Open problems on compact constant mean curvature surfaces with boundary

Rafael López

ABSTRACT. We present a collection of easily stated open problems in the theory of compact constant mean curvature surfaces with boundary. We also survey the current status of answering them.

1. Introduction and preliminaries

In this paper, we collect a number of open problems that we have found along the years during my research, have struggled with, and which are still open. At the same time, we present the current state of knowledge of these problems, indicating what the difficulties in answering them are. We hope this collection of open questions may stimulate younger minds, especially because most of the problems are easily formulated and any mathematician with a basic background in classical differential geometry can understand them.

Surfaces with constant mean curvature (cmc surfaces, in short; or H-surfaces, where H is the constant value of the mean curvature) belongs to the field of classical differential geometry of surfaces. A cmc surface in Euclidean 3-dimensional space $\mathbb{R}^3$ has the property its area is stationary under all volume-preserving variations in compact domains. For this reason, cmc surfaces are models of soap films and soap bubbles ([**27**]), hence its use in physics, chemistry, biology, and engineering. From another perspective, the mean curvature equation is a quasilinear elliptic equation and it serves as a model in the theory of PDEs of elliptic type [**20**]. Hence, surfaces with constant mean curvature are of interest in other fields of mathematics. As a recent sample button, overdetermined problems in PDEs initiated by Serrin, Berestycki, Cafarelli, and Nirenberg [**10**, **19**, **52**] have recently received a great boost using cmc surfaces; we refer to [**16**] and references therein for the interested reader.

The main open problems proposed in this paper come from the three celebrated characterizations of the sphere in the family of the closed cmc surfaces.

THEOREM 1.1.
  (1) *Spheres are the only* **embedded** *closed cmc surfaces (Alexandrov [**2**]).*
  (2) *Spheres are the only* **genus zero** *cmc surfaces (Hopf [**25**]).*
  (3) *Spheres are the only* **stable** *closed cmc surfaces (Barbosa-do Carmo [**8**]).*

There exist closed cmc surfaces others than spheres. It was Wente [**55**] who first proved the existence of an immersed torus in $\mathbb{R}^3$ with constant mean curvature. After that work, new examples of closed cmc surfaces with higher genus were discovered [**1, 11, 29, 30, 46**].

It is natural to ask for the boundary versions of these characterizations. The simplest case of boundary curve is a circle $\mathbb{S}^1$. If we intersect a sphere by a plane, we obtain two surfaces, called spherical caps, whose common boundary curve is a circle $\mathbb{S}^1$ (Figure 1.1). Both surfaces have nonzero constant mean curvature. Furthermore, the planar disk determined by $\mathbb{S}^1$ is a minimal ($H = 0$) surface. Thus, planar disks and spherical caps are examples of cmc surfaces spanning a circle.
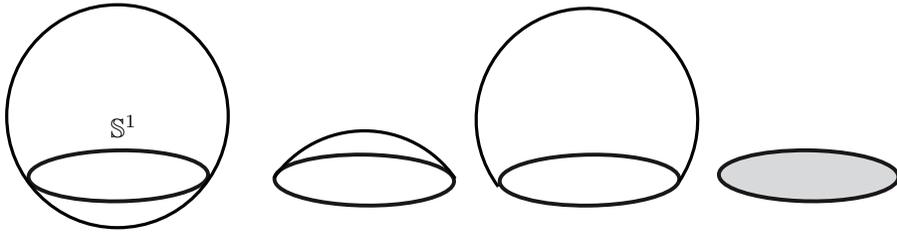

FIGURE 1.1. Surfaces with constant mean curvature spanning a circle $\mathbb{S}^1$ obtained by intersecting a sphere with a plane (left). The middle surfaces are (small and big) spherical caps; the right surface is a planar disk.

Following Theorem 1.1, the analogous hypothesis to consider in the boundary version is whether the surface is embedded, is a topological disk (genus zero), or is stable. There are the following three conjectures.

CONJECTURE 1. *Planar disks and spherical caps are the only compact embedded cmc surfaces in $\mathbb{R}^3$ with circular boundary.*

CONJECTURE 2. *Planar disks and spherical caps are the only cmc disks immersed in $\mathbb{R}^3$ with circular boundary.*

CONJECTURE 3. *Planar disks and spherical caps are the only compact stable cmc surfaces in $\mathbb{R}^3$ with circular boundary.*

As far as we know, the first time when similar questions appeared was in [**15**]; see Problem 1.4 proposed by Gulliver and Kusner in [**15**]. They asked about the existence of cmc surfaces spanning a circle. They proposed the hypothesis that the surface is embedded (Conjecture 1) or it is a topological disk (Conjecture 2). For the initial question of existence of examples, Kapouleas proved that there are compact cmc surfaces of genus $g > 2$ spanning a circle [**29**]. These examples appeared in the mentioned context of finding closed cmc surfaces.

Some of the problems proposed in this paper around these conjectures are well known by people working in cmc surfaces, particularly around the 1980s and '90s. In one way or another they appear in the works of Barbosa, Brito, Kapouleas, Koiso, Korevaar, Kusner, Meeks, Palmer, Sa Earp, and Rosenberg, among others. Perhaps because of their difficulty, these problems were gradually abandoned and forgotten but I think that they deserve a wider dissemination. It is also clear that the simplicity in the formulations will make famous—in this small world of differential geometry—the one who can prove or disprove the above conjectures.

We use the following notations. If $\Psi : \Sigma \to \mathbb{R}^3$ is an isometric immersion of a compact surface Σ with nonempty boundary $\partial\Sigma$ and $C \subset \mathbb{R}^3$ is a closed curve, we say that Σ has C as a boundary if $\Psi_{|\partial\Sigma} : \partial\Sigma \to C$ is a diffeomorphism. If C is a planar curve, we denote by P the plane containing C. We will assume that P is horizontal, that is, parallel to the xy-plane when we consider (x, y, z) the standard coordinates of $\mathbb{R}^3$. In the case where C is a round circle, we use $\mathbb{S}^1$ instead of C. If the boundary is planar, then any minimal surface spanning C is an open set of the plane containing C. Hence the situation $H = 0$ can be viewed as "trivial". In this paper we will consider surfaces in Euclidean 3-dimensional space $\mathbb{R}^3$, although many of the problems can be formulated in other space forms as well as for hypersurfaces in arbitrary dimensions.

