods.

The proposed MCDM framework highlights several key managerial implications for the circular economy:

(1) Improved decision-making under uncertainty. The framework transforms fuzzy linguistic evaluations into continuous cloud information, helping managers better assess circular economy projects under uncertain conditions such as market fluctuations or material availability. This supports scenario-based planning and risk-informed decisions.

(2) Comprehensive evaluation of impacts. By covering grid technology, green energy, and composite benefits, the evaluation system ensures well-rounded project assessment. It allows

decision-makers to balance technical, environmental, and social factors, improving alignment with circular economy goals.

(3) Prioritization of high-impact projects: The MGCRS method enables precise ranking based on energy efficiency, carbon reduction, and resource use. It helps identify projects with strong sustainability potential, guiding investment toward initiatives with the greatest long-term impact.

(4) Efficient resource allocation. The framework highlights which EI projects best meet circular economy objectives, ensuring that funding and support are directed to the most effective solutions. It also helps reduce investment risk and promotes low-carbon transformation.

(5) Actionable insights for implementation. The framework turns complex data into practical strategies, helping managers adjust projects based on real-time feedback. This supports continuous improvement, better resource use, and long-term project sustainability.

8 Conclusions

In this section, we present the results obtained from the research carried out during this Ph.D. dissertation. The primary goal of this thesis is to advance the field of GDM by integrating CRP with multi-criteria 3WD models. In addition to the theoretical advancements, the proposed models have been applied and validated in real-world contexts, thereby enhancing both the methodological foundation and practical utility of GDM. The research results obtained in this thesis are described as follows:

The first topic has focused on the evolution of group consensus in social trust networks by introducing opinion reliability into opinion dynamics. We have developed a novel SNGDM framework that incorporates expert reliability into both trust propagation and opinion aggregation processes. Opinion reliability has been quantitatively defined using stability similarity and weighted similarity, reflecting individual consistency and alignment with others' views. These reliability measures have been integrated into an improved trust mechanism and an extended HK model to better simulate the opinion evolution in real-life networks. This approach has allowed us to avoid over-reliance on high-weight but potentially biased leadership opinions, and to highlight the value of grassroots expert contributions. Simulation analyses have confirmed the framework's effectiveness and parameter sensitivity, supporting its applicability in practical decision scenarios such as production planning.

The second topic has addressed MCDM problems with uncertain and preference-driven information by proposing a preference-based regret 3WD model under LZNs environments. We have introduced the Z-LINMAP method to determine criteria weights while capturing individual preference coefficients. Two preference-based equivalence classes have been defined to derive conditional probabilities, allowing the model to reflect preference differences across experts or datasets. Additionally, regret loss functions based on regret theory have been constructed to capture psychological behaviors such as regret and rejoice. This framework has been applied to classification and ranking problems in MDISs, especially in image recognition tasks. Sensitivity and comparative analyses have verified the model's wider applicability and stronger psychological realism compared to existing 3WD methods.

The third topic has explored consensus reaching in a multi-criteria sequential three-state 3WD context by proposing a new consensus model based on SPA theory. We have constructed a consensus set pair probability space to distinguish experts with different levels of consensus, and proposed a TS3WD model that extends the classic two-state framework. Based on SPA, we have derived complete and simplified decision rules using extended loss functions, which can reduce to traditional models when necessary. Furthermore, we have designed a sequential consensus feedback mechanism that enables non-consensus experts to adjust their opinions dynamically through coefficient-based strategies. Seven different adjustment cases have been tested to demonstrate the model's effectiveness in supporting adaptive and iterative consensus processes.

The fourth topic has enhanced the practical application of GDM models by proposing a fuzzy linguistic MCDM framework for the evaluation of EI projects. Using FLEs and MGCRS, we have built a model to address both discrete and continuous uncertainty in sustainable project evaluation. This framework incorporates sustainability and circular economy indicators into a tailored evaluation index system, helping managers prioritize projects that contribute most effectively to resource efficiency and sustainable transformation. The proposed model has facilitated the conversion of complex evaluation data into actionable strategies, supporting informed investment and policy decisions in industrial sustainability planning.

In summary, this dissertation has made meaningful theoretical and practical contributions to GDM by developing integrated models that combine CRP with multi-criteria 3WD approaches. By incorporating social influence, opinion reliability, psychological factors, preference information, and sustainability considerations, the proposed methods enhance the realism, flexibility, and applicability of GDM frameworks. Validated through simulations and real-world applications, these models provide effective tools for addressing complex and uncertain decision problems across diverse domains.

9 Future works

Although we have done some research on the integration of CRP and multi-criteria 3WD in GDM, there are still promising avenues for future research. In this section, we propose three potential directions that align with the core concepts of consensus and 3WD theory, focusing on their combination and diversified applications.

9.1 The consensus models in complex social networks

The first important research direction in the future is to expand the CRP model by integrating more complex social network structures and multilingual environments. The current consensus models mainly focus on structured social networks, in which the relationships among experts are stable and homogeneous. However, social networks in the real world are often dynamic and heterogeneous, with varying levels of opinion reliability and constantly fluctuating social relationships.

Therefore, it is crucial to develop models that can consider the reliability of opinions in more diverse network topologies, such as random networks or scale-free networks. Furthermore, integrating language variables into the consensus model will enhance the model's adaptability to the multilingual decision-making environment. Further research can focus on exploring the dissemination of opinions in random networks, including how the reliability of opinions affects the evolution of consensus. Another promising direction is to explore the spread of rumors in a consensus environment, emphasizing the impact of reliable information dissemination. These advancements will enhance the robustness and applicability of consensus models in the real-world social decision-making environment.

9.2 The 3WD and TS3WD models for heterogeneous information systems

The second research direction focuses on improving the 3WD and TS3WD models for handling heterogeneous and incomplete decision information systems in MCGDM problems. In practical applications, especially in large-scale decision-making scenarios, decision-making information is often inconsistent and incomplete, which poses a challenge to reaching the reliable group consensus.

Future research can be dedicated to developing adaptive multi-criteria 3WD and TS3WD models, integrating the determination of subjective and objective criterion weights. This involves exploring methods that combine expert judgment with data-driven criterion weight estimation, thereby enhancing the flexibility of the model. Furthermore, the integration of heterogeneous information fusion methods to unify decision-making systems with different information structures remains a key challenge. Exploring the mechanism of dynamically adjusting the consensus coefficient based on historical decision-making data will also provide valuable insights for optimizing the consensus process. Addressing these challenges can significantly expand the practical usability of the 3WD and TS3WD models in GDM.

9.3 The application of integrated consensus and 3WD Models in real-world scenarios

The third research direction is to explore and develop the application of integrated consensus models and 3WD models in various real-world decision-making scenarios. The effectiveness of combining the consensus process with 3WD has been confirmed in theoretical models. However, there is still great potential to transform these models into practical applications, especially in areas where group consensus is crucial under uncertain and conflicting information.

The assessment of sustainable energy projects is a promising application field. In this field, the combination of consensus models and multi-criteria 3WD methods can optimize project selection by balancing environmental, economic, and social factors. Furthermore, in social commerce and online platforms, analyzing user-generated content and emotions through consensus-based 3WD models can enhance the accuracy of recommendations, as this approach can take into account both the diversity of user preferences and the dynamics of viewpoints.

Furthermore, the study of the integration of consensus coefficients and conflict coefficients in real-time decision support systems will provide practical and feasible solutions for adaptive consensus construction in dynamic environments, such as policy decision-making, collaborative filtering in e-commerce, and strategic planning under uncertainties. By focusing on these applications, future research can bridge the gap between the development of theoretical models and their actual deployment.

Chapter II

Publications: Published Papers

1 A social network group decision making framework with opinion dynamics considering opinion reliability

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  - Subject Category: Computer Science, Interdisciplinary Applications, Ranking 21/170 (Q1).
  - Subject Category: Engineering, Industrial, Ranking 11/69 (Q1).

A social network group decision making framework with opinion dynamics considering opinion reliability

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Abstract

Opinion dynamics play an important role in the consensus reaching process (CRP) when tackling social network group decision making (SNGDM) problems. Opinion reliability can be regarded as an important characteristic of expert to indicate the reliability of experts' opinions in social trust network. However, it has rarely been considered into the opinion evolution process in SNGDM. To explore the impact of opinion reliability on group consensus reaching, a SNGDM framework with opinion dynamics considering opinion reliability is proposed in this paper. Firstly, opinion reliability of experts in social trust network is defined, and the comprehensive trust degree based on the social network structure and individual characteristic is proposed. Secondly, trust propagation and aggregation mechanisms are designed to obtain the social trust matrix. Thirdly, considering opinion similarity and opinion reliability, social network evolution rules and opinion evolution rules based on the extended Hegselmann-Krause (HK) model are presented. Finally, the proposal is applied to two numerical experiments about supplier performance evaluation and Zachary's karate club, and the simulation and comparison analyses are provided to demonstrate the convergence and illustrate the feasibility and effectiveness of the proposed model.

Keywords: Opinion reliability, Group decision making, Opinion dynamics, Social trust network

1. Introduction

The development of digital transformation has increased the complexity of decision-making problems in the real world, which makes it more difficult for a single decision maker to get a reliable solution. To overcome this challenge, group decision making (GDM) has attracted widespread attention, as it can deal with complex decision-making environment effectively [1–4]. Usually, multiple experts with different backgrounds are involved in decision-making process to provide their opinions for a set of alternatives, and the best alternative is finally selected based on the wisdom of crowds [1, 5–7].

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Generally, GDM process is divided into two steps: the consensus reaching process (CRP) and the selection process. CRP is necessary to motivate individuals to adjust their opinions for improving the group consensus, where individuals negotiate with each other to get a reliable group consensus solution. In the previous literature, a series of CRP models have been proposed [3, 4, 8–10], which can be divided into two categories from the perspective of the feedback mechanism: (i) With feedback mechanism: some CRP models incorporate the feedback mechanisms guiding experts to modify opinions to realize the group consensus [11–14]. (ii) Without feedback mechanism: other CRP models only apply the designed mechanisms that automatically update opinions without considering human intervention [15–18]. The former pays more attention to the behavior and psychology of experts, while the latter focuses on automation of feedback mechanisms considering budget constraints. Obviously, CRP models with feedback mechanism are more appropriate to solve practical GDM problems.

With the increasing complexity of decision-making problems, more and more individuals are involved in GDM problems. Meanwhile, social media like Facebook, WeChat and Twitter strengthens the communication between individuals, which leads to more rapid and complex interactions among individuals, further generating the social network group decision making (SNGDM) problems [19–23]. In SNGDM, individuals tend to negotiate with their friends or neighbors based on their trust relationships. Therefore, individual relationships can be regarded as key factors to influence the final solution. Dong et al. [24] have summarized the CRP paradigm for trust relationships where trust propagation and trust aggregation are regarded as two critical steps. Most researchers estimated unknown trust value using t-norm operator [25–27] or product method [22, 28]. Some studies developed CRP models based on trust relationships with unknown trust values [29, 30]. However, the existing literature about trust relationships ignored that the opinion reliability is an important source of trust relationship, which should be considered when estimating the trust values. From the perspective of social network characteristics [27, 31–34], there are two kinds of CRP models in SNGDM problems: (i) CRP models in static social networks [27, 28] and (ii) CRP models in dynamic social networks [35–38]. Actually, the social networks change dynamically so that CRP models in dynamic social networks have been extensively studied [24]. To conquer the complexity of dynamic social networks, opinion dynamics are widely developed which is regarded as a common and useful tool to solve SNGDM problems [39], which are often used to describe the opinion evolution process for a group of individuals about decision-making problems [40]. Many well-known opinion dynamics models have been proposed: (i) Continuous models: the French model [41], the DeGroot model [42], the Friedkin and Johnsen model [43], the Deffuant-Weisbuch (DW) model [44, 45] and Hegselmann-Krause (HK) model [46]. (ii) Discrete models: the Ising model [47], the Sznajd model [48], the voter model [49–51] and the majority rule model [52–55].

Most prior studies about SNGDM problems assume that experts are completely reliable, where the reliability of expert has rarely been considered. Although system reliability has been widely considered into industry and engineering domains to improve the safety and robustness of system, human reliability analysis is still a new topic by incorporating human factor into the reliability analysis to mitigate the human errors [56]. Fu et al. [57] claimed that reliability of experts in GDM problems is different from that in human reliability analysis and proposed the definition of expert reliability in multiple attribute group decision analysis. Furthermore, Xue et al. [58] presented a CRP model based on expert reliability proposed by Fu et al. [57]. Liu et al. [59] measured the expert reliability combining the similarity degree and the hesitate degree based on Dempster–Shafer evidence theory.

Although previous studies have made significant contributions on SNGDM problems, there are still some research challenges as follows:

(i) Reliability is an important characteristic of experts in SNGDM problems. However, most previous researchers rarely consider the reliability of experts, or simply assume that all experts are completely reliable. In reality, reliability of experts may significantly affect the opinion evolution process, which needs to be further explored in SNGDM problems.

(ii) Most SNGDM studies assume that trust values between experts in social networks are provided by experts. However, unreliable experts may not accurately quantify the trust values due to the impacts of subjective and objective factors, which may lead to unreasonable predefined trust values given by experts. Therefore, it deserves to explore how to obtain the trust values and derive the complete trust matrix and weights of experts in social network accurately when considering the opinion reliability.

(iii) In CRP, experts are more willing to accept opinions of the experts they trust, which can be affected by opinion dynamics. The existing models paid little attention to the evolution of social networks and ignored the characteristic of experts like opinion reliability, which needs to be further extended using opinion dynamics.

Motivated by the above research gaps, a GDM framework with opinion dynamics in social trust network considering opinion reliability is developed. The main contributions of the proposed model are listed as follows:

(i) Opinion reliability of experts is proposed based on stability similarity and weighted similarity, which is measured from the perspective of individual and others, respectively. The stability similarity reflects the stability of experts' opinions from the perspective of individual, and the weighted similarity reflects the opinion closeness between the expert and reliable experts from the perspective of others.

(ii) Trust propagation and aggregation in social networks based on opinion reliability is considered in opinion evolution process. The comprehensive trust degree is derived from the characteristics of experts and social network structure. Considering the propagation efficiency affected by opinion similarity, we improve the T-norm aggregation operator, and further propose a trust aggregation optimization model considering the number of mediators to obtain the complete trust matrix and weights of experts.

(iii) A novel GDM framework of trust propagation and opinion dynamics in social networks is proposed. Opinion reliability is considered as an important factor affecting the social network. Therefore, the connections of experts in social networks depend not only on opinion similarity but also on opinion reliability of experts. Meanwhile, the extended HK model is proposed to design the opinion evolution rules combining opinion similarity with opinion reliability of experts.

The research paradigm of this paper is shown in Fig. 1. The remainder of this paper is structured as follows. Section 2 introduces the basic knowledge necessary related to graph theory, social network analysis (SNA) and opinion dynamics. Section 3 proposes the definition and properties of opinion reliability, and designs the trust propagation mechanism based on opinion reliability. Section 4 presents a GDM framework of trust propagation and opinion dynamics in social networks. Section 5 provides two numerical experiments to illustrate the usefulness of the proposed model, and the simulation and comparison analyses are provided to illustrate the feasibility and effectiveness of the proposal in Section 6. Finally, Section 7 presents the concluding remarks.

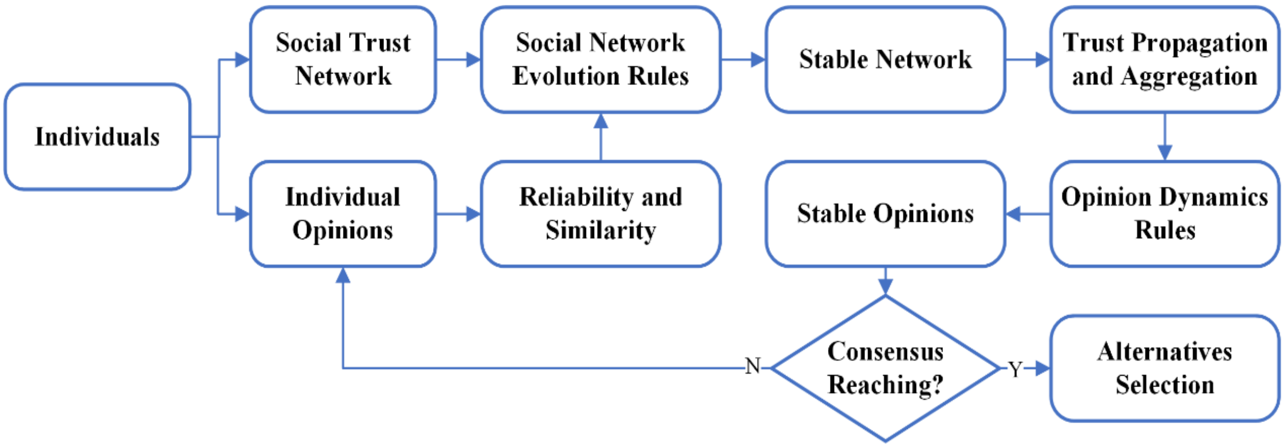


Fig. 1. The research paradigm of the proposal.

2. Preliminaries

This section briefly introduces some basic knowledge necessary to develop and understand our proposals, regarding graphs, SNA, the DeGroot model, the DW and HK models.

2.1. Graph theory

In GDM problems, individuals tend to be dependent on each other, and there may exist trust or distrust relationships between them. Generally, individuals and the trust relationships between them can be essentially demonstrated by graph theory. The basic definitions and notations regarding graphs are provided as follows.

Definition 1. [53] A directed graph is denoted by $G(V, E)$, where $V = \{v_1, v_2, ..., v_k\}$ is a set of finite nodes, and $E = \{(v_i, v_j)|v_i, v_j \in V; i \neq j\}$ denotes the ordered edges connecting two nodes. In this paper, the sets V and E are assumed to be finite and V is assumed to be nonempty.

Definition 2. [53] The adjacency matrix of $G(V, E)$ is denoted by $A = (a_{ij})_{k \times k}$. $a_{ij} = 1$ indicates that there is an edge from v_i to v_j, and $a_{ij} = 0$ indicates that there is no edge from v_i to v_j, i.e.,

$$a_{ij} = \begin{cases} 1, (v_i, v_j) \in E \\ 0, (v_i, v_j) \notin E \end{cases}.$$

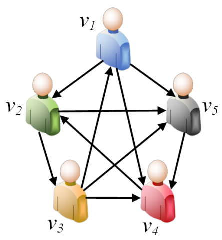
Definition 3. [52] The indegree and outdegree of v_i are denoted by $d^+_{v_i}$ and $d^-_{v_i}$, respectively. The indegree $d^+_{v_i}$ is defined as the sum of the incoming edges, i.e., $d^+_{v_i} = \sum_{j=1}^{k} a_{ji}$, and the outdegree $d^-_{v_i}$ is defined as the sum of the outcoming edges of v_i, i.e., $d^-_{v_i} = \sum_{j=1}^{k} a_{ij}$.

Definition 4. [53] In the directed graph $G(V, E)$, a sequence of edges $\{(v_i, v_{k_1}), (v_{k_1}, v_{k_2}), (v_{k_2}, v_{k_3}), ..., (v_{k_s}, v_j)\}$ is called a directed path from v_i to v_j, which is denoted as $v_i \rightarrow v_j$.The length of the path depends on the number of the edges in the directed graph.

2.2. Social network analysis

SNA [60] contains a series of structural approaches based on ties linking social actors to characterize relationships between entities using the graphic imagery and mathematical models. SNA studies are used to analyze the structure and location properties including centrality, prestige, structural balance and trust relationships [61, 62]. The three main elements in a social network are represented as: set of individuals, relationships between them and individual attributes. Therefore, we refer to important network concepts in a unified manner using three representation schemes, which is shown in Table 1 [16].

Table 1
Three representation schemes in social networks [16].

Graph theoretical	Sociometric	Algebraic
	$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$v_1 R v_2,\ v_1 R v_4$ $v_1 R v_5,\ v_2 R v_3$ $v_2 R v_5,\ v_3 R v_1$ $v_3 R v_4,\ v_3 R v_5$ $v_4 R v_2,\ v_5 R v_4$

(i) Graph theoretical: the social network is viewed as a directed graph $G(V, E)$, which consists of the set of experts $V = \{v_1, v_2, ..., v_k\}$ and the set of edges $E = \{(v_i, v_j)|v_i, v_j \in V; i \neq j\}$. $v_i \rightarrow v_j$ denotes that there is a relationship from v_i to v_j, which may be trust, distrust, conflict or communicate relationships.

(ii) Sociometric: the adjacency matrix $A = (a_{ij})_{k\times k}$ is used to represent the relationships among experts $V = \{v_1, v_2, ..., v_k\}$, which is called sociometric.

(iii) Algebraic: this notation distinguishes several distinct relationships and presents the combinations of these relationships.

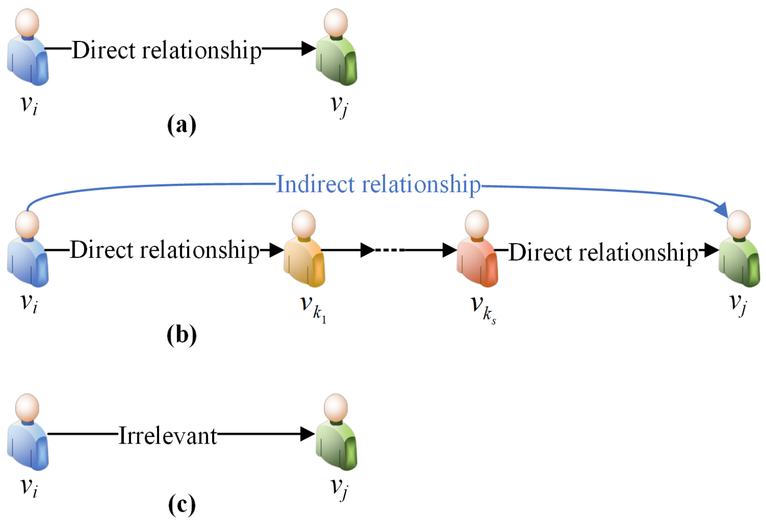


Fig. 2. Three types of relationships in social networks [16].

These representations show whether there is a direct relationship between experts. However, the relationships between experts tend to be actually uncertain. To identify the uncertain relationship, the indirect relationships between experts are considered in SNA. Considering the transitivity of information, if expert v_i trusts v_k and expert v_k trusts v_j, then there may be an indirect trust from expert v_i to v_j even though they do not know each other. Three types of relationships in social networks are shown in Fig. 2 [33]. Taking the trust relationship for example, three kinds of trust relationships are interpreted as follows:

(i) Direct relationship. As Fig. 2 (a) shows, there is an edge from expert v_i to v_j. Therefore, expert v_i has a direct relationship with v_j, which means that expert v_i trusts v_j to some extent and may interact with v_j in the real life.

(ii) Indirect relationship. As Fig. 2 (b) shows, there is no edge from expert v_i to v_j but expert v_i can establish a connection with v_j through many mediators $\{v_{k_1}, v_{k_2}, v_{k_3}, ..., v_{k_s}\}$. Therefore, expert v_i has an indirect relationship with v_j. Even though experts v_i and v_j may not know each other, a reliable relationship from expert v_i to v_j can be established by some direct relationships among these mediators.

(iii) Irrelevant. As Fig. 2 (c) shows, there is neither a direct relationship nor an indirect relationship from expert v_i to v_j. Therefore, experts v_i and v_j are irrelevant, which means that expert v_i does not trust v_j and has no chance to communicate with v_j.

Any social network contains at least one of three relationships, in which experts are identified as nodes and relationships among them are considered as edges. A social network cannot be constructed when all experts are irrelevant each other. Therefore, this case that all experts are irrelevant will not be considered into in this paper. In this paper, the edges do not represent the trust relationships but the communicate relationships between experts. In other words, the edge from expert v_i to v_j indicates that there is a communication relationship between them, which does not reflect that expert v_i trusts v_j.

The reason is that people do not always trust someone with whom they interact in reality, but may distrust or resist them to a certain extent. However, when individuals trust someone like a celebrity but cannot communicate with them, the trust relationship does not propagate further in social networks. Therefore, the communication relationships and trust relationships between experts are two necessary conditions for trust propagation process in social networks.

2.3. Opinion dynamics

Individual will neither simply agree nor completely refuse others' opinions in social networks, but consider these opinions to some extent to update his/her own opinion. Generally, individual opinions will evolve dynamically when they communicate with each other. Opinion dynamics are a fusion process of individual opinions through interactions among a group of individuals, in which continuously update and fuse their opinions through the fusion rules and reach a consensus, polarization, or fragmentation in the final stage [61]. The general framework of fusion process in opinion dynamics is shown in Fig. 3. The consensus opinion of a group means that there is a fused opinion remaining unchanged over time [63]. Different continuous opinion dynamics models have been proposed to model the opinion evolution process, such as the DeGroot model and the bounded confidence model which are briefly introduced in the following section.

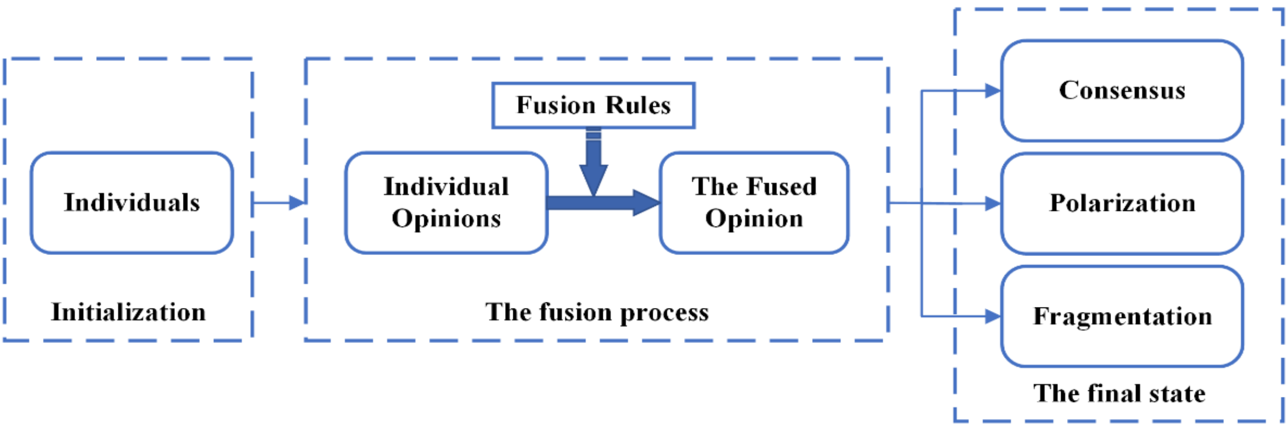


Fig. 3. The general framework of fusion process in opinion dynamics.

2.3.1. The DeGroot model

The DeGroot model [42] is one of the classical opinion dynamics models, which refers to the phenomenon that individual opinions will be influenced by opinions of different individuals with fixed weights. The DeGroot model is defined as follows.

Definition 6. [42] Suppose that $V = \{v_1, v_2, ..., v_k\}$ is a set of experts, and $x_i^t \in [0, 1]$ is expert v_i's opinion at time t, where $t \in \mathrm{N}$. $w_{ij} \in [0, 1]$ is the expert v_j's weight assigned by v_i, and $\sum_{j=1}^{k} w_{ij} = 1$. The expert v_i's opinion in the $(t+1)$th iteration can be described as:

$$x_i^{t+1} = w_{i1}x_1^t + w_{i2}x_2^t + ... + w_{ik}x_k^t, i = 1, 2, ..., k; t = 0, 1, 2, ... \tag{1}$$

Eq. (1) can be equally written as:

$$X^{t+1} = W \times X^t, t = 0, 1, 2, ... \tag{2}$$

where the weight matrix is $W = (w_{ij})_{k\times k}$ and the opinion vector at time t is $X^t = (x_1^t, x_2^t, ..., x_k^t)^T$. When W does not change over time, Eq. (1) or Eq. (2) is called the DeGroot model.

2.3.2. The bounded confidence models

Generally, the weight matrix W in Eq. (2) may change over time or with opinions. Individuals may not accept other individual opinions completely, but consider these opinions in part and remain some of their own opinions to form new opinions. The bounded confidence models assume that each expert only communicates with some experts sharing similar opinions and ignores other experts with completely different opinions. Therefore, the bounded confidence models including the DW model [44] and the HK model [46] have been proposed.

In the DW model, two experts whose opinion distance is smaller than a threshold τ_1 will be chosen randomly to communicate and update new opinions in each round. The updated rules for opinions x_i^{t+1} and x_j^{t+1} of a pair of experts v_i and v_j are defined as follows:

$$\begin{cases} x_i^{t+1} = \alpha x_i^t + (1-\alpha)x_j^t \\ x_j^{t+1} = \alpha x_j^t + (1-\alpha)x_i^t \end{cases}, i, j = 1, 2, ..., k, i \neq j; t = 0, 1, 2, ... \tag{3}$$

where α is a predetermined convergence parameter. The bounded confidence threshold τ_1 shows that experts will exchange their opinions effectively with other experts who share similar opinions, and the convergence parameter α reflects the degree to which experts reserve their opinions.

In the HK model, the updated opinion of expert v_i is obtained by averaging opinions of neighbors whose opinion distance with expert v_i is smaller than a threshold τ_2. The updated opinion x_i^{t+1} is denoted as follows:

$$x_i^{t+1} = \frac{1}{\left\| N_i^t \right\|} \sum_{j \in N_i^t} x_j^t, i = 1, 2, ..., k; t = 0, 1, 2, ... \tag{4}$$

where $N_i^t = \{j | \left| x_i^t - x_j^t \right| \leq \tau_2\}$ is the confidence set and $\left\| N_i^t \right\|$ is the cardinality of the set N_i^t.

The main difference between the DW model and the HK model is the ways of opinion fusion. The former claims that whether a pair of individuals compromise with each other, while the latter holds that individual opinions move towards the average opinions of their neighbors.

3. Trust propagation mechanism in social networks based on opinion reliability

In this section, a trust propagation mechanism in social networks considering opinion reliability is proposed. Section 3.1 details the motivation for opinion reliability and proposes the definition and properties of opinion reliability. The trust propagation and aggregation methods based on opinion reliability in social networks are presented in Section 3.2.

3.1. Opinion reliability

3.1.1. Motivation for opinion reliability

In GDM problems, opinions of leaders with high status and power have greater impact on the decision result due to larger weights assigned to them. Meanwhile, opinions of grassroots staff with smaller weight and more expertise are not well considered. Taking a production planning GDM problem as an example, four experts including general manager, manufacturing manager, purchasing manager and workshop manager are invited to determine the optimal production planning for the production workshop. The four experts belong to three management levels and are defined as v_1, v_2, v_3 and v_4, as shown in Fig. 4. Generally, the weights assigned to experts depend on power, status and other factors. The higher the management level of an expert is, the larger the weight assigned to the expert is. Therefore, we assume that the weights of four experts meet the condition: $w_1¿w_2=w_3¿w_4$, i.e., w_1 is much higher than w_4. Meanwhile, we assume that general manager is trusted by the other experts, not vice versa. Manufacturing manager and purchasing manager trust each other due to the same management level.

The general manager makes overall strategy by obtaining information mostly from the changing market. General manager's opinion will be affected by the changing market environment or competitors' strategy, so that his/her opinion changes constantly and varies greatly. Therefore, general manager is likely to make an unreasonable decision for the production planning due to lack of knowledge about the actual capacity of the workshop. However, the change of workshop manager's opinion may be

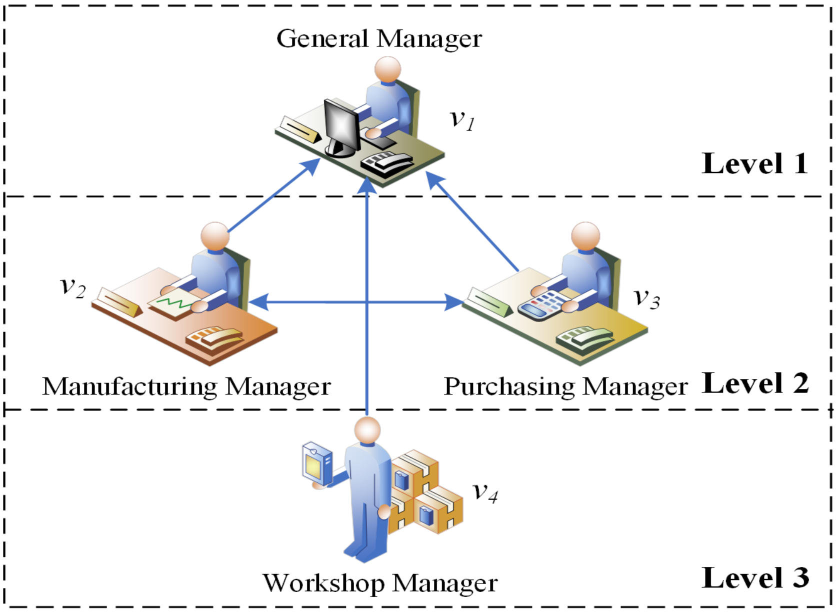


Fig. 4. The example for the optimal production planning decision.

small and quite different from that of general manager. The workshop manager knows more about the characteristics of the workshop due to the rich experience in the product manufacturing. Therefore, the workshop manager is confident to determine the optimal production planning, such as product types and quantities, which means that workshop manager's opinion may be more reliable than those of other three experts.

In the opinion evolution process, there are two possible situations for the production planning decision:

(i) When there is a large opinion distance between general manager and workshop director: although workshop manager trusts general manager, the workshop manager would reserve opinion more in each round due to the high self-confidence level. Meanwhile, manufacturing manager and purchasing manager not only trust general manager, but also trust each other. However, their self-confidence levels are not as high as workshop director, so that their opinions will gradually move closer to general manager's opinion. In this case, the final opinions are likely to become polarized and the consensus cannot be reached.

(ii) When there is little difference between general manager's opinion and workshop director's opinion: general manager with the larger weight is trusted by other three experts, causing that other three experts' opinions will converge towards general manager's opinion. On the one hand, general manager's opinion will change greatly when the market trend or competitors' strategy changes greatly, resulting in the slow convergence of opinions or the failure to reach a consensus. On the other hand, although opinions may evolve at a fast speed, the final stable opinion may be very close to the original opinion of the general manager, which leads to the generation of unreliable opinions.

Therefore, most studies have only considered the weights of experts or ignored the impact of opinion reliability on making decisions, which caused the unreliable solution. Based on the above mentioned, opinion stability of experts and communication with more reliable grassroots staff should be taken into account in GDM problems. In this paper, we provide the definition of opinion reliability and discuss the properties of opinion reliability.

3.1.2. The definition and properties of opinion reliability

Reliability is commonly defined as the probability that a product, system, or service will perform its intended function adequately for a certain period [56]. In other words, the ability of a system to operate in a defined environment without failure is called reliability. However, the definition of reliability in engineering and mathematics is fundamentally different from that in human behavior, which refers to the stability and validity of human behavior. Expert reliability tends to be mistaken for the weights of experts in the existing studies. In GDM problems, the weights of experts mean that the relative importance to other experts and the sum of weights is 1, that is, experts' weights need to be normalized. However, opinion reliability mainly depends on the individual themselves, which should not be normalized. In general, the weight is often related to status, power and reputation of the expert, while opinion reliability depends on the expertise, intelligence and experience of experts. The definition of opinion reliability and the relative properties are as follows.

Definition 7. The opinion reliability of an expert at time $(t+1)$ in a group is defined as a combination of the stability similarity and the weighted similarity. The stability similarity is the difference between the experts' opinions at time $(t+1)$ and at time t. Meanwhile, the weighted similarity is the distance between the expert's opinion at time $(t+1)$ and opinions of other reliable experts at time t.

The former indicates the stability of the expert's opinion in the feedback mechanism, and the latter similarity reflects the social acceptability of the expert by other experts in a group. Definition 7 shows that reliability of an expert depends not only on the stability of the expert's opinion, but also opinions of other experts in this group. It is worth noting that the rumors may be also reliable for rumormongers. This paper mainly considers the evolution of non-rumor opinions.

To describe quantitatively the opinion reliability in Definition 7, the two similarities are constructed as follows.

Definition 8. The stability similarity S_i^{t+1} is denoted by the similarity between the opinion at time $(t+1)$ and the opinion at time t for expert v_i, which is defined as

$$S_i^{t+1} = 1 - |x_i^{t+1} - x_i^t|, i = 1, 2, ..., k; t = 0, 1, 2, ... \tag{5}$$

where $|x_i^{t+1} - x_i^t|$ represents the distance representing the difference between x_i^{t+1} and x_j^t. The larger the value S_i^{t+1} is, the more similar the opinions of v_i at time $(t+1)$ and time t are.

Definition 9. The weighted similarity WS_i^{t+1} between expert v_i's opinion at time $(t+1)$ and opinions of other reliable experts at time t is denoted as

$$WS_i^{t+1} = 1 - \sqrt{\sum_{j\in H, H=\{j|R_j^{t-1}\geq r*\}} \frac{w_{ij}}{\sum\limits_{j\in H} w_{ij}} (x_i^{t+1} - x_j^t)^2}, i = 1,2,...,k; t = 0,1,2,... \tag{6}$$

where H is a set of more reliable experts whose opinion reliability are not less than the reliability threshold $r*$, i.e., $H = \{j|R_j^{t-1} \geq r*\}$, and $\frac{w_{ij}}{\sum_{j\in H} w_{ij}}$ is the normalized weight of expert v_j assigned by experts v_i.

Definition 10. The opinion reliability of x_i^t provided by expert v_i at time t is denoted as

$$R_i^t = \theta S_i^t + (1-\theta) WS_i^t, i = 1,2,...,k; t = 1,2,3,... \tag{7}$$

where the adaptive coefficient $\theta \in [0,1]$ represents the proportion of stability similarity S_i^t to opinion reliability. The larger the value θ is, the greater the impact of experts' opinion changes on opinion reliability is. The initial reliability R_i^0 is determined by self-confidence value ε_i of v_i.

Property 1. When $x_i^t \in [0,1]$, R_i^t satisfies that:

(i) $0 \leq R_i^t \leq 1$.

(ii) If expert v_i do not modify his/her opinion in the tth iteration and the opinion is exactly the same as opinions of the reliable experts in the $(t-1)$th iteration, then $R_i^t = 1$.

(iii) If there's the largest distance between the opinion of experts v_i in tth iteration and opinions of experts including H and v_i in the $(t-1)$th iteration, then $R_i^t = 0$.

Proof. (i) When $x_i^t \in [0,1]$, then we can get the two inequalities $0 \leq \left|x_i^t - x_i^{t-1}\right| \leq 1$ and $0 \leq S_i^t \leq 1$. For $\forall i (i = 1,2,...,k)$, the following two inequalities hold: $0 \leq \sqrt{\min_j\{(x_i^t - x_j^{t-1})^2\}} \leq \sqrt{\sum_{j\in H} \frac{w_{ij}}{\sum_{j\in H} w_{ij}} (x_i^t - x_j^{t-1})^2}$ and $\sqrt{\sum_{j\in H} \frac{w_{ij}}{\sum_{j\in H} w_{ij}} (x_i^t - x_j^{t-1})^2} \leq \sqrt{\max_j\{(x_i^t - x_j^{t-1})^2\}} \leq 1$, According to Eq. (6), we have $0 \leq WS_i^t \leq 1$. Furthermore, we get the inequality $0 \leq \min\{S_i^t, WS_i^t\} \leq \theta S_i^t + (1-\theta)WS_i^t \leq \max\{S_i^t, WS_i^t\} \leq 1$, i.e., $R_i^t \in [0,1]$.

(ii) If expert v_i remains his/her opinion in the tth iteration, which is exactly the same as opinions of the reliable experts in the $(t-1)$th iteration, the final state is available because they reach a consensus. In this case, two expressions hold: $x_i^t = x_j^{t-1}$ where $j \in H$ and $x_i^t = x_i^{t-1}$, which means that $R_i^t = \theta(1-0) + (1-\theta)(1-0) = \theta + (1-\theta) = 1$.

(iii) There are two cases: one is $x_i^t = 1$ ($x_i^{t-1} = 0$ and $x_j^{t-1} = 0$ where $j \in H$), and the other is $x_i^t = 0$

($x_i^{t-1} = 1$ and $x_j^{t-1} = 1$). However, neither of them will happen in the real life. When expert v_i has the same opinion as other experts in the $(t-1)$ th iteration, he/she does not revise the opinion too much in the t th iteration.

3.2. Trust propagation and aggregation methods in social networks

The trust relationships among experts may not only change the social network structure, but also affect the final opinion evolved from experts' opinions. This subsection presents the trust propagation and aggregation methods considering opinion reliability to obtain the complete social trust network between experts.

Most SNGDM studies have paid more attention to the trust propagation and aggregation methods and ignored the composition of the trust value, in which the trust values between experts are predetermined in the initial state. However, the predefined trust may not reasonable because experts do not accurately quantify the trust degrees. Meanwhile, the trust value may be affected by opinion similarity between experts, leadership, and reliability of experts. Therefore, we define the comprehensive trust degree integrating opinion similarity, reliability and centrality degree.

Definition 11. The opinion similarity Sm_{ij}^t between two experts v_i and v_j at time t is defined as

$$Sm_{ij}^t = 1 - D(x_i^t, x_j^t) = 1 - |x_i^t - x_j^t|, i, j = 1, 2, ..., k, i \neq j; t = 0, 1, 2, ... \quad (8)$$

Definition 12. The centrality degree CD_i^t of expert v_i at time t is defined as

$$CD_i^t = \frac{d_{v_i^+}^t}{d_{v_i^+}^t + d_{v_i^-}^t}, i = 1, 2, ..., k; t = 0, 1, 2, ... \quad (9)$$

where $d_{v_i^+}^t$ is the indegree and $d_{v_i^-}^t$ is the outdegree of expert v_i at time t, and CD_i^t represents the proportion of the number of experts who communicate with v_i to the total number of experts who has the direct and indirect relationships with v_i.

Definition 13. In a social trust network, the comprehensive trust degree t_{ij}^t of expert v_i on expert v_j at time t depends on the leadership of v_j, the reliability of v_j and the opinion similarity between them, is defined as

$$t_{ij}^t = \alpha Sm_{ij}^t + \beta R_j^t + (1 - \alpha - \beta) CD_j^t, i, j = 1, 2, ..., k, i \neq j, t = 0, 1, 2, ... \quad (10)$$

where $\alpha \in [0, 1], \beta \in [0, 1]$ and $(1 - \alpha - \beta) \in [0, 1]$ are the similarity coefficient, reliability coefficient and centrality coefficient, respectively. R_j^t and CD_j^t reflect the opinion reliability and centrality degree of expert v_j.

In a social trust network, trust relationship exists between any two experts v_i and v_j, even though there's no path from v_i and v_j. Celebrity effect is a case in point: celebrities have strong public influence due to higher visibility in society so that other people may trust them in GDM problems, even though they do not have the opportunity to interact directly with celebrities. However, when there's a trust value but not a direct connecting edge between experts, the trust relationship cannot propagate in social networks. Therefore, the direct connecting edge between two experts is a necessary condition for the trust propagation. Then we define the updated trust matrix to derive the trust propagation method.

Definition 14. The real comprehensive trust value in the trust propagation process is updated according to the trust degree t^t_{ij} and the adjacency matrix $A^t = (a^t_{ij})_{k\times k}$. The updated trust matrix is denoted by $T^t = (\tilde{t}^t_{ij})_{k\times k}$, and $\tilde{t}^t_{ij} = t^t_{ij} \times a^t_{ij}$ is the real trust value of expert v_i on expert v_j which is used in the trust propagation process, where $\tilde{t}^t_{ij} = t^t_{ij}$ when $a^t_{ij} = 1$, otherwise $\tilde{t}^t_{ij} = 0$ when $a^t_{ij} = 0$.

In GDM problems, experts often communicate with several people in the group rather than all of them. There may be an indirect relationship or irrelevant between two experts. In other words, one expert may not directly communicate with all experts in a real social network, which means that the trust matrix $T^t = (\tilde{t}^t_{ij})_{k\times k}$ is incomplete. The indirect trust relationship between any two experts can be established by some mediators. Therefore, we design the trust propagation and aggregation mechanism to obtain a complete trust matrix.

Triangular norm [64] (briefly t-norm) is one of the most popular aggregation operators, which is often used in the trust propagation process. A binary function $T : [0,1]^2 \rightarrow [0,1]$ is called a t-norm if neutrality, commutativity, associativity and monotonicity are satisfied. In this paper, we use the Einstein product $\otimes_\varepsilon$ as the t-norm to derive the trust propagation process. The Einstein product operator aggregating elements a and b is as follows:

$$E_\otimes(a,b) = a \otimes_\varepsilon b = \frac{a\cdot b}{1+(1-a)\cdot(1-b)}, \forall (a,b) \in [0,1]^2. \tag{11}$$

And a property of Einstein product is that the greatest of all t-norms is the minimum operator, i.e.,

$$E_\otimes(a,b) \leq \min(a,b). \tag{12}$$

$E_\otimes(a,b)$ is used to aggregate two elements, and the associativity property is extended to the case of n elements $\{a_1, a_2, ..., a_n\}$:

$$E_\otimes(a_1,a_2,...,a_n) = \frac{\prod_{i=1}^n (1+a_i) - \prod_{i=1}^n (1-a_i)}{\prod_{i=1}^n (1+a_i) + \prod_{i=1}^n (1-a_i)}, \tag{13}$$

or

$$E_{\otimes}(a_1, a_2, ..., a_n) = \frac{2\prod_{i=1}^{n} a_i}{\prod_{i=1}^{n}(2-a_i) + \prod_{i=1}^{n} a_i}. \tag{14}$$

The monotonicity property of t-norm also holds with n elements:

$$E_{\otimes}(a_1, a_2, ..., a_n) \leq \min(a_1, a_2, ..., a_n). \tag{15}$$

The above formula about Einstein product is proposed on the assumption that the trust information is fully propagated to the next node. However, the information may be lost in the propagation process so that the propagation efficiency of trust information may not be as high as expected, as shown in Fig. 5. Some researchers claimed that the relationship strength may affect the propagation efficiency [65, 66]: the stronger the relationship strength between experts is, the higher the propagation efficiency is. However, the relationship strength is often predefined or presupposed, which is not well quantified. The opinion similarity represents the relationship strength quantitatively. If experts share the more similar opinions, the willingness of sharing information and the acceptance for information will be higher to experts. Meanwhile, Tan et al. [66] argued that the number of mediators between experts will decrease the propagation efficiency of information. Therefore, we use opinion similarity between experts to measure the relationship strength and define the propagation efficiency function. The number of mediators between experts is considered as an important factor affecting the trust aggregation of multiple paths.

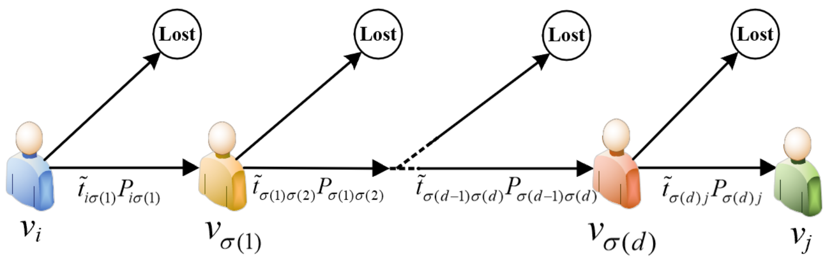


Fig. 5. The trust propagation process.

Inspired by Liu et al. [16], we define the propagation efficiency based on opinion similarity using the sinusoidal function:

Definition 15. The trust propagation efficiency from v_i to v_j at time t is defined as:

$$P_{ij}^t(Sm_{ij}^t) = \sin(\frac{\pi Sm_{ij}^t}{2}), i, j = 1, 2, ..., k, i \neq j, t = 0, 1, 2, ... \tag{16}$$

where Sm_{ij}^t is the opinion similarity between experts v_i and v_j at time t.

From $Sm_{ij}^t \in [0, 1]$, we have $0 \leq \frac{\pi \times Sm_{ij}^t}{2} \leq \frac{\pi}{2}$. The function $f(x) = \sin x$ is a monotonic increasing function with decreasing slope when $x \in [0, \frac{\pi}{2}]$. Therefore, the function $P_{ij}^t(Sm_{ij}^t)$ keeps the ordinate of $f(x) = \sin x$ unchanged and narrows the abscissa by $\frac{2}{\pi}$ times, which is also a monotonic increasing curve. When $Sm_{ij}^t = 0$, $P_{ij}^t(0) = 0$; and when $Sm_{ij}^t = 1$, $P_{ij}^t(1) = 1$. Obviously, we can conclude that $P_{ij}^t(Sm_{ij}^t) \in [0, 1]$, as shown in Fig. 6. The function $P_{ij}^t(Sm_{ij}^t)$ reflects the impact of opinion similarity on trust propagation efficiency: the marginal growth value of trust propagation efficiency decreases with the increase of opinion similarity. In addition, the decrease of similarity will make the trust propagation efficiency decrease rapidly.

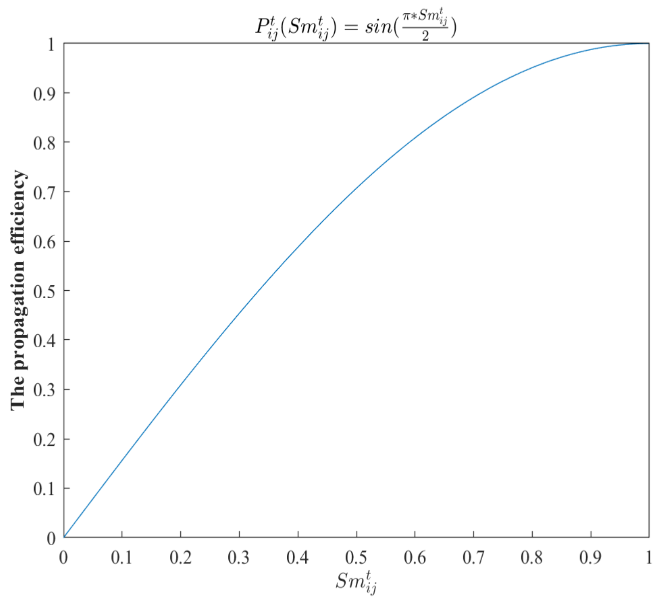


Fig. 6. The trust propagation efficiency function curve.

However, the existing studies only roughly multiply propagation efficiency by Einstein product operator, which has limited influence on the final trust information and cannot truly reflect the actual propagation process. Therefore, we define $\vec{t}_{ij}$ as the actual trust value propagated from v_i and v_j at time t as follows:

$$\vec{t}_{ij} = \tilde{t}_{ij} \cdot P_{ij}^t(Sm_{ij}^t), i, j = 1, 2, ..., k, i \neq j, t = 0, 1, 2, ... \tag{17}$$

Suppose that $\left\{(v_i, v_{\sigma(1)}), (v_{\sigma(1)}, v_{\sigma(2)}), (v_{\sigma(2)}, v_{\sigma(3)}), ..., (v_{\sigma(d)}, v_j)\right\}$ is a directed path from v_i to v_j, then t-norm-based trust propagation value $\hat{t}_{ij}$ at time t considering the propagation efficiency can be written as:

$$\begin{aligned} \hat{t}_{ij} &= E_{\otimes}(\vec{t}_{i\sigma(1)}, \vec{t}_{\sigma(1)\sigma(2)}, ..., \vec{t}_{\sigma(d)j}) \\ &= \frac{2 \cdot \vec{t}_{i\sigma(1)} \cdot \vec{t}_{\sigma(d)j} \prod_{z=1}^{d} \vec{t}_{\sigma(z-1)\sigma(z)}}{(2-\vec{t}_{i\sigma(1)}) \cdot (2-\vec{t}_{\sigma(d)j}) \prod_{z=1}^{d} \vec{t}_{\sigma(z-1)\sigma(z)} + \vec{t}_{i\sigma(1)} \cdot \vec{t}_{\sigma(d)j} \prod_{z=1}^{d} \vec{t}_{\sigma(z-1)\sigma(z)}}. \end{aligned} \tag{18}$$

Specially, if there is only a mediator from v_i to v_j, the trust propagation value $\hat{t}_{ij}$ at time t is computed by:

$$\hat{t}_{ij} = E_{\otimes}(\vec{t}_{i\sigma(1)}, \vec{t}_{\sigma(1)j}) = \frac{\vec{t}_{i\sigma(1)} \cdot \vec{t}_{\sigma(1)j}}{1 + (1 - \vec{t}_{i\sigma(1)}) \cdot (1 - \vec{t}_{\sigma(1)j})}, i, j = 1, 2, ..., k, i \neq j, t = 0, 1, 2, ... \tag{19}$$

In a social network, there may be several paths between experts, which means that we can get several different trust propagation values. Therefore, we need to aggregate these values to obtain the final trust value from v_i to v_j. The Order Weighted Averaging (OWA) operator is often used to compute the trust aggregation value. The larger the trust propagation value obtained by a path is, the higher the weight assigned to the path is. However, the number of mediators in a path may impact the trust aggregation process: the smaller the number of mediators is, the more accurate the obtained trust value is. Therefore, the path with less mediators should be assigned larger weight. The following optimization model is used to obtain the trust aggregation value:

$$\max t_{ij}^{t'}$$
$$s.t. \begin{cases} t_{ij}^{t'} = \sum_{k=1}^{h} w_{ij}^{kt} \times \hat{t}_{ij}^{kt} \\ \sum_{k=1}^{h} w_{ij}^{kt} = 1 \\ w_{ij}^{1t} \geq w_{ij}^{2t} \geq \cdots \geq w_{ij}^{ht} \\ 0 \leq w_{ij}^{kt} \leq 1 (k = 1, 2, ..., h) \\ i, j = 1, 2, 3, ..., k, i \neq j, t = 0, 1, 2, ... \end{cases}, \tag{20}$$

where w_{ij}^{kt} and $\hat{t}_{ij}^{kt}$ is the weight and trust aggregation value of the path with the k^{th} least number of mediators at time t. $t_{ij}^{t'}$ is the final trust value aggregating all paths from v_i to v_j. Then we can get the trust matrix $T^{t'} = (t_{ij}^{t'})_{k\times k}$ and the weight matrix $W^t = (w_{ij}^t)_{k\times k}$ after normalization procedure.

Definition 16. The weight w_{ij}^t assigned to expert v_j by expert v_i at time t is obtained by the trust matrix $T^{t'} = (t_{ij}^{t'})_{k\times k}$ as follows:

$$w_{ij}^t = \frac{t_{ij}^{t'}}{\sum_{i=1}^{k} t_{ij}^{t'}}, i, j = 1, 2, ..., k, i \neq j, t = 0, 1, 2, ... \tag{21}$$

Furthermore, the weight matrix $W^t = (w_{ij}^t)_{k\times k}$ is obtained.

4. A novel group decision making framework of trust propagation and opinion dynamics in social networks

In this section, a novel GDM framework of trust propagation and opinion dynamics in social networks is proposed. Firstly, we design the social network evolution rules to illustrate the dynamic process of social networks in Section 4.1. Section 4.2 extends the traditional HK model based on opinion reliability to get the opinion evolution rules. Based on the trust propagation mechanism in Section 3, the general algorithm of the proposed model is presented in Section 4.3.

4.1. Social network evolution rules

In the CRP, social networks among experts are critical. In most studies, social networks are assumed to be static or predetermined for convenience to reach the consensus. However, social networks in the real life are dynamic, which means that the connection between experts may be broken or a new connection between experts who do not know each other may be established due to opinion changes.

Social network evolution rules mainly depend on two ways: random connection [37] and opinion distance [36]. Random connection reflects the stochastic characteristic of social networks but ignore the psychological characteristic of people. People tend to be willing to communicate with others who share a similar opinion, and that's the main idea of the latter way about opinion difference. Experts involved in the GDM usually have rich social experience and knowledge background. When an expert is unreliable, other experts may be reluctant to communicate with the expert even though they share similar opinions due to lower reliability of the expert. Experts involved in decision making usually have a wealth of social experience and knowledge background, and they may pay more attention to the reliability of other experts than other people. Therefore, the reliability is also the critical factor affecting the social network evolution.

Therefore, we design the social network evolution rules considering two factors: (i) opinion similarity Sm_{ij}^t between opinions of experts v_i and v_j, and (ii) reliability R_j^t of expert v_j. We set the opinion similarity threshold $Sm*$ and the reliability threshold $R*$ as the judgment rules. Without loss of bias, 0.5 is considered to be the fairest threshold for $Sm*$ and $R*$. In real problems, the thresholds can be determined according to the actual situation or previous experience. The social network evolution rules are divided into two cases:

(i) When there is no connection from expert v_i to v_j in the $(t-1)$th iteration, that is $a_{ij}^{t-1} = 0$.

(a) If $Sm_{ij}^t < Sm*$ and $R_j^t < R*$ hold, then expert v_i still will not communicate with the unreliable expert who has different opinions. In this case, expert v_i will keep disconnecting with expert v_j in the tth iteration, i.e., $a_{ij}^t = 0$.

(b) If $Sm_{ij}^t \geq Sm*$ and $R_j^t \geq R*$ hold, then expert v_i is willing to communicate with expert v_j because v_i not only has a similar opinion but is reliable enough. In this case, there will be an edge between expert v_i and v_j in next round, i.e., $a_{ij}^t = 1$.

(c) If neither (a) nor (b) is satisfied, then expert v_i may be willing to communicate with expert v_j due to the similar opinions or the reliability of v_j. In this case, the connection from expert v_i to v_j will be established with a probability γ_{ij} in the tth iteration, i.e., $a_{ij}^t = \gamma_{ij}$. The probability γ_{ij} will be provided in the initial state.

(ii) When there is an edge from expert v_i to v_j in the $(t-1)$th iteration, that is $a_{ij}^{t-1} = 1$.

(a) If $Sm_{ij}^t \geq Sm*$ and $R_j^t \geq R*$ hold, then expert v_i is willing to communicate with expert v_j

because not only has a similar opinion but is reliable enough. In this case, keep connecting from expert v_i to v_j in next round, i.e., $a_{ij}^t = 1$.

(b) If $Sm_{ij}^t < Sm*$ and $R_j^t < R*$ hold, then expert v_i will not communicate with the expert who is unreliable and has the different opinions. In this case, the connection from expert v_i to v_j will be broken in the t th iteration, i.e., $a_{ij}^t = 0$.

(c) If neither (a) nor (b) is satisfied, then the robustness of the connection from expert v_i to v_j will be questioned. Expert v_i may be refuse to communicate with expert v_j because the distance of their opinions is large or v_j is not reliable enough. In this case, the connection may be broken with a probability $(1 - \gamma_{ij})$ in the tth iteration, i.e., $a_{ij}^t = 1 - \gamma_{ij}$.

The initial social network evolution rules are expressed as follows:

$$a_{ij}^t = \begin{cases} 1, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t \geq R^* \\ \gamma_{ij}, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t < R^* \\ \gamma_{ij}, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t \geq R^* \\ 0, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t < R^* \end{cases} ;when \;\; a_{ij}^{t-1} = 0 \tag{22}$$

$$a_{ij}^t = \begin{cases} 1, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t \geq R^* \\ 1 - \gamma_{ij}, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t < R^* \\ 1 - \gamma_{ij}, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t \geq R^* \\ 0, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t < R^* \end{cases} ;when \;\; a_{ij}^{t-1} = 1 \tag{23}$$

Remarkably, the element a_{ij}^t of the updated adjacency matrix might not be 0 or 1 due to $\gamma_{ij} \in [0, 1]$. Therefore, we take 0.5 as the bound and divide the social network evolution rules into two cases: $a_{ij}^{t-1} \geq 0.5$ and $a_{ij}^{t-1} < 0.5$. The updated social network evolution rules are as follows:

$$a_{ij}^t = \begin{cases} 1, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t \geq R^* \\ \gamma_{ij}, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t < R^* \\ \gamma_{ij}, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t \geq R^* \\ 0, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t < R^* \end{cases} ;when \;\; a_{ij}^{t-1} < 0.5 \tag{24}$$

$$
a_{ij}^t = \begin{cases} 1, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t \geq R^* \\ 1-\gamma_{ij}, & \text{if } Sm_{ij}^t \geq Sm^* \wedge R_j^t < R^* \\ 1-\gamma_{ij}, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t \geq R^* \\ 0, & \text{if } Sm_{ij}^t < Sm^* \wedge R_j^t < R^* \end{cases} ; when \;\; a_{ij}^{t-1} \geq 0.5 \tag{25}
$$

4.2. Opinion evolution rules

In the opinion evolution process, expert's opinions evolved from opinions of other experts in the group, like the DW model and the HK model. The DW model is an asynchronous updating process that randomly selects a pair of experts according to the opinion distance for opinion evolution, which leads to a high number of iterations affecting the efficiency of consensus reaching. In contrast, the HK model adopts a synchronous approach to evolve opinions, where all experts' opinions will be updated in each round. Therefore, we adopt the HK model to design the opinion evolution rules.

However, the traditional HK model assume that each expert only depends on individuals whose opinions differ at least within a certain boundary. Individuals will also consider their own original opinions as well as more reliable experts' opinions when communicating with other individuals. Therefore, we propose an extended HK model based on opinion reliability to solve the above defects. The opinion evolution rule is as follows:

$$
x_i^{t+1} = \rho \frac{\sum_{j_1 \in \{N_1^t \cup i\}} w_{ij_1}^t x_{j_1}^t}{\sum_{j_1 \in \{N_1^t \cup i\}} w_{ij_1}^t} + (1-\rho) \frac{\sum_{j_2 \in \{N_2^t \cup i\}} w_{ij_2}^t x_{j_2}^t}{\sum_{j_2 \in \{N_2^t \cup i\}} w_{ij_2}^t}, i = 1, 2, ..., k; t = 1, 2, 3, ... \tag{26}
$$

where $\rho \in [0, 1]$ is an adaptive coefficient. For convenience, this paper assumes all adaptive coefficients of experts are equal in each round. $w_{ij_1}^t$ and $w_{ij_2}^t$ is the expert v_{j_1}'s weight and expert v_{j_2}'s weight assigned by expert v_i at time t, respectively. N_1^t is the similarity set of experts pairs who meet the condition that the opinion distance with expert v_i is less than $(1 - Sm^*)$ i.e., $N_1^t = \{(i, j_1) \,\|x_i^t - x_{j_1}^t| \leq (1 - Sm^*)\,, i, j_1 = 1, 2, ..., k; t = 1, 2, 3, ...\}$. N_2^t is the reliability set of expert whose reliability is more than $R*$, i.e., $N_2^t = \{j_2 | R_{j_2}^t \geq R*, j_2 = 1, 2, ..., k; t = 1, 2, 3, ...\}$. The opinion of expert depends on his/her own opinion and opinions of other experts belonging to N_1^t or N_2^t. The extended HK model will be degenerated into the traditional HK model when $\rho = 1$ and all $w_{ij_1}^t$ are equal.

Compared to the HK model, the extended HK model not only takes the expert's own opinion into account, but also pays attention to the impact of the reliability expert on the opinion evolution. Therefore, the extended HK model is closer to the interaction between individuals in reality.

4.3. The algorithm of the proposed model

Based on the evolution rules, we propose a novel GDM framework of trust propagation and opinion dynamics, which is shown in Fig. 7, and the steps of the proposed model are shown in Algorithm 1.

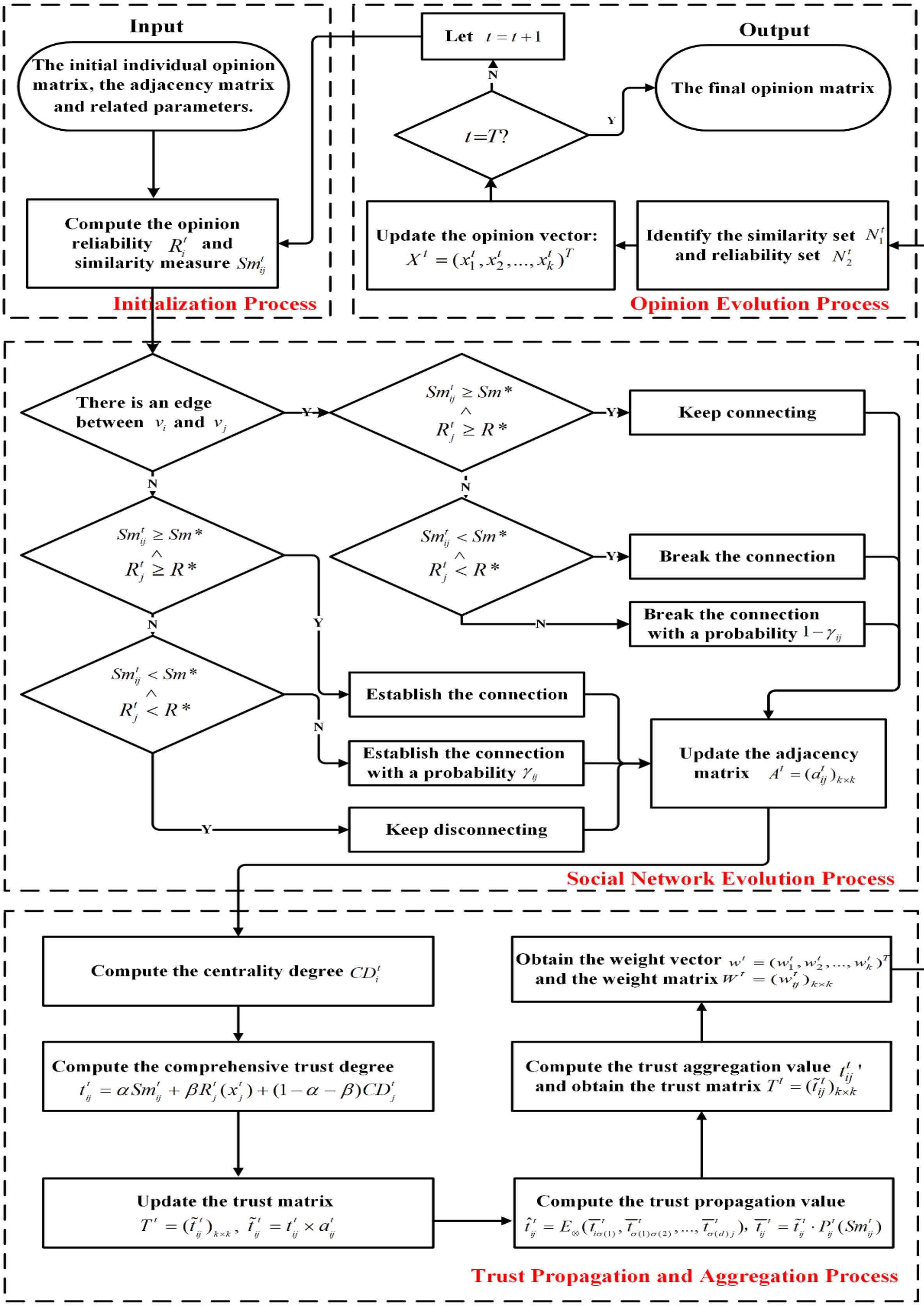


Fig. 7. Framework of trust propagation and opinion dynamics in social networks.

