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PLATEAU-RAYLEIGH INSTABILITY OF SINGULAR MINIMAL SURFACES

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Abstract. We prove a Plateau-Rayleigh criterion of instability for singular minimal surfaces, providing explicit bounds on the amplitude and length of the surface. More generally, we study the stability of α-singular minimal hypersurfaces considered as hypersurfaces in weighted manifolds. If $\alpha < 0$ and the hypersurface is a graph, then we prove that the hypersurface is stable. If $\alpha > 0$ and the surface is cylindrical, we give numerical evidences of the instability of long cylindrical α-singular minimal surfaces.

1. **Introduction and results.** The phenomenon of the Plateau-Rayleigh instability shows that long circular cylinders are unstable as surfaces of constant mean curvature. This was investigated by Plateau in the second half of the 19th century ([20]). To be precise, fixed a radius $r > 0$, we ask what lengths L of cylindrical columns of radius r determine stable cylinders. Rayleigh derived a relation between the r and L, proving that the length L of stable circular cylinders must be less than $2\pi r$ ([21]). Analogous results in other contexts are [6, 13, 14, 16, 19]. The aim of this work is to investigate the Plateau-Rayleigh instability of singular minimal surfaces. In this Introduction, we recall the definition of singular minimal surfaces, we then present the notion of stability and finally, we formulate the question on the Plateau-Rayleigh instability.

Let $\alpha \in \mathbb{R}$ be a real number and let $\mathbb{R}^{n+1}_+ = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ be the upper halfspace of $\mathbb{R}^{n+1}$, where $x = (x_1, \ldots, x_{n+1})$ stand for the canonical coordinates of Euclidean space. An (orientable) hypersurface Σ of $\mathbb{R}^{n+1}_+$ is said to be an α-*singular minimal hypersurface* if its mean curvature H satisfies

$$H(x) = \frac{\alpha}{n}\frac{N_{n+1}(x)}{x_{n+1}}, \quad x \in \Sigma, \tag{1}$$

where $N = (N_1, \ldots, N_{n+1})$ is the unit normal vector field of Σ. In a non-parametric form, if Σ is locally the graph of a function $u = u(x_1, \ldots, x_n)$, then (1) writes as

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\alpha}{u\sqrt{1+|Du|^2}}$$

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where u is defined in a domain of $\mathbb{R}^n$. Some interesting cases of α are the following. If $\alpha = 0$ then Σ is a minimal hypersurface. If $\alpha = -n$ then Σ is a minimal hypersurface of the hyperbolic space when $\mathbb{R}^{n+1}_+$ is endowed with the hyperbolic metric. If $\alpha = 1$, Σ is the n-dimensional analogue of the catenary ([3, 9]). For this situation, and following Dierkes, we say that Σ is a *singular minimal hypersurface* ([7, 8]). The case $\alpha = 0$ will be discarded in this paper.

Hypersurfaces that satisfy (1) are solutions of a variational problem in the theory of weighted manifolds pioneered by Gromov ([12]) and consequently, it makes sense to consider the stability of these hypersurfaces. Let ψ be a density function on $\mathbb{R}^{n+1}$ and consider the weighted area and weighted volume elements in $\mathbb{R}^{n+1}$ defined by

$$d\mathcal{A}_\psi = e^\psi d\mathcal{A}, \quad d\mathcal{V}_\psi = e^\psi d\mathcal{V},$$

respectively, where $\mathcal{A}$ and $\mathcal{V}$ denote the Euclidean area and volume, respectively. The formulas of the first variation of weighted area and the weighted volume are

$$\mathcal{A}'_\psi(0) = -\int_\Sigma u\left(nH - \langle\overline{\nabla}\psi, N\rangle\right)\, d\mathcal{A}_\psi, \quad \mathcal{V}'_\psi(0) = \int_\Sigma u\, d\mathcal{A}_\psi, \tag{2}$$

where $\langle,\rangle$ is the Euclidean metric, $\overline{\nabla}$ is the Euclidean gradient on $\mathbb{R}^{n+1}$ and u is the normal component of the variational vector field associated to the variation. The function H_ψ defined by

