# STABILITY OF A CAPILLARY CIRCULAR CYLINDER BETWEEN TWO PARALLEL CYLINDERS

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**Abstract.** Consider a system of two parallel solid cylinders of equal radius made of a homogenous material. We study the stability of a liquid bridge of circular cylinder shape between both solid cylinders. It is proved that if the circular cylinder liquid is concave, then it is stable. If the circular cylinder liquid is convex, we establish conditions on the radius of the cylinder liquid and the contact angle that ensure that long convex circular cylinders are not stable. Estimates for the length of these convex cylinders are given.

**Key words.** capillarity, surface tension, Sturm-Liouville problem, Morse index, stability

**MSC codes.** 76B45, 53A10, 49Q10, 58E12

## 1. Introduction.

It is of interest in technology, e. g. textile and paper industries or composite materials, the understanding of the wetting and repellency properties of a liquid in contact with a set of parallel circular fibers [3, 7, 14, 21, 23, 36]. These properties of the liquid are deduced by the shape of the liquid interface connecting the fibers. This shape is governed by the capillary and adhesion effects of the liquid with the walls of the fibers. Usually when the number of fibers is great, these fibers form a symmetric system of parallel circular cylinders obtained by repeating a given pattern.

In the simplest configuration, consider two parallel solid circular cylinders $\mathsf{C}_1$ and $\mathsf{C}_2$ of equal radius, which can be assumed to be 1, and separated by a distance $2d > 0$. We are interested in the shape of a finite amount of liquid which is spread out between $\mathsf{C}_1$ and $\mathsf{C}_2$. We assume ideal conditions for the materials and the liquid: homogeneity, constant density, isotropy and no hysteresis. It is also assumed that the gravity acting on the liquid is negligible. This scenario of zero gravity can be thought on microscopic scale where the interaction between the liquid and the walls of $\mathsf{C}_1 \cup \mathsf{C}_2$ is only due to capillarity and wetting.

In equilibrium the air-liquid interface is an extremal of an energy functional $E$ which it is now defined. Up to constants, the total free energy is

$$E(\Omega) = |\Sigma| - c_1|\widetilde{\Sigma}_1| - c_2|\widetilde{\Sigma}_2|, \tag{1.1}$$

where $\widetilde{\Sigma}_i = \overline{\Omega} \cap \mathsf{C}_i$ is the wetted region by the liquid $\Omega$ on $\mathsf{C}_i$, $i = 1, 2$. The term $|\Sigma|$ represents the energy of the free surface $\Sigma$ and the terms $c_i|\widetilde{\Sigma}_i|$ are the wetting energies of $\widetilde{\Sigma}_i$. The constants $c_i$ depend on the wetting properties in each support $\mathsf{C}_i$ and they are given by quotient

$$c_i = \frac{\sigma_{\mathsf{C}_i A} - \sigma_{\mathsf{C}_i L}}{\sigma_{LA}},$$

where $\sigma_{ij}$ denotes the surface tensions along the solid ($\mathsf{C}_i$), liquid (L) and air (A) phases. The inequality $|\sigma_{\mathsf{C}_i A} - \sigma_{\mathsf{C}_i L}| < \sigma_{LA}$ is assumed in order to ensure the existence of wetting in a position of equilibrium. It is assumed that the volume of $\Omega$ is preserved for small perturbations (for example, no evaporation of the liquid), and that the contact lines $\partial\widetilde{\Sigma}_i$ can freely move on $\mathsf{C}_i$.

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A critical point of the total energy is governed by the Laplace-Young equations. These equations describe the balance between the hydrostatic pressure within the liquid and the pressure jump across the interface due to surface tension. First, the Laplace equation asserts that $\Sigma$, viewed as a two-dimensional surface of Euclidean space $\mathbb{R}^3$, is a surface of constant mean curvature. On the other hand, the Young equation says that the contact angle $\gamma_i \in (0,\pi)$ between $\Sigma$ and the supports $\mathsf{C}_i$ is constant, with $\cos\gamma_i = c_i$, $i = 1, 2$. Under such situation, we say that $\Sigma$ is a *capillary surface* on $\mathsf{C}_1 \cup \mathsf{C}_2$. Pioneering works on capillary surfaces on cylinders are [24, 25, 26], which have generated a considerable literature in physics: see for example, [4, 6, 11, 12, 15, 20, 28]. The mathematical theory of capillary surfaces can be seen in [8].

The geometry of the surfaces with constant mean curvature is difficult to describe in all its generality. However the family of axisymmetric surfaces are known and called Delaunay surfaces [5]. They are planes, catenoids, spheres, cylinders, unduloids and nodoids. The profile curves of the Delaunay surfaces are the trace of the focus of a conic which is rolled without slipping along a line and rotating this curve around that line. Planes and catenoids are the only ones with zero mean curvature everywhere. Here we only focus in those configurations of planes and cylinders that make a constant angle with two parallel cylinders $\mathsf{C}_1 \cup \mathsf{C}_2$. In this context, it is assumed that the above perturbations to deduce the Laplace equation are of compact support and the definitions in (1.1) are taken only on bounded domains of $\Sigma$.

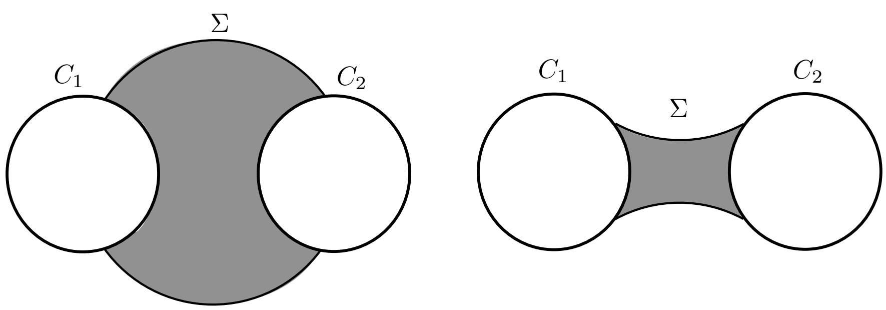


FIG. 1. *Top view of a circular cylinder liquid column $\Sigma$ between two cylinders $\mathsf{C}_1 \cup \mathsf{C}_2$. The dark region represents the liquid supported on the walls of $\mathsf{C}_1 \cup \mathsf{C}_2$. There are two types of circular cylinder liquids: convex (left) and concave (right).*

From now on, we will assume that the contact angle between $\Sigma$ and $\mathsf{C}_1$ and $\mathsf{C}_2$ coincide, $\gamma_1 = \gamma_2 = \gamma$. There are two types of capillary circular cylinders on $\mathsf{C}_1 \cup \mathsf{C}_2$: the cylinder $\Sigma$ is convex (Fig. 1, left) and the cylinder $\Sigma$ is concave (Fig. 1, right). Another situation occurs when $\Sigma$ is a plane (Fig. 2 in Sect. 3) which it can be seen as the limit case of infinity radius of $\Sigma$ in Fig. 1.

In general, capillary surfaces are not minimizers of the energy $E$. Deciding whether a surface is a minimizer is a difficult question to solve. Instead, we will consider the stability condition, which means that the surface is a second order minimum of the area under deformations with fixed volume. In this context, we say then that $\Sigma$ is *stable*. Stable capillary surfaces are of special interest because stable configurations would be physically observed [1].

The purpose of this paper is the study of the stability of planes and cylinders as capillary surfaces in contacting with $\mathsf{C}_1 \cup \mathsf{C}_2$. In the context of liquid bridges between parallel circular fibers, some simulations on stability have been obtained [2, 13]. In capillary theory, the stability of cylindrical interfaces dates back to studies of Plateau and Rayleigh in the late of 1800s [22, 27]. They proved that a stream of liquid of circular radius $r > 0$ is not stable if its length $L$ satisfies $L > 2\pi r$. If the capillary surface $\Sigma$ is a circular cylinder and the support surface is formed by one circular cylinder, the author studied the stability of long capillary cylinders on the exterior of a solid circular cylinder [16]. It was proved that the liquid bifurcates to create a family of periodic constant mean curvature surfaces. Following with the case that the support is one circular cylinder, Vogel investigated the stability and energy minimality of some types of Delaunay surfaces [35]. In [19], the authors studied capillary surfaces on a given circular cylinder proving some results of symmetry when the surface lies inside of a hollow circular cylinder. For example, if the mean curvature of the capillary surface is zero, sufficient conditions were proved to conclude that the surface is part of a plane or a catenoid. In case that the support is a set of two spheres or between a sphere and plane, see [17]. Stability of cylinders supported in arbitrary cylindrical surface has been recently studied by the author in [18].

The main the objective of this paper is to determine a criterion of instability for long capillary cylinders contacting two given solid cylinders. Taking in mind the classical setting of the Plateau-Rayleigh phenomenon, we want to give an estimate of the length $L$ of the capillary cylinder that ensures the instability of the liquid. Our theoretical approach aims to be as general as possible, i.e. to consider the case where the capillary circular cylinder is convex and concave. Also we allow for any separation between the two cylinders $\mathsf{C}_1 \cup \mathsf{C}_2$ and any contact angle between the liquid and the two cylinders.