Algorithm 1. The algorithm of the proposed model.

Input: The initial individual opinion vector $X^0 = (x_1^0, x_2^0, ..., x_k^0)^T$, the initial weight matrix W^0, the initial adjacency matrix $A^0 = (a_{ij})_{k\times k}$, the reliability factor r^*, the opinion similarity threshold $Sm*$, the reliability threshold $R*$, the self-confidence vector $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)^T$, the adaptive coefficient θ, the similarity coefficient α, the reliability coefficient β, the adaptive coefficient ρ, the initial iteration time $t = 0$ and the maximum iteration T.
Output: The final opinion vector $X^* = (x_1^*, x_2^*, ..., x_k^*)^T$ in a stable state.
Step 1. Initialization. Compute the S_i^t, WS_i^t, R_i^t and Sm_{ij}^t by using Eqs. (5)-(8).
Step 2. Social network evolution. Based on the adjacency matrix $A^t = (a_{ij}^t)_{k\times k}$, the opinion similarity Sm_{ij}^t and the opinion reliability R_i^t obtained by **Step 1**, update the social network according to the social network evolution rules by using Eqs. (24) and (25).
Step 3. Trust propagation and aggregation. Compute the centrality degree CD_i^t and the comprehensive trust degree t_{ij}^t by using Eqs. (9) and (10). According to Definition 16, we obtain the real trust value $\tilde{t}_{ij}^t$ and the updated trust matrix $T^t = (\tilde{t}_{ij}^t)_{k\times k}$. The trust propagation value $\hat{t}_{ij}^t$ is computed by using Eqs. (16)-(18). Furthermore, solve the optimization model in Eq. (20) to get the trust aggregation matrix $T^{t'} = (t_{ij}^{t'})_{k\times k}$. Finally, the weight matrix $W^t = (w_{ij}^t)_{k\times k}$ is derived by using Eq. (21).
Step 4. Opinion evolution. According to the opinion similarity threshold $Sm*$ and the reliability threshold $R*$, identify the similarity set N_1^t and reliability set N_2^t and update the opinions by using Eq. (26). Then we get the opinion vector $X^t = (x_1^t, x_2^t, ..., x_k^t)^T$.
Step 5. Update the iteration time. Let $t = t + 1$. If $t \neq T$, then return to **Step 1**. Otherwise, go to the next step.
Step 6. End.

5. Numerical analysis

This section applies the proposed model to two numerical experiments about supplier performance evaluation and Zachary's karate club. The case of supplier performance evaluation is to describe the steps of our proposal in detail and the real application about Zachary's karate club in real-world is to illustrate the usefulness of our proposal.

5.1. The case of supplier performance evaluation

A numerical experiment about supplier performance evaluation is conducted to the application of the proposed approach. To effectively manage and supervise suppliers effectively, a company invites six experts $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ with different professional backgrounds to evaluate the performance of the current supplier. The relative social trust network is shown in Fig. 8. According to the initial social trust network, the initial adjacency matrix is:

$$A^0 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The initial opinions $X^0 = (x_1^0, x_2^0, ..., x_6^0)^T$ are randomly generated within several specified interval as follows: $x_1^0 = [0.05, 0.10]$, $x_2^0 = [0.10, 0.15]$, $x_3^0 = [0.30, 0.35]$, $x_4^0 = [0.50, 0.55]$, $x_5^0 = [0.70, 0.75]$, and $x_6^0 = [0.90, 0.95]$. The initial individual opinion vector is $X^0 = (0.07, 0.14, 0.33, 0.51, 0.71, 0.93)^T$. The self-confidence vector is $\varepsilon = (0.81, 0.12, 0.17, 0.35, 0.55, 0.91)^T$, and the relative parameters are as follows: $\theta = 0.5$, $\alpha = \gamma = 0.6$, $\beta = 0.3$, $r* = R* = 0.8$, $\rho = 0.9$, $Sm* = 0.6$ and $T = 1000$. The initial weight matrix W^0 is generated randomly in the interval [0, 1], which is as follows:

$$W^0 = \begin{bmatrix} 0.2865 & 0.0231 & 0.1409 & 0.1823 & 0.2276 & 0.1396 \\ 0.1753 & 0.2330 & 0.2478 & 0.1930 & 0.1078 & 0.0432 \\ 0.1294 & 0.1973 & 0.0933 & 0.2795 & 0.2931 & 0.0074 \\ 0.0896 & 0.2486 & 0.2832 & 0.1543 & 0.0779 & 0.1464 \\ 0.2442 & 0.2085 & 0.2562 & 0.0195 & 0.2119 & 0.0598 \\ 0.0618 & 0.1133 & 0.2305 & 0.2344 & 0.1636 & 0.1963 \end{bmatrix}.$$

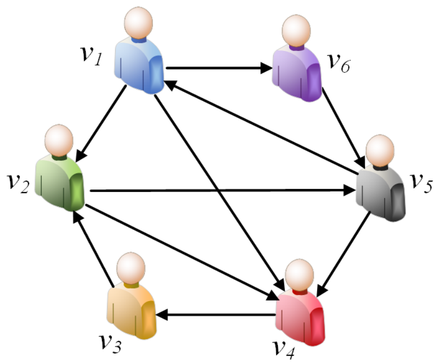


Fig. 8. The social network among experts.

The opinion evolution process is shown in **Steps 1-5**.

Step 1. Initialization.

Since the initial reliabilities and initial opinions are provided, we need to perform initial opinion evolution before proceeding to formal evolution. Based on the initial reliability R^0 and the initial opinion vector X^0, we get the similarity matrix Sm_{ij}^0 and obtain the opinion vector X^1. Firstly, we can get the following similarity matrix $SM^0 = (Sm^0_{ij})_{6\times6}$:

$$SM^0 = \begin{bmatrix} 1 & \mathbf{0.93} & \mathbf{0.74} & 0.56 & 0.36 & 0.14 \\ & 1 & \mathbf{0.81} & \mathbf{0.63} & 0.43 & 0.21 \\ & & 1 & \mathbf{0.82} & \mathbf{0.62} & 0.40 \\ & & & 1 & \mathbf{0.80} & 0.58 \\ & & & & 1 & \mathbf{0.78} \\ & & & & & 1 \end{bmatrix},$$

then we identify the similarity set $N^0_1 = \{(1,2),(1,3),(2,3),(2,4),(3,4),(3,5),(4,5),(5,6)\}$ due to $Sm* = 0.6$. According to the initial reliability $R^0 = \varepsilon = (\mathbf{0.81}, 0.12, 0.17, 0.35, 0.55, \mathbf{0.91})^T$ and the reliability threshold $R* = 0.8$, we can determine the reliability set $N^0_2 = \{v_1, v_6\}$. Therefore, we get the opinion vector $X^1 = (0.2216, 0.1835, 0.2158, 0.5609, 0.4603, 0.8247)^T$ by using Eq. (26). The initial opinion evolution is finished and let $t = t + 1$, i.e., $t = 1$.

Subsequently, the formal opinion evolution begins: we use Eqs. (5)-(8) to get the reliability $R^1 = (0.7121, 0.6050, 0.5858, 0.7900, 0.6403, \mathbf{0.8947})^T$ and the following similarity matrix $SM^1 = (Sm^1_{ij})_{6\times6}$:

$$SM^1 = \begin{bmatrix} 1 & 0.9619 & 0.9941 & 0.6607 & 0.7614 & \mathbf{0.3970} \\ & 1 & 0.9678 & 0.6226 & 0.7232 & \mathbf{0.3589} \\ & & 1 & 0.6549 & 0.7555 & \mathbf{0.3911} \\ & & & 1 & 0.8994 & 0.7363 \\ & & & & 1 & 0.6356 \\ & & & & & 1 \end{bmatrix}.$$

Step 2. Social network evolution.

Based on the adjacency matrix $A^1 = A^0$, opinion similarity Sm^1_{ij} and opinion reliability R^1_i obtained by **Step 1**, we can update the social network according to the social network evolution rules by using Eqs. (24) and (25). The adjacency matrix A^2 in the 2nd iteration is updated by A^1, which is as follows:

$$A^1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0.4 & 0.4 & 0.6 & 0.4 & 0.6 & 0 \\ 0.6 & 0.4 & 0.6 & 0.4 & 0.4 & 0 \\ 0.6 & 0.4 & 0.4 & 0.6 & 0.6 & 0 \\ 0.6 & 0.6 & 0.4 & 0.4 & 0.6 & 0.6 \\ 0.4 & 0.6 & 0.6 & 0.4 & 0.4 & 0.6 \\ 0.6 & 0.6 & 0.6 & 0 & 1 & 1 \end{bmatrix}.$$

Step 3. Trust propagation and aggregation.

First, we use Eqs. (9) and (10) to compute the centrality degree $CD^2 = (0.25, 0.50, 0.50, 0.75, 0.50, 0.50)^T$ and the following updated trust matrix T^2:

$$T^2 = \begin{bmatrix} 0.3355 & 0.3263 & 0.5011 & 0.2540 & 0.4173 & 0 \\ 0.4852 & 0.3326 & 0.4873 & 0.2420 & 0.2662 & 0 \\ 0.4933 & 0.3226 & 0.3303 & 0.3712 & 0.4074 & 0 \\ 0.4251 & 0.4113 & 0.2820 & 0.3648 & 0.5110 & 0.4523 \\ 0.2796 & 0.4056 & 0.4172 & 0.3127 & 0.3368 & 0.3741 \\ 0.3340 & 0.3202 & 0.3318 & 0 & 0.6998 & 0.9184 \end{bmatrix}.$$

Then the trust propagation value $\hat{t}_{ij}^2$ is computed by using Eqs. (16)-(18). Furthermore, we can solve the optimization model in Eq. (20) to get the following trust aggregation value matrix $T^{2'}$:

$$T^{2'} = \begin{bmatrix} 0.3355 & 0.3263 & 0.5011 & 0.2540 & 0.4173 & 0.0860 \\ 0.4852 & 0.3326 & 0.4873 & 0.2420 & 0.2662 & 0.0566 \\ 0.4933 & 0.3226 & 0.3303 & 0.3712 & 0.4074 & 0.0940 \\ 0.4251 & 0.4113 & 0.2820 & 0.3648 & 0.5110 & 0.4523 \\ 0.2796 & 0.4056 & 0.4172 & 0.3127 & 0.3368 & 0.3741 \\ 0.3340 & 0.3202 & 0.3318 & 0.7602 & 0.6998 & 0.9184 \end{bmatrix}.$$

Finally, the following weight matrix W^2 is derived by using Eq. (21):

$$W^2 = \begin{bmatrix} 0.3314 & 0.1171 & 0.1660 & 0.0864 & 0.1264 & 0.0417 \\ 0.1608 & 0.3589 & 0.1614 & 0.0823 & 0.0806 & 0.0274 \\ 0.1635 & 0.1158 & 0.3312 & 0.1263 & 0.1234 & 0.0456 \\ 0.1409 & 0.1476 & 0.0934 & 0.3401 & 0.1548 & 0.2192 \\ 0.0927 & 0.1456 & 0.1382 & 0.1064 & 0.3029 & 0.1813 \\ 0.1107 & 0.1149 & 0.1099 & 0.2585 & 0.2119 & 0.4847 \end{bmatrix}.$$

Step 4. Opinion evolution.

According to the opinion similarity threshold $Sm*$ and the reliability threshold $R*$, we can identify the similarity set $N_1^2 = \{(1,6),(2,6),(3,6)\}$ and reliability set $N_2^2 = \{6\}$ and update the opinions by using Eq. (26). Then we obtain the opinion vector $X^2 = (0.2870, 0.2299, 0.2880, 0.6474, 0.5816, 0.8146)^T$.

Step 5. Update the iteration time.

Let $t = t + 1$ and loop through **Steps 1-5** until reaching the consensus. We can get the final consensus opinion vector $X^* = (0.5855, 0.5855, 0.5855, 0.5855, 0.5855, 0.5855)^T$ when $t = 7$. The opinion evolution process from $t = 0$ to $t = 15$ is shown in Fig. 9, and the reliability evolution process from $t = 0$ to $t = 15$ is shown in Fig. 10. For the sake of convenience, the evolution process when $t > 15$ is omitted.

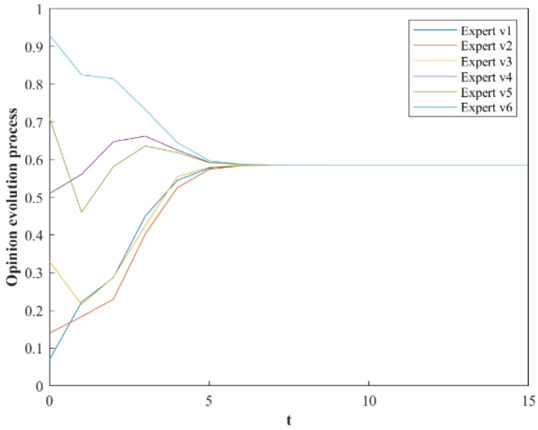


Fig. 9. The opinion evolution process about supplier performance evaluation.

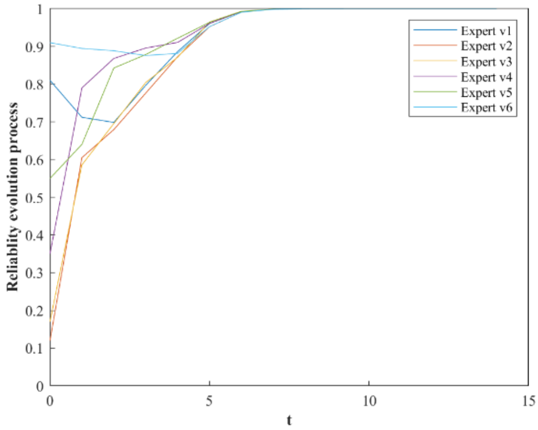


Fig. 10. The reliability evolution process about supplier performance evaluation.

It can be seen from Fig. 9 that six experts reach the consensus in the 8th iteration and the consensus opinion is 0.5855. Expert v_3 and $v_5$revise their opinions in $t = 1$ in the opposite direction of the consensus opinion, which do not contribute to reach the consensus in the first iteration. The reason is that expert v_3 is more similar to v_1 and assigns bigger weight to expert v_1 than v_5 so that expert v_1 will have a greater influence than v_5. Hence, it shows that considering opinion reliability of experts is reasonable, which not only avoids blindly revising opinions due to herd mentality, but also selectively modifies opinions according to the relationship between different reliable experts.

In Fig. 10, we find that although the reliability is closely related to the revised opinions of experts and other reliable experts, the reliability of experts as a whole shows an upward trend with the evolution of opinions. Moreover, the reliability of experts tends to be 1 in the consensus state along with social network evolution.

5.2. The real application example of Zachary's karate club

To illustrate the usefulness of our proposal, we apply the proposed model in the real application about Zachary's karate club in real-world [67]. The social network in Zachary's karate club is as shown in Fig. 11, which describes 78 social relationships of 34 members in a karate club. The Zachary's karate club data set contains only one social network graph, so the variables including initial opinions, initial weights and initial opinion reliabilities are obtained through multiple random sampling tests in order to ensure the independence and randomness of the samples. The remaining parameters are the same as in the numerical experiment of supplier performance evaluation.

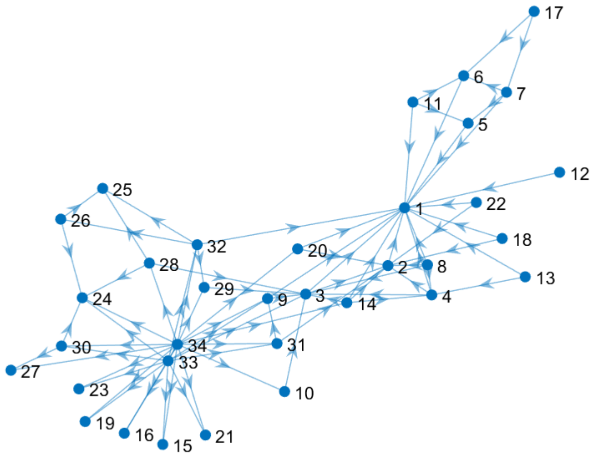


Fig. 11. The social network in Zachary's karate club.

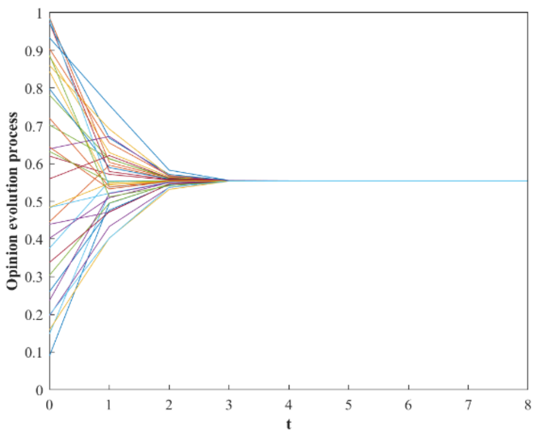


Fig. 12. The opinion evolution process about Zachary's karate club.

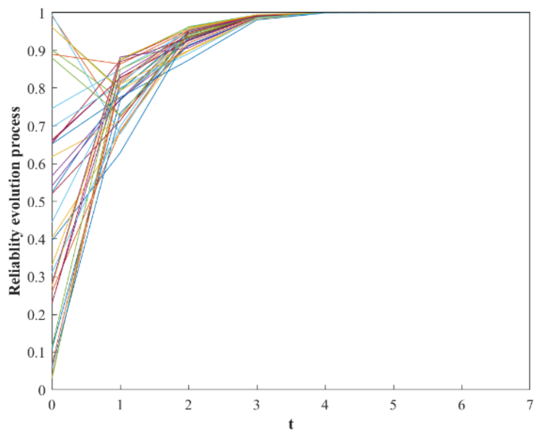


Fig. 13. The reliability evolution process about Zachary's karate club.

The opinion evolution process and the reliability evolution process are shown in Fig. 12 and Fig. 13, respectively. It can be seen from Fig. 12 that the karate club members reach the consensus in 5th iteration and the consensus opinion is a compromise of opinions of all members. The opinion modifications in first iteration are relatively large, which leads to the change of opinion similarity and reliability increase. The cumulative increase of opinion similarity and reliability promotes the rise of trust value and thus improves the convergence speed of opinions. From Fig. 13, we can find that the opinion reliabilities of experts are effectively corrected. The initial opinion reliabilities vary greatly due to the different experience backgrounds of experts, which have strong subjectivity. The structure of social networks and expert opinions can effectively adjust and correct the reliability in the group, which helps to speed up the reaching of consensus.

6. Simulation and Comparison analyses

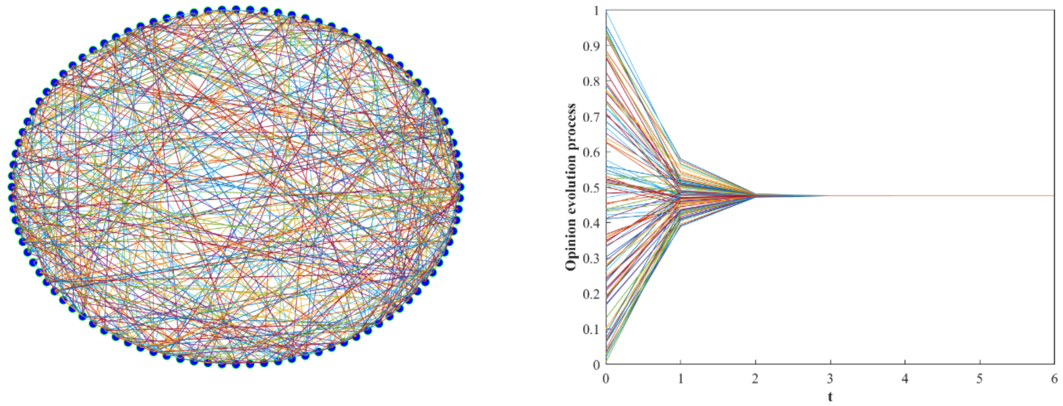
In this section, we apply the proposed model to two random networks to show its effectiveness and conduct three simulation experiments to explore the impact of different parameters on the results. In addition, the comparison and discussion between our proposed model and the existing SNGDM methods are carried out.

6.1. Random simulation

To prove the validity and rationality of the proposed approach, we design two simulation experiments about the complex network——Erdos–Renyi (ER) random network [38] and Watts–Strogatz (WS) small-world network [68]. The ER random network introduced by Erdos and Renyi provides the $G(n, p)$ model to create the random graph with n nodes and the link probability of a constant n. The WS small-world network proposed by Watts and Strogatz in 1998 is a one-dimensional regular network with certain randomness, which provides the $p(0 < p < 1)$ model. The transition from regular network to random network can be realized by adjusting random reconnection probability. The main idea is that n nodes in the network are connected with k neighbors, and other connections are made through connection probability $p(0 < p < 1)$. This model reflects the characteristics of social relationships in the real life: people who do not know each other can be connected by a chain of acquaintances, which is a small-world phenomenon.

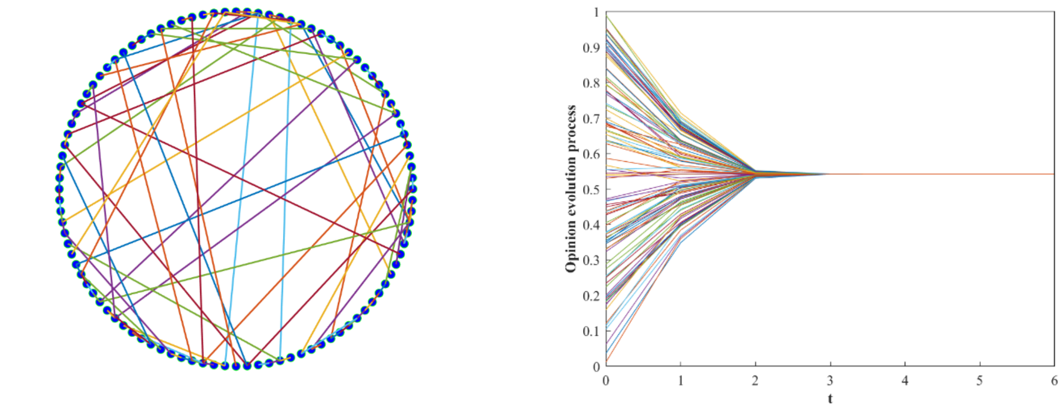
For the simulation, the result of ER random network in which $n = 100$, $p = 0.1$ and $\rho = 0.9$ is shown in Fig. 14. The result of WS small-world network in which $n = 100$, $k = 2$, $p = 0.5$ and $\rho = 0.65$ is obtained, which is shown in Fig. 15. The remaining parameters are the same as in the numerical experiment in Section 5. The initial opinions X^0, the self-confidence vector ε and the weight matrix W^0 are randomly generated from $[0, 1]$ for each experiment as shown in Fig. 7.

Fig. 14 shows the initial ER random social network and the opinion evolution process of ER random social network, and 15 shows the initial WS random social network and the opinion evolution process of WS random social network. Experts are represented by blue dots in social network graphs, and the connections between them are represented by different colored lines. From the opinion evolution process graphs, we find that the consensus opinions are reached within 5 iterations, which indicates that the proposed method is effective and universal. The evolution trend of opinions varies in different value of ρ. Since the value of ρ is larger in the ER random network than that in the WS random network, the opinions in the ER random network change more and converge faster than those in the WS random network.



(a) The initial ER random social network. (b) The opinion evolution process of ER.

Fig. 14. The social network and opinion evolution process of ER model.



(a) The initial WS random social network. (b) The opinion evolution process of WS.

Fig. 15. The social network and opinion evolution process of WS model.

6.2. Simulation experiment

6.2.1. Simulation experiment I: The impact of ρ on reaching the consensus

For the numerical experiment about supplier performance evaluation in Section 5, we obtain 11 opinion evolution results with different values of $\rho \in [0, 1]$ and compare with the opinion evolution process with the traditional HK model, which are shown in Fig. 16. The opinion evolution process when $\rho = 0.9$ is corresponding to the original supplier performance evaluation using the extended HK model, and the opinion evolution processes when $\rho = 1$ and $\rho = 0$ are corresponding to the supplier performance evaluation only considering the opinion similarity or opinion reliability.

Compared to the results of $\rho = 1$ and $\rho = 0$, the consensus opinion obtained by the extended HK model when $\rho = 0.9$ is a compromise of the consensus opinion that only considers opinion similarity or opinion reliability. Although the evaluation using the traditional HK model can quickly reach a consensus, the consensus opinion is very close to the opinions of a few reliable experts in the group like v_6. However, the opinions of equally reliable experts like v_1 have not been fully considered. The consensus opinions obtained by the extended HK model are compromises, which demonstrates the rationality of our proposal. From Fig. 16, we find that the iteration number of reaching consensus decreases with the increase of ρ. The larger ρ means that the greater the proportion of people with more similar opinions will be, which raises the opinion modifications and accelerates the consensus. However, the higher consensus opinion may be unrealistic, and it is unreasonable to blindly increase the value of ρ to accelerate consensus and obtain higher consensus. In order to balance consensus opinion and consensus speed, the choice of ρ in practical problems is very important.

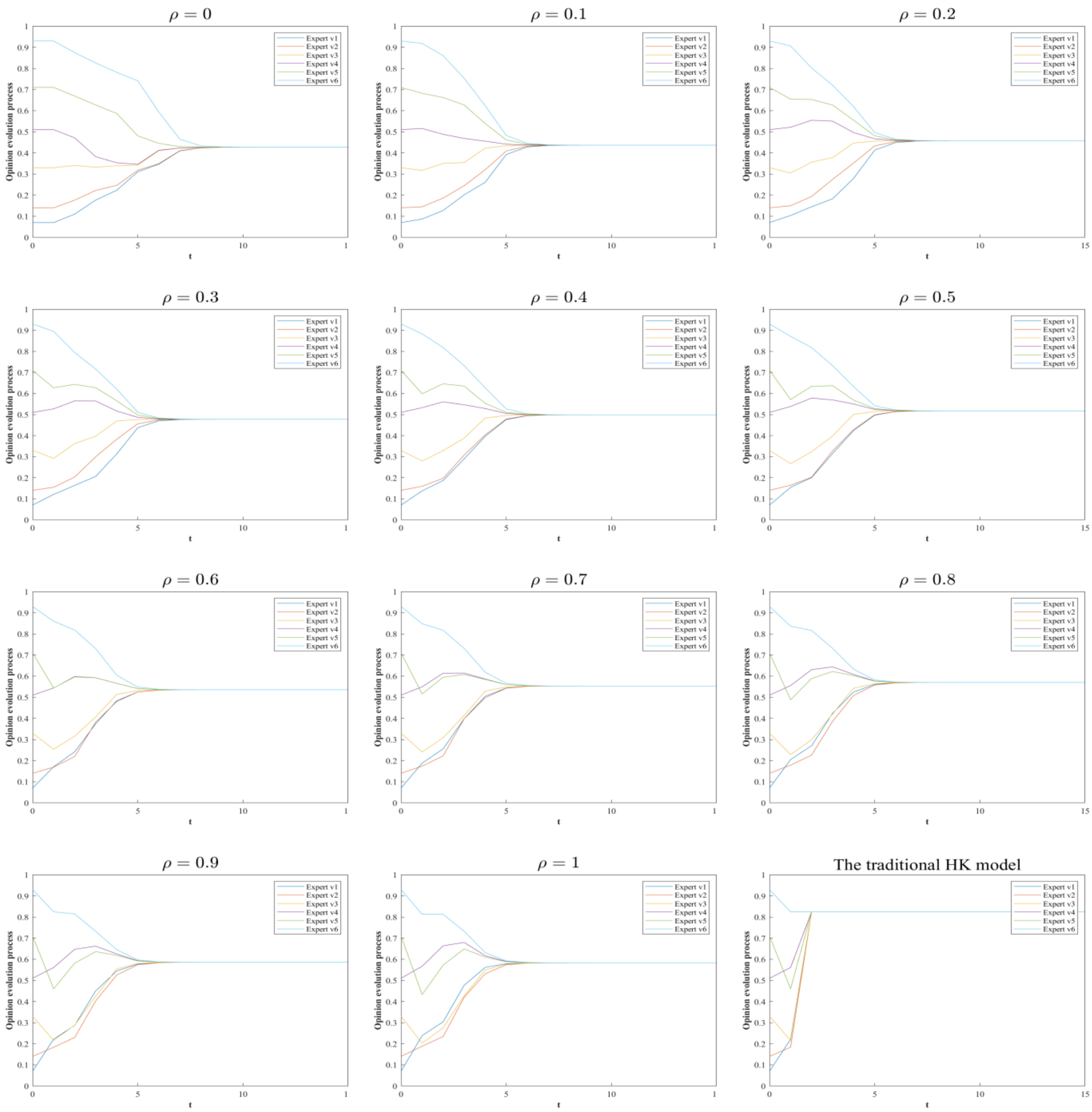


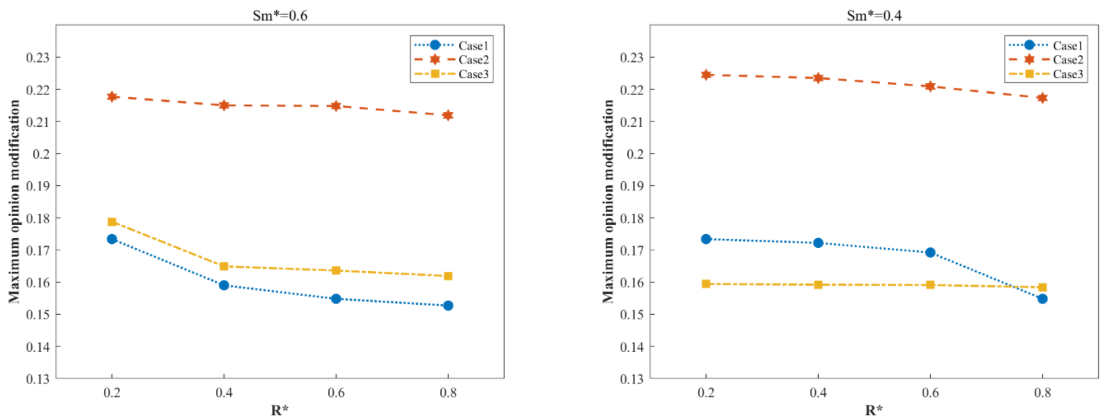
Fig. 16. The impact of the adaptive coefficient ρ.

6.2.2. *Simulation experiment II: The impact of $R*$ on opinion modification*

To quantify the impact of some parameters on opinion modification, we define the maximum opinion modification (MOM), which is denoted by the average of the maximum opinion modification of all experts in the CRP:

$$MOM = \frac{\sum_{i=1}^{k} \max_{t} \left| x_i^t - x_i^{t-1} \right|}{k}. \tag{27}$$

For convenience, the numerical experiment about supplier performance evaluation in Section 5, the ER random network experiment and the WS random network experiment are called Case 1, Case 2 and Case 3, respectively. Fig. 17 shows that the impact of the reliability threshold $R*$ on MOM in three cases when $Sm* = 0.6$ and $Sm* = 0.6$, and the remaining parameters are the same as in the numerical experiment in Section 5.



(a) $Sm* = 0.6$ (b) $Sm* = 0.4$

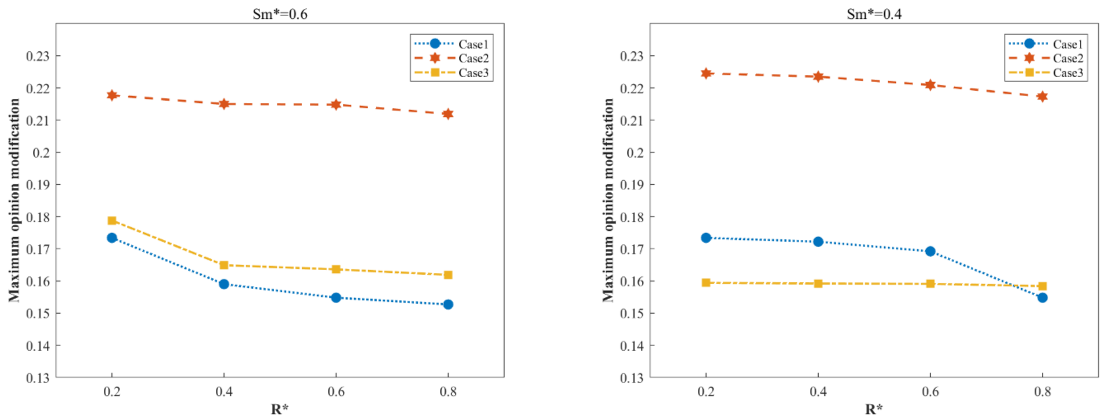
Fig. 17. The impact of the reliability threshold $R*$.

From Fig. 17, we can find that the MOM of Case 2 is obviously larger than that of Cases 1 and 3. The reason is that ER random network in Case 2 is significantly denser than networks in Cases 1 and 3, so the opinion change of ER random network is larger than that of other two networks at time t. In the three cases, the MOM decreases gently with the increase of $R*$, which means that the higher the reliability threshold $R*$ is, the smaller the maximum opinion modification is. This can be explained in another way: a higher reliability threshold means that a more stringent reliability condition, so fewer experts meet the consensus conditions, and the range of opinions modification is relatively small during the opinion evolution process. On the contrary, if the reliability threshold $R*$ is too low, the consensus conditions will be relaxed. Therefore, an appropriate reliability threshold should be determined in practical application.

6.2.3. Simulation experiment III: The impact of $Sm*$ on opinion modification

Similar to simulation experiment II, we investigate the impact of the opinion similarity threshold $Sm*$ on opinion modification MOM when the reliability threshold $R*$ is fixed, and the remaining parameters are the same as in the numerical experiment in Section 5, which are shown in Fig. 18.

From Fig. 18, we can find that the MOM decreases as the opinion similarity threshold $Sm*$ increases in Case 1. The result of Case 2 is similar to that of Case 3, that is, the MOM decreases very little or even stays the same. Compared with Case 1, Cases 2 and 3 are characterized by strong randomness, which may lead to great fluctuations in the modification value of experts' opinions in each iteration, thus resulting the MOM to be affected by extreme values. In general, the opinion similarity threshold $Sm*$ has a certain impact on consensus reaching, but the impact is not as obvious as the reliability threshold $R*$. The reason is that the stability similarity and the weighted similarity are included in opinion reliability, whose influences will accumulate and amplify the influence of opinion reliability.



(a) $R* = 0.8$ (b) $R* = 0.4$

Fig. 18. The impact of the opinion similarity threshold $Sm*$.

6.3. Comparison and discussion

In this subsection, we compare three SNGDM methods including Zhang et al. [69]'s method, Liu et al. [32] 's method and Wu et al. [70]'s method with our proposal and discuss the advantages and limitations of our proposal.

The comparisons of the four SNGDM methods are shown in Table **??**. Zhang et al. [69]'s method claimed that the trust values would evolve with the time and proposed a CRP model with trust evolution, where the feedback mechanism included trust adjustment and opinion adjustment. However, the social network was considered be static.Liu et al. [32] 's method addressed the defect of not considering the dynamic nature of social network and proposed a dual-path feedback mechanism by preference and weight adjustments, in which they did not use the opinion dynamics model but make artificial adjustments. Wu et al. [70]'s method not only considered dynamic social networks, but also applied opinion dynamics, which focused primarily on the interaction between trust and consensus networks. However, the trust value in three methods was derived from the opinion similarity or network centrality, where the opinion reliability was not taken into account. Meanwhile, the edges in social networks were considered as the trust values between experts, which is sometimes unrealistic. The advantages of our proposal are as follows: (i) We distinguish between social networks and trust networks, where trust relationship spread through the edges of social networks. (ii) When estimating trust values, we not only consider opinion similarity and network centrality, but also take into account opinion reliability. (iii) The feedback mechanism with social network evolution and opinion dynamics is proposed to reach the group consensus. The limitations of our proposal are as follows: (i) The opinion is real value in this paper and more opinion types like triangular fuzzy number, trapezoidal fuzzy number or intuitionistic fuzzy number can be considered further. (ii) The interaction between trust value caused by social network evolution and consensus is worthy of further exploration.

Table 2
Comparisons of our proposal and the relevant SNGDM methods.

References	Opinion type	Opinion dynamics	Dynamic social network	Trust evolution	Social network is equivalent to Trust network?	Source of trust value			Feedback mechanism	
						Similarity	Centrality	Reliability	Opinion adjustment	Other adjustment
Zhang et al. (2022)	Real value	Yes	No	Yes	Yes	Yes	No	No	Yes	Trust adjustment
Liu et al. (2022)	Intuitionistic fuzzy value	No	Yes	Yes	Yes	Yes	No	No	Yes	Weight adjustment
Wu et al. (2022)	Real value	Yes	Yes	Yes	Yes	No	Yes	No	Yes	Consensus adjustment
Our proposal	Real value	Yes	Yes	Yes	No	Yes	Yes	Yes	Yes	Social network adjustment

7. Conclusions

In SNGDM problems, the reliability is the key characteristic of experts which has rarely been considered into the opinion dynamics. To fully consider the reliability of experts in social networks, this paper proposes a novel GDM framework with opinion dynamics in social trust network based on opinion reliability. Firstly, we define the opinion reliability of experts based on stability similarity and weighted similarity, which is measured from the perspective of individual and others. Secondly, by incorporating opinion reliability into the trust propagation and aggregation in social networks, we propose an improved trust propagation and aggregation mechanism. Furthermore, opinion reliability based on opinion similarity and opinion reliability of experts affects the social networks. Meanwhile, the extended HK model is proposed to design the opinion evolution rules combining opinion similarity with opinion reliability of experts. Finally, a numerical example is presented to illustrate the effectiveness of the proposed model. The simulation analyses are conducted to show the feasibility of the GDM framework and the sensitivity of some parameters.

The proposed SNGDM method can not only deal with the defect that ignoring the opinion reliability in practical problems, but also provide a dynamic decision-making framework based on the social trust network evolution and opinion dynamics. The definition of opinion reliability takes into account the opinions of grassroots employees with low weight but better understanding of practical decision-making issues, which can avoid unreasonable decisions caused by too high weight of the leadership in practical problems like the optimal production planning decision. The improved HK model not only considers the opinions of experts with similar opinions to decision-makers, but also considers the opinions of reliable experts and retains some opinions of individuals. Compared with previous studies, it is closer to the psychological behavior of people, so as to obtain more reliable managerial decision-making results.

For further study, opinion dynamics model in social networks considering opinion reliability could be extended to different linguistic environments and the impacts of reliability on a variety of stochastic networks in SNGDM problems could be further explored. Besides, the rumor propagation considering the reliability is also a focus of our future research.

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2 Preference-based regret three-way decision method on multiple decision information systems with linguistic z-numbers

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Preference-based regret three-way decision method on multiple decision information systems with linguistic Z-numbers

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Abstract

Three-way decision theory (3WD) provides a reasonable solution to solve multi-criteria decision-making (MCDM) problems and reduces more decision risks than two-way decision (2WD). However, the impact of preferences on decision results has not been reasonably considered in 3WD. Thus, this paper proposes a preference-based regret 3WD model on multiple decision information systems (MDISs) with linguistic Z-numbers (LZNs) to address the defect. First of all, a Z-LINMAP method is proposed by extending LINMAP method into the LZNs environment to derive criteria weights, Z-number ideal solutions and preference coefficients. On the basis of preference coefficients, the consistency-order and inconsistency-order are defined and further the consistency and inconsistency equivalence classes are presented to derive the conditional probabilities of alternatives. Then, the regret loss functions based on regret theory are presented and the regret 3WD rules of a single decision information system is designed. Furthermore, the weighted expected regret loss function and loss score function are defined to classify and rank all alternatives for MDISs. Finally, the proposed 3WD model is applied to the image recognition case with human-computer interaction to verify the effectiveness, and the comparative and sensitivity analyses are carried out to demonstrate the feasibility of our model.
Keywords: Preference-based three-way decision; Multi-criteria decision-making; LINMAP method; Regret theory; Multiple decision information systems

1. Introduction

With the rapid development of computer vision technology, image recognition technology has shown a strong value in online shopping, face recognition and other fields. When there are multiple image recognition tasks, multi-criteria decision-making method (MCDM) can help people determine the recognition order of tasks according to the features of images. However, the low quality pictures in real life require the human-computer technology and even the human to be recognized effectively.

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Three-way decision (3WD) theory proposed by Yao [1] divides a problem into three decisions: acceptance, delay and rejection, which corresponds to three recognition types, namely computer recognition, human-computer recognition and human recognition. Consequently, based on MCDM methods, a preference-based regret three-way decision method on multiple decision information systems (MDISs) with linguistic Z-numbers is proposed to classify and rank image recognition tasks. In what follows, we will briefly review the related studies of MCDM and 3WD.

1.1. A brief review of MCDM

MCDM plays an important role in decision theories and aims to rank all alternatives and select the optimal one considering multiple criteria of alternatives. The application of MCDM methods has been presented in different fields, including supplier selection [2–4], financial market [5–7], blockchain technology [8–10], artificial intelligence [11–14], and so forth. Up to now, a great number of approaches have been explored to solve different MCDM problems, such as weighted averaging aggregation (WAA) method [15], grey relational analysis (GRA) method [16], VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method [17], the technique for order preference by similarity to an ideal solution (TOPSIS) method [18], linear programming techniques (LINMAP) method [19], etc. Most MCDM methods like TOPSIS, GRA and VIKOR methods have been used to rank alternatives via introducing positive ideal solution (PIS) and negative ideal solution (NIS), which are often predetermined or directly determined based on evaluation information. The LINMAP method has an advantage of scientifically obtaining the PIS and weights under different criteria based on the evaluation and preference information. However, the traditional LINMAP method ignores the NIS and its effect on criteria weights.

To measure the uncertainty of evaluation, the above MCDM methods have been extended with different types of data, such as triangular fuzzy numbers [20, 21], trapezoidal fuzzy numbers [22, 23], interval fuzzy numbers [24, 25], hesitate fuzzy numbers [26–28], etc. Most studies have assumed that crisp or fuzzy decision information is completely reliable and ignored the influence of realistic factors on evaluation values. In fact, the evaluation information given is not always reliable due to the environmental and psychological factors of decision-makers. Z-numbers proposed by Zadeh [29] can not only express the decision maker's assessment information, but also measure the information reliability. The linguistic Z-numbers (LZNs) proposed by Wang et al. [30], which are considered to be more in line with people's expression habits than Z-numbers, apply a series of linguistic term elements to express the evaluation value and reliability. Liu et al. [31] presented a gained and lost dominance score (GLDS) method based on LZNs to rank five Internet hospitals and select the best one. Huang et al. [32] established a new failure mode and effect analysis (FMEA) model with LZNs and TOPSIS method to obtain the ranking result of ten failure modes. Obviously, the traditional MCDM methods are two-way decision (2WD) corresponding acceptance and rejection two actions, which can only rank alternatives but not classify alternatives. However, the practical problems like medical diagnosis and paper review may exist three possible actions: acceptance, delay and rejection. In light of this perspective, 3WD theory can effectively deal with the defect of MCDM problems.

1.2. A brief review of 3WD

The 3WD theory originates from the decision-theoretic rough sets (DTRSs) and Bayesian theory, which is initiated by Yao [1] in 2009. The core idea of 3WD theory is to divide the domain of the universe into three independent regions including the positive region, the boundary region and the negative region, which correspond acceptance, delay and rejection actions, respectively. Due to the concept of trisecting-acting-outcome (TAO) in 3WD theory, different 3WD models have been applied in medical diagnosis [33, 34], engineering investment [35, 36], face recognition [37, 38] and other fields. The conditional probabilities and loss functions are two key topics of the 3WD theory, so the existing research are briefly reviewed from the two aspects.

Determining the conditional probabilities of alternatives is one of cores for 3WD theory, which has been widely explored. Jia and Liu [39] proposed a new 3WD model where the conditional probability was given subjectively by decision-makers. Liu et al. [40] used the grey relation degree to compute the conditional probability based on condition attributes. Wang et al. [41] derived the conditional probability via hesitate fuzzy operators according to the characteristics of hesitate fuzzy information. Wang et al. [28] applied the TOPSIS and GRA method to estimate the conditional probability. However, these existing 3WD models ignore the preference information of decision information system, which may effect the classification result of alternatives. Thus, this paper establishes two equivalence relations considering the preference to derive the conditional probability.

The loss functions refers to the losses of a object taking three actions in two states in 3WD models, which is another core of 3WD theory. Yao [1] deemed that all objects have the same losses when taking the same actions. Furthermore, Yao [42] proposed the relative loss functions where the loss functions corresponding to three actions in the same state are proportional. Liu et al. [43] proposed a 3WD model in which each object has different losses when taking different actions. The loss functions under different criteria have been considered in some studies [39, 40]. However, the psychological behaviors of taking different actions will have more or less influence on the loss functions. The regret theory (RT) can measure the regret-rejoice value of decision-makers for taking one action instead of another, which can well simulate the psychological behaviors in real problems. Thus, RT is considered to derive the loss functions for different objects in this paper.

Most 3WD models can only classify alternatives rather than rank alternatives in a single decision information system (DIS). However, practical decision-making problems sometimes require not only the classification result of alternatives but also the ranking result of alternatives. For instance, the medical team in hospital can determine the treatment area, delayed area and no treatment area through the 3WD method. The order of treatment in the treatment area and the order of further examination in the delayed area are very critical to improve the efficiency of treatment and diagnosis. Furthermore, different medical teams may be invited to classify and rank the same group of patients, which have different preferences due to different experience and knowledge. Consequently, combining the perspectives of MCDM and 3WD, a preference-based 3WD method with LZNs based on MDISs is proposed, which can not only classify but also rank alternatives.

1.3. Motivations, innovations and the structure of the paper

Based on the above descriptions, the motivations of this paper are summarized as follows:

(1) Most 3WD models have been classified data of DIS in databases where preference information has not been involved. However, preference information generated from the historical data of different DISs may affect the decision results. Meanwhile, different experts recorded in databases may have the subjective preferences for different DISs. Therefore, the preference information is necessary to be considered and solved in 3WD model.

(2) Experts often express evaluation values using natural language in practical decision-making problems. Some existing MCDM or 3WD methods have used fuzzy information to measure the uncertainty of evaluation information which might be assumed to be totally reliable. Besides, experts feel regret or rejoice due to psychological factors when comparing three actions in 3WD models, which has not been well studied.

(3) The existing 3WD research have only focused on how to determine the conditional probabilities or loss functions of single DIS. In complex decision-making scenarios, multi-source DISs may be collected to make integrated decisions to obtain more reasonable results. Therefore, the classification and ranking method for MDISs are worth exploring.

The innovations and significance of our proposal corresponding to the above motivations are described as follows:

(1) The consistency-order and inconsistency-order based on the preference information are defined, and further the consistency and inconsistency equivalence classes are proposed to derive the conditional probability.

(2) Considering the reliability of decision information, a Z-LINMAP method is proposed and involved into the proposed 3WD model to derive the criteria weights, positive and negative ideal

Z-numbers based on experts' preferences. With the support of RT, this paper constructs a group of regret loss functions based on positive and negative ideal Z-numbers and presents the preference-based regret 3WD rules.

(3) This paper proposes a 3WD model on MDISs which can not only classify but also rank all alternatives. The comprehensive decision rules based on the weighted expected regret loss functions are presented to obtain the final classification results for all experts and the loss score function is defined to rank all alternatives.

The structure of this paper is shown in Fig. 1 and the remainder of this paper is organized as follows. Section 2 introduces the basic knowledge necessary related to linguistic scale function (LSF), LZNs, RT and 3WD model. Section 3 proposes a preference-based regret 3WD method on MDISs with LZNs. Section 4 provides a numerical analysis for image recognition based on human-computer interaction. The Comparative and sensitivity analyses in Section 5 are used to demonstrate the feasibility and effectiveness of the proposed regret 3WD model. Finally, Section 6 presents the concluding remarks.

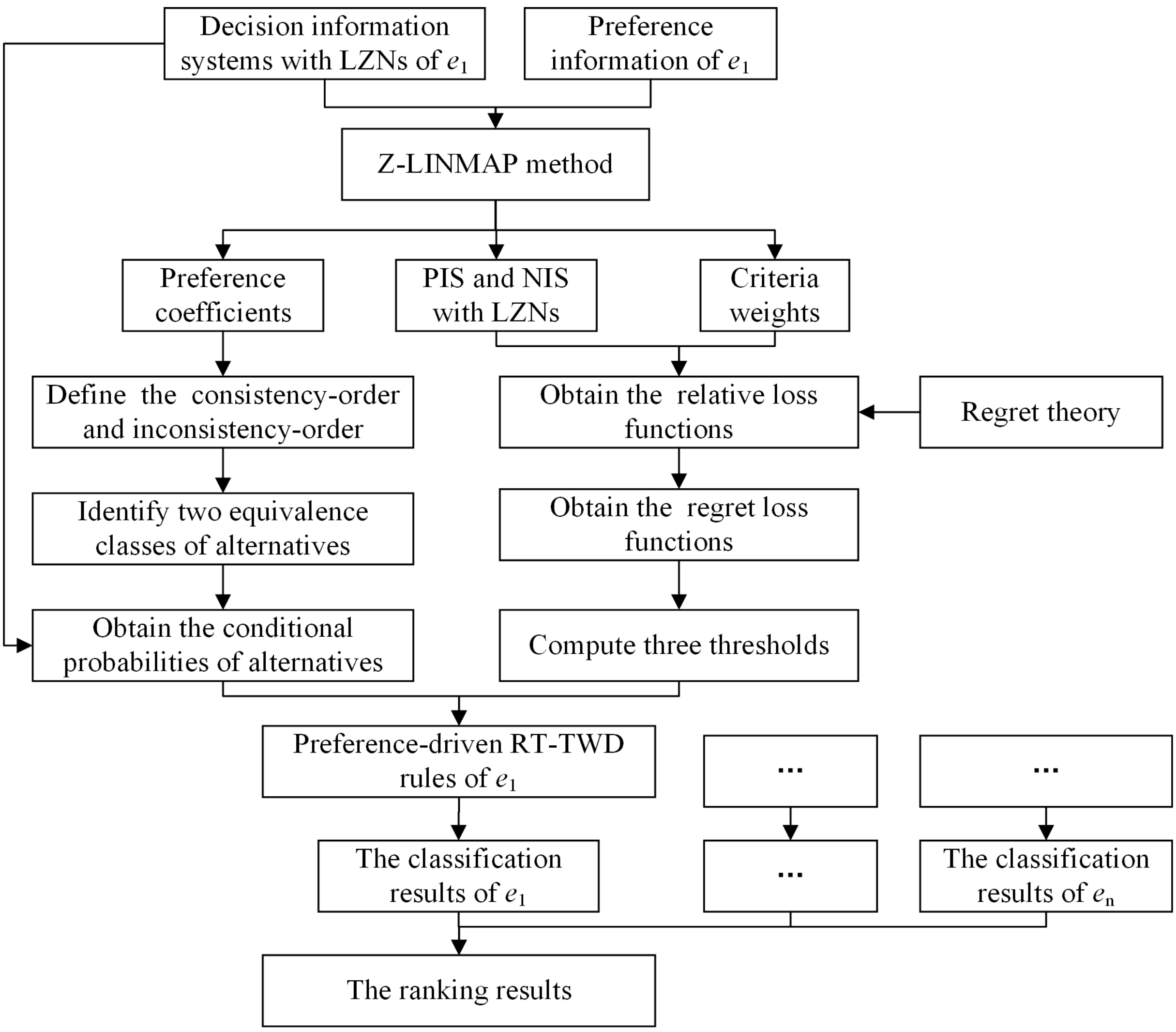

Fig. 1. The flowchart of the proposed model.

2. Preliminaries

This section briefly reviews some basic concepts, regarding LSF [44], LZNs [30], RT [45] and 3WD [42].

2.1. Linguistic scale function (LSF)

Definition 1. [44] Given a linguistic term set $L = \{l_0, l_1, ..., l_{2g}\}$, the LSF H mapping from l_i to ς_i is defined as follows:

$$H : l_i \rightarrow \varsigma_i \ (i = 0, 1, ..., 2g),$$

where the symbol ς_i reflects the preference of expert using the linguistic term l_i and $0 \leq \varsigma_0 < \varsigma_1 < \cdots < \varsigma_{2g} \leq 1$. The possible LSFs $H_o(l_i)$ $(o = 1, 2, 3, 4)$ can be given as follows [30]:

$$H_1(l_i) = \frac{i}{2g} \ (0 \leq i \leq 2g), \qquad H_2(l_i) = (\frac{i}{2g})^g \ (0 \leq i \leq 2g),$$
$$H_3(l_i) = (\frac{i}{2g})^{\frac{1}{g}} \ (0 \leq i \leq 2g), \quad H_4(l_i) = \begin{cases} \dfrac{a^g - a^{g-i}}{2a^g - 2} & (0 \leq i \leq g) \\ \dfrac{a^g + a^{i-g} - 2}{2a^g - 2} & (g + 1 \leq i \leq 2g) \end{cases}, \tag{1}$$

where the value of a can be determined using a subjective method. Assuming the indicator A is far more important than indicator B and the important ratio is m, then $a^k = m$ (k represents the scale level) and $a = \sqrt[k]{m}$. The vast majority of researchers believe that $m = 9$ is the upper limit of the important ratio. If the scale level is 7, then $a = \sqrt[7]{9} \approx 1.37$ can be obtained.

2.2. Linguistic Z-numbers (LZNs)

To characterize the reliability of evaluation information, Zadeh [29] proposed the concept of Z-number with two components including a restriction on the values and the reliability of the restriction. The two components in Z-numbers can be described in natural languages. Wang et al.[30] proposed the concept of linguistic Z-number (LZNs) as a subclass of Z-numbers, where the fuzziness and randomness inherent in linguistic terms exactly correspond to the restriction and probability measure of Z-numbers.

Definition 2. [30] Let X be a universe discourse, $A = \{A_0, A_1, ..., A_{2g_1}\}$ and $B = \{B_0, B_1, ..., B_{2g_2}\}$ be two finite and totally ordered linguistic term sets, where g_1 and g_2 are non-negative integers. A LZN for $x \in X$ is defined as $Z = (A_{\varphi(x)}, B_{\phi(x)})$, where $A_{\varphi(x)} \in A$ is a restriction of the value that uncertain variables allow to take and $B_{\phi(x)} \in B$ is the measure of reliability of the first component $A_{\varphi(x)}$. $\varphi(x)$ and $\phi(x)$ are the subscripts of linguistic terms A and B, respectively.

Definition 3. [30] Let $Z_i = (A_{\varphi(i)}, B_{\phi(i)})$ and $Z_j = (A_{\varphi(j)}, B_{\phi(j)})$ be two LZNs. f^* and g^* are possible LSFs of linguistic term sets A and B from $H_o(\cdot)$ $(o = 1, 2, 3, 4)$. f^{*-1} and g^{*-1} are the inverse functions of f^* and g^*, respectively. The symbols $\varphi(i)$ $(i = 0, 1, ..., 2g_1)$ and $\phi(j)$ $(j = 0, 1, ..., 2g_2)$ represent the linguistic term A_i and B_j. Some operations of LZNs are defined as follows:

(1) $Z_i \oplus Z_j = (f^{*-1}(f^*(A_{\varphi(i)}) + f^*(A_{\varphi(j)})), g^{*-1}(\frac{f^*(A_{\varphi(i)})\times g^*(B_{\phi(i)})+f^*(A_{\varphi(j)})\times g^*(B_{\phi(j)})}{f^*(A_{\varphi(i)})+f^*(A_{\varphi(j)})}))$;

(2) $Z_i \otimes Z_j = (f^{*-1}(f^*(A_{\varphi(i)}) \times f^*(A_{\varphi(j)})), g^{*-1}(g^*(B_{\phi(i)}) \times g^*(B_{\phi(j)})))$;

(3) $\lambda Z_i = (f^{*-1}(\lambda f^*(A_{\varphi(i)})), B_{\phi(i)})$,where $\lambda \geq 0$;

(4) $Z_i^\lambda = (f^{*-1}(f^*(A_{\varphi(i)})^\lambda), g^{*-1}(g^*(B_{\phi(i)})^\lambda))$, where $\lambda \geq 0$.

In a Z-number, part B measures the reliability of part A. Generally, A is more important than B. Based on this view, Chai et al. [25] defined the weighted Euclidean distance of Z-numbers.

Definition 4. [25] The weighted Euclidean distance between $Z_i = (A_{\varphi(i)}, B_{\phi(i)})$ and $Z_j = (A_{\varphi(j)}, B_{\phi(j)})$ is defined as follows:

$$d(Z_i, Z_j) = \sqrt{\rho_1(f^*(A_{\varphi(i)}) - f^*(A_{\varphi(j)}))^2 + \rho_2(g^*(B_{\phi(i)}) - g^*(B_{\phi(j)}))^2}, \tag{2}$$

where $\rho_1 \in [0, 1]$ and $\rho_2 \in [0, 1]$ are parameters to reflect the importance of A and B in LZNs and $\rho_1 + \rho_2 = 1$. Part A is usually considered more important than part B, then $\rho_1 \geq \rho_2$. In particular, A and B are equally important when $\rho_1 = \rho_2 = 0.5$.

2.3. *Regret theory (RT)*

RT proposed by Bell [45] is a critical behavior theory, which takes into account the psychology of regret when making risky decisions. Expert compares the outcome of the chosen alternative with other alternatives to select the best choice that expert will not regret. Let y_1 and y_2 be the outcomes of choosing alternative x_1 and x_2, the perceived utility of expert to choose x_1 is obtained as follows:

$$U(y_1, y_2) = u(y_1) + r(u(y_1) - u(y_2)), \tag{3}$$

where $u(\cdot)$ is the utility function and $r(\cdot)$ is the regret-rejoice function. Let the utility difference be $\Delta u = u(y_1) - u(y_2)$, then expert prefers to x_1 than x_2 when $r(\Delta u) > 0$. The regret-rejoice function $r(\Delta u)$ is calculated as follows:

$$r(\Delta u)=\begin{cases} 1 - e^{-\delta\times\Delta u}, & u(y_1) \geq u(y_2) \\ 0, & u(y_1) < u(y_2) \end{cases}, \tag{4}$$

where $\delta \in [0 + \infty]$ is the regret aversion coefficient. The bigger δ is, the higher the regret aversion degree of expert is.

2.4. *Three-way decision (3WD)*

Yao [42] proposed the 3WD theory based on the Bayesian theory. Generally, the 3WD model includes two states $\Omega = \{X, \neg X\}$ and three actions $\mathrm{A} = \{a_P, a_B, a_N\}$. X and $\neg X$ represent that an object x belongs to X and do not belong to X. a_P, a_B and a_N stand for acceptance, delay and rejection actions, which denote $x \in POS(X)$, $x \in BND(X)$ and $x \in NEG(X)$, respectively. R is an equivalence relation on X and $[x]_R$ is the equivalence class of R including the object x. $\Pr(X|[x]_R)$ and $\Pr(\neg X|[x]_R)$ are the conditional probabilities of the object x belonging to and do not belonging to X. In specific, $\lambda_{\bullet\circ}(\bullet = P, B, N; \circ = P, N)$ are denoted to measure the losses of taking three actions, which are presented in Table 1. λ_{PP}, λ_{BP} and λ_{NP} represent the losses taking actions a_P, a_B and a_N when $x \in X$. Similarly, λ_{PN}, λ_{BN} and λ_{NN} denote the losses taking actions a_P, a_B and a_N when $x \in \neg X$. A reasonable assumption is considered in the 3WD model: $0 \leq \lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}, 0 \leq \lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$.

The expected loss function $R(a_\bullet|[x]_R)$ of taking three actions can be calculated as:

(1) $R(a_P|[x]_R) = \lambda_{PP}\Pr(X|[x]_R) + \lambda_{PN}\Pr(\neg X|[x]_R)$;

(2) $R(a_B|[x]_R) = \lambda_{BP}\Pr(X|[x]_R) + \lambda_{BN}\Pr(\neg X|[x]_R)$;

(3) $R(a_N|[x]_R) = \lambda_{NP}\Pr(X|[x]_R) + \lambda_{NN}(\Pr(\neg X|[x]_R)$.

According to the Bayesian theory, the following decision rules of minimum losses can be demonstrated as follows:

(P1) If $R(a_P|[x]_R) \leq R(a_B|[x]_R)$ and $R(a_P|[x]_R) \leq R(a_N|[x]_R)$, then $x \in POS(X)$;

(B1) If $R(a_B|[x]_R) \leq R(a_P|[x]_R)$ and $R(a_B|[x]_R) \leq R(a_N|[x]_R)$, then $x \in BND(X)$;

(N1) If $R(a_N|[x]_R) \leq R(a_P|[x]_R)$ and $R(a_N|[x]_R) \leq R(a_B|[x]_R)$, then $x \in NEG(X)$.

The above rules (P1)-(N1) relate to the loss function $\lambda_{\bullet\circ}$ and the conditional probability $\Pr(X|[x]_R)$, then the decision rules can be simplified as:

(P2) If $\Pr(X|[x]_R) \geq \alpha$ and $\Pr(X|[x]_R) \geq \gamma$, then $x \in POS(X)$;

(B2) If $\Pr(X|[x]_R) \leq \alpha$ and $\Pr(X|[x]_R) \geq \beta$, then $x \in BND(X)$;

(N2) If $\Pr(X|[x]_R) \leq \gamma$ and $\Pr(X|[x]_R) \leq \beta$, then $x \in NEG(X)$;

where $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$, $\gamma = \frac{\lambda_{PN}-\lambda_{NN}}{(\lambda_{PN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{PP})}$ and $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$.

Table 1
The loss functions in 3WD theory.

Actions	$X(P)$	$\neg X(N)$
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

3. Preference-based regret 3WD model on MDISs with linguistic Z-numbers

This section proposes a preference-based regret 3WD model on MDISs with LZNs. Section 3.1 proposes a Z-LINMAP method to derive PIS vector, NIS vector and criteria weights. The consistency-order and inconsistency-order are defined to determine two equivalence classes in Section 3.2, and the conditional probabilities based on LZNs are derived. Section 3.3 presents the regret loss functions and designs two 3WD decision rules. In Section 3.4, we discuss the 3WD decision rules on MDISs and define a loss score function to rank all alternatives.