Just like Google reviews allows one to rate the quality of a site, we have rated— from 1 to 5 stars—the importance, in the author's opinion, of each of the problems. The problems will be classified in three categories: the circular boundary case, the arbitrary boundary curve, and the Dirichlet problem.

2. Part I: Circular boundary

In this section we will investigate the case when the boundary of the surface is a circle $\mathbb{S}^1$. Spheres are rotational surfaces. Rotational surfaces of constant mean curvature are called Delaunay surfaces and they are well known: planes, catenoids, spheres, cylinders, unduloids, and nodoids [17]. The profile curves of the Delaunay surfaces are the trace of the focus of a conic which is rolled without slipping along a line and rotating this curve around that line. Planes and catenoids are the only ones with $H = 0$. Taking compact pieces of Delaunay surfaces, planar disks and spherical caps are the only compact rotational cmc surfaces spanning a circle.

The first problem is the boundary version of the Alexandrov theorem.

PROBLEM 2.1 (5*). Is a compact embedded cmc surface of $\mathbb{R}^3$ spanning a circle a planar disk or a spherical cap?

Let us show the difference from the closed case. The technique employed in the Alexandrov theorem is called the reflection method [2]. If $\Sigma \subset \mathbb{R}^3$ is a closed embedded surface, then Σ encloses a 3-domain $W \subset \mathbb{R}^3$. Let N be the orientation pointing towards W. Fix an arbitrary direction of $\mathbb{R}^3$, say, the x-direction. Let Π_t be the plane of equation $x = t$. Denote by $\Sigma_t^+ = \Sigma \cap \{x > t\}$, $\Sigma_t^- = \Sigma \cap \{x < t\}$, and Σ_t^* the reflection of Σ_t^+ with respect to Π_t. Coming from $t = +\infty$ towards Σ, let Π_{t_0} be the first plane touching Σ. By embeddedness of Σ, $\Sigma_t^* \subset W$ for all $t \in (t_0 - \epsilon, t_0)$ and for some $\epsilon > 0$. Decreasing t and by compactness, let $t_1 < t_0$ be the first time where $\Sigma_{t_1}^*$ touches $\Sigma_{t_1}^-$. If $p \in \Sigma_{t_1}^* \cap \Sigma_{t_1}^-$, then the surfaces $\Sigma_{t_1}^*$ and $\Sigma_{t_1}^-$ are H-surfaces with the same orientations at p. Moreover, $\Sigma_{t_1}^*$ lies in one side of $\Sigma_{t_1}^-$ around p. Then, the maximum principle implies $\Sigma_{t_1}^* = \Sigma_{t_1}^-$ in an open

set around p [20]. By connectedness, $\Sigma_{t_1}^* = \Sigma_{t_1}^-$. This shows that Π_{t_1} is a plane of symmetry of Σ. Since the fixed orientation was arbitrary, then Σ is a sphere.

Now consider a compact embedded cmc surface spanning a circle $\mathbb{S}^1$. Suppose that $\mathbb{S}^1 \subset P$ is centered at the origin. If we repeat the reflection argument, a first problem we face is the existence of the domain W. For example, if we simply attach to Σ the planar disk D bounded by $\mathbb{S}^1$, the existence of W is not assured because the surface may cross D.

A situation where the reflection method applies is when Σ lies in one side of P. This hypothesis is not only because now $\Sigma \cup D$ encloses a 3-domain W, but also because Σ does not intersect the plane P; see Figure 2.1 left. By repeating the argument as in the closed case, we now fix an arbitrary horizontal direction, say, the x-direction. The time t_1 can occur is when $p \notin \mathbb{S}^1$ or $p \in \mathbb{S}^1$. In the first case, Π_{t_1} is a plane of symmetry, in particular, of symmetry of $\mathbb{S}^1$. This implies $t_1 = 0$. The second case implies $t_1 = 0$. Let us follow this situation. If Σ_0^* is tangent to Σ_0^-, then the maximum principle applies proving that Π_0 is a plane of symmetry. On the contrary, Σ_0^* is contained in W and it does not touch Σ_0^-. We see that this situation is not possible. For this we repeat the argument but coming from $t = -\infty$. Then there is a time $t_2 < 0$ such that the reflection of $\Sigma_{t_2}^+$ about Π_{t_2} tangentially touches $\Sigma_{t_2}^-$. By the maximum principle, the plane Π_{t_2} would be a plane of symmetry of Σ. This is a contradiction because $t_2 < 0$ and Π_{t_2} cannot be a plane of symmetry of $\mathbb{S}^1$. Summarizing, we have proved that for any horizontal direction, the plane orthogonal to this direction and containing the z-axis is a plane of symmetry of Σ. This proves that Σ is rotationally symmetric about the z-axis, hence a spherical cap.

Thus Problem 2.1 can be affirmatively answered if one proves that a compact embedded cmc surface spanning a circle is contained in one side of P.

PROBLEM 2.2 (3*). Under what conditions on the boundary is a compact embedded cmc surface of $\mathbb{R}^3$ spanning a planar closed curve C contained in one side of the boundary plane? And if C is convex?

A partial answer to Problem 2.2 was obtained by Koiso [31] assuming that $\Sigma \cap (P \setminus \overline{D}) = \emptyset$. In fact, in [31] it is not necessary that H is constant but only $H \neq 0$ on Σ.

The existence of the domain W under other circumstances does not solve Problem 2.1. In Figure 2.1 right, the surface Σ together with D defines W but int(Σ) intersects the plane P. Then the reflection process may find a touching point p between Σ_t^* and $\partial\Sigma$, where Σ_t^* and Σ_t^- are not tangent: the maximum principle cannot be applied.

Let us observe that the surface of Figure 2.1 right is star-shaped with respect to the point q, that is, any ray from q meets the surface once at most. If Σ is compact, the star-shaped property implies that Σ is embedded. This is equivalent to say that Σ is a radial graph on some domain of a sphere centered at q. In the nineteenth century, Jellet proved that a closed star-shaped cmc surface is a sphere [28]. This motivates the boundary version of this result.

PROBLEM 2.3 (5*). Is a compact star-shaped cmc surface of $\mathbb{R}^3$ spanning a circle a planar disk or a spherical cap?