In Sect. 2 we introduce the Sturm-Liouville eigenvalue problem that facilitates the study of the stability of capillary surfaces. The case that $\Sigma$ is a planar strip is studied in Sect. 3 proving that $\Sigma$ is stable (Theorem 3.1). Section 4 is devoted to the case that $\Sigma$ is a convex cylinder and in Sect. 5 when it is a concave cylinder. In the convex cylinder case, the study of stability depends on the contact angle $\gamma$ and the radius $r$ of the circular cylinder liquid. We prove that if $r > \cos\gamma$, then the liquid is stable, independently of the length of the liquid (Th. 4.8). This situation includes the case $\gamma \geq \pi/2$. In contrast, if $r < \cos\gamma$, then $\Sigma$ is not stable (Th. 4.2). In this case we are able to establish different estimates for the length $L$ of a piece of convex cylinder liquid $\Sigma_L$ that ensures instability of $\Sigma_L$. Some of these estimates can be given in terms of the initial data (Cor. 4.3). Other sharp estimates are obtained after the calculation of the roots for some equations (Cors. 4.4 and 4.5). Particular configurations of liquids have been presented to show the power of our results: see Examples 4.6 and 4.7. In each of them, we have established specific lengths $L$ to ensure instability. With these examples it is shown that the methods and results presented in this paper can be applied easily in explicit configurations of circular cylinder liquids.

**2. The Sturm-Liouville eigenvalue problem.** In this section we introduce the notation for the stability of a capillary surface and we will define the Morse index associated to a Sturm-Liouville eigenvalue problem. Let $\langle , \rangle$ be the Euclidean metric of $\mathbb{R}^3$. Let $W$ be the domain of $\mathbb{R}^3$ whose boundary is $\mathsf{C}_1 \cup \mathsf{C}_2$. Let $\Sigma$ be a surface with boundary $\partial\Sigma$ and let $\Psi \colon \Sigma \to \mathbb{R}^3$ be an immersion of $\Sigma$ in $\mathbb{R}^3$. Assume that $\Psi(\text{int}(\Sigma)) \subset W$, $\Psi(\partial\Sigma) \subset \mathsf{C}_1 \cup \mathsf{C}_2$ and that $\Sigma$ meets transversally $\mathsf{C}_1 \cup \mathsf{C}_2$. The surface $\Sigma$ together some domains $\widetilde{\Sigma}_i$, $i = 1, 2$, enclose a 3-domain $\Omega$, which represents the volume of the liquid connecting $\mathsf{C}_1$ and $\mathsf{C}_2$. Let $N$ be the unit normal vector field on $\Sigma$ along $\Psi$ pointing towards $\Omega$. Denote by $\nu$ the unit exterior conormal vector to $\partial\Sigma$

in $\Sigma$. On $\mathsf{C}_1\cup\mathsf{C}_2$, let $\overline{N}$ be the unit normal pointing outwards $\Omega$ and let $\overline{\nu}$ be the unit exterior conormal vector to $\partial\Sigma$ in $\mathsf{C}_1\cup\mathsf{C}_2$.

In the case that is the subject of this article, the surface $\Sigma$ is an unbounded circular cylinder. Then the definition of $E(\Omega)$ in (2.1) (or (1.1)) does not make sense. Although infinitely long columns of liquid are not (ever) observed, the critical points of the energy $E$ will be obtained by doing perturbations of compact support of the initial surface $\Sigma$, that is, by considering a one-parameter family of surfaces which coincide with $\Sigma$ outside of a compact set of $\Sigma$.

An admissible variation of $\Sigma$ is defined as a smooth map $\overline{\Psi}\colon(-\epsilon,\epsilon)\times\Sigma$ satisfying: (i) $\overline{\Psi}_t(p)=\overline{\Psi}(t,p)$ is an immersion such that $\overline{\Psi}_t(\text{int}(\Sigma)\subset W$ and $\overline{\Psi}_t(\partial\Sigma)\subset\mathsf{C}_1\cup\mathsf{C}_2$; (ii) $\overline{\Psi}_0=\Psi$ and; (iii) there is a compact set $K\subset\Sigma$ such that $\overline{\Psi}_t=\Psi$ in $\Sigma\setminus K$. The last condition says that the variation is of compact support. Let $E(t)$ be the energy of $\overline{\Psi}_t$ defined in (1.1) where $c_1=c_2=\cos\gamma$. Denote by $V(t)$ the volume enclosed by $\overline{\Psi}_t$. The surface $\Sigma$ is called a *capillary surface* on $\mathsf{C}_1\cup\mathsf{C}_2$ if $E'(0)=0$ for any volume-preserving admissible variation of $\Sigma$.

The formulas of the first variation of $E$ and the volume $V$ are

$$E'(0)=-2\int_\Sigma Hu+\int_{\partial\Sigma}\langle\nu-\cos\gamma\,\overline{\nu},\xi\rangle,\quad V'(0)=\int_\Sigma u, \tag{2.1}$$

where $H$ is the mean curvature of the immersion $\Psi$, $\xi$ is the variation vector field of $\overline{\Psi}$ and $u=\langle N,\xi\rangle$. As consequence of Eq. (2.1), $\Sigma$ is a capillary surface if and only if $\Sigma$ has constant mean curvature $H$ and the angle of $\Sigma$ with $\mathsf{C}_1\cup\mathsf{C}_2$ is $\gamma$. The latter condition writes as $\cos\gamma=\langle N,\overline{N}\rangle=\langle\nu,\overline{\nu}\rangle$.

If $\Sigma$ is a capillary surface, the second variation of $E$ is

$$E''(0)=-\int_\Sigma u\left(\Delta u+|A|^2u\right)+\int_{\partial\Sigma}u\left(\frac{\partial u}{\partial\nu}-\mathbf{b}u\right), \tag{2.2}$$

where $\Delta$ is the Laplacian on $\Sigma$ and $A$ is the second fundamental form of $\Sigma$. Recall that $|A|^2=4H^2-2K$, where $K$ is the Gaussian curvature of $\Sigma$. The function $\mathbf{b}$ in (2.2) is given by

$$\mathbf{b}=\frac{\overline{A}(\overline{\nu},\overline{\nu})}{\sin\gamma}+\cot\gamma A(\nu,\nu), \tag{2.3}$$

where $\overline{A}$ is the second fundamental form of $\mathsf{C}_1\cup\mathsf{C}_2$ with respect to $-\overline{N}$: see details in [29]. A capillary surface $\Sigma$ is called *stable* if $E''(0)\geq 0$ for all volume-preserving variation of $\Sigma$.

The expression of $E''(0)$ in (2.2) defines a quadratic form $Q$ in the space $C_0^\infty(\Sigma)$ given by $Q[u]=E''(0)$. Stability of $\Sigma$ is equivalent to have $Q[u]\geq 0$ in the space $\mathcal{M}=\{u\in C_0^\infty(\Sigma):\int_\Sigma u=0\}$. If $\Sigma$ is not stable, the weak Morse index, denoted by $\text{index}_w(\Sigma)$, is defined as the maximal dimension of a subspace of functions of mean zero of $C_0^\infty(\Sigma)$ where $Q$ is negative definite, that is,

$$\text{index}_w(\Sigma)=\dim\ \max\{V\subset\mathcal{M}:V\text{ is a subspace and }Q_{|V\times V}\text{ is negative definite}\}. \tag{2.4}$$

The weak Morse index indicates the number of directions whose associated variations decrease the energy $E$ of $\Sigma$.

In order to understand the Morse index, we establish a Sturm-Liouville eigenvalue problem. The parenthesis in the first integral of (2.2) defines the *Jacobi operator*

$J = \Delta + |A|^2$. Consider the eigenvalue problem

$$\begin{cases} J[u] + \lambda u = 0 \text{ in } \Sigma, \\ \dfrac{\partial u}{\partial \nu} = \mathbf{b}u \text{ in } \partial\Sigma, \\ u \in \mathcal{M}. \end{cases} \tag{2.5}$$

The second equation in (2.5) is called the Robin boundary condition. By the ellipticity of $J$, the eigenvalues of (2.5) are ordered as a discrete spectrum $\lambda_1 < \lambda_2 \leq \lambda_3 \cdots \uparrow \infty$ counting multiplicities. The weak Morse index of $\Sigma$ coincides with the number of negative eigenvalues of (2.5). Thus $\Sigma$ is stable if $\text{index}_w(\Sigma) = 0$. In general, it is difficult to work with the mean zero condition $\int_\Sigma u = 0$. Instead of (2.5), we drop the condition $u \in \mathcal{M}$ and consider the Sturm-Liouville eigenvalue problem

$$\begin{cases} J[u] + \lambda u = 0 \text{ in } \Sigma, \\ \dfrac{\partial u}{\partial \nu} = \mathbf{b}u \text{ in } \partial\Sigma. \end{cases} \tag{2.6}$$

The *Morse index* of $\Sigma$, denoted by $\text{index}(\Sigma)$, is defined similarly

$$\text{index}(\Sigma) = \dim\ \max\{V \subset C_0^\infty(\Sigma) : V \text{ is a subspace and } Q_{|V\times V} \text{ is negative definite}\}. \tag{2.7}$$

In the literature, the Morse index is also called the strong Morse index. This number coincides with the number of negative eigenvalues of (2.6). The numbers $\text{index}_w(\Sigma)$ and $\text{index}(\Sigma)$ cannot differ by more than one. In fact, the relation between both indices is

$$\text{index}_w(\Sigma) \leq \text{index}(\Sigma) \leq \text{index}_w(\Sigma) + 1. \tag{2.8}$$

Our objective is to find conditions that ensure that $\text{index}(\Sigma) \geq 2$ because from (2.8), we have $\text{index}_w(\Sigma) \geq 1$ and then $\Sigma$ is not stable. Only when $\text{index}(\Sigma) = 1$, a criteria on the stability of $\Sigma$ is not immediate. In this situation, we refer to [33, 34] to know when to decide whether $\Sigma$ is stable when $\text{index}(\Sigma) = 1$: see also the recent contributions [30, 31]. If $\text{index}(\Sigma) = 0$, we say that $\Sigma$ is *strongly stable*. In such a case, $E''(0) \geq 0$ for any admissible variation of $\Sigma$ regarding if the variation preserves or not the volume of $\Sigma$.