3.1. Z-LINMAP method

Assume $X = \{x_1, x_2, ..., x_n\}$ is the alternative set and $C = \{c_1, c_2, ..., c_m\}$ is the criteria set. The LZNs evaluation matrix can be expressed as $Z = (Z_{ij})_{n\times m}$, where $Z_{ij} = (A_{ij}, B_{ij})$. The preference relation set $Q = \{(k,l)|x_k, x_l \in X\}$ can be provided by an expert, where (k,l) represents that the expert prefers to alternative x_k rather than x_l. A LZN $Z_{ij} = (A_{ij}, B_{ij})$ can be transformed as a crisp Z-number $\overline{Z}_{ij} = (f(A_{ij}), g(B_{ij}))$, where $f(\cdot)$ and $g(\cdot)$ are possible LSFs of linguistic term sets A and B. Then we can obtain the crisp Z-number matrix $\overline{Z} = (\overline{Z}_{ij})_{n\times m}$. The PIS vector and NIS vector with LZNs can be denoted as $Z^+ = (Z_1^+, Z_2^+, ..., Z_m^+)$ and $Z^- = (Z_1^-, Z_2^-, ..., Z_m^-)$, where $Z_j^+ = (A_j^+, B_j^+)$ and $Z_j^- = (A_j^-, B_j^-)(j = 1, 2, ..., m)$. Then the positive weighted distance and negative weighted distance of alternative x_i are denoted as $S_i^+ = \sum_{j=1}^{m} w_j d(Z_{ij}, Z_j^+)^2$ and $S_i^- = \sum_{j=1}^{m} w_j d(Z_{ij}, Z_j^-)^2$, which are calculated as follows:

$$
\begin{aligned}
S_i^+ &= \sum_{j=1}^{m} w_j(\rho_1(f(A_{ij}) - f(A_j^+))^2 + \rho_2(g(B_{ij}) - g(B_j^+))^2); \\
S_i^- &= \sum_{j=1}^{m} w_j(\rho_1(f(A_{ij}) - f(A_j^-))^2 + \rho_2(g(B_{ij}) - g(B_j^-))^2).
\end{aligned} \tag{5}
$$

Traditional LINMAP method [19] only considers the PIS with each alternative and ignores the NIS. We discuss how to obtain the PIS vector firstly. For a preference relation (k,l), $S_k^+ \leq S_l^+$ is regarded as satisfying the consistency. The positive inconsistency degree $(S_l^+ - S_k^+)^-$ and the positive consistency degree $(S_l^+ - S_k^+)^+$ under PIS are denoted as:

$$
\begin{aligned}
(S_l^+ - S_k^+)^- &= \max\{0, (S_k^+ - S_l^+)\} = \begin{cases} 0, & if\ S_l^+ \geq S_k^+ \\ S_k^+ - S_l^+, & if\ S_l^+ < S_k^+ \end{cases}; \\
(S_l^+ - S_k^+)^+ &= \max\{0, (S_l^+ - S_k^+)\} = \begin{cases} S_l^+ - S_k^+, & if\ S_l^+ \geq S_k^+ \\ 0, & if\ S_l^+ < S_k^+ \end{cases}.
\end{aligned} \tag{6}
$$

The total positive inconsistency degree B and the total positive consistency degree G under PIS can be calculated as $B = \sum_{(k,l)\in Q} (S_l^+ - S_k^+)^-$ and $G = \sum_{(k,l)\in Q} (S_l^+ - S_k^+)^+$. Generally, the total positive consistency degree G is expected to be higher than the total positive inconsistency degree B, then the condition $G - B = \varepsilon$ should be satisfied, where ε is a real number greater than 0.

To derive the weight vector $w = (w_1, w_2, ..., w_m)$ under PIS and PIS vector Z^+, the minimum of inconsistency degree under all preference relations is considered as the goal. Let the positive preference coefficient $\mu_{kl} = \max\{0, (S_k^+ - S_l^+)\}$ and the objective function be $\min \sum_{(k,l)\in Q} \mu_{kl}$, then the PIS programming model can be established as follows:

$$\min \sum_{(k,l)\in Q} \mu_{kl}$$
$$s.t. \begin{cases} S_k^+ = \sum_{j=1}^{m} w_j(\rho_1(f(A_{kj}) - f(A_j^+))^2 + \rho_2(g(B_{kj}) - g(B_j^+))^2), \\ S_l^+ = \sum_{j=1}^{m} w_j(\rho_1(f(A_{lj}) - f(A_j^+))^2 + \rho_2(g(B_{lj}) - g(B_j^+))^2), \\ \mu_{kl} + (S_l^+ - S_k^+) \geq 0, \forall (k,l) \in Q, \\ \sum_{(k,l)\in Q} (S_l^+ - S_k^+) = \varepsilon, \\ \mu_{kl} \geq 0, \forall (k,l) \in Q. \end{cases} \tag{M-1}$$

Then we discuss how to obtain the NIS vector. Similarly, $S_k^- \geq S_l^-$ is considered to satisfy the consistency. The negative inconsistency degree $(S_l^- - S_k^-)^-$ and the negative consistency degree $(S_l^- - S_k^-)^+$ under NIS are denoted as:

$$(S_l^- - S_k^-)^- = \max\{0, (S_l^- - S_k^-)\} = \begin{cases} 0, & if\ S_l^- \leq S_k^- \\ S_l^- - S_k^-, & if\ S_l^- > S_k^- \end{cases};$$
$$(S_l^- - S_k^-)^+ = \max\{0, (S_k^- - S_l^-)\} = \begin{cases} S_k^- - S_l^-, & if\ S_l^- \leq S_k^- \\ 0, & if\ S_l^- > S_k^- \end{cases}. \tag{7}$$

The total negative inconsistency degree $B' = \sum_{(k,l)\in Q} (S_l^- - S_k^-)^-$ and the total negative consistency degree $G' = \sum_{(k,l)\in Q} (S_l^- - S_k^-)^+$ under NIS satisfy the condition $G' - B' = \varepsilon'$, where ε' is a real number greater than 0. Let the negative preference coefficient be $\mu'_{kl} = \max\{0, (S_l^- - S_k^-)\}$ and the objective function be $\min \sum_{(k,l)\in Q} \mu'_{kl}$, the NIS programming model to obtain weights and NIS vetctor can be established as follows:

$$\min \sum_{(k,l)\in Q} \mu'_{kl}$$
$$s.t. \begin{cases} S_k^- = \sum_{j=1}^{m} w_j(\rho_1(f(A_{kj}) - f(A_j^-))^2 + \rho_2(g(B_{kj}) - g(B_j^-))^2), \\ S_l^- = \sum_{j=1}^{m} w_j(\rho_1(f(A_{lj}) - f(A_j^-))^2 + \rho_2(g(B_{lj}) - g(B_j^-))^2), \\ \mu'_{kl} + (S_k^- - S_l^-) \geq 0, \forall (k,l) \in Q, \\ \sum_{(k,l)\in Q} (S_k^- - S_l^-) = \varepsilon', \\ w_j \in [0,1], j = 1,2,...,m, \\ \mu'_{kl} \geq 0, \forall (k,l) \in Q. \end{cases} \tag{M-2}$$

The $f(\cdot)$ and $g(\cdot)$ by solving models (M-1) and (M-2) may go beyond the boundary [0, 1] of LSFs. Therefore, the constraint on LSF boundaries is required: $f(\cdot), g(\cdot) \in [0, 1]$. To obtain the comprehensive weights considering PIS and NIS, we use a parameter θ to aggregate all preference coefficients μ_{kl} and μ'_{kl}. The objective function is updated as $\min \sum_{(k,l)\in Q} \theta\mu_{kl} + (1-\theta)\mu'_{kl}$, and the comprehensive Z-LINMAP model is as follows:

$$
\begin{aligned}
&\min \sum_{(k,l)\in Q} \theta\mu_{kl} + (1-\theta)\mu'_{kl} \\
&s.t. \begin{cases}
\sum_{(k,l)\in Q} (S_l^+ - S_k^+) = \varepsilon, \\
\mu_{kl} + (S_l^+ - S_k^+) \geq 0, \forall (k,l) \in Q, \\
\sum_{(k,l)\in Q} (S_k^- - S_l^-) = \varepsilon', \\
\mu'_{kl} + (S_k^- - S_l^-) \geq 0, \forall (k,l) \in Q, \\
S_l^+ - S_k^+ = \sum_{j=1}^{m} w_j(\rho_1(f(A_{lj}) - f(A_j^+))^2 + \rho_2(g(B_{lj}) - B(b_j^+))^2 - \\
\qquad \rho_1(f(A_{kj}) - f(A_j^+))^2 - \rho_2(g(B_{kj}) - g(B_j^+))^2), \forall (k,l) \in Q, \\
S_k^- - S_l^- = \sum_{j=1}^{m} w_j(\rho_1(f(A_{lj}) - f(A_j^-))^2 + \rho_2(g(B_{lj}) - g(B_j^-))^2 - \\
\qquad \rho_1(f(A_{kj}) - f(A_j^-))^2 - \rho_2(g(B_{kj}) - g(B_j^-))^2), \forall (k,l) \in Q, \\
w_j, f(A_j^+), f(A_j^-), g(B_j^+), g(B_j^-) \in [0,1], j = 1, 2, ..., m, \\
\mu_{kl} \geq 0, \mu'_{kl} \geq 0, \forall (k,l) \in Q.
\end{cases}
\end{aligned}
\tag{M-3}
$$

To transform the model (M-3) into a linear programming model, four variables w_j^{A+}, w_j^{A-}, w_j^{B+} and w_j^{B-} are added into model (M-3), where $w_j^{A+} = w_j * f(A_j^+)$, $w_j^{A-} = w_j * f(A_j^-)$, $w_j^{B+} = w_j * g(B_j^+)$ and $w_j^{B-} = w_j * g(B_j^-)$. Then the linear programming model (M-4) is as follows:

$$
\begin{aligned}
&\min \sum_{(k,l)\in Q} \theta\mu_{kl} + (1-\theta)\mu'_{kl} \\
&s.t. \begin{cases}
\sum_{(k,l)\in Q} (S_l^+ - S_k^+) = \varepsilon, \\
\mu_{kl} + (S_l^+ - S_k^+) \geq 0, \forall (k,l) \in Q, \\
\sum_{(k,l)\in Q} (S_k^- - S_l^-) = \varepsilon', \\
\mu'_{kl} + (S_k^- - S_l^-) \geq 0, \forall (k,l) \in Q, \\
S_l^+ - S_k^+ = \sum_{j=1}^{m} w_j * (\rho_1 * (f(A_{lj})^2 - f(A_{kj})^2) + \rho_2 * (g(B_{lj})^2 - g(B_{kj})^2)) - 2\rho_1 * \\
\qquad \sum_{j=1}^{m} w_j^{A+}(f(A_{lj}) - f(A_{kj})) - 2\rho_2 * \sum_{j=1}^{m} w_j^{B+}(g(B_{lj}) - g(B_{kj})), \forall (k,l) \in Q, \\
S_k^- - S_l^- = \sum_{j=1}^{m} w_j * (\rho_1 * (f(A_{kj})^2 - f(A_{lj})^2) + \rho_2 * (g(B_{kj})^2 - g(B_{lj})^2)) - 2\rho_1 * \\
\qquad \sum_{j=1}^{m} w_j^{A-}(f(A_{kj}) - f(A_{lj})) - 2\rho_2 * \sum_{j=1}^{m} w_j^{B-}(g(B_{kj}) - g(B_{lj})), \forall (k,l) \in Q, \\
w_j \in [0,1], w_j^{A+}, w_j^{A-}, w_j^{B+}, w_j^{B-} \in [0, w_j], \forall j \in [1, m], \\
\mu_{kl} \geq 0, \mu'_{kl} \geq 0, \forall (k,l) \in Q.
\end{cases}
\end{aligned}
\tag{M-4}
$$

where w_j, w_j^{A+}, w_j^{B+}, w_j^{A-} and w_j^{B-} $(j = 1, 2, ..., m)$ are decision variables. Due to $f(A_j^+) = w_j^{A+}/w_j$,

$f(A_j^-) = w_j^{A-}/w_j$, $g(B_j^+) = w_j^{B+}/w_j$ and $g(B_j^-) = w_j^{B-}/w_j$, then PIS Z-number $Z_j^+ = (f^{-1}(A_j^+), g^{-1}(B_j^+))$ and NIS Z-number $Z_j^- = (f^{-1}(A_j^-), g^{-1}(B_j^-))$ can be obtained by solving the model (M-4) using CPLEX or LINGO. The normalized weight vector $\overline{w} = (\overline{w}_1, \overline{w}_2, ..., \overline{w}_m)$ can be calculated using $\overline{w}_j = w_j/\sum_{j=1}^m w_j$. Meanwhile, μ_{kl} and μ'_{kl} for $\forall(k,l) \in Q$ can be obtained, which are the foundation of the proposed preference-based 3WD model.

3.2. Consistency-order and inconsistency-order

In this section, we define the consistency-order and inconsistency-order based on Z-LINMAP method and determine the conditional probabilities of alternatives to explore the impact of expert preference on classification results.

Definition 5. The quad $\Theta = \{X, C, Z, Q\}$ describes a decision information system. $X = \{x_1, x_2, ..., x_n\}$ is a nonempty finite alternative set and $C = \{c_1, c_2, ..., c_m\}$ is nonempty finite criteria set. $Z = (Z_{ij})_{n\times m}$ is the evaluation matrix with linguistic Z-numbers. $Q = \{(k,l)|x_k, x_l \in X\}$ is the preference relation set provided by an expert. μ_{kl} and μ'_{kl} for Θ obtained by Z-LINMAP method represent the inconsistency degrees of preference order, which can be considered as the condition of determining the equivalence class. The consistency-order P_{con} and inconsistency-order P_{incon} are defined as follows:

$$\begin{aligned} P_{con} &= \{(k,l) \in Q | (\mu_{kl} \le 0) \cap (\mu'_{kl} \le 0)\} \\ P_{incon} &= \{(k,l) \in Q | (\mu_{kl} > 0) \cup (\mu'_{kl} > 0)\} \end{aligned}, \tag{8}$$

where $\mu_{kl} \le 0$ and $\mu'_{kl} \le 0$ are two necessary conditions of consistency-order. The inconsistency degree of preference relation (k,l) under PIS or NIS case is the minumum value when the two conditions are satisfied, then (k,l) is considered to be contained in the consistency-order P_{con}. Meanwhile, the inconsistency degree at least one case (PIS or NIS) does not meet the minimum requirement when $\mu_{kl} \ge 0$ or $\mu'_{kl} \ge 0$, then (k,l) is contained in the inconsistency-order set P_{incon}.

Definition 6. Based on the definition of consistency-order and inconsistency-order, the consistency equivalence class $[x_k]_{P_{con}}$ and inconsistency equivalence class $[x_k]_{P_{incon}}$ of alternative x_k are defined as follows:

$$\begin{aligned} [x_k]_{P_{con}} &= \{x_l | (x_k P_{con} x_l) \vee (x_l P_{con} x_k) \vee x_k, x_l \in X)\} \\ [x_k]_{P_{incon}} &= \{x_l | (x_k P_{incon} x_l) \vee (x_l P_{incon} x_k) \vee x_k, x_l \in X)\} \end{aligned}, \tag{9}$$

where $x_k P_{con} x_l$ represents the preference relation $(k,l) \in P_{con}$ and $x_l P_{con} x_k$ shows the preference relation $(l,k) \in P_{con}$. The consistency equivalence class $[x_k]_{P_{con}}$ contains all elements of preference relations about x_k that satisfy the consistency-order. For example, alternatives x_l and x_h are contained in $[x_k]_{P_{con}}$ when $(k,l) \in P_{con}$ and $(h,k) \in P_{con}$. Similarly, the inconsistency equivalence class $[x_k]_{P_{incon}}$ contains all elements of preference relations about x_k that satisfy the inconsistency-order. If $(k,l) \in P_{incon}$

and $(h, k) \in P_{incon}$, then x_l and x_h belong to $[x_k]_{P_{incon}}$. Significantly, the two equivalence classes of x_k contains the alternative x_k.

Proposition 3.1. *For a non-emepty alternative set X, let $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$ be the consistency equivalence class and inconsistency equivalence class of alternative x_k, then:*

(1) The two equivalence classes $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$ satisfy the reflexivity.

(2) The two equivalence classes $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$ satisfy the symmetry.

(3) The two equivalence classes $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$ may not satisfy the transitivity.

Proof.

(1) For any $x_k \in X$, we have $x_k \in [x_k]_{P_{con}}$ and $x_k \in [x_k]_{P_{incon}}$. Therefore, $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$ satisfy the reflexivity.

(2) For any $x_k, x_l \in X$, if $x_l \in [x_k]_{P_{con}}$, then we have $x_k P_{con} x_l$ or $x_l P_{con} x_k$, and we can further get $x_k \in [x_l]_{P_{con}}$. Similarly, if $x_l \in [x_k]_{P_{incon}}$, then we have $x_k P_{incon} x_l$ or $x_l P_{incon} x_k$, and we can further get $x_k \in [x_l]_{P_{incon}}$. Therefore, $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$ satisfy the symmetry.

(3) For any $x_k, x_l, x_h \in X$, we have $x_k P_{con} x_l$ or $x_l P_{con} x_k$ if $x_l \in [x_k]_{P_{con}}$, and we have $x_k P_{con} x_h$ or $x_h P_{con} x_k$ if $x_h \in [x_k]_{P_{con}}$. Since the relation of x_l and x_h is unknown, $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$ may not satisfy the transitivity, which is consistent with the characteristic of LINMAP method. ■

Table 2
The preference coefficients in Q.

$Q(k, l)$	μ_{kl}	μ'_{kl}	$Q(k, l)$	μ_{kl}	μ'_{kl}
$Q(1, 2)$	+	+	$Q(4, 3)$	-	+
$Q(1, 3)$	-	-	$Q(3, 5)$	-	-
$Q(4, 1)$	+	-	$Q(5, 4)$	+	+
$Q(1, 5)$	-	-	$Q(6, 7)$	-	-
$Q(2, 3)$	-	-	$Q(6, 2)$	-	-
$Q(4, 2)$	+	-	$Q(7, 3)$	+	-
$Q(2, 5)$	-	+	$Q(5, 6)$	-	+

Example 3.1. Assume the preference relation set is $Q = \{Q(1, 2), Q(1, 3), Q(1, 5), Q(2, 3), Q(2, 5), Q(3, 5), Q(4, 1), Q(4, 2), Q(4, 3), Q(5, 4), Q(5, 6), Q(6, 2), Q(6, 7), Q(7, 3)\}$. Based on Z-LINMAP method, the preference coefficients μ_{kl} and μ'_{kl} can be solved. For convenience, $u_{kl} > 0$ or $u'_{kl} > 0$ are replaced by symbol "+", and symbol "-" replaces the cases of $u_{kl} \leq 0$ or $u'_{kl} \leq 0$, which are shown in Table 2. According to Eq. (8), we can obtain $P_{con} = \{Q(1, 3), Q(1, 5), Q(2, 3), Q(3, 5), Q(6, 7), Q(6, 2)\}$ and $P_{incon} = \{Q(1, 2), Q(4, 1), Q(4, 2), Q(2, 5), Q(4, 3), Q(5, 4), Q(7, 3), Q(5, 6)\}$. For alternative x_1, $Q(1, 3)$ and $Q(1, 5)$ are contained into P_{con}, then the consistency equivalence class of x_1 is $[x_1]_{P_{con}} = \{x_1, x_3, x_5\}$. Similarly, $Q(1, 2)$ and $Q(4, 1)$ belong to P_{incon}, then the inconsistency equivalence class of x_1 is

$[x_1]_{P_{incon}} = \{x_1, x_2, x_4\}$. Therefore, we have:

$$
\begin{aligned}
&[x_1]_{P_{con}} = \{x_1, x_3, x_5\}; && [x_1]_{P_{incon}} = \{x_1, x_2, x_4\};\\
&[x_2]_{P_{con}} = \{x_2, x_3, x_6\}; && [x_2]_{P_{incon}} = \{x_2, x_1, x_4, x_5\};\\
&[x_3]_{P_{con}} = \{x_3, x_1, x_2, x_5\}; && [x_3]_{P_{incon}} = \{x_3, x_4, x_7\};\\
&[x_4]_{P_{con}} = \{x_4\}; && [x_4]_{P_{incon}} = \{x_4, x_1, x_2, x_3\};\\
&[x_5]_{P_{con}} = \{x_5, x_1, x_3\}; && [x_5]_{P_{incon}} = \{x_5, x_2, x_4, x_6\};\\
&[x_6]_{P_{con}} = \{x_6, x_2, x_7\}; && [x_6]_{P_{incon}} = \{x_6, x_5\};\\
&[x_7]_{P_{con}} = \{x_7, x_6\}; && [x_7]_{P_{incon}} = \{x_7, x_3\}.
\end{aligned}
$$

Definition 7. The conditional probability $\Pr(X|[x_k]_{P_{con}})$ of alternative x_k determined by the consistency-order P_{con} and the conditional probability $\Pr(X|[x_k]_{P_{incon}})$ of alternative x_k determined by the inconsistency-order P_{incon} are denoted as follows:

$$\Pr(X|[x_k]_{P_{con}}) = \frac{\left(\sum_{x_a\in[x_k]_{P_{con}}} Z_a\right)/\#[x_k]_{P_{con}}}{\left(\sum_{x_a\in[x_k]_{P_{con}}} Z_a\right)/\#[x_k]_{P_{con}} + \left(\sum_{x_b\in[x_k]_{P_{incon}}} Z_b\right)/\#[x_k]_{P_{incon}}}, \tag{10}$$

$$\Pr(X|[x_k]_{P_{incon}}) = \frac{\left(\sum_{x_b\in[x_k]_{P_{incon}}} Z_b\right)/\#[x_k]_{P_{incon}}}{\left(\sum_{x_a\in[x_k]_{P_{con}}} Z_a\right)/\#[x_k]_{P_{con}} + \left(\sum_{x_b\in[x_k]_{P_{incon}}} Z_b\right)/\#[x_k]_{P_{incon}}}, \tag{11}$$

where $Z_a = \sum_{j=1}^{m} \overline{w}_j Z_{aj}$ and $Z_b = \sum_{j=1}^{m} \overline{w}_j Z_{bj}$ are the aggregated Z-numbers of alternatives x_a and x_b, respectively. x_a is the element in the consistency equivalence class $[x_k]_{P_{con}}$ and x_b is the element in the inconsistency equivalence class $[x_k]_{P_{incon}}$. $\#[x_k]_{P_{con}}$ and $\#[x_k]_{P_{incon}}$ are the numbers of $[x_k]_{P_{con}}$ and $[x_k]_{P_{incon}}$.

Remark 3.1. For the sake of convenience, let $Z_{[x_k]_{P_{con}}} = \sum_{x_a\in[x_k]_{P_{con}}} Z_a$ and $Z_{[x_k]_{P_{incon}}} = \sum_{x_b\in[x_k]_{P_{incon}}} Z_b$ here. According to operators of Z-numbers, $Z_{[x_k]_{P_{con}}}$ and $Z_{[x_k]_{P_{incon}}}$ should be Z-numbers. However, the part $f(A)$ of the two crisp Z-numbers $\bar{Z}_{[x_k]_{P_{con}}}$ and $\bar{Z}_{[x_k]_{P_{incon}}}$ may be beyond the boundary of $[0, 1]$. In Example 3.1, if $\{\overline{Z}_1, \overline{Z}_2, \overline{Z}_3, \overline{Z}_4, \overline{Z}_5\} = \{(0.4789, 0.2904), (0.3185, 0.8626), (0.7228, 0.4744), (0.6750, 0.3889), (0.6743, 0.6489)\}$, then we can get $\overline{Z}_{[x_1]_{P_{con}}} = (1.8760, 0.4902)$ and $\overline{Z}_{[x_1]_{P_{incon}}} = (1.4724, 0.4593)$ when $f(\cdot) = H_4(\cdot)$ and $g(\cdot) = H_1(\cdot)$. Since $1.4724 > 1$ and $1.8760 > 1$, $\overline{Z}_{[x_1]_{P_{con}}}$ and $\overline{Z}_{[x_1]_{P_{incon}}}$ cannot be converted to Z-numbers by $f^{-1}(\cdot)$. Therefore $\#[x_k]_{P_{con}}$ and $\#[x_k]_{P_{incon}}$ in Definition 7 are introduced to average the part $f(A)$. Then $\overline{Z}_{[x_1]_{P_{con}}} = (\frac{1.8760}{3}, 0.4902) = (0.6253, 0.4902)$ and $\overline{Z}_{[x_1]_{P_{incon}}} = (\frac{1.4724}{3}, 0.4593) = (0.4908, 0.4593)$.

Direct division of Z-numbers may cause information loss, so we introduce a relative close degree $CL(Z)$ transforming Z-numbers into crisp numbers to calculate these conditional probabilities, which are as follows:

$$CL(Z) = \frac{d^2(\bar{Z}, \bar{Z}^-)}{d^2(\bar{Z}, \bar{Z}^+) + d^2(\bar{Z}, \bar{Z}^-)}, \tag{12}$$

where $\bar{Z}^+$ and $\bar{Z}^-$ are the crisp Z-numbers corresponding to the best linguistic Z-number $Z^+ = (A_{2g_1}, B_{2g_2})$ and the worst linguistic Z-number $Z^- = (A_0, B_0)$, respectively. Moreover, the larger the relative close degree $CL(Z)$ is, the better the Z-number Z is.

Therefore, the conditional probabilities of alternative x_k can be calculated as follows:

$$\Pr(X|[x_k]_{P_{con}}) = \frac{CL(Z_{[x_k]_{P_{con}}})}{CL(Z_{[x_k]_{P_{con}}}) + CL(Z_{[x_k]_{P_{incon}}})}, \tag{13}$$

$$\Pr(X|[x_k]_{P_{incon}}) = \frac{CL(Z_{[x_k]_{P_{incon}}})}{CL(Z_{[x_k]_{P_{con}}}) + CL(Z_{[x_k]_{P_{incon}}})}. \tag{14}$$

where $Z_{[x_k]_{P_{con}}} = \left(\sum_{x_a \in [x_k]_{P_{con}}} Z_a\right) / \#[x_k]_{P_{con}}$ and $Z_{[x_k]_{P_{incon}}} = \left(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b\right) / \#[x_k]_{P_{incon}}$ are the aggregated Z-numbers based on the consistency and inconsistency equivalence classes.

Proposition 3.2. *The conditional probabilities determined by consistency-order and inconsistency-order meet the following conditions:*

(1) $\Pr(X|[x_k]_{P_{con}}) + \Pr(X|[x_k]_{P_{incon}}) = 1$;

(2) $\Pr(\neg X|[x_k]_{P_{con}}) + \Pr(\neg X|[x_k]_{P_{incon}}) = 1$.

Proof.

(1) $\Pr(X|[x_k]_{P_{con}}) + \Pr(X|[x_k]_{P_{incon}}) = \frac{CL(\sum_{x_a \in [x_k]_{P_{con}}} Z_a) + CL(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b)}{CL(\sum_{x_a \in [x_k]_{P_{con}}} Z_a) + CL(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b)} = 1$.

(2) $\Pr(\neg X|[x_k]_{P_{con}}) = 1 - \Pr(X|[x_k]_{P_{con}}) = \frac{CL(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b)}{CL(\sum_{x_a \in [x_k]_{P_{con}}} Z_a) + CL(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b)}$; and $\Pr(\neg X|[x_k]_{P_{incon}}) = 1 - \Pr(X|[x_k]_{P_{incon}}) = \frac{CL(\sum_{x_a \in [x_k]_{P_{con}}} Z_a)}{CL(\sum_{x_a \in [x_k]_{P_{con}}} Z_a) + CL(\sum_{x_b \in [x_k]_{P_{incon}}} Z_b)}$. Therefore, $\Pr(\neg X|[x_k]_{P_{con}}) + \Pr(\neg X|[x_k]_{P_{incon}}) = 1$.■

Example 3.2. Assume $g_1 = 3$ and $g_2 = 2$ and let $A = \{A_0 : very\ poor,\ A_1 : poor,\ A_2 : slightlypoor,\ A_3 : fair,\ A_4 : slightlygood, A_5 : good,\ A_6 : very\ good\}$ and $B = \{B_0 : more\ unlikely,\ B_1 : unlikely,\ B_2 : fairly,\ B_3 : likely,\ B_4 : more\ likely\}$. Then $Z^+ = (A_6, B_4)$ and $Z^- = (A_0, B_0)$ can be obtained. The possible LSFs of A and B are $f(\cdot) = H_4(\cdot)$ and $g(\cdot) = H_1(\cdot)$. The alternative set is $X = \{x_1, x_2, x_3, x_4, x_5\}$ and the criteria set is $C = \{c_1, c_2\}$. The weight of criteria is $\overline{w} = (0.35, 0.65)$. The parameters are assumed as $\rho_1 = 0.8$ and $\rho_2 = 0.2$. The LZNs matrix Z is as shown in Table 3. The equivalence classes of all alternatives are shown in Table 4.

Firstly, the LZNs matrix Z can be transformed into the crisp Z-numbers matrix $\overline{Z} = (\overline{Z}_{ij})_{n\times m}$, which is as shown in Table 5. The best LZN and the worst LZN can be transformed into $\overline{Z}^+ = (1, 1)$ and $\overline{Z}^- = (0, 0)$. The aggregated Z-numbers $\overline{Z}_i = \sum_{j=1}^{m} \overline{w}_j \overline{Z}_{ij}$ $(i = 1, 2, 3, 4, 5)$ can be calculated as follows:

$\overline{Z}_1 = \overline{w}_1 * \overline{Z}_{11} + \overline{w}_2 * \overline{Z}_{12} = (0.35 * 0.2208, 0.5000) + (0.65 * 0.6179, 0.2500) = (0.0773, 0.5000) + (0.4016, 0.2500) = (0.0773 + 0.4016, \frac{0.0773*0.5000+0.4016*0.2500}{0.0773+0.4016}) = (0.4789, 0.2904)$;

$\overline{Z}_2 = \overline{w}_1 * \overline{Z}_{21} + \overline{w}_2 * \overline{Z}_{22} = (0.3185, 0.8626);$

$\overline{Z}_3 = \overline{w}_1 * \overline{Z}_{31} + \overline{w}_2 * \overline{Z}_{32} = (0.7228, 0.4744);$

$\overline{Z}_4 = \overline{w}_1 * \overline{Z}_{41} + \overline{w}_2 * \overline{Z}_{42} = (0.6750, 0.3889);$

$\overline{Z}_5 = \overline{w}_1 * \overline{Z}_{51} + \overline{w}_2 * \overline{Z}_{52} = (0.6743, 0.6489).$

Table 3
The LZNs matrix Z.

x_i	x_1	x_2	x_3	x_4	x_5
c_1	(A_1, B_2)	(A_3, B_3)	(A_4, B_4)	(A_6, B_3)	(A_5, B_2)
c_2	(A_4, B_1)	(A_1, B_4)	(A_5, B_1)	(A_3, B_0)	(A_4, B_3)

According to the given equivalence classes in Table 4, we can calculate $\overline{Z}_{[x_k]_{P_{con}}}$ and $\overline{Z}_{[x_k]_{P_{incon}}}$ as follows:

$\overline{Z}_{[x_1]_{P_{con}}} = (0.6253, 0.4902); \overline{Z}_{[x_1]_{P_{incon}}} = (0.4908, 0.4593);$

$\overline{Z}_{[x_2]_{P_{con}}} = (0.5207, 0.5931); \overline{Z}_{[x_2]_{P_{incon}}} = (0.5367, 0.5189);$

$\overline{Z}_{[x_3]_{P_{con}}} = (0.5486, 0.5442); \overline{Z}_{[x_3]_{P_{incon}}} = (0.6989, 0.4331);$

$\overline{Z}_{[x_4]_{P_{con}}} = (0.6750, 0.3889); \overline{Z}_{[x_4]_{P_{incon}}} = (0.5488, 0.4643);$

$\overline{Z}_{[x_5]_{P_{con}}} = (0.6253, 0.4902); \overline{Z}_{[x_5]_{P_{incon}}} = (0.5559, 0.5845).$

Finally, the conditional probabilities of X are calculated using Eqs. (13) and (14) as follows:

$\Pr(X|[x_1]_{P_{con}}) = \frac{0.6872}{0.6872+0.4691} = 0.5943; \Pr(X|[x_1]_{P_{incon}}) = \frac{0.4691}{0.6872+0.4691} = 0.4057;$

$\Pr(X|[x_2]_{P_{con}}) = 0.5017; \Pr(X|[x_2]_{P_{incon}}) = 0.4983;$

$\Pr(X|[x_3]_{P_{con}}) = 0.4396; \Pr(X|[x_3]_{P_{incon}}) = 0.5604;$

$\Pr(X|[x_4]_{P_{con}}) = 0.5585; \Pr(X|[x_4]_{P_{incon}}) = 0.4415;$

$\Pr(X|[x_5]_{P_{con}}) = 0.5251; \Pr(X|[x_5]_{P_{incon}}) = 0.4749.$

Table 4
The equivalence classes in X.

x_i	$[x_i]_{P_{con}}$	$[x_i]_{P_{incon}}$
x_1	x_1, x_3, x_5	x_1, x_2, x_4
x_2	x_2, x_3	x_2, x_1, x_4, x_5
x_3	x_3, x_1, x_2, x_5	x_3, x_4
x_4	x_4	x_4, x_1, x_2, x_3
x_5	x_5, x_1, x_3	x_5, x_2, x_4

Table 5
The crisp Z-numbers matrix $\overline{Z}$.

X	x_1	x_2	x_3	x_4	x_5
c_1	(0.2208, 0.5000)	(0.5000, 0.7500)	(0.6179, 1.0000)	(1.0000, 0.7500)	(0.7792, 0.5000)
c_2	(0.6179, 0.2500)	(0.2208, 1.0000)	(0.7792, 0.2500)	(0.5000, 0)	(0.6179, 0.7500)

3.3. The loss functions and individual 3WD decision rules

Yao [42] proposed the relative loss functions, i.e., the loss functions corresponding to three actions in the same state are proportional, and that of the optimal action is 0. However, the relative loss functions ignored the experts' psychology of regret when taking different actions. Therefore, we take into account the RT when determining the loss functions of different experts. Let the expert set be $E = \{e_1, e_2, ..., e_\Gamma\}$ and e_h represents the hth expert. The relative loss functions $\lambda^{hk}_{\bullet\circ}(\bullet = P, B, N; \circ = P, N)$ provided by e_h for alternative x_k is shown in Table 6, where σ_h is the risk avoidance coefficient of e_h.

Table 6
The relative loss functions provided by e_h.

Actions	$X(P)$	$\neg X(N)$
a_P	$\lambda^{hk}_{PP} = 0$	$\lambda^{hk}_{PN} = \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z^-_j)^2$
a_B	$\lambda^{hk}_{BP} = \sigma_h \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z^+_j)^2$	$\lambda^{hk}_{BN} = \sigma_h \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z^-_j)^2$
a_N	$\lambda^{hk}_{NP} = \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z^+_j)^2$	$\lambda^{hk}_{NN} = 0$

The conditional loss function of taking three actions a_P, a_B and a_N under two states $\Omega = \{X, \neg X\}$ is denoted as $l(a_\bullet|\circ)(\bullet = P, B, N; \circ = P, N)$, which represents the loss that taking the action $\bullet$ when the object is under the state $\circ$. Then the set of conditional loss function is $\Upsilon = \{l(a_P|P), l(a_B|P), l(a_N|P), l(a_P|N), l(a_B|N), l(a_N|N)\}$, where $l(a_\bullet|\circ)$ can be represented by $\lambda^{hk}_{\bullet\circ}$ in Table 6. Assume $a_\bullet$ and a_* $(\bullet, * = P, B, N)$ represent two different actions. Based on the RT, we define a perceived loss function $L(a_\bullet|\circ, a_*|\circ)$ taking action $a_\bullet$ instead of action a_* as follows:

$$L(a_\bullet|\circ, a_*|\circ) = l(a_\bullet|\circ) + r(l(a_\bullet|\circ) - l(a_*|\circ)); \bullet, * = P, B, N; \circ = P, N;\ \bullet \neq *; \tag{15}$$

where $r(\Delta l)$ is the regret-rejoice function. Let the loss difference be $\Delta l = l(a_\bullet|\circ) - l(a_*|\circ)$, then the regret-rejoice function $r(\Delta l)$ is calculated as follows:

$$r(\Delta l) = \begin{cases} 1 - e^{-\delta_h \times \Delta l}, & l(a_\bullet|\circ) \geq l(a_*|\circ) \\ 0, & l(a_\bullet|\circ) < l(a_*|\circ) \end{cases}. \tag{16}$$

Then the regret loss function $\overline{\lambda}^{hk}_{\bullet\circ}$ is considered as the total losses of taking action $\bullet$ instead of the other two actions under the state $\circ$, which is denoted as follows:

$$\overline{\lambda}^{hk}_{\bullet\circ} = L(a_\bullet|\circ, a_*|\circ) + L(a_\bullet|\circ, a_\star|\circ); \bullet = P, B, N; \circ = P, N; \tag{17}$$

where $\bullet$, $*$ and $\star$ represent three different actions. Let $\overline{S}_k^+ = \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z_j^+)^2$ and $\overline{S}_k^- = \sum_{j=1}^{m} \overline{w}_j d(Z_{kj}, Z_j^-)^2$, then we can obtain all regret loss functions shown in Table 7, which are as follows:

$\overline{\lambda}_{PP}^{hk} = \lambda_{PP}^{hk}+r(\lambda_{PP}^{hk} - \lambda_{BP}^{hk})+\lambda_{PP}^{hk}+r(\lambda_{PP}^{hk} - \lambda_{NP}^{hk}) = 0;$

$\overline{\lambda}_{BP}^{hk} = \lambda_{BP}^{hk}+r(\lambda_{BP}^{hk} - \lambda_{PP}^{hk})+\lambda_{BP}^{hk}+r(\lambda_{BP}^{hk} - \lambda_{NP}^{hk}) = 2\sigma_h\overline{S}_k^+ +1 - e^{-\delta_h\sigma_h\overline{S}_k^+};$

$\overline{\lambda}_{NP}^{hk} = \lambda_{NP}^{hk}+r(\lambda_{NP}^{hk} - \lambda_{PP}^{hk})+\lambda_{NP}^{hk}+r(\lambda_{NP}^{hk} - \lambda_{BP}^{hk}) = 2\overline{S}_k^+ +1 - e^{-\delta_h\overline{S}_k^+} +1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^+};$

$\overline{\lambda}_{PN}^{hk} = \lambda_{PN}^{hk}+r(\lambda_{PN}^{hk} - \lambda_{BN}^{hk})+\lambda_{PN}^{hk}+r(\lambda_{PN}^{hk} - \lambda_{NN}^{hk}) = 2\overline{S}_k^- +1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-} +1 - e^{-\delta_h\overline{S}_k^-};$

$\overline{\lambda}_{BN}^{hk} = \lambda_{BN}^{hk}+r(\lambda_{BN}^{hk} - \lambda_{PN}^{hk})+\lambda_{BN}^{hk}+r(\lambda_{BN}^{hk} - \lambda_{NN}^{hk}) = 2\sigma_h\overline{S}_k^- +1 - e^{-\delta_h\sigma_h\overline{S}_k^-};$

$\overline{\lambda}_{NN}^{hk} = \lambda_{NN}^{hk}+r(\lambda_{NN}^{hk} - \lambda_{PN}^{hk})+\lambda_{NN}^{hk}+r(\lambda_{NN}^{hk} - \lambda_{BN}^{hk}) = 0.$

Table 7
The regret loss functions.

Actions	$X(P)$	$\neg X(N)$
a_P	$\overline{\lambda}_{PP}^{hk}$	$\overline{\lambda}_{PN}^{hk}$
a_B	$\overline{\lambda}_{BP}^{hk}$	$\overline{\lambda}_{BN}^{hk}$
a_N	$\overline{\lambda}_{NP}^{hk}$	$\overline{\lambda}_{NN}^{hk}$

Proposition 3.3. *The regret loss function $\overline{\lambda}_{\bullet\circ}^{hk}(\bullet = P, B, N; \circ = P, N)$ meets the following conditions: (1) $0 \le \overline{\lambda}_{PP}^{hk} \le \overline{\lambda}_{BP}^{hk} \le \overline{\lambda}_{NP}^{hk}$; (2) $0 \le \overline{\lambda}_{NN}^{hk} \le \overline{\lambda}_{BN}^{hk} \le \overline{\lambda}_{PN}^{hk}$.*

Proof.

(1) For $\overline{\lambda}_{\bullet P}$, we can obtain $\overline{\lambda}_{NP}^{hk} - \overline{\lambda}_{BP}^{hk} = 2\overline{S}_k^+ +1 - e^{-\delta_h\overline{S}_k^+} +1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^+} - (2\sigma_h\overline{S}_k^+ +1 - e^{-\delta_h\sigma_h\overline{S}_k^+}) = 2(1-\sigma_h)\overline{S}_k^+ + (e^{-\delta_h\sigma_h\overline{S}_k^+} - e^{-\delta_h\overline{S}_k^+}) + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^+}$. Due to $\sigma_h \in [0, 1]$, $\delta_h \ge 0$ and $\overline{S}_k^+ \ge 0$, we can get two inequalities: $2(1-\sigma_h)\overline{S}_k^+ \ge 0$, $e^{-\delta_h\sigma_h\overline{S}_k^+} - e^{-\delta_h\overline{S}_k^+} \ge 0$ and $1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^+} \ge 0$. Then we have $\overline{\lambda}_{NP}^{hk} - \overline{\lambda}_{BP}^{hk} \ge 0$. Meanwhile, we have $0 \le \overline{\lambda}_{PP}^{hk} \le \overline{\lambda}_{BP}^{hk} \le \overline{\lambda}_{NP}^{hk}$ due to $\overline{\lambda}_{PP}^{hk} = 0$;

(2) For $\overline{\lambda}_{\bullet N}$, we can obtain $\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{BN}^{hk} = 2\overline{S}_k^- +1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-} +1 - e^{-\delta_h\overline{S}_k^-} - (2\sigma_h\overline{S}_k^- +1 - e^{-\delta_h\sigma_h\overline{S}_k^-}) = 2(1-\sigma_h)\overline{S}_k^- + (e^{-\delta_h\sigma_h\overline{S}_k^-} - e^{-\delta_h\overline{S}_k^-}) + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-}$. Due to $\sigma_h \in [0, 1]$, $\delta_h \ge 0$ and $\overline{S}_k^- \ge 0$, we can get two inequalities: $2(1-\sigma_h)\overline{S}_k^- \ge 0$, $e^{-\delta_h\sigma_h\overline{S}_k^-} - e^{-\delta_h\overline{S}_k^-} \ge 0$ and $1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-} \ge 0$. Then we have $\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{BN}^{hk} \ge 0$. Meanwhile, we have $0 \le \overline{\lambda}_{NN}^{hk} \le \overline{\lambda}_{BN}^{hk} \le \overline{\lambda}_{PN}^{hk}$ due to $\overline{\lambda}_{NN}^{hk} = 0$. ■

The expected regret loss function $R_h(a_\bullet|[x_k]_{P_{con}})$ of taking three actions for expert e_h can be calculated as:

$$\begin{aligned} R_h(a_P|[x_k]_{P_{con}}) &= \overline{\lambda}_{PP}^{hk}\Pr_h(X|[x_k]_{P_{con}}) + \overline{\lambda}_{PN}^{hk}\Pr_h(X|[x_k]_{P_{incon}}); \\ R_h(a_B|[x_k]_{P_{con}}) &= \overline{\lambda}_{BP}^{hk}\Pr_h(X|[x_k]_{P_{con}}) + \overline{\lambda}_{BN}^{hk}\Pr_h(X|[x_k]_{P_{incon}}); \\ R_h(a_N|[x_k]_{P_{con}}) &= \overline{\lambda}_{NP}^{hk}\Pr_h(X|[x_k]_{P_{con}}) + \overline{\lambda}_{NN}^{hk}\Pr_h(X|[x_k]_{P_{incon}}). \end{aligned} \tag{18}$$

According to the Bayesian theory, the following decision rules of minimum regret losses can be demonstrated as follows:

(P3) If $R_h(a_P|[x_k]_{P_{con}}) \le R_h(a_B|[x_k]_{P_{con}})$ and $R_h(a_P|[x_k]_{P_{con}}) \le R(a_N|[x_k]_{P_{con}})$, then $x \in POS^h(X)$;

(B3) If $R_h(a_B|[x_k]_{P_{con}}) \leq R_h(a_P|[x_k]_{P_{con}})$ and $R_h(a_B|[x_k]_{P_{con}}) \leq R_h(a_N|[x_k]_{P_{con}})$, then $x \in BND^h(X)$;

(N3) If $R_h(a_N|[x_k]_{P_{con}}) \leq R_h(a_P|[x_k]_{P_{con}})$ and $R_h(a_N|[x_k]_{P_{con}}) \leq R_h(a_B|[x_k]_{P_{con}})$, then $x \in NEG^h(X)$.

To simplify the decision rules (P3)-(N3), the following thresholds of e_h for alternative x_k can be obtained as follows:

$$\alpha^{hk} = \frac{\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{BN}^{hk}}{(\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{BN}^{hk}) + (\overline{\lambda}_{BP}^{hk} - \overline{\lambda}_{PP}^{hk})} = \frac{2(1-\sigma_h)\overline{S}_k^- + (e^{-\delta_h\sigma_h\overline{S}_k^-} - e^{-\delta_h\overline{S}_k^-}) + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-}}{2(1-\sigma_h)\overline{S}_k^- + (e^{-\delta_h\sigma_h\overline{S}_k^-} - e^{-\delta_h\overline{S}_k^-}) + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-} + 2\sigma_h\overline{S}_k^+ + 1 - e^{-\delta_h\sigma_h\overline{S}_k^+}};$$

$$\gamma^{hk} = \frac{\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{NN}^{hk}}{(\overline{\lambda}_{PN}^{hk} - \overline{\lambda}_{NN}^{hk}) + (\overline{\lambda}_{NP}^{hk} - \overline{\lambda}_{PP}^{hk})} = \frac{2\overline{S}_k^- + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-} + 1 - e^{-\delta_h\overline{S}_k^-}}{2\overline{S}_k^- + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^-} + 1 - e^{-\delta_h\overline{S}_k^-} + 2\overline{S}_k^+ + 1 - e^{-\delta_h\overline{S}_k^+} + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^+}};$$

$$\beta^{hk} = \frac{\overline{\lambda}_{BN}^{hk} - \overline{\lambda}_{NN}^{hk}}{(\overline{\lambda}_{BN}^{hk} - \overline{\lambda}_{NN}^{hk}) + (\overline{\lambda}_{NP}^{hk} - \overline{\lambda}_{BP}^{hk})} = \frac{2\sigma_h\overline{S}_k^- + 1 - e^{-\delta_h\sigma_h\overline{S}_k^-}}{2\sigma_h\overline{S}_k^- + 1 - e^{-\delta_h\sigma_h\overline{S}_k^-} + 2(1-\sigma_h)\overline{S}_k^+ + (e^{-\delta_h\sigma_h\overline{S}_k^+} - e^{-\delta_h\overline{S}_k^+}) + 1 - e^{-\delta_h(1-\sigma_h)\overline{S}_k^+}}.$$

Then the simplified decision rules of expert e_h related to three thresholds $\{\alpha^{hk}, \beta^{hk}, \gamma^{hk}\}$ and the conditional probability $\Pr_h(X|[x_k]_{P_{con}})$ are as follows:

(P4) If $\Pr_h(X|[x_k]_{P_{con}}) \geq \alpha^{hk}$ and $\Pr_h(X|[x_k]_{P_{con}}) \geq \gamma^{hk}$, then $x_k \in POS^h(X)$;

(B4) If $\Pr_h(X|[x_k]_{P_{con}}) < \alpha^{hk}$ and $\Pr_h(X|[x_k]_{P_{con}}) > \beta^{hk}$, then $x_k \in BND^h(X)$;

(N4) If $\Pr_h(X|[x_k]_{P_{con}}) < \gamma^{hk}$ and $\Pr_h(X|[x_k]_{P_{con}}) \leq \beta^{hk}$, then $x_k \in NEG^h(X)$.

3.4. Preference-based regret 3WD method on MDISs

Based on the above methods, if expert e_h gives the preference set Q_h and Z-number decision information system Z^h, then we can obtain three regions $POS^h(X), BND^h(X)$ and $NEG^h(X)$ for e_h according to the decision rules (P4)-(N4). For the expert set $E = \{e_1, e_2, ..., e_\Gamma\}$ and expert weight set $V = \{v_1, v_2, ..., v_\Gamma\}$ where $\sum_{h=1}^{\Gamma} v_h = 1$, we design the comprehensive decision rules integrating MDISs and preferences, which can be divided into two scenarios:

- If alternative x_k belongs to the same region ($POS^h(X)$, $BND^h(X)$ or $NEG^h(X)$) for all Z-numbers DISs, which means that the classification results of x_k by all experts are consistent. Then x_k is contained in the region all experts support.

- If alternative x_k is divided into different regions under different MDISs, which means that the classification results of x_k for all experts may be inconsistent. Then the weighted expected regret loss function $\overline{R}(a_\bullet|[x]_{P_{con}})$ can be calculated as follows:

$$\overline{R}(a_\bullet|[x_k]_{P_{con}}) = \sum_{h=1}^{\Gamma} v_h R_h(a_\bullet|[x_k]_{P_{con}}); \bullet = P, B, N; k = 1, 2, ..., n. \tag{19}$$

Then the comprehensive decision rules are as follows:

(P5) If $\overline{R}(a_P|[x_k]_{P_{con}}) \leq \overline{R}(a_B|[x_k]_{P_{con}})$ and $\overline{R}(a_P|[x_k]_{P_{con}}) \leq \overline{R}(a_N|[x_k]_{P_{con}})$, then $x \in POS(X)$;

(B5) If $\overline{R}(a_B|[x_k]_{P_{con}}) \leq \overline{R}(a_P|[x_k]_{P_{con}})$ and $\overline{R}(a_B|[x_k]_{P_{con}}) \leq \overline{R}(a_N|[x_k]_{P_{con}})$, then $x \in BND(X)$;

(N5) If $\overline{R}(a_N|[x_k]_{P_{con}}) \leq \overline{R}(a_P|[x_k]_{P_{con}})$ and $\overline{R}(a_N|[x_k]_{P_{con}}) \leq \overline{R}(a_B|[x_k]_{P_{con}})$, then $x \in NEG(X)$.

To rank these alternatives, we define a loss score function $LS(x_k)$ for alternative x_k as follows:

$$LS(x_k) = \frac{|POS_k|\bar{R}(a_P|[x_k]_{P_{con}}) + |BND_k|\bar{R}(a_B|[x_k]_{P_{con}}) + |NEG_k|\bar{R}(a_N|[x_k]_{P_{con}})}{|POS_k| + |BND_k| + |NEG_k|}, \tag{20}$$

where $|POS_k|$ ($|BND_k|$ or $|NEG_k|$) is the number of x_k divided into the region $POS^h(X)$ ($BND^h(X)$ or $NEG^h(X)$) for all experts and $|POS_k| + |BND_k| + |NEG_k| = \Gamma$, which means that the smaller the loss score $LS(x_k)$ is, the better the alternative x_k is.

3.5. The time complexity analysis of the proposed 3WD model

The preference-based regret 3WD method on MDISs with LZNs is shown in Algorithm 1. Suppose that n, m and Γ are the numbers of alternatives, criteria and experts, respectively. $|Q_{e_h}|$ stands for the number of preference relations (k, l) given by experts. Obviously, the Algorithm 1 is divided into two parts including lines 2-18 and lines 19-28. For the first part, the lines 3-5 have the time complexity with $O(\Gamma mn)$. The lines 6-10 have the time complexity with $O(\Gamma|Q_{e_h}|)$, and the maximum of complexity is $O(\frac{\Gamma n^2-\Gamma n}{2})$ due to $max\{|Q_{e_h}|\} = \frac{n^2-n}{2}$. The lines 11-14 calculate the probabilities of n alternatives with a time complexity of $O(\Gamma n)$. The lines 15-17 have the time complexity of $O(3\Gamma)$ to obtain the classification results of Γ experts. For the second part, the comprehensive regions for all alternatives can be obtained in lines 19-27, whose time complexity is $O(n)$. Finally, the line 28 have a time complexity of $O(1)$. Therefore, the time complexity of the first part is $O(\Gamma mn + \frac{\Gamma n^2-\Gamma n}{2} + \Gamma n + 3\Gamma) \approx O(\Gamma mn + \Gamma n^2)$, and the second part has the time complexity of $O(n + 1) \approx O(n)$. In short, the overall time complexity of Algorithm 1 for the proposed 3WD model is $O(\Gamma mn + \Gamma n^2) = max\{O(\Gamma mn), O(\Gamma n^2)\}$.

4. Numerical analysis

4.1. The description of image recognition case based on human-computer interaction

Image recognition is an important field of artificial intelligence and pattern recognition, which uses computer technology to process and analyze images to identify targets and objects of different patterns. High-precision image recognition technology often relies on high-quality pictures, which are difficult to obtain in real life. Therefore, image recognition sometimes needs the aid of human empirical knowledge, i.e., human-computer recognition. Low quality images sometimes even require complete human recognition.

For a image recognition case of human-computer interaction, two states $\Omega = \{X, \neg X\}$ represent that the task is suitable for machine or human, and three actions a_P, a_B and a_N indicate the computer recognition, human-computer recognition and human recognition, respectively. The image task set is $X = \{x_1, x_2, ..., x_7\}$ and four kinds of image features are used to evaluate the image tasks including color feature (c_1), texture feature (c_2), shape feature (c_3) and spatial relationship feature (c_4). Four experts $E = \{e_1, e_2, e_3, e_4\}$ with different preferences Q_{e_h} are invited to evaluate the image tasks by providing LZNs information. The linguistic sets are $A = \{A_0 : \textit{very poor},\ A_1 : \textit{poor},\ A_2 : \textit{slightlypoor},\ A_3 : \textit{fair},\ A_4 : \textit{slightlygood}, A_5 : \textit{good},\ A_6 : \textit{very good}\}$ and $B = \{B_0 : \textit{more unlikely},\ B_1 : \textit{unlikely},\ B_2 : \textit{fairly},\ B_3 : \textit{likely},\ B_4 : \textit{more likely}\}$. The possible LSFs of A and B are $f(\cdot) = H_4(\cdot)$ and $g(\cdot) = H_1(\cdot)$. The MDISs with LZNs of four experts are as shown in Table 8. For convenience, the preference sets Q_{e_i} provided by E are rewritten as as the preference matrixes $\overline{Q}_{e_i}$, which are as follows:

$$\overline{Q}_{e_1} = \begin{bmatrix} - & (2,1) & (1,3) & (1,4) & (5,1) & (6,1) & (7,1) \\ - & - & (3,2) & (2,4) & (5,2) & (2,6) & (7,2) \\ - & - & - & (3,4) & (3,5) & (3,6) & (3,7) \\ - & - & - & - & (4,5) & (6,4) & (7,4) \\ - & - & - & - & - & (5,6) & (7,5) \\ - & - & - & - & - & - & (6,7) \\ - & - & - & - & - & - & - \end{bmatrix}, \overline{Q}_{e_2} = \begin{bmatrix} - & (2,1) & (1,3) & (4,1) & (5,1) & (6,1) & (7,1) \\ - & - & (2,3) & (4,2) & (5,2) & (6,2) & (7,2) \\ - & - & - & (3,4) & (3,5) & (3,6) & (3,7) \\ - & - & - & - & (5,4) & (6,4) & (7,4) \\ - & - & - & - & - & (6,5) & (5,7) \\ - & - & - & - & - & - & (7,6) \\ - & - & - & - & - & - & - \end{bmatrix};$$

$$\overline{Q}_{e_3} = \begin{bmatrix} - & (1,2) & (3,1) & (4,1) & (5,1) & (1,6) & (1,7) \\ - & - & (2,3) & (2,4) & (2,5) & (2,6) & (2,7) \\ - & - & - & (3,4) & (3,5) & (3,6) & (3,7) \\ - & - & - & - & (5,4) & (6,4) & (7,4) \\ - & - & - & - & - & (5,6) & (7,5) \\ - & - & - & - & - & - & (7,6) \\ - & - & - & - & - & - & - \end{bmatrix}, \overline{Q}_{e_4} = \begin{bmatrix} - & (2,1) & (3,1) & (1,4) & (5,1) & (6,1) & (1,7) \\ - & - & (3,2) & (2,4) & (2,5) & (6,2) & (7,2) \\ - & - & - & (4,3) & (5,3) & (6,3) & (7,3) \\ - & - & - & - & (4,5) & (4,6) & (4,7) \\ - & - & - & - & - & (6,5) & (7,5) \\ - & - & - & - & - & - & (7,6) \\ - & - & - & - & - & - & - \end{bmatrix}.$$

For the selection of parameters, $\rho_1 = 0.6$ and $\rho_2 = 0.4$ are assumed which indicates the part A is more important that part B in Z-numbers. Let $\varepsilon = \varepsilon' = 1$ and $\theta = 0.5$ in Z-LINMAP method. Assume that four expert have equal weights $v_h = 0.25$, and risk avoidance coefficients are $\sigma_h = 0.45$ and regret aversion coefficients are $\delta_h = 0.8$, $h = 1, 2, 3, 4$.

Algorithm 1: preference-based 3WD method on MDISs with LZNs

Input: Alternative set $X = \{x_1, x_2, ..., x_n\}$, criteria set $C = \{c_1, c_2, ..., c_m\}$, expert set $E = \{e_1, e_2, ..., e_\Gamma\}$, expert weight $V = \{v_1, v_2, ..., v_\Gamma\}$, LZNs evaluation matrix $Z^h = (Z^h_{ij})_{n\times m}$ $(h = 1, 2, ..., \Gamma)$ and preference relation set $Q_{e_h} = \{(k,l)|x_k, x_l \in X\}$ of expert e_h. The possible LSFs $f(\cdot)$ and $g(\cdot)$ and the parameters $\rho_1, \rho_2, \theta, \sigma_h$ and δ_h.

Output: The ranking and classification results of alternative set X.

begin
  for $h = 1, 2, ..., \Gamma$ **do**
    for $i = 1$ *to* n, $j = 1$ *to* m **do**
      Transform $Z^h = (Z^h_{ij})_{n\times m}$ into the crisp Z-number matrix $\overline{Z}^h = (\overline{Z}^h_{ij})_{n\times m}$ using Eq. (1).
    end
    for $(k,l) \in Q_{e_h}$ **do**
      Compute $B = \sum_{(k,l)\in Q} (S^+_l - S^+_k)^-$ and $G = \sum_{(k,l)\in Q} (S^+_l - S^+_k)^+$ using Eqs. (5) and (6);
      Compute $B' = \sum_{(k,l)\in Q} (S^-_l - S^-_k)^-$ and $G' = \sum_{(k,l)\in Q} (S^-_l - S^-_k)^+$ using Eqs. (5) and (7);
      Solve the linear programming model (M-4) to obtain the PIS Z-number Z^{h+}_j, NIS Z-numbers Z^{h-}_j and normalized weight vector $\overline{w}^h = (\overline{w}^h_1, \overline{w}^h_2, ..., \overline{w}^h_m)$.
    end
    for $i = 1, 2, ..., n$ **do**
      Determine the consistency equivalence class $[x_i]_{P_{con}}$ and inconsistency equivalence class $[x_i]_{P_{incon}}$ using Eq. (9);
      Calculate the conditional probabilities $\Pr_h(X|[x_i]_{P_{con}})$ and $\Pr_h(X|[x_i]_{P_{incon}})$ using Eqs. (13) and (14).
    end
    Compute relative loss functions $\lambda^h_{\bullet\circ}(\bullet = P, B, N; \circ = P, N)$ in Table 6 and regret loss functions $\overline{\lambda}^h_{\bullet\circ}$ using Eq. (17);
    Calculate the expected regret loss function $R_h(a_\bullet|[x]_{P_{con}})$ using Eq. (18);
    Obtain the three region $POS^h(X)$, $BND^h(X)$ and $NEG^h(X)$ using decision rules (P3)-(N3) or (P4)-(N4).
  end
  for $i = 1, 2, ..., n$ **do**
    if $x_i \in POS^h(X)\left(BND^h(X) \text{ or } NEG^h(X)\right)$, $\forall h \in [1, \Gamma]$ **then**
      $x_i \in POS(X)\,(BND(X) \text{ or } NEG(X))$;
    else
      Calculate the weighted expected regret loss function $\overline{R}(a_\bullet|[x_i]_{P_{con}})$ using Eq. (19);
      Determine the region that x_i belongs to according to the decision rules (P5)-(N5).
    end
    Calculate the loss score $LS(x_i)$ using Eq. (20).
  end
  Rank the loss scores of all alternatives in ascending order.
  return Three region $POS(X)$, $BND(X)$ and $NEG(X)$ and ranking result of all alternatives.
end

4.2. Decision analysis of the proposed 3WD model

In this subsection, the proposed MCDM framework is used to classify and rank these image recognition tasks of human-computer interaction. The decision steps are as follows:

Table 8
The MDISs with LZNs of four experts.

Experts	Criteria	x_1	x_2	x_3	x_4	x_5	x_6	x_7
e_1	c_1	(A_2,B_4)	(A_0,B_3)	(A_4,B_4)	(A_2,B_3)	(A_5,B_1)	(A_3,B_4)	(A_3,B_1)
	c_2	(A_4,B_1)	(A_5,B_2)	(A_3,B_3)	(A_3,B_1)	(A_5,B_3)	(A_6,B_0)	(A_2,B_3)
	c_3	(A_1,B_1)	(A_4,B_3)	(A_6,B_3)	(A_0,B_1)	(A_3,B_1)	(A_5,B_4)	(A_1,B_3)
	c_4	(A_6,B_4)	(A_4,B_2)	(A_6,B_4)	(A_5,B_2)	(A_6,B_3)	(A_5,B_4)	(A_6,B_4)
e_2	c_1	(A_3,B_0)	(A_1,B_2)	(A_1,B_2)	(A_6,B_3)	(A_6,B_2)	(A_0,B_1)	(A_5,B_4)
	c_2	(A_5,B_1)	(A_4,B_1)	(A_1,B_0)	(A_1,B_1)	(A_6,B_4)	(A_3,B_1)	(A_6,B_2)
	c_3	(A_1,B_2)	(A_2,B_2)	(A_4,B_3)	(A_5,B_1)	(A_6,B_3)	(A_1,B_2)	(A_3,B_4)
	c_4	(A_5,B_3)	(A_2,B_2)	(A_2,B_3)	(A_5,B_2)	(A_4,B_4)	(A_1,B_1)	(A_6,B_1)
e_3	c_1	(A_4,B_4)	(A_6,B_2)	(A_6,B_2)	(A_3,B_1)	(A_3,B_2)	(A_0,B_4)	(A_0,B_2)
	c_2	(A_5,B_1)	(A_4,B_4)	(A_6,B_4)	(A_1,B_1)	(A_6,B_2)	(A_3,B_3)	(A_5,B_2)
	c_3	(A_1,B_2)	(A_2,B_4)	(A_4,B_3)	(A_5,B_1)	(A_3,B_2)	(A_1,B_2)	(A_4,B_2)
	c_4	(A_5,B_3)	(A_2,B_2)	(A_2,B_3)	(A_5,B_2)	(A_4,B_4)	(A_5,B_3)	(A_0,B_1)
e_4	c_1	(A_5,B_1)	(A_2,B_4)	(A_5,B_2)	(A_2,B_3)	(A_6,B_4)	(A_4,B_1)	(A_5,B_2)
	c_2	(A_6,B_0)	(A_3,B_3)	(A_1,B_1)	(A_1,B_1)	(A_2,B_2)	(A_0,B_2)	(A_1,B_1)
	c_3	(A_1,B_0)	(A_6,B_4)	(A_5,B_3)	(A_1,B_1)	(A_1,B_0)	(A_2,B_3)	(A_4,B_2)
	c_4	(A_3,B_1)	(A_3,B_3)	(A_5,B_2)	(A_5,B_3)	(A_0,B_2)	(A_2,B_0)	(A_1,B_0)

Table 9
The crisp Z-number MDISs of four experts.

E	x_1	x_2	x_3	x_4	x_5	x_6	x_7
e_1	(0.3821,1)	(0,0.75)	(0.6179,1)	(0.3821,0.75)	(0.7792,0.25)	(0.5,1)	(0.5,0.25)
	(0.6179,0.25)	(0.7792,0.5)	(0.5,0.75)	(0.5,0.25)	(0.7792,0.75)	(1,0)	(0.3821,0.75)
	(0.2208,0.25)	(0.6179,0.75)	(1,0.75)	(0,0.25)	(0.5,0.25)	(0.7792,1)	(0.2208,0.75)
	(1,1)	(0.6179,0.5)	(1,1)	(0.7792,0.5)	(1,0.75)	(0.7792,1)	(1,1)
e_2	(0.5,0)	(0.2208,0.5)	(0.2208,0.5)	(1,0.75)	(1,0.5)	(0,0.25)	(0.7792,1)
	(0.7792,0.25)	(0.6179,0.25)	(0.2208,0)	(0.2208,0.25)	(1,1)	(0.5,0.25)	(1,0.5)
	(0.2208,0.5)	(0.3821,0.5)	(0.6179,0.75)	(0.7792,0.25)	(1,0.75)	(0.2208,0.5)	(0.5,1)
	(0.7792,0.75)	(0.3821,0.5)	(0.3821,0.75)	(0.7792,0.5)	(0.6179,1)	(0.2208,0.25)	(1,0.25)
e_3	(0.6179,1)	(1,0.5)	(1,0.5)	(0.5,0.25)	(0.5,0.5)	(0,1)	(0,0.5)
	(0.7792,0.25)	(0.6179,1)	(1,1)	(0.2208,0.25)	(1,0.5)	(0.5,0.75)	(0.7792,0.5)
	(0.2208,0.5)	(0.3821,1)	(0.6179,0.75)	(0.7792,0.25)	(0.5,0.5)	(0.2208,0.5)	(0.6179,0.5)
	(0.7792,0.75)	(0.3821,0.5)	(0.3821,0.75)	(0.7792,0.5)	(0.6179,1)	(0.7792,0.75)	(0,0.25)
e_4	(0.7792,0.25)	(0.3821,1)	(0.7792,0.5)	(0.3821,0.75)	(1,1)	(0.6179,0.25)	(0.7792,0.5)
	(1,0)	(0.5,0.75)	(0.2208,0.25)	(0.2208,0.25)	(0.3821,0.5)	(0,0.5)	(0.2208,0.25)
	(0.2208,0)	(1,1)	(0.7792,0.75)	(0.2208,0.25)	(0.2208,0)	(0.3821,0.75)	(0.6179,0.5)
	(0.5,0.25)	(0.5,0.75)	(0.7792,0.5)	(0.7792,0.75)	(0,0.5)	(0.3821,0)	(0.2208,0)

Step 1. Transformation from LZNs to crisp Z-numbers. The possible LSFs of A and B are $\{0, 0.2208, 0.3821, 0.5000, 0.6179, 0.7792, 1\}$ and $\{0, 0.2500, 0.5000, 0.7500, 1.0000\}$, corresponding to $\{A_0, A_1, A_2, A_3, A_4, A_5, A_6\}$ and $\{B_0, B_1, B_2, B_3, B_4\}$, respectively. The DISs with LZNs in Table 8 can be

transformed into the crisp Z-number DISs in Table 9.