Although a positive answer is a partial answer to Problem 2.1, I rate it as a 5* problem because it is the analog of a classical result. The proof in the closed case
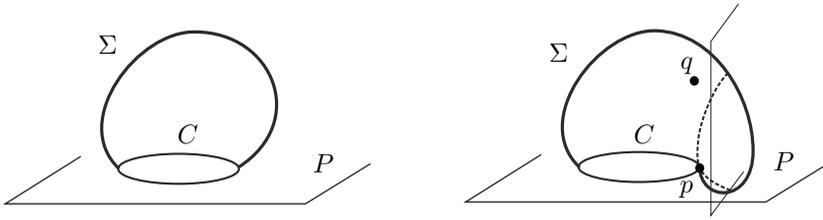
FIGURE 2.1. Left: a compact embedded surface over the boundary plane P. Right: an embedded surface where the Alexandrov reflection method fails at p. Notice that the surface Σ is star-shaped from the point q.

is the following. Assume that Σ is star-shaped from the origin of $\mathbb{R}^3$. If x is the position vector of Σ, the support function $h = \langle N, x \rangle$ has sign, say $h < 0$ with the inward orientation (and thus, $H > 0$). Moreover, we have

$$\Delta |x|^2 = 4 + 4Hh$$
$$\Delta h = -2H - |\sigma|^2 h,$$

(2.1)

where Δ is the Beltrami–Laplace operator and σ is the second fundamental form of Σ. The first equation holds for any surface but the second one holds because H is constant. Integrating and using that Σ is closed, we have $A + H \int_\Sigma h = 0$ and $2HA + \int_\Sigma |\sigma|^2 h = 0$, where $A = \text{area}(\Sigma)$. Combining both equations and using $|\sigma|^2 \geq 2H^2$, we have

$$0 = 2HA + \int_\Sigma |\sigma|^2 h \leq 2HA + 2H^2 \int_\Sigma h = 2H(A + H \int_\Sigma h) = 0.$$

Thus $|\sigma|^2 = 2H^2$ on Σ, concluding that the surface is totally umbilical, hence a sphere. In the boundary case, the troubles appear in the boundary terms coming from the divergence theorem for both equations (2.1).

Returning to the difficulty to find the domain W, it was proved in [**14**] that if $\partial \Sigma = C$ is convex and Σ is transverse to the boundary plane P along C, then Σ lies in one side of P. Thus in Problem 2.1 it would remain to study that it is not possible that along $\mathbb{S}^1$, the surface have parts in both sides of P. In the same paper, the authors propose the following problem.

PROBLEM 2.4 (4*). Let C be a simple closed convex curve in a plane P. Let Σ be a compact embedded cmc surface in $\mathbb{R}^3$ with $\partial \Sigma = C$. Assuming that Σ lies in one side of P, is Σ a topological disk?

A partial affirmative answer is when Σ is a graph on D in a neighborhood of C. Under this hypothesis, Σ is a graph on D. Here $D \subset P$ is the domain bounded by C [**34**]. In this result is not necessary that C is convex. The proof is an application of the Alexandrov method by means of parallel planes to P. Returning to Problem 2.4 we propose Problem 2.5.

PROBLEM 2.5 (4*). Under the same hypothesis of Problem 2.4, if Σ is a graph on $P \setminus \overline{D}$ around C, is Σ a bigraph? that is, is there a planar domain Ω, $\overline{D} \subset \Omega$,

such that $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is a graph on D and Σ_2 is an annulus which it is a graph on $\Omega \setminus \overline{D}$?

Related to the above two problems, Ros and Rosenberg answered Problem 2.4 affirmatively when the volume of Σ is sufficiently large [**49**].

A further related question is when the boundary $\partial\Sigma$ is formed of two curves.

PROBLEM 2.6 (4*). Let C_1 and C_2 be two convex curves in parallel planes. Is a compact embedded cmc surface spanning $C_1 \cup C_2$ a topological annulus?

If $H = 0$, this problem was proposed by Meeks [**45**] and it is still open. Problem 2.6 has an interesting version when the boundary is formed by two coaxial circles.

PROBLEM 2.7 (4*). Let C_1 and C_2 be two coaxial circles in parallel planes. Is a compact embedded cmc surface spanning $C_1 \cup C_2$ a surface of revolution?

In order to apply the Alexandrov reflection method, if the surface is contained in the slab determined by the two boundary planes, then the surface is rotational [**35**]. However, and in contrast to Problem 2.2, one cannot expect that the surface is included in the slab: there are embedded pieces of nodoids spanning two coaxial circles with parts outside of the slab.

We now continue with Conjecture 2.

PROBLEM 2.8 (5*). Is a topological cmc disk spanning a circle a planar disk or a spherical cap?

In the closed case, the proof is based in the properties of the Hopf differential form. Assume that $\Psi = \Psi(z)$ is a conformal parametrization of the surface, where $z = u + iv$. Now we denote by $\langle , \rangle$ the metric in $\mathbb{R}^3$ as well as in $\mathbb{C}^3$. Then $2\langle \Psi_z, \Psi_{\bar{z}} \rangle = E > 0$ and $4HE = \langle N, \Psi_{uu} + \Psi_{vv} \rangle$. Hopf defined the quadratic form $Q(dz)^2$ where $Q = \langle N, \Psi_{zz} \rangle = \frac{1}{4}(e - g + 2if)$, being $\{e, f, g\}$ the coefficients of σ with respect to Ψ. Note that the zeroes of Q are the umbilical points of Σ. Using the Codazzi equations, one gets $Q_{\bar{z}} = EH_z/2$ [**25**]. Consequently, Q is holomorphic if and only if H is constant. Moreover, either the umbilical points of the surface are isolated or the surface is totally umbilical. Suppose now that Ψ is an immersion of a topological sphere. Then we can assume that the immersion is $\Psi \colon \overline{\mathbb{C}} \to \mathbb{R}^3$. Because $Q \colon \overline{\mathbb{C}} \to \mathbb{C}$ is holomorphic, then $Q \equiv 0$. Thus Ψ is totally umbilical, hence it is a sphere. In the boundary version, if we have a cmc topological disk Ψ, the zeroes of Q coincide again with the umbilical points of Ψ. However, we cannot assure $\Psi \equiv 0$ on the surface because the existence of boundary of Σ.