The notion of index for unbounded surfaces can be extended by taking a limit of the indices of an exhaustion of the surface. Here we follow similar steps as in the theory of complete minimal surfaces [9, 10]. If $\Sigma$ is an unbounded capillary surface, consider any bounded open subset $\Omega \subset \Sigma$. Then we can define the weak Morse index and the Morse index of $\Omega$ as in (2.4) and (2.7), respectively. Notice that for two bounded open sets $\Omega_1 \subset \Omega_2$, if $\Omega_1 \subset \Omega_2$, then $\text{index}_w(\Omega_1) \geq \text{index}_w(\Omega_2)$ and $\text{index}(\Omega_1) \geq \text{index}(\Omega_2)$. If $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Sigma$ is an exhaustion of $\Sigma$ by bounded open sets, the weak Morse index and the Morse index of $\Sigma$ are defined by

$$\text{index}_w(\Sigma) = \lim_{n\to\infty} \text{index}_w(\Omega_n), \quad \text{index}(\Sigma) = \lim_{n\to\infty} \text{index}(\Omega_n).$$

Both definitions are independent of the choice of the exhaustion of $\Sigma$ [10]. Both numbers can be infinite, but if they are finite, then the relation (2.8) holds too.

**3. The planar case.** In this section, we will study the stability of a planar strip as capillary surface supported on two parallel cylinders (Fig. 2). This setting can be viewed as the limit case of Fig. 1 when the capillary circular cylinder has infinity radius. First, we introduce the following notation, which it will be valid for the rest of the paper. Let $\mathsf{C}_1$ and $\mathsf{C}_{-1}$ be the cylinders of radius 1 separated $2d > 0$ and with axes parallel to the $z$-axis. We can assume that the

$$\mathsf{C}_\epsilon : (x - c_\epsilon)^2 + y^2 = 1, \quad c_\epsilon = \epsilon(1 + d), \quad \epsilon \in \{-1, 1\}. \tag{3.1}$$

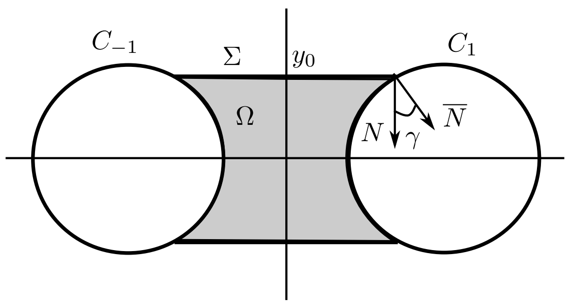


FIG. 2. *Top view of a planar strip $\Sigma$ as capillary surface on two cylinders $\mathsf{C}_1$ and $\mathsf{C}_{-1}$.*

Since the angle that makes $\Sigma$ is the same with $\mathsf{C}_1$ and $\mathsf{C}_{-1}$, then $\Sigma$ is a parallel plane to the plane containing the axes of $\mathsf{C}_1$ and $\mathsf{C}_{-1}$. We also assume that the liquid $\Omega$ is the domain determined by $\mathsf{C}_1 \cup \mathsf{C}_{-1}$ with the vertical planes of equations $y = \pm y_0$, $y_0 \in (0, 1)$: see the dark region in Fig. 2. Consider a connected component $\Sigma$ of the air-liquid interface, which we can assume the domain of the plane $y = y_0$, with $|x| \leq x_0$, where $x_0 = 1+d-\sqrt{1 - y_0^2}$. The parametrization of $\Sigma$ is $\Psi(x, z) = (x, y_0, z)$, $z \in \mathbb{R}$. The contact angle $\gamma$ is given by $\cos\gamma = y_0$. Notice that $\gamma \in (0, \frac{\pi}{2})$. Here $\overline{A}(\overline{\nu}, \overline{\nu}) = -1$ and $\mathbf{b} = -\frac{1}{\sin\gamma}$ because $A = 0$ on $\Sigma$. Then (2.6) is

$$\begin{cases} u_{xx} + u_{zz} + \lambda u = 0, \text{ in } \Sigma, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{1}{\sqrt{1 - y_0^2}} u = 0, \text{ in } \partial\Sigma. \end{cases} \tag{3.2}$$

Since $\Sigma$ is not bounded, and in order to make sense the volume constraint, for the unbounded planar strip $\Sigma$, consider the cut-off planes $\Pi_0$ and $\Pi_L$ of equations $z = 0$ and $z = L$, respectively, with $L > 0$. Take the part of $\Sigma$ between both planes and we study the stability of rectangles of $\Sigma$ determined by $\Sigma$ and the planes $\Pi_0$ and $\Pi_L$. Then the surface $\Sigma$ is replaced by the rectangle $\Sigma_L = \Psi([-x_0, x_0] \times [0, L])$. In this context, the admissible variations are those that preserve the volume of the liquid between $\Sigma$ and the planes $\Pi_0$ and $\Pi_L$. This approach has been used in the literature, for example by Vogel in [32, Sec. 2]. In consequence, the part of $\Sigma_L$ given by $\Sigma_L \setminus (\partial\Sigma_L \cap (\mathsf{C}_1 \cup \mathsf{C}_2))$ must be fixed in the admissible variations. Thus $u = 0$ along $\Sigma_L \setminus (\partial\Sigma_L \cap (\mathsf{C}_1 \cup \mathsf{C}_2))$, that is, $u = 0$ when $z = 0$ and $z = L$. Recall that $\Sigma_L$ makes a contact angle $\gamma$ along $\partial\Sigma \cap (\mathsf{C}_1 \cup \mathsf{C}_2)$.

To simplify the notation, let

$$\varpi = \frac{1}{\sqrt{1-y_0^2}} = \frac{1}{\sin\gamma}.$$

For the study of the eigenvalue problem (3.2), consider separation of variables, $u(x,z) = f(x)g(z)$. The Robin boundary condition in (3.2) are

$$\frac{\partial u}{\partial \nu}(x_0,z) = f'(x_0)g(z), \quad \frac{\partial u}{\partial \nu}(-x_0,z) = -f'(x_0)g(z).$$

Thus the eigenvalue problem to investigate is

$$\text{(3.3)} \qquad \begin{cases} f''(x)g(z) + f(x)g''(z) + \lambda f(x)g(z) = 0 \\ f'(x_0) + \varpi f(x_0) = 0, \\ f'(-x_0) - \varpi f(-x_0) = 0, \\ g(0) = g(L) = 0. \end{cases}$$

The last condition in (3.3) reflects that $u$ vanishes on $[-x_0, x_0] \times \{0, L\}$. If (3.3) has two or more negative eigenvalues, then there is a volume-preserving admissible variation of $\Sigma$ such that $E''(0) < 0$, proving that $\Sigma$ is not stable. If, on the other hand, all eigenvalues of (3.3) are positive, then $\Sigma$ is strongly stable.

Dividing the first equation of (3.3) by $f(x)g(z)$, we deduce that there is $C \in \mathbb{R}$ such that

$$\text{(3.4)} \qquad \frac{f''(x)}{f(x)} + \lambda = C = -\frac{g''(z)}{g(z)}.$$

Solving for $g$ together the boundary condition $g(0) = g(L) = 0$, we get

$$g(z) = d\sin(\frac{n\pi}{L}z), \quad d \in \mathbb{R},$$

where $n \in \mathbb{N}$ and $C = \frac{n^2\pi^2}{L^2}$. From (3.4), the function $f$ satisfies

$$f''(x) + \left(\lambda - \frac{n^2\pi^2}{L^2}\right) f(x) = 0.$$

The solutions depend on the sign of the parenthesis.

1. Case $\lambda - \frac{n^2\pi^2}{L^2} = 0$. The solution is $f(x) = ax + b$, $a, b \in \mathbb{R}$. Imposing the boundary conditions of (3.3), we have

$$\begin{aligned} \varpi x_0 a + (1+\varpi)\, b = 0, \\ \varpi x_0 a + (1-\varpi)\, b = 0. \end{aligned}$$

   There are non trivial solutions $a$ and $b$ if and only if the determinant of the coefficients of $a$ and $b$ is zero. This is equivalent to $\frac{x_0}{\varpi^2} = 0$. This shows that this case is not possible.
2. Case $\lambda - \frac{n^2\pi^2}{L^2} = -\delta^2$ with $\delta > 0$. The solution is

$$f(x) = a\cosh(\delta x) + b\sinh(\delta x), \quad a, b \in \mathbb{R}.$$

The boundary conditions of (3.3) give

$$(\varpi\cosh(\delta x_0)+\delta\sinh(\delta x_0))a+(\delta\cosh(\delta x_0)+\varpi\sinh(\delta x_0)b=0$$
$$(-\varpi\cosh(\delta x_0)-\delta\sinh(\delta x_0))a+(\delta\cosh(\delta x_0)+\varpi\sinh(\delta x_0)b=0.$$

Again, there are non trivial solutions if and only if the determinant of the coefficients of $a$ and $b$ is zero, that is, if

$$2\varpi\delta\cosh(2\delta x_0)+(\varpi^2+\delta^2)\sinh(2\delta x_0)=0. \tag{3.5}$$

This equation has no solutions because $\varpi>0$.