Step 2. Z-LINMAP method. According to crisp Z-number information and preference set Q_{e_h} provided by expert e_h, the normalized weight vector $\overline{w}^h = (\overline{w}_1^h, \overline{w}_2^h, ..., \overline{w}_m^h)$, PIS Z-number Z_j^{h+}, NIS Z-number Z_j^{h-} and preference coefficients including μ_{kl} and μ'_{kl} can be calculated by solving the linear programming model (M-4).

Step 3. Obtain the conditional probabilities $\Pr_h(X|[x_i]_{P_{con}})$ of x_i for e_h. For each expert e_h ($h = 1, 2, 3, 4$), the consistency equivalence class $[x_i]_{P_{con}}$ and inconsistency equivalence class $[x_i]_{P_{incon}}$ can be obtained using Eq. (9). The conditional probabilities $\Pr_h(X|[x_i]_{P_{con}})$ can be calculated using Eq. (13), which are shown in Table 10.

Table 10
The conditional probabilities of all alternatives for four experts.

$\Pr_h(X\|[x_i]_{P_{con}})$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
$\Pr_1(X\|[x_i]_{P_{con}})$	0.4151	0.4538	0.4801	0.7636	0.7765	0.4036	0.5863
$\Pr_2(X\|[x_i]_{P_{con}})$	0.5929	0.6316	0.2968	0.5874	0.4415	0.7745	0.5033
$\Pr_3(X\|[x_i]_{P_{con}})$	0.4958	0.4600	0.4205	0.4889	0.4705	0.5877	0.6566
$\Pr_4(X\|[x_i]_{P_{con}})$	0.5467	0.4553	0.4544	0.5144	0.4180	0.7010	0.5693

Fig. 2 describes that the conditional probabilities of all alternatives for all experts, where the larger the green area and the darker the color is, the greater the corresponding conditional probability is. All conditional probabilities are within the interval of $[0.2968, 0.7765]$ and there's no extreme value, which means that the acquisition method of conditional probability is reasonable. The maximum and minimum conditional probabilities are $\Pr_1(X|[x_5]_{P_{con}})$ and $\Pr_2(X|[x_3]_{P_{con}})$ corresponding to alternative x_5 and x_3. The conditional probabilities of x_5 for all experts are not the minimum, so x_5 is less likely to be assigned to the negative region $NEG(X)$. Similarly, the conditional probabilities of x_3 for all experts are not the maximum, so x_3 is less likely to be assigned to the positive region $POS(X)$. However, the classification result of alternatives depends not only on the conditional probability but also on the loss functions.

Step 4. Calculate the regret loss functions or three threshold values of x_i for e_h. According to $\overline{w}^h$, Z_j^{h+} and Z_j^{h-} in Step 2, we can compute relative loss functions $\lambda_{\bullet\circ}^h (\bullet = P, B, N; \circ = P, N)$ and regret loss functions $\overline{\lambda}_{\bullet\circ}^h$ using Eq. (17). The expected regret loss function $R_h(a_\bullet|[x_i]_{P_{con}})$ can be obtained by using Eq. (18), and three threshold values α^{hi}, γ^{hi} and β^{hi} for e_h also can be calculated which are as shown in Table 11.

Fig. 3 reflects that three regions for four experts and conditional probabilities of x_i for e_h according to Tables 10-11. Dark green, green and light green represent the three areas $POS^h(X)$, $BND^h(X)$ and $NEG^h(X)$. From Fig. 3, the area of $BND^h(X)$ is smaller than that of $POS^h(X)$ and $NEG^h(X)$, and the areas of $POS^h(X)$ and $NEG^h(X)$ overlap greatly. Under different alternatives, the three region changes

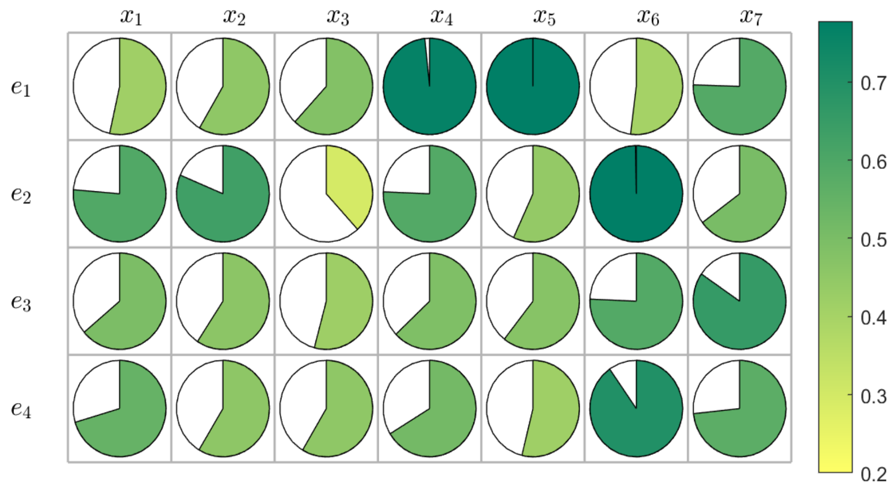

Fig. 2. The conditional probabilities of different alternatives for experts.

Table 11
Three thresholds of four experts under different alternatives.

Experts	Thresholds	x_1	x_2	x_3	x_4	x_5	x_6	x_7
e_1	α^{1i}	0.6423	0.6177	0.7459	0.4614	0.7896	0.5834	0.3635
	γ^{1i}	0.5476	0.5222	0.6681	0.3669	0.7168	0.4892	0.2787
	β^{1i}	0.4495	0.4251	0.5809	0.2812	0.6313	0.3957	0.2068
e_2	α^{2i}	0.2922	0.5228	0.6535	0.6518	0.5963	0.5805	0.5187
	γ^{2i}	0.2193	0.4253	0.5597	0.5577	0.5060	0.4830	0.4263
	β^{2i}	0.1598	0.3330	0.4616	0.4594	0.4154	0.3867	0.3384
e_3	α^{3i}	0.5872	0.6626	0.6398	0.4931	0.5902	0.5600	0.7236
	γ^{3i}	0.4914	0.5735	0.5491	0.3979	0.4944	0.4628	0.6387
	β^{3i}	0.3961	0.4797	0.4554	0.3095	0.3989	0.3683	0.5445
e_4	α^{4i}	0.1950	0.6782	0.4028	0.6359	0.5337	0.4354	0.4829
	γ^{4i}	0.1410	0.5888	0.3145	0.5422	0.4387	0.3423	0.3868
	β^{4i}	0.0997	0.4936	0.2374	0.4456	0.3477	0.2597	0.2985

for experts e_2 and e_3 are relatively gentle, while the other two experts show great region differences. According to the yellow point of conditional probability in Fig. 3, the region results assigned to for alternatives can be obtained. For example, the alternative x_4 for expert e_1 should be classified into the region $POS^1(X)$ since the conditional probability point is within the dark green area.

Step 5. Obtain the three region $POS^h(X)$, $BND^h(X)$ and $NEG^h(X)$ for expert e_h using decision rules (P3)-(N3) or (P4)-(N4). The classification results are shown in Table 12.

In Table 12, the alternative x_5 for four experts are all classified into the $BND^h(X)$. Although $\Pr_1(X|[x_5]_{P_{con}}) = 0.7765$ is the maximum conditional probability, the first threshold α^{15} is 0.7896 which is larger than the conditional probability, so alternative x_5 for expert e_1 is assigned to $BND^1(X)$. For alternative x_3 with minimum conditional probability, the first three experts assign it to the region

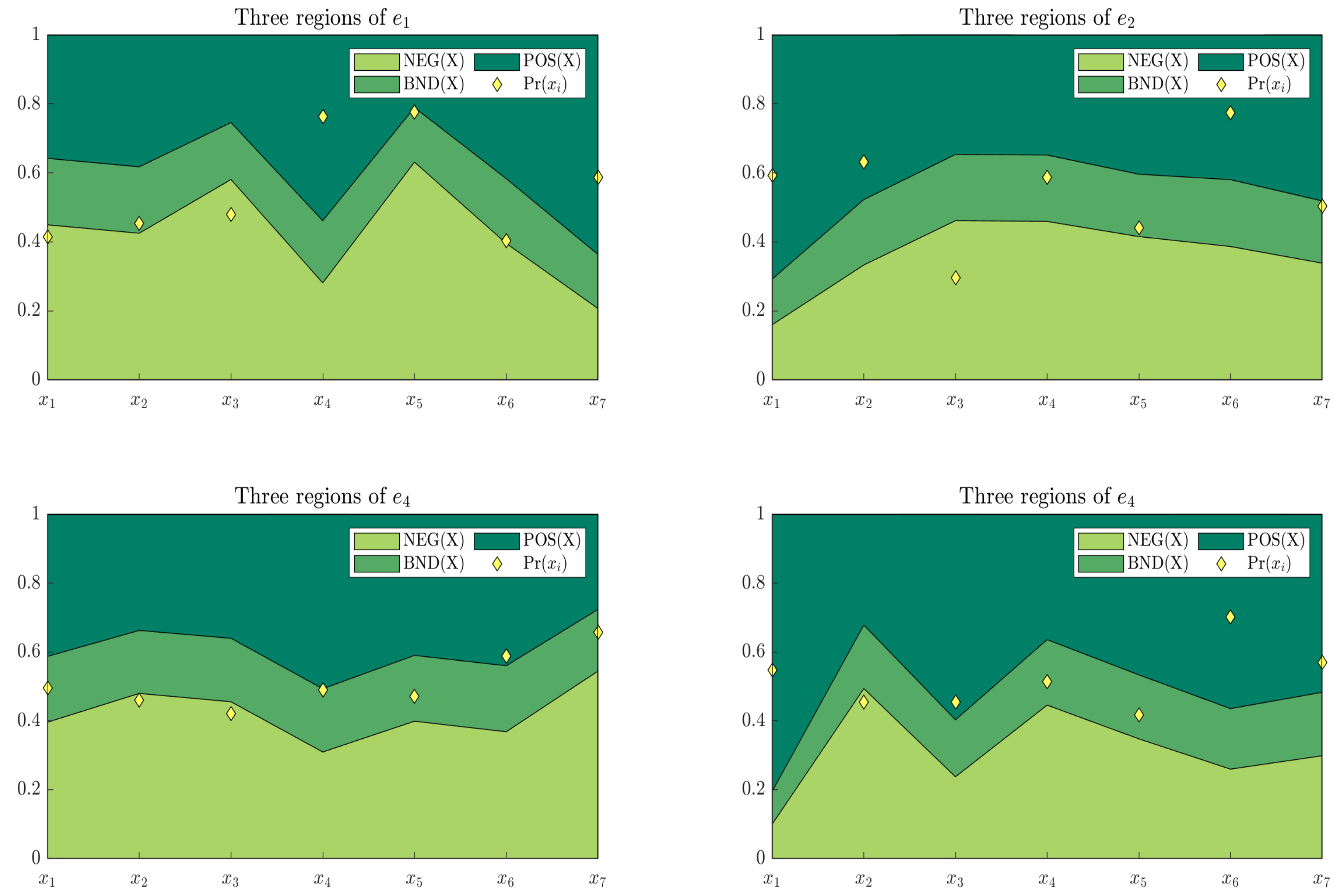

Fig. 3. Three regions of all alternatives for experts.

$NEG^h(X)$ and expert e_4 assign it to $POS(X)$. The reason is that the three thresholds for expert e_4 are smaller than $\Pr_4(X|[x_3]_{P_{con}}) = 0.4544$. Therefore, the classification results are effected by regret loss functions.

To intuitively demonstrate the classification result, Fig. 4 shows that the classification color distribution, where dark green, green and light green represent the three areas $POS^h(X)$, $BND^h(X)$ and $NEG^h(X)$, respectively. The classification results varies for different experts, but some alternatives are assigned to the same region more than half of experts. For example, alternatives $\{x_6\}$, $\{x_4, x_5\}$ and $\{x_3\}$ are assigned to $POS^h(X)$, $BND^h(X)$ and $NEG^h(X)$ with three times, respectively.

Table 12
The classification result of four experts.

Region	e_1	e_2	e_3	e_4
$POS^h(X)$	x_4, x_7	x_1, x_2, x_6	x_6	x_1, x_3, x_6, x_7
$BND^h(X)$	x_2, x_5, x_6	x_4, x_5, x_7	x_1, x_4, x_5, x_7	x_4, x_5
$NEG^h(X)$	x_1, x_3	x_3	x_2, x_3	x_2

Step 6. Obtain the comprehensive classification result and ranking result of all alternatives. Calculate the weighted expected regret loss function $\overline{R}(a_\bullet|[x_i]_{P_{con}})$ using Eq. (19) and determine the

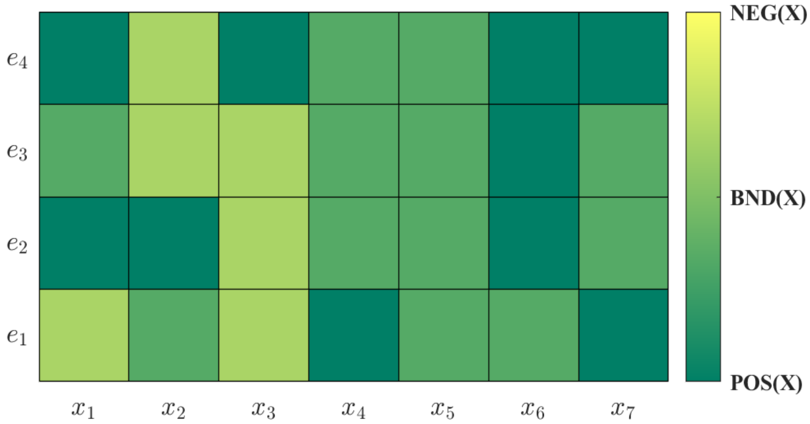

Fig. 4. The classification of all alternatives for experts.

region result that x_i belongs to according to the decision rules (P5)-(N5). Calculate the loss score $LS(x_i)$ using Eq. (20) and rank the loss scores of all alternatives in ascending order to obtain the ranking result. The final result of four experts is shown in Table 13. Take x_1 as an example, we can get $|POS_1| = 2$, $|BND_1 = 1$ and $|NEG_1| = 1$ according to Table 12, then $LS(x_1) = \frac{2\times0.2360+1\times0.2892+1\times0.4778}{4} = 0.3907$. From Table 13, the three regions are $POS(X) = \{x_1, x_4, x_6, x_7\}$, $BND(X) = \{x_2, x_5\}$ and $NEG(X) = \{x_3\}$, and the ranking result is $x_4 \succ x_6 \succ x_7 \succ x_1 \succ x_2 \succ x_5 \succ x_3$.

Table 13
The final result of four experts.

Alternatives	$\overline{R}(a_P\|[x_i]_{P_{con}})$	$\overline{R}(a_B\|[x_i]_{P_{con}})$	$\overline{R}(a_N\|[x_i]_{P_{con}})$	Region	$LS(x_i)$	Rank
x_1	0.2360	0.2891	0.4778	$POS(X)$	0.3097	4
x_2	0.4454	0.3267	0.3611	$BND(X)$	0.3736	5
x_3	0.5732	0.3828	0.3687	$NEG(X)$	0.4198	7
x_4	0.2592	0.2704	0.4101	$POS(X)$	0.2676	1
x_5	0.5008	0.3855	0.4470	$BND(X)$	0.3855	6
x_6	0.2732	0.2906	0.4455	$POS(X)$	0.2775	2
x_7	0.2887	0.3153	0.4888	$POS(X)$	0.3020	3

Therefore, tasks set $\{x_1, x_4, x_6, x_7\}$ are recognized by the computer technology, and tasks x_2 and x_5 should be recognized using human-computer technology and task x_3 is assigned by the human recognition. The recognition order is $\{x_4, x_6, x_7, x_1, x_2, x_5, x_3\}$.

5. Comparative and sensitivity analyses

In this section, we compare the final result under different preferences in Section 5.1. The impacts of risk avoidance coefficients and regret aversion coefficients are analyzed in Sections 5.2 and 5.3, respectively. The comparative analysis in Section 5.4 is used to demonstrate the effectiveness of the proposed method.

5.1. Comparative analysis of different preferences

Preference plays an important role in the proposed regret 3WD model, we discuss the classification results under different preference sets. To explore the impact of preference sets, these parameters remain the same except for the preference set for the case in Section 4. Four new preference matrixes $\overline{Q}_{e_h}$ $(h = 1, 2, 3, 4)$ for expert e_h are given as follows:

$$\overline{Q}'_{e_1} = \begin{bmatrix} - & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) & (7,1) \\ - & - & (2,3) & (4,2) & (5,2) & (6,2) & (2,7) \\ - & - & - & (3,4) & (3,5) & (3,6) & (3,7) \\ - & - & - & - & (5,4) & (6,4) & (7,4) \\ - & - & - & - & - & (6,5) & (5,7) \\ - & - & - & - & - & - & (7,6) \\ - & - & - & - & - & - & - \end{bmatrix}, \overline{Q}'_{e_2} = \begin{bmatrix} - & (2,1) & (3,1) & (1,4) & (5,1) & (1,6) & (7,1) \\ - & - & (3,2) & (4,2) & (2,5) & (6,2) & (2,7) \\ - & - & - & (4,3) & (3,5) & (6,3) & (3,7) \\ - & - & - & - & (4,5) & (4,6) & (4,7) \\ - & - & - & - & - & (5,6) & (5,7) \\ - & - & - & - & - & - & (6,7) \\ - & - & - & - & - & - & - \end{bmatrix};$$

$$\overline{Q}'_{e_3} = \begin{bmatrix} - & (1,2) & (1,3) & (1,4) & (1,5) & (6,1) & (7,1) \\ - & - & (3,2) & (4,2) & (2,5) & (2,6) & (2,7) \\ - & - & - & (3,4) & (5,3) & (3,6) & (7,3) \\ - & - & - & - & (4,5) & (4,6) & (4,7) \\ - & - & - & - & - & (6,5) & (5,7) \\ - & - & - & - & - & - & (6,7) \\ - & - & - & - & - & - & - \end{bmatrix}, \overline{Q}'_{e_4} = \begin{bmatrix} - & (1,2) & (1,3) & (4,1) & (1,5) & (1,6) & (1,7) \\ - & - & (3,2) & (4,2) & (2,5) & (2,6) & (7,2) \\ - & - & - & (3,4) & (3,5) & (6,3) & (7,3) \\ - & - & - & - & (5,4) & (6,4) & (4,7) \\ - & - & - & - & - & (6,5) & (7,5) \\ - & - & - & - & - & - & (7,6) \\ - & - & - & - & - & - & - \end{bmatrix}.$$

The comparative results under different preference sets for four experts are shown in Fig. 5. All experts remain the same LZNs evaluation matrixes and change their preference matrix $\overline{Q}_{e_h}$ to $\overline{Q}'_{e_h}$, whose classification results have obvious differences. For example, expert e_1 with $\overline{Q}'_{e_1}$ classify seven image recognition tasks into two classes $BND'(X)$ and $NEG'(X)$, which means that no task should be recognized only by using computer technology. The number of tasks using computer technology increases to 5 from 3 for expert e_2 with $\overline{Q}'_{e_2}$, which means that the preferences provided by experts have significant impact on the results in the proposed regret 3WD model.

5.2. Sensitivity analysis of risk avoidance coefficients

The risk avoidance coefficient σ_h of expert e_h $(h = 1, 2, ..., \Gamma)$ is the important parameter affecting the relative loss functions $\lambda^h_{B\circ}(\circ = P, N)$, which has indirect impacts on the regret loss functions $\overline{\lambda}^h_{\bullet\circ}(\bullet = P, B, N; \circ = P, N)$. Those parameters remain the same except for the risk avoidance coefficient σ_h for the case in Section 4. For each expert, we carry out 101 numerical experiments within $\sigma_h \in [0, 1]$ and obtain 101 groups of classification results, which are shown in Fig. 6. Dark green, green and light green represent the three areas $POS^h(X)$, $BND^h(X)$ and $NEG^h(X)$.

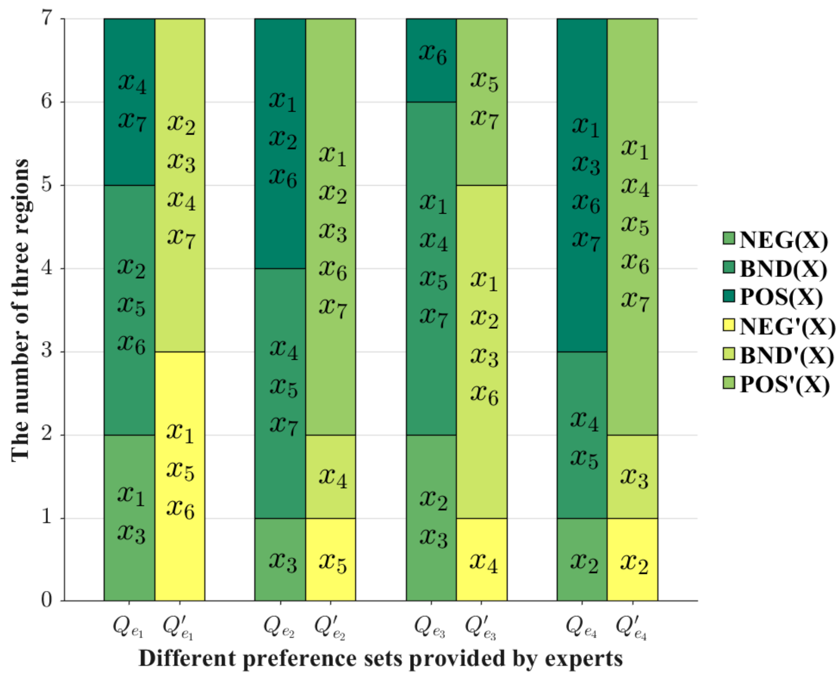

Fig. 5. The results under different preference sets for four experts.

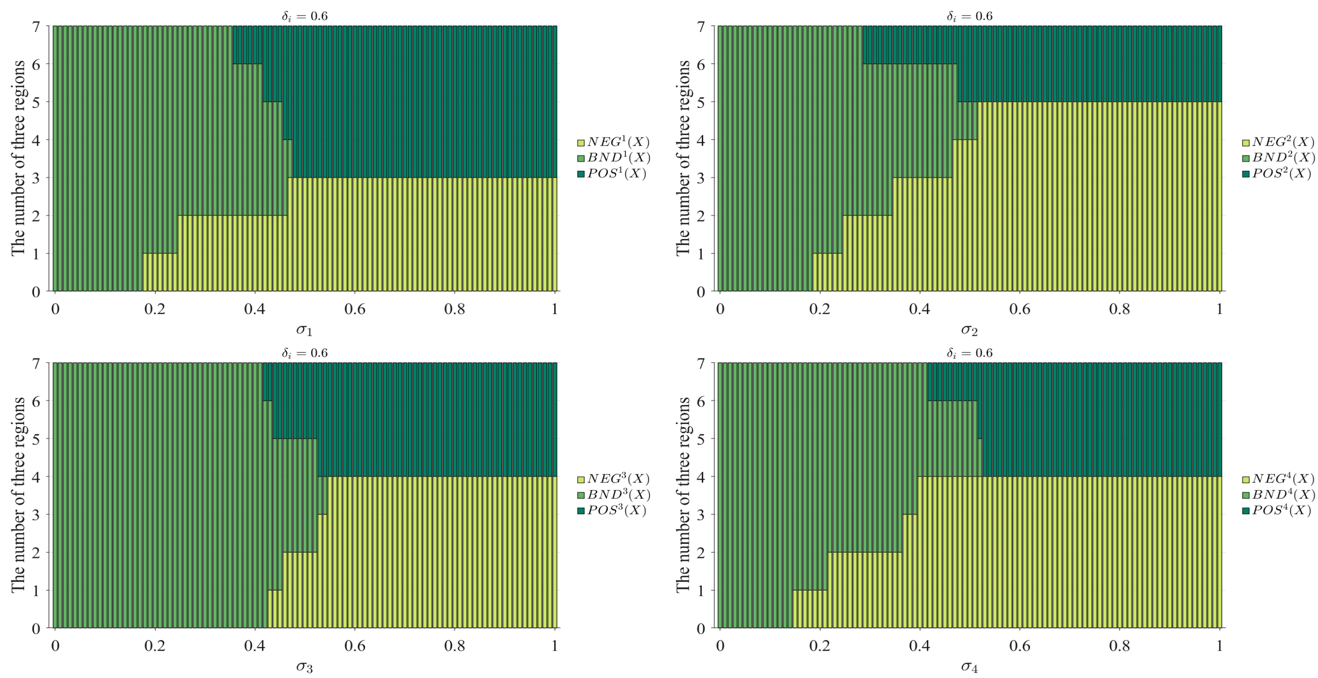

Fig. 6. The results under different risk avoidance coefficients σ_h.

From Fig. 6, with the increase of risk avoidance coefficient σ_h, the number of image tasks assigned to $POS(X)$ and $NEG(X)$ increases, while the number of image tasks assigned to $BND(X)$ decreases. A small risk avoidance coefficient σ_h may lead to all image tasks being recognized by human-computer recognition technology. The reason is that the smaller the risk avoidance coefficient σ_h is, the larger the α^h is and the smaller the β^h is. Then the region $BND(X)$ becomes larger, so more tasks will be assigned to this region. A large risk avoidance coefficient σ_h may lead to 2WD, in which no image task will be assigned to human-computer recognition. The larger risk avoidance coefficient σ_h will lead to smaller α^h and larger β^h. Then the region $BND(X)$ becomes smaller, so more tasks will be assigned to the other two regions. Therefore, determining appropriate risk avoidance coefficients for each expert is very important according to the characteristics of the actual problems.

5.3. Sensitivity analysis of regret aversion coefficients

The regret aversion coefficient δ_h of expert e_h $(h = 1, 2, ..., \Gamma)$ is another important parameter affecting the regret loss functions $\overline{\lambda}^h_{\bullet\circ}(\bullet = P, B, N; \circ = P, N)$. Those parameters remain the same except for the regret aversion coefficients δ_h for the case in Section 4. For each expert, we carry out 11 numerical experiments within $\delta_h \in [0, 1]$ and obtain 11*7 groups of threshold results. To explore the results when 2WD appears, we also carry out 11 numerical experiments within $\delta_h \in [0, 1]$ when $\sigma_h = 0.8$ and obtain 11*7 groups of threshold results. For the sake of convenience, three average thresholds are proposed to demonstrate the trend of results, which are as follows:

$$\bar{\alpha}^h = \frac{\sum_{i=1}^{n} \alpha_i^h}{n}, \bar{\gamma}^h = \frac{\sum_{i=1}^{n} \bar{\gamma}_i^h}{n}, \bar{\beta}^h = \frac{\sum_{i=1}^{n} \bar{\beta}_i^h}{n}, h = 1, 2, ..., \Gamma. \tag{21}$$

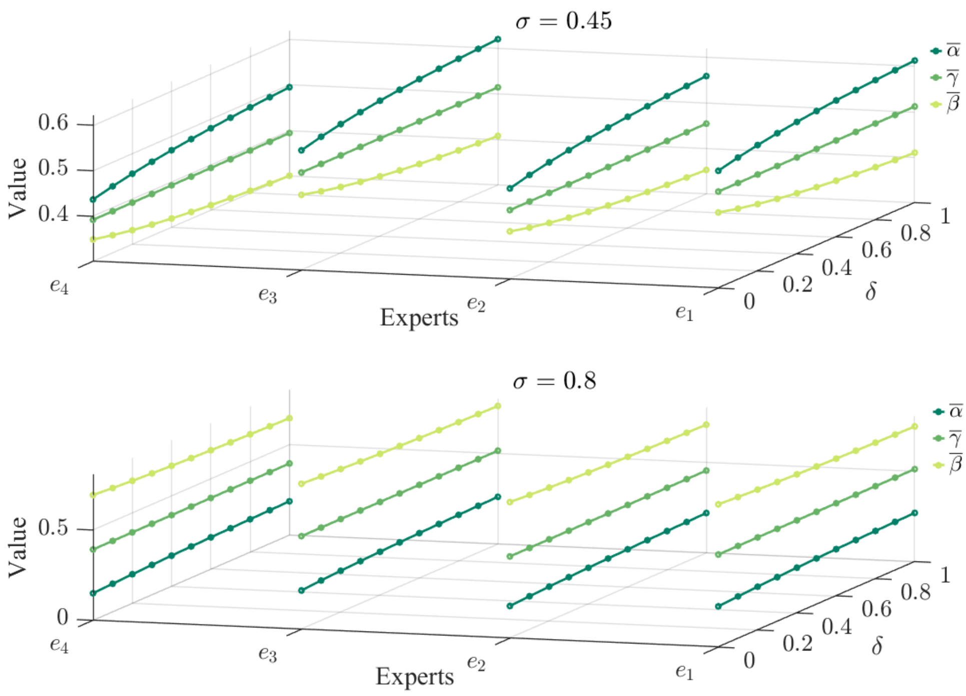

Fig. 7. The results under different regret aversion coefficients δ_h.

The results under different regret aversion coefficients δ_h are shown in Fig. 7. Fig. 7 reflects four experts' three average thresholds including $\bar{\alpha}^h$, $\bar{\gamma}^h$ and $\bar{\beta}^h$ for e_h under different regret aversion coefficients δ_h when $\sigma_h = 0.45$ and $\sigma_h = 0.8$. Dark green, green and light green lines represent $\bar{\alpha}^h$,

$\bar{\gamma}^h$ and $\bar{\beta}^h$, respectively. From Fig. 7, we have $\bar{\alpha}^h > \bar{\beta}^h$ when $\sigma_h = 0.45$ and $\bar{\alpha}^h \leq \bar{\beta}^h$ when $\sigma_h = 0.8$, corresponding to the 3WD and 2WD, respectively. Three average thresholds show the similar trend: $\bar{\alpha}^h$ increases and $\bar{\beta}^h$ decreases with the increase of δ_h, and $\bar{\gamma}^h$ changes little with only a slight decrease. For a 3WD with $\sigma_h = 0.45$, a larger regret aversion coefficients $\bar{\delta}_h$ will lead to larger $\bar{\alpha}^h$ is and smaller $\bar{\beta}^h$. Then the region $BND(X)$ becomes larger, so more tasks will be assigned to the region. For a 2WD with $\sigma_h = 0.8$, $\bar{\gamma}^h$ is the only threshold to determine whether an alternative belongs to $POS(X)$ and $NEG(X)$. When the conditional probabilities of multiple alternatives are concentrated near the threshold $\bar{\gamma}^h$, more alternatives will be assigned to region $POS(X)$ and fewer to region $NEG(X)$ with the increase of δ_h. Otherwise, δ_h affects slightly the classification results. Therefore, the regret aversion coefficient in RT has significant impact on three thresholds and classification results in the proposed regret 3WD model.

5.4. Comparative analysis of different MCDM methods

To verify the effectiveness of the proposed regret 3WD model, five MCDM methods are chosen to carry out the comparative analysis, including Zou et al's method [46], Wang et al.'s method [28], Chu and Xiao's method [47], Büyüközkan and Tüfekçi's method [48] and Yager's method [15]. Considering the effect of preferences, we combine the LINMAP method with the five methods which are rewritten as LINMAP-TOPSIS [46], Wang et al.'s method [28], LINMAP-GRA [47], LINMAP-VIKOR [48] and LINMAP-WAA [15] in turn. The ranking results of six methods is shown in Table 14. From Table 14, the ranking results of our proposal are generally consistent with those of the other methods, that is, x_1, x_4, x_6 and x_7 ranked in the top four, and x_2, x_3 and x_5 ranked in the bottom three, which demonstrates the effectiveness of our proposal. Alternative x_4 is the best choice for six methods, which reflects the reliability of the obtained ranking results.

Table 14
The ranking results of six methods.

Approach	Ranking results
The proposed regret 3WD model	$x_4 \succ x_6 \succ x_7 \succ x_1 \succ x_2 \succ x_5 \succ x_3$
LINMAP-TOPSIS [46]	$x_4 \succ x_7 \succ x_6 \succ x_1 \succ x_3 \succ x_5 \succ x_2$
Wang et al.'s method [28]	$x_4 \succ x_1 \succ x_6 \succ x_7 \succ x_2 \succ x_3 \succ x_5$
LINMAP-GRA [47]	$x_4 \succ x_7 \succ x_1 \succ x_6 \succ x_2 \succ x_5 \succ x_3$
LINMAP-VIKOR [48]	$x_4 \succ x_7 \succ x_1 \succ x_6 \succ x_5 \succ x_2 \succ x_3$
LINMAP-WAA [15]	$x_4 \succ x_6 \succ x_1 \succ x_7 \succ x_2 \succ x_3 \succ x_5$

The Spearman's correlation coefficient (SCC) [49] is used to characterize the correlations between the above six methods, and the SCC between two ranking results can be calculated as follows:

$$SCC = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{\zeta(\zeta^2 - 1)}, \tag{22}$$

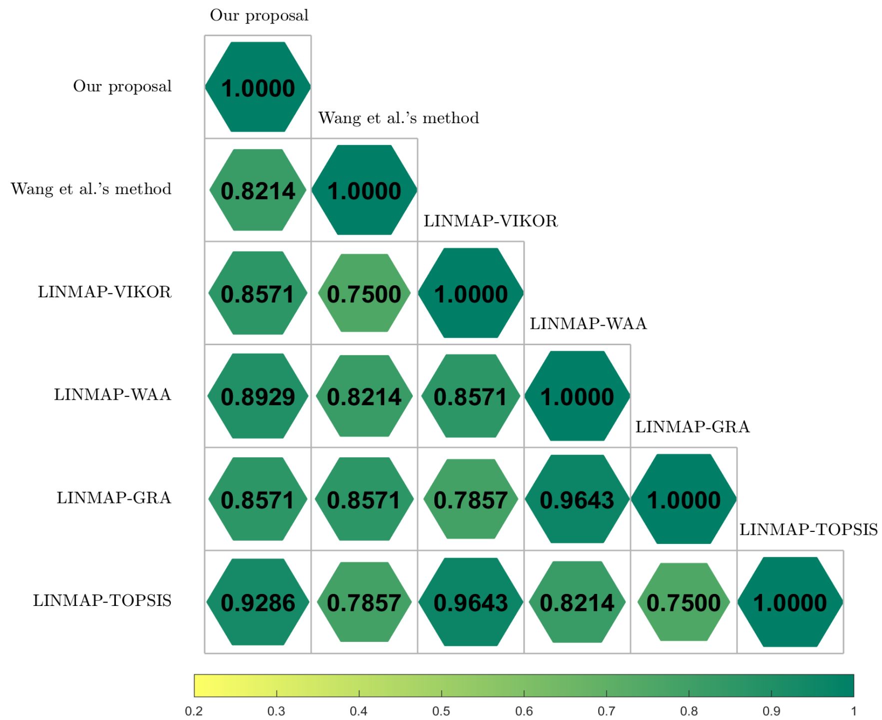

Fig. 8. The Spearman's correlations among six methods.

where ζ is the number of alternatives and d_i denotes the grade difference corresponding to the ith position of the two set of ranking results.

Generally speaking, the larger the SCC is, the more similar the ranking results of two methods are. If the SCC between two methods is greater than 0.8, the correlation degree is significant. The Spearman's correlations among six method are shown Fig. 8. It can be seen from Fig. 8 that the correlation coefficients of the proposed regret 3WD model and the other five methods are: 0.8214, 0.8571, 0.8929, 0.8571 and 0.9286, which means that the proposed regret 3WD model has high consistency with the other five methods.

6. Discussion

To demonstrate the advantages of our proposed 3WD model, we compare our proposal with other 3WD methods from four dimensions in theory and summarize the main advantages of our proposal in this section.

6.1. The theoretical differences between the proposed model and other 3WD models

The theoretical differences between the proposed model with other 3WD models are summarized from four aspects as follows:

(1) The difference of the decision environment. The evaluation information has been used different types of fuzzy numbers to measure the uncertainty in different literature. Zhu et al. [34] and Wang et

al. [35] assumed that experts provided evaluation information with fuzzy numbers. Wang et al. [33] and Wang et al. [41] used hesitant fuzzy numbers considering the hesitation of expressions. Triangular fuzzy numbers [20], interval fuzzy numbers [36] and intuitionistic fuzzy numbers [40] have been applied to 3WD models. However, the reliability of evaluations is ignored which can be solved by Z-numbers. Wang et al. [28] utilized LZNs to depict the decision information where LZNs can be transformed triangular fuzzy numbers, which might result in information loss. In this paper, the LZNs are calculated directly without the conversion of other fuzzy numbers, which can effectively solve the problem of information loss. Meanwhile, the above studies only focus on the classification and ranking of single DIS and ignore the influence of preference information on the results. Our method not only examines the effect of preference information on 3WD results, but also addresses the classification and ranking of MDISs.

(2) The difference of determining the criteria weights. In previous some literature, the 3WD model is the research focus so criteria weights are usually given directly [28, 36, 40, 41]. Entropy weighting method [33], SMAA method [20] and the deviation maximization method [34] have used to obtain the weights of criteria objectively. Wang et al. [35] derived the criteria wights involved the cumulative prospect theory. The influence of preference information on criteria weights are ignored in previous literature. Therefore, we use Z-LINMAP method to synthesize preference information and evaluation information to determine the weights of criteria from the perspective of consistency and inconsistency.

(3) The difference of obtaining the conditional probability. The conditional probability usually is obtained by involving the outranking methods like PROMETHEE method [33, 34], ELECTRE method [35, 41] and stochastic dominance [20]. Jiang and Hu [36] applied the TOPSIS method to compute the evaluation function which is equal to the conditional probability. Liu et al.[40] used the GRA method to obtain the conditional probability. Combining TOPSIS with GRA methods, Wang et al.[28] proposed two MCDM methods with Z-numbers to determine the conditional probability. In this paper, we construct two equivalences based on consistency and inconsistency to obtain the conditional probability considering the preference information.

(4) The difference of constructing the loss functions. The loss functions are given directly in Wang et al.'s [41] method. Most literature have provided the methods constructing the loss functions. The prospect theory has been applied into the 3WD models by transforming the loss functions to prospect values in Wang et al.'s [28] method and Wang et al.'s [35] method. Wang et al. [33] and Zhu et al. [34] obtained the expected utility of alternatives using the utility theory and RT. Meanwhile, the relative loss functions have been computed in different methods [20, 36, 40]. Based on the RT, our paper considers three actions of 3WD as three choices and proposes the regret relative loss functions by comparing the regret and rejoice values of the three choices, where the relative distances based on the LZNs are

calculated.

6.2. The main advantages of the proposed 3WD model

Based on the above analysis, the main advantages of the proposed 3WD model are summarized as follows:

(1) In light of the traditional preference method, we extend the LINMAP method into the linguistic Z-numbers environment, which reflects the uncertainty and reliability of decision information and involves the 3WD model to measure the effect of preferences.

(2) Based on the preference coefficients obtained by Z-LINMAP method, we define the consistency-order and inconsistency-order, and further propose two equivalence classes to derive the conditional probabilities of alternatives, which shows the impact of experts' preferences on decision result.

(3) Considering the RT, we construct a group of regret relative loss functions and present the preference-based regret 3WD rules, which overcomes the weakness of the existing methods ignoring the regret psychology of experts.

(4) The preference-based regret 3WD model on MDISs with LZNs not only classifies different alternatives into three regions but also ranks all alternatives by introducing the weighted expected regret loss function and loss score function, which solves the classification and ranking problems of MDISs.

7. Conclusions

This paper proposes a preference-based regret 3WD model on MDISs with LZNs to classify and rank results for MDISs, which is applied into the image recognition based on human-computer interaction. In the proposed 3WD model, the proposed Z-LINMAP method determining the criteria weights not only deals with the uncertainty of MDISs with LZNs but also obtains the preference coefficients to measure the effect of preferences. Two preference-based equivalence classes are defined to derive the conditional probabilities of alternatives which shows the impact of experts' preferences on decision results. The regret loss functions based on RT are constructed and the preference-based regret 3WD rules are presented which overcome the weakness of the existing methods ignoring the regret psychology of experts. The feasibility and effectiveness of the proposed 3WD model have been verified by sensitivity analyses of preference and two parameters and comparative analysis with the existing methods.

In what follows, we firstly summarize the practical advantages of this paper are listed as follows:

(1) When classifying and ranking the data of real databases, different databases or experts recorded in different databases may exist different preferences for the same data bars, which is often ignored by

the existing 3WD research. Considering the differences of preference information among different databases or experts recorded in different databases, our method can handle MDISs with preference information which has wider applicability than other methods.

(2) In the actual decision-making scenario, experts tend to express the evaluation information in natural language, where the evaluation information may be not completely determined and accurate. Meanwhile, experts are easily affected by psychological factors and may feel regret or rejoice by comparing different alternatives or actions. Therefore, we extend the 3WD method to the LZNs environment and apply RT to determine classification rules, which is more practical than other methods and reflects the actual situation more accurately.

Secondly, we demonstrate some limitations of the proposed 3WD model, which are described as follows:

(1) The Z-LINMAP method is a subjective method for determining criteria weights based on preference information. However, when the inconsistency of the preference information given by experts is much greater than the consistency, or when experts only provide too little preference information, the criteria wights obtained by our proposal may be unreasonable.

(2) Our proposal assumes that multiple DISs are homogeneous where all evaluation information can be expressed by LZNs. However, different DISs may be heterogeneous with different types of evaluation information in complex decision-making problems.

For future research, the preference-based 3WD model for multiple heterogeneous incomplete DISs is worth studying. Firstly, obtaining reasonable criteria weights combining subjective with objective methods is a research topic. Secondly, the completion of heterogeneous decision information is also a focus of future research. Finally, it's meaningful to explore the unification of multiple complete heterogeneous information systems.

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3 Multi-criteria sequential three-state three-way decision consensus model based on set pair analysis theory

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Multi-criteria sequential three-state three-way decision consensus model based on set pair analysis theory

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Abstract

The three-way decision (3WD) theory involves two states of objects corresponding to consensus and conflict states in multi-criteria group decision-making (MCGDM) problems. However, the set pair analysis (SPA) theory provides a three-state perspective by incorporating the uncertainty state. Inspired of this, we propose a multi-criteria sequential three-state three-way decision (TS3WD) consensus model to reach the group consensus. Firstly, we propose the concept of consensus set pair probability space. Drawing upon SPA theory, the set pair consensus rules are designed to obtain the consensus set pair probability space which is categorized into three states including consensus, uncertainty and conflict. Secondly, the TS3WD model is presented by extending two states of classic 3WD model into three states. Related to the consensus set pair probability space and extended loss functions, the three-state decision rules and simplified decision rules are derived. Furthermore, the consensus feedback mechanism and rules based on the sequential TS3WD consensus model are proposed to adjust uncertainty and conflict opinions, in which the consensus adjustment model and coefficient adjustment strategy play essential roles. Finally, a connection number (CN) is applied to rank all alternatives and some experimental analyses are conducted to demonstrate the effectiveness and feasibility of the proposed model.

Keywords: Sequential three-way decision; three-state three-way decision; multi-criteria group decision making; consensus reaching process; set pair analysis

1. Introduction

Multi-criteria decision-making (MCDM) [1, 2] is to evaluate different alternatives under multiple criteria and select the best one from the available alternatives. In recent years, MCDM has been applied into several practical scenarios, such as supplier selection [3], site selection [4], energy evaluation [5],

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engineering management [6] and medical management [7]. In complex decision-making problems, a group of decision-makers (DMs) engages in MCDM problems by utilizing their expertise to derive optimal solutions, which significantly contributes to the development of multi-criteria group decision-making (MCGDM) [8, 9]. Generally, MCGDM generally consists of the consensus reaching process (CRP) and the selection process [10]. CRP is to motivate DMs to revise their opinions through the negotiation and feedback among DMs in order to reach the group consensus [11]. Once the group consensus is reached, the subsequent selection process is implemented to assist DMs in ranking all alternatives and identifying the most desirable alternative. Group consensus is a critical prerequisite for tackling the group decision-making (GDM) problems, so CRP plays a prominent role in MCGDM problems.

The Three-way decision(3WD) theory, proposed by Yao [12, 13], offers a reasonable solution for MCDM problems, deriving its foundations from the decision-theoretic rough sets (DTRSs) and Bayesian theory. The main idea of 3WD theory is to classify all objects into three disjoint regions corresponding to acceptance, delay and rejection actions [14]. The delay action acts as a buffer for the Two-way decision (2WD) problems, which only include acceptation and rejection actions. This buffering effect proves to be more effective in reducing decision risks compared to the 2WD theory [15]. Most scholars have extensively explored the application of the 3WD theory research across different MCDM problems [16, 17]. For instance, Jia and Liu [18] extended the 3WD theory into the multi-criteria environment, introducing diverse loss functions for all alternatives. Zhan et al. [15] proposed a new 3WD model based on outranking relations to address the enterprise project investment target selection problem. Jiang and Hu [19] formulated a 3WD model in the intuitionistic fuzzy environment for group investment decision-making. Wang et al. [20] presented a 3WD-MCDM method based on additive consistency and fuzzy preference relations, aiming to classify and select air training pilots. Li et al. [21] considered the requirements and targets of DMs, exploring a 3WD model that integrates both qualitative and quantitative information for the selection of various algorithm engineers.

Simultaneously, 3WD theory has been applied to address the CRP in the existing research. Liu et al. [22] defined a consensus degree by measuring the similarity of intuitionistic fuzzy sets (IFSs) and improved the group consensus of loss functions in the 3WD model based on a convex combination strategy. Liang et al. [23] employed 3WD theory to identify three types of followers in social network GDM problems and constructed the minimum adjustment model to reach the group consensus. Wang et al. [24] proposed a CRP model based on 3WD theory to distinguish three types of experts taking acceptance decision, reject decision and no-commitment decision for opinion adjustment. Zhu et al. [25] developed a three-way consensus model based on regret theory using probabilistic linguistic term sets, where 3WD theory was utilized to determine the range of parameter adjustments. Wang et al.

[26] constructed a consensus model minimizing decision loss differences to enhance the quality of consensus in three-way GDM problems. Guo et al. [27] designed a three-way cluster mechanism based on the consensus measure for fuzzy large-scale GDM problems. Liang and Duan [28] presented an optimization CRP model for the evaluation adjustment of large-scale group 3WD problems. However, most of the existing research followed the two-state principle of classic 3WD model and promoted the group consensus under the guidance of consensus degree.

Sequential three-way decision (S3WD) [29, 30] is a typical branch of three-way granular computing to solve the classification problems across multiple levels of granularity. The S3WD theory has evolved from the foundational concept of trisecting-acting-outcome (TAO) in 3WD model, including trisecting module, acting module and outcome module [31, 32]. A universal set can be divided into three regions using the trisecting module. The acting module provides three action strategies corresponding these regions. The effectiveness of the trisecting and acting modules is assessed through the outcome module. In MCGDM problems, non-commitment or delay decision of DMs brings great risks for alternatives. The S3WD theory effectively addresses this issue. For S3WD theory, elements in delay region can be further refined by incorporating new information and iterating the process until the acceptance or reject decision is determined. Nowadays, S3WD model has been applied to solve uncertain decision-making problems [33–35]. CRP is an iterative process that adjusts opinions of non-consensus experts in each round until the group consensus is reached, which coincides with the idea of S3WD theory. Hence, S3WD theory can effectively contribute to enhancing the CRP in MCGDM problems. Wang et al. [36] proposed an S3WD model incorporating individual attributes for MCDM problems, employing an opinion dynamics model to adjust opinions. Han and Zhan [37] introduced the concept of attribute discriminant based on S3WD theory and presented an S3WD group consensus method under probabilistic linguistic environment.

Set pair analysis (SPA) theory [38] is recognized to be an effective technique to analyze set pairs of system from identity, discrepancy and contrary perspectives. The connection number (CN) is employed to quantitatively measure and analyze the same, different and opposite features between two sets. Different MCDM models have been developed based on SPA theory under multiple evaluation environments. Kumar and Chen [39] proposed the CN for interval-valued intuitionistic fuzzy sets (IVIFSs) to determine criteria weights in MCDM problems. Cao et al. [40] developed a stochastic MCDM method for IVIFVs based on SPA theory. Garg and Kumar [41] presented a MCDM method using a possibility measure of IVIFSs and SPA theory. Both SPA theory and 3WD theory can deal with decision-making problems from three perspectives, and exhibit significant commonality and similarity. As previously mentioned, 3WD theory includes three actions and two states. The operations related to SPA theory have been used to determine three actions in 3WD models. Zhang et al. [42] proposed a

3WD model based on the space of set pair information granule. Li et al. [43] quantified the uncertainty of 3WD model using SPA theory and established the decision rules using the set pair connection entropy. Zhan et al. [44] constructed a 3WD model based on fuzzy set pair dominance degrees and the behavioral decision theory for incomplete information systems. SPA theory has undergone significant development with the introduction of uncertainty, where the three aspects—identity, discrepancy, and contrary—offer a pathway for extending the two-state of the 3WD model from a novel perspective.

Following the aforementioned challenges, the motivations of this paper are summarized as follows.

(1) In traditional MCGDM problems, experts are typically classified as either consensus experts or conflict experts, the latter of whom adjust their opinions to facilitate the group consensus. However, experts may encounter a state of uncertainty between consensus and conflict in the complex decision-making environment, that is, they are unsure about whether they should revise their opinions and need to make further judgements. For the two types of non-consensus experts——uncertainty experts and conflict experts, the consensus reaching strategies are worth studying.

(2) The 3WD theory extends two actions of 2WD theory to three actions by introducing the delay action. However, the classic 3WD theory includes only two states of consensus and conflict in MCGDM problems. By introducing the discrepancy degree, SPA theory provides a three-state perspective, which opens up a new view to expand the two states of classic 3WD theory.

(3) The idea of S3WD can be leveraged to guide the CRP in MCGDM problems. However, the challenge of granularity being too coarse for further classification in the S3WD process remains unresolved. Meanwhile, the methods for reclassifying the delay region and adjusting the opinions of conflict experts need further exploration. Designing the adjustment and feedback strategies for uncertainty experts and conflict experts based on S3WD theory has a significant effect on the CRP in MCGDM problems.

Based on the above motivations, we propose a multi-criteria sequential three-state three-way decision (TS3WD) consensus model based on SPA theory to address the challenges in the exiting research. The proposed model is an extended 3WD consensus model combining the TS3WD model with SPA theory from the perspective of group consensus, which not only enriches the 3WD theory from two states to three states, but also distinguishes three types of experts with different levels of consensus and effectively contributes to the group consensus reaching. The innovations and significance of our proposal corresponding to the above motivations are described as follows.

(1) A construction method for the consensus set pair probability space is proposed, incorporating three levels of consensus. Drawing upon SPA theory, the set pair consensus rules are designed, categorizing opinion differences between experts into three levels: consensus, uncertainty and conflict. The aggregation function is adopted to aggregate three types of consensus CNs to obtain the consensus

pair probability space, which corresponds to the state set in 3WD theory.

(2) A TS3WD model based on SPA theory is presented. By extending two states of classic 3WD model to three states, the TS3WD model incorporates the identity degree, discrepancy degree, and contrary degree within the consensus set pair probability space, aligning with the conditional probabilities of the three states. The extended loss functions are formulated to account for these factors. The corresponding three-state decision rules, along with the simplified decision rules related to the identity degree and discrepancy degree, are derived and validated.

(3) A consensus feedback mechanism based on sequential TS3WD consensus model is proposed to reach the group consensus. For conflict experts, a consensus adjustment model is conducted to modify their opinions under all criteria in order to reach the consensus. The uncertainty expert set is further classified through granularity refinement, in which the coefficient adjustment strategy is designed to solve the coarse-granularity problem. The consensus feedback rules are proposed to provide effective solutions with respect to different classification results.

The remainder of this paper is organized as follows. Section 2 introduces the basic knowledge necessary related to SPA theory, 3WD and S3WD theory. Section 3 proposes a multi-criteria set pair TS3WD consensus model. Section 4 presents a consensus feedback mechanism of the sequential TS3WD consensus model and summarizes the framework of the proposed model. The experimental analyses in Section 5 including two illustrative examples, the sensitive and comparative analysis are conducted to demonstrate the feasibility and effectiveness of the proposed model. Finally, Section 6 presents the concluding remarks.

2. Preliminaries

2.1. Set pair analysis theory

Set pair analysis (SPA) proposed by Zhao [38] is an uncertainty theory to deal with the interaction between certainty and uncertainty of system. The main idea of the SPA theory is to form a set pair $H = (A, B)$ of two sets A and B with certain connection for a specific problem W with N features, and make the opposite and identical analyses for the characteristics of the two sets. The connection number (CN) is an important tool for characterizing features of two sets in the SPA theory.

Definition 1. [38] For a problem W with N features, a set pair including two related set A and B is denoted as $H(A, B)$, in which S features are mutual, P features are opposite and $F(F = N - S - P)$ features are neither mutual nor opposite between A and B. The connection number (CN) $u(A, B)$ of the set pair $H(A, B)$ is defined as:

$$u(A, B) = \frac{S}{N} + \frac{F}{N}\delta_1 + \frac{P}{N}\delta_2, \tag{1}$$

where $\frac{S}{N}$, $\frac{F}{N}$ and $\frac{P}{N}$ denote the identity degree, discrepancy degree and contrary degree of the set pair $H(A, B)$, respectively. N is the total number of features. $\delta_1 \in [-1, 1]$ is the discrepancy coefficient of the set pair $H(A, B)$. $\delta_2 = -1$ is the contrary coefficient of the set pair $H(A, B)$. Suppose $\frac{S}{N} = a$, $\frac{F}{N} = b$ and $\frac{P}{N} = c$, then the CN can be rewritten as:

$$u(A, B) = a + b\delta_1 + c\delta_2, \tag{2}$$

where $0 \leq a, b, c \leq 1$ and $a + b + c = 1$.

Definition 2. [38] Let $u(A_1, B_1) = a_1 + b_1\delta_1 + c_1\delta_2$ and $u(A_2, B_2) = a_2 + b_2\delta_1 + c_2\delta_2$ be two CNs, then the related operation rules are shown as follows:

(1) $u(A_1, B_1) \oplus u(A_2, B_2) = (a_1 + a_2) + (b_1 + b_2)\delta_1 + (c_1 + c_2)\delta_2$;

(2) $u(A_1, B_1) \otimes u(A_2, B_2) = a_1a_2 + (a_1b_2 + a_2b_1)\delta_1 + (a_1c_2 + a_2c_1)\delta_2 + (b_1c_2 + b_2c_1)\delta_1\delta_2 + b_1b_2\delta_1^2 + c_1c_2\delta_2^2$;

(3) $u(A_1, B_1) \sim u(A_2, B_2)$ if and only if $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$, which means that $u(A_1, B_1)$ and $u(A_2, B_2)$ are non-differential.

2.2. *Three-way decision and sequential three-way decision theory*

The 3WD theory proposed by Yao [13] based on the Bayesian theory is aim to divide the universal set into three disjoint regions: the positive region, the boundary region and the negative region. The three regions correspond to acceptance action, delay action and rejection action, respectively. Generally, the 3WD model includes two states $\Omega = \{X, \neg X\}$ and three actions $\mathrm{A} = \{a_P, a_B, a_N\}$. X and $\neg X$ represent that an object x belongs to X and does not belong to X. a_P, a_B and a_N stand for acceptance, delay and rejection actions, which are denoted as $x \in POS(X)$, $x \in BND(X)$ and $x \in NEG(X)$, respectively. R is an equivalence relation on X and $[x]_R$ is the equivalence class of R including the object x. $\Pr(X|[x]_R)$ and $\Pr(\neg X|[x]_R)$ are the conditional probabilities of the object x belonging to and not belonging to X. In particular, $\lambda_{\bullet\circ}(\bullet = P, B, N; \circ = P, N)$ are used to measure the losses of taking three actions, which are shown in Table 1. λ_{PP}, λ_{BP} and λ_{NP} represent the losses taking actions a_P, a_B and a_N when $x \in X$. Similarly, λ_{PN}, λ_{BN} and λ_{NN} denote the losses taking actions a_P, a_B and a_N when $x \in \neg X$. A reasonable assumption is considered in the 3WD model: $0 \leq \lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}$ and $0 \leq \lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$.

Table 1
The loss functions in 3WD theory.

Actions	$X(P)$	$\neg X(N)$
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

The expected loss function $R(a_\bullet|[x]_R)$ of taking three actions can be calculated as follows:

(1) $R(a_P|[x]_R) = \lambda_{PP} \Pr(X|[x]_R) + \lambda_{PN} \Pr(\neg X|[x]_R)$;

(2) $R(a_B|[x]_R) = \lambda_{BP} \Pr(X|[x]_R) + \lambda_{BN} \Pr(\neg X|[x]_R)$;

(3) $R(a_N|[x]_R) = \lambda_{NP} \Pr(X|[x]_R) + \lambda_{NN} \Pr(\neg X|[x]_R)$.

According to the Bayesian theory, the following decision rules of minimum losses can be demonstrated as follows:

(P) If $R(a_P|[x]_R) \leq R(a_B|[x]_R)$ and $R(a_P|[x]_R) \leq R(a_N|[x]_R)$, then $x \in POS(X)$;

(B) If $R(a_B|[x]_R) \leq R(a_P|[x]_R)$ and $R(a_B|[x]_R) \leq R(a_N|[x]_R)$, then $x \in BND(X)$;

(N) If $R(a_N|[x]_R) \leq R(a_P|[x]_R)$ and $R(a_N|[x]_R) \leq R(a_B|[x]_R)$, then $x \in NEG(X)$.

The above rules (P)-(N) relate to the loss function $\lambda_{\bullet\circ}$ and the conditional probability $\Pr(X|[x]_R)$, then the decision rules can be rewritten as:

(P1) If $\Pr(X|[x]_R) \geq \alpha$ and $\Pr(X|[x]_R) \geq \gamma$, then $x \in POS(X)$;

(B1) If $\Pr(X|[x]_R) \leq \alpha$ and $\Pr(X|[x]_R) \geq \beta$, then $x \in BND(X)$;

(N1) If $\Pr(X|[x]_R) \leq \gamma$ and $\Pr(X|[x]_R) \leq \beta$, then $x \in NEG(X)$;

where $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$, $\gamma = \frac{\lambda_{PN}-\lambda_{NN}}{(\lambda_{PN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{PP})}$ and $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$.

When the condition $(\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN}) \leq (\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN})$ holds, we can obtain $\alpha \geq \beta$. In this case, the simplified decision rules (P1)-(N1) are listed as follows:

(P2) If $\Pr(X|[x]_R) \geq \alpha$, then $x \in POS(X)$;

(B2) If $\beta < \Pr(X|[x]_R) < \alpha$, then $x \in BND(X)$;

(N2) If $\Pr(X|[x]_R) \leq \beta$, then $x \in NEG(X)$.

Based on the granular computing theory, Yao and Deng [29] proposed the S3WD model, extending the classic 3WD model from coarse granularity to fine granularity and from one step to multiple steps. In cases of coarse granularity, the precise classification of objects cannot be obtained when information is insufficient. S3WD is used to gradually classify the delay objects into the positive and negative regions through granularity refinement until the decision problem transforms into a 2WD problem, thereby minimizing the additional cost of misclassification. In the S3WD process, the boundary region gradually shrinks until the definitive decision results are obtained as information accumulates and granularity refines, which conforms to the basic principles of human cognition and decision-making. Therefore, this paper applies the S3WD theory to deal with the CRP in MCGDM problems.

3. The multi-criteria set pair three-way decision consensus model

In this section, we design the set pair consensus rules by dividing opinion differences into three states. Subsequently, we construct the consensus set pair probability space by integrating the evaluation matrices using SPA theory operations in MCGDM problems. Leveraging the consensus set pair probability space, the classic 3WD model is extended to the TS3WD model involved SPA theory to construct the multi-criteria set pair TS3WD consensus model.

3.1. Construct the consensus set pair probability space

For a MCGDM problem, a group of experts give the evaluation information under different criteria for a series of alternatives. The alternative set is denoted as $X = \{x_1, x_2, ..., x_m\}$. The expert set and the expert weight set are denoted as $E = \{e_1, e_2, ..., e_K\}$ and $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_K\}$ where $\sum_{k=1}^{K} \lambda_k = 1$, respectively. Assume that each expert assigns weights to all other experts, then the interactive weight matrix for experts is denoted as $V = (v_{kh})_{K\times K}$, where $v_{kh} \in [0, 1]$ is the weight assigned to expert e_h by expert e_k and $\sum_{h=1}^{K} v_{kh}= 1$. The criteria set is denoted as $C = \{c_1, c_2, ..., c_n\}$, and the criteria weight set is denoted as $W = \{w_1, w_2, ..., w_n\}$, where $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. The evaluation matrix given by expert e_k $(k = 1, 2, ..., K)$ is $M^k = (m^k_{ij})_{m\times n}$, where $m^k_{ij} \in [0, 1]$ is the evaluation value of expert e_k under criterion c_j for alternative x_i. Based on SPA theory, evaluation information of each pair of experts under the specific criterion for an alternative can be distinguished as identity, discrepancy or contrary, corresponding to three states of consensus, between consensus and conflict or conflict. To obtain the CN of a pair of experts, the set pair consensus rules are defined as follows.

Definition 3. For a pair of experts (e_k, e_h), the evaluation matrices provided by two experts are $M^k = (m^k_{ij})_{m\times n}$ and $M^h = (m^h_{ij})_{m\times n}$, then the consensus CN under criteria c_j for alternative x_i is defined as $u_{ij}(e_k, e_h) = S^{kh}_{ij} + F^{kh}_{ij}\delta_1 + P^{kh}_{ij}\delta_2$ where S^{kh}_{ij}, F^{kh}_{ij} and P^{kh}_{ij} represent the identity degree, discrepancy degree and contrary degree of expert pair (e_k, e_h) under criteria c_j for alternative x_i, respectively. S^{kh}_{ij}, F^{kh}_{ij} and P^{kh}_{ij} satisfy the condition $S^{kh}_{ij} + F^{kh}_{ij} + P^{kh}_{ij} = 1$. The set pair consensus rules are designed as follows:

$$u_{ij}(e_k, e_h) = \begin{cases} 1 + 0\delta_1 + 0\delta_2, \left|m^k_{ij} - m^h_{ij}\right| \le \tau_1 \\ 0 + 1\delta_1 + 0\delta_2, \tau_1 < \left|m^k_{ij} - m^h_{ij}\right| < \tau_2 \\ 0 + 0\delta_1 + 1\delta_2, \left|m^k_{ij} - m^h_{ij}\right| \ge \tau_2 \end{cases} ; i = 1, 2, ..., m; j = 1, 2, ..., n; \forall e_k, e_h \in E, \quad (3)$$

where τ_1 and τ_2 are the consensus coefficient and the conflict coefficient, respectively. Obviously, $0 \le \tau_1 < \tau_2 \le 1$ is satisfied, and we have $u_{ij}(e_k, e_k) = 1 + 0\delta_1 + 0\delta_2$ for the same expert e_k.

Example 3.1. Let the expert set and alternative set be $E = \{e_1, e_2, e_3\}$ and $X = \{x_1, x_2, x_3\}$. The criteria set is $C = \{c_1, c_2, c_3, c_4, c_5\}$ and the weight set of criteria is $W = \{0.2, 0.2, 0.2, 0.2, 0.2\}$. The consensus coefficient and conflict coefficient are assumed as $\tau_1 = 0.1$ and $\tau_2 = 0.3$. The evaluation matrices $M^k = (m^k_{ij})_{3\times 5}$ $(k = 1, 2, 3)$ given by three experts are shown in Table 2. According to the set pair consensus rules in Eq. (3), the differences $\left|m^k_{ij} - m^h_{ij}\right|$ of evaluation values for each pair of experts can be computed, which are shown in Table 3. For experts e_1 and e_2, the evaluation difference under criterion c_1 for alternative x_1 can be computed as: $\left|m_{11}^1 - m_{11}^2\right| = |0.49 - 0.22| = 0.27$. Due to $\tau_1 < \left|m_{11}^1 - m_{11}^2\right| < \tau_2$, we get $u_{11}(e_1, e_2) = 0 + 1\delta_1 + 0\delta_2$ using Eq. (3). All consensus CNs $u_{ij}(e_k, e_h)$ are shown in Table 4.

Table 2
The evaluation matrices $M^k = (m_{ij}^k)_{m\times n}$ provided by $e_k (k = 1, 2, 3)$.

E	X	c_1	c_2	c_3	c_4	c_5
e_1	x_1	0.49	0.29	0.56	0.24	0.35
	x_2	0.56	0.40	0.49	0.37	0.46
	x_3	0.59	0.49	0.52	0.45	0.36
e_2	x_1	0.22	0.43	0.48	0.16	0.28
	x_2	0.25	0.40	0.46	0.44	0.52
	x_3	0.34	0.45	0.38	0.24	0.35
e_3	x_1	0.54	0.47	0.28	0.33	0.42
	x_2	0.51	0.37	0.35	0.36	0.45
	x_3	0.40	0.35	0.45	0.33	0.42

Table 3
The differences of evaluation values among experts in Example 3.1.