A result related to Problem 2.8 is [**42**]. For immersed disks Σ in $\mathbb{R}^3$ Barbosa and do Carmo proved the isoperimetric inequality $L^2 \geq 4\pi A(1 - \frac{k_0 A}{4\pi})$, where L is the length of $\partial\Sigma$, A is the area of Σ and $k_0 \in \mathbb{R}$ is an upper bound of K [**7**]. The equality holds if and only if Σ is a totally geodesic disk with curvature $K \equiv k_0$. If Σ has constant mean curvature H, we can choose $k_0 = H^2$. In case that $\partial\Sigma$ is a circle $\mathbb{S}^1(r)$ of radius r, this inequality implies that either $A \leq A_-$ or $A \geq A_+$, where

$$A_- = \frac{2\pi}{H^2}(1 - \sqrt{1 - r^2 H^2}), \quad A_+ = \frac{2\pi}{H^2}(1 - \sqrt{1 + r^2 H^2}).$$

The values A_- and A_+ are the areas of the small and the big H-spherical caps with boundary $\mathbb{S}^1(r)$, respectively (Figure 1.1). In [**42**] it is proved that if Σ is a topological H-disk with $A \leq A_-$, then Σ is the small spherical cap.

The following problem is Conjecture 3.

PROBLEM 2.9. Is a stable compact cmc surface spanning a circle a circle a planar disk or a spherical cap?

Stability means that the second derivative of the area functional is nonnegative for any volume preserving variation of the surface. This is equivalent to

$$(2.2) \qquad -\int_\Sigma u(\Delta u + |\sigma|^2 u) \geq 0$$

for all $u \in C^2(\Sigma)$ such that

$$(2.3) \qquad \int_\Sigma u = 0, \quad u_{|\partial\Sigma} = 0.$$

The mean zero integral $\int_\Sigma u = 0$ is a consequence of the fact that all admissible variations of Σ preserve the volume of the surface. The boundary condition $u_{|\partial\Sigma} = 0$ says that the variation fixes the boundary of the surface. In the closed case the condition $u_{|\partial\Sigma} = 0$ makes no sense and it is dropped. In such a case, Barbosa and do Carmo consider the function $u = 1 + H\langle N, x\rangle$ in (2.2) as from (2.1), and we know that $\int_\Sigma u = 0$. After some computations, and using the second identity of (2.1), the stability inquality (2.2) becomes

$$-\int_\Sigma (|\sigma|^2 - 2H^2) \geq 0.$$

Thus $|\sigma|^2 = 2H^2$, hence Σ is totally umbilical, and thus it is a sphere.

Coming back to the boundary case, the extra condition $u_{|\partial\Sigma} = 0$ adds difficulties to find a suitable test function to put in (2.2). Notice that the above function $u = 1 + H\langle N, x\rangle$ does not satisfy $\int_\Sigma u = 0$ neither $u_{|\partial\Sigma} = 0$. In [**3**] it was proved that if Σ is a stable topological cmc disk, then it is a planar disk or a spherical cap. This gives a partial answer to Conjecture 3. The proof uses the Hopf differential 2-form $Q(dz)^2$ and the behavior of the first and second Dirichlet eigenfunctions of the Jacobi equation.

To finish this section, we return to Kapouleas's examples [**29**]. These examples have self-intersections, genus $g > 2$, and they are included in one of the half-spaces determined by the boundary plane. The methods employed in [**29**] are the same that for the construction of closed cmc surfaces. There are a great number of pictures of closed cmc surfaces; see, for example, [**56**–**58**]. However, up to today, we do not have any numerical pictures of the Kapouleas's examples with circular boundary.

PROBLEM 2.10 (3*). Give numerical pictures of the Kapouleas's examples with circular boundary.

As far as we know, graphics of cmc surfaces with circular boundary a circle appear in [**12**], where the authors solve the Björling problem for H-surfaces. In [**12**, Figure 1], the authors take a circle $\mathbb{S}^1$ in the Björling data and the graphic shows the corresponding Björling solution. Of course, the result is local in the sense that the surface is defined in a neighborhood of $\mathbb{S}^1$: it is unknown if the surface closes to be compact.

Related to the three conjectures, an easier approach is assuming two of the three hypothesis. Recall that only if the surface is a topological disk and stable, it is known that the surface is a planar disk or a spherical cap [**3**].

PROBLEM 2.11 (4*). Is a compact stable embedded cmc surface of $\mathbb{R}^3$ spanning a circle a planar disk or a spherical cap?

PROBLEM 2.12 (4*). Is a compact embedded cmc disk of $\mathbb{R}^3$ spanning a circle a planar disk or a spherical cap?

The proposed problems in this section can be formulated in space forms. Notice that examples of compact cmc surfaces spanning a circle are geodesic spheres of totally umbilical surfaces. The hyperbolic space $\mathbb{H}^3$ is special because the behavior of the H-surfaces in $\mathbb{H}^3$ depends on the value of H in relation with the value 1, the absolute value of the sectional curvature of $\mathbb{H}^3$. If $|H| = 1$, then 1-surfaces are similar to minimal surfaces of $\mathbb{R}^3$. If $|H| < 1$, then H-surfaces have not a counterpart in $\mathbb{R}^3$. In this case, the three conjectures are true. Even more, any compact H-surface in $\mathbb{H}^3$ spanning a circle is umbilical without any further assumption on the surface [**9**, **36**]. However, if $|H| > 1$, then the properties of H-surfaces are similar to cmc surfaces of $\mathbb{R}^3$: the same results that hold in $\mathbb{R}^3$ are valid in $\mathbb{H}^3$; the same open questions in $\mathbb{R}^3$ are open in $\mathbb{H}^3$.

In Lorentzian space forms, the mean curvature is defined for surfaces with non-degenerate induced metric. However the behavior of spacelike surfaces and timelike surfaces is very different. If the ambient space is the Lorentz–Minkowski space, it was proved that the only immersed compact spacelike cmc surfaces spanning a circle are planar disks and hyperbolic caps: no further assumptions are needed [**4**].