3. Case $\lambda-\frac{n^2\pi^2}{L^2}=\delta^2$ with $\delta>0$. The solution is

$$f(x)=a\cos(\delta x)+b\sin(\delta x),\quad a,b\in\mathbb{R}.$$

The boundary conditions of (3.3) give

$$(\varpi\cos(\delta x_0)-\delta\sin(\delta x_0))a+(\delta\cos(\delta x_0)+\varpi\sin(\delta x_0)b=0$$
$$(-\varpi\cos(\delta x_0)-\delta\sin(\delta x_0))a+(\delta\cos(\delta x_0)+\varpi\sin(\delta x_0)b=0.$$

There are non trivial solutions if and only if the determinant of the coefficients of $a$ and $b$ is zero, that is, if

$$2\varpi\delta\cos(2\delta x_0)+(\varpi^2-\delta^2)\sin(2\delta x_0)=0.$$

We have two subcases:

1. Subcase $\delta=\varpi$. Then $\cos(2\delta x_0)=0$. This implies

$$2\varpi x_0=\frac{\pi}{2}+m\pi,\quad m\in\mathbb{Z}.$$

2. Subcase $\delta\neq\varpi$. Then

$$\tan(2\delta x_0)=2\varpi\frac{\delta}{\delta^2-\varpi^2}.$$

This equation in $\delta$ has infinitely many solutions because of the behavior of the functions $\delta\mapsto\tan(2\delta x_0)$ and $\delta\mapsto 2\varpi\frac{\delta}{\delta^2-\varpi^2}$.

In both subcases, the eigenvalues are

$$\lambda_{m,n}=\delta^2+\frac{n^2\pi^2}{L^2}.$$

Since all $\lambda_{m,n}$ are positive, the surface $\Sigma_L$ is strongly stable for all $L>0$.

We summarize the above arguments.

THEOREM 3.1. *Any capillary planar strip supported on two parallel cylinders of the same radius is stable.*

*Remark* 3.2. We compare the result of Th. 3.1 with the situation of a capillary planar strip $\Sigma$ supported in the interior of a cylinder $\mathsf{C}$. This case was studied by Vogel proving that $\Sigma$ is not stable [35]. The difference is that now $\overline{A}(\overline{\nu},\overline{\nu})=1$ because the normal vector of $\mathsf{C}$ points towards the interior of the cylinder. Then $\mathbf{b}=1/\sin\gamma$. Repeating the computations, the case $\lambda-\frac{n^2\pi^2}{L^2}=-\delta^2$ with $\delta>0$, cannot be discarded because now $\varpi$ is negative and Eq. (3.5) has at least a solution $\delta_0$. With this value $\delta_0$, the eigenvalues are $\lambda_n=-\delta_0^2+\frac{n^2\pi^2}{L^2}$. Thus for $L$ sufficiently large (for example, $L>\frac{2\pi}{\delta_0}$), the number of negative eigenvalues is at least 2. Then index$\,(\Sigma_L)\geq 2$ and $\Sigma_L$ is not stable. This proves that $\Sigma$ is not stable.

**4. The convex cylinder case.** In this section we study capillary convex cylinders supported on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$. Following the notation (3.1), let $\Sigma$ be the cylinder of radius $r > 0$, $d < r < d + 2$ given by

$$\Sigma : x^2 + y^2 = r^2.$$

A parametrization of $\Sigma$ is

$$\Psi(\theta, t) = (r\cos\theta, r\sin\theta, t), \quad \theta, t \in \mathbb{R}. \tag{4.1}$$

The cylinder $\Sigma$ intersects $\mathsf{C}_1 \cup \mathsf{C}_{-1}$ in four vertical straight lines determined when the variable $\theta$ is $\theta_1$, $\pi - \theta_1$, $\pi + \theta_1$ and $2\pi - \theta_1$. By the sine formula, we have

$$\sin\theta_1 = \frac{\sin\gamma}{c}, \tag{4.2}$$

and $c = c_1 = 1 + d$. See Fig. 3. Notice that the angle $\theta_1$ satisfies $\theta_1 < \pi/2$. We only consider the parts of the cylinder $x^2 + y^2 = r^2$ that are capillary surfaces on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$, that is, when $\theta \in [\theta_1, \pi - \theta_1] \cup [\pi + \theta_1, 2\pi - \theta_1]$. Thus the capillary surface is formed by two connected components. By symmetry, it is enough to take one of such as components. Without loss of generality, the piece of cylinder, which we denote by $\Sigma$ again, will be the component corresponding to $\theta \in [\theta_1, \pi - \theta_1]$. Let $\Sigma = \Psi([\theta_1, \pi - \theta_1] \times \mathbb{R})$: see Fig. 3.

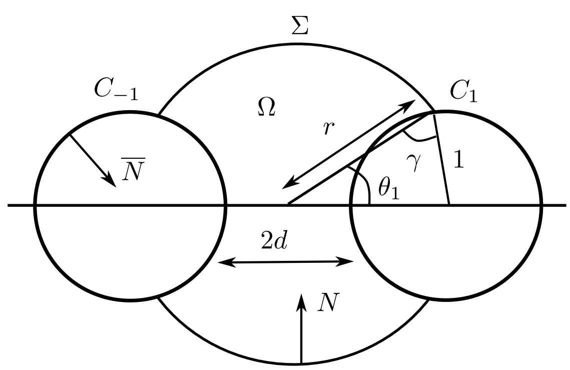


FIG. 3. *The convex cylinder case.*

The orientations $N$ towards $\Omega$ and $\overline{N}$ outwards $\Omega$ are

$$N(x, y, z) = -\frac{1}{r}(x, y, 0), \quad \overline{N}(x, y, z) = -(x - c_\epsilon, y, 0).$$

By the cosine formula we have

$$\cos\gamma = \frac{r^2 + 1 - c^2}{2r}. \tag{4.3}$$

Notice that the inequality $1 + r^2 < c^2$ is equivalent to $\gamma > \pi/2$

We now express the eigenvalue problem (2.6) in $(\theta, t)$-coordinates. The mean curvature of $\Sigma$ is $H = 1/(2r)$. Since the Gaussian curvature $K$ of $\Sigma$ is 0, then $|A|^2 = 1/r^2$. The Laplace operator is $\Delta = \frac{1}{r^2}\partial_\theta^2 + \partial_t^2$ because the first fundamental form of $\Sigma$ is $r^2d\theta^2+dt^2$. Let $u(\theta,t)=f(\theta)g(t)$ be a function on $\Sigma$ expressed by separable variables. The Jacobi operator of $u$ is

$$J[u]=\frac{1}{r^2}f''g+fg''+\frac{1}{r^2}fg. \tag{4.4}$$

The first equation of (2.6) is

$$\frac{1}{r^2}f''g+fg''+\left(\frac{1}{r^2}+\lambda\right)fg=0.$$

We now calculate the Robin boundary condition (2.6). In the cylinder $\Sigma$, we have $A(\nu,\nu)=\frac{1}{r}$. In the support cylinders $\mathsf{C}_1\cup\mathsf{C}_{-1}$ the second fundamental form is $\overline{A}(\overline{\nu},\overline{\nu})=-1$. Thus

$$\mathbf{b}=-\frac{1}{\sin\gamma}+\frac{\cos\gamma}{r\sin\gamma}=\frac{\cos\gamma-r}{r\sin\gamma}. \tag{4.5}$$

We calculate $\frac{\partial u}{\partial\nu}$. The unit exterior conormal vector $\nu$ is parallel to $\Psi_\theta$. Thus $\nu(\theta,t)=\pm\frac{1}{r}\Psi_\theta(\theta,t)$. We also have

$$\nabla u=\frac{1}{r^2}\langle\nabla u,\Psi_\theta\rangle\Psi_\theta+\langle\nabla u,\Psi_t\rangle\Psi_t=\frac{f'g}{r^2}\Psi_\theta+fg'\Psi_t.$$

Thus

$$\frac{\partial u}{\partial\nu}(\theta,t)=\langle\nabla u,\pm\frac{1}{r}\Psi_\theta\rangle=\pm\frac{1}{r}f'(\theta)g(t).$$

Since the unit exterior conormal $\nu$ is computed when $\theta=\theta_1$ and $\theta=\pi-\theta_1$, we have

$$\nu(\theta_1,t)=-\frac{1}{r}\Psi_\theta(\theta_1,t),\quad \nu(\pi-\theta_1,t)=\frac{1}{r}\Psi_\theta(\pi-\theta_1,t).$$

The Robin boundary condition $\frac{\partial u}{\partial\nu}-\mathbf{b}u=0$ in (2.6) is

$$\begin{aligned}-\frac{1}{r}f'(\theta_1)-\mathbf{b}f(\theta_1)&=0,\\ \frac{1}{r}f'(\pi-\theta_1)-\mathbf{b}f(\pi-\theta_1)&=0.\end{aligned}$$

Let

$$\mathbf{q}=r\mathbf{b}=\frac{\cos\gamma-r}{\sin\gamma}.$$

A first observation is that $\mathbf{q}\neq 0$. Indeed, if $\mathbf{q}=0$ then $\cos\gamma=r$, in particular $r<1$. From (4.3), we have $r^2+c^2=1$, hence $\sin\gamma=c$. Then the sine formula (4.2) yields $\theta_1=\pi/2$, which it is not possible.