$\left\|m_{ij}^k - m_{ij}^h\right\|$	X	c_1	c_2	c_3	c_4	c_5
$\left\|m_{ij}^1 - m_{ij}^2\right\|$	x_1	0.27	0.14	0.08	0.08	0.07
	x_2	0.31	0	0.03	0.07	0.06
	x_3	0.25	0.04	0.14	0.21	0.01
$\left\|m_{ij}^1 - m_{ij}^3\right\|$	x_1	0.05	0.18	0.28	0.09	0.07
	x_2	0.05	0.03	0.14	0.01	0.01
	x_3	0.19	0.14	0.07	0.12	0.06
$\left\|m_{ij}^2 - m_{ij}^3\right\|$	x_1	0.32	0.04	0.2	0.17	0.14
	x_2	0.26	0.03	0.11	0.08	0.07
	x_3	0.06	0.1	0.07	0.09	0.07

Definition 4. For a pair of experts (e_k, e_h), the weighted consensus CN $u_i(e_k, e_h)$ of alternative x_i aggregating all consensus CNs $u_{ij}(e_k, e_h)(j = 1, 2, ..., n)$ is denoted as follows:

$$u_i(e_k, e_h) = S_i^{kh} + F_i^{kh}\delta_1 + P_i^{kh}\delta_2; i = 1, 2, ..., m; \forall e_k, e_h \in E, \tag{4}$$

Table 4
The consensus CNs $u_{ij}(e_k, e_h)$ in Example 3.1.

$u_{ij}(e_k, e_h)$	X	c_1	c_2	c_3	c_4	c_5
$u_{ij}(e_1, e_2)$	x_1	$0 + 1\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
	x_2	$0 + 0\delta_1 + 1\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
	x_3	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
$u_{ij}(e_1, e_3)$	x_1	$1 + 0\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
	x_2	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
	x_3	$0 + 1\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
$u_{ij}(e_2, e_3)$	x_1	$0 + 0\delta_1 + 1\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$
	x_2	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
	x_3	$1 + 0\delta_1 + 0\delta_2$	$0 + 1\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$	$1 + 0\delta_1 + 0\delta_2$
w_j		0.2	0.2	0.2	0.2	0.2

where $S_i^{kh} \in [0, 1]$, $F_i^{kh} \in [0, 1]$ and $P_i^{kh} \in [0, 1]$ represent the identity degree, discrepancy degree and contrary degree of expert pair (e_k, e_h) for alternative x_i, respectively. The condition $S_i^{kh} + F_i^{kh} + P_i^{kh} = 1$ holds.

By introducing an aggregation function $f(\cdot)$, the weighted consensus CN $u_i(e_k, e_h)$ can be computed as $u_i(e_k, e_h) = f(u_{i1}(e_k, e_h), u_{i2}(e_k, e_h), ..., u_{in}(e_k, e_h))$. If the aggregation function $f(\cdot)$ is assumed as the weight averaging (WA) operator, then we have $f(u_{i1}(e_k, e_h), u_{i2}(e_k, e_h), ..., u_{in}(e_k, e_h)) = (w_1 \otimes u_{i1}(e_k, e_h)) \oplus (w_2 \otimes u_{i2}(e_k, e_h)) \oplus \cdots \oplus (w_n \otimes u_{in}(e_k, e_h))$, where w_j is the weight of criterion c_j and $w_j \in [0, 1]$. For the convenience of calculation, the real number w_j can be rewritten as a CN $w_j + 0\delta_1 + 0\delta_2$, then $w_j \otimes u_{ij}(e_k, e_h) = w_j S_{ij}^{kh} + w_j F_{ij}^{kh}\delta_1 + w_j P_{ij}^{kh}\delta_2$ based on Definition 2. The weighted consensus CN $u_i(e_k, e_h)$ can be rewritten as:

$$u_i(e_k, e_h) = \sum\nolimits_{j=1}^{n} w_j S_{ij}^{kh} + \sum\nolimits_{j=1}^{n} w_j F_{ij}^{kh}\delta_1 + \sum\nolimits_{j=1}^{n} w_j P_{ij}^{kh}\delta_2; i = 1, 2, ..., m; \forall e_k, e_h \in E, \tag{5}$$

where $\sum_{j=1}^{n} w_j S_{ij}^{kh}$, $\sum_{j=1}^{n} w_j F_{ij}^{kh}$ and $\sum_{j=1}^{n} w_j P_{ij}^{kh}$ are used to obtain the identity degree S_i^{kh}, discrepancy degree F_i^{kh} and contrary degree P_i^{kh} of expert pair (e_k, e_h) for alternative x_i, respectively. Obviously, $u_i(e_k, e_k) = 1 + 0\delta_1 + 0\delta_2$ holds.

Example 3.2. (Continued from Example 3.1). In Example 3.1, we take the criteria weight set as $\{w_1, w_2, w_3, w_4, w_5\} = \{0.2, 0.2, 0.2, 0.2, 0.2\}$. For the expert pair (e_k, e_h), we can aggregate all consensus CNs $u_{ij}(e_k, e_h)$ under five criteria in Table 4 by using Eq. (5) to obtain the weighted consensus CNs $u_i(e_k, e_h)$ for three alternatives. Taking the expert pair (e_1, e_2) as an example, the weighted consensus CNs $u_1(e_1, e_2)$ for alternative x_1 can be computed as: $u_1(e_1, e_2) = (0.2 \otimes (0 + 1\delta_1 + 0\delta_2)) \oplus (0.2 \otimes (0 +$

$1\delta_1+0\delta_2))\oplus(0.2\otimes(1+0\delta_1+0\delta_2))\oplus(0.2\otimes(1+0\delta_1+0\delta_2))\oplus(0.2\otimes(1+0\delta_1+0\delta_2)) = 0.6+0.4\delta_1+0\delta_2$, i.e., the identity degree, discrepancy degree and contrary degree are $S_1^{12}=0.6$, $F_1^{12}=0.4$ and $P_1^{12}=0$. Similarly, other weighted consensus CNs $u_i(e_k,e_h)$ can also be obtained, as shown in Table 5.

Table 5
The weighted consensus CNs $u_i(e_k,e_h)$ in Example 3.2.

(e_k,e_h)	X	$u_i(e_k,e_h)$	S_i^{kh}	F_i^{kh}	P_i^{kh}
(e_1,e_2)	x_1	$0.6+0.4\delta_1+0\delta_2$	0.6	0.4	0
	x_2	$0.8+0\delta_1+0.2\delta_2$	0.8	0	0.2
	x_3	$0.4+0.6\delta_1+0\delta_2$	0.4	0.6	0
(e_1,e_3)	x_1	$0.6+0.4\delta_1+0\delta_2$	0.6	0.4	0
	x_2	$0.8+0.2\delta_1+0\delta_2$	0.8	0.2	0
	x_3	$0.4+0.6\delta_1+0\delta_2$	0.4	0.6	0
(e_2,e_3)	x_1	$0.2+0.6\delta_1+0.2\delta_2$	0.2	0.6	0
	x_2	$0.6+0.4\delta_1+0\delta_2$	0.6	0.4	0
	x_3	$0.8+0.2\delta_1+0\delta_2$	0.8	0.2	0

Definition 5. For an alternative x_i, the individual consensus CN $u_i(e_k)$ of expert e_k under the alternative x_i aggregating all weighted consensus CNs $u_i(e_k,e_h)(h=1,2,...,K)$ is denoted as follows:

$$u_i(e_k) = S_i^k + F_i^k\delta_1 + P_i^k\delta_2; i=1,2,...,m; \forall e_k\in E, \tag{6}$$

where $S_i^k\in[0,1]$, $F_i^k\in[0,1]$ and $P_i^k\in[0,1]$ represent the identity degree, discrepancy degree and contrary degree of expert e_k under the alternative x_i, respectively. The condition $S_i^k+F_i^k+P_i^k=1$ holds.

Similarly, the individual consensus CN $u_i(e_k)$ can be obtained as $u_i(e_k)=f(u_i(e_k,e_1),u_i(e_k,e_2),..., u_i(e_k,e_K))$. When $f(\cdot)$ is assumed as WA operator, then we have $f(u_i(e_k,e_1),u_i(e_k,e_2),...,u_i(e_k,e_K)) = (v_{k1}\otimes u_i(e_k,e_1))\oplus(v_{k2}\otimes u_i(e_k,e_2))\oplus\cdots\oplus(v_{kK}\otimes u_i(e_k,e_K))$, where v_{kh} is the weight assigned to expert e_h by expert e_k and $v_{kh}\in[0,1]$. The real number v_{kh} can be rewritten as a CN $v_{kh}+0\delta_1+0\delta_2$, then we have $v_{kh}\otimes u_i(e_k,e_h) = v_{kh}S_i^{kh}+v_{kh}F_i^{kh}\delta_1+v_{kh}P_i^{kh}\delta_2$ based on Definition 2. Thus, the individual consensus CN $u_i(e_k)$ can be rewritten as:

$$u_i(e_k) = \sum_{h=1}^{K} v_{kh}S_i^{kh} + \sum_{h=1}^{K} v_{kh}F_i^{kh}\delta_1 + \sum_{h=1}^{K} v_{kh}P_i^{kh}\delta_2; i=1,2,...,m; \forall e_k\in E, \tag{7}$$

where $\sum_{h=1}^{K} v_{kh}S_i^{kh}$, $\sum_{h=1}^{K} v_{kh}F_i^{kh}$ and $\sum_{h=1}^{K} v_{kh}P_i^{kh}$ are used to represent the identity degree S_i^k, discrepancy degree F_i^k and contrary degree P_i^k of expert e_k for alternative x_i, respectively.

The three coefficients S_i^k, F_i^k and P_i^k can be regarded as the probabilities of three states including 'Consensus', 'Between consensus and conflict' and 'Conflict'. For convenience, the state 'Between consensus and conflict' is described as 'Uncertainty' in what follows. The consensus set pair probability space is defined based on three basic events corresponding to three states.

Definition 6. We define the five-tuple $(\Omega, \Lambda, Pr, X, E)$ as the consensus set pair probability space, where $\Omega = \left\{C, \widetilde{C}, \overline{C}\right\}$ is a state sample space corresponding to three states $\{Consensus, Uncertainty, Conflict\}$, respectively. $\Lambda = \left\{\emptyset, \{C\}, \{\widetilde{C}\}, \{\overline{C}\}, \Omega\right\}$ is the state set consisting of subsets of Ω. $X = \{x_1, x_2, ..., x_m\}$ is the alternative set and $E = \{e_1, e_2, ..., e_K\}$ is the expert set. $Pr(\cdot)$ is the probability measure of the measurable space $\{\Omega, \Lambda\}$ when the following conditions are satisfied:

(1) $Pr(\emptyset) = 0$;

(2) $Pr(\Omega) = 1$;

(3) For any disjointed event $A_1, A_2, ... \in \Lambda$, $Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} Pr(A_i)$.

For the consensus set pair probability space $(\Omega, \Lambda, Pr, X, E)$, the three probabilities $Pr_k^i(C) = S_k^i$, $Pr_k^i(\widetilde{C}) = F_k^i$ and $Pr_k^i(\overline{C}) = P_k^i$ reflect the state likelihood of expert e_k under alternative x_i where $Pr_k^i(C) + Pr_k^i(\widetilde{C}) + Pr_k^i(\overline{C}) = 1$. For an alternative x_i, the probability measure triple of expert e_k is described as $\left\{Pr_k^i(C), Pr_k^i(\widetilde{C}), Pr_k^i(\overline{C})\}\right\} = \{S_k^i, F_k^i, P_k^i\}$. Then the probability measure matrix $\mathcal{P} = \left(\left\{Pr_k^i(C), Pr_k^i(\widetilde{C}), Pr_k^i(\overline{C})\}\right\}\right)_{K\times m}$ is as follows:

$$\mathcal{P} = \begin{bmatrix} \{S_1^1, F_1^1, P_1^1\} & \{S_1^2, F_1^2, P_1^2\} & \cdots & \{S_1^m, F_1^m, P_1^m\} \\ \{S_2^1, F_2^1, P_2^1\} & \{S_2^2, F_2^2, P_2^2\} & \cdots & \{S_2^m, F_2^m, P_2^m\} \\ \vdots & \vdots & \ddots & \vdots \\ \{S_K^1, F_K^1, P_K^1\} & \{S_K^2, F_K^2, P_K^2\} & \cdots & \{S_K^m, F_K^m, P_K^m\} \end{bmatrix}. \tag{8}$$

Example 3.3. (Continued from Example 3.2). Let the interactive weight matrix be $V = \{$ 0.4, 0.3, 0.3; 0.3, 0.4, 0.3; 0.3, 0.3, 0.4 $\}$. Based on the weighted consensus CNs $u_i(e_k, e_h)$ in Table 5, the individual consensus CN $u_i(e_k)$ can be obtained by using Eq. (7). Taking the individual consensus CN $u_1(e_1)$ for example, we have $u_1(e_1) = (0.4 \otimes u_1(e_1, e_1)) \oplus (0.3 \otimes u_1(e_1, e_2)) \oplus (0.3 \otimes u_1(e_1, e_3)) = 0.76 + 0.24\delta_1 + 0\delta_2$, i.e., $S_1^1 = 0.76, P_1^1 = 0.24, F_1^1 = 0$. The individual consensus CNs $u_i(e_k)$ are shown in Table 6. Then the probability measure matrix $\mathcal{P} = \left(\left\{Pr_k^i(C), Pr_k^i(\widetilde{C}), Pr_k^i(\overline{C})\}\right\}\right)_{3\times 3}$ is obtained as follows:

$$\mathcal{P} = \begin{bmatrix} \{0.76, 0.24, 0.00\} & \{0.88, 0.06, 0.06\} & \{0.64, 0.36, 0.00\} \\ \{0.64, 0.30, 0.06\} & \{0.82, 0.12, 0.06\} & \{0.76, 0.24, 0.00\} \\ \{0.64, 0.30, 0.06\} & \{0.82, 0.18, 0.00\} & \{0.76, 0.24, 0.00\} \end{bmatrix}. \tag{9}$$

Table 6
The individual consensus CNs $u_i(e_k)$ in Example 3.3.

E	X	$u_i(e_k)$	S_i^k	F_i^k	P_i^k
e_1	x_1	$0.76 + 0.24\delta_1 + 0\delta_2$	0.76	0.24	0
	x_2	$0.64 + 0.3\delta_1 + 0.06\delta_2$	0.64	0.3	0.06
	x_3	$0.64 + 0.3\delta_1 + 0.06\delta_2$	0.64	0.3	0.06
e_2	x_1	$0.88 + 0.06\delta_1 + 0.06\delta_3$	0.88	0.06	0.06
	x_2	$0.82 + 0.12\delta_1 + 0.06\delta_2$	0.82	0.12	0.06
	x_3	$0.82 + 0.18\delta_1 + 0\delta_2$	0.82	0.18	0
e_3	x_1	$0.64 + 0.36\delta_1 + 0\delta_4$	0.64	0.36	0
	x_2	$0.76 + 0.24\delta_1 + 0\delta_2$	0.76	0.24	0
	x_3	$0.76 + 0.24\delta_1 + 0\delta_2$	0.76	0.24	0

3.2. The three-state three-way decision model based on set pair analysis

The classic 3WD theory assumes that an object has two states: belonging or not belonging to state C, corresponding to the consensus and conflict state among experts in MCGDM problems. However, the state among experts is not an either-or situation, and the third state may emerge in the decision-making process——uncertainty state, which coincides with the idea of SPA theory. Based on SPA theory, the classic two-state 3WD model is extended to the TS3WD model involved in uncertainty state.

The TS3WD model includes three states $\Omega = \left\{C, \widetilde{C}, \overline{C}\right\}$ and three actions $\mathrm{A} = \{a_P, a_B, a_N\}$. C, $\widetilde{C}$ and $\overline{C}$ represent that an object x_i is the consensus state, the uncertainty state and the conflict state, respectively. a_P, a_B and a_N stand for acceptance, delay and rejection actions, which are denoted as $x_i \in POS(C)$, $x_i \in BND(C)$ and $x_i \in NEG(C)$, respectively. For an alternative-expert pair (x_i, e_k) consisting of alternative x_i and expert e_k, $Pr_k(C|[x_i])$, $Pr_k(\widetilde{C}|[x_i])$ and $Pr_k(\overline{C}|[x_i])$ are the conditional probabilities of the three states of the alternative-expert pair (x_i, e_k), which can be obtained that $Pr_k(C|[x_i]) = S_k^i$, $Pr_k(\widetilde{C}|[x_i]) = F_k^i$ and $Pr_k(\overline{C}|[x_i]) = 1 - S_k^i - F_k^i$ based on the consensus set pair probability space. The extended loss functions $\lambda_{\bullet\circ}(\bullet = P, B, N; \circ = P, B, N)$ are denoted to measure the losses of taking three actions, as shown in Table 7. λ_{PP}, λ_{BP} and λ_{NP} represent the losses taking actions a_P, a_B and a_N when $x \in C$. λ_{PB}, λ_{BB} and λ_{NB} denote the losses taking actions a_P, a_B and a_N when $x \in \widetilde{C}$. λ_{PN}, λ_{BN} and λ_{NN} denote the losses taking actions a_P, a_B and a_N when $x \in \overline{C}$. A reasonable assumption is considered in the TS3WD model: $0 \le \lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $0 \le \lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$. When determining the optimistic extended loss functions, the loss of taking action a_P is less than the loss of taking action a_N if $x \in \widetilde{C}$, then $0 \le \lambda_{BB} \le \lambda_{PB} < \lambda_{NB}$ is assumed. Similarly, $0 \le \lambda_{BB} \le \lambda_{NB} \le \lambda_{PB}$ is considered when determining the pessimistic extended loss functions.

Table 7
The extended loss functions in TS3WD model.

Actions	$C(P)$	$\widetilde{C}(B)$	$\overline{C}(N)$
a_P	λ_{PP}	λ_{PB}	λ_{PN}
a_B	λ_{BP}	λ_{BB}	λ_{BN}
a_N	λ_{NP}	λ_{NB}	λ_{NN}

The expected extended loss functions $R_k(a_\bullet|[x_i])$ of taking three actions for expert e_k under alternative x_i can be calculated as:

(1) $R_k(a_P|[x_i]) = \lambda_{PP}\Pr_k(C|[x_i]) + \lambda_{PB}\Pr_k(\widetilde{C}|[x_i]) + \lambda_{PN}\Pr_k(\overline{C}|[x_i])$;

(2) $R_k(a_B|[x_i]) = \lambda_{BP}\Pr_k(C|[x_i]) + \lambda_{BB}\Pr_k(\widetilde{C}|[x_i]) + \lambda_{BN}\Pr_k(\overline{C}|[x_i])$;

(3) $R_k(a_N|[x_i]) = \lambda_{NP}\Pr_k(C|[x_i]) + \lambda_{NB}\Pr_k(\widetilde{C}|[x_i]) + \lambda_{NN}\Pr_k(\overline{C}|[x_i])$.

According to the Bayesian theory, the decision rules of minimum losses can be demonstrated as follows:

(P3) If $R_k(a_P|[x_i]) \leq R_k(a_B|[x_i])$ and $R_k(a_P|[x_i]) \leq R_k(a_N|[x_i])$, then $(x_i, e_k) \in POS(C)$;

(B3) If $R_k(a_B|[x_i]) \leq R_k(a_P|[x_i])$ and $R_k(a_B|[x_i]) \leq R_k(a_N|[x_i])$, then $(x_i, e_k) \in BND(C)$;

(N3) If $R_k(a_N|[x_i]) \leq R_k(a_P|[x_i])$ and $R_k(a_N|[x_i]) \leq R_k(a_B|[x_i])$, then $(x_i, e_k) \in NEG(C)$.

Proposition 3.1. *For an alternative-expert pair (x_i, e_k), the decision rules (P3)-(N3) can be rewritten as the decision rules (P4)-(N4), where α, γ and β are three thresholds in (P1)-(N1). The decision rules (P4)-(N4)* are listed as follows:

(P4) If $S_k^i \geq \frac{(\lambda_{PB}-\lambda_{BB}-\lambda_{PN}+\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})}F_k^i + \alpha$ *and* $S_k^i \geq \frac{(\lambda_{PB}-\lambda_{NB}+\lambda_{NN}-\lambda_{PN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}F_k^i + \gamma$*, then* $(x_i, e_k) \in POS(C)$;

(B4) If $S_k^i \leq \frac{(\lambda_{PB}-\lambda_{BB}-\lambda_{PN}+\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})}F_k^i + \alpha$ *and* $S_k^i \geq \frac{(\lambda_{BB}-\lambda_{NB}+\lambda_{NN}-\lambda_{BN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}F_k^i + \beta$*, then* $(x_i, e_k) \in BND(C)$;

(N4) If $S_k^i \leq \frac{(\lambda_{PB}-\lambda_{NB}+\lambda_{NN}-\lambda_{PN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}F_k^i + \gamma$ *and* $S_k^i \leq \frac{(\lambda_{BB}-\lambda_{NB}+\lambda_{NN}-\lambda_{BN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}F_k^i + \beta$*, then* $(x_i, e_k) \in NEG(C)$.

Proof. *For the condition $R_k(a_P|[x_i]) \leq R_k(a_B|[x_i])$ in (P3), we can obtain $\lambda_{PP}S_k^i + \lambda_{PB}F_k^i + \lambda_{PN}(1 - S_k^i - F_k^i) \leq \lambda_{BP}S_k^i + \lambda_{BB}F_k^i + \lambda_{BN}(1 - S_k^i - F_k^i)$, which can be converted into $(\lambda_{PB} - \lambda_{BB} - \lambda_{PN} + \lambda_{BN})F_k^i + (\lambda_{PN} - \lambda_{BN}) \leq (\lambda_{BP} - \lambda_{PP} + \lambda_{PN} - \lambda_{BN})S_k^i$ by merging similar items. Due to $\lambda_{BP} \geq \lambda_{PP}$ and $\lambda_{PN} > \lambda_{BN}$, then $\lambda_{BP} - \lambda_{PP} + \lambda_{PN} - \lambda_{BN} > 0$ is satisfied and we can obtain $S_k^i \geq \frac{(\lambda_{PB}-\lambda_{BB}-\lambda_{PN}+\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})}F_k^i + \frac{(\lambda_{PN}-\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})}$. Similarly, $R_k(a_P|[x_i]) \leq R_k(a_N|[x_i])$ in (P3) can be replaced by $S_k^i \geq \frac{(\lambda_{PB}-\lambda_{NB}+\lambda_{NN}-\lambda_{PN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}F_k^i + \frac{(\lambda_{PN}-\lambda_{NN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}$ and $R_k(a_B|[x_i]) \leq R_k(a_N|[x_i])$ in (B3) can be converted as $S_k^i \geq \frac{(\lambda_{BB}-\lambda_{NB}+\lambda_{NN}-\lambda_{BN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}F_k^i + \frac{(\lambda_{BN}-\lambda_{NN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$. For the other three inequalities including $R_k(a_B|[x_i]) \leq R_k(a_P|[x_i])$, $R_k(a_N|[x_i]) \leq R_k(a_P|[x_i])$ and $R_k(a_N|[x_i]) \leq R_k(a_B|[x_i])$, we can transform them using the similar way and the relative conversions are omitted here for convenience. In Section 2.1, three thresholds in classic 3WD theory are $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$, $\gamma = \frac{\lambda_{PN}-\lambda_{NN}}{(\lambda_{PN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{PP})}$ and $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$ which can be replaced by the above inequalities, then we can update the decision rules (P3)-(N3) to (P4)-(N4).* ■

Proposition 3.2. *Let the three thresholds in TS3WD model be* $\alpha_1 = \frac{(\lambda_{PB}-\lambda_{BB}-\lambda_{PN}+\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})}F_k^i + \alpha$, $\beta_1 = \frac{(\lambda_{BB}-\lambda_{NB}+\lambda_{NN}-\lambda_{BN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}F_k^i + \beta$ *and* $\gamma_1 = \frac{(\lambda_{PB}-\lambda_{NB}+\lambda_{NN}-\lambda_{PN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}F_k^i + \gamma$. *For an alternative-expert pair* (x_i, e_k), *when* $(\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN}) \leq (\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN})$ *is satisfied, the decision rules (P4)-(N4) can be simplified as the following decision rules (P5)-(N5):*

(P5) If $S_k^i \geq \alpha_1$, *then* $(x_i, e_k) \in POS(C)$;

(B5) If $\beta_1 < S_k^i < \alpha_1$, *then* $(x_i, e_k) \in BND(C)$;

(N5) If $S_k^i \leq \beta_1$, *then* $(x_i, e_k) \in NEG(C)$.

Proof. *The condition* $(\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN}) \leq (\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN})$ *can be rewritten as* $\frac{\lambda_{BP}-\lambda_{PP}}{\lambda_{PN}-\lambda_{BN}} \leq \frac{\lambda_{NP}-\lambda_{BP}}{\lambda_{BN}-\lambda_{NN}}$, *then we can obtain* $\frac{(\lambda_{PN}-\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} \geq \frac{(\lambda_{BN}-\lambda_{NN})}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$, *i.e.,* $\alpha \geq \beta$, *then the inequality* $1 \geq \alpha \geq \gamma \geq \beta \geq 0$ *is satisfied in classic 3WD model. For* α_1, β_1 *and* γ_1, *we compare the three thresholds in pairwise as follows:*

(1) For α_1 *and* β_1, *we can obtain* $\alpha_1 - \beta_1 = \frac{(\lambda_{PN}-\lambda_{BN})(1-F_k^i)}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} - \frac{(\lambda_{BN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})} + \frac{(\lambda_{PB}-\lambda_{BB})F_k^i}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} - \frac{(\lambda_{BB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$. *Due to* $\alpha \geq \beta$ *and* $(1 - F_k^i) \geq 0$, *then we have* $\frac{(\lambda_{PN}-\lambda_{BN})(1-F_k^i)}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} - \frac{(\lambda_{BN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})} \geq 0$. *Due to* $(\lambda_{PB} - \lambda_{BB}) \geq 0$ *and* $(\lambda_{BB} - \lambda_{NB}) \leq 0$, *we have* $\frac{(\lambda_{PB}-\lambda_{BB})F_k^i}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} \geq \frac{(\lambda_{BB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$. *Therefore, we have* $\alpha_1 - \beta_1 \geq 0$.

(2) For α_1 *and* γ_1, *we can obtain* $\alpha_1 - \gamma_1 = \frac{(\lambda_{PN}-\lambda_{BN})(1-F_k^i)}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} - \frac{(\lambda_{PN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} + \frac{(\lambda_{PB}-\lambda_{BB})F_k^i}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} - \frac{(\lambda_{PB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}$. *Due to* $\alpha \geq \gamma$ *and* $(1 - F_k^i) \geq 0$, *then we have* $\frac{(\lambda_{PN}-\lambda_{BN})(1-F_k^i)}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} - \frac{(\lambda_{PN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} \geq 0$. *Due to* $(\lambda_{PB} - \lambda_{BB}) \geq (\lambda_{PB} - \lambda_{NB})$ *and* $0 \leq (\lambda_{BP} - \lambda_{PP} + \lambda_{PN} - \lambda_{BN}) \leq (\lambda_{NP} - \lambda_{PP} + \lambda_{PN} - \lambda_{NN})$, *we have* $\frac{(\lambda_{PB}-\lambda_{BB})F_k^i}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} \geq \frac{(\lambda_{PB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})}$. *Therefore, we have* $\alpha_1 - \gamma_1 \geq 0$.

(3) For γ_1 *and* β_1, *we can obtain* $\gamma_1 - \beta_1 = \frac{(\lambda_{PN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} - \frac{(\lambda_{BN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})} + \frac{(\lambda_{PB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} - \frac{(\lambda_{BB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$. *Due to* $\alpha \geq \gamma$ *and* $(1 - F_k^i) \geq 0$, *then we have* $\frac{(\lambda_{PN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} - \frac{(\lambda_{BN}-\lambda_{NN})(1-F_k^i)}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})} \geq 0$. *In this case, the optimistic and pessimistic TS3WD model should be discussed as follows. 1) For optimistic TS3WD model,* $0 \leq \lambda_{BB} \leq \lambda_{PB} < \lambda_{NB}$ *is satisfied and then* $(\lambda_{NB} - \lambda_{PB}) \leq (\lambda_{NB} - \lambda_{BB})$. *Due to* $(\lambda_{NP} - \lambda_{PP} + \lambda_{PN} - \lambda_{NN}) \geq (\lambda_{NP} - \lambda_{BP} + \lambda_{BN} - \lambda_{NN})$, *we have* $\frac{(\lambda_{NB}-\lambda_{PB})F_k^i}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} \leq \frac{(\lambda_{NB}-\lambda_{BB})F_k^i}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$. *Therefore, we can obtain* $-\frac{(\lambda_{NB}-\lambda_{PB})F_k^i}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} \geq \frac{(\lambda_{BB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$, *i.e.,* $\frac{(\lambda_{PB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} \geq \frac{(\lambda_{BB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$. *2) For pessimistic TS3WD model,* $0 \leq \lambda_{BB} \leq \lambda_{NB} \leq \lambda_{PB}$ *is satisfied, then* $(\lambda_{PB} - \lambda_{NB}) \geq 0$ *and* $(\lambda_{BB} - \lambda_{NB}) \leq 0$ *can be satisfied. Therefore, we have* $\frac{(\lambda_{PB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} \geq \frac{(\lambda_{BB}-\lambda_{NB})F_k^i}{(\lambda_{NP}-\lambda_{BP}+\lambda_{BN}-\lambda_{NN})}$. *In short, we have* $\gamma_1 - \beta_1 \geq 0$ *for both the optimistic and pessimistic TS3WD models.*

According to the pairwise discussions, when $(\lambda_{BP} - \lambda_{PP})(\lambda_{BN} - \lambda_{NN}) \leq (\lambda_{NP} - \lambda_{BP})(\lambda_{PN} - \lambda_{BN})$, *we have* $1 \geq \alpha_1 \geq \gamma_1 \geq \beta_1 \geq 0$. *Therefore, the decision rules (P4)-(N4) can be simplified as the decision rules (P5)-(N5).* ■

Proposition 3.3. *When the conditional probability of uncertainty state is* $\Pr_k(\widetilde{C}|[x_i]) = 0$, *the TS3WD model degenerates into the classic two-state 3WD model.*

Proof. *When* $\Pr_k(\widetilde{C}|[x_i]) = 0$, *the expected extended loss functions* $R_k(a_\bullet|[x_i])$ *are reduced as follows:*

(1) $R_k(a_P|[x_i]) = \lambda_{PP}\Pr_k(C|[x_i]) + \lambda_{PN}\Pr_k(\overline{C}|[x_i])$;

(2) $R_k(a_B|[x_i]) = \lambda_{BP}\Pr_k(C|[x_i]) + \lambda_{BN}\Pr_k(\overline{C}|[x_i])$;

(3) $R_k(a_N|[x_i]) = \lambda_{NP}\Pr_k(C|[x_i]) + \lambda_{NN}\Pr_k(\overline{C}|[x_i])$.

In this case, the expected extended loss functions $R_k(a_\bullet|[x_i])$ only relate the conditional probabilities and the extended loss functions under states C and $\overline{C}$, which are consistent with classic 3WD model. ■

4. The consensus feedback mechanism of the sequential three-state three-way consensus model

In this section, a consensus adjustment model for alternative-expert pairs (x_i, e_k) in the uncertainty state is proposed to modify the experts' evaluation under all criteria. Based on the sequential TS3WD consensus model, a consensus feedback mechanism with several classification results is designed to refine the classification of alternative-expert pairs.

4.1. The consensus adjustment model based on the three-state three-way decision consensus model

According to the multi-criteria set pair TS3WD consensus model, all alternative-expert pairs (x_i, e_k) can be divided into positive region $POS(C)$, boundary region $BND(C)$ and negative region $NEG(C)$. All elements in the positive region are regarded to be in the consensus state, while elements in the other two regions require improvement to reach the consensus. Based on S3WD theory, the boundary region can be further divided into three new regions as the information increases. In this subsection, we specifically discuss how to modify the elements in the negative region.

For a sequential TS3WD consensus problem, all alternative-expert pairs (x_i, e_k) would be constantly divided into three regions until the boundary region is empty. The consensus adjustment model is used to locate the alternative-expert pairs (x_i, e_k) in the conflict state and modify the opinions in these positions. This ensures that the adjusted opinions can reach the consensus in next iteration, effectively placing the updated alternative-expert pairs $(x_i, \bar{e}_k)$ into the positive region. For convenience, the three regions can be represented by $POS^t(C)$, $BND^t(C)$ and $NEG^t(C)$ in the tth iteration. $POS(x_i^t) = \{e_k|(x_i, e_k) \in POS^t(C)\}$, $BND(x_i^t) = \{e_k|(x_i, e_k) \in BND^t(C)\}$ and $NEG(x_i^t) = \{e_k|(x_i, e_k) \in NEG^t(C)\}$ are used to represent the three types of experts set for alternative x_i in the tth iteration, namely the consensus expert set, uncertainty expert set and conflict expert set, respectively.

Based on the principle of minimum opinion adjustment, a consensus adjustment model (M-1) for all experts belonging to $NEG(x_i^t)$ in the tth iteration is proposed to obtain the opinion modifications under all criteria. Considering that the minimum opinion modifications, the main idea of the model (M-1) is to modify opinions of experts in the conflict state under alternative x_i according to the opinions of experts in the consensus state, in which the experts modifying the opinions would be divided into the consensus state in next iteration. In model (M-1), $e_k \in NEG(x_i^t)$ defines the conflict experts with the conflict state and $e_{o(\varepsilon)} \in POS(x_i^t)$ represents the consensus experts with the consensus state for alternative x_i in the tth iteration, respectively. The number of the consensus expert set for alternative x_i in the tth iteration is ε_i^t and $o(\cdot)$ is the index of the consensus experts. The modified experts $\bar{e}_k$ would reach the consensus in the $(t+1)$th iteration, which is denoted as $\bar{e}_k \in POS(x_i^{t+1})$. The objective function is to minimize the total opinion modifications under all criteria, denoted as $\min \sum_{j=1}^{n} w_j (m_{ij}^k - \bar{m}_{ij}^k)^2$. $\bar{m}_{ij}^k$ is the modified opinion of expert e_k under criterion c_j for alternative x_i. The constraints show the construction rules of consensus set pair probability space for conflict experts $e_k \in NEG(x_i^t)$, which include the set pair consensus rules, aggregation rules of CNs and the decision rules of consensus set pair probability space.

$$
\begin{aligned}
&\min \sum_{j=1}^{n} w_j (m_{ij}^k - \bar{m}_{ij}^k)^2 \\
&s.t. \begin{cases}
u_{ij}(e_k, e_h) = \begin{cases} 1 + 0\delta_1 + 0\delta_2, \left|m_{ij}^k - m_{ij}^h\right| \le \tau_1 \\ 0 + 1\delta_1 + 0\delta_2, \tau_1 < \left|m_{ij}^k - m_{ij}^h\right| < \tau_2, j = 1, 2, ..., n \\ 0 + 0\delta_1 + 1\delta_2, \left|m_{ij}^k - m_{ij}^h\right| \ge \tau_2 \end{cases} \\
u_{ij}(\bar{e}_k, e_h) = \begin{cases} 1 + 0\delta_1 + 0\delta_2, \left|\bar{m}_{ij}^k - m_{ij}^h\right| \le \tau_1 \\ 0 + 1\delta_1 + 0\delta_2, \tau_1 < \left|\bar{m}_{ij}^k - m_{ij}^h\right| < \tau_2, j = 1, 2, ..., n \\ 0 + 0\delta_1 + 1\delta_2, \left|\bar{m}_{ij}^k - m_{ij}^h\right| \ge \tau_2 \end{cases} \\
u_i(e_k, e_h) = f\left(u_{i1}(e_k, e_h), u_{i2}(e_k, e_h), ..., u_{in}(e_k, e_h)\right), \\
u_i(\bar{e}_k, e_h) = f\left(u_{i1}(\bar{e}_k, e_h), u_{i2}(\bar{e}_k, e_h), ..., u_{in}(\bar{e}_k, e_h)\right), \\
u_i(e_k) = f\left(u_i(e_k, e_{o(1)}), u_i(e_k, e_{o(2)}), ..., u_i(e_k, e_{o(\varepsilon_i^t)})\right), \\
u_i(\bar{e}_k) = f\left(u_i(\bar{e}_k, e_{o(1)}), u_i(\bar{e}_k, e_{o(2)}), ..., u_i(\bar{e}_k, e_{o(\varepsilon_i^t)})\right), \\
\bar{e}_k \in POS(x_i^{t+1}), \\
\forall e_k \in NEG(x_i^t); \forall e_{o(\cdot)} \in POS(x_i^t).
\end{cases}
\end{aligned}
\tag{M-1}
$$

The model (M-1) is a nonlinear programming model including the logical constraints, which can be transformed into a linear programming model (M-2). The transformation steps are shown as follows.

(1) For the set pair consensus rules, the logical constraints in model (M-1) can be rewritten as four inequalities in model (M-2), where $S_{ij}^{ko(\varepsilon)}$, $F_{ij}^{ko(\varepsilon)}$ and $P_{ij}^{ko(\varepsilon)}$ denote the consensus CN of expert pair $(e_k, e_{o(\varepsilon)})$ under criteria c_j for alternative x_i and M is an infinite constant. Obviously, the four inequalities involving absolute value operations can be easily linearized by transforming each inequality into two inequalities without absolute value operations. For convenience, the linearization steps of the absolute value operation are omitted and only the four inequalities containing the absolute value operation are retained. Meanwhile, the model (M-2) is referred to as linear programming model in what follows.

(2) For the aggregation rules of CNs, the three coefficients S_i^k, F_i^k, P_i^k of the individual consensus CN $u_i(e_k)$ can be calculated based on Definitions 4 and 5. Due to $S_i^k + F_i^k + P_i^k = 1$, the operations about P_i^k are omitted. In particular, the weights $v_{ko(\varepsilon)}$ should be normalized as $\bar{v}_{ko(\varepsilon)}$ when aggregating all $u_i(\bar{e}_k, e_{o(\varepsilon)})$, i.e., $\bar{v}_{ko(\varepsilon)} = \frac{v_{ko(\varepsilon)}}{\sum_{\varepsilon=1}^{\varepsilon_i^t} v_{ko(\varepsilon)}}$.

(3) For the decision rules of probability space, $\bar{e}_k \in POS(x_i^{t+1})$ can be replaced by the rule P4 in decision rules (P4)-(N4).

Based on the linear programming model (M-2), the modified opinions $\bar{m}_{ij}^k$ under all criteria can be obtained for all conflict experts $e_k \in NEG(x_i^t)$ in the tth iteration. The consensus adjustment model not only allows conflict experts to revise their opinions under all criteria to reach the consensus, but also guarantees the minimization of total opinion modifications.

$$\min \sum_{j=1}^{n} w_j (m_{ij}^k - \bar{m}_{ij}^k)^2$$
$$s.t.\begin{cases} \left|m_{ij}^{o(\varepsilon)} - \bar{m}_{ij}^k\right| \leq \tau_1 + (1 - S_{ij}^{ko(\varepsilon)})M, \\ \left|m_{ij}^{o(\varepsilon)} - \bar{m}_{ij}^k\right| \geq \tau_2 - (1 - F_{ij}^{ko(\varepsilon)})M, \\ \left|m_{ij}^{o(\varepsilon)} - \bar{m}_{ij}^k\right| < \tau_2 + (1 - P_{ij}^{ko(\varepsilon)})M, \\ \left|m_{ij}^{o(\varepsilon)} - \bar{m}_{ij}^k\right| > \tau_1 - (1 - P_{ij}^{ko(\varepsilon)})M, \\ S_{ij}^{ko(\varepsilon)} + F_{ij}^{ko(\varepsilon)} + P_{ij}^{ko(\varepsilon)} = 1, \\ \bar{v}_{ko(\varepsilon)} = \frac{v_{ko(\varepsilon)}}{\sum_{\varepsilon=1}^{\varepsilon_i^t} v_{ko(\varepsilon)}}, \\ S_i^k = \sum_{\varepsilon=1}^{\varepsilon_i^t} \bar{v}_{ko(\varepsilon)} \sum_{j=1}^{n} w_j S_{ij}^{ko(\varepsilon)}, F_i^k = \sum_{\varepsilon=1}^{\varepsilon_i^t} \bar{v}_{ko(\varepsilon)} \sum_{j=1}^{n} w_j F_{ij}^{ko(\varepsilon)}, \\ \frac{(\lambda_{PB}-\lambda_{BB}-\lambda_{PN}+\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} F_i^k + \frac{(\lambda_{PN}-\lambda_{BN})}{(\lambda_{BP}-\lambda_{PP}+\lambda_{PN}-\lambda_{BN})} \leq S_i^k, \\ \frac{(\lambda_{PB}-\lambda_{NB}+\lambda_{NN}-\lambda_{PN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} F_i^k + \frac{(\lambda_{PN}-\lambda_{NN})}{(\lambda_{NP}-\lambda_{PP}+\lambda_{PN}-\lambda_{NN})} \leq S_i^k, \\ j = 1, 2, ..., n; \varepsilon = 1, 2, ..., \varepsilon_i^t, \\ S_{ij}^{ko(\varepsilon)}, F_{ij}^{ko(\varepsilon)}, P_{ij}^{ko(\varepsilon)} \in \{0, 1\}; \forall e_k \in NEG(x_i^t), \forall e_{o(\varepsilon)} \in POS(x_i^t). \end{cases} \tag{M-2}$$

4.2. The consensus feedback rules with several sequential three-way classification results

The conflict experts $e_k \in NEG(x_i^t)$ can reach the consensus by modifying opinions towards opinions of consensus experts $e_{o(\varepsilon)} \in POS(x_i^t)$. However, the consensus adjustment model (M-2) can not be established in some specific cases like $POS(x_i^t) = \emptyset$. In addition, the boundary regions of two successive iterations may completely coincide due to the coarser granularity in the sequential TS3WD consensus classification process, i.e., the case $BND(x_i^{t+1}) = BND(x_i^t)$ may exist in some cases. Therefore, some classification results for three regions under alternative x_i in the tth iteration are worth exploring to process the sequential TS3WD consensus classification.

According to the number of nonempty regions, all possible classification results for three regions $POS(x_i^t)$, $BND(x_i^t)$ and $NEG(x_i^t)$ are listed in Table 8. The discussions about seven classification cases under alternative x_i in the tth iteration are given as follows.

Table 8
The classification results for $POS(x_i^t)$, $BND(x_i^t)$ and $NEG(x_i^t)$.

Cases	*Nonempty number*	$POS(x_i^t)$	$BND(x_i^t)$	$NEG(x_i^t)$
Case 1	3	*nonempty*	*nonempty*	*nonempty*
Case 2	2	*nonempty*	*nonempty*	$\emptyset$
Case 3	2	*nonempty*	$\emptyset$	*nonempty*
Case 4	2	$\emptyset$	*nonempty*	*nonempty*
Case 5	1	*nonempty*	$\emptyset$	$\emptyset$
Case 6	1	$\emptyset$	$\emptyset$	*nonempty*
Case 7	1	$\emptyset$	*nonempty*	$\emptyset$

Case 1. When three region are all nonempty in the tth iteration, experts in the conflict expert set $NEG(x_i^t)$ can adjust their opinions towards opinions of the consensus expert set $POS(x_i^t)$, which can be obtained by solving the consensus adjustment model (M-2) directly. The uncertainty expert set $BND(x_i^t)$ would be divided into three new regions including $POS(x_i^{t+1})$, $BND(x_i^{t+1})$ and $NEG(x_i^{t+1})$ in the $(t+1)$th iteration.

Case 2. When $NEG(x_i^t)$ is empty and the other two regions are nonempty in the tth iteration, no opinion should be adjusted in this iteration. The uncertainty expert set $BND(x_i^t)$ would be divided into three new regions including $POS(x_i^{t+1})$, $BND(x_i^{t+1})$ and $NEG(x_i^{t+1})$ in the $(t+1)$th iteration.

Case 3. When $BND(x_i^t)$ is empty and the other two regions are nonempty in the tth iteration, the empty region $BND(x_i^t)$ would not be divided in next iteration. In this case, experts in the conflict expert set $NEG(x_i^t)$ can adjust their opinions towards opinions of the consensus expert set $POS(x_i^t)$, which can be obtained by solving the consensus adjustment model (M-2) directly. In the $(t+1)$th iteration, all experts in $NEG(x_i^t)$ would be divided into the consensus expert set $POS(x_i^t)$. Therefore, the procedure of the sequential TS3WD consensus model terminates in the $(t+1)$th iteration.

Case 4. When $POS(x_i^t)$ is empty and the other two regions are nonempty in the tth iteration, conflict experts in $NEG(x_i^t)$ can not adjust their opinions towards the empty set $POS(x_i^t)$ due to $POS(x_i^t) = \emptyset$. In this case, opinions of consensus experts in the $(t-1)$th iteration are used as the reference opinions, i.e., experts in the conflict expert set $NEG(x_i^t)$ can adjust their opinions towards opinions of the consensus expert set $POS(x_i^{t-1})$. In this case, $\forall e_{o(\varepsilon)} \in POS(x_i^t)$ in the consensus adjustment model (M-2) should be replaced by $\forall e_{o(\varepsilon)} \in POS(x_i^{t-1})$. The uncertainty expert set $BND(x_i^t)$ would be divided into three new regions including $POS(x_i^{t+1})$, $BND(x_i^{t+1})$ and $NEG(x_i^{t+1})$ in the $(t+1)$th iteration.

Case 5. When only $POS(x_i^t)$ is nonempty, all uncertainty experts $e_k \in BND(x_i^{t-1})$ in the $(t-1)$th iteration are all divided into the consensus region $POS(x_i^t)$ in tth iteration. Obviously, all experts reach the consensus in the tth iteration. The procedure of the sequential TS3WD consensus model terminates in the tth iteration.

Case 6. When only $NEG(x_i^t)$ is nonempty, all experts in $NEG(x_i^t)$ should adjust their opinions towards opinions of consensus expert set $POS(x_i^t)$. However, the consensus expert set $POS(x_i^t)$ is empty. Similar to *Case 4*, experts in the conflict expert set $NEG(x_i^t)$ can adjust their opinions towards opinions of the consensus expert set $POS(x_i^{t-1})$. After adjusting opinions for $NEG(x_i^t)$, the procedure of the sequential TS3WD consensus model terminates in the $(t+1)$th iteration.

Case 7. When only $BND(x_i^t)$ is nonempty, all uncertainty experts $e_k \in BND(x_i^{t-1})$ in the last iteration are divided again into the uncertainty region $BND(x_i^t)$ in this iteration, which means that the granularity in the tth iteration is too coarse to further refine the classification results. According to the decision rules (P4)-(N4), the classification results relate to the extended loss functions $\lambda_{\bullet\circ}(\bullet = P, B, N; \circ = P, B, N)$ and two coefficients S_k^i and F_k^i, in which S_k^i and F_k^i depend on the consensus coefficient τ_1 and the conflict coefficient τ_2. Therefore, a coefficient adjustment strategy is proposed to refine the classification results.

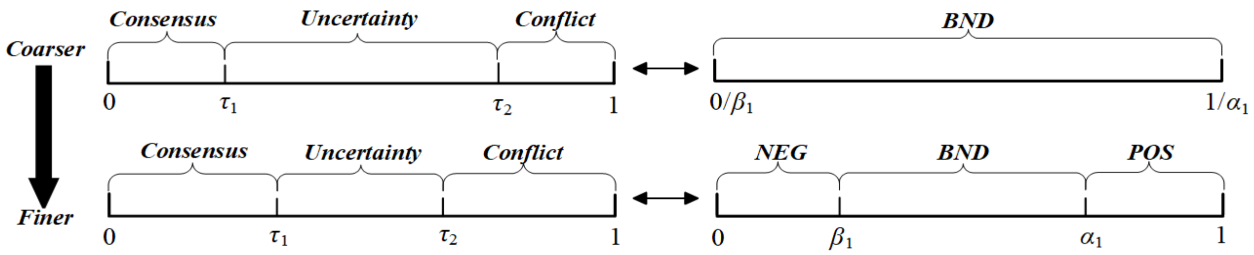

Fig. 1. The classification results under different granularities.

Fig. 1 illustrates that the classification results of both coarse granularity and fine granularity. In the case of coarse granularity, the uncertainty region is more extensive than the other two regions, potentially yielding a classification outcome similar to *Case 7*, i.e., BND is the universal set and the remaining two regions are empty. The classification result of *Case 7* implies that the two thresholds α_1 and β_1 in decision rules (P5)-(N5) are set to 1 and 0, respectively. When increasing τ_1 and decreasing τ_2, the uncertainty region becomes smaller and the granularity is finer, which means that the two regions NEG and POS have larger proportions. Based on the above analysis, the main idea of the coefficient adjustment strategy is to increase τ_1 and decrease τ_2 in the set pair consensus rules. The adjusted coefficients $\overline{\tau}_1$ and $\overline{\tau}_2$ are updated by introducing a penalty factor θ as follows:

$$\begin{aligned} \overline{\tau}_1 &= \tau_1 \times (2-\theta), \\ \overline{\tau}_2 &= \tau_2 \times \theta, \end{aligned} \tag{10}$$

where θ is a penalty factor and $\theta \in (\frac{2\tau_1}{\tau_1+\tau_2}, 1)$. The updated coefficient $\overline{\tau}_1$ is larger than the original coefficient τ_1, and the updated coefficient $\overline{\tau}_2$ is smaller than the original coefficient τ_2 due to $\theta < 1$. The condition $\overline{\tau}_1 < \overline{\tau}_2$ should be satisfied for the two updated coefficients, i.e., $\tau_1 \times (2 - \theta) < \tau_2 \times \theta$. Therefore, the penalty factor θ satisfies $\theta \in (\frac{2\tau_1}{\tau_1+\tau_2}, 1)$.

In summary, $BND(x_i^t)$ would be divided into three new regions in the $(t+1)$th iteration when $BND(x_i^t) \neq \emptyset$ and $BND(x_i^t) \neq BND(x_i^{t-1})$ hold. When $BND(x_i^t) = BND(x_i^{t-1})$, the coefficient adjustment strategy is used to update τ_1 and τ_2, and $BND(x_i^t)$ would be divided into three new regions in the $(t+1)$th iteration after obtaining new coefficients. For $NEG(x_i^t)$, all experts in $NEG(x_i^t)$ should adjust their opinions towards opinions of consensus expert set $POS(x_i^t)$ when $NEG(x_i^t) \neq \emptyset$ and $POS(x_i^t) \neq \emptyset$. When $POS(x_i^t) \neq \emptyset$, all experts in $NEG(x_i^t)$ should adjust their opinions towards opinions of consensus expert set $POS(x_i^{t-1})$. The consensus feedback rules for two regions $BND(x_i^t)$ and $NEG(x_i^t)$ under seven cases are summarized in Table 9.

Table 9
The consensus feedback rules for $BND(x_i^t)$ and $NEG(x_i^t)$.

Cases	The adjustment strategy for $BND(x_i^t)$	The adjustment strategy for $NEG(x_i^t)$
Case 1	Classify again in next iteration	Adjust opinions toward $POS(x_i^t)$
Case 2	Classify again in next iteration	No adjustment
Case 3	END	Adjust opinions toward $POS(x_i^t)$
Case 4	Classify again in next iteration	Adjust opinions toward $POS(x_i^{t-1})$
Case 5	END	END
Case 6	END	Adjust opinions toward $POS(x_i^{t-1})$
Case 7	Update the coefficients and classify again in next iteration	No adjustment

When the procedure of sequential TS3WD consensus model terminates, all experts for each alternative reach the group consensus. The updated evaluation matrix for expert e_k is denoted as $\overline{M}^k = (\overline{m}_{ij}^k)_{m\times n}$. To rank all alternatives, the alternative consensus CN $u_i(E)$ of alternative x_i for all experts is denoted as:

$$u_i(E) = S_i + F_i\delta_1 + P_i\delta_2; i = 1, 2, ..., m. \tag{11}$$

The alternative consensus CN $u_i(E)$ can be obtained as $u_i(E) = f(u_i(e_1), u_i(e_2), ..., u_i(e_K))$. When $f(\cdot)$ is assumed as WA operator, the alternative consensus CN $u_i(E)$ can be rewritten as:

$$u_i(E) = \sum_{k=1}^{K} \lambda_k S_i^k + \sum_{k=1}^{K} \lambda_k F_i^k \delta_1 + \sum_{k=1}^{K} \lambda_k P_i^k \delta_2; i = 1, 2, ..., m, \tag{12}$$

where $\sum_{k=1}^{K} \lambda_k S_i^k$, $\sum_{k=1}^{K} \lambda_k F_i^k$ and $\sum_{k=1}^{K} \lambda_k P_i^k$ reflect the identity degree S_i, discrepancy degree F_i and contrary degree P_i of alternative x_i. The bigger the identity degree S_i is and the smaller the discrepancy degree F_i is, the better the alternative x_i is. The ranking rules for two alternatives x_i and x_j are listed as follow:

(1) $u_i(E) \sim u_j(E)$ when $S_i = S_j$, $F_i = F_j$ and $P_i = P_j$ hold;

(2) $u_i(E) \succ u_j(E)$ when $S_i > S_j$ holds;

(3) $u_i(E) \succ u_j(E)$ when $S_i = S_j$ and $F_i < F_j$ hold;

(4) $u_i(E) \succ u_j(E)$ when $S_i = S_j$, $F_i = F_j$ and $P_i < P_j$ hold.

4.3. The framework of the proposed model

The whole decision-making process is divided into four steps, where the first three steps are used to reach the consensus, and the last step is to obtain the ranking results of alternatives. Steps 1-4 of the proposed model are listed as follows.

Step 1. Construct the consensus set pair probability space. Different types of consensus CNs can be computed based the set pair consensus rules and WA operator by using Eqs. (3) - (7). The probability measure matrix $\mathcal{P} = \left(\left\{Pr_k^i(C), Pr_k^i(\widetilde{C}), Pr_k^i(\overline{C})\right\}\right)_{K\times m}$ is obtained by using Eq. (8).

Step 2. Classify all alternative-expert pairs (x_i, e_k) into three regions. Based on the probability measure matrix $\mathcal{P} = \left(\left\{Pr_k^i(C), Pr_k^i(\widetilde{C}), Pr_k^i(\overline{C})\right\}\right)_{K\times m}$, all alternative-expert pairs (x_i, e_k) can be divided into three regions based on decision rules (P5)-(N5), and we can obtain the three types of experts set $POS(x_i^t)$, $BND(x_i^t)$ and $NEG(x_i^t)$.

Step 3. Adjust the two regions $BND(x_i^t)$ and $NEG(x_i^t)$. The consensus feedback rules in Table 9 is used to adjust two regions $BND(x_i^t)$ and $NEG(x_i^t)$. Conflict experts in the region $NEG(x_i^t)$ can modify their opinions using the model (M-2). The uncertainty expert set $BND(x_i^t)$ is divided into new regions in the next iteration where a coefficient adjustment strategy is adopted to refine the classification results.

Step 4. Rank all alternatives. After reaching the consensus, the alternative consensus CN $u_i(E)$ of alternative x_i can be computed by using Eq. (12). The ranking rules are used to obtain the order of all alternatives.

The multi-criteria sequential TS3WD consensus model based on SPA theory is shown in Algorithm 1. The structure of our proposal is shown in Fig. 2.

5. Experimental analyses

To demonstrate the validity of our algorithm, some numerical experiments are conducted by adopting the Ecoli data set (http://archive.ics.uci.edu/dataset/39/ecoli) and the Iris data set (http://archive.ics.uci.edu/dataset/53/iris) in the UCI database to simulate the expert evaluation systems in our proposal. The sensitive and comparative analyses are presented to demonstrate the feasibility and effectiveness of the proposed model. All experiments are carried out on

Algorithm 1: The multi-criteria sequential TS3WD consensus model.

Input: Alternative set $X = \{x_1, x_2, ..., x_m\}$, criteria set $C = \{c_1, c_2, ..., c_n\}$, criteria weight set $W = \{w_1, w_2, ..., w_n\}$, expert set $E = \{e_1, e_2, ..., e_K\}$, expert weight $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_K\}$, interactive weight matrix $V = (v_{kh})_{K\times K}$, evaluation matrix $M^k = (m_{ij}^k)_{m\times n}$, consensus coefficient τ_1, conflict coefficient τ_2, penalty factor θ, the extended loss functions $\lambda_{\bullet\circ}(\bullet = P, B, N; \circ = P, B, N)$ and the initial iteration $t = 1$.

Output: The updated evaluation matrix $\overline{M}^k = (\overline{m}_{ij}^k)_{m\times n}$, the ranking result of alternative set X and maximum iterations T.

1 **begin**
2   **for** $i = 1$ *to* m, $k = 1$ *to* K **do**
3   **for** $h = 1$ *to* K **do**
4   **for** $j = 1$ *to* n **do**
5   Compute the consensus CN $u_{ij}(e_k, e_h)$ by using Eq. (3).
6   **end**
7   Compute the weighted consensus CN $u_i(e_k, e_h)$ by using Eq. (5).
8   **end**
9   Compute the individual consensus CN $u_i(e_k)$ by using Eq. (7) and obtain the probability measure matrix $\mathcal{P} = \left(\left\{Pr_k^i(C), Pr_k^i(\widetilde{C}), Pr_k^i(\overline{C})\right\}\right)_{K\times m}$ by using Eq. (8).
10   Classify all alternative-expert pairs (x_i, e_k) into three regions based on decision rules (P5)-(N5) and obtain the three types of experts set $POS(x_i^t)$, $BND(x_i^t)$ and $NEG(x_i^t)$.
11   Apply the consensus feedback rules in Table 9 to improve $NEG(x_i^t)$ and $BND(x_i^t)$ and update the evaluation matrix $\overline{M}^k = (\overline{m}_{ij}^k)_{m\times n}$.
12   **if** *'END' appears* **then**
13   **if** $NEG(x_i^t) = \emptyset$ **then**
14         $T = t$;
15   **else**
16         $T = t + 1$;
17   **end**
18   Execute the lines 4-7.
19   Compute the alternative consensus CN $u_i(E)$ by using Eq. (12).
20   Apply the ranking rules and obtain the order of alternatives.
21   **else**
22   Let $t = t + 1$.
23   Return to line 4.
24   **end**
25   **end**
26   **return** the updated evaluation matrix $\overline{M}^k = (\overline{m}_{ij}^k)_{m\times n}$, the ranking result of alternative set X and maximum iterations T.
27 **end**

a PC with Microsoft Windows 10, AMD Ryzen 7 5800HS Creator Edition 3.20 GHz and 16.0 GB of memory. The programming languages used are MATLAB R2020b and LINGO 18.0 x64.

5.1. Example A: Ecoli data set

We apply the Ecoli data set to construct the first example of our proposal. The Ecoli data set consists of 336 escherichia coli protein data, which are described by using seven input criteria including

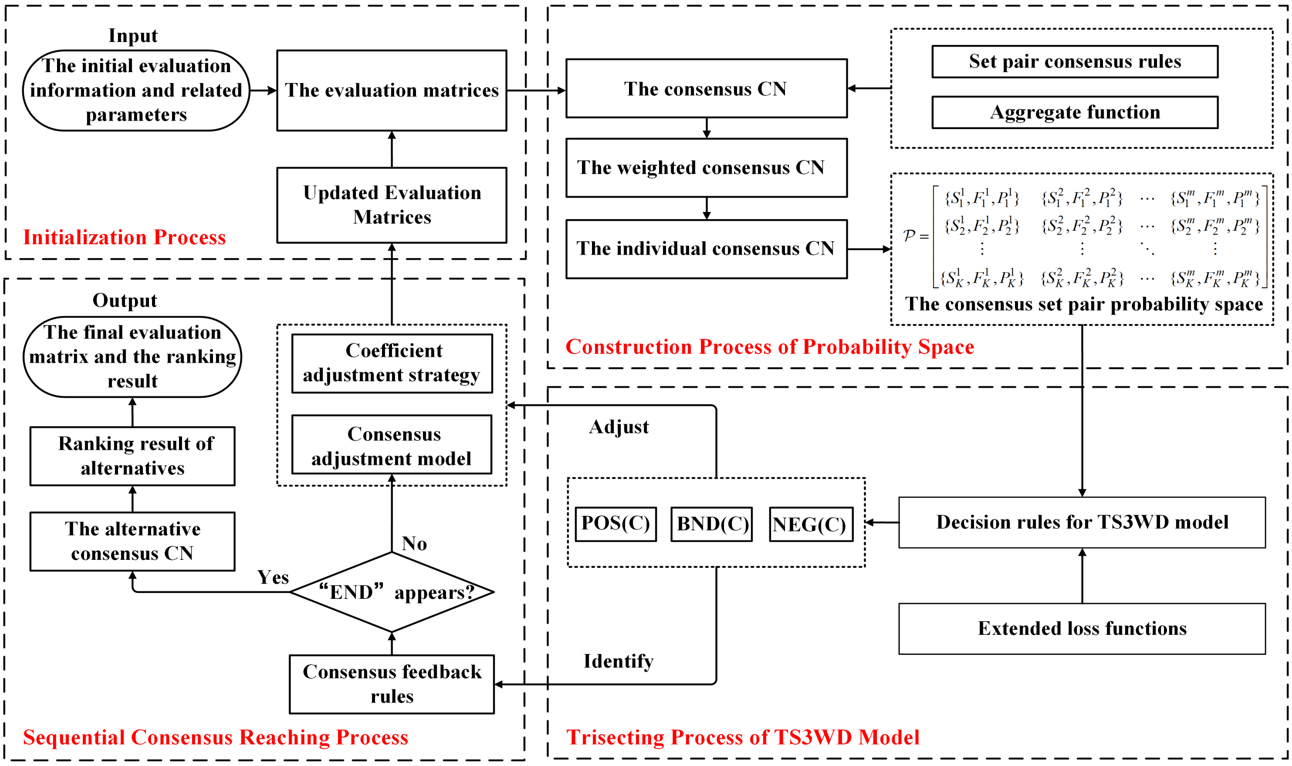

Fig. 2. The flowchart of the proposed model.

'mcg', 'gvh', 'lip', 'chg', 'aac', 'alm1' and 'alm2'. The criteria type of the Ecoli data set is real number within [0,1]. To process the Ecoli data, we eliminate 'lip' and 'chg' criteria due to their little differences between values, and eleven pieces of escherichia coli protein data with extreme values to obtain 325 pieces of data. These 325 pieces of data with five criteria are divided into twenty-five data matrices $M^k = (m_{ij}^k)_{13\times 5}$, corresponding to the initial evaluation matrices of twenty-five experts on thirteen alternatives. Through the processing of the Ecoli data set, there are twenty-five experts $E = \{e_1, e_2, ..., e_{25}\}$ and thirteen alternatives $X = \{x_1, x_2, ..., x_{13}\}$. For the evaluation information of experts, five criteria $C = \{c_1, c_2, ..., c_5\}$ are described by escherichia coli protein data and all criteria weights are assumed to be equal, i.e., $W = \{0.2, 0.2, 0.2, 0.2, 0.2\}$. The weights of all experts are assumed to be equal, i.e., $\lambda = \{0.04, 0.04, ..., 0.04\}$. The elements in the interactive weight matrix $V = (v_{kh})_{25\times 25}$ are generated randomly within [0,1], where $\sum_{h=1}^{25} v_{kh} = 1$. The consensus coefficient and conflict coefficient are assumed as $\tau_1 = 0.1$ and $\tau_2 = 0.3$. The penalty factor is assumed as $\theta = 0.8$. The extended loss functions are shown in Table 10, where the risk attitude is assumed as be neutral when $x \in \widetilde{C}$, i.e., $\lambda_{PB} = \lambda_{NB} = 0.3$. The initial iteration is assumed as $t = 1$. The aggregation function $f(\cdot)$ is assumed as WA operator.

The decision rules (P4)-(N4) can be updated as (P6)-(N6) as follows:

(P6) If $S_k^i \geq 0.6 - 0.4 * F_k^i$ and $S_k^i \geq 0.625 - 0.625 * F_k^i$, then $(x_i, e_k) \in POS(C)$;

(B6) If $S_k^i \leq 0.6 - 0.4 * F_k^i$ and $S_k^i \geq 0.6667 - F_k^i$, then $(x_i, e_k) \in BND(C)$;

(N6) If $S_k^i \leq 0.625 - 0.625 * F_k^i$ and $S_k^i \leq 0.6667 - F_k^i$, then $(x_i, e_k) \in NEG(C)$.

Table 10
The extended loss functions in the illustrative examples.

Actions	$C(P)$	$\widetilde{C}(B)$	$\overline{C}(N)$
a_P	0.1	0.3	0.6
a_B	0.3	0.2	0.3
a_N	0.4	0.3	0.1

To illustrate the process of the algorithm in detail, we describe the steps of the model using alternative x_2 as an example. For alternative x_2, $\{m^k_{21}, m^k_{22}, m^k_{23}, m^k_{24}, m^k_{25}\}$ represent the evaluation values of expert e_k under five criteria including 'mcg', 'gvh', 'aac', 'alm1' and 'alm2'. The evaluation values of twenty-five expert for alternative x_2 are shown in Table 11, where 'Ecoli name' is the index of data in the Ecoli data set and has no other meanings.