$$H_\psi = H - \frac{1}{n}\langle\overline{\nabla}\psi, N\rangle$$

is called the *weighted mean curvature* of Σ. From (2), we deduce that Σ is a critical point of the weighted area functional $\mathcal{A}_\psi$ if and only if $H_\psi = 0$ on Σ. In the case that the family of admissible variations preserve the weighted volume of Σ, then Σ is a critical point of $\mathcal{A}_\psi$ if and only if H_ψ is constant on Σ. If we particularize the density to be

$$\psi(x) = \alpha \log x_{n+1},$$

then the equation $H_\psi = 0$ coincides with (1). In this context, it is natural to ask if these hypersurfaces are stable in the sense to be local minimum of the weighted area. We say that Σ is *strongly ψ-stable* (resp. *ψ-stable*) if $\mathcal{A}''_\psi(0) \geq 0$ for any variation (resp. weighted volume preserving variation) of Σ. The study of stability α-singular minimal hypersurfaces from the viewpoint of weighted minimal hypersurfaces has not received attention in literature, and in this paper we aim to make a first approximation.

A first result on stability that we will prove in Section 3 is that if $\alpha < 0$, then any α-singular minimal hypersurface which is a graph on a horizontal hyperplane is strongly ψ-stable (Theorem 3.2). This result is analogous in other weighted manifolds: minimal graphs ([11]), self-shrinkers ([5]), translators ([23]) or Lagrangian translators ([24]). However, for α-singular minimal hypersurfaces, the sign of α plays a key role, and the result does not hold if α is positive.

As expected, the expression $\mathcal{A}''_\psi(0)$ is not easy to manage so we have to turn to a type of hypersurfaces whose geometry is simple and easy to handle. It is here when we return to the initial motivation of the Plateau-Rayleigh instability phenomenon and we will focus on surfaces ($n = 2$) that are invariant in one direction of $\mathbb{R}^3$, also called cylindrical surfaces.

Let Σ be a cylindrical surface of $\mathbb{R}^3_+$ parametrized by $X(s,t) = \gamma(s) + t\vec{a}$, where $t \in \mathbb{R}$, $\gamma = \gamma(s)$, $s \in I \subset \mathbb{R}$, is a planar curve and $\vec{a}$ is a unit orthogonal vector to the plane containing γ. If Σ is an α-singular minimal surface, then either Σ is a vertical plane or the vector $\vec{a}$ is parallel to the horizontal plane $P_0 = \{x \in \mathbb{R}^3 : x_3 = 0\}$. In the latter case, the curvature of γ satisfies the one-dimensional version of (1) and γ must be a symmetric graph on P_0 ([15]). After a change of coordinates, Σ can be written as $X(s,t) = (s, t, h(s))$, where $h\colon (-m, m) \to \mathbb{R}$ is a smooth function and $(-m, m)$ is its maximal domain. In particular, Σ is a graph on the plane P_0 and if, in addition, $\alpha < 0$, then Σ is strongly ψ-stable by Theorem 3.2.

Suppose now $\alpha > 0$. Consider a bounded symmetric piece of Σ,

$$\Sigma(a, L) = \{X(s,t) : -a \leq s \leq a,\ 0 \leq t \leq L\},$$

where $a < m$. We call a the *amplitude* and L the *length* of $\Sigma(a, L)$, respectively. This is just the same scenario that columns of circular cylinders in the classical setting of the Plateau-Rayleigh instability. Following this, we pose the next

> **Question.** Let $a > 0$ be a given amplitude. Does exist a length $L_0 > 0$ such that the surface $\Sigma(a, L)$ is unstable if $L > L_0$?