As in the planar case (Sect. 3) and because $\Sigma$ is a non compact surface, we need to truncate $\Sigma$ in bounded pieces determined by the $z$-coordinate. Let $\Sigma_L=\Psi([\theta_1,\pi-\theta_1]\times[0,L])$. Then the eigenvalue problem to study is

$$\left\{\begin{aligned} f''(\theta)g(t)+\frac{1}{r^2}f(\theta)g''(t)+\left(\frac{1}{r^2}+\lambda\right)f(\theta)g(t)&=0,\\ f'(\theta_1)+\mathbf{q}f(\theta_1)&=0,\\ f'(\pi-\theta_1)-\mathbf{q}f(\pi-\theta_1)&=0,\\ g(0)=g(L)&=0.\end{aligned}\right. \tag{4.6}$$

*Remark* 4.1. This stability analysis of the capillary cylinder between two cylinders of equal radius can be extended to a configuration of three or more equidistant parallel cylinders situated in a symmetric position [26]. Under this situation, the air-liquid interface is formed by a finite number of connected components, all them making the same contact angle. By the symmetry of the configuration, the investigation of stability can be reduced to the case of two cylinders which it is the object of this paper.

We now solve (4.6). Dividing the first equation by $fg$, there must be a constant $C$ such that

$$\frac{f''(\theta)}{r^2 f(\theta)} + \frac{1}{r^2} + \lambda = C = -\frac{g''(t)}{g(t)}.$$

By the boundary conditions $g(0) = g(L) = 0$, the solution for $g$ is

$$g(t) = d \sin(\frac{n\pi}{L} t), \quad d \in \mathbb{R}, \tag{4.7}$$

where $C = \frac{n^2\pi^2}{L^2}$, $n \in \mathbb{N}$, $n \neq 0$. Using this value for $C$, the function $f$ satisfies

$$f'' + \left(1 + \lambda r^2 - \frac{n^2\pi^2 r^2}{L^2}\right) f = 0. \tag{4.8}$$

The solution of this equation depends on the sign of the parenthesis. We now do a general discussion of the solutions of (4.8) according this sign which it will allow later to get results under specific assumptions on the configuration of $\Sigma$.

1. Case $1 + \lambda r^2 - \frac{n^2\pi^2 r^2}{L^2} = 0$. The solution is $f(\theta) = a\theta + b$, $a, b \in \mathbb{R}$. The evaluation of the boundary conditions of (4.6) yields

$$(1 + \mathbf{q}\theta_1)a + \mathbf{q}b = 0,$$

$$(1 - \mathbf{q}(\pi - \theta_1))a - \mathbf{q}b = 0.$$

There are non trivial solutions $a$ and $b$ if and only if the determinant of the coefficients of $a$ and $b$ is 0. This becomes

$$2(\mathbf{q}\theta_1 + 1) = \pi \mathbf{q}.$$

Then

$$\theta_1 = \frac{\pi}{2} - \frac{1}{\mathbf{q}} = \frac{\pi}{2} - \frac{\sin\gamma}{\cos\gamma - r}. \tag{4.9}$$

In particular, $\mathbf{q} > 0$ because $\theta_1 < \pi/2$. Identity (4.9) establishes a particular relation between $\gamma$ and $r$. Indeed, from the sine formula (4.2), we have

$$\theta_1 = \sin^{-1}(\frac{\sin\gamma}{c}) = \frac{\pi}{2} - \frac{\sin\gamma}{\cos\gamma - r}.$$

The eigenvalues are

$$\lambda_n = -\frac{1}{r^2} + \frac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N}. \tag{4.10}$$

If $L$ is sufficiently large, then there are many natural numbers $n$ such that $\lambda_n$ is negative. This proves that index $(\Sigma_L) \geq 2$ and thus, $\Sigma$ is not stable.

2. Case $1+\lambda r^2-\frac{n^2\pi^2 r^2}{L^2}=-\delta^2$, where $\delta>0$. The solution is $f(\theta)=a\cosh(\delta\theta)+b\sinh(\delta\theta)$, $a,b\in\mathbb{R}$. Evaluating the boundary conditions of (4.6), we get

$$\begin{aligned}(\delta\sinh(\delta\theta_1)+\mathbf{q}\cosh(\delta\theta_1))a+(\delta\cosh(\delta\theta_1)+\mathbf{q}\sinh(\delta\theta_1))b&=0,\\ (\delta\sinh(\delta(\pi-\theta_1))-\mathbf{q}\cosh(\delta(\pi-\theta_1)))a+(\delta\cosh(\delta(\pi-\theta_1))-\mathbf{q}\sinh(\delta(\pi-\theta_1)))b&=0.\end{aligned}$$

Vanishing the determinant of the coefficients of $a$ and $b$, we have

$$2\mathbf{q}\delta\cosh(\delta(\pi-2\theta_1))-(\mathbf{q}^2+\delta^2)\sinh(\theta(\pi-2\theta_1))=0. \tag{4.11}$$

As a consequence, if $\mathbf{q}$ is negative, there are not solutions of this equation. If $\mathbf{q}$ is positive, then (4.11) gives

$$\tanh(\delta(\pi-2\theta_1))=2\frac{\mathbf{q}\delta}{\mathbf{q}^2+\delta^2}. \tag{4.12}$$

We see that this equation has one or two solutions. For $\delta>0$, define the functions

$$\begin{aligned}q(\delta)&=\tanh(\delta(\pi-2\theta_1)),\\ p(\delta)&=2\frac{\mathbf{q}\delta}{\mathbf{q}^2+\delta^2}.\end{aligned}$$

The function $q$ is increasing in $(0,\infty)$ with values from 0 at $\delta=0$ to 1 as $\delta\to\infty$. The function $p$ satisfies

$$\lim_{\delta\to 0^+}p(\delta)=\lim_{\delta\to\infty}p(\delta)=0.$$

Moreover, $p(\delta)$ is increasing in $(0,\mathbf{q})$ and decreasing in $(\mathbf{q},\infty)$ with a maximun at $\delta=\mathbf{q}$ where $p(\mathbf{q})=1$. Then (4.12) has one solution $\eta_1$ or two solutions $\eta_1<\eta_2$. This will depend whether $q'(0)\leq p'(0)$ or $q'(0)>p'(0)$ respectively. For each of these roots, the eigenvalues are

$$\mu_{j,n}=-\frac{\eta_j^2+1}{r^2}+\frac{n^2\pi^2}{L^2},\quad j=1,2,n\in\mathbb{N}. \tag{4.13}$$

If $L$ is sufficiently large, there are many natural numbers $n$ where $\mu_{j,n}<0$. This implies again $\mathrm{index}\,(\Sigma_L)\geq 2$ and thus $\Sigma$ is not stable.

3. Case $1+\lambda r^2-\frac{n^2\pi^2 r^2}{L^2}=\delta^2$, where $\delta>0$. The solution is now $f(\theta)=a\cos(\delta\theta)+b\sin(\delta\theta)$, $a,b\in\mathbb{R}$. Evaluating the boundary conditions of (4.6), we get

$$\begin{aligned}(-\delta\sin(\delta\theta_1)+\mathbf{q}\cos(\delta\theta_1))a+(\delta\cos(\delta\theta_1)+\mathbf{q}\sin(\delta\theta_1))b&=0,\\ (\delta\sin(\delta(\pi-\theta_1))+\mathbf{q}\cos(\delta(\pi-\theta_1)))a-(\delta\cos(\delta(\pi-\theta_1))-\mathbf{q}\sin(\delta(\pi-\theta_1)))b&=0.\end{aligned}$$

There are non trivial solutions if and only if the determinant of the coefficients of $a$ and $b$ is 0. This yields

$$\left(\delta^2-\mathbf{q}^2\right)\sin(\delta(\pi-2\theta_1))+2\delta\mathbf{q}\cos(\delta(\pi-2\theta_1))=0. \tag{4.14}$$

The eigenvalues are

$$\lambda=\frac{\delta^2-1}{r^2}+\frac{n^2\pi^2}{L^2}.$$

Because we are interesting to find negative eigenvalues, we need the condition $\delta<1$.

(a) Subcase $\delta = |\mathbf{q}|$. Then $\cos(\delta(\pi - 2\theta_1)) = 0$. Since $\delta$ is positive, there are $m \in \mathbb{N}$ such that Eq. (4.14) becomes

$$\delta(\pi - 2\theta_1) = \frac{\pi}{2} + m\pi. \tag{4.15}$$

Let $\delta_m = \delta$. The eigenvalues are

$$\lambda_{m,n} = \frac{\frac{(2m+1)^2\pi^2}{(2\pi-4\theta_1)^2} - 1}{r^2} + \frac{n^2\pi^2}{L^2}, \quad m, n \in \mathbb{N}.$$

From (4.15), it is clear that $\delta_m \geq \frac{3\pi}{2(\pi-2\theta_1)} > 1$ for $m \geq 1$ and thus $\lambda_{m,n} > 0$. Consider now $m = 0$ in (4.15). Inequality $\delta_0 < 1$ implies $\theta_1 < \frac{\pi}{4}$ provided $\delta = |\mathbf{q}|$. This yields

$$\theta_1 = \frac{\pi(2|\mathbf{q}| - 1)}{4|\mathbf{q}|}.$$

The condition $|\mathbf{q}| < 1$ is equivalent to $|\cos\gamma - r| < \sin\gamma$.