Table 11
The evaluation values of twenty-five experts under five criteria for alternative x_2.

e_k	Ecoli name	mcg	gvh	aac	alm1	alm2	e_k	Ecoli name	mcg	gvh	aac	alm1	alm2
e_1	ACEK	0.56	0.4	0.49	0.37	0.46	e_{14}	LNT	0.6	0.61	0.54	0.67	0.71
e_2	CHEA	0.25	0.4	0.46	0.44	0.52	e_{15}	CYOE	0.67	0.37	0.54	0.64	0.68
e_3	KDSA	0.51	0.37	0.35	0.36	0.45	e_{16}	PNTA	0.33	0.37	0.46	0.65	0.69
e_4	MURF	0.35	0.48	0.56	0.4	0.48	e_{17}	PTOA	0.35	0.51	0.61	0.71	0.74
e_5	PHOH	0.29	0.47	0.41	0.23	0.34	e_{18}	EMRB	0.71	0.52	0.64	1	0.99
e_6	DDLA	0.43	0.39	0.47	0.31	0.41	e_{19}	MELB	0.47	0.46	0.62	0.74	0.77
e_7	GCVA	0.4	0.5	0.45	0.39	0.47	e_{20}	FADL	0.78	0.68	0.83	0.4	0.29
e_8	PTKB	0.64	0.76	0.45	0.35	0.38	e_{21}	FECA	0.52	0.81	0.72	0.38	0.38
e_9	SERC	0.49	0.43	0.49	0.3	0.4	e_{22}	MEPA	0.75	0.84	0.35	0.52	0.33
e_{10}	SYK2	0.17	0.39	0.53	0.3	0.39	e_{23}	ECPD	0.64	0.72	0.49	0.42	0.19
e_{11}	UGPQ	0.26	0.4	0.36	0.26	0.37	e_{24}	AMY1	0.7	0.61	0.56	0.52	0.43
e_{12}	BETT	0.52	0.39	0.65	0.71	0.73	e_{25}	RBSB	0.64	0.81	0.37	0.39	0.44
e_{13}	FRDC	0.33	0.56	0.33	0.78	0.8							

Step 1. Construct the consensus set pair probability space. Different types of consensus CNs including $u_{2j}(e_k, e_h)$, $u_2(e_k, e_h)$ and $u_2(e_k)$ can be computed based on the set pair consensus rules and WA operator by using Eqs. (3) - (7). According to Definition 6, the probability measure triples $\left\{Pr^2_k(C), Pr^2_k(\widetilde{C}), Pr^2_k(\overline{C})\right\}$ can be obtained corresponding to the condition probabilities $\left\{Pr_k(C|[x_2]), Pr_k(\widetilde{C}|[x_2]), Pr_k(\overline{C}|[x_2])\right\}$, i.e., $\left\{\{S^2_1, F^2_1, P^2_1\}, \{S^2_2, F^2_2, P^2_2\}, ..., \{S^2_{25}, F^2_{25}, P^2_{25}\}\right\}$. The condition probabilities of twenty-five experts under three states for alternative x_2 are shown in Table 12.

Step 2. Classify all alternative-expert pairs (x_2, e_k) into three regions. Based on the extended loss functions in Table 10 and conditional probabilities in Table 12, we can calculate the expected extended loss functions $R_k(a_\bullet|[x_2])$ of taking three actions for expert e_k under alternative x_2, which are shown in Table 13. According to the decision rules (P3)-(N5), the minimum losses $\min R_k(a_\bullet|[x_2])$ of expert e_k are recognized to classify alternative-expert pairs (x_2, e_k), which are indicated in bold in Table 13. Therefore, the initial classification results in the first iteration for alternative-expert pairs (x_2, e_k) related to x_2 can be obtained, i.e., $\{(x_2, e_1), (x_2, e_2), (x_2, e_4), (x_2, e_6), (x_2, e_7), (x_2, e_8), (x_2, e_9)\} \subset POS^1(C)$; $\{(x_2, e_3), (x_2, e_5), (x_2, e_{10}), (x_2, e_{11}), (x_2, e_{12}), (x_2, e_{13}), (x_2, e_{14}), (x_2, e_{15}), (x_2, e_{16}), (x_2, e_{17}), (x_2, e_{19}), (x_2, e_{20}), (x_2, e_{21}), (x_2, e_{22}), (x_2, e_{23}), (x_2, e_{24}), (x_2, e_{25})\} \subset BND^1(C)$ and $\{(x_2, e_{18})\} \subset NEG^1(C)$. Then three types of expert set $POS(x_2^1)$, $BND(x_2^1)$ and $NEG(x_i^1)$ for alternative x_2 can be obtained as follows.

(1) The consensus expert set: $POS(x_2^1)$={e_1, e_2, e_4, e_6, e_7, e_8, e_9};

(2) The uncertainty expert set: $BND(x_2^1)$={e_3, e_5, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{19}, e_{20}, e_{21}, e_{22}, e_{23}, e_{24}, e_{25}};

(3) The conflict expert set: $NEG(x_2^1)$={e_{18}}.

Table 12
The conditional probabilities of experts under three states for alternative x_2.

e_k	$Pr_k(C\|[x_2])$	$Pr_k(\widetilde{C}\|[x_2])$	$Pr_k(\overline{C}\|[x_2])$	e_k	$Pr_k(C\|[x_2])$	$Pr_k(\widetilde{C}\|[x_2])$	$Pr_k(\overline{C}\|[x_2])$
e_1	0.5490	0.3347	0.1163	e_{14}	0.3538	0.4769	0.1693
e_2	0.4657	0.3942	0.1401	e_{15}	0.3562	0.3890	0.2548
e_3	0.4025	0.4371	0.1604	e_{16}	0.3706	0.4088	0.2206
e_4	0.4602	0.4255	0.1143	e_{17}	0.3041	0.3731	0.3228
e_5	0.4270	0.3296	0.2434	e_{18}	0.2611	0.3380	0.4010
e_6	0.5574	0.2726	0.1700	e_{19}	0.3477	0.3846	0.2677
e_7	0.4722	0.3712	0.1565	e_{20}	0.2463	0.4787	0.2750
e_8	0.5044	0.2597	0.2358	e_{21}	0.3694	0.3777	0.2529
e_9	0.5284	0.2863	0.1853	e_{22}	0.2164	0.4507	0.3330
e_{10}	0.4348	0.2933	0.2719	e_{23}	0.3081	0.4770	0.2149
e_{11}	0.4574	0.3020	0.2407	e_{24}	0.3896	0.5127	0.0977
e_{12}	0.3614	0.4099	0.2288	e_{25}	0.4751	0.2770	0.2480
e_{13}	0.2879	0.3893	0.3228				

Step 3. Adjust the two regions $BND(x_2^1)$ and $NEG(x_2^1)$. Obviously, three types of expert set $POS(x_2^1)$, $BND(x_2^1)$ and $NEG(x_i^1)$ are nonempty, corresponding to the *Case 1* in Table 8. Therefore, we

use the consensus feedback rules for *Case 1* in Table 9 to adjust two regions $BND(x_2^1)$ and $NEG(x_2^1)$. Expert e_{18} in $NEG(x_2^1)$ should modify the opinion towards opinions of experts in $POS(x_2^1)$ where the interactive weights v_{kh} assigned by expert x_{18} are shown in Table 14. Since the consensus expert set $POS(x_2^1)$ contains seven experts, we have $\varepsilon_2^1 = 7$, $\varepsilon = 1, 2, ..., 7$ and the index of consensus expert $o(\varepsilon) = 1, 2, 4, 6, 7, 8, 9$ corresponding to $\{e_1, e_2, e_4, e_6, e_7, e_8, e_9\}$.

$$\min \sum_{j=1}^{5} w_j (m_{2j}^{18} - \bar{m}_{2j}^{18})^2$$
$$s.t. \begin{cases} \left| m_{2j}^{o(\varepsilon)} - \bar{m}_{2j}^{18} \right| \leq 0.1 + (1 - S_{2j}^{18o(\varepsilon)})M, \\ \left| m_{2j}^{o(\varepsilon)} - \bar{m}_{2j}^{18} \right| \geq 0.3 - (1 - F_{2j}^{18o(\varepsilon)})M, \\ \left| m_{2j}^{o(\varepsilon)} - \bar{m}_{2j}^{18} \right| < 0.3 + (1 - P_{2j}^{18o(\varepsilon)})M, \\ \left| m_{2j}^{o(\varepsilon)} - \bar{m}_{2j}^{18} \right| > 0.1 - (1 - P_{2j}^{18o(\varepsilon)})M, \\ S_{2j}^{18o(\varepsilon)} + F_{2j}^{18o(\varepsilon)} + P_{2j}^{18o(\varepsilon)} = 1, \\ \bar{v}_{18o(\varepsilon)} = \frac{v_{18o(\varepsilon)}}{\sum_{\varepsilon=1}^{7} v_{18o(\varepsilon)}}, \\ S_2^{18} = \sum_{\varepsilon=1}^{7} \bar{v}_{18o(\varepsilon)} \sum_{j=1}^{5} w_j S_{2j}^{18o(\varepsilon)}, \\ F_2^{18} = \sum_{\varepsilon=1}^{7} \bar{v}_{18o(\varepsilon)} \sum_{j=1}^{5} w_j F_{2j}^{18o(\varepsilon)}, \\ 0.6 - 0.4 * F_2^{18} \leq S_2^{18}, \\ 0.625 - 0.625 * F_2^{18} \leq S_2^{18}, \\ S_{2j}^{18o(\varepsilon)}, F_{2j}^{18o(\varepsilon)}, P_{2j}^{18o(\varepsilon)} \in \{0, 1\}, \\ j = 1, 2, ..., 5; \varepsilon = 1, 2, ..., 7, \\ o(\varepsilon) = 1, 2, 4, 6, 7, 8, 9. \end{cases} \tag{M-3}$$

By solving model (M-3), the modified evaluation values $\{\overline{m}_{21}^{18}, \overline{m}_{22}^{18}, \overline{m}_{23}^{18}, \overline{m}_{24}^{18}, \overline{m}_{25}^{18}\}$ of expert e_{18} under five criteria are shown in Table 15, where the modified values $\overline{m}_{2j}^{18} \neq m_{2j}^{18} (j = 1, 2, ..., 5)$ are indicated in bold.

The uncertainty expert set $BND(x_2^1)$ is divided into new regions in the next iteration where a coefficient adjustment strategy is used to refine the classification results. Meanwhile, the iteration is updated as $t = 1 + 1 = 2$. The whole decision process of x_2 is shown in Fig. 3. Due to space limitation, the decision process for other twenty-four alternatives are omitted. When “END” appears in some cases like *Cases 3*, *5* and *6* in Table 9, the procedure of the sequential TS3WD consensus model stops and the final updated evaluation matrix $\overline{M}^k = (\overline{m}_{ij}^k)_{13\times5}$ can be obtained. The opinion adjustments of twenty-five experts under five criteria for thirteen alternatives can be deduced by computing the differences between the modified opinion $\overline{m}_{ij}^k$ and the initial opinion m_{ij}^k, which are shown in Fig. 4.

Fig. 4 reflects that all opinion adjustments of twenty-five experts under thirteen alternatives. Twenty-five colors in ‘Jet’ colormap array are used to distinguish different experts and each kind

Table 13
The expected losses of experts taking three actions for alternative x_2.

e_k	$R_k(a_P\|[x_2])$	$R_k(a_B\|[x_2])$	$R_k(a_N\|[x_2])$	e_k	$R_k(a_P\|[x_2])$	$R_k(a_B\|[x_2])$	$R_k(a_N\|[x_2])$
e_1	**0.2251**	0.2665	0.3316	e_{14}	0.2800	**0.2523**	0.3015
e_2	**0.2489**	0.2606	0.3185	e_{15}	0.3052	**0.2611**	0.2847
e_3	0.2676	**0.2563**	0.3082	e_{16}	0.2920	**0.2591**	0.2930
e_4	**0.2423**	0.2575	0.3232	e_{17}	0.3360	**0.2627**	0.2659
e_5	0.2876	**0.2670**	0.2940	e_{18}	0.3681	0.2662	**0.2459**
e_6	**0.2395**	0.2727	0.3217	e_{19}	0.3108	**0.2615**	0.2812
e_7	**0.2525**	0.2629	0.3159	e_{20}	0.3332	**0.2521**	0.2696
e_8	**0.2699**	0.2740	0.3033	e_{21}	0.3020	**0.2622**	0.2864
e_9	**0.2499**	0.2714	0.3158	e_{22}	0.3566	**0.2549**	0.2550
e_{10}	0.2946	**0.2707**	0.2891	e_{23}	0.3028	**0.2523**	0.2878
e_{11}	0.2807	**0.2698**	0.2976	e_{24}	0.2514	**0.2487**	0.3194
e_{12}	0.2964	**0.2590**	0.2904	e_{25}	0.2794	**0.2723**	0.2979
e_{13}	0.3393	**0.2611**	0.2642				

Table 14
The interactive weights v_{kh} assigned by expert x_{18}.

e_h	v_{18h}	e_h	v_{18h}	e_h	v_{18h}	e_h	v_{18h}	e_h	v_{18h}
e_1	0.0471	e_6	0.0151	e_{11}	0.0060	e_{16}	0.0037	e_{21}	0.0144
e_2	0.0474	e_7	0.0163	e_{12}	0.0103	e_{17}	0.0426	e_{22}	0.0649
e_3	0.0143	e_8	0.0847	e_{13}	0.0675	e_{18}	0.0119	e_{23}	0.0250
e_4	0.0363	e_9	0.0549	e_{14}	0.0886	e_{19}	0.0485	e_{24}	0.0763
e_5	0.0885	e_{10}	0.0141	e_{15}	0.0367	e_{20}	0.0145	e_{25}	0.0702

of colored square indicates that the opinion adjustments of an expert. For convenience, the legend of twenty-five colors in 'Jet' colormap array is omitted and all colored squares are labeled by e_k to distinguish different experts. The red dashed line indicates the 0-scale line. Colored squares above it represent that experts have modified opinions in the positive direction, while those below it represent opinions modified in the negative direction. It can be seen that from Fig. 4 that opinions under each criterion have been modified by some experts for thirteen alternatives, which demonstrates the validity of the model (M-3). For alternatives x_3, x_6, x_{10} and x_{13}, more experts have modified their opinions in the positive direction. For the remaining alternatives, more experts have modified their opinions in the negative direction.

Step 4. Rank all alternatives. For the obtained updated evaluation matrix $\overline{M}^k = (\overline{m}_{ij}^k)_{13\times5}$, $u_i(e_k)$

Table 15
The modified evaluation values of twenty-five experts under five criteria for alternative x_2.

e_k	Ecoli name	$\overline{mcg}$	$\overline{gvh}$	$\overline{aac}$	$\overline{alm1}$	$\overline{alm2}$	e_k	Ecoli name	$\overline{mcg}$	$\overline{gvh}$	$\overline{aac}$	$\overline{alm1}$	$\overline{alm2}$
e_1	ACEK	0.56	0.4	0.49	0.37	0.46	e_{14}	LNT	0.6	**0.55**	0.54	0.67	0.71
e_2	CHEA	0.25	0.4	0.46	0.44	0.52	e_{15}	CYOE	0.67	**0.38**	0.54	0.64	**0.7**
e_3	KDSA	0.51	0.37	0.35	0.36	0.45	e_{16}	PNTA	0.33	0.37	0.46	0.65	**0.7**
e_4	MURF	0.35	0.48	0.56	0.4	0.48	e_{17}	PTOA	0.35	**0.49**	**0.55**	0.71	**0.76**
e_5	PHOH	0.29	0.47	0.41	0.23	0.34	e_{18}	EMRB	**0.59**	**0.5**	**0.55**	1	**0.56**
e_6	DDLA	0.43	0.39	0.47	0.31	0.41	e_{19}	MELB	0.47	0.46	0.62	0.74	0.77
e_7	GCVA	0.4	0.5	0.45	0.39	0.47	e_{20}	FADL	**0.76**	0.68	0.83	0.4	**0.36**
e_8	PTKB	0.64	0.76	0.45	0.35	0.38	e_{21}	FECA	0.52	0.81	0.72	0.38	0.38
e_9	SERC	0.49	0.43	0.49	0.3	0.4	e_{22}	MEPA	0.75	0.84	**0.37**	**0.49**	**0.34**
e_{10}	SYK2	0.17	0.39	0.53	0.3	0.39	e_{23}	ECPD	0.64	0.72	0.49	0.42	0.19
e_{11}	UGPQ	0.26	0.4	0.36	0.26	0.37	e_{24}	AMY1	0.7	0.61	0.56	0.52	0.43
e_{12}	BETT	0.52	0.39	**0.59**	0.71	0.73	e_{25}	RBSB	0.64	0.81	0.37	0.39	0.44
e_{13}	FRDC	**0.44**	**0.52**	**0.44**	0.78	0.8							

can be calculated again using Eqs. (3)-(7). The alternative consensus CN $u_i(E) = S_i + F_i\delta_1 + P_i\delta_2$ of alternative x_i can be computed by using Eq. (12). Table 16 reflects the identity degree S_i, the discrepancy degree F_i and the contrary degree P_i for thirteen alternatives. The ranking rules in Section 4.2 are used to derive the priority of all alternatives. For example, we can obtain $x_2 \succ x_1$ due to $S_2 > S_1$. The ranking result of thirteen alternatives is $x_7 \succ x_{10} \succ x_{12} \succ x_2 \succ x_3 \succ x_5 \succ x_8 \succ x_{11} \succ x_1 \succ x_4 \succ x_6 \succ x_{13} \succ x_9$.

Table 16
The expected losses of experts taking three actions for alternative x_2.

x_i	S_i	F_i	P_i	x_i	S_i	F_i	P_i
x_1	0.4081	0.3846	0.2073	x_8	0.4164	0.3861	0.1975
x_2	0.4320	0.3586	0.2094	x_9	0.3735	0.3793	0.2472
x_3	0.4231	0.3849	0.1920	x_{10}	0.4777	0.2936	0.2287
x_4	0.3967	0.4079	0.1954	x_{11}	0.4097	0.3772	0.2131
x_5	0.4187	0.4061	0.1752	x_{12}	0.4639	0.3224	0.2137
x_6	0.3938	0.3735	0.2327	x_{13}	0.3810	0.4109	0.2082
x_7	0.4903	0.2832	0.2265				

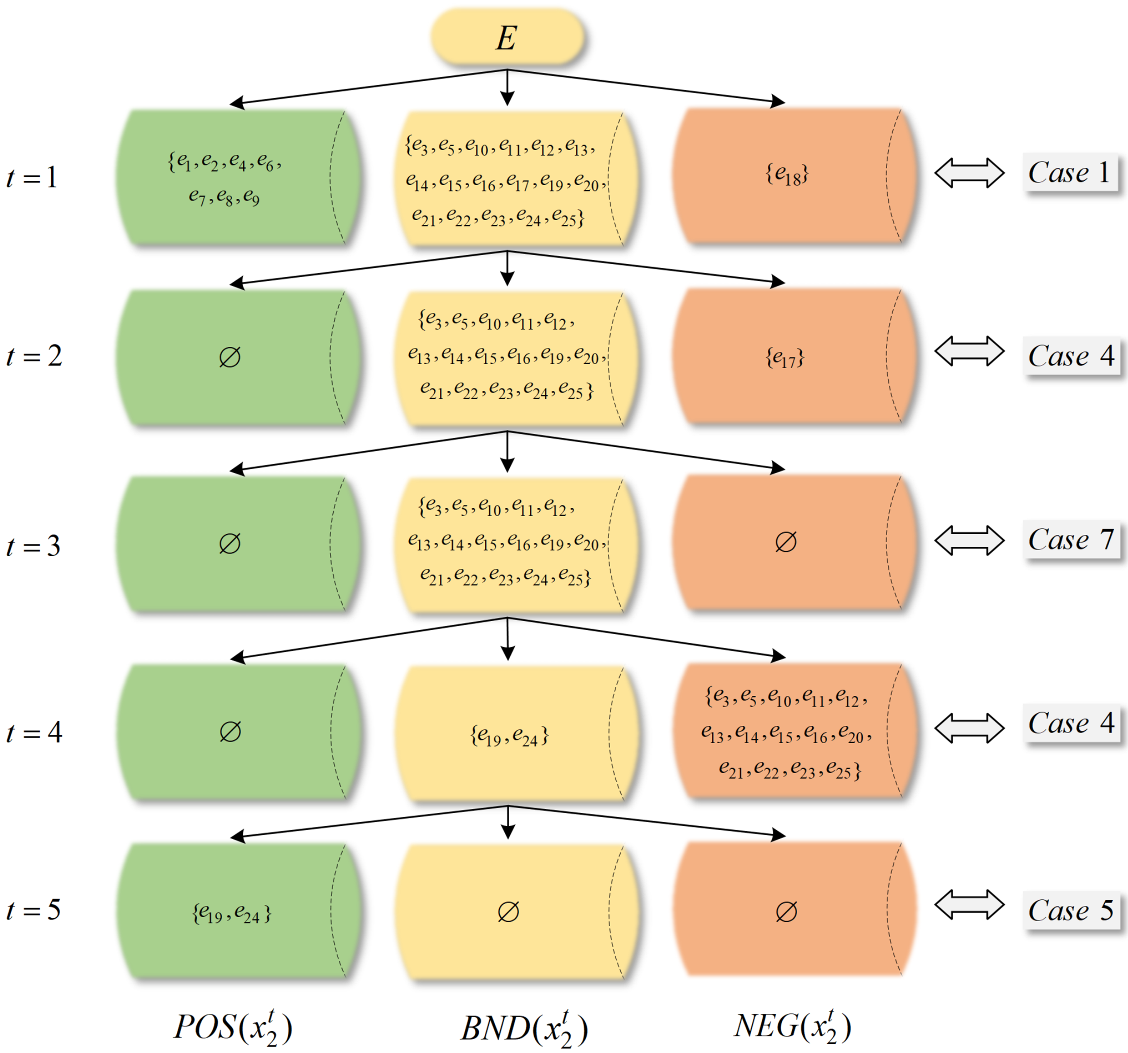

Fig. 3. The whole decision process of alternative x_2.

5.2. Example B: Iris data set

The Iris data set is used to construct the second example of our proposal. The Iris data set consists of 150 plant data, which are described by using four input criteria including ‘sepal length’, ‘sepal width’, ‘petal length’ and ‘petal width’. All values in the Iris data set are normalized as the real numbers within [0,1] for the convenience of calculation. The 150 pieces of data with four criteria are divided into fifteen data matrices $M^k = (m_{ij}^k)_{10\times4}$ ($k = 1, 2, ..., 15$), corresponding to the initial evaluation matrices of fifteen experts on ten alternatives. Through the processing of the data, there are fifteen experts $E = \{e_1, e_2, ..., e_{15}\}$ and ten alternatives $X = \{x_1, x_2, ..., x_{10}\}$. For the evaluation information of experts, five criteria $C = \{c_1, c_2, c_3, c_4\}$ are described as plant data and all criteria weights are assumed to be equal, i.e., $W = \{0.25, 0.25, 0.25, 0.25\}$. The weights of all experts are assumed to be equal and the interactive weight matrix $V = (v_{kh})_{15\times15}$ is generated randomly within [0,1], where $\sum_{h=1}^{15} v_{kh} = 1$. The consensus coefficient and conflict coefficient are assumed as $\tau_1 = 0.1$ and $\tau_2 = 0.3$. The penalty factor is assumed as $\theta = 0.8$. The extended loss functions are shown in Table 10, where the risk attitude is assumed as to be neutral when $x \in \widetilde{C}$, i.e., $\lambda_{PB} = \lambda_{NB} = 0.3$. The initial iteration is assumed as $t = 1$.

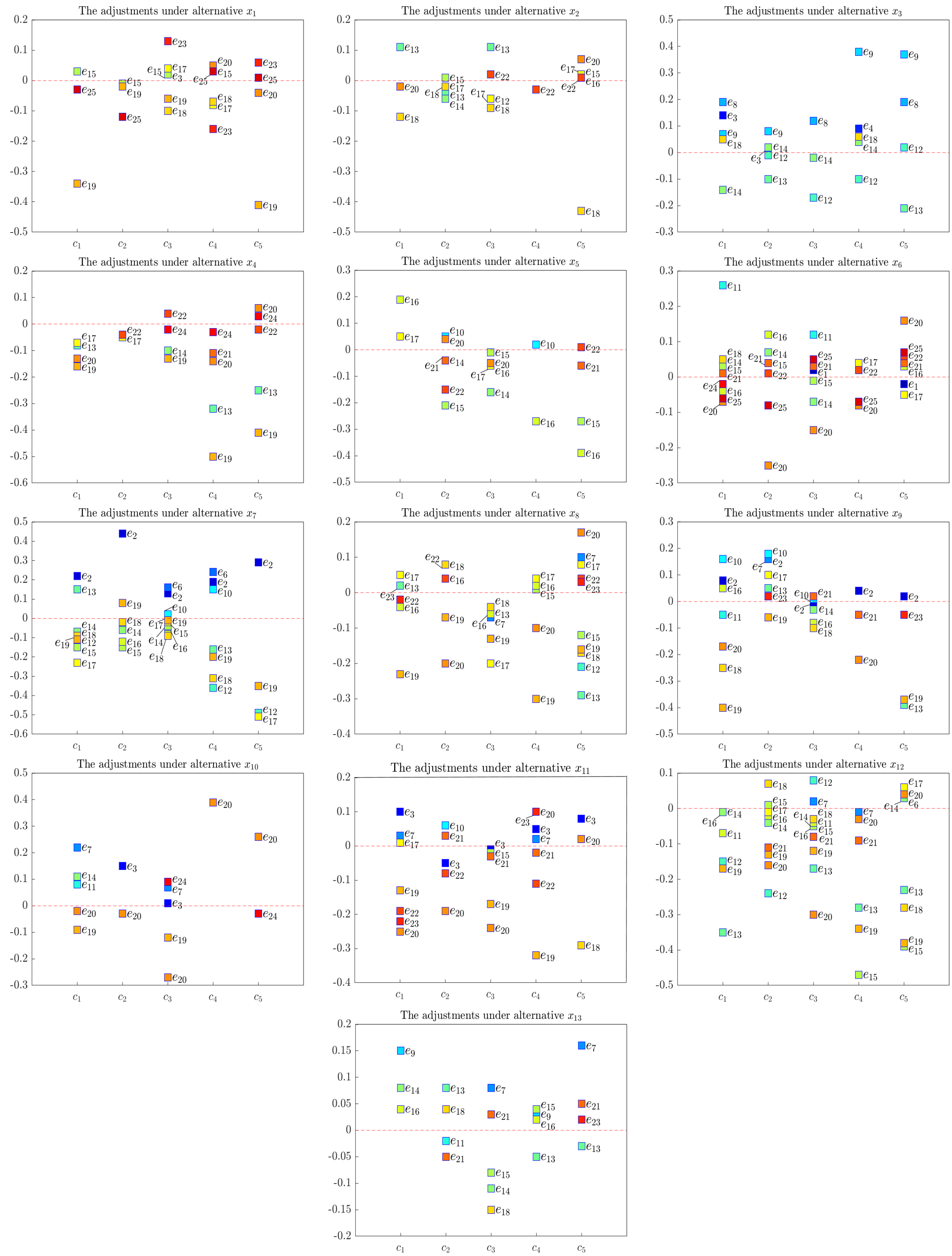

Fig. 4. The opinion adjustments in Ecoli data set.

The aggregation function $f(\cdot)$ is assumed as WA operator.

Due to space limitations, the decision steps of the Iris data set are omitted, which are similar to those of the Ecoli data set in Section 5.1. The opinion adjustments of fifteen experts under four criteria for ten alternatives are shown in Fig. 5. From Fig. 5, it can be observed that fewer experts adjust their

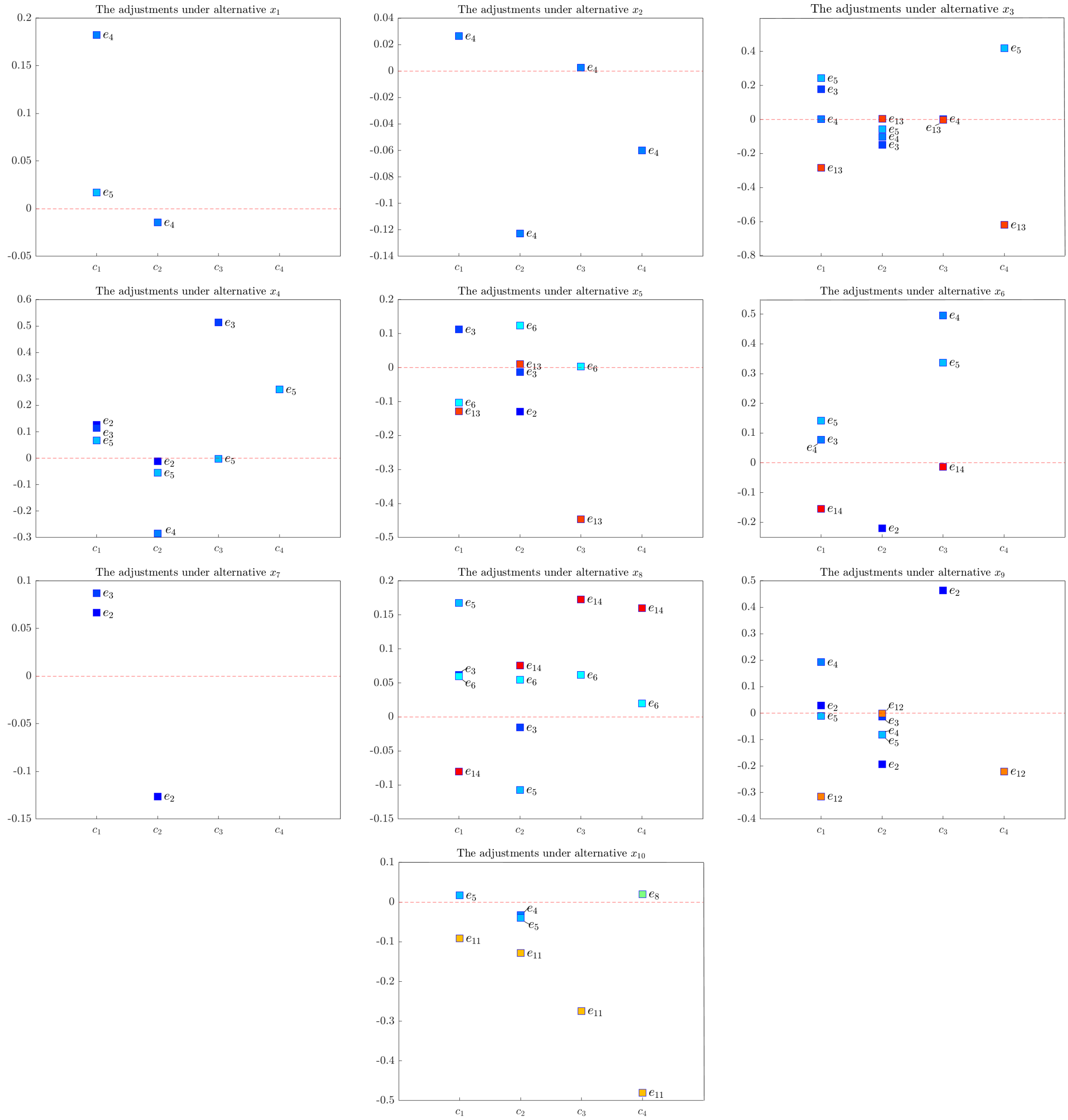

Fig. 5. The opinion adjustments in Iris data set.

opinions for alternatives x_1, x_2 and x_7, while more experts modify opinions for other alternatives. For alternatives x_1 and x_7, all opinions under criteria c_3 and c_4 remain unchanged. For alternatives x_2, x_4, x_8 and x_{10}, some experts adjust their opinions under each criterion. The above observations can reflect the flexibility and inclusiveness of our proposal. Based on these opinion adjustments, the ranking result of thirteen alternatives in Iris data set is $x_5 \succ x_6 \succ x_7 \succ x_{10} \succ x_2 \succ x_1 \succ x_4 \succ x_9 \succ x_3 \succ x_8$.

5.3. Sensitivity analysis

In the set pair consensus rules, the consensus coefficient τ_1 and the conflict coefficient τ_2 are two essential parameters for the group consensus process. In this subsection, we conduct the sensitive analysis of consensus coefficient τ_1 and the conflict coefficient τ_2 to explore their effect on the classification results.

For the above two examples in subsections 5.1 and 5.2, the consensus coefficient τ_1 and the conflict coefficient τ_2 are assumed as $\tau_1 = 0.1$ and $\tau_2 = 0.3$. Based on the given examples, we discuss the initial classification results $POS(x_i^1)$, $BND(x_i^1)$ and $NEG(x_i^1)$ with different τ_1 when $\tau_2 = 0.3$ and those with different τ_2 when $\tau_1 = 0.1$, where $1 \geq \tau_2 > \tau_1 \geq 0$ holds. For $\tau_2 = 0.3$, we firstly conduct the classification process when $\tau_1 \in [0, 0.3]$ and obtain the numbers of three regions $POS(x_i^1)$, $BND(x_i^1)$ and $NEG(x_i^1)$ for thirteen alternatives in Ecoli and Iris data sets, which are shown in Figs. 6 and 7. For $\tau_1 = 0.1$, we also conduct the classification process when $\tau_2 \in [0.15, 0.5]$ and obtain the numbers of three regions $POS(x_i^1)$, $BND(x_i^1)$ and $NEG(x_i^1)$ for thirteen alternatives in Ecoli and Iris data sets, which are shown in Figs. 8 and 9. We adjust the perspective of the 3D figures for viewing in Figs. 6-9.

From Figs. 6 and 7, the number of $POS(x_i^1)$ increases and the number of $BND(x_i^1)$ decreases as the consensus coefficient τ_1 increases. The larger τ_1 loosens the consensus condition, so more experts are assigned to the consensus set $POS(x_i^1)$, and experts who were previously in the conflict or uncertainty state under strict consensus condition may be regarded in the consensus state under the relaxed consensus condition. The number of $NEG(x_i^1)$ firstly flats and then decreases with the increase of τ_1. This reflects that the increase rate of $POS(x_i^1)$ would outpace the decrease rate of $BND(x_i^1)$ as the consensus coefficient τ_1 increases, so that the number of $NEG(x_i^1)$ gradually gets smaller later.

From Figs. 8 and 9, the number of $POS(x_i^1)$ increases and the number of $NEG(x_i^1)$ decreases as the conflict coefficient τ_2 increases. The larger τ_2 tightens the conflict condition, so more experts are assigned to the conflict set $NEG(x_i^1)$, and experts who were previously in the conflict or uncertainty state under relaxed conflict condition may be regarded in the consensus state under the strict consensus condition. The number of $BND(x_i^1)$ firstly increases and then decreases with the increase of τ_2. This reflects that the increase rate of $POS(x_i^1)$ is firstly inferior to and then exceeds the decrease rate of $NEG(x_i^1)$ as the conflict coefficient τ_2 increases, so that the number of $BND(x_i^1)$ firstly increases and then decreases.

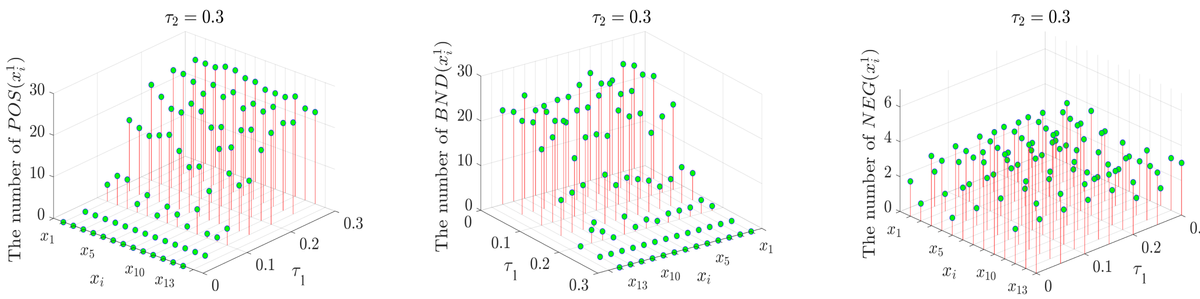

Fig. 6. The classification results with different τ_1 when $\tau_2 = 0.3$ in Ecoli dataset.

Based on the above analyses, the consensus coefficient τ_1 and conflict coefficient τ_2 have the significant effect on the classification results. Therefore, determining appropriate consensus and

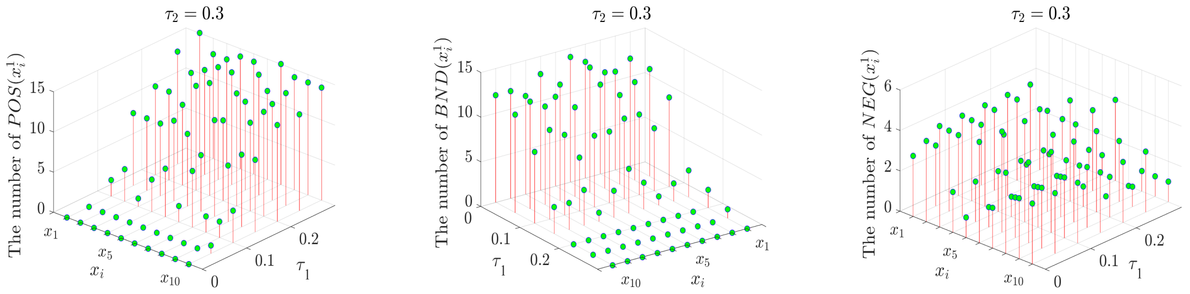

Fig. 7. The classification results with different τ_1 when $\tau_2 = 0.3$ in Iris dataset.

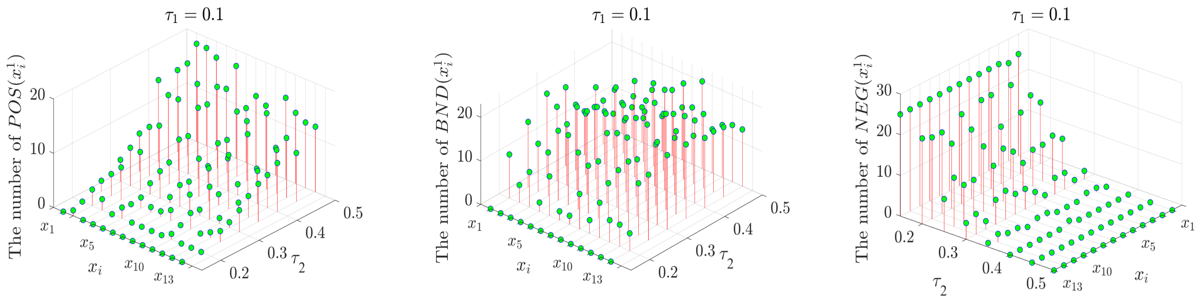

Fig. 8. The classification results with different τ_2 when $\tau_1 = 0.1$ in Ecoli dataset.

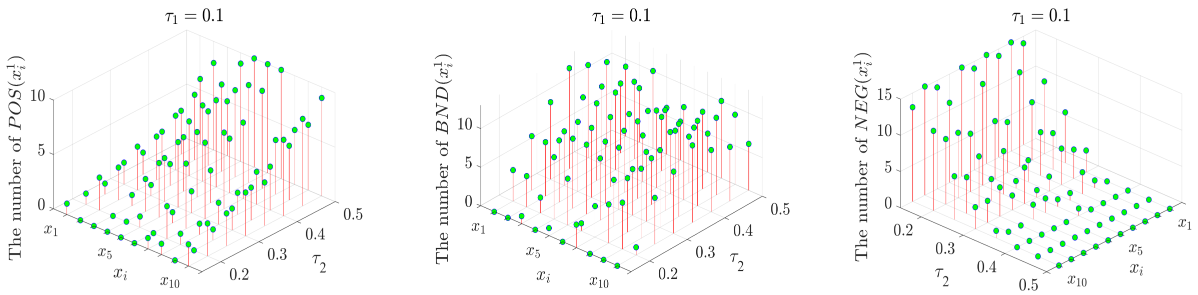

Fig. 9. The classification results with different τ_2 when $\tau_1 = 0.1$ in Iris dataset.

conflict coefficients is crucial for practical problems. Several suggestions to determine the consensus and conflict coefficients are listed: (1) Given a fixed conflict coefficient τ_2, select a larger consensus coefficient τ_1 if the decision-maker aims to obtain coarse-grained optimal solutions or quickly reach the consensus. Otherwise, select a smaller consensus coefficient τ_1. (2) Given a fixed consensus coefficient τ_1, select a larger conflict coefficient τ_2 when most experts are not willing to cooperate or have high self-confidence degrees. Otherwise, select a smaller conflict coefficient τ_2. (3) Select a larger consensus coefficient τ_1 and a larger conflict coefficient τ_2 when most experts are not willing to cooperate and the decision-maker aims to quickly reach the consensus. (4) Select a smaller consensus coefficient τ_1 and a smaller conflict coefficient τ_2 when most experts are willing to cooperate and the decision-maker aims to obtain fine-grained optimal solutions. (5) Select a larger consensus coefficient τ_1 and a smaller conflict coefficient τ_2 when most experts are willing to cooperate and the decision-maker aims to quickly reach the consensus. (6) Select a smaller consensus coefficient τ_1 and a larger conflict coefficient τ_2 when most experts are not willing to cooperate and the decision-maker aims to obtain fine-grained optimal solutions.

5.4. Comparative analysis and discussion

To demonstrate the advantages of our proposal, we compare it with the existing methods from the performance aspect and the theoretical aspect. From the perspective of performance, three 3WD methods are compared with our proposal, including the classic 3WD method, S3WD-HK method and Wang et al.'s [26] method. From the theoretical perspective, six CRP methods including Wang et al.'s [36] method, Liang and Duan's [28] method, Wang et al.'s [26] method, Liu et al.'s [45] method, Han and Zhan's [37] method and Guo et al.'s [27] method are analyzed with our proposal from five dimensions.

Three criteria are introduced to compare the performance of four methods, i.e., iterations T, average deferment rate ADR and average comprehensive score ACS. The average deferment rate ADR is used to measure the average deferment rate for T iterations, which can be calculated as:

$$ADR = \frac{1}{T}\sum_{t=1}^{T}\frac{\left|BND^t(C)\right|}{m\times K}, \tag{13}$$

where m and K reflect the number of alternatives and the number of experts, respectively. $\left|BND^t(C)\right|$ is the number of the region $BND^t(C)$ in the tth iteration and T is the maximum iterations. To obtain the average comprehensive score ACS, the accuracy rate AR^t in the tth iteration is calculated as follows:

$$AR^t = \frac{\left|POS^t(C) \to POS^t(C)\right| + \left|NEG^t(C) \to NEG^t(C)\right|}{|POS^t(C)| + |NEG^t(C)|}, \tag{14}$$

where $|\bullet \to *|$ denotes the number of the alternative-expert pairs which are actually in the $\bullet$ region but are judged to be in the $*$ region in the tth iteration, $\left|POS^t(C)\right|$ and $\left|NEG^t(C)\right|$ are the number of the region $POS^t(C)$ and the number of the region $NEG^t(C)$ in the tth iteration, respectively. Furthermore, the average comprehensive score ASC is denoted as:

$$ASC = \frac{1}{T}\sum_{t=1}^{T}\frac{2\times AR^t\times(1-\frac{\left|BND^t(C)\right|}{m\times K})}{AR^t+(1-\frac{|BND^t(C)|}{m\times K})}. \tag{15}$$

For convenience, we apply the three existing 3WD methods and our proposal into the Breast Cancer data set (`http://archive.ics.uci.edu/dataset/14/breast+cancer`) in the UCI database, which has been used in Wang et al.'s [26] method. By calculating the above three criteria, the performance comparisons of these four methods are shown in Table 17. Table 17 indicates that our proposal requires fewer iterations compared to the other three 3WD methods, demonstrating the ability to quickly reach the consensus. Additionally, our proposal exhibits a smaller average deferment rate and a larger average comprehensive score than the other methods, highlighting its effectiveness and comprehensiveness in addressing decision-making problems.

Table 17
Performance comparisons of our proposal and the existing 3WD methods.

Methods	Iterations	Average deferment rate	Average comprehensive score
Classic 3WD [13] method	–	0.8558	0.7853
S3WD-HK method	5	0.1547	0 .8928
Wang et al.'s [26] method	5	0.1473	0.8952
Our proposal	3	0.1450	0.8971

In theory, we compare our proposal with six based-3WD CRP methods and summarize the main advantages of our proposal, as shown in Table 18. From Table 18, it can be seen that our proposal not only deals with the CRP in MCDM problems based on S3WD theory, but also extends the classic 3WD theory from two states to three states. In addition, most existing methods use the 3WD theory to classify alternatives, criteria or experts. The classification objects of our proposal are all pairs including experts and alternatives, which can recognize more precisely the positions of non-consensus experts and alternatives than other methods. Furthermore, compared to the other methods using consensus degree or competition degree to measure the group consensus, we integrate 3WD theory into the CRP and use the classification results to judge the group consensus.

Table 18
Theoretical comparisons of our proposal and the existing methods.

Methods	MCDM	S3WD	The classification objects	The number of states	Consensus judgement
Wang et al.'s [36] method	✓	✓	Alternatives	2	Consensus degree
Liang and Duan's [28] method	✓	×	Experts	2	Consensus level
Wang et al.'s [26] method	✓	×	Alternatives	2	Consensus level
Liu et al.'s [45] method	×	×	Alternatives	2	Consensus index
Han and Zhan's [37] method	✓	✓	Criteria	2	Consensus level
Guo et al.'s [27] method	×	×	Alternatives	2	Competition degree
Our proposal	✓	✓	Experts and alternatives	3	Positive region

6. Conclusions

In this paper, a multi-criteria sequential TS3WD consensus model based on SPA theory has been proposed. The feasibility and effectiveness of the proposal have been verified by sensitivity and comparative analyses with existing methods. The main contributions of this paper are listed as follows.

(1) Based on the set pair consensus rules, we have proposed and constructed the consensus set pair probability space, which can effectively distinguish three types of experts with different levels of consensus.

(2) Based on SPA theory, we have presented the TS3WD model and derived the decision rules and simplified decision rules by constructing the extended loss functions, which extends the classic two-state 3WD theory. The TS3WD model can be reduced to the classic two-state 3WD model, which proves the rationality of the proposal.

(3) The consensus feedback mechanism based on the sequential TS3WD consensus model has been designed to indicate the CRP, where two non-consensus experts can adjust opinions by using the consensus adjustment model and the coefficient adjustment strategy. The consensus feedback rules for different classification results are presented, which provides effective solutions under seven cases.

For the future research, the adaptive multi-criteria sequential TS3WD consensus model under different linguistic expression environments is worth studying. Firstly, normalization of different linguistic expressions is emerging as a research topic. Secondly, how to learn the consensus and conflict coefficients from the historical data to reach the group consensus effectively is also desired for further exploring. Finally, it's also meaningful to design the adaptive consensus adjustment rules.

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4 Energy internet project evaluation in circular economy practices: A novel multi-criteria decision-making framework with flexible linguistic expressions based on multi-granularity cloud-rough set

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Energy internet project evaluation in circular economy practices: A novel multi-criteria decision-making framework with flexible linguistic expressions based on multi-granularity cloud-rough set

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Abstract

Energy internet (EI) serves as a practical platform for implementing the principles of the circular economy. As the process of realizing the transformation of industrial systems to a sustainable circular economy accelerates, the importance of evaluating and selecting the most effective EI projects becomes increasingly apparent. This paper proposes a new multi-criteria decision-making (MCDM) framework based on the multi-granularity cloud-rough set (MGCRS) and flexible linguistic expressions (FLEs) to evaluate and select different EI projects. Firstly, a comprehensive index system is established from three aspects including grid technology, green energy, and composite benefits. Secondly, FLE is used to express experts' preferences and a transformation method converting discrete FLEs into continuous cloud information is proposed. To select the best EI project, the optimistic MGCRS and pessimistic MGCRS over two universes are presented to deal with the continuous cloud information in the decision-making process. Furthermore, the comprehensive multi-granularity lower approximation and upper approximation based on the cloud model are proposed to rank different EI projects. Finally, a case study of China's Beijing-Tianjin-Hebei region is analyzed to illustrate the proposed model, and the simulation and comparative analyses are provided to demonstrate the effectiveness of the proposed framework.

Keywords: Circular economy, Energy internet, Flexible linguistic expression, Multi-granularity could-rough set, Cloud model, Multi-criteria decision-making

1. Introduction

Facing the dual challenges of decarbonization and sustainable development, industrial systems are evolving towards a sustainable circular economy [1, 2]. The core of circular economy lies in optimizing resource utilization, reducing waste generation, and improving material re-utilization to achieve sustainable development [3]. Therefore, industrial systems must shift from a traditional linear economy to a resource-efficient and waste-minimizing circular system to meet modern sustainability needs by reducing raw material consumption and focusing on material reuse and energy recycling. Energy plays an important role in industrial production, directly impacting the efficiency of resource utilization and waste generation. To achieve circular economy goals,

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energy flow management is crucial for industrial systems, which involves improving every stage of the energy supply chain to maximize energy use and reduce waste from generation and transmission to consumption and recycling. As a cutting-edge concept in energy research, the energy internet (EI) occupies an important position in the transformation process [4]. EI connects each generator set through multiple communication platforms and builds an interconnected energy ecosystem with the power system as the core, which is not only a crucial tool for driving the transformation of industrial systems to a circular economy but also a concrete implementation of circular economy concepts at both technical and operational levels. By integrating technologies such as smart grids, real-time data analysis, and distributed energy resources, the EI provides comprehensive energy optimization solutions for industrial systems [5]. These technologies enhance energy efficiency and support resource recycling and waste reduction, helping industrial systems achieve the core goals of a circular economy.

The GEIDCO (Global Energy Interconnection Development and Cooperation Organization) highlights the active promotion of strategic and technological innovations to expedite the development of a global EI, aiming to achieve intra-continental interconnection by 2030 and intercontinental interconnection by 2050 [6]. The blueprint of the global EI in 2050 is shown in Fig. 1. With the rapid development of EI, more and more countries and regions have begun to pay attention to the theoretical research and practical application in this field. IEEE (Institute of Electrical and Electronics Engineers) promotes standardization research in the field of EI. The technical council of the IEEE Power and Energy Society approved the establishment of the IEEE-EICC (Energy Internet Coordinating Committee) in July 2020 [7]. The NIST (National Institute of Standards and Technology) has released version 4.0 of NIST Framework and Roadmap for Smart Grid Interoperability Standards in 2021. Europe countries and Japan have also developed relevant standards and technical frameworks [8]. To advance the transition of industrial systems towards a circular economy, the NEA (National Energy Administration) in China has approved 55 green energy demonstration projects, covering urban energy, energy, electric cars, flexible energy, and other types of energy [9]. These efforts have promoted the implementation of circular economy policies and deepened the exploration of technological innovation and operational models of industrial systems.

However, current research about EI has focused on concepts, standards, and technology systems of EI [10, 11]. Evaluating different EI projects is essential to understanding the practical effects of driving the transformation of industrial systems to a circular economy, which not only contributes to an in-depth understanding of the actual effect of EI in promoting the transformation of industrial systems to a sustainable circular economy, but also provides data support and decision-making basis for further technology optimization and policy-making. Employing a comprehensive and strategic approach to evaluate EI projects will provide a clear insight into how the EI specifically supports circular economy objectives, thereby more effectively driving industrial systems towards a greener and more sustainable future [12]. To guarantee the accuracy and comprehensiveness of the evaluation, adopting a scientific and systematic approach is essential. Evaluating EI projects involves a multi-criteria decision-making (MCDM) problem, encompassing the comprehensive analysis of various dimensions including technological, environmental, social, and economic factors. By applying comprehensive

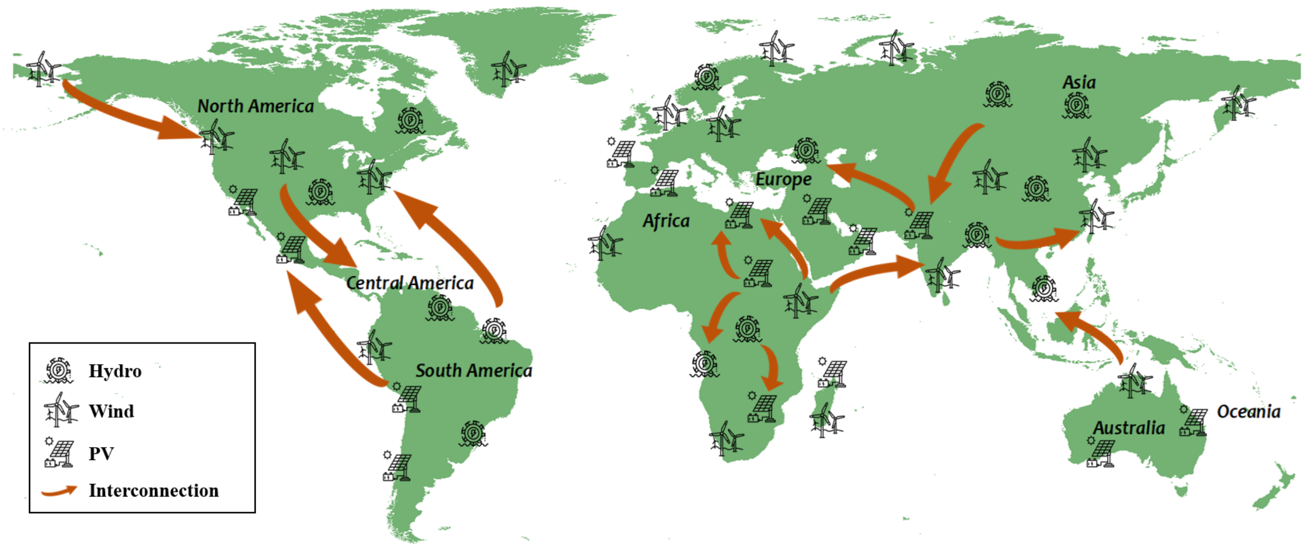

Fig. 1. The blueprint for global EI in 2050.

criteria and MCDM methods [13, 14], companies and technology developers can identify and prioritize EI projects with technological innovation and positive environmental and social impact, thereby driving the transformation of industrial systems to a more sustainable and circular economy [15]. The evaluation of EI projects promotes green innovation and low-carbon development of industrial systems and ensures that these projects play a key role in the sustainable circular transformation of industrial systems, laying a solid foundation for achieving long-term environmental and social benefits [16].

For multiple EI projects, providing evaluation information in an easily understandable format is essential. The linguistic term is commonly used to convey complex information, particularly when dealing with intricate EI systems or multidimensional environmental factors. In the face of complex systems or multi-dimensional factors, EI project evaluation often relies on multiple discrete linguistic terms to accurately convey evaluation information. These linguistic terms typically describe various levels of evaluation and are accompanied by probabilities. For example, the sustainable influence of an EI project might be characterized with a 70% probability as 'high' or 'medium' and a 30% probability as 'low', i.e., $\{(\{high, medium\}, 0.7), (\{low\}, 0.3)\}$. This expression, referred to as flexible linguistic expressions (FLEs) [17], integrates the intuitive nature of linguistic terms with the quantitative precision of probabilities, which effectively addresses the ambiguity and uncertainty inherent in evaluations and provides a consistent framework for experts and stakeholders, thereby facilitating robust decision-making. However, due to the subjectivity and fuzziness of discrete linguistic terms, the evaluation results are often uncertain and difficult to support accurate decision-making and analysis. Additionally, discrete evaluation information is difficult to compare and integrate, which limits the effective interaction and integration between different evaluation indicators and information. Previous research on FLEs has focused on the normalization method of symbolic proportion and qualitative analysis of linguistic terms, ignoring the quantitative analysis of linguistic terms. To minimize the loss of information while preserving the fuzziness and uncertainty of the original evaluation, how to convert discrete FLEs into continuous information is still a research gap. Aiming at this problem, this paper proposes an optimization method that transforms discrete FLEs into continuous cloud model information, which provides a more precise representation of evaluation continuity while retaining the inherent fuzziness and uncertainty of the original evaluation.

The multi-granularity rough set (MGRS) theory provides an efficient and flexible tool for handling and analyzing complex and uncertain evaluation information within the context of EI project evaluation [18]. Evaluating EI systems requires a comprehensive consideration of various factors, including diverse technological and environmental variables, as well as dynamic influences such as policy changes, market demands, and operational conditions. Consequently, these evaluations often encounter significant uncertainty and vagueness. The core advantage of MGRS lies in its ability to process information at different levels of granularity, effectively accommodating the diversity of factors involved. By constructing upper and lower approximation sets, this method captures the inherent vagueness in the data and provides classification and ranking of systems or projects [19]. This structured analytical framework for handling uncertain and fuzzy data supports comprehensive evaluation and comparative analysis of EI projects. However, traditional MGRS methods are primarily designed for handling discrete information, which poses limitations when dealing with continuous data or highly fuzzy situations. These constraints may impede the effectiveness of conventional methods in addressing the diverse and continuous information often encountered in EI evaluations. To overcome these challenges, this paper integrates MGRS with the cloud model and proposes a novel multi-granularity cloud-rough set (MGCRS). This approach not only retains the inherent vagueness and uncertainty of the original data but also allows the final evaluation results to be presented in a continuous format and ranked accordingly. This innovative method significantly enhances the accuracy and practicality of the evaluation process, providing more comprehensive and precise support for the selection and optimization of EI projects.

To solve the above research gaps in the evaluation of EI projects and advance the goal of circular economy effectively, we propose an MCDM framework based on FLEs and MGCRS. The contributions of this paper are as follows:

(1) A novel method to convert discrete FLEs into continuous cloud information is proposed. FLEs are first aggregated into floating clouds using a series of basic clouds. Then an programming model is constructed to minimize the difference to derive a comprehensive cloud to obtain continuous cloud information for each discrete FLE, effectively achieving the conversion from discrete to continuous representation. This approach preserves the uncertainty of evaluation information while making it easier to integrate to support accurate decision-making and analysis for EI projects.

(2) Based on the relevant literature and analyses, we develop an evaluation index system tailored to evaluate various EI projects, which is designed to capture the multifaceted impacts of these projects on sustainable development. To assign appropriate weights to the evaluation criteria, we utilize the Shannon entropy method, leveraging the cloud information derived from our novel transformation technique. This approach ensures that the evaluation framework accurately reflects the EI projects' effectiveness in advancing energy efficiency and circular economy goals.

(3) To select the best EI project, we define the MGCRS and present the optimistic and pessimistic MGCRSs over two universes to deal with the continuous cloud information in the decision-making process. Sequentially, the comprehensive multi-granularity lower approximation and upper approximation based on the cloud model are proposed to rank different EI projects. This approach allows the final evaluation results to be presented in a continuous format and ranked accordingly and significantly enhances the accuracy and practicality of the evaluation process, offering a more precise and reliable basis for optimizing EI projects in the context of the circular economy.

The remainder of this paper is structured as follows. Section 2 reviews the literature on the research of EI evaluation. Section 3 introduces the basic knowledge necessary related to FLE, cloud model, and rough sets. Section 4 proposes an MCDM framework for EI project evaluation in the FLE environment based on the Shannon entropy method and MGCRS. Section 5 provides a numerical analysis of the EI project evaluation in China's Beijing-Tianjin-Hebei region and the relevant analyses including the simulation analysis and comparative analysis to demonstrate the feasibility and effectiveness of the proposal. Section 6 discusses the results. The final section presents the conclusions.

2. Literature review

This section reviews the existing literature on the MCDM methods and evaluation index system for energy systems. The evaluation index system for EI projects is established based on the literature review and analyses.

2.1. Energy internet evaluation approaches

To better characterize the influence of various energy sources in advancing circular economy and sustainable development, numerous MCDM approaches have been applied to the evaluation and selection of different integrated energy systems, including AHP (Analytic hierarchy process) [20–22], ANP (Analytic network process) [23], TOPSIS (The technique for order preference by similarity to an ideal solution) [24–27], VIKOR (VlseKriterijumska Optimizacija I Kompromisno Resenje) [28, 29], DEMATEL (Decision Making Trail and Evaluation Laboratory) [26, 30, 31], ELECTRE(ELimination and Choice Expressing REality) [23], etc. The summary of evaluation approaches for energy systems is shown in Table 1, in which the research goal, evaluation approaches, and criteria dimensions are listed. The relevant approaches can be divided into two categories: single MCDM method and hybrid MCDM methods.

1) Single MCDM method: Kong et al. [20] analyzed the internationalization implementation effect of technical standards using the fuzzy AHP method. Jiang et al. [24] used the TOPSIS method to evaluate the operational performance of community-integrated energy systems. Wang et al. [25] established an evaluation model based on the improved TOPSIS method to select an urban integrated energy station. Shang [28] proposed the fuzzy VIKOR method to select a distributed energy storage system. To explore the difference between these single MCDM methods, Dagtekin et al. [23] compared the ranking results of different methods including AHP, ANP, TOPSIS, ELECTRE, PROMETHEE and VIKOR for distributed energy systems selection.

2) Hybrid MCDM methods: Otay et al. [27] combined the BWM (Best Worst Method) and TOPSIS for multiple experts to evaluate sustainable energy systems in smart cities. Ke et al. [32] proposed a hybrid method integrating BWM and CRITIC (Criteria importance though inter-criteria correlation) to determine the urban integrated energy systems site. Esangbedo et al. [33] employed a hybrid method in subjective and objective aspects to determine the weight of criteria in the subcontractor selection of the photothermal power station problem. Zhao et al. [26] proposed multiple decision-making methods including TOPSIS, anti-entropy weight method, grey-DEMATEL, and quotient grey relation analysis to select the best one from eight building-typed microgrid systems. Bagherian et al. [30] adopted the ISM-MICMAC and DEMATEL method to analyze the energy sustainability and digitalization. Bac et al. [34] developed a hybrid framework integrating modified SWARA (Stepwise Weight evaluation Ratio Analysis) and WASPAS (Weighted Additive Sum Product evaluation) methods to evaluate air-conditioning systems.

Since experts may be irrational, the preferences of experts are characterized by uncertainty and fuzziness. To deal with the judgment uncertainty, many fuzzy discrete expressions like HFS (Hesitant fuzzy set), TFN (Triangular fuzzy number), and TrFN (Trapezoid fuzzy number) have been used to characterize the preference information of experts. Liu et al. [21] proposed a new decision-making method under interval type-2 fuzzy numbers to evaluate the multi-energy transaction performance, and presented a novel integrated performance evaluation method with flexible fuzzy boundaries to deal with linguistic imprecision and ambiguity of expert judgments. Qin et al. [35] extended a fuzzy AHP method based on the cloud model and TFN to evaluate the performance of regional EI, where fuzzy linguistic terms are converted into the cloud model by the aggregated weight method. Tan et al. [36] proposed a probabilistic hesitant fuzzy MCDM method considering prospect theory to evaluate different rural EI scenarios without the information transformation. Wu et al. [37] constructed a fuzzy evaluation framework based on interval type-2 TrFN and applied the Choquet integral fuzzy synthetic model to fuse the evaluation information. Zhou et al. [38] proposed a probabilistic evaluation approach based on the Dirichlet mixture model to evaluate an integrated energy supply system, where expectation and variance values are used to process probability evaluation information. Xu et al. [31] applied the DEMATEL HFS method to evaluate the risk of integrated energy systems, where a hesitant fuzzy entropy is used to aggregate the HES information. Otay et al. [27] developed a novel interval-valued Pythagorean fuzzy method where multi-expert fuzzy BWM and TOPSIS methodology to better handle uncertainty and vagueness in experts' linguistic assessments. Although the existing literature has developed different evaluation approaches for integrated energy systems, few studies have focused on the EI project evaluation considering the transformation between discrete information and continuous information to integrate the information smoothly and prevent information loss. Therefore, this paper proposes an MCDM approach with FLEs based on MGCRS to evaluate and select the best EI project.

Table 1. Summary of evaluation approaches for energy systems.

References	Research goal	Evaluation approaches	Criteria dimensions
Kong et al. [20]	Evaluation of the internationalization implementation effect	Fuzzy AHP	Standard adoption, activities, benefit, compilation, and text internationalization
Dagtekin et al. [23]	Evaluation of distributed energy storage system	AHP, ANP, TOPSIS, ELECTRE, PROMETHEE and VIKOR	Energy, cost, emissions, and investment
Jiang et al. [24]	Performance evaluation of community integrated energy systems	TOPSIS	System efficiency, renewable energy penetration and operation cost
Shang [28]	Evaluation of distributed energy storage systems	Fuzzy measure and VIKOR	Environment, society and business
Wang et al. [25]	Site selection for urban integrated energy station	GIS and improved TOPSIS	Nature, economy and society
Ke et al. [32]	Site selection for urban integrated energy systems	BWM and CRITIC	Economy, energy, environment and society
Wang et al. [29]	Evaluation of distributed energy systems	DEMATEL and VIKOR	Technique, economy, environment and society
Zhou et al. [38]	Evaluation of integrated energy supply system	Probabilistic approach and Dirichlet mixture model	Resource, economy, and environment
Zhao et al. [26]	Evaluation of building-typed microgrid systems	TOPSIS, anti-entropy weight method, grey-DEMATEL and quotient grey relation analysis	Economy, environment and energy
Bac et al. [34]	Evaluation of HVAC system	SWARA and WASPAS	Ergonomics, environment, reliability, technique and economy
Liu et al. [21]	Performance evaluation of multi-energy transaction	Fuzzy comprehensive evaluation and AHP	Suppliers, transaction attributes, consumers and distributors
Qin et al. [35]	Performance evaluation of regional EI	Fuzzy AHP and cloud model	Technique, economy, society and engineering
Tan et al. [36]	Feasibility evaluation of rural EI	Probability hesitation fuzzy method and prospect theory	Economy
Wu et al. [37]	Investment evaluation of regional EI	Interval type-2 trapezoid fuzzy number and Choquet integral	Internal and external attributes
Zhou et al. [39]	Evaluation of park-level integrated energy systems	DEMATEL and the extended TODIM	Economy, environment, energy utilization, reliability and sustainability
Xu et al. [31]	Risk evaluation of integrated energy systems	DEMATEL and HFS	Economy, technology, politics, society and management

2.2. *Evaluation index system for EI project*

The evaluation criteria of the integrated energy system have been investigated in some previous works, including but not limited to aspects of economy, society, environment, resources, reliability, etc. Economy, society and environment are the most commonly used dimensions when determining evaluation index systems. Otay et al. [27] considered six criteria to evaluate the energy system in a smart city involving environmental, economic, social, and technical factors. Pamucar et al. [40] claimed that environmental factors were more essential than social and economic factors from the perspective of green energy. Zhao et al. [26] constructed a performance evaluation index system for the microgrid system from the economy, environment, and energy dimensions. Wang et al. [25] established a comprehensive index system for nature, economy, and society to select the final optimal urban integrated energy station. Ke et al. [32] proposed a comprehensive evaluation index system from the economy, energy, environment, and society for urban integrated energy systems selection. Bac et al. [34] prioritized transformation of the energy market and smart manufacturing technologies based on the critical measurements in Europe's energy domain. To select the subcontractor for the photothermal power station, Esangbedo et al. [33] included enterprise reputation as a criterion in addition to the commonly used evaluation system mentioned above.