Following the classical situation of circular cylinders, one can think that if the amplitude a is firstly fixed, then $\Sigma(a, L)$ is ψ-stable for small values of L, but if $L \to \infty$, there is a first length L_0 where $\Sigma(a, L)$ is not ψ-stable for $L > L_0$. The function h can only be explicitly integrated from (1) in a few particular cases of α. An example is $\alpha = 1$ where, up to integration constants, h is the catenary $h(s) = \cosh(s)$. Under this situation, we give an answer to the Question in Section 4. We prove in Theorem 4.1 that *there exists a value* $a_0 > 0$ *such that for each* $a > a_0$*, there is a critical length* L_0*, depending on* a*, such that* $\Sigma(a, L)$ *is not strong* ψ*-stable for all* $L > L_0$. In Theorem 4.2, we also prove a similar result for the problem of ψ-stability. Finally in Theorem 4.3 we extend both theorems to a stability problem with mixed boundary conditions.

All these results can be interpreted in the field of architecture. As we said, singular minimal surfaces are surfaces that generalize the property of the catenary of having the lowest center of gravity. The shape of a flexible hanging surface of uniform mass acting upon solely by gravity already attracted the interest of mathematicians such as Beltrami, Germain, Jellet, Lagrange and Poisson. The reader can see a historical account in [8]. So, a singular minimal surface is, after upside down, a model for cupolas in the sense that loads and tensions act tangentially on the roof, giving solidity to the construction. These surfaces are models of the shape of a 'hanging roof' for the construction of perfect cupolas according to the architect Frei Otto ([18]). Years earlier to the Otto's ideas, the Spanish architect Antonio Gaudí used cylindrical singular minimal surfaces in the construction of roofs of corridors. A clear example appears in the Colegio de las Teresianas, Barcelona, (1888-1890), where Gaudí constructed a corridor taking the shape of an inverted catenary and repeating in a horizontal direction of the space (Figure 1, left). Compare with the singular minimal surface of cylindrical type constructed with the catenary $h(s) = \cosh(s)$ in Figure 1, right.

As a consequence of Theorems 4.1 and 4.2, long corridors constructed by catenaries should be unstable under the mathematical viewpoint.

For the case $\alpha > 0$ in general, the function h is given in terms of elliptic functions. In Section 5 we make use of numerical methods and we will implement functions of the `Mathematica`© software to investigate if the Plateau-Rayleigh instability is true for general positive values of α. We will show particular cases of α where Theorems 4.1 and 4.2 are equally valid, proving numerically that for values of the amplitude bigger than a certain number a_0, there is a critical length L_0 such that $\Sigma(a, L)$ is strong ψ-stable for all $L > L_0$. We conclude with an appendix (Section 6) showing

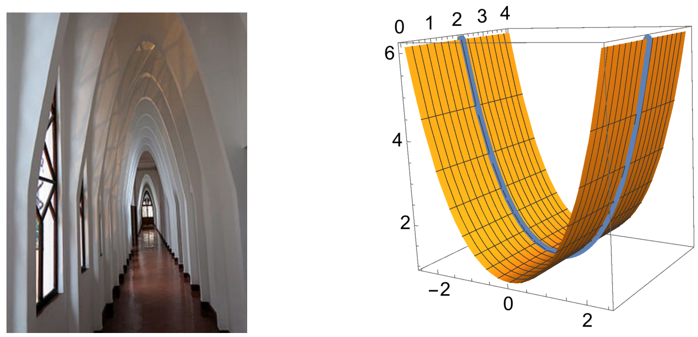

FIGURE 1. Left: corridor in the Colegio de las Teresianas, Barcelona ([25]). Right: the singular minimal surface $\{(s, t, \cosh(s)) : s, t \in \mathbb{R}\}$ constructed by repeating a catenary (blue) in a horizontal direction.

that α-minimal singular hypersurfaces are minimal hypersurfaces in $\mathbb{R}^{n+1}_+$ with a conformal metric to the Euclidean one and calculating the sectional curvatures of this space.