3. Part II: Arbitrary boundary curve

In this section we will consider that the boundary of a compact cmc surface is not necessarily a circle. Let $C \subset \mathbb{R}^3$ be a closed curve. If Σ is a compact H-surface bounded by C, then the value of H is not arbitrary and it depends on the geometry of C. This is due to the following argument ([**43**]; also in [**22**, **32**, **33**] for particular cases). The 1-form $v \mapsto (Hp + N) \times v$, $v \in T_p\Sigma$, is closed because H is constant. The divergence theorem yields $H \int_{\partial\Sigma} \alpha \times \alpha' + \int_{\partial\Sigma} \nu = 0$, where α is a parametrization by arc-length of $\partial\Sigma$ and $\nu = N \times \alpha'$ is the unit conormal vector along $\partial\Sigma$. If $\vec{v} \in \mathbb{R}^3$ is a unit vector, then

$$(3.1) \qquad H \int_{\partial\Sigma} \langle \alpha \times \alpha', \vec{v} \rangle + \int_{\partial\Sigma} \langle \nu, \vec{v} \rangle = 0.$$

This identity is know as the flux formula of cmc surfaces because $\int_{\partial\Sigma} \langle \nu, \vec{v} \rangle$ is the flux of the vector field $\vec{v}$ along Σ. For example, on a compact minimal surface, the the sum of the flux along all the boundary components must vanish.

The first integral $\int_{\partial\Sigma} \langle \alpha \times \alpha', \vec{v} \rangle$ is twice the algebraic area $A_{\vec{v}}$ of the projection of C on a plane Π orthogonal to $\vec{v}$. Then $2|H||A_{\vec{v}}| \leq \int_{\partial\Sigma} |\langle \nu, \vec{v} \rangle| \leq L$, where L is the length of C. If $H \neq 0$, we conclude

$$|H| \leq \frac{L}{2|A_{\vec{v}}|}.$$

This implies that H is not arbitrary and that the value of H only depends on C. For example, suppose that C is a simple closed curve contained in a plane P and let D be the bounded domain by C in P. Then $\max\{|A_{\vec{v}}| : |\vec{v}| = 1\}$ is attained when $\vec{v}$ is orthogonal to P and its value is the area of D. Therefore $|H| \leq \frac{L}{2|D|}$. If $C = \mathbb{S}^1(r)$, then $|H| \leq \frac{1}{r}$. The case $\mathbb{S}^1(r)$ as boundary curve is special because

for each $-\frac{1}{r} \leq H \leq \frac{1}{r}$ there exists an H-surface spanning $\mathbb{S}^1(r)$: planar disks and spherical caps. This phenomenon is unknown when C is an arbitrary curve.

PROBLEM 3.1 (3*). Let C be a simple planar closed curve. Is there an H-surface bounded by C for each H of the interval $[0, \frac{L}{2|D|}]$? Particularize to the case that C is a convex curve or C is an ellipse.

If J_C is the subset of $\mathbb{R}$ formed by the numbers H such that there is an H-surface bounded by C, Problem 3.1 asks the relation between J_C and the interval $[0, \frac{L}{2|D|}]$, where we know that $J_C \subset [0, \frac{L}{2|D|}]$. It is also known that for values close to 0, there exist H-graphs on D bounded by C (independently whether C is convex or not). So, there is $\epsilon > 0$ such that $[0, \epsilon) \subset J_C$. Nothing is known about the distribution of values H in the set J_C. The last value $\frac{L}{2|D|}$ is of interest. If $C = \mathbb{S}^1(r)$, this value is $1/r$. It was proved that hemispheres are the only $1/r$-surfaces bounded by $\mathbb{S}^1(r)$: not further hypotheses are needed [13]. For a simple planar closed curve, if $H = \frac{L}{2|D|}$, the flux formula (3.1) says that Σ is orthogonal to P along C. In particular, C is a line of curvature of Σ by the Joachimsthal theorem. With the extra hypothesis that C is convex and Σ is embedded, we know from [14] that Σ lies in one side of P. Then the Alexandrov reflection method can be employed obtaining that C is a circle and Σ is a spherical cap. This result also holds assuming that the angle between Σ and P is constant (but not necessarily $\pi/2$). In the general case, if Σ is an immersed H-disk and C is a simple closed curve, then $|H| = \frac{L}{2|D|}$ implies that Σ is a hemisphere and C is a circle [37].

PROBLEM 3.2 (3*). Let C be a simple planar closed curve. Is there a H-surface bounded by C where $H = \frac{L}{2|D|}$? What can be said about the geometry of this surface?

A method to construct cmc surfaces is by solving the Plateau problem: given a boundary curve C and $H \in \mathbb{R}$, find an immersed disk of mean curvature H and boundary C. The techniques of functional analysis look for a critical point of the functional $X \mapsto \int_\Omega \frac{1}{2}|\nabla X|^2 + \frac{2H}{3} \int_\Omega \langle X, X_u \times X_v \rangle$, where $X = X(u,v): \Omega \to \mathbb{R}^3$ is a conformal map from the unit disk Ω and $X_{|\partial\Omega}$ a parametrization of C. Critical points of this functional are H-surfaces. Hildebrandt proved that if C lies included in a ball $\mathcal{B}_R$ of radius $R > 0$, then for any H such that $|H| < 1/R$, there is an H-surface bounded by C. Moreover, this surface is included in the closure of $\mathcal{B}_R$ [23]. Later, Gulliver and Spruck proved a similar result replacing $\mathcal{B}_R$ by a solid cylinder $\mathcal{C}_R$ of radius R [21]. It is natural to ask if it is possible to get a similar result bounding only one spatial coordinate. Notice that given a circle $\mathbb{S}^1(r)$, H-spherical caps bounded by $\mathbb{S}^1(r)$ lies in a slab of width $2/|H|$.

PROBLEM 3.3 (3*). Let $H \neq 0$ and let C be a closed curve contained in a slab $\mathcal{S}_w \subset \mathbb{R}^3$ of width w such that $|H| < 2/w$. Does an H-topological disk exist spanning C and included in $\mathcal{S}_w$?

It is crucial in the results of [21, 23] that the boundary of the sphere or of the cylinder has positive mean curvature: this is not the case of a slab. Related with the results of Hildebrandt, Gulliver and Spruck, and Barbosa considered the particular case that C is a circle. He proved that an H-surface bounded by a circle $\mathbb{S}^1(r)$ and included in a solid cylinder of radius $1/|H|$ must be a spherical cap [5, 6]: here there is no further hypothesis on the surface. Thus we have the following problem.

PROBLEM 3.4 (3*). Let $H \neq 0$ and let Σ be a compact H-surface spanning a circle $\mathbb{S}^1(r)$. If Σ included in the slab $\mathcal{S}_w$, with $|H| < \frac{2}{w}$, is Σ a spherical cap? If we replace $\frac{2}{w}$ by $w\frac{1}{w}$, is Σ a small spherical cap?