In addition, security, technique, and politics are also used to measure the performance of energy systems. Qin et al. [35] selected sixteen criteria of region EI about technical, economic, social, and engineering dimensions. Zhou et al. [38] proposed an evaluation system from the economy, efficiency, environment, and security. Xu et al. [31] summarized sixteen risk factors from the economy, technology, politics, society, and management to evaluate the risk of integrated energy systems. Bac et al. [34] selected twenty-seven criteria under several categories including ergonomic, environmental, reliability, technical, and economical aspects to evaluate seven air-conditioning systems. Lu and Liu [41] constructed an MCDM framework using cost, reliability, energy consumption, and environmental factors.

The idea of systems engineering is gradually applied in the establishment of index systems. Wu et al. [37] constructed a criteria system from internal and external attributes to evaluate the regional EI investment. Berjawi et al. [42] summarized six characteristics for evaluating the integrated energy systems multidimensional, multivectorial, systemic, applicability, futuristic, and systematic. Liu et al. [21] established a multi-energy transaction index system about suppliers, transaction attributes, consumers, and distributors. Dagtekin et al. [23] determined five criteria primary energy utilization rate, operating cost, primary energy consumption, carbon emissions, and investment cost for distributed energy systems. Despite the above research, the existing index system often only focuses on a few dimensions, which is a lack of comprehensiveness and objectivity. As the foundation for the evaluation and selection of EI, a comprehensive and reasonable index system is urgently needed to extract objectively and thoroughly.

Considering the power interoperability, environmental protection, and efficiency of the EI, we propose a new evaluation index system according to existing literature and analyses, which is shown in Fig. 2. The evaluation index system includes 10 criteria from grid technology, green energy, and composite benefits aspects. The grid technology mainly indicates the internal and external technical level of power grids in the EI, including grid interconnection, reliability of grid structure, grid informatization, and power transmission capacity 4 sub-criteria. Corresponding to the core part of the EI——energy, green energy measures the clean energy usage, waste utilization, and waste emissions of the EI, which are indicated by the proportion of clean energy, the utilization rate of waste, and the emission of waste gas, respectively. To explore the benefits of EI projects, social, economic, and environmental benefits are used to measure the composite benefits. Details of the evaluation index system are specifically summarized and explained in Table 2.

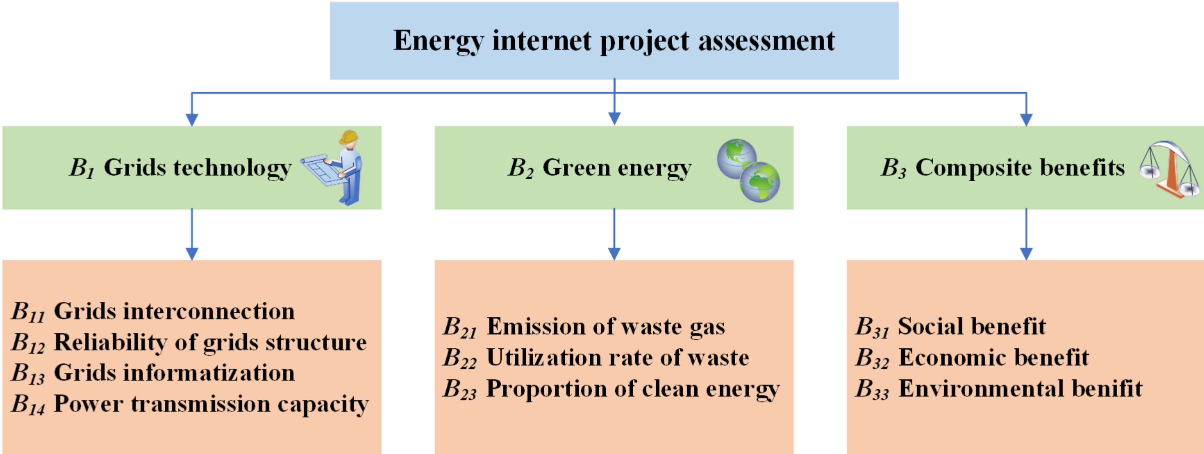

Fig. 2. The evaluation index system for EI project.

Table 2. A detailed description of the evaluation index system.

Dimensions	Criteria	Criteria description
Grid technology (B_1)	Grids interconnection(B_{11})	It indicates the degree of interconnection between internal and external power grids in a region.
	Reliability of grids structure(B_{12})	It measures the resilience and reliability of power grids structure.
	Grids informatization(B_{13})	It refers to the application of smart technologies such as modern information, communication, and control to the power grids.
	Power transmission capacity(B_{14})	It indicates the transmission capacity of the grid, the length of the line, and the amount of power.
Green energy (B_2)	Emission of waste gas(B_{21})	It is the reciprocal of the level of waste emissions per unit of electricity generated.
	Utilization rate of waste(B_{22})	It measures the utilization rate of waste generated per unit of electricity generation.
	Proportion of clean energy(B_{23})	It is the ratio of clean energy to the total energy.
Composite benefits (B_3)	Social benefit(B_{31})	It indicates the benefit that the EI brings to society.
	Economic benefit(B_{32})	It refers to the benefit of the EI to the macroeconomy.
	Environmental benefit(B_{33})	It measures the environmental benefit of an EI.

3. Preliminaries

This section briefly introduces some basic definitions, regarding linguistic scale function [43], FLEs [17], cloud model [44–46]), Pawlak rough set [47], and MGRS [18, 19].

3.1. Linguistic scale function and flexible linguistic expression

Definition 1. [43] Given a linguistic term set $L = \{l_0, l_1, ..., l_g\}$, the linguistic scale function H mapping from l_i to δ_i is defined as follows:

$$H : l_i \rightarrow \delta_i \ (i = 0, 1, ..., g),$$

where $0 < \delta_0 < \delta_2 < \cdots < \delta_g < 1$. The symbol δ_i reflects the preference of experts using the linguistic term l_i. The function H is a strictly monotonically increasing function with respect to l_i, which is denoted as follows:

$$H(l_i) = \delta_i = \begin{cases} \dfrac{a^{\frac{g}{2}} - a^{\frac{g}{2}-i}}{2a^{\frac{g}{2}} - 2} & (i = 0, 1, ..., \frac{g}{2}) \\ \dfrac{a^{\frac{g}{2}} + a^{i-\frac{g}{2}} - 2}{2a^{\frac{g}{2}} - 2} & (i = \frac{g}{2} + 1, \frac{g}{2} + 2, ..., g) \end{cases}, \tag{1}$$

the value of a can be determined using a subjective method. Assuming the indicator A is far more important than indicator B and the important ratio is m, then $a^k = m$ (k represents the scale level) and $a = \sqrt[k]{m}$. The vast majority of researchers believe that $m = 9$ is the upper limit of the important ratio. If the scale level is 7, then $a = \sqrt[7]{9} \approx 1.37$ can be obtained.

Definition 2. [17] Let $L = \{l_0, l_1, ..., l_g\}$ be a fixed linguistic term set with odd cardinality. $\hat{S}_L$ is a set composed of the subsets s_L of L and individuals express preferences by providing the distribution information of s_L. Then, the individual's preference is FLE, denoted as

$$m_L = \{s_L, p(s_L) | s_L \in \hat{S}_L, p(s_L) \in [0, 1]\}, \tag{2}$$

where $p(s_L)$ is the symbolic proportion assigned to the subset s_L. The negation operator of an FLE m_L is $Neg(\{s_L, p(s_L)|s_L \in \hat{S}_L\}) = \{Neg(s_L), p(s_L)|s_L \in \hat{S}_L\}$, where $Neg(s_L) = \{l_{g-t}|l_t \in s_L, t \in \{0, 1, ..., g\}\}$.

The set L is not fixed because individuals may use different subsets to express preferences for special decision-making problems. Wu et al. [17] argued that the sum of symbolic proportions shouldn't be restricted to be one or less than one because the constraint is hard to satisfy.

3.2. Cloud model

Definition 3. [44] Let U be the universe of discourse and $\tilde{A}$ be a qualitative concept in U. If $x \in U$ is a random instantiation of the qualitative concept $\tilde{A}$ that satisfies $x \sim N(Ex, En'^2)$ and $En' \sim N(En, He^2)$, and the certainty degree y of x belonging to concept $\tilde{A}$ is a probability distribution, which satisfies

$$y = e^{-\frac{(x-Ex)^2}{2En'^2}}, \tag{3}$$

then the distribution of x in the universe U is called a normal cloud, and the cloud drop can be denoted as (x, y). The overall quantitative properties of concept $\tilde{A}$ can be perfectly depicted in cloud C with three numerical features: expectation Ex, entropy En, and hyper entropy He. Cloud C can be described as $C = (Ex, En, He)$.

For a cloud $C = (Ex, En, He)$, Ex is the mathematical expectation that the cloud drops belong to a concept in the universe, which can be regarded as the most typical sample of the qualitative concept. En represents the numerical range of qualitative concepts which reflects the uncertainty measurement of the concept. The larger En is, the fuzzier the concept is. He is the second-entropy of entropy En, which represents the uncertainty degree of En. The larger He reflects that cloud drops are more random which means the cloud is thicker. For example, a normal cloud generated by $C(5, 0.38, 0.03)$ with 3000 cloud drops is shown in Fig. 3.

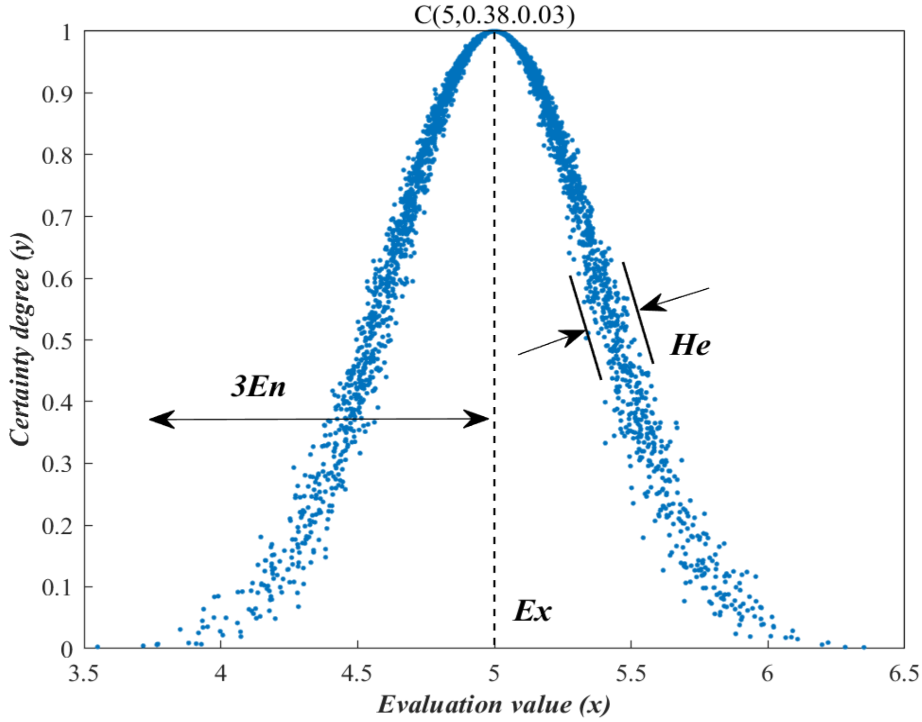

Fig. 3. The normal cloud generated by $C(5, 0.38, 0.03)$ with 3000 cloud drops.

To measure the difference between clouds, the definition of distance between two clouds is proposed, which is essential for decision-making problems.

Definition 4. [45] If $C_1 = (Ex_1, En_1, He_1)$ and $C_2 = (Ex_2, En_2, He_2)$ are two arbitrary normal clouds, then the Euclid distance $d(C_1, C_2)$ between C_1 and C_2 is denoted as follows:

$$d(C_1, C_2) = \sqrt{\frac{1}{2}((Ex_1 - Ex_2)^2 + (En_1 - En_2)^2 + (He_1 - He_2)^2)}. \tag{4}$$

Definition 5. [46] If $C_1 = (Ex_1, En_1, He_1)$ and $C_2 = (Ex_2, En_2, He_2)$ are two arbitrary basic normal clouds, then the floating cloud $C = (Ex, En, He)$ between C_1 and C_2 is calculated as follows:

$$\begin{cases} Ex = \alpha Ex_1 + (1-\alpha)Ex_2 \\ En = \dfrac{\alpha Ex_1 En_1 + (1-\alpha)Ex_2 En_2}{\alpha Ex_1 + (1-\alpha)Ex_2} \\ He = \sqrt{He_1^2 + He_2^2} \end{cases}, \tag{5}$$

where $\alpha \in [0, 1]$ is the adjustment coefficient affecting three numerical features of the floating cloud C.

3.3. Pawlak rough set and multi-granularity rough set

Definition 6. [47] Let U be a non-empty finite universe and $R \in U \times U$ be a binary equivalence relation over universe U, then (U, R) is Pawlak approximation space. The lower and upper approximations for $X \in U$ are defined as follows:

$$\begin{aligned} \underline{R}(X) &= \cup\{[x]_R | [x]_R \subseteq X, X \in U\}, \\ \overline{R}(X) &= \cup\{[x]_R | [x]_R \cap X \neq \emptyset, X \in U\}, \end{aligned} \tag{6}$$

where $[x]_R$ is the equivalence class of x under the binary equivalence relation R. $X(X \in U)$ is called Pawlak rough set if $\overline{R}(X) \neq \underline{R}(X)$.

The Pawlak rough set only contains a binary relation over the universe, which is regarded as a single granularity rough set. Qian et al. [18, 19] extended the Pawlak rough set concerning multiple binary relations over two universes, which is called multi-granularity rough sets (MGRSs). The MGRSs contain the optimistic MGRS and the pessimistic MGRS corresponding to risk preference decision-making and risk-averse decision-making, respectively.

Definition 7. [19] Let U and V be two non-empty finite universes and $\{R_1, R_2, ..., R_m\} \in U \times V$ be m generalized binary equivalence relations over universe $U \times V$, then $(U, V, \{R_i\}_{i=1,2,...,m})$ is multi-granularity approximation space over two universes. The optimistic multi-granularity lower approximation $\underline{R}^O_{\sum_{i=1}^m R_i}(X)$ and upper approximation $\overline{R}^O_{\sum_{i=1}^m R_i}(X)$ for $X \subseteq V$ are defined as follows:

$$\begin{aligned} &\underline{R}^O_{\sum_{i=1}^m R_i}(X) = \{x \in U | R_1(x) \subseteq X \vee R_2(x) \subseteq X \vee \cdots \vee R_m(x) \subseteq X\}, \\ &\overline{R}^O_{\sum_{i=1}^m R_i}(X) = \{x \in U | R_1(x) \cap X \neq \emptyset \wedge R_2(x) \cap X \neq \emptyset \wedge \cdots \wedge R_m(x) \cap X \neq \emptyset\}, \end{aligned} \tag{7}$$

where $R_i(x) = \{y \in V | x \in U, (x, y) \in R_i\}$. The interval set $(\underline{R}^O_{\sum_{i=1}^m R_i}(X), \overline{R}^O_{\sum_{i=1}^m R_i}(X))$ is called optimistic MGRS over two universes if $\underline{R}^O_{\sum_{i=1}^m R_i}(X) \neq \overline{R}^O_{\sum_{i=1}^m R_i}(X)$.

Definition 8. [18] Let U and V be two non-empty finite universes and $\{R_1, R_2, ..., R_m\} \in U \times V$ be m generalized binary equivalence relations over universe $U \times V$, then $(U, V, \{R_i\}_{i=1,2,...,m})$ is multi-granularity approximation space over two universes. The pessimistic multi-granularity lower approximation $\underline{R}^P_{\sum_{i=1}^m R_i}(X)$ and upper approximation $\overline{R}^P_{\sum_{i=1}^m R_i}(X)$ for $X \subseteq V$ are defined as follows:

$$\begin{aligned} &\underline{R}^P_{\sum_{i=1}^m R_i}(X) = \{x \in U | R_1(x) \subseteq X \wedge R_2(x) \subseteq X \wedge \cdots \wedge R_m(x) \subseteq X\}, \\ &\overline{R}^P_{\sum_{i=1}^m R_i}(X) = \{x \in U | R_1(x) \cap X \neq \emptyset \vee R_2(x) \cap X \neq \emptyset \vee \cdots \vee R_m(x) \cap X \neq \emptyset\}, \end{aligned} \tag{8}$$

where $R_i(x) = \{y \in V | x \in U, (x, y) \in R_i\}$. The interval set $(\underline{R}^P_{\sum_{i=1}^m R_i}(X), \overline{R}^P_{\sum_{i=1}^m R_i}(X))$ is called pessimistic MGRS over two universes if $\underline{R}^P_{\sum_{i=1}^m R_i}(X) \neq \overline{R}^P_{\sum_{i=1}^m R_i}(X)$.

4. The MCDM framework with FLEs based on MGCRS for EI project evaluation

4.1. Problem description for EI project evaluation

For the EI project evaluation problem, $E = \{e_1, e_2, ..., e_K\}$ is the set of experts, and $V = \{v_1, v_2, ..., v_K\}$ is the set of experts' weights. $B = \{b_1, b_2, ..., b_m\}$ is the criteria set of EI projects, and the criteria weight set is $W = \{W_1, W_2, ..., W_m\}$. There are n EI projects $X = \{x_1, x_2, ..., x_n\}$ to be evaluated. Experts express their preference through providing linguistic terms set $L = \{l_0, l_1, ..., l_g\}$ with symbolic proportions, i.e., FLEs. The evaluation matrix provided by expert e_k using FLEs is $M^k = (m^k_{ij})_{n\times m}$, where m^k_{ij} is an FLE provided by expert e_k over the EI project x_i under criterion b_j.

The proposed MCDM framework for EI project evaluation contains three parts: (1) The transformation between FLE and cloud model. To quantify linguistic in FLEs and obtain a normalized FLE, the FLEs in $M^k = (m^k_{ij})_{n\times m}$ are converted into corresponding clouds according to the cloud model. Therefore, the cloud matrix $R^k = (\tilde{C}^k_{ij})_{n\times m}$ can be obtained. (2) Determine weights of criteria using the Shannon entropy method. Based on the cloud matrix, using the Shannon entropy method to obtain the criteria weights. (3) The ranking method is based on MGCRS over two universes. The multiple decision-making cloud information system over two universes is presented, and the optimistic and pessimistic MGCRSs over two universes are proposed. Furthermore, the comprehensive MGCRS is proposed to rank these EI projects. The MCDM framework for EI project evaluation is shown in Fig. 4.

4.2. Transformation between FLE and cloud model

An FLE is composed of a series of linguistic terms and symbolic proportions. To obtain the corresponding cloud model, the first step is to convert these linguistic terms into the basic clouds. Inspired by Wang et al. [43], the method transforming linguistic terms into the basic clouds is as follows:

(1) Calculate δ_i. Experts tend to be risk-sensitive when evaluating different EI projects. Therefore, $\delta_i = H(l_i)$ can be obtained using Eq. (1). The absolute deviation between adjacent linguistic terms also increases with the extension from the middle of the linguistic term to both ends.

(2) Calculate Ex_i. According to the effective domain $U = [U^L, U^U]$, $Ex_i = U^L + \delta_i(U^U - U^L)$ can be calculated. Therefore, we can obtain $Ex_0 = U^L$ and $Ex_g = U^U$.

(3) Calculate En_i. For a cloud drop (x, y), $x \sim N(Ex, En'^2)$ means that the '3σ principle' of the normal distribution curve should be satisfied, i.e., $3En'_i = \max\{U^U - Ex_i, Ex_i - U^L\}$. Since $En' \sim N(En, He^2)$, En_i can be regarded as the expectation of En' corresponding to the ith cloud and its adjacent clouds. Then En'_i and En_i can be determined by the following two equations:

$$En'_i = \begin{cases} \dfrac{(1-\delta_i)(U^U - U^L)}{3} & (i = 0, 1, ..., \frac{g}{2}) \\ \dfrac{\delta_i(U^U - U^L)}{3} & (i = \frac{g}{2}+1, \frac{g}{2}+2, ..., g) \end{cases}, \text{ and } En_i = \begin{cases} \dfrac{En'_{i+1}+En'_i}{2} & (i = 0) \\ \dfrac{En'_{i-1}+En'_i+En'_{i+1}}{3} & (0 < i < g) \\ \dfrac{En'_{i-1}+En'_i}{2} & (i = g) \end{cases}.$$

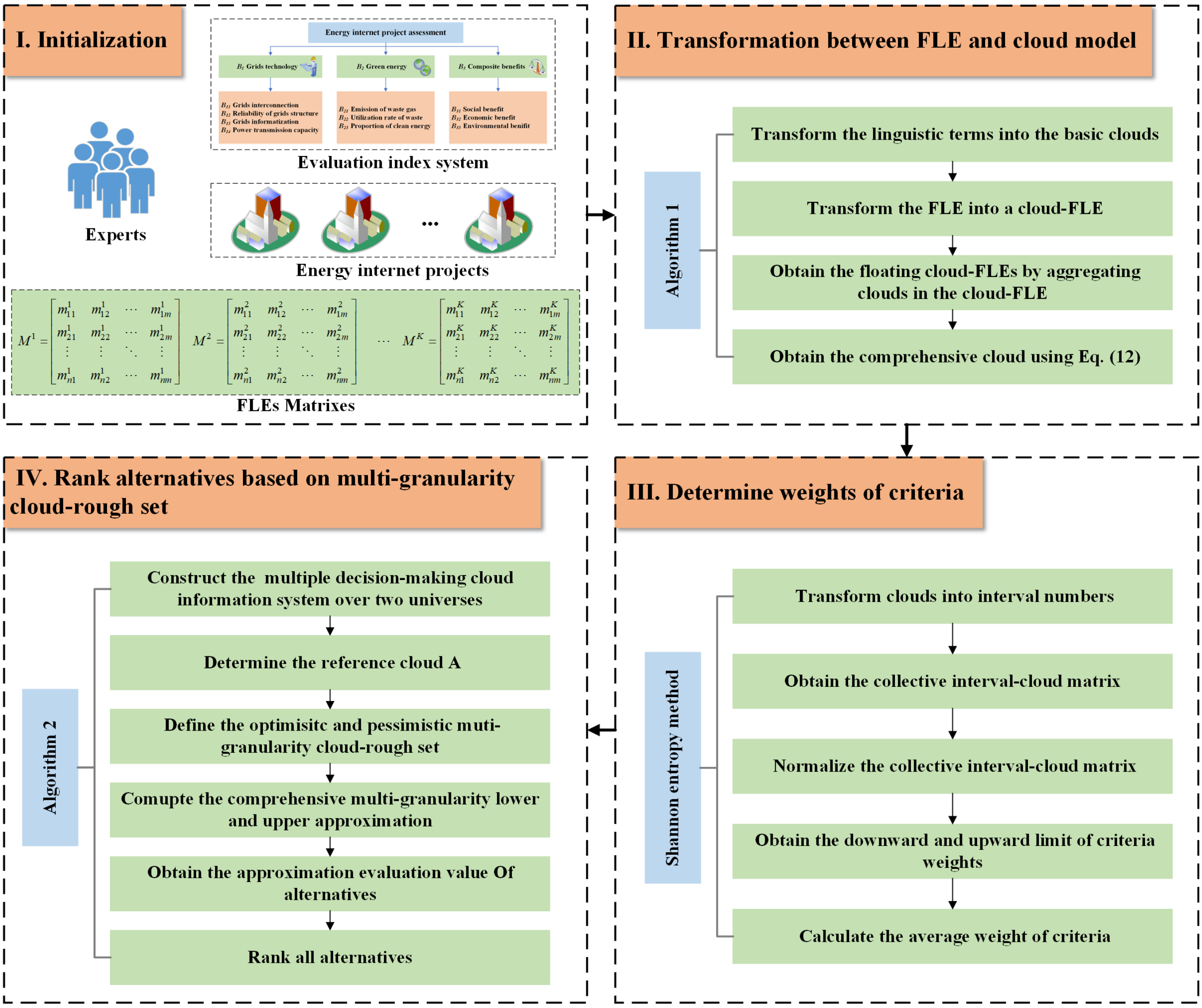

Fig. 4. Flowchart of the proposed MCDM framework for EI project evaluation.

(4) Calculate He_i. Due to $En' \sim N(En, He^2)$, He_i should obey the '3σ principle' of the normal distribution curve, then $He_i = \frac{\max\{\max_k\{En'_k\} - En_i, En_i - \min_k\{En'_k\}\}}{3}$ $(i = 0, 1, ..., g)$.

(5) Obtain the basic clouds $C_i = (Ex_i, En_i, He_i)$. The basic cloud C_i corresponding to the linguistic term l_i can be obtained, which is composed of the three numerical features Ex_i, En_i, and He_i.

Based on the above method, all linguistic terms in s_L can be transformed into the corresponding basic clouds. In other words, an FLE $m_L = \{(s_L^1, p^1), (s_L^2, p^2), ..., (s_L^T, p^T)\}$ can be transformed into a cloud-FLE $m'_L = \{(C_L^1, p^1), (C_L^2, p^2), ..., (C_L^T, p^T)\}$, where $C_L^t (t = 1, 2, ..., T)$ is the set of basic clouds corresponding to linguistic term set s_L^t.

Example 1. Given the domain $U = [0, 10]$ and the linguistic evaluation set $L = \{l_0 : very\ poor, l_1 : poor, l_2 : fair, l_3 : good, l_4 : very\ good\}$, then $\delta = (\delta_0, \delta_1, \delta_2, \delta_3, \delta_4) = (0, 0.2890, 0.5, 0.71097, 1)$. These linguistic variables can be converted into asymmetric normal clouds, which are as follows: $C_0 = (0, 2.8518, 0.3950)$, $C_1 = (2.8892, 2.4568, 0.2922)$, $C_2 = (5, 2.1357, 0.3992)$, $C_3 = (7.1108, 2.4568, 0.2922)$, $C_4 = (10, 2.8518, 0.3950)$. Therefore, an FLE $m_L = \{(\{s_0, s_1\}, 0.3), (\{s_2, s_3\}, 0.2), (\{s_4\}, 0.2)\}$ can be transformed into a cloud-FLE $m'_L =$

$\{(C_L^1, 0.3), (C_L^2, 0.2), (C_L^3, 0.2)\}$, i.e., $m'_L = \{(\{C_0, C_1\}, 0.3), (\{C_2, C_3\}, 0.2), (\{C_4\}, 0.2)\}$.

After obtaining the basic clouds, the approximate cloud between these basic clouds can be obtained through the definition of floating cloud using Eq. (5), in which the calculation of float hyper entropy ignored the effect of α on He. In this way, the floating cloud may be thicker than two basic clouds because the float hyper entropy may be larger than the hyper entropies of two clouds. For example, the floating cloud between $C_0 = (0, 2.8518, 0.3950)$ and $C_1 = (2.8892, 2.4568, 0.2922)$ is $C = (1.4446, 2.4568, 0.4913)$ when $\alpha = 0.5$. In this case, the uncertainty of the floating cloud will increase as the number of basic clouds. To obtain the float hyper entropy He closer to the basic normal clouds, the adjustment coefficient α is considered in the process of fusing two hyper entropies. Furthermore, to aggregate more than two basic clouds, the floating cloud is extended into the case of n clouds $C_i (i = 1, 2, ..., n)$, which is shown in Definition 9.

Definition 9. If there are n basic clouds $C_i (i = 1, 2, ..., n)$ in the universe, then the floating cloud $C = (Ex, En, He)$ between n clouds is calculated as follows:

$$\begin{cases} Ex = \sum_{i=1}^{n} \alpha_i Ex_i \\ En = \dfrac{\sum_{i=1}^{n} \alpha_i Ex_i En_i}{\sum_{i=1}^{n} \alpha_i Ex_i} \\ He = \sqrt{\sum_{i=1}^{n} \alpha_i He_i^2} \end{cases}, \tag{9}$$

where $\alpha_i \in [0, 1]$ is the adjustment coefficient corresponding to the basic cloud C_i, and $\sum_{i=1}^{n} \alpha_i = 1$.

Based on the concept of floating cloud, all basic clouds in $C_L^t (t = 1, 2, ..., T)$ can be formed as a floating cloud which can be regarded as an approximate cloud between these basic clouds. Therefore, a cloud-FLE $m'_L = \{(C_L^1, p^1), (C_L^2, p^2), ..., (C_L^T, p^T)\}$ can be converted into a floating-cloud-FLE $m^f = \{(C^1, p^1), (C^2, p^2), ..., (C^T, p^T)\}$, where $C^t (t = 1, 2, ..., T)$ is a floating cloud element between these clouds in C_L^t.

Example 2. For the cloud-FLE $m'_L = \{(\{C_0, C_1\}, 0.3), (\{C_2, C_3\}, 0.2), (\{C_4\}, 0.2)\}$ in Example 1, the clouds in s_L can be transformed into the floating clouds using Eq. (9). If clouds have the same adjustment coefficients, i.e., $\alpha_1 = \alpha_2 = ... = \alpha_n = \frac{1}{n}$, then the first floating cloud C^1 between two clouds C_0 and C_1 is $C^1 = (1.4446, 2.4568, 0.3474)$ and the second floating cloud C^2 between two clouds C_2 and C_3 is $C^2 = (6.0554, 2.3242, 0.3498)$. Since the third element $(\{C_4\}, 0.2)$ in m'_L contains only one cloud C_4, the third floating cloud is $C^3 = C_4 = (10, 2.8518, 0.3950)$. Therefore, the cloud-FLE m'_L can be transformed into a floating-cloud-FLE, i.e., $m^f = \{(C^1, 0.3), (C^2, 0.2), (C^3, 0.2)\}$.

The floating cloud-FLE can be regarded as the approximate cloud set with symbolic proportions between these basic clouds. However, the sum of these symbolic proportions may not be 1, which should be reassigned to each floating-cloud-FLE element. To obtain a normalized cloud-FLE according to the floating cloud-FLE, a programming model is established to convert linguistic variables in a cloud-FLE into several clouds with a normalized probability distribution.

Since the sum of symbolic proportions $p^t (t = 1, 2, ..., T)$ in a floating-cloud-FLE $m^f = \{(C^1, p^1), (C^2, p^2), ..., (C^T, p^T)\}$ might not be 1, the programming model in Eq. (10) is proposed to obtain a normalized cloud-FLE

$\tilde{m}^f = \{(\tilde{C}^1, \tilde{p}^1), (\tilde{C}^2, \tilde{p}^2), ..., (\tilde{C}^T, \tilde{p}^T)\}$, where $\sum_{t=1}^{T} \tilde{p}^t = 1$. The main idea of the programming model is to obtain the normalized cloud-FLE $\tilde{m}^f$ that is as close to the floating-cloud-FLE m^f as possible. Therefore, the minimum objective function includes two parts: (1) The weighted distance between the floating-cloud-FLE m^f and the normalized cloud-FLE $\tilde{m}^f$, denoted as $\sum_{t=1}^{T} d(\tilde{C}^t, C^t) * \tilde{p}^t$. (2) The sum of distances between symbolic proportions, denoted as $\sum_{t=1}^{T} \left|\tilde{p}^t - p^t\right|$. Then the programming model is as follows:

$$\min \sum_{t=1}^{T} d(\tilde{C}^t, C^t) * \tilde{p}^t + \sum_{t=1}^{T} \left|\tilde{p}^t - p^t\right|$$
$$s.t. \begin{cases} \sum_{t=1}^{T} \tilde{p}^t = 1 \\ 0 \le \tilde{p}^t \le 1 \\ d(\tilde{C}^t, C^t) = \sqrt{\frac{1}{2}((Ex^t - \widetilde{Ex}^t)^2 + (En^t - \widetilde{En}^t)^2 + (He^t - \widetilde{He}^t)^2)} \\ \widetilde{Ex}^t - 3\widetilde{En}^t - 9\widetilde{He}^t > 0 \\ \widetilde{En}^t \ge 3\widetilde{He}^t \\ \widetilde{Ex}^t - 3\widetilde{En}^t \ge U^L \\ \widetilde{Ex}^t + 3\widetilde{En}^t \le U^U \\ \widetilde{En}^t \ge 0, \widetilde{En}^t \ge 0, \widetilde{He}^t \ge 0 \end{cases}, \quad (10)$$

where the tth cloud in the floating-cloud-FLE m^f is $C^t = (Ex^t, En^t, He^t)$, and the tth cloud in the normalized cloud-FLE is $\tilde{C}^t = (\widetilde{Ex}^t, \widetilde{En}^t, \widetilde{He}^t)$.

In the programming model in Eq. (10), the objective function is to obtain the normalized cloud-FLE $\tilde{m}^f = \{(\tilde{C}^1, \tilde{p}^1), (\tilde{C}^2, \tilde{p}^2), ..., (\tilde{C}^T, \tilde{p}^T)\}$. Some extra constraints should be considered: (1) $\tilde{p}^t (t = 1, 2, ..., T)$ is the tth normalized symbolic proportion, then $\sum_{t=1}^{T} \tilde{p}^t = 1$. (2) $\widetilde{Ex}^t$, $\widetilde{En}^t$ and $\widetilde{He}^t$ are non-negative numbers. (3) Based on the '3σ principle', it also satisfies that $3\widetilde{En'}_i = \max\{U^U - \widetilde{Ex}_i, \widetilde{Ex}_i - U^L\}$, which can be rewritten as $(\widetilde{Ex}^t \pm 3\widetilde{En}^t) \in [U^L, U^U]$ since En_i can be regarded as an approximate value of En'_i. (4) Based on the '$3En$ principle', $\widetilde{En}^t \ge 3\widetilde{He}^t$ should also be satisfied.

Remark 1. The above programming model in Eq. (10) is guaranteed to have at least one optimal solution under the following conditions: (1) the objective function is continuous, and (2) the feasible region is non-empty, closed, and bounded. Firstly, the objective function consists of terms involving Euclidean distances, absolute values, and linear combinations, which are well-known to be continuous functions. Since continuity is preserved under summation and scalar multiplication, the entire objective function is continuous. Secondly, the constraint $\sum_{t=1}^{T} \tilde{p}^t = 1$ and $0 \le \tilde{p}^t \le 1$ define a standard simplex due to $T > 0$, and the constraints related to $\widetilde{Ex}^t, \widetilde{En}^t, \widetilde{He}^t$ are compatible, then the feasible region satisfies the non-emptiness. Thirdly, all constraints are continuous functions of the decision variables. The feasible region is defined by these constraints as the preimage of closed intervals under continuous functions, which ensures it is closed. Finally, boundedness is guaranteed by the simplex constraint $\sum_{t=1}^{T} \tilde{p}^t = 1$ and $0 \le \tilde{p}^t \le 1$, as this restricts all $\tilde{p}^t$ to lie within a finite region, and the remaining constraints involving $\widetilde{Ex}^t, \widetilde{En}^t, \widetilde{He}^t$ further confine the feasible region to a bounded subset of the decision space. The Weierstrass Theorem states that a continuous function achieves its minimum on a non-empty, closed, and bounded set. Given that the objective function is continuous and the feasible region is non-empty, closed, and bounded, the optimization problem satisfies the conditions of the Weierstrass Theorem. Therefore, the model is guaranteed to have at least one optimal solution.

Furthermore, a cloud gathering the normalized cloud-FLE element can be obtained by Definition 10.

Definition 10. If there are n elements in a normalized cloud-FLE $\tilde{m}^f = \{(\tilde{C}^1, \tilde{p}^1), (\tilde{C}^2, \tilde{p}^2), ..., (\tilde{C}^T, \tilde{p}^T)\}$, then a comprehensive cloud $\tilde{C} = (\widetilde{Ex}, \widetilde{En}, \widetilde{He})$ is calculated as follows:

$$\begin{cases} \widetilde{Ex} = \sum_{t=1}^{T} \widetilde{Ex}^t * \tilde{p}_{ij}^t \\ \widetilde{En} = \sqrt{\sum_{t=1}^{T} (\widetilde{En}^t)^2 * \tilde{p}^t} \\ \widetilde{He} = \sqrt{\sum_{t=1}^{T} (\widetilde{He}^t)^2 * \tilde{p}^t} \end{cases}. \tag{11}$$

Based on the above methods, an FLE m_L can be converted into a comprehensive cloud $\tilde{C} = (\widetilde{Ex}, \widetilde{En}, \widetilde{He})$ using the transformation model which is shown in Algorithm 1. Furthermore, each FLE matrix $M = (m_{ij})_{n\times m}$ can be converted into a comprehensive cloud matrixes $R = (\tilde{C}_{ij})_{n\times m}$ using the programming model in Eq. (12), where $\tilde{C}_{ij} = (\widetilde{Ex}_{ij}, \widetilde{En}_{ij}, \widetilde{He}_{ij})$. Obviously, the programming model in Eq. (12), which incorporates the aggregation formula Eq. (11), is guaranteed to have at least one optimal solution because the original programming model in Eq. (10) already satisfies the conditions for the existence of an optimal solution. The inclusion of the aggregation formula does not alter the continuity of the objective function or the compactness, i.e., non-emptiness, closedness, and boundedness of the feasible region, as it is formulated in a manner consistent with the original constraints. Thus, the programming model in Eq. (12) inherits the existence of an optimal solution from the original model.

$$\min \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{T} d(\tilde{C}_{ij}^t, C^t) * \tilde{p}_{ij}^t + \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{T} \left|\tilde{p}_{ij}^t - p_{ij}^t\right|}{mn}$$
$$s.t. \begin{cases} \sum_{t=1}^{T} \tilde{p}_{ij}^t = 1 \\ 0 \le \tilde{p}_{ij}^t \le 1 \\ d(\tilde{C}_{ij}^t, C_{ij}^t) = \sqrt{\frac{1}{2}((Ex_{ij}^t - \widetilde{Ex}_{ij}^t)^2 + (En_{ij}^t - \widetilde{En}_{ij}^t)^2 + (He_{ij}^t - \widetilde{He}_{ij}^t)^2)} \\ \widetilde{Ex}_{ij}^t - 3\widetilde{En}_{ij}^t - 9\widetilde{He}_{ij}^t > 0 \\ \widetilde{En}_{ij}^t \ge 3\widetilde{He}_{ij}^t \\ \widetilde{Ex}_{ij}^t - 3\widetilde{En}_{ij}^t \ge U^L \\ \widetilde{Ex}_{ij}^t + 3\widetilde{En}_{ij}^t \le U^U \\ \widetilde{En}_{ij}^t \ge 0, \widetilde{En}_{ij}^t \ge 0, \widetilde{He}_{ij}^t \ge 0 \\ \widetilde{Ex}_{ij} = \sum_{t=1}^{T} \widetilde{Ex}_{ij}^t * \tilde{p}_{ij}^t \\ \widetilde{En}_{ij} = \sqrt{\sum_{t=1}^{T} (\widetilde{En}_{ij}^t)^2 * \tilde{p}_{ij}^t} \\ \widetilde{He}_{ij} = \sqrt{\sum_{t=1}^{T} (\widetilde{He}_{ij}^t)^2 * \tilde{p}_{ij}^t} \\ i = 1, 2, ..., n; j = 1, 2, ..., m. \end{cases} \tag{12}$$

Algorithm 1. The algorithm of transformation between FLE and cloud model

Input: A fixed linguistic term set $L = \{l_0, l_1, ..., l_g\}$ and the FLE $m_L = \{(s_L^1, p^1), (s_L^2, p^2), ..., (s_L^T, p^T)\}$.
Output: The comprehensive cloud $\tilde{C} = (\widetilde{Ex}, \widetilde{En}, \widetilde{He})$.
Step 1. Transform the linguistic terms into the basic clouds. Compute $\delta_i = H(l_i)(i = 0, 1, ..., g)$ using Eq. (1), and then compute expectation Ex_i, entropy En_i and hyper entropy He_i to obtain g asymmetric basic clouds $C_i = (Ex_i, En_i, He_i)$.
Step 2. Transform the FLE into a cloud-FLE. For the FLE $m_L = \{(s_L^1, p^1), (s_L^2, p^2), ..., (s_L^T, p^T)\}$, linguistic variables in $s_L^t(t = 1, 2, ..., T)$ can be replaced by their corresponding basic clouds. Therefore, a cloud-FLE $m'_L = \{(C_L^1, p^1), (C_L^2, p^2), ..., (C_L^T, p^T)\}$ can be obtained, where C_L^t is the set of basic clouds corresponding to linguistic term set s_L^t.
Step 3. Obtain the floating cloud-FLEs by aggregating clouds in the cloud-FLE. The cloud-FLE $m'_L = \{(C_L^1, p^1), (C_L^2, p^2), ..., (C_L^T, p^T)\}$ can be converted into a floating-cloud-FLE $m^f = \{(C^1, p^1), (C^2, p^2), ..., (C^T, p^T)\}$ using Eq. (9).
Step 4. Compute the normalized cloud-FLE. Based on the floating cloud-FLEs, solve the programming model in Eq. (10) to obtain the normalized cloud-FLE $\tilde{m}^f = \{(\tilde{C}^1, \tilde{p}^1), (\tilde{C}^2, \tilde{p}^2), ..., (\tilde{C}^T, \tilde{p}^T)\}$.
Step 5. Aggregate all elements in the normalized cloud-FLE. For a normalized cloud-FLE $\tilde{m}^f = \{(\tilde{C}^1, \tilde{p}^1), (\tilde{C}^2, \tilde{p}^2), ..., (\tilde{C}^T, \tilde{p}^T)\}$, aggregate T elements using the weighted average (WA) operators in Eq. (11) or (12) to obtain a comprehensive cloud $\tilde{C} = (\widetilde{Ex}, \widetilde{En}, \widetilde{He})$.

4.3. Determine weights of criteria using Shannon entropy method

For the sake of calculation, clouds can be converted into other forms like interval numbers. Zhou et al. [48] proposed a transformation method between cloud model and interval number $[\underline{C}, \overline{C}]$. Therefore, a comprehensive cloud $\tilde{C} = (\widetilde{Ex}, \widetilde{En}, \widetilde{He})$ can be converted into interval number $[\underline{C}, \overline{C}]$, which is calculated as follows:

$$\begin{cases} \overline{C} = \widetilde{Ex} + 3(\widetilde{En} + 3\widetilde{He}) \\ \underline{C} = \widetilde{Ex} - 3(\widetilde{En} + 3\widetilde{He}) \end{cases}. \tag{13}$$

The evaluation matrix provided by expert e_k using FLEs is $M^k = (m_{ij}^k)_{n\times m}$, and the comprehensive cloud matrix $R^k = (\tilde{C}_{ij}^k)_{n\times m}$ where $\tilde{C}_{ij}^k = (\widetilde{Ex}_{ij}^k, \widetilde{En}_{ij}^k, \widetilde{He}_{ij}^k)$ can be obtained by Algorithm 1. The comprehensive cloud matrix $R^k(k = 1, 2, ..., K)$ can be transformed into interval-cloud matrix $RI^k = (r_{ij}^k)_{n\times m}$, where $r_{ij}^k = [\underline{C}_{ij}^k, \overline{C}_{ij}^k]$ using Eq. (13). Based on the weighted average (WA) operator, the collective interval-cloud matrix is $RI^c = (r_{ij})_{n\times m}$, where $r_{ij} = [\underline{C}_{ij}, \overline{C}_{ij}]$, $\underline{C}_{ij} = \sum_{k=1}^K v_k \underline{C}_{ij}^k$ and $\overline{C}_{ij} = \sum_{k=1}^K v_k \overline{C}_{ij}^k$. The weights of criteria can be determined by the Shannon entropy method [49, 50] which are as follows:

Step 1. Normalize the collective interval-cloud matrix by calculating $\underline{h}_{ij}$ and $\overline{h}_{ij}$.

$$\underline{h}_{ij} = \frac{\underline{C}_{ij}}{\sum_{i=1}^n \underline{C}_{ij}}, \overline{h}_{ij} = \frac{\overline{C}_{ij}}{\sum_{i=1}^n \overline{C}_{ij}},\ i = 1, 2, ..., n;\ j = 1, 2, ..., m. \tag{14}$$

Step 2. Calculate the lower entropy $\underline{g}_j$ of $\underline{h}_{ij}$ and the upper entropy $\overline{g}_j$ of $\overline{h}_{ij}$.

$$\begin{aligned} \underline{g}_j &= -\frac{1}{\ln n} \sum_{i=1}^n \underline{h}_{ij} \ln \underline{h}_{ij}, j = 1, 2, ..., m, \\ \overline{g}_j &= -\frac{1}{\ln n} \sum_{i=1}^n \overline{h}_{ij} \ln \overline{h}_{ij}, j = 1, 2, ..., m. \end{aligned} \tag{15}$$

Step 3. Obtain the downward limit $\underline{W}_j$ and upward limit $\overline{W}_j$ of criteria weights.

$$\underline{W}_j = \frac{1 - \underline{g}_j}{\sum_{j=1}^{m}(1 - \underline{g}_j)}, \overline{W}_j = \frac{1 - \overline{g}_j}{\sum_{j=1}^{m}(1 - \overline{g}_j)}, j = 1, 2, ..., m. \tag{16}$$

Step 4. Calculate the average weight W_j of criterion b_j.

$$W_j = \frac{\overline{W}_j + \underline{W}_j}{\sum_{j=1}^{m}(\overline{W}_j + \underline{W}_j)}, j = 1, 2, ..., m. \tag{17}$$

Based on the above methods, the weight set $W = \{W_1, W_2, ..., W_m\}$ can be obtained by the Shannon entropy method based on all evaluation matrixes using FLEs.

4.4. The ranking method based on multi-granularity cloud-rough set over two universes for EI project evaluation

We call five-tuple (X, E, F, R, B) a multiple decision-making cloud information system over two universes, where $X = \{x_1, x_2, ..., x_n\}$ is the set of EI projects, $E = \{e_1, e_2, ..., e_K\}$ is the expert set and $B = \{b_1, b_2, ..., b_m\}$ is the criteria set. $F = \{f^1, f^2, ..., f^m\}$ is a family of mapping set between X and E. For the criterion $b_j \in B$, the comprehensive cloud evaluation value provided by expert $e_k \in E$ over the EI project x_i can be mapped as f^j: $X \times E \to \Gamma^j$, where Γ^j is the range of cloud evaluation value. $R = \{R^1, R^2, ..., R^m\}$ is the comprehensive cloud evaluation matrix set provided by expert set $E = \{e_1, e_2, ..., e_K\}$, and $R^j = (C^j_{ik})_{n\times K}$ is the comprehensive cloud evaluation matrix over the EI project x_i under criterion b_j, which can be obtained using Algorithm 1. Based on the definition of multiple decision-making cloud information systems over two universes, the optimistic MGCRS over two universes and pessimistic MGCRS over two universes are defined as follows, respectively.

Definition 11. Let (X, E, F, R, B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j = 1, 2, ..., m)$ is the binary cloud relation between universe X and E. For any $A \in F(E)$, $e \in E$ and $x \in X$, the optimistic multi-granularity lower approximation $\underline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x)$ and upper approximation $\overline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x)$ of A with respect to (X, E, F, R, B) are as follows:

$$\begin{aligned} \underline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x) &= \bigvee_{j=1}^{m} \bigwedge_{e\in E} \max(N(R^j(x, e)), A(e)), x \in X, \\ \overline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x) &= \bigwedge_{j=1}^{m} \bigvee_{e\in E} \min(R^j(x, e), A(e)), x \in X, \end{aligned} \tag{18}$$

where $\vee$ and $\wedge$ are the maximum operator and minimum operator, respectively. The binary cloud relation under criterion b_j is $R^j(x, e) = (Ex_{R^j}(x, e), En_{R^j}(x, e), He_{R^j}(x, e))$ and $N(R^j(x, e))$ is the negation operator of $R^j(x, e)$. The interval set $(\underline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x), \overline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x))$ is called optimistic MGCRS over two universes if $\underline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x) \neq \overline{R}^{O}_{\sum_{j=1}^{m} R^j}(A)(x)$.

Remark 2. Let $Ex_{R^j}(x, e)$ and $N(Ex_{R^j}(x, e))$ be symmetric about the middle point in the effective domain $U = [U^L, U^U]$, then $N(Ex_{R^j}(x, e)) = U^U + U^L - Ex_{R^j}(x, e)$. Since entropy En and hyper entropy He reflect the uncertainty degree, let the negations of entropy and hyper entropy be $N(En_{R^j}(x,e)) = En_{R^j}(x,e)$ and $N(He_{R^j}(x,e)) = He_{R^j}(x,e)$. For a cloud $C = (Ex, En, He)$, large Ex, small En, and small He are expected numerical features. Therefore, the negation of $\overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max(N(En_{R^j}(x,e)), En_A(e))$ is $\overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(En_{R^j}(x,e), En_A(e))$, and the negation of $\overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max(N(He_{R^j}(x,e)), He_A(e))$ is $\overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(He_{R^j}(x,e), He_A(e))$. The upper approximation $\overline{R}^O_{\sum_{j=1}^m R^j}(A)(x)$ can be calculated in the same way.

Therefore, Eq. (18) can be rewritten as follows:

$$\begin{aligned} \underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) &= \{x|x \in X, \left\langle x, \underline{Ex}^O_A(x), \underline{En}^O_A(x), \underline{He}^O_A(x)\right\rangle\}, \\ \overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) &= \{x|x \in X, \left\langle x, \overline{Ex}^O_A(x), \overline{En}^O_A(x), \overline{He}^O_A(x)\right\rangle\}, \end{aligned} \tag{19}$$

where $\underline{Ex}^O_A(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_A(e))$, $\underline{En}^O_A(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(En_{R^j}(x,e), En_A(e))$ and $\underline{He}^O_A(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(He_{R^j}(x,e), He_A(e))$ in the lower approximation $\underline{R}^O_{\sum_{j=1}^m R^j}(A)(x)$. And $\overline{Ex}^O_A(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min (Ex_{R^j}(x,e), Ex_A(e))$, $\overline{En}^O_A(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max(En_{R^j}(x,e), En_A(e))$ and $\overline{He}^O_A(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max(He_{R^j}(x,e), He_A(e))$ in the upper approximation $\overline{R}^O_{\sum_{j=1}^m R^j}(A)(x)$.

Definition 12. Let (X, E, F, R, B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j = 1, 2, ..., m)$ is the binary cloud relation between universe X and E. For any $A \in F(E)$, $e \in E$ and $x \in X$, the pessimistic multi-granularity lower approximation $\underline{R}^P_{\sum_{j=1}^m R^j}(A)(x)$ and upper approximation $\overline{R}^P_{\sum_{j=1}^m R^j}(A)(x)$ of A with respect to (X, E, F, R, B) are as follows:

$$\begin{aligned} \underline{R}^P_{\sum_{j=1}^m R^j}(A)(x) &= \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\wedge} \max(N(R^j(x,e)), A(e)), x \in X, \\ \overline{R}^P_{\sum_{j=1}^m R^j}(A)(x) &= \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\vee} \min(R^j(x,e), A(e)), x \in X, \end{aligned} \tag{20}$$

where $R^j(x,e) = (Ex_{R^j}(x,e), En_{R^j}(x,e), He_{R^j}(x,e))$ and $N(R^j(x,e))$ is the negation operator of $R^j(x,e)$. The interval set $(\underline{R}^P_{\sum_{j=1}^m R^j}(A)(x), \overline{R}^P_{\sum_{j=1}^m R^j}(A)(x))$ is called pessimistic MGCRS over two universes if $\underline{R}^P_{\sum_{j=1}^m R^j}(A)(x) \neq \overline{R}^P_{\sum_{j=1}^m R^j}(A)(x)$.

Similarly, Eq. (20) can be rewritten as follows:

$$\begin{aligned} \underline{R}^P_{\sum_{j=1}^m R^j}(A)(x) &= \{x|x \in X, \left\langle x, \underline{Ex}^P_A(x), \underline{En}^P_A(x), \underline{He}^P_A(x)\right\rangle\}, \\ \overline{R}^P_{\sum_{j=1}^m R^j}(A)(x) &= \{x|x \in X, \left\langle x, \overline{Ex}^P_A(x), \overline{En}^P_A(x), \overline{He}^P_A(x)\right\rangle\}, \end{aligned} \tag{21}$$

where $\underline{Ex}^P_A(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_A(e))$, $\underline{En}^P_A(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\vee} \min(En_{R^j}(x,e), En_A(e))$ and $\underline{He}^P_A(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\vee} \min((He_{R^j}(x,e), He_A(e))$ in the lower approximation $\underline{R}^P_{\sum_{j=1}^m R^j}(A)(x)$. And $\overline{Ex}^P_A(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\vee} \min (Ex_{R^j}(x,e), Ex_A(e))$, $\overline{En}^P_A(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\wedge} \max(En_{R^j}(x,e), En_A(e))$ and $\overline{He}^P_A(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\wedge} \max(He_{R^j}(x,e), He_A(e))$ in the upper approximation $\overline{R}^P_{\sum_{j=1}^m R^j}(A)(x)$.

Due to space constraints, the relevant theorems and proofs for the optimistic and pessimistic MGCRSs can be found in Appendix A. Based on the optimistic and pessimistic MGCRSs, the comprehensive MGCRS is defined as follows.

Definition 13. Let (X, E, F, R, B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j = 1, 2, ..., m)$ is the binary cloud relation between universe X and E. For any $A \in F(E)$, $e \in E$ and $x \in X$, the comprehensive multi-granularity lower approximation $\underline{R}_{\sum_{j=1}^m R^j}(A)(x_i)$ and upper approximation $\overline{R}_{\sum_{j=1}^m R^j}(A)(x_i)$ of A with respect to (X, E, F, R, B) are as follows:

$$\begin{aligned} \underline{R}_{\sum_{j=1}^m R^j}(A)(x_i) &= \sum_{j=1}^{m} W_j \bigwedge_{e \in E} \max(N(R^j(x_i, e_k)), A(e_k)), x_i \in X, \\ \overline{R}_{\sum_{j=1}^m R^j}(A)(x_i) &= \sum_{j=1}^{m} W_j \bigvee_{e \in E} \min(R^j(x_i, e_k), A(e_k)), x_i \in X. \end{aligned} \tag{22}$$

Eq. (22) can be rewritten as follows:

$$\begin{aligned} \underline{R}_{\sum_{j=1}^m R^j}(A)(x_i) &= \{x_i | x_i \in X, \left\langle x_i, \underline{Ex}_A(x_i), \underline{En}_A(x_i), \underline{He}_A(x_i) \right\rangle\}, \\ \overline{R}_{\sum_{j=1}^m R^j}(A)(x_i) &= \{x_i | x_i \in X, \left\langle x_i, \overline{Ex}_A(x_i), \overline{En}_A(x_i), \overline{He}_A(x_i) \right\rangle\}, \end{aligned} \tag{23}$$

where $\underline{Ex}_A(x_i) = \sum_{j=1}^{m} W_j \bigwedge_{e_k \in E} \max((U^U + U^L - Ex_{R^j}(x_i, e_k)), Ex_A(e_k))$, $\underline{En}_A(x_i) = \sqrt{\sum_{j=1}^{m} W_j (\bigvee_{e_k \in E} \min(Ex_{R^j}(x_i, e_k), En_A(e_k)))^2}$ and $\underline{He}_A(x_i) = \sqrt{\sum_{j=1}^{m} W_j (\bigvee_{e_k \in E} \min(He_{R^j}(x_i, e_k), He_A(e_k)))^2}$ in the lower approximation $\underline{R}_{\sum_{j=1}^m R^j}(A)(x)$. And $\overline{Ex}_A(x_i) = \sum_{j=1}^{m} W_j \bigvee_{e_k \in E} \min(Ex_{R^j}(x_i, e_k), Ex_A(e_k))$, $\overline{En}_A(x_i) = \sqrt{\sum_{j=1}^{m} W_j (\bigwedge_{e_k \in E} \max(En_{R^j}(x_i, e_k), En_A(e_k)))^2}$ and $\overline{He}_A(x_i) = \sqrt{\sum_{j=1}^{m} W_j (\bigwedge_{e_k \in E} \max(He_{R^j}(x_i, e_k), He_A(e_k)))^2}$ in the upper approximation $\overline{R}_{\sum_{j=1}^m R^j}(A)(x)$.

Therefore, the approximation evaluation value $R_{\sum_{j=1}^m R^j}(A)(x_i)$ of A for x_i using MGCRS over two universes is as follows:

$$R_{\sum_{j=1}^m R^j}(A)(x_i) = \theta \overline{R}_{\sum_{j=1}^m R^j}(A)(x_i) + (1 - \theta) \underline{R}_{\sum_{j=1}^m R^j}(A)(x_i), \tag{24}$$

where θ is the preference coefficient and $\theta \in [0, 1]$.

The reference cloud $A = (Ex_A(E), En_A(E), He_A(E))$ where $E = \{e_1, e_2, ..., e_K\}$ can be determined by aggregating all cloud matrixes $R^j(j = 1, 2, ..., m)$ under criterion b_j. The positive ideal solution (PIS) and the negative ideal solution (NIS) under criterion b_j are defined as follows:

$$\begin{aligned} C^{j+} &= \{C_1^{j+}, C_2^{j+}, ..., C_K^{j+}\}, j = 1, 2, ..., m, \\ C^{j-} &= \{C_1^{j-}, C_2^{j-}, ..., C_K^{j-}\}, j = 1, 2, ..., m, \end{aligned} \tag{25}$$

where $C_k^{j+} = \max\{C_k^j | k = 1, 2, ..., K\}$ and $C_k^{j-} = \min\{C_k^j | k = 1, 2, ..., K\}$. The jth reference cloud $C^j =$

$\{C_1^j, C_2^j, ..., C_K^j\}$ under criterion $b_j (j = 1, 2, ..., m)$ can be calculated as follows:

$$C^j = \gamma C^{j+} + (1 - \gamma) C^{j-}, j = 1, 2, ..., m, \tag{26}$$

where γ is the risk preference coefficient and $\gamma \in [0, 1]$.

Therefore, the reference cloud $A = (Ex_A(E), En_A(E), He_A(E))$ is obtained by $A = \sum_{j=1}^{m} W_j C^j$, which is as follows:

$$\begin{cases} Ex_A(E) = \sum_{j=1}^{m} W_j Ex^j \\ En_A(E) = \sqrt{\sum_{j=1}^{m} W_j (En^j)^2} \\ He_A(E) = \sqrt{\sum_{j=1}^{m} W_j (He^j)^2} \end{cases}. \tag{27}$$

The ranking method for EI projects using MGCRS over two universes is shown in Algorithm 2.

Algorithm 2. The ranking method for EI projects using MGCRS over two universes

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Input: The cloud evaluation matrixes $R^j = (C_{ik}^j)_{n \times K}$ where $r_{ik}^j = (Ex_{ik}^j, En_{ik}^j, He_{ik}^j)$, weight set $W = \{W_1, W_2, ..., W_m\}$, preference coefficient θ, risk preference coefficient γ and the effective domain $U = [U^L, U^U]$.
Output: The approximation evaluation value $R_{\sum_{j=1}^{m} R^j}(A)(x_i)$ with respect to x_i and the ranking of EI projects.

Step 1. Determine the reference cloud A. For $R^j = (C_{ik}^j)_{n \times K}$, determine the PIS C^{j+} and NIS C^{j-} using Eq. (25) and obtain the jth reference cloud $C^j = \{C_1^j, C_2^j, ..., C_K^j\}$ using Eq. (26). The reference cloud A can be calculated by aggregating $C^j (j = 1, 2, ..., m)$ using Eq. (27).

Step 2. Obtain the multi-granularity lower and upper approximation of EI projects. Compute the comprehensive multi-granularity lower approximation $\overline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i)$ and upper approximation $\underline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i)$ of with respect to x_i by Eqs. (22) and (23).

Step 3. Obtain the approximation evaluation value of EI projects. Compute the approximation evaluation value $R_{\sum_{j=1}^{m} R^j}(A)(x_i)$ of A with respect to x_i using Eq. (24).

Step 4. Obtain the ranking of EI projects. Determine the final ranking by ranking the approximation evaluation values $R_{\sum_{j=1}^{m} R^j}(A)(x_i)$ of all EI projects in descending order.

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5. Case study: the EI project evaluation in the Beijing-Tianjin-Hebei region

5.1. Problem description for EI project evaluation in the Beijing-Tianjin-Hebei region

China's Beijing-Tianjin-Hebei (BTH) region is actively promoting the construction of the green EI based on the principles of the circular economy, aiming to facilitate the circular energy transformation of the traditional power grid industrial system. This transformation focuses on the recycling of resources and sustainable development, enabling the efficient and green operation of energy systems. As a key economic development zone and a renewable energy demonstration area in China, the BTH region faces the dual challenges of upgrading its industrial system and protecting the environment. In June 2024, the BTH Energy Collaboration Task Force was officially established, and a key work plan was formulated. The objectives include promoting the interconnection of energy infrastructure across provinces and regions, cultivating a green, low-carbon energy consumption model, constructing a diversified energy supply system, and driving the innovation and application of key energy technologies. To support this transition, four representative EI projects were selected for evaluation. These projects cover solar power generation, wind power generation, multi-energy integration system optimization, and smart microgrid management, which are the Beijing Haidian North EI Project, the Zhangbei "Internet+ Smart Energy" Wind Power Demonstration Project, the Tianjin Binhai Smart Energy Demonstration Project, and the Xiong'an New Area Green Smart Microgrid Demonstration Project. The aim of evaluating these EI projects is to provide important references for the green energy transition in the BTH region and help achieve its green and low-carbon development goals. Against this backdrop, four experts from universities, the Power Grid Federation, State Grid Corporation, and other institutions will conduct a comprehensive evaluation and analysis of these four EI projects based on ten evaluation criteria. The four EI demonstration projects are represented by the EI project set $X = \{x_1, x_2, x_3, x_4\}$. The four experts are proficient in the fields of technology, management, environmental protection, and policy in the field of EI, and have strong strategic thinking and decision-making consulting ability for the development of EI. Let $E = \{e_1, e_2, e_3, e_4\}$ be the set of experts, and $V = \{v_1, v_2, v_3, v_4\}$ is the set of experts' weights, where four experts are given exactly equal weight. $B = \{B_1, B_2, B_3\}$ is the set of criteria where $B_1 = \{b_{11}, b_{12}, b_{13}, b_{14}\}$, $B_2 = \{b_{21}, b_{22}, b_{23}\}$ and $B_3 = \{b_{31}, b_{32}, b_{33}\}$ in Table 2. The evaluation matrixes provided by expert e_k using FLEs are $M^k = (m^k_{ij})_{n\times m}$, which are shown in Table B.1 (See Appendix B). Given the domain $U = [0, 10]$, the linguistic evaluation set is $L = \{l_0 : \mathit{very\ poor}, l_1 : \mathit{poor}, l_2 : \mathit{fair}, l_3 : \mathit{good}, l_4 : \mathit{very\ good}\}$. The relative parameters are assumed that θ=0.5 and γ=0.5.

5.2. The decision process

In this section, the proposed MCDM framework is used to evaluate several EI demonstration projects. The decision steps are demonstrated as follows.

Step 1. For the domain $U = [0, 10]$ and the fixed linguistic set $L = \{l_0 : \mathit{very\ poor}, l_1 : \mathit{poor}, l_2 : \mathit{fair}, l_3 : \mathit{good}, l_4 : \mathit{very\ good}\}$, the basic clouds C_i $(i = 0, 1, ..., 4)$ can be calculated using the transforming method shown in Example 1: $C_0 = (0, 2.8518, 0.3950)$, $C_1 = (2.8892, 2.4568, 0.2922)$, $C_2 = (5, 2.1357, 0.3992)$, $C_3 = (7.1108, 2.4568, 0.2922)$, $C_4 = (10, 2.8518, 0.3950)$. Therefore, the FLEs evaluation matrixes $M^k = (m^k_{ij})_{n\times m}$ in Table B.1 (See Appendix B) can be transformed into the cloud-FlE matrixes $M'^k = (m'^k_{ij})_{n\times m}$ by replacing linguistic term l_i with C_i.

Step 2. Each cloud-FLE m'^k_{ij} in cloud-FLE matrixes M'^k can be transformed into a floating-cloud-FlE $m^f_{ij} = \{(C^1_{ij}, p^1_{ij}), (C^2_{ij}, p^2_{ij}), ..., (C^T_{ij}, p^T_{ij})\}$, Therefore, the floating-cloud-FLE matrixes $M^{fk} = (m^{fk}_{ij})_{n\times m}$ are obtained.