(b) Subcase $\delta \neq |\mathbf{q}|$. Then Eq. (4.14) becomes

$$\tan(\delta(\pi - 2\theta_1)) = 2\frac{\delta\mathbf{q}}{\mathbf{q}^2 - \delta^2}. \tag{4.16}$$

We see that this equation has many infinitely solutions. For $\delta > 0$, define the functions

$$\rho(\delta) = \tan(\delta(\pi - 2\theta_1)),$$
$$\varphi(\rho) = 2\frac{\delta\mathbf{q}}{\mathbf{q}^2 - \delta^2}.$$

The function $\rho$ is increasing in each interval $J_m := (\frac{(2m-1)\pi}{2(\pi-2\theta_1)}, \frac{(2m+1)\pi}{2(\pi-2\theta_1)})$, $m \in \mathbb{N}$, $m \geq 1$, with values from $-\infty$ to $\infty$. In the interval $J_0 := (0, \frac{\pi}{2(\pi-2\theta_1)})$, the function $\rho$ is increasing from 0 to $\infty$. The function $\rho$ only vanishes at the points $\frac{m\pi}{\pi-2\theta_1}$, $m \in \mathbb{N}$. The behavior of the function $\varphi$ depend on the sign of $\mathbf{q}$:

i. Case $\mathbf{q} > 0$. Then $\varphi$ is increasing. Moreover, $\varphi > 0$ in the interval $(0, \mathbf{q})$ with values from 0 to $\infty$. In the interval $(\mathbf{q}, \infty)$ we have $\varphi < 0$ with values from $-\infty$ to 0.

ii. Case $\mathbf{q} < 0$. The function $\varphi$ is decreasing. In the interval $(0, -\mathbf{q})$, we have $\varphi < 0$ with values from 0 to $-\infty$. In the interval $(-\mathbf{q}, \infty)$, we have $\varphi > 0$ with values from $\infty$ to 0.

By the properties of the functions $\rho$, there is unique solution $\delta_m$ of Eq. (4.16) in the interval $J_m$ for $m \geq 1$. The uniqueness of a solution $\delta_0$ also holds in $J_0$ if the solution exists.

If $\delta_m$ is a root of (4.16), the eigenvalues are

$$\lambda_{m,n} = \frac{\delta_m^2 - 1}{r^2} + \frac{n^2\pi^2}{L^2}, \quad m, n \in \mathbb{N}. \tag{4.17}$$

If $\delta_m < 1$, then $\lambda_{m,n}$ is negative for sufficiently large $L$. Since the instability of $\Sigma$ is assured if $\operatorname{index}(\Sigma_L) \geq 2$, then the condition $\lambda_{m,2} < 0$ gives instability of $\Sigma$. This inequality is equivalent to

$$L > \frac{2\pi r}{\sqrt{1 - \delta_m^2}}. \tag{4.18}$$

Since $\delta_m < \delta_k$ if $m < k$, then we have

$$\lambda_{m,n} < \lambda_{k,n}, \quad \lambda_{m,n} < \lambda_{m,r}, \tag{4.19}$$

if $m < k$ and $n < r$. In order to look a sharp inequality (4.18), we need to find the least root $\delta_m$ of (4.16) that satisfies the inequality $\delta_m < 1$.

Until here we have discussed the solutions of (4.8) according the sign of $1 + \lambda r^2 - \frac{n^2\pi^2 r^2}{L^2}$. Now we will deduce Plateau-Rayleigh results for long convex cylinders depending on the sign of $\mathbf{q}$. A type of these results will say that under some hypothesis, convex cylinders are not stable. A second type of results will be more interesting because we will get estimates of the length $L$ of unstable convex cylinders.

We distinguish the sign of $\mathbf{q}$. Let us observe that $\mathbf{q} < 0$ (resp. $\mathbf{q} > 0$) if and only if $\cos\gamma < r$ (resp. $\cos\gamma > r$).

THEOREM 4.2. *Let $\Sigma$ be a capillary convex cylinder of radius $r$ on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$. If $r < \cos\gamma$, then $\Sigma$ is not stable. Moreover, $\Sigma_L$ is not stable if*

$$L > \frac{2\pi r}{\sqrt{1+\eta_2^2}}, \tag{4.20}$$

*where $\eta_2$ is the largest root of* (4.12).

*Proof.* The assumption $r < \cos\gamma$ implies that $\mathbf{q}$ is positive. We have then seen that Eq. (4.12) has one or two solutions and this implies that $\Sigma$ is not stable.

We now estimate the length $L$ that ensures instability of $\Sigma_L$. We know that there are one or two solutions $\eta_1$, $\eta_2$ of (4.12), $\eta_1 < \eta_2$. To simplify the notation, if there is only one solution, we denote it by $\eta_2$. In (4.13), if $\mu_{2,2} < 0$, then we have at least two negative eigenvalues, and thus $\text{index}(\Sigma_L) \geq 2$, hence $\Sigma_L$ is not stable. The inequality $\mu_{2,2} < 0$ is equivalent to the inequality (4.20). In case that there are two solutions and because $\eta_1 < \eta_2$, then the inequality $L > \frac{2\pi r}{\sqrt{1+\eta_2^2}}$ is sharp than $L > \frac{2\pi r}{\sqrt{1+\eta_1^2}}$. ∎

It is possible to estimate the largest solution of (4.20). Notice that the function $p = p(\delta)$ has a maximum at $\delta = \mathbf{q}$. If there is only one solution, then $\eta_2 > \mathbf{q}$; if there are two solutions $\eta_1$ and $\eta_2$, then $\eta_1 < \mathbf{q} < \eta_2$. In both cases, $\eta_2 > \mathbf{q}$. Thus we have proved the following consequence.

COROLLARY 4.3. *Let $\Sigma$ be a capillary convex cylinder of radius $r$ on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$ such that $r < \cos\gamma$. If*

$$L > \frac{2\pi r \sin\gamma}{\cos\gamma - r}, \tag{4.21}$$

*then $\Sigma_L$ is not stable.*

*Proof.* Inequality (4.21) is just $L > \frac{2\pi r}{\sqrt{1+\mathbf{q}^2}}$. On the other hand $\frac{2\pi r}{\sqrt{1+\mathbf{q}^2}} \geq \frac{2\pi r}{\sqrt{1+\eta_2^2}}$ and we now apply Th. 4.2. ∎

The advantage of the estimate (4.21) is that it is given only in terms of the initial conditions and there is no need to solve numerically Eq. (4.12).

If $\mathbf{q} > 0$, a special case is when the angle $\theta_1$ satisfies (4.9). In such a case, the condition $\lambda_2 < 0$ in (4.10) implies $\text{index}(\Sigma_L) \geq 2$ and $\Sigma_L$ is not stable. The condition $\lambda_2 < 0$ is equivalent to $L > 2\pi r$ and this estimate for $L$ is worse than of Th. 4.2. However, the negative eigenvalues of $\Sigma_L$ comes from the solutions of equations (4.10) and (4.13). So, it is possible $\lambda_1 < 0$ and $\lambda_2 \geq 0$ which provides one negative eigenvalue. Now $\lambda_1 < 0$ is equivalent to $L > \pi r$. On the other hand, Eq. (4.12) has only one solution because (4.9) implies that $q'(0) = p'(0) = 2/\mathbf{q}$. Then other required negative eigenvalue to ensure that the index is bigger than or equal 2 comes from the inequality $\mu_{2,1} < 0$. This is equivalent to $L > \pi r/\sqrt{1+\eta_2^2})$. Since this inequality is consequence of $L > \pi r$, we have proved the following result.

COROLLARY 4.4. *Let $\Sigma$ be a capillary convex cylinder of radius $r$ on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$ such that $r < \cos\gamma$. Suppose that $\theta_1$ satisfies* (4.9). *If*

$$L > \pi r,$$

*then $\Sigma_L$ is not stable.*

The estimate (4.20) has been obtained by imposing that $\mu_{2,2}$ is negative. However, possible negative eigenvalues may appear by solving Eq. (4.16). Recall that this equation has many infinite solutions $\delta = \delta_m$ in each interval $J_m$, $m \geq 1$. The existence of a possible solution $\delta_0$ in the interval $J_0 = (0, \frac{\pi}{2(\pi-2\theta_1)})$ depends of the behaviour of the functions $p$ and $q$ in this interval. Recall that the corresponding eigenvalues $\lambda_{m,n}$ are negative if $\delta_m < 1$. In such a case, the inequality $\lambda_{m,n} < 0$ is equivalent to $L > \frac{n\pi r}{\sqrt{1-\delta_m^2}}$. Since we are looking a sharp estimate of $L$ for the existence of instability of $\Sigma_L$, then we focus in the smallest solution $\delta_m$ of (4.16) satisfying $\delta_m < 1$. If we only work with the eigenvalues $\lambda_{m,n}$, then index $(\Sigma_L) \geq 2$ is assured if $\lambda_{m,2} < 0$. This inequality is equivalent to $L > \frac{2\pi}{\sqrt{1-\delta_m^2}}$, which is worse than (4.20). Therefore the only case of interest to investigate is that a negative eigenvalue comes is $\mu_{2,1}$ and the other is $\lambda_{m,1}$. In other words, the situation that we have is the following:

1. If $L > \frac{\pi r}{\sqrt{1+\eta_2^2}}$, then $\mu_{2,1} < 0$. This provides a negative eigenvalues to the index of $\Sigma_L$.
2. If $L > \frac{2\pi r}{\sqrt{1+\eta_2^2}}$, then $\mu_{2,2} < 0$. This provides another negative eigenvalue to the index of $\Sigma_L$ and thus the index is at least 2.
3. If $L > \frac{\pi r}{\sqrt{1-\delta_m^2}}$, then $\lambda_{m,1}$ and this provides a negative eigenvalue. Since $\frac{\pi r}{\sqrt{1-\delta_m^2}} > \frac{\pi r}{\sqrt{1+\eta_2^2}}$, then $\mu_{2,1}$ is negative and the index of $\Sigma_L$ is at least 2.