The results proved in [**5**, **6**] also hold for closed cmc surfaces. However, in the case that the surface is included in a slab, we can propose:

PROBLEM 3.5 (3*). Is the sphere of radius $1/H$ the only closed H-surface included in the closure of a slab of width $2/|H|$?

Notice that small spherical caps bounded by $\mathbb{S}^1(r)$ are included in the solid cylinder, where $D \subset P$ is the domain bounded by $\mathbb{S}^1(r)$.

PROBLEM 3.6 (3*). Let $C \subset P$ be a simple planar convex curve. Let Σ be a compact cmc surface spanning C. If Σ is included in $D \times \mathbb{R}$, is Σ a graph on D?

If Σ is embedded, then Σ is a graph [**34**]. On the other hand, if there exists an H-graph Σ^* on D, where H is the mean curvature of Σ, then $\Sigma = \Sigma^*$ up to reflections about P [**37**]. This is a consequence of the flux formula (3.1) and the maximum principle. In particular, this is the situation when H is sufficiently small because the existence of H-graphs on D with boundary C is assured when H varies in an interval $[0, \epsilon)$ for some $\epsilon > 0$. The above two results does not use the convexity of C.

In the same direction of the results in [**21**, **23**], Wente proved that given a real number $V > 0$, there is a topological disk Σ bounded by C and with volume V [**54**]. This surface is a critical point of the Dirichlet energy $\int_\Omega \frac{1}{2}|\nabla X|^2$ in a suitable family of conformal immersions $X : \Omega \to \mathbb{R}^3$ with boundary C and fixed volume V. In particular, the mean curvature of Σ must be constant. In case that C is a convex planar curve, it was proved in [**42**] that a cmc surface bounded by C is a graph if the volume of Σ is sufficiently small. Recall here the result of Ros and Rosenberg for large volume [**49**], mentioned in Section 2.

The following problem was first formulated by Rosenberg [**24**, p. 21].

PROBLEM 3.7 (4*). Let C be a non-circular closed convex planar curve. Prove that there is $V_C > 0$ depending only on C such that, if Σ is a compact cmc embedded disk spanning C and with volume $V > 0$, it holds:

(1) If $V \leq V_C$, then Σ lies in one side of P.
(2) If $V = V_C$, then Σ is tangent to P at the points of minimum curvature of C.
(3) If $V > V_C$, then Σ has points in both sides of P.

Hoffman used the Brakke's Surface Evolver to show numerical graphics that support that (3) is true [**24**]. Notice that C is not a circle in Problem 3.7.

When the volume is sufficiently small and motivated by the result of [**42**], we can ask the following problem.

PROBLEM 3.8 (4*). Let $C_1, C_2 \subset \mathbb{R}^2$ be two simple planar convex curves with disjoint interiors. Does $V_0 > 0$ exist such that if Σ is a cmc surface with boundary $C_1 \cup C_2$ and volume $V < V_0$, then Σ is not connected.

4. The Dirichlet problem for the cmc equation

A method to find H-surfaces spanning a given curve is by solving the Dirichlet problem

$$(4.1) \qquad \begin{cases} \operatorname{div} \dfrac{Du}{\sqrt{1+|Du|^2}} & = 2H, \quad \text{in } \Omega \\ u & = \varphi, \quad \text{in } \partial\Omega, \end{cases}$$

where $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Here Ω is a domain of $\mathbb{R}^2$, $H \in \mathbb{R}$ and $\varphi : \partial\Omega \to \mathbb{R}$ is a continuous function. The graph $\Sigma = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$ is an H-surface whose boundary C is the graph of φ. Notice that integrating the first equation of (4.1) and using the divergence theorem, we obtain the flux formula (3.1). By the maximum principle, the solution of (4.1), if exists, is unique. The solvability of (4.1) requires to establish *a priori* C^1 estimates for the solutions of (4.1). This means finding estimates $|u|$ in Ω and gradient estimates $|Du|$ along $\partial\Omega$. In general, results in the literature employ suitable H-surfaces as barriers to get these estimates. A result of Serrin shows that (2.2) has a solution for *any* function φ if and only if Ω is convex and the curvature κ of $\partial\Omega$ satisfies $\kappa > 2|H|$ [51].

From now on we assume $\varphi = 0$ (planar boundary curve). Then it is natural to relax the Serrin's condition on H. For example, if Ω is a convex domain, a first result asserts that if $\kappa > |H|$ then (4.1) is solvable. Here the condition $\kappa > |H|$ allows the use of suitable small H-spherical caps as barriers in establishing the C^1-estimates [37]. In [43] it was proved an estimate of the height of an H-surface Σ with planar boundary, namely,

$$(4.2) \qquad h \le \frac{2\pi}{|H|} A,$$

where h is the height of Σ with respect to the boundary plane and A is the area of Σ. Thanks to (4.2), the Dirichlet problem (4.1) was solved under a variety of assumptions, namely, $2|\Omega|H^2 < \pi$, $LH^2 < \sqrt{3}\pi$ and $\operatorname{diam}(\Omega) < 1/|H|$ [37–39, 42]; see also [40, 47, 48] for other results of existence. However, if Ω is a round disk, the above inequalities are not sharp such as small spherical caps show.

PROBLEM 4.1 (4*). Let Ω be a bounded convex domain. Solve the Dirichlet problem (4.1) for $\varphi = 0$ under each one of the following hypothesis:
   (1) $|\Omega|H^2 < \pi$.
   (2) $LH^2 < 2\pi$.
   (3) $\operatorname{diam}(\Omega) < 2/|H|$.

If Ω is a nonbounded convex domain, it was proved in [38] that (4.1) has a solution for $\varphi = 0$ if and only if Ω is included in a strip of width $1/|H|$. Notice that a half-cylinder of radius $\frac{1}{2|H|}$ is a graph on a strip of width $1/|H|$. Interesting questions on cmc graphs on convex domains are about the convexity of the solutions or of its level curves. Here we are assuming planar boundary curve ($\varphi = 0$ on $\partial\Omega$). It is known that the level curves are convex if the mean curvature H or the volume V of the graph are sufficiently small [44, 50]. However, Wang gave an example of a cmc graph on a convex domain whose level curves are not all convex [53].