2. **Preliminaries.** In this section, we recall the formula of the second variation of the area in weighted manifolds and the properties of α-singular minimal surfaces of cylindrical type. Let $\psi \in C^\infty(\mathbb{R}^{n+1})$ be a density function, which is now arbitrary, but later we will take $\psi(x) = \alpha \log x_{n+1}$ defined in $\mathbb{R}^{n+1}_+$. Let Σ be a hypersurface with $H_\psi = 0$ or constant H_ψ depending if we consider arbitrary variations or only weighted volume-preserving variations, respectively. The second derivative of $\mathcal{A}_\psi$ computed for compactly supported normal variations of Σ is

$$\mathcal{A}''_\psi(0) = -\int_\Sigma u\left(\Delta u + \langle \nabla\psi, \nabla u\rangle + (|A|^2 - \overline{\nabla}^2\psi(N,N))u\right)\, d\mathcal{A}_\psi.$$

Here $|A|^2$ denotes the square of the norm of the second fundamental form A of Σ, ∇ and Δ are the gradient and the Laplacian computed on Σ with the induced Euclidean metric and $\overline{\nabla}^2$ is the Euclidean Hessian operator in $\mathbb{R}^{n+1}$. In the case that the admissible variations preserve the weighted volume of Σ, from (2), the function u satisfies the extra condition $\int_\Sigma u\, d\mathcal{A}_\psi = 0$. We refer to [1, 4] for details.

For a vector field Z on Σ, the *ψ-divergence* of Z is defined by

$$\mathrm{div}_\psi Z = \mathrm{div} Z + \langle \nabla\psi, Z\rangle,$$

where div is the Euclidean divergence in Σ. Hence, the *ψ-Laplacian* of a function $u \in C^\infty(\Sigma)$ is

$$\Delta_\psi u = \mathrm{div}_\psi \nabla u = \Delta u + \langle \nabla u, \nabla\psi\rangle.$$

The *ψ-Jacobi operator* is defined by

$$L_\psi[u] = \Delta_\psi u + (|A|^2 - \overline{\nabla}^2\psi(N,N))u,$$

acting on the space $C_0^\infty(\Sigma)$ of all compactly supported functions on Σ. Both Δ_ψ and L_ψ are not self-adjoint with respect to the L^2-inner product, but they are with respect to the weighted inner product $\int_\Sigma uv\, d\mathcal{A}_\psi$. The expression of $\mathcal{A}''_\psi(0)$ allows to define the quadratic form

$$Q_\psi[u] = -\int_\Sigma u \cdot L_\psi[u]\, d\mathcal{A}_\psi, \quad u \in C_0^\infty(\Sigma). \tag{3}$$

Integrating by parts and using the ψ-divergence theorem for weighted manifolds ([4]), Q_ψ can be written as

$$Q_\psi[u] = \int_\Sigma \left(|\nabla u|^2 - (|A|^2 - \overline{\nabla}^2 \psi(N,N))u^2 \right) d\mathcal{A}_\psi.$$

We point out that α-singular minimal hypersurfaces are also minimal hypersurfaces in the Riemannian manifold $(\mathbb{R}^{n+1}, e^{2\psi/n}\langle,\rangle)$ which is conformal to the Euclidean space $(\mathbb{R}^{n+1}, \langle,\rangle)$. However a density is not equivalent to conformally scaling the metric by e^ψ because the area and the volume would change with different scaling factors. See details in Section 6.

We apply these preliminaries to α-singular minimal hypersurface where now $\psi = \alpha \log x_{n+1}$. We will use the affine concepts of horizontal and vertical to indicate parallel to the hyperplane $P_0 = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ and parallel to the vector e_{n+1}, respectively, where $\{e_1, \ldots, e_{n+1}\}$ is the canonical basis of $\mathbb{R}^{n+1}$. Let us observe that any horizontal translation, any rotation about a vertical axis and any dilation with respect to points of P_0 preserve the value of H_ψ.

We compute each one of the terms of the ψ-Jacobi operator. Since

$$\Delta_\psi u = \Delta u + \frac{\alpha}{x_{n+1}}\langle \nabla u, e_{n+1}\rangle, \quad \overline{\nabla}^2\psi(N,N) = -\alpha \frac{N_{n+1}^2}{x_{n+1}^2},$$

the ψ-Jacobi operator is

$$L_\psi[u] = \Delta u + \frac{\alpha}{x_{n+1}}\langle \nabla u, e_{n+1}\rangle + \left(|A|^2 + \alpha \frac{N_{n+1}^2}{x_{n+1}^2} \right) u. \tag{4}$$

Standard spectral theory ensures that L_ψ has