As a conclusion, we have the next Plateau-Rayleigh result for convex cylinders.

COROLLARY 4.5. *Let $\Sigma$ be a capillary convex cylinder of radius $r$ on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$ such that $r < \cos\gamma$. Suppose that Eq.* (4.16) *has solutions less than* 1 *and denote by $\delta_m$ the smallest solution of* (4.16).

1. *If $3 - 4\delta_m^2 \leq \eta_2^2$, then $\Sigma_L$ is not stable provided $L > \frac{2\pi r}{\sqrt{1+\eta_2^2}}$.*
2. *If $3 - 4\delta_m^2 > \eta_2^2$, then $\Sigma_L$ is not stable provided $L > \frac{\pi r}{\sqrt{1-\delta_m^2}}$.*

*Proof.* It is immediate from the above arguments because the inequality $3-4\delta_m^2 \leq \eta_2^2$ is equivalent to $\frac{2\pi r}{\sqrt{1+\eta_2^2}} \leq \frac{\pi r}{\sqrt{1-\delta_m^2}}$. ∎

We give two numerical examples of Th. 4.2 where the solutions of (4.16) are all bigger than 1.

*Example* 4.6. 1. Suppose $\gamma = \frac{\pi}{10}$ and $r = 0.2$. From the cosine formula (4.3), the value of $c$ that defines the cylinders $\mathsf{C}_1$ and $\mathsf{C}_{-1}$ is $c = 0.812$. Then the sine formula (4.2) yields $\theta_1 = 0.390$. Now Eq. (4.12) has two roots whose numerical values are $\delta_1 = 2.414$ and $\delta_2 = 2.445$: see Fig. 4. Taking $\delta_2$, we deduce that $\Sigma_L$ is not stable if $L > \frac{2\pi r}{\sqrt{1+\delta_2^2}} = 0.475$.

In this case, the first solution of Eq. (4.16) is $\delta_1 = 1.890$ and thus all eigenvalue $\lambda_{m,n}$ in (4.17) are positive: see Fig. 4.

2. Consider $\gamma = \pi/3$ and the cylinder $\Sigma$ of radius $r = 1/4$. The cylinders $\mathsf{C}_1$ and $\mathsf{C}_{-1}$ are determined by the cosine formula (4.3), obtaining $c = \sqrt{13}/4$. Then the parameter $\theta$ belongs to the interval $[\theta_1, \pi - \theta_1]$ where $\theta_1$ is given by the sine formula (4.2), yielding $\theta_1 = 1.289$. With this information, we have $\mathbf{q} = \frac{1}{2\sqrt{3}}$. The equation (4.12) only has one root whose numerical value is $\delta_2 = 1.027$. Thus $\Sigma_L$ is not stable if $L > \frac{2\pi r}{\sqrt{1+\delta_2^2}} = 1.095$.

   Looking for solutions of Eq. (4.16), the first solution is $\delta_1 = 5.339$. This implies that all eigenvalue $\lambda_{m,n}$ in (4.17) are positive.

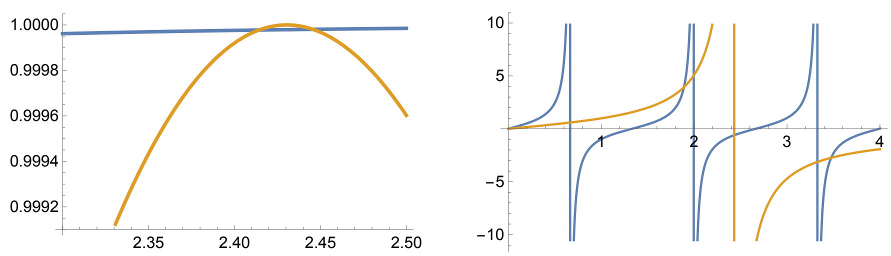


FIG. 4. *Case (1) of Example 4.6. Left: the solutions of Eq.* (4.12) *with two solutions. Right: the solutions of* (4.16) *where it can be observe that the first solution is bigger than* 1.

Now we show an example where the roots of Eq. (4.16) are less than 1. Following with Cor. 4.5, we give the two cases that appear in the statement.

*Example* 4.7. 1. Let $\gamma = \pi/5$ and $r = 0.2$. Then $c = 0.846$ and (4.2) gives $\theta_1 = 1.036$. The value of $\mathbf{q}$ is 1.03. Equation (4.12) only has one solution, $\eta_2 = 1.319$: see Fig. 5. However, Eq. (4.16) has a first solution $\delta_0$ in the interval $J_0$, $\delta_0 = 0.86 < 1$: see Fig. 5. Then $\lambda_{0,1} < 1$ if and only

$$L > \frac{\pi r}{\sqrt{1-\delta_0^2}} = 0.474.$$

   The condition $\mu_{2,1} < 0$ is equivalent to $L > \frac{\pi r}{\sqrt{1+\mu_{2,1}^2}} = 1.269$. Notice that the estimate in (4.20) is $L > \frac{2\pi r}{\sqrt{1+\delta_2^2}} = 0.759$ which it is better than $\frac{\pi r}{\sqrt{1-\delta_0^2}}$. Definitively, if $L > 0.759$, then $\Sigma_L$ is not stable. In terms of the statement of Cor. 4.5, we have $3 - 4\delta_0^2 = -0.019$ and $\eta_2^2 = 1.740$, so $3 - 4\delta_0^2 < \eta_2^2$.

2. Let $\gamma = \pi/10$ take $r = 0.6$. Then $c = 0.846$ and $\theta_1 = 1.036$. The value of $\mathbf{q}$ is 1.136. Equation (4.12) has only one solution, $\eta_2 = 1.378$. On the other hand, Eq. (4.16) has a first solution $\delta_0 = 0.383 \in J_0$. Now $3 - 4\delta_0^2 = 2.412$ and $\eta_2^2 = 1.899$. Thus if $L > \frac{\pi r}{\sqrt{1-\delta_0^2}} = 2.040$, then $\Sigma_L$ is not stable. Notice that in this case, we have $\frac{2\pi r}{\sqrt{1+\eta_2^2}} = 2.213$.

Consider now the case that $\mathbf{q}$ is negative. This is equivalent to $r > \cos\gamma$. In particular, this occurs if $r \geq 1$ or if $\gamma \in [\frac{\pi}{2}, \pi)$.

THEOREM 4.8. *Let $\Sigma$ be a capillary convex cylinder of radius $r$ on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$. If $r > \cos\gamma$ then $\Sigma$ is stable.*

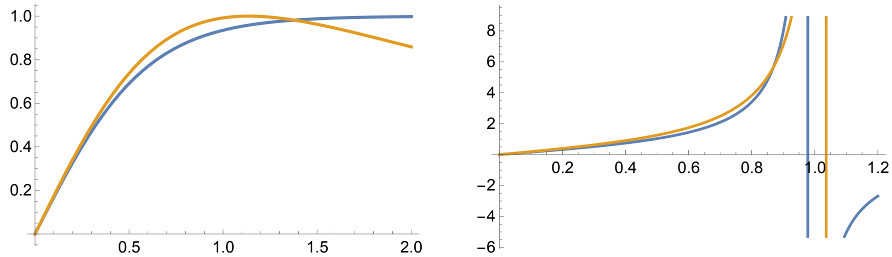


FIG. 5. *Case (1) of Example 4.7. Left: the solutions of Eq.* (4.12) *with one solutions. Right: the first solution of* (4.16) *is less than* 1.

*Proof.* Since $\mathbf{q}$ is negative, the cases $1+\lambda r^2-\frac{n^2\pi^2 r^2}{L^2}\leq 0$ are not possible. Let $1+\lambda r^2-\frac{n^2\pi^2 r^2}{L^2}=\delta^2$ and $\delta>0$. From the sine formula (4.2) we have

$$\tan^2\theta_1=\frac{\sin^2\gamma}{c^2-\sin^2\gamma}.$$

Since $r>\cos\gamma$ and $\theta_1\in(0,\pi/2)$, then $\tan\theta_1=\frac{\sin\gamma}{\sqrt{c^2-\sin^2\gamma}}$. The cosine formula (4.3) gives

$$c^2-\sin^2\gamma=(r-\cos\gamma)^2.$$

Thus

$$\tan\theta_1=\frac{\sin\gamma}{r-\cos\gamma}=-\frac{1}{\mathbf{q}} \tag{4.22}$$

and

$$\sin\theta_1=-\frac{\mathbf{q}}{\sqrt{1+\mathbf{q}^2}}. \tag{4.23}$$

A first case is $\delta=-\mathbf{q}$. From (4.15), we have $-\mathbf{q}<1$ and $-\mathbf{q}(\pi-2\theta_1)=\frac{\pi}{2}$. Thus $\theta_1=\frac{\pi}{2}+\frac{\pi}{4\mathbf{q}}$, in particular, $\mathbf{q}>1/2$. On the other hand, $\sin\theta_1=\cos(-\frac{\pi}{4\mathbf{q}})$ and using (4.23), we have

$$\cos(-\frac{\pi}{4\mathbf{q}})=\frac{-\mathbf{q}}{\sqrt{1+\mathbf{q}^2}}.$$

This equation on $\mathbf{q}$ has no solutions in the interval $(\frac{1}{2},1)$. This proves that this case is not possible.