PROBLEM 4.2 (3*). Suppose that Ω is a bounded convex domain. Let u be a solution (4.1) for $\varphi = 0$ on $\partial\Omega$. Give a control of the distribution of the convex level curves in terms of the value H. Extend the Wang's example, if possible, when Ω is not bounded.

Following (4.1) and $\varphi = 0$ along $\partial\Omega$, it is natural to ask if a solution of the Dirichlet problem is convex in the sense that the Gauss curvature K of the graph is positive everywhere. Notice that there exist pieces of unduloids obtained by cutting-off an unduloid for some particular planes parallel to the rotation axis that are graphs on convex domains but the surface has parts with negative Gauss curvature. See related results in [**18**, **26**, **41**].

PROBLEM 4.3 (4*). Let Ω be a bounded convex domain. Give conditions that assure the convexity of a cmc graph on Ω and with planar boundary

References

[1] U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew. Math. **374** (1987), 169–192, DOI 10.1515/crll.1987.374.169. MR876223

[2] A. D. Aleksandrov, *Uniqueness theorems for surfaces in the large. V*, Amer. Math. Soc. Transl. (2) **21** (1962), 412–416. MR150710

[3] L. J. Alías, R. López, and B. Palmer, *Stable constant mean curvature surfaces with circular boundary*, Proc. Amer. Math. Soc. **127** (1999), no. 4, 1195–1200, DOI 10.1090/S0002-9939-99-04950-3. MR1618705

[4] L. J. Alías, R. López, and J. A. Pastor, *Compact spacelike surfaces with constant mean curvature in the Lorentz-Minkowski 3-space*, Tohoku Math. J. (2) **50** (1998), no. 4, 491–501, DOI 10.2748/tmj/1178224893. MR1653414

[5] J. L. M. Barbosa, *Hypersurfaces of constant mean curvature on $\mathbf{R}^{n+1}$ bounded by a Euclidean sphere*, Geometry and topology of submanifolds, II (Avignon, 1988), World Sci. Publ., Teaneck, NJ, 1990, pp. 1–9. MR1068731

[6] J. L. M. Barbosa, *Constant mean curvature surfaces bounded by a plane curve*, Mat. Contemp. **1** (1991), 3–15. VIIth School on Differential Geometry (Portuguese) (Campinas, 1990). MR1304297

[7] J. L. Barbosa and M. do Carmo, *A proof of a general isoperimetric inequality for surfaces*, Math. Z. **162** (1978), no. 3, 245–261, DOI 10.1007/BF01186367. MR508841

[8] J. L. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **185** (1984), no. 3, 339–353, DOI 10.1007/BF01215045. MR731682

[9] J. L. M. Barbosa and R. Sa Earp, *Prescribed mean curvature hypersurfaces in $H^{n+1}(-1)$ with convex planar boundary. I*, Geom. Dedicata **71** (1998), no. 1, 61–74, DOI 10.1023/A:1005046021921. MR1624726

[10] H. Berestycki, L. A. Caffarelli, and L. Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), no. 11, 1089–1111, DOI 10.1002/(SICI)1097-0312(199711)50:11¡1089::AID-CPA2¿3.0.CO;2-6. MR1470317

[11] A. I. Bobenko, *All constant mean curvature tori in $\mathbf{R}^3$, S^3, H^3 in terms of theta-functions*, Math. Ann. **290** (1991), no. 2, 209–245, DOI 10.1007/BF01459243. MR1109632

[12] D. Brander and J. F. Dorfmeister, *The Björling problem for non-minimal constant mean curvature surfaces*, Comm. Anal. Geom. **18** (2010), no. 1, 171–194, DOI 10.4310/CAG.2010.v18.n1.a7. MR2660462

[13] F. Brito and R. Sa Earp, *Geometric configurations of constant mean curvature surfaces with planar boundary*, An. Acad. Brasil. Ciênc. **63** (1991), no. 1, 5–19. MR1115489

[14] R. Earp, F. Brito, W. H. Meeks III, and H. Rosenberg, *Structure theorems for constant mean curvature surfaces bounded by a planar curve*, Indiana Univ. Math. J. **40** (1991), no. 1, 333–343, DOI 10.1512/iumj.1991.40.40017. MR1101235

[15] J. E. Brothers (ed.), *Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute*, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 441–464, DOI 10.1090/pspum/044/840292. MR840292

[16] M. Del Pino, F. Pacard, and J. Wei, *Serrin's overdetermined problem and constant mean curvature surfaces*, Duke Math. J. **164** (2015), no. 14, 2643–2722, DOI 10.1215/00127094-3146710. MR3417183

[17] C. Delaunay, *Sur la surface de révolution dont la courbure moyenne est constante*, J. Math. Pures Appl. **6**, 309–315 (1841).

[18] H. Deng, H. Liu, and L. Tian, *Critical points of solutions for the mean curvature equation in strictly convex and nonconvex domains*, Israel J. Math. **233** (2019), no. 1, 311–332, DOI 10.1007/s11856-019-1906-2. MR4013976

[19] B. Gidas, W. M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), no. 3, 209–243. MR544879

[20] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364

[21] R. Gulliver and J. Spruck, *The Plateau problem for surfaces of prescribed mean curvature in a cylinder*, Invent. Math. **13** (1971), 169–178, DOI 10.1007/BF01390099. MR298515

[22] E. Heinz, *On the nonexistence of a surface of constant mean curvature with finite area and prescribed rectifiable boundary*, Arch. Rational Mech. Anal. **35** (1969), 249–252, DOI 10.1007/BF00248159. MR246237

[23] S. Hildebrandt, *On the Plateau problem for surfaces of constant mean curvature*, Comm. Pure Appl. Math. **23** (1970), 97–114, DOI 10.1002/cpa.3160230105. MR256276

[24] D. Hoffman, *Comment utiliser un ordinateur pour trouver de nouvelles surfaces minimales et des bulles de savon*, Surfaces minimales et solutions de problèmes variationnels, SMF Journ. Annu., vol. 1993, Soc. Math. France, Paris, 1993, pp. 29. MR1483381

[25] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics, vol. 1000, Springer-Verlag, Berlin, 1983. Notes taken by Peter Lax and John Gray; With a preface by S. S. Chern, DOI 10.1007/978-3-662-21563-0. MR707850

[26] W.-H. Huang and C.-C. Lin, *Negatively curved sets on surfaces of constant mean curvature in $\mathbf{R}^3$ are large*, Arch. Rational Mech. Anal. **141** (1998), no. 2, 105–116, DOI 10.1007/s002050050074. MR1615516

[27] C. Isenberg, *The science of soap films and soap bubbles*, Tieto Ltd., Clevedon, 1978. With a foreword by George Porter. MR620322

[28] J. H. Jellet, *Sur la surface dont la courbure moyenne est constant*, J. Math. Pures Appl. **18**, 163–167 (1853).