Step 3. The cloud evaluation matrixes can be obtained using the programming model in Eq. (12) $R^k = (\tilde{C}^k_{ij})_{n\times m}$ are shown in Tables B.2-B.4 (See Appendix B).

Step 4. To determine the weights of 10 criteria, the cloud evaluation matrixes R^k are aggregating into a collective interval-cloud matrix $RI^c = (r_{ij})_{n\times m}$ in Table 3, and the weights of criteria are obtained by Shannon entropy method in Table 4.

Step 5. To rank these EI demonstration projects, we rewrite the cloud evaluation matrixes $R^k = (\tilde{C}^k_{ij})_{n\times m}$

Table 3. The collective interval-cloud matrix RI^c.

RI^c	b_{11}	b_{12}	b_{13}	b_{14}	b_{21}
x_1	[1.4836,11.5958]	[1.9801,12.9266]	[1.8983,11.7933]	[1.6833,10.9297]	[0.6927,10.7101]
x_2	[1.9044,11.0021]	[1.0387,12.7268]	[0.8694,10.6341]	[1.7942,10.9980]	[1.0403,10.9977]
x_3	[1.8147,10.4929]	[0.8492,11.1752]	[1.8525,11.5137]	[1.5980,12.2486]	[1.7216,10.9731]
x_4	[0.8004,11.1575]	[0.6544,12.1694]	[0.5135,10.7098]	[1.3608,11.7684]	[0.3003,12.0447]
	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}
x_1	[0.0702,11.4337]	[0.8755,11.0035]	[1.3222,11.1971]	[1.5835,11.3499]	[0.4760,10.9631]
x_2	[1.8908,11.5570]	[1.8207,11.7525]	[0.3386,11.2949]	[0.9073,11.0291]	[0.8684,11.5762]
x_3	[1.5049,11.4905]	[1.2635,10.7625]	[1.3292,12.0433]	[1.2775,11.1359]	[1.5842,11.3049]
x_4	[0.8082,11.1398]	[1.0702,11.2476]	[1.4845,10.8348]	[0.4088,11.3638]	[0.5489,11.2885]

Table 4. The weights of criteria.

Criteria	b_{11}	b_{12}	b_{13}	b_{14}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}
Weight	0.0673	0.1551	0.1284	0.0837	0.1488	0.1375	0.0560	0.1024	0.0517	0.0691

as $R^j = (\tilde{C}^j_{ik})_{n\times K}$. We can determine the PIS and NIS to obtain the jth reference cloud $C^j = \{C^j_1, C^j_2, ..., C^j_K\}$ in Table 5. The reference cloud A is shown in Table 6.

Step 6. The comprehensive multi-granularity lower approximation $\underline{R}_{\sum_{j=1}^m R^j}(A)(x_i)$ and upper approximation $\overline{R}_{\sum_{j=1}^m R^j}(A)(x_i)$ of A can be calculated as follows:

$$\underline{R}_{\sum_{j=1}^m R^j}(A)(x_1) = (6.1895, 1.1633, 0.2106), \quad \underline{R}_{\sum_{j=1}^m R^j}(A)(x_2) = (6.1895, 1.1775, 0.2099),$$
$$\underline{R}_{\sum_{j=1}^m R^j}(A)(x_3) = (6.1936, 1.1398, 0.2092), \quad \underline{R}_{\sum_{j=1}^m R^j}(A)(x_4) = (6.1895, 1.1902, 0.2111);$$
$$\overline{R}_{\sum_{j=1}^m R^j}(A)(x_1) = (6.1903, 1.1008, 0.2070), \quad \overline{R}_{\sum_{j=1}^m R^j}(A)(x_2) = (6.2601, 1.0932, 0.2071),$$
$$\overline{R}_{\sum_{j=1}^m R^j}(A)(x_3) = (6.2682, 1.1021, 0.2056), \quad \overline{R}_{\sum_{j=1}^m R^j}(A)(x_4) = (6.2520, 1.1249, 0.2040).$$

Table 5. The reference cloud $C^j = \{C^j_1, C^j_2, ..., C^j_K\}$.

C^j	e_1	e_2	e_3	e_4
C^1	(6.8995,0.8928,0.2233)	(6.4304,0.8788,0.1922)	(6.9237,0.8935,0.2247)	(4.6237,1.1250,0.1773)
C^2	(6.1120,1.9820,0.2606)	(6.8995,0.8928,0.2233)	(6.3618,1.0629,0.179)	(6.3880,1.8607,0.2694)
C^3	(6.2483,0.9755,0.2240)	(5.4142,1.0856,0.1468)	(5.8581,0.8637,0.1467)	(6.6236,0.9644,0.2127)
C^4	(5.4142,1.0856,0.1468)	(6.6236,0.9644,0.2127)	(6.2483,0.9755,0.2240)	(6.8057,0.8734,0.1789)
C^5	(5.9653,1.1751,0.1589)	(5.5752,1.3359,0.2278)	(6.9237,0.8935,0.2247)	(5.7710,0.9768,0.1692)
C^6	(6.9237,0.8935,0.2247)	(6.5242,0.8992,0.2341)	(5.5888,1.0880,0.2046)	(6.8995,0.8928,0.2233)
C^7	(6.2483,0.9755,0.2240)	(6.9237,0.8935,0.2247)	(5.9653,1.1751,0.1589)	(6.2483,0.9755,0.2240)
C^8	(5.7710,0.9768,0.1692)	(6.6236,0.9644,0.2127)	(6.9237,0.8935,0.2247)	(5.7832,1.2888,0.1961)
C^9	(6.6236,0.9644,0.2127)	(6.4304,0.8788,0.1922)	(6.2483,0.9755,0.2240)	(5.5888,1.0880,0.2046)
C^{10}	(5.7710,0.9768,0.1692)	(5.7832,1.2888,0.1961)	(5.7832,1.2888,0.1961)	(6.9237,0.8935,0.2247)

Table 6. The reference cloud A.

	e_1	e_2	e_3	e_4
A	(6.1895,1.2100,0.2071)	(6.2772,1.0374,0.2099)	(6.2922,1.0045,0.2011)	(6.2389,1.1818,0.2130)

Step 7. The approximation evaluation value $R_{\sum_{j=1}^m R^j}(A)(x_i)$ of A with respect to x_i are calculated using Eq. (24) as follows:

$$R_{\sum_{j=1}^m R^j}(A)(x_1) = (6.1899, 1.1325, 0.2088), \quad R_{\sum_{j=1}^m R^j}(A)(x_2) = (6.2248, 1.1362, 0.2085),$$
$$R_{\sum_{j=1}^m R^j}(A)(x_3) = (6.2309, 1.1211, 0.2074), \quad R_{\sum_{j=1}^m R^j}(A)(x_4) = (6.2207, 1.1580, 0.2076).$$

The clouds generated by four EI projects with 3000 cloud drops are shown in Fig. 5. Therefore, the ranking of four EI projects is: $x_3 \succ x_2 \succ x_4 \succ x_1$. Based on the decision results, the following policy recommendations are served as a reference for the future energy integration of the BTH region. The government can prioritize supporting multi-energy integration projects, such as the Tianjin Binhai Smart Energy Demonstration Project, which demonstrates the great potential of integrating wind energy, solar energy, and energy storage systems with smart grids, aligning with the principles of resource efficiency in the circular economy. This outcome is highly consistent with the strategic goal of promoting cross-provincial and cross-regional energy infrastructure interconnection in the BTH region, further validating the feasibility of multi-energy integration and connectivity. The government can promote the research, development, and application of energy storage and smart grid technologies through financial support and tax incentives, thereby improving system efficiency and flexibility and promoting the recycling of energy resources. Additionally, the success of the Zhangbei "Internet+ Smart Energy" wind power project indicates that policy should strengthen support for the wind energy industry and the integration of smart grid technologies, promoting the efficient integration of wind power into the grid and increasing utilization rates. The abundant wind energy resources in Hebei should be fully utilized to promote the sustainable development of the wind power industry and strengthen coordination between the smart grid and wind power for improved energy efficiency and system stability. From the experience of the Xiong'an New Area green smart microgrid project, policy should encourage the integration of green energy solutions with microgrid technologies, promoting the green and low-carbon development of the New Area and remote regions. Due to its unique geographical and policy advantages, Xiong'an New Area can serve as a demonstration area for green smart microgrids, utilizing solar and wind energy to promote zero-carbon energy supply systems, further advancing regional energy circular economy development. Although the Beijing Energy Haidian project ranks lower, it still provides valuable experience in the application of smart grid technologies. The government should continue to support the construction of smart grid infrastructure, especially in the renovation of old grids, to enhance renewable energy integration capabilities and promote the transformation of energy systems toward a circular economy model, achieving efficient use of resources.

5.3. Sensitivity analysis

The expectation Ex in the cloud model is the most representative numerical feature, which can reflect the average level of cloud drops. Therefore, the impact of preference coefficient θ and risk preference coefficient γ on the mathematical expectation Ex should be considered. For the EI projects evaluation, the mathematical expectation Ex of four EI projects varies with different preference coefficients and with different risk preference coefficients, which are as shown in Fig. 6.

In Fig. 6(a), it can be seen that the mathematical expectation Ex of four EI projects grows in a straight line with the preference coefficient θ when $\gamma = 0.5$. The expectation gap between the four EI projects becomes wider as the preference coefficient increases. The reason is that the increase of θ makes the proportion of the upper approximation bigger in the approximate evaluation value. The gap of the upper approximation is more

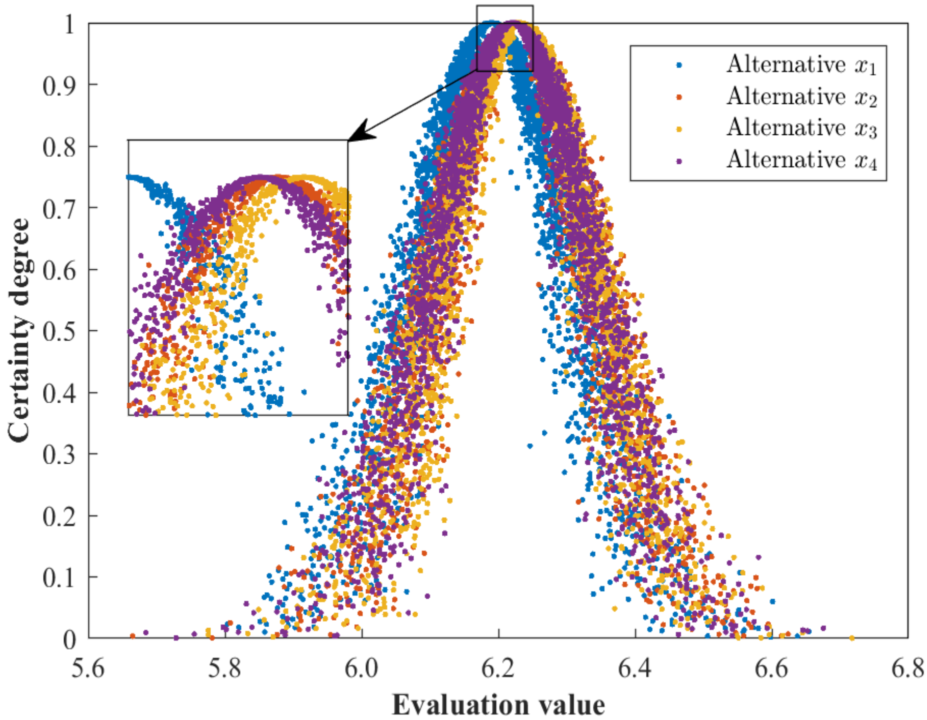
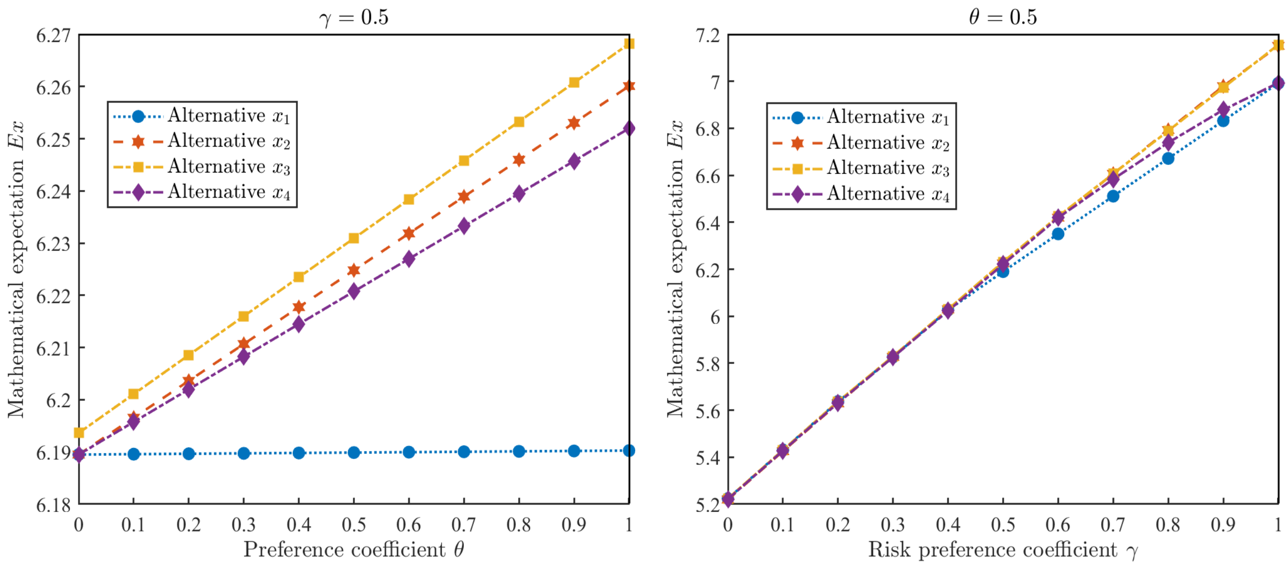


Fig. 5. The clouds generated by four EI projects with 3000 cloud drops.



(a) The results with different preference coefficients θ

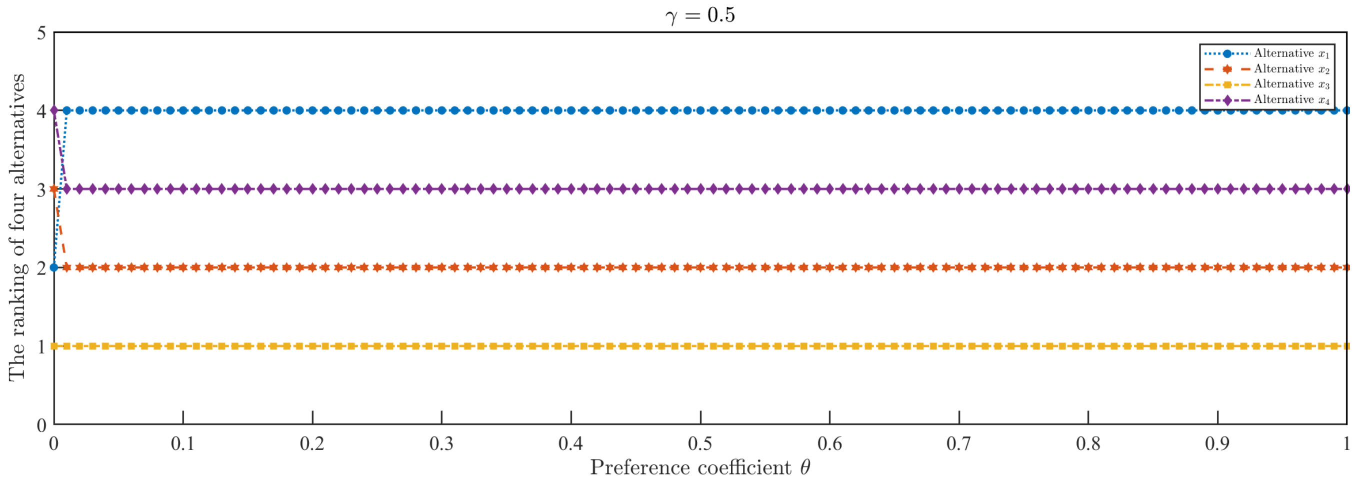
(b) The results with different risk preference coefficients γ

Fig. 6. The expectation Ex results with different coefficients.

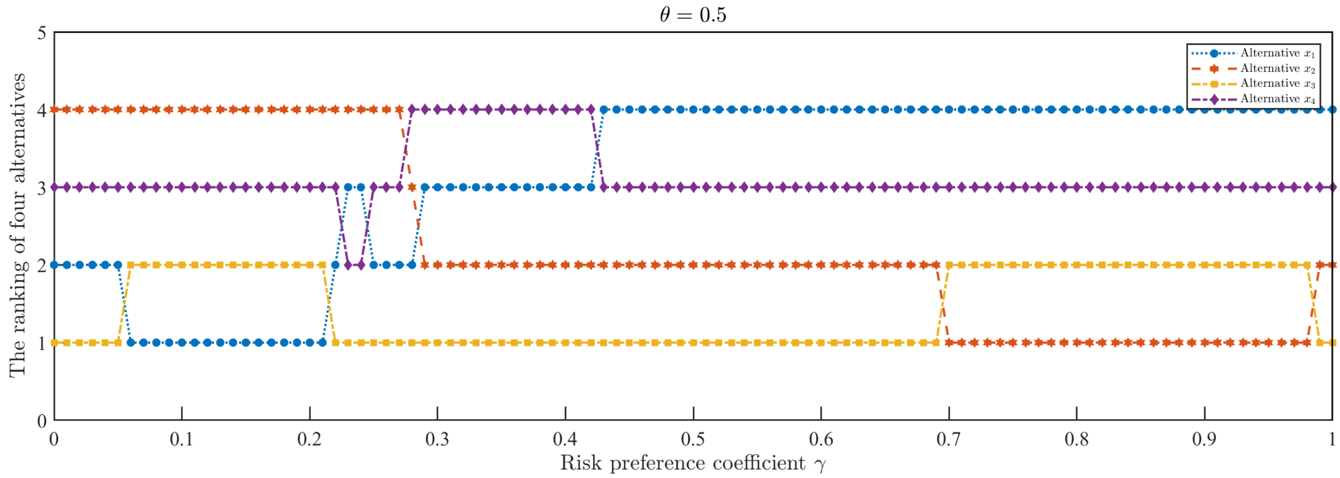
obvious than that of the lower approximation, which leads to the difference of the expectations Ex under four EI projects increasing with θ.

In Fig. 6(b), when the preference coefficient $\theta = 0.5$, the mathematical expectation Ex of four EI projects grows in a straight line with the risk preference coefficient γ. The gap of the four EI projects is not obvious when $\gamma \leq 0.4$ and begins to widen when $\gamma > 0.4$. The expectation curves of EI projects x_2 and x_3 coincide, which means that the difference between the two expectations is small. The gap between EI project x_1 and x_4 gradually increases and then decreases until the expectations of the two basically coincide when $\gamma = 1$.

The ranking results of four EI projects with different coefficients are shown in Fig. 7. Fig. 7(a) shows the ranking results with different preference coefficients when $\gamma = 0.5$, and it can be seen that the ranking of EI projects is always $x_3 \succ x_2 \succ x_4 \succ x_1$ when $\theta \neq 0$, and the ranking is $x_3 \succ x_1 \succ x_2 \succ x_4$ when $\theta = 0$. This is because the approximation evaluation value $R_{\sum_{j=1}^{m} R^j}(A)(x_i) = \underline{R}_{\sum_{j=1}^{m} R^j}(A)(x_i)$ when $\theta = 0$. Therefore, the ranking of EI project when $\theta = 0$ is consistent with the ranking of the comprehensive multi-granularity lower approximation. For different preference coefficients, the ranking of EI projects is always $x_3 \succ x_2 \succ x_4 \succ x_1$, which reflects that the change of preference coefficient has little effect on the ranking of EI projects. Fig. 7(b) shows the ranking results with different risk preference coefficients when $\theta = 0.5$. The ranking of EI projects is changing dynamically with the risk preference coefficient. EI project x_3 basically ranks first or second and EI project x_4 ranks third or fourth when $\theta \in [0, 1]$. Meanwhile, the ranking of EI project x_2 is gradually higher and that of EI project x_1 is gradually lower with the increase of the risk preference coefficient.



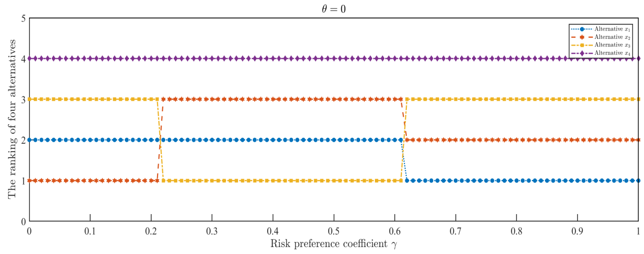
(a) The results with different preference coefficients θ



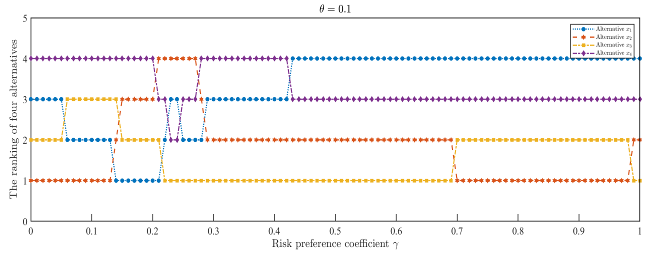
(b) The results with different risk preference coefficients γ

Fig. 7. The ranking results with different coefficients.

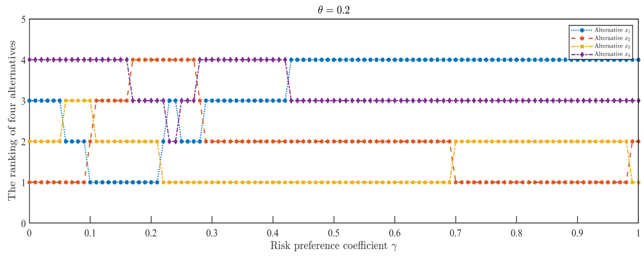
Based on the above analyses, the risk preference coefficient has a significant effect on the ranking of EI projects. To explore the impact under different preference coefficients, the ranking results when $\theta \in [0, 1]$ are shown in Fig. 8. It can be seen from Fig. 8(a) that there are two inflection points $\gamma_1 = 0.22$ and $\gamma_2 =$



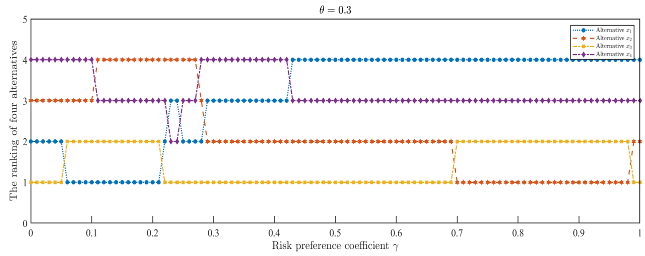
(a) The ranking results when $\theta = 0$



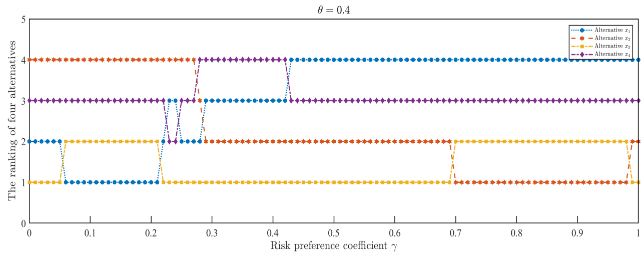
(b) The ranking results when $\theta = 0.1$



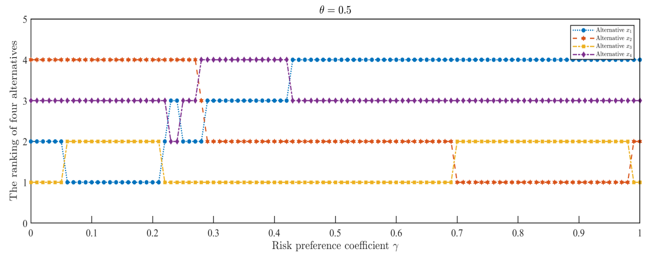
(c) The ranking results when $\theta = 0.2$



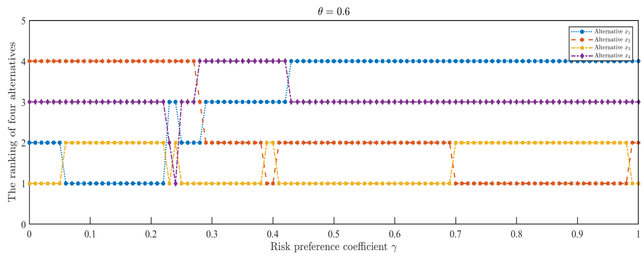
(d) The ranking results when $\theta = 0.3$



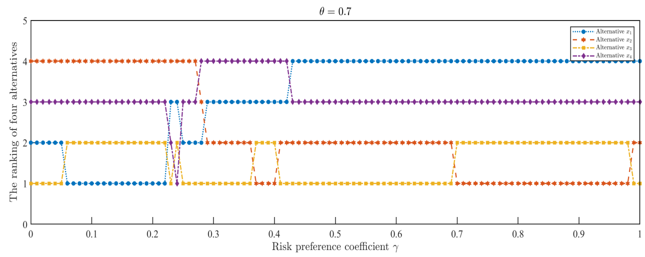
(e) The ranking results when $\theta = 0.4$



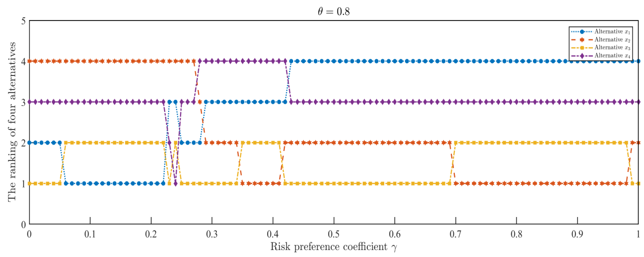
(f) The ranking results when $\theta = 0.5$



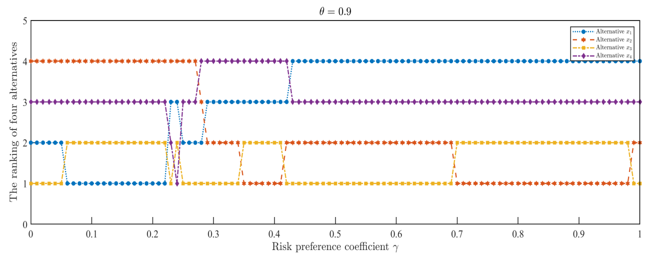
(g) The ranking results when $\theta = 0.6$



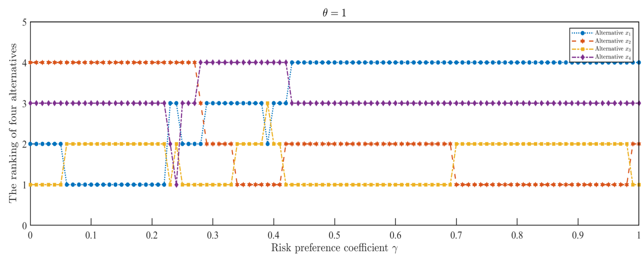
(h) The ranking results when $\theta = 0.7$



(i) The ranking results when $\theta = 0.8$



(j) The ranking results when $\theta = 0.9$



(k) The ranking results when $\theta = 1$

Fig. 8. The ranking results with different θ and γ.

0.62 affecting the ranking of EI projects when $\theta = 0$. For $\gamma < \gamma_1$, $\gamma_1 \leq \gamma \leq \gamma_2$ and $\gamma > \gamma_2$, the best EI project is x_2, x_3 and x_1, respectively. From Fig. 8(b)-(f), the ranking of EI projects is different when selecting different risk preference coefficients, in which $\gamma_3 = 0.43$ is a critical point affecting the ranking of x_1 and x_4, i.e., x_1 ranks fourth and x_4 ranks third when $\gamma > 0.43$. When $\gamma \leq 0.43$, the ranking of EI projects changes fluctuate greatly. When $\theta > 0.5$, a special point $\gamma_4 = 0.24$ appears when x_4 ranks first. In total, Fig. 8 (a)-(f) differ from each other to some extent, which reflects that the preference coefficient has an impact on the ranking of EI projects and some inflection points should be concerned. Therefore, determining an appropriate risk preference coefficient and preference coefficient is critical for ranking these EI projects. In practical applications, decision-makers can determine the appropriate range for the risk preference coefficient through extensive discussions or surveys, or estimate the range of risk preference coefficient using existing case data. Simultaneously, through multiple simulations and experiments, the risk preference coefficient can be gradually adjusted based on the results under different scenarios. This process can involve expert evaluations, historical data analysis, and practical experience related to the project, providing decision-making with more contextually relevant parameter guidance.

5.4. Comparative analyses

To further demonstrate the effectiveness of the proposed method, we conduct a comparison between our method and the other four MCDM methods for energy system evaluation, including Jiang et al. [24]'s method, Shang [28]'s method, Zhou et al. [39]'s method and Wu et al. [17]'s method. The ranking results of five methods for energy system evaluation are shown in Fig. 9 and Table 7.

It can be seen from Fig. 9 that the ranking of x_1 and x_2 is more stable than that of x_3 and x_4. The ranking of x_3 and x_4 is sensitive to parameters and MCDM methods, that is, using different parameters and MCDM methods can lead to two extreme states of the best or worst EI project. From Table 7, the ranking result of our proposal is not completely consistent with that of the other four methods. However, x_1 and x_2 rank second or third under five methods, and the ranking of x_3 and x_4 are always first or fourth. The ranking of x_1 and x_2 under our proposed method is consistent with that of Jiang et al.'s method, and the ranking of x_3 and x_4 under our proposed method is the same as that of Shang's method and Zhou et al.'s method. This reflects that there's basically no difference between our method and the other four methods, which demonstrates the effectiveness of our proposed method when determining extreme or intermediate EI projects. Based on the above analyses of preference coefficient θ, the ranking can also be exactly the same as the four methods by selecting the appropriate preference coefficient. Therefore, our proposed method not only has the effectiveness but also has a strong flexibility as the parameters change.

6. Discussion

This section examines the theoretical and management implications of the EI project evaluation framework in the context of circular economy practice, while also discussing the limitations of the proposal.

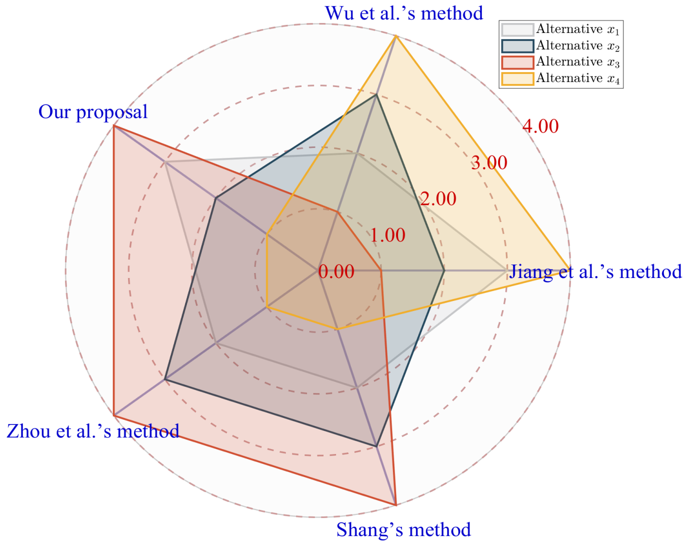


Fig. 9. The comparative results of five MCDM methods for energy system evaluation.

Table 7. The ranking results of five MCDM methods for energy system evaluation.

EI projects	Our proposal	Jiang et al. [24]'s method	Shang [28]'s method	Zhou et al. [39]'s method	Wu et al. [17]'s method
x_1	3	3	2	2	2
x_2	2	2	3	3	3
x_3	4	1	4	4	1
x_4	1	4	1	1	4

6.1. Theoretical implications

The proposed MCDM framework provides a substantial theoretical advance in the circular economy practice. First, a new method for transforming discrete FLEs into continuous cloud information enhances the management of uncertainty by transforming discrete data into a continuous format, allowing for a more nuanced and precise integration of assessment criteria. This theoretical advance contributes to a deeper understanding of how various EI projects impact sustainability, leading to a more accurate representation of their contribution to circular economy. Secondly, with the support of the Shannon entropy method, a tailored evaluation index system is developed to expand the theoretical framework for evaluating the different impacts of EI projects. By using this approach to assign appropriate weight to assessment criteria, the framework ensures that assessments reflect the multifaceted nature of green innovation and sustainability in the circular economy. This theoretical refinement improves the ability to capture the effectiveness of projects in improving energy efficiency and achieving sustainable development goals. Finally, the application of MGCRS and integrated multi-granularity approximations has introduced significant theoretical advances to the decision-making process. This approach provides a powerful mechanism for ranking and optimizing EI projects by managing ongoing cloud information. The use of optimistic and pessimistic MGCRSs in both areas can improve the accuracy and reliability of project evaluations. In theory, this innovation provides a more precise and practical basis for optimizing energy grid connections, thereby supporting the circular economy and advancing energy integration. Totally, these theoretical contributions collectively strengthen the understanding and evaluation of green innovation and circular economy by improving uncertainty management, refining evaluation frameworks, and advancing decision-making methods.

6.2. Management implications

The proposed MCDM framework highlights several key managerial implications for the circular economy. (1) Enhanced decision-making accuracy and adaptability for the circular economy: Uncertainty is a key challenge in circular economy projects, particularly when assessing feasibility and long-term sustainability. Factors like fluctuating market conditions, technological variability, and unpredictable availability of recyclable materials add complexity to evaluations. The proposed MCDM framework addresses these uncertainties by converting FLEs into continuous cloud information, offering a more reliable foundation for decision-making. For example, uncertainty about the availability of recyclable materials can lead to over-optimistic projections in material recycling projects. The framework allows managers to assess projects under various scenarios, identify potential risks, and make informed adjustments. By modeling supply chain variations, it ensures resources are allocated to projects with higher chances of success and sustainability, helping meet circular economy goals like waste reduction and material recycling. (2) Consistent evaluation of circular economy impact: The tailored evaluation system provides a comprehensive assessment across three key dimensions: grid technology, green energy, and composite benefits. This multi-dimensional evaluation index system ensures managers can holistically evaluate projects, addressing both technical performance and broader sustainability impacts. For example, when assessing a project that incorporates renewable energy, the framework does not only focus on energy efficiency but also includes factors like carbon emission reductions, the use of recycled materials, and socio-economic benefits, such as local job creation. This integrated evaluation system enables managers to understand the full range of a project's impact, ensuring that decisions are based on a well-rounded assessment of both short-term feasibility and long-term sustainability. By incorporating these diverse dimensions, the framework helps identify projects that offer the most balanced and impactful contributions to the circular economy, aligning technical, environmental, and social goals. (3) Optimized selection of high-impact circular economy projects: The MGCRS method enables precise ranking of circular economy projects based on key factors such as energy efficiency, carbon reduction, and resource utilization. This method allows managers to identify and prioritize projects with the highest potential for advancing circular economy objectives, ensuring that resources are allocated where they will have the greatest impact. For example, the ranking system helps identify initiatives that reduce energy consumption while also utilizing renewable or recycled materials in projects aimed at improving energy efficiency, maximizing the project's environmental and economic benefits. By applying this approach, managers can select projects that not only contribute to energy savings but also promote the circularity of materials, ensuring that investments are directed toward projects that deliver the most significant and sustainable long-term outcomes. (4) Strategic resource allocation for the circular economy: The framework provides clear and actionable insights into which EI projects align best with circular economy goals, enabling managers to allocate resources efficiently. By focusing on projects with the highest potential for carbon reduction and resource efficiency, the framework ensures that resources are directed to initiatives that yield the greatest long-term impact. For example, in projects focused on energy recovery from waste, the framework helps managers prioritize initiatives that maximize the reuse of materials and reduce carbon emissions, ensuring that available resources are not wasted on less effective projects. This approach not only minimizes investment risks but also accelerates the transition towards a sustainable circular economy by ensuring that resources are efficiently allocated to projects that support both environmental and economic sustainability goals. (5) Practical application in circular economy initiatives: The framework enables the translation of complex evaluation data into practical, actionable strategies by systematically processing and analyzing key metrics. This allows managers to make informed decisions and implement carbon-neutral initiatives based on real-world performance. For example, when evaluating a project that uses recycled materials for product manufacturing, the framework helps managers track actual material inputs and outputs, enabling adjustments to improve material efficiency or reduce energy consumption over time. This adaptability ensures that managers can optimize projects in response to performance feedback, enhancing their long-term sustainability and impact. Ultimately, the framework supports the continuous improvement of circular economy initiatives by providing managers with the tools to adjust strategies as needed, driving more effective and sustained progress towards circular economy goals. In summary, this framework equips managers with the necessary tools to make informed decisions, standardize project evaluations, optimize resource allocation, and ultimately, drive significant progress toward achieving circular economy through the effective management of EI projects.

6.3. Limitations of the proposal

The proposed framework faces some notable limitations that need to be addressed. Firstly, the dynamic nature of the circular economy, driven by the rapid advancement of technologies, policies, and market conditions, presents a challenge. The framework may need continuous updates to stay relevant and effective in accommodating new trends and changes in sustainability goals. Secondly, scalability issues may arise when applying the framework to larger and more complex EI projects or managing a large number of evaluation criteria. As project scale and criteria expand, the computational demands and complexity of the evaluation process may increase, potentially affecting the framework's feasibility and efficiency.

7. Conclusions

EI represents an advanced stage in the development of sustainable industrial systems, focusing on enhancing resource efficiency and supporting circular economy principles through an interconnected energy network centered around electricity. We propose an MCDM framework with FLEs based on MGCRS to evaluate multiple EI projects. Firstly, converting discrete and continuous information effectively handles uncertainty and ambiguity in evaluating EI projects, mitigating risks associated with inefficiencies, and ensuring the reliable promotion of sustainable industrial practices. Secondly, by incorporating relevant sustainability and circular economy factors, a tailored evaluation index system helps managers identify and prioritize resources for projects that most effectively contribute to sustainable industrial transformation, enhancing circular economy practices and accelerating progress toward sustainability goals. The MGCRS method ranks EI projects in detail, ensuring that resources are allocated to the most impactful initiatives for improving resource efficiency and supporting circular economy principles. This approach prevents resource waste and maximizes the potential of sustainable innovation. By systematically processing complex evaluation data, the framework helps managers translate theoretical evaluations into practical strategies, ensuring effective implementation and adaptation of plans based on realistic performance, thus significantly advancing the goals of sustainable industrial transformation and circular economy. This data-driven management approach supports decision-making and investment strategies, facilitating the transition to a more sustainable and circular industrial system.

Future development will focus on overcoming the limitations of the proposal to better align with circular economy principles. Specifically, the evaluation index system could be extended to different EI sub-networks, allowing for tailored assessments that address the specific service characteristics and demands of various energy types, in line with the circular economy's emphasis on resource efficiency and minimizing waste. This extension would support the upgrading of EI systems to meet evolving needs, ensuring that energy flows and resources are utilized optimally across sectors, promoting material recycling and reducing waste in the process. Additionally, further research should explore the network and layout planning of energy systems based on the ranking of different alternatives. This includes considering investment costs and the energy supply range of sub-networks to enhance the synergistic interaction between different energy sub-networks. Such optimization can facilitate closed-loop energy systems, where energy, materials, and by-products are reused and recycled, ultimately improving the overall efficiency and sustainability of energy integration in line with circular economy goals.

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Appendix A. Proof of Theorems

Theorem 1. Let (X, E, F, R, B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j = 1, 2, ..., m)$ is the binary cloud relation between universe X and E. For any $A, B \in F(E)$, $e \in E$ and $x \in X$, the optimistic MGCRS over two universes satisfies the following theorems:

(1) $\underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) = \left(\overline{R}^O_{\sum_{j=1}^m R^j}(A^C)(x)\right)^C$ and $\overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) = \left(\underline{R}^O_{\sum_{j=1}^m R^j}(A^C)(x)\right)^C$ when A is the set of cloud model.

(2) $\overline{R}^O_{\sum_{j=1}^m R^j}(\emptyset_E)(x) = \emptyset_X$, $\underline{R}^O_{\sum_{j=1}^m R^j}(E)(x) = X$.

(3) If $A \subseteq B$, then $\underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \subseteq \underline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$ and $\overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \subseteq \overline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$.

(4) $\underline{R}^O_{\sum_{j=1}^m R^j}(A \cup B)(x) = \underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \cup \underline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$;
$\overline{R}^O_{\sum_{j=1}^m R^j}(A \cup B)(x) = \overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \cup \overline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$.

(5) $\underline{R}^O_{\sum_{j=1}^m R^j}(A \cap B)(x) = \underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \cap \underline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$;
$\overline{R}^O_{\sum_{j=1}^m R^j}(A \cap B)(x) = \overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \cap \overline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$.

Proof.

(1) $\left(\overline{R}^O_{\sum_{j=1}^m R^j}(A^C)(x)\right)^C = \left(\overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(R^j(x,e), A^C(e))\right)^C = \left\langle x, (\overline{Ex}^O_{A^C}(x))^C, (\overline{En}^O_{A^C}(x))^C, (\overline{He}^O_{A^C}(x))^C \right\rangle$, where $(\overline{Ex}^O_{A^C}(x))^C = \left(\overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(Ex_{R^j}(x,e), Ex_{A^C}(e))\right)^C = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max(\left(Ex_{R^j}(x,e)\right)^C, (Ex_{A^C}(e))^C) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_A(e)) = \underline{Ex}^O_A(x)$. Similarly, $(\overline{En}^O_{A^C}(x))^C = \underline{En}^O_A(x)$ and $(\overline{He}^O_{A^C}(x))^C = \underline{He}^O_A(x)$ can be proved. Therefore, $\underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) = \left(\overline{R}^O_{\sum_{j=1}^m R^j}(A^C)(x)\right)^C$ is proved. Similarly, we have $\overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) = \left(\underline{R}^O_{\sum_{j=1}^m R^j}(A^C)(x)\right)^C$.

(2) $\overline{R}^O_{\sum_{j=1}^m R^j}(\emptyset_E)(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(R^j(x,e), \emptyset_E(e))$, where $\overline{Ex}^O_{\emptyset_E}(x) = \overset{m}{\underset{j=1}{\wedge}} \underset{e\in E}{\vee} \min(Ex_{R^j}(x,e), Ex_{\emptyset_E}(e)) = Ex_{\emptyset_E}(e)$, $\overline{En}^O_{\emptyset_E}(x) = En_{\emptyset_E}(e)$ and $\overline{He}^O_{\emptyset_E}(x) = He_{\emptyset_E}(e)$. Therefore, $\overline{R}^O_{\sum_{j=1}^m R^j}(\emptyset_E)(x) = (Ex_{\emptyset_E}(e), En_{\emptyset_E}(e), He_{\emptyset_E}(e)) = \emptyset_X$. Similarly, $\underline{R}^O_{\sum_{j=1}^m R^j}(E)(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max(N(R^j(x,e)), E(e)) = (Ex_E(e), En_E(e), He_E(e)) = X$, therefore $\underline{R}^O_{\sum_{j=1}^m R^j}(E)(x) = X$ can be proved.

(3) Due to $A \subseteq B$, then $Ex_A(x) \le Ex_B(x)$, $En_A(x) \ge En_B(x)$ and $He_A(x) \ge He_B(x)$. We can obtain $\overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_A(e)) \le \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_B(e))$, then $\underline{Ex}^O_A(x) \le \underline{Ex}^O_B(x)$. Similarly, $\underline{En}^O_A(x) \le \underline{En}^O_B(x)$ and $\underline{He}^O_A(x) \le \underline{He}^O_B(x)$. Therefore, $\underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \subseteq \underline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$. Similarly, $\overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \subseteq \overline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$.

(4) For $\underline{R}^O_{\sum_{j=1}^m R^j}(A \cup B)(x)$, $\underline{Ex}^O_{A\cup B}(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), max(Ex_A(e), Ex_B(e))) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_A(e) \vee Ex_B(e))$. Meanwhile, $\underline{Ex}^O_A(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)),$
$Ex_A(e))$, $\underline{Ex}^O_B(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_B(e))$, then $\underline{Ex}^O_A(x) \cup \underline{Ex}^O_B(x) = \overset{m}{\underset{j=1}{\vee}} \underset{e\in E}{\wedge} \max((U^U + U^L - Ex_{R^j}(x,e)), Ex_A(e) \vee Ex_B(e)) = \underline{Ex}^O_{A\cup B}(x)$. Similarly, $\underline{En}^O_A(x) \cup \underline{En}^O_B(x) = \underline{En}^O_{A\cup B}(x)$ and $\underline{He}^O_A(x) \cup \underline{He}^O_B(x) = \underline{He}^O_{A\cup B}(x)$. Therefore, $\underline{R}^O_{\sum_{j=1}^m R^j}(A \cup B)(x) = \underline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \cup \underline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$. Similarly, $\overline{R}^O_{\sum_{j=1}^m R^j}(A \cup B)(x) = \overline{R}^O_{\sum_{j=1}^m R^j}(A)(x) \cup \overline{R}^O_{\sum_{j=1}^m R^j}(B)(x)$ can be proved.

(5) The proof is similar to that of (4) in **Theorem 1**, so it's omitted.

Theorem 2. Let (X, E, F, R, B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j = 1, 2, ..., m)$ is the binary cloud relation between universe X and E. For any $A_k \in F(E)(k = 1, 2, ..., K)$, $e \in E$ and $x \in X$, the optimistic MGCRS over two universes satisfy the following theorems:

(1) $\underline{R}^O_{\sum_{j=1}^m R^j}(\overset{K}{\underset{k=1}{\cap}} A_k)(x) = \overset{K}{\underset{k=1}{\cap}} \underline{R}^O_{\sum_{j=1}^m R^j}(A_k)(x)$, and $\overline{R}^O_{\sum_{j=1}^m R^j}(\overset{K}{\underset{k=1}{\cap}} A_k)(x) = \overset{K}{\underset{k=1}{\cap}} \overline{R}^O_{\sum_{j=1}^m R^j}(A_k)(x)$.

(2) $\underline{R}^O_{\sum_{j=1}^m R^j}(\overset{K}{\underset{k=1}{\cup}} A_k)(x) = \overset{K}{\underset{k=1}{\cup}} \underline{R}^O_{\sum_{j=1}^m R^j}(A_k)(x)$, and $\overline{R}^O_{\sum_{j=1}^m R^j}(\overset{K}{\underset{k=1}{\cup}} A_k)(x) = \overset{K}{\underset{k=1}{\cup}} \overline{R}^O_{\sum_{j=1}^m R^j}(A_k)(x)$.

(3) $\underline{R}^O_{\sum_{j=1}^m R^j}(\overset{K}{\underset{k=1}{\cap}} A_k)(x) = \overset{m}{\underset{j=1}{\cup}} \left(\overset{K}{\underset{k=1}{\cap}} \underline{R}^j(A_k)(x)\right)$ and $\overline{R}^O_{\sum_{j=1}^m R^j}(\overset{K}{\underset{k=1}{\cup}} A_k)(x) = \overset{m}{\underset{j=1}{\cap}} \left(\overset{K}{\underset{k=1}{\cup}} \overline{R}^j(A_k)(x)\right)$.

Proof.

(1) $\underline{R}^{O}_{\sum_{j=1}^{m} R^j}(\bigcap_{k=1}^{K} A_k)(x) = \bigvee_{j=1}^{m} \bigwedge_{e\in E} \max\left(N(R^j(x,e)), min\{A_1(e), A_2(e), ..., A_K(e)\}\right)$, where $\underline{Ex}^{O}_{\bigcap_{k=1}^{K} A_k}(x) = \bigvee_{j=1}^{m} \bigwedge_{e\in E} \max((U^U + U^L - Ex_{\underline{R}^j}(x,e)), Ex_{A_1}(e) \wedge Ex_{A_2}(e) \wedge ... \wedge Ex_{A_K}(e))$. For $\bigcap_{k=1}^{K} \underline{R}^{O}_{\sum_{j=1}^{m} R^j}(A_k)(x)$, $\bigcap_{k=1}^{K} \underline{Ex}^{O}_{A_k}(x) = \bigcap_{k=1}^{K} \left(\bigvee_{j=1}^{m} \bigwedge_{e\in E} \max((U^U + U^L - Ex_{\underline{R}^j}(x,e)), Ex_{A_k}(e))\right) = \bigvee_{j=1}^{m} \bigwedge_{e\in E} \max((U^U + U^L - Ex_{\underline{R}^j}(x,e)), Ex_{A_1}(e) \wedge Ex_{A_2}(e) \wedge ... \wedge Ex_{A_K}(e))$, i.e., $\underline{Ex}^{O}_{\bigcap_{k=1}^{K} A_k}(x) = \bigcap_{k=1}^{K} \underline{Ex}^{O}_{A_k}(x)$. Similarly, $\underline{En}^{O}_{\bigcap_{k=1}^{K} A_k}(x) = \bigcap_{k=1}^{K} \underline{En}^{O}_{A_k}(x)$ and $\underline{He}^{O}_{\bigcap_{k=1}^{K} A_k}(x) = \bigcap_{k=1}^{K} \underline{He}^{O}_{A_k}(x)$. Therefore, $\underline{R}^{O}_{\sum_{j=1}^{m} R^j}(\bigcap_{k=1}^{K} A_k)(x) = \bigcap_{k=1}^{K} \underline{R}^{O}_{\sum_{j=1}^{m} R^j}(A_k)(x)$. Similarly, $\overline{R}^{O}_{\sum_{j=1}^{m} R^j}(\bigcap_{k=1}^{K} A_k)(x) = \bigcap_{k=1}^{K} \overline{R}^{O}_{\sum_{j=1}^{m} R^j}(A_k)(x)$ can be proved.

(2) The proof is similar to that of (1) in **Theorem 2**, so it's omitted.

(3) For $\underline{R}^j(A_k)(x)$, $\underline{Ex}_{R^j(A_k)}(x) = \bigwedge_{e\in E} \max((U^U + U^L - Ex_{\underline{R}^j}(x,e)), Ex_{A_k}(e))$, then $\bigcap_{k=1}^{K} \underline{Ex}_{R^j(A_k)}(x) = \bigwedge_{e\in E} \max\left((U^U + U^L - Ex_{\underline{R}^j}(x,e)), Ex_{A_1}(e) \wedge Ex_{A_2}(e) \wedge ... \wedge Ex_{A_K}(e)\right)$. Afterwards, we have $\bigcup_{j=1}^{m} \left(\bigcap_{k=1}^{K} \underline{Ex}_{R^j(A_k)}(x)\right) = \bigvee_{j=1}^{m} \bigwedge_{e\in E} \max((U^U + U^L - Ex_{\underline{R}^j}(x,e)), Ex_{A_1}(e) \wedge Ex_{A_2}(e) \wedge ... \wedge Ex_{A_K}(e)) = \underline{Ex}^{O}_{\bigcap_{k=1}^{K} A_k}(x)$. Similarly, $\bigcup_{j=1}^{m} \left(\bigcap_{k=1}^{K} \underline{En}_{R^j(A_k)}(x)\right) = \underline{En}^{O}_{\bigcap_{k=1}^{K} A_k}(x)$ and $\bigcup_{j=1}^{m} \left(\bigcap_{k=1}^{K} \underline{He}_{R^j(A_k)}(x)\right) = \underline{He}^{O}_{\bigcap_{k=1}^{K} A_k}(x)$. Therefore, $\underline{R}^{O}_{\sum_{j=1}^{m} R^j}(\bigcap_{k=1}^{K} A_k)(x) = \bigcup_{j=1}^{m} \left(\bigcap_{k=1}^{K} \underline{R}^j(A_k)(x)\right)$. Similarly, $\overline{R}^{O}_{\sum_{j=1}^{m} R^j}(\bigcup_{k=1}^{K} A_k)(x) = \bigcap_{j=1}^{m} \left(\bigcup_{k=1}^{K} \overline{R}^j(A_k)(x)\right)$ can be proved.

Theorem 3. Let (X, E, F, R, B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j = 1, 2, ..., m)$ is the binary cloud relation between universe X and E. For any $A, B \in F(E)$, $e \in E$ and $x \in X$, the pessimistic MGCRS over two universes satisfies the following theorems:

(1) $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) = \left(\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A^C)(x)\right)^C$ and $\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) = \left(\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A^C)(x)\right)^C$ when A is the set of cloud model.

(2) $\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(\emptyset_E)(x) = \emptyset_X$, $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(E)(x) = X$.

(3) If $A \subseteq B$, then $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) \subseteq \underline{R}^{P}_{\sum_{j=1}^{m} R^j}(B)(x)$ and $\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) \subseteq \overline{R}^{P}_{\sum_{j=1}^{m} R^j}(B)(x)$.

(4) $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A \cup B)(x) = \underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) \cup \underline{R}^{P}_{\sum_{j=1}^{m} R^j}(B)(x)$;
$\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A \cup B)(x) = \overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) \cup \overline{R}^{P}_{\sum_{j=1}^{m} R^j}(B)(x)$.

(5) $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A \cap B)(x) = \underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) \cap \underline{R}^{P}_{\sum_{j=1}^{m} R^j}(B)(x)$;
$\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A \cap B)(x) = \overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A)(x) \cap \overline{R}^{P}_{\sum_{j=1}^{m} R^j}(B)(x)$.

Proof. The proof is similar to that of **Theorem 1**, so it's omitted.

Theorem 4. Let (X, E, F, R, B) be a multiple decision-making cloud information system over two universes and $R^j \in F(X \times E)(j = 1, 2, ..., m)$ is the binary cloud relation between universe X and E. For any $A_k \in F(E)(k = 1, 2, ..., K)$, $e \in E$ and $x \in X$, the pessimistic MGCRS over two universes satisfy the following theorems:

(1) $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(\bigcap_{k=1}^{K} A_k)(x) = \bigcap_{k=1}^{K} \underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A_k)(x)$, and $\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(\bigcap_{k=1}^{K} A_k)(x) = \bigcap_{k=1}^{K} \overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A_k)(x)$.

(2) $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(\bigcup_{k=1}^{K} A_k)(x) = \bigcup_{k=1}^{K} \underline{R}^{P}_{\sum_{j=1}^{m} R^j}(A_k)(x)$, and $\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(\bigcup_{k=1}^{K} A_k)(x) = \bigcup_{k=1}^{K} \overline{R}^{P}_{\sum_{j=1}^{m} R^j}(A_k)(x)$.

(3) $\underline{R}^{P}_{\sum_{j=1}^{m} R^j}(\bigcap_{k=1}^{K} A_k)(x) = \bigcap_{j=1}^{m} \left(\bigcap_{k=1}^{K} \underline{R}^j(A_k)(x)\right)$ and $\overline{R}^{P}_{\sum_{j=1}^{m} R^j}(\bigcup_{k=1}^{K} A_k)(x) = \bigcup_{j=1}^{m} \left(\bigcup_{k=1}^{K} \overline{R}^j(A_k)(x)\right)$.

Proof. The proof is similar to that of **Theorem 2**, so it's omitted.

Appendix B. The Evaluation information of Experts

Table B.1. The FLEs evaluation matrixes of four experts on 10 criteria.

e_1	b_{11}	b_{12}	b_{13}	b_{14}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}
x_1	$\{(l_2,0.6),(\{l_3,l_4\},0.3)\}$	$\{(l_3,1)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$
x_2	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_2,0.3),(l_3,0.5)\}$
x_3	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_1,0.9),(\{l_2,l_3\},0.1)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$
x_4	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.3),(l_3,0.7)\}$
e_2	b_{11}	b_{12}	b_{13}	b_{14}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}
x_1	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$
x_2	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_1,0.9),(\{l_2,l_3\},0.1)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_1,0.2),(l_3,0.5)\}$
x_3	$\{(l_0,0.1),(l_1,l_2,0.5),(\{l_3,l_4\},0.3)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.2),(l_3,0.5)\}$
x_4	$\{(l_2,0.2),(\{l_3,l_4\},0.5)\}$	$\{(l_3,1)\}$	$\{(l_2,0.2),(\{l_3,l_4\},0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$
e_3	b_{11}	b_{12}	b_{13}	b_{14}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}
x_1	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(l_2,0.2),(\{l_3,l_4\},0.5)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$
x_2	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_1,0.2),(l_3,0.5)\}$
x_3	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_3,1)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$
x_4	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.9),(\{l_2,l_3\},0.1)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$
e_4	b_{11}	b_{12}	b_{13}	b_{14}	b_{21}	b_{22}	b_{23}	b_{31}	b_{32}	b_{33}
x_1	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_1,0.2),(l_3,0.5)\}$
x_2	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_3,1)\}$	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$
x_3	$\{(l_1,0.9),(\{l_2,l_3\},0.1)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(l_2,0.2),(l_3,0.3),(l_4,0.5)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_3,0.4),(l_4,0.6)\}$	$\{(\{l_0,l_2\},0.5),(\{l_3,l_4\},0.5)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.3),(l_3,0.7)\}$	$\{(l_3,0.4),(l_4,0.6)\}$
x_4	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_1,0.3),(l_2,0.2),(l_3,0.5)\}$	$\{(l_0,l_2,0.5),(\{l_3,l_4\},0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$	$\{(\{l_2,l_3\},0.8),(l_4,0.2)\}$	$\{(l_1,0.4),(\{l_3,l_4\},0.6)\}$	$\{(l_0,0.1),(\{l_1,l_2\},0.5),(\{l_3,l_4\},0.3)\}$	$\{(l_1,0.2),(l_3,0.5)\}$	$\{(l_2,0.3),(l_3,0.5)\}$

Table B.2. The cloud evaluations of experts under B_1.

e_1	b_{11}	b_{12}	b_{13}	b_{14}
x_1	(5.8647,1.4470,0.1289)	(7.1108,2.4568,0.2922)	(5.1133,1.3217,0.2247)	(5.4776,1.0717,0.1551)
x_2	(8.1339,0.7398,0.1999)	(6.0644,0.8910,0.1823)	(6.4530,1.2628,0.1626)	(7.2460,1.0563,0.2001)
x_3	(7.2460,1.0563,0.2001)	(5.4776,1.0717,0.1551)	(7.3832,0.7357,0.2232)	(3.5823,1.1450,0.0553)
x_4	(5.6651,1.1124,0.2445)	(5.1133,1.3217,0.2247)	(6.0371,1.3480,0.2308)	(7.2460,1.0563,0.2001)
e_2	b_{11}	b_{12}	b_{13}	b_{14}
x_1	(7.3832,0.7357,0.2232)	(8.1339,0.7398,0.1999)	(7.2460,1.0563,0.2001)	(5.1133,1.3217,0.2247)
x_2	(6.0644,0.8910,0.1823)	(5.6651,1.1124,0.2445)	(3.5823,1.1450,0.0553)	(5.4776,1.0717,0.1551)
x_3	(5.4776,1.0717,0.1551)	(6.4530,1.2628,0.1626)	(5.1133,1.3217,0.2247)	(8.1339,0.7398,0.1999)
x_4	(7.0596,1.0719,0.1989)	(7.1108,2.4568,0.2922)	(7.0596,1.0719,0.1989)	(6.0371,1.3480,0.2308)
e_3	b_{11}	b_{12}	b_{13}	b_{14}
x_1	(7.2460,1.0563,0.2001)	(7.0596,1.0719,0.1989)	(8.1339,0.7398,0.1999)	(6.0644,0.8910,0.1823)
x_2	(6.0644,0.8910,0.1823)	(7.2460,1.0563,0.2001)	(5.1133,1.3217,0.2247)	(7.3832,0.7357,0.2232)
x_3	(8.1339,0.7398,0.1999)	(6.4530,1.2628,0.1626)	(5.6651,1.1124,0.2445)	(7.1108,2.4568,0.2922)
x_4	(5.7135,1.1124,0.2470)	(5.4776,1.0717,0.1551)	(3.5823,1.1450,0.0553)	(5.1133,1.3217,0.2247)
e_4	b_{11}	b_{12}	b_{13}	b_{14}
x_1	(5.6651,1.1124,0.2445)	(6.0644,0.8910,0.1823)	(6.4530,1.2628,0.1626)	(8.1339,0.7398,0.1999)
x_2	(5.1133,1.3217,0.2247)	(7.1108,2.4568,0.2922)	(7.2460,1.0563,0.2001)	(5.4776,1.0717,0.1551)
x_3	(3.5823,1.1450,0.0553)	(5.6651,1.1124,0.2445)	(8.1339,0.7398,0.1999)	(7.2460,1.0563,0.2001)
x_4	(5.4776,1.0717,0.1551)	(6.0644,0.8910,0.1823)	(5.1133,1.3217,0.2247)	(7.3832,0.7357,0.2232)

Table B.3. The cloud evaluations of experts under B_2.

e_1	b_{21}	b_{22}	b_{23}
x_1	(5.4776,1.0717,0.1551)	(5.7135,1.1124,0.2470)	(5.1133,1.3217,0.2247)
x_2	(5.6651,1.1124,0.2445)	(8.1339,0.7398,0.1999)	(7.3832,0.7357,0.2232)
x_3	(6.0644,0.8910,0.1823)	(6.0371,1.3480,0.2308)	(5.4776,1.0717,0.1551)
x_4	(6.4530,1.2628,0.1626)	(5.7135,1.1124,0.2470)	(5.6651,1.1124,0.2445)
e_2	b_{21}	b_{22}	b_{23}
x_1	(5.1133,1.3217,0.2247)	(6.0371,1.3480,0.2308)	(6.0644,0.8910,0.1823)
x_2	(5.1133,1.3217,0.2247)	(7.3832,0.7357,0.2232)	(8.1339,0.7398,0.1999)
x_3	(5.4776,1.0717,0.1551)	(6.0644,0.8910,0.1823)	(5.7135,1.1124,0.2470)
x_4	(6.0371,1.3480,0.2308)	(5.6651,1.1124,0.2445)	(6.4530,1.2628,0.1626)
e_3	b_{21}	b_{22}	b_{23}
x_1	(5.7135,1.1124,0.2470)	(5.1133,1.3217,0.2247)	(6.0644,0.8910,0.1823)
x_2	(7.3832,0.7357,0.2232)	(5.6651,1.1124,0.2445)	(6.0371,1.3480,0.2308)
x_3	(8.1339,0.7398,0.1999)	(5.7135,1.1124,0.2470)	(5.4776,1.0717,0.1551)
x_4	(6.0371,1.3480,0.2308)	(6.0644,0.8910,0.1823)	(6.4530,1.2628,0.1626)
e_4	b_{21}	b_{22}	b_{23}
x_1	(6.0644,0.8910,0.1823)	(5.6651,1.1124,0.2445)	(6.0371,1.3480,0.2308)
x_2	(5.4776,1.0717,0.1551)	(5.7135,1.1124,0.2470)	(5.1133,1.3217,0.2247)
x_3	(5.7135,1.1124,0.2470)	(8.1339,0.7398,0.1999)	(7.3832,0.7357,0.2232)
x_4	(6.0371,1.3480,0.2308)	(6.4530,1.2628,0.1626)	(6.0644,0.8910,0.1823)

Table B.4. The cloud evaluations of experts under B_3.

e_1	b_{31}	b_{32}	b_{33}
x_1	(5.4776,1.0717,0.1551)	(5.1133,1.3217,0.2247)	(6.0644,0.8910,0.1823)
x_2	(5.7135,1.1124,0.2470)	(5.6651,1.1124,0.2445)	(6.0371,1.3480,0.2308)
x_3	(6.0371,1.3480,0.2308)	(8.1339,0.7398,0.1999)	(5.4776,1.0717,0.1551)
x_4	(6.0644,0.8910,0.1823)	(6.4530,1.2628,0.1626)	(5.6651,1.1124,0.2445)
e_2	b_{31}	b_{32}	b_{33}
x_1	(5.7135,1.1124,0.2470)	(7.3832,0.7357,0.2232)	(5.1133,1.3217,0.2247)
x_2	(5.1133,1.3217,0.2247)	(6.0371,1.3480,0.2308)	(5.7135,1.1124,0.2470)
x_3	(8.1339,0.7398,0.1999)	(5.4776,1.0717,0.1551)	(5.7135,1.1124,0.2470)
x_4	(7.3832,0.7357,0.2232)	(5.7135,1.1124,0.2470)	(6.4530,1.2628,0.1626)
e_3	b_{31}	b_{32}	b_{33}
x_1	(8.1339,0.7398,0.1999)	(7.3832,0.7357,0.2232)	(5.1133,1.3217,0.2247)
x_2	(6.4530,1.2628,0.1626)	(6.0644,0.8910,0.1823)	(5.7135,1.1124,0.2470)
x_3	(6.0371,1.3480,0.2308)	(5.1133,1.3217,0.2247)	(6.4530,1.2628,0.1626)
x_4	(5.7135,1.1124,0.2470)	(5.6651,1.1124,0.2445)	(5.4776,1.0717,0.1551)
e_4	b_{31}	b_{32}	b_{33}
x_1	(5.7135,1.1124,0.2470)	(5.1133,1.3217,0.2247)	(5.7135,1.1124,0.2470)
x_2	(5.1133,1.3217,0.2247)	(6.0644,0.8910,0.1823)	(7.3832,0.7357,0.2232)
x_3	(6.4530,1.2628,0.1626)	(5.6651,1.1124,0.2445)	(8.1339,0.7398,0.1999)
x_4	(5.4776,1.0717,0.1551)	(5.7135,1.1124,0.2470)	(6.0371,1.3480,0.2308)

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