Suppose now $\delta\neq-\mathbf{q}$. The study of stability requires the analysis of the solutions of Eq. (4.16). We will prove that the first solution $\delta_m$ of this equation is bigger than or equal to 1. Consequently from (4.17) and (4.19), the eigenvalues $\lambda_{k,n}$ are non-negative for all $k,n\in\mathbb{N}$. This shows that $\Sigma_L$ is strongly stable for all $L>0$, hence $\Sigma$ is stable.

First we see that $\delta=1$ is a solution of (4.16). From (4.22), we have

$$\rho(1)=\tan(\pi-2\theta_1)=-\tan(2\theta_1)=\frac{2\mathbf{q}}{\mathbf{q}^2-1}=\varphi(1).$$

This proves that $\delta=1$ is a solution of Eq. (4.16).

In a second step, we prove that the first solution of Eq. (4.16) is bigger than 1. We distinguish three cases.

1. Case $\theta_1 < \pi/4$. Since $\theta_1 < \pi/4$, then $1 > \frac{\pi}{2(\pi-2\theta_1)}$. Because the inequality $1 < \frac{3\pi}{2(\pi-2\theta_1)}$ always holds, we have $1 \in J_1$. On the other hand, and using (4.23), the condition $\theta_1 < \pi/4$ is equivalent to $-\mathbf{q} > 1$. Thus $\frac{\pi}{2(\pi-2\theta_1)} < -\mathbf{q}$. By the behaviour of the functions $\rho$ and $\varphi$, the first root of (4.16) must belong to $J_1$. By unicity of the solutions of (4.16) in each interval $J_m$, we have proved $1 = \delta_1$ and $\delta_1$ is the first solution of Eq. (4.16).
2. Case $\theta_1 > \pi/4$. Now this inequality is equivalent to $1 < \frac{\pi}{2(\pi-2\theta_1)}$. By using (4.23), the condition $\theta_1 > \pi/4$ is equivalent to $-\mathbf{q} < 1$. Thus $-\mathbf{q} < \frac{\pi}{2(\pi-2\theta_1)}$ and this implies that the first root of (4.16) must belong to $J_0$. By unicity $1 = \delta_0$ and $\delta_0$ is the first solution of Eq. (4.16).
3. Case $\theta_1 = \pi/4$. Then $1 = \frac{\pi}{2(\pi-2\theta_1)}$ and (4.23) gives $-\mathbf{q} = 1$. Then the first root of (4.16) lies in $J_1$, in particular, $\delta_1 > \frac{\pi}{2(\pi-2\theta_1)} = 1$. Thus the first solution of Eq. (4.16) is bigger than 1. □

From this theorem, it deserves to highlight three special cases of the inequality $r > \cos\gamma$.

COROLLARY 4.9. *Let $\Sigma$ be a capillary convex cylinder on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$. In the next three situations, $\Sigma$ is stable:*
1. *The intersection of $\Sigma$ with $\mathsf{C}_1 \cup \mathsf{C}_{-1}$ is orthogonal.*
2. *The contact angle $\gamma$ satisfies $\gamma > \pi/2$.*
3. *The radius $r$ of $\Sigma$ satisfies $r \geq 1$.*

**5. The concave cylinder case.** Suppose now that $\Sigma$ is a concave cylinder supported on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$. As in the convex cylinder case (Sect. 4), the air-liquid interface formed between the cylinders $\mathsf{C}_1$ and $\mathsf{C}_{-1}$ has two connected components. We will consider one of such component $\Sigma$. The notation will be the following. Without loss of generality, $\Sigma$ is the concave cylinder of radius $r > 0$ of equation

$$\Sigma : x^2 + (y + y_0)^2 = r^2,$$

with $y_0 > 0$ and $0 < r < y_0$. A parametrization of $\Sigma$ is

$$\Psi(\theta, t) = (r\cos\theta, -y_0 + r\sin\theta, t), \quad \theta \in [\theta_1, \pi - \theta_1], t \in \mathbb{R}, \tag{5.1}$$

where the value of $\theta_1$ is determined by the contact of the cylinder with the supports $\mathsf{C}_1 \cup \mathsf{C}_{-1}$. See Fig. 6. Given $\mathsf{C}_1 \cup \mathsf{C}_{-1}$, the existence of a concave cylinder liquid is not assured for any value of $y_0$. For this, it is necessary that $\sqrt{c^2 + y_0^2} - 1 < y_0$, where $c = c_1 = 1 + d$. This is equivalent to $c^2 - 1 < 2y_0$. Once we get the value of $y_0$, let $R = \sqrt{c^2 + y_0^2}$. Then the value of the radius of $r$ of $\Sigma$ varies in the interval $(R-1, y_0)$. If $r \to R - 1$, then $\gamma \to 0$. Notice that $0 < \gamma < \pi/2$.

The domain $\Omega$ (liquid) is bounded by $\Sigma$, the reflection of $\Sigma$ about the $xz$-plane and $\mathsf{C}_1 \cup \mathsf{C}_{-1}$. The orientations $N$ towards $\Omega$ and $\overline{N}$ outwards $\Omega$ are, respectively,

$$N(x, y, z) = \frac{1}{r}(x, y + y_0, 0), \quad \overline{N}_\epsilon(x, y, z) = -(x - c_\epsilon, y, 0).$$

The cosine formula and sine formulas give

$$\cos\gamma = \frac{R^2 - r^2 - 1}{2r}, \quad \frac{R}{\sin\gamma} = \frac{1}{\sin(\theta_1 - \phi)}, \tag{5.2}$$

where the angle $\phi$ is given by $\sin\phi = y_0/R$ (Fig. 6).

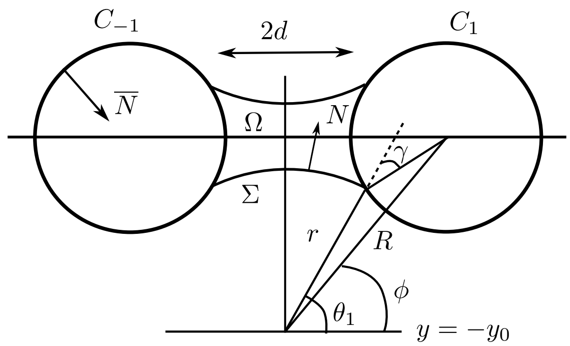


FIG. 6. *The concave cylinder case.*

The difference in the eigenvalue problem (2.6) with respect to the convex case is the computation of $\mathbf{b}$. Now $A(\nu,\nu) = -\frac{1}{r}$ and $\overline{A}(\overline{\nu},\overline{\nu}) = -1$. Thus

$$\mathbf{b} = -\frac{\cos\gamma + r}{r\sin\gamma}.$$

Let

$$\mathbf{q} = r\mathbf{b} = -\frac{\cos\gamma + r}{\sin\gamma}.$$

Notice that $\mathbf{q}$ is negative because $\gamma < \pi/2$.

We now truncate $\Sigma$ by compact pieces of length $L > 0$ in the $z$-coordinate. Let $\Sigma_L = \Psi([\theta_1, \pi - \theta_1] \times [0, L])$. First we have

$$g(t) = d\sin(\frac{n\pi}{L}t), \quad d \in \mathbb{R},$$

where $C = \frac{n^2\pi^2}{L^2}$, $n \in \mathbb{N}$, $n \neq 0$. The discussion for $f$ is similar as in the convex case given in Th. 4.8. Following the same arguments, the cases $1 + \lambda r^2 - \frac{n^2\pi^2 r^2}{L^2} \leq 0$ are not possible because the constant $\mathbf{q}$ is negative.

THEOREM 5.1. *Any capillary concave cylinder on $\mathsf{C}_1 \cup \mathsf{C}_{-1}$ is stable.*

*Proof.* As we have seen, the discussion of the eigenvalues of the Sturm-Liouville eigenvalue problem in $\Sigma_L$ reduces to the case $1 + \lambda r^2 - \frac{n^2\pi^2 r^2}{L^2} = \delta^2$ and $\delta > 0$. We will prove that all eigenvalues $\lambda$ are non-negative. As in the proof of Th. 4.8, the cosine and the sine formulas (5.2) give

$$\tan(\theta_1 - \phi) = -\frac{1}{\mathbf{q}}, \quad \sin(\theta_1 - \phi) = -\frac{\mathbf{q}}{\sqrt{1 + \mathbf{q}^2}}. \tag{5.3}$$

The first case to consider is $\delta = -\mathbf{q}$. As in the proof of Th. 4.8, this case is not possible.

Suppose now $\delta \neq -\mathbf{q}$. We need to study the solutions of Eq. (4.16). However, from (5.3) and as in the proof of Th. 4.8, we know that the solutions of

$$\tan(\delta(\pi - 2(\theta_1 - \phi))) = 2\frac{\delta\mathbf{q}}{\mathbf{q}^2 - \delta^2} \tag{5.4}$$

□

are bigger than or equal to 1. Notice that the vertical asymptotes of the graphic of the function $\delta \mapsto \tan(\delta(\pi - 2(\theta_1 - \phi)))$ are displaced to the left direction with respect to that of the graphic of the function $\rho(\delta) = \tan(\delta(\pi - 2\theta_1))$. This proves that the first root of (4.16) is bigger than the first root of (5.4) and consequently, the first root of Eq. (5.4) is strictly bigger than 1. Again all eigenvalues $\lambda_{m,n}$ are positive for all $m, n \in \mathbb{N}$. This proves that $\Sigma_L$ is strongly stable and thus $\Sigma$ is stable.

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