[29] N. Kapouleas, *Compact constant mean curvature surfaces in Euclidean three-space*, J. Differential Geom. **33** (1991), no. 3, 683–715. MR1100207

[30] N. Kapouleas, *Constant mean curvature surfaces constructed by fusing Wente tori*, Proc. Nat. Acad. Sci. U.S.A. **89** (1992), no. 12, 5695–5698, DOI 10.1073/pnas.89.12.5695. MR1165926

[31] M. Koiso, *Symmetry of hypersurfaces of constant mean curvature with symmetric boundary*, Math. Z. **191** (1986), no. 4, 567–574, DOI 10.1007/BF01162346. MR832814

[32] N. Korevaar and R. Kusner, *The global structure of constant mean curvature surfaces*, Invent. Math. **114** (1993), no. 2, 311–332, DOI 10.1007/BF01232673. MR1240641

[33] R. Kusner, *Bubbles, conservation laws, and balanced diagrams*, Geometric analysis and computer graphics (Berkeley, CA, 1988), Math. Sci. Res. Inst. Publ., vol. 17, Springer, New York, 1991, pp. 103–108, DOI 10.1007/978-1-4613-9711-3_11. MR1081331

[34] R. López, *A note on H-surfaces with boundary*, J. Geom. **60** (1997), no. 1-2, 80–84, DOI 10.1007/BF01252219. MR1477073

[35] R. López, *Surfaces of constant mean curvature bounded by two planar curves*, Ann. Global Anal. Geom. **15** (1997), no. 3, 201–210, DOI 10.1023/A:1006545410182. MR1456507

[36] R. López, *Constant mean curvature surfaces with boundary in hyperbolic space*, Monatsh. Math. **127** (1999), no. 2, 155–169, DOI 10.1007/s006050050030. MR1669875

[37] R. López, *Constant mean curvature surfaces with boundary in Euclidean three-space*, Tsukuba J. Math. **23** (1999), no. 1, 27–36, DOI 10.21099/tkbjm/1496163774. MR1693145

[38] R. López, *Constant mean curvature graphs on unbounded convex domains*, J. Differential Equations **171** (2001), no. 1, 54–62, DOI 10.1006/jdeq.2000.3830. MR1816793

[39] R. López, *An existence theorem of constant mean curvature graphs in Euclidean space*, Glasg. Math. J. **44** (2002), no. 3, 455–461, DOI 10.1017/S0017089502030100. MR1956553

[40] R. López, *Constant mean curvature graphs in a strip of $\mathbb{R}^2$*, Pacific J. Math. **206** (2002), no. 2, 359–373, DOI 10.2140/pjm.2002.206.359. MR1926782

[41] R. López, *Gradient estimates for the constant mean curvature equation in hyperbolic space*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 6, 3216–3230, DOI 10.1017/prm.2019.71. MR4190109

[42] R. López and S. Montiel, *Constant mean curvature discs with bounded area*, Proc. Amer. Math. Soc. **123** (1995), no. 5, 1555–1558, DOI 10.2307/2161147. MR1286001

[43] R. López and S. Montiel, *Constant mean curvature surfaces with planar boundary*, Duke Math. J. **85** (1996), no. 3, 583–604, DOI 10.1215/S0012-7094-96-08522-1. MR1422358

[44] J. McCuan, *Continua of H-graphs: convexity and isoperimetric stability*, Calc. Var. Partial Differential Equations **9** (1999), no. 4, 297–325, DOI 10.1007/s005260050143. MR1731469

[45] W. H. Meeks III, *Lectures on Plateau's Problem*, Instituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, 1978.

[46] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Ann. of Math. (2) **130** (1989), no. 2, 407–451, DOI 10.2307/1971425. MR1014929

[47] J. Ripoll, *Some characterization, uniqueness and existence results for Euclidean graphs of constant mean curvature with planar boundary*, Pacific J. Math. **198** (2001), no. 1, 175–196, DOI 10.2140/pjm.2001.198.175. MR1831977

[48] J. Ripoll, *Some existence results and gradient estimates of solutions of the Dirichlet problem for the constant mean-curvature equation in convex domains*, J. Differential Equations **181** (2002), no. 1, 230–241, DOI 10.1006/jdeq.2001.4050. MR1900466

[49] A. Ros and H. Rosenberg, *Constant mean curvature surfaces in a half-space of $\mathbf{R}^3$ with boundary in the boundary of the half-space*, J. Differential Geom. **44** (1996), no. 4, 807–817. MR1438193

[50] S. Sakaguchi, *Uniqueness of critical point of the solution to the prescribed constant mean curvature equation over convex domain in $\mathbf{R}^2$*, Recent topics in nonlinear PDE, IV (Kyoto, 1988), North-Holland Math. Stud., vol. 160, North-Holland, Amsterdam, 1989, pp. 129–151, DOI 10.1016/S0304-0208(08)70510-3. MR1041379

[51] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A **264** (1969), 413–496, DOI 10.1098/rsta.1969.0033. MR282058

[52] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318, DOI 10.1007/BF00250468. MR333220

[53] X.-J. Wang, *Counterexample to the convexity of level sets of solutions to the mean curvature equation*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 6, 1173–1182, DOI 10.4171/JEMS/457. MR3226739

[54] H. C. Wente, *A general existence theorem for surfaces of constant mean curvature*, Math. Z. **120** (1971), 277–288, DOI 10.1007/BF01117500. MR282300

[55] A. M. de Souza Doering, *The depth conjecture: a counterexample*, J. Algebra **77** (1982), no. 2, 443–448, DOI 10.1016/0021-8693(82)90265-4. MR673127

[56] The Center for Geometry, Analysis, Numerics & Graphics (GANG). `http://www.gang.umass.edu/gallery/cmc/`

[57] GeometrieWerkstatt Surface Gallery. `https://www.math.uni-tuebingen.de/user/nick/gallery/`

[58] m:iv research network. `https://www.miv-gallery.net/`

Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

Email address: `rcamino@ugr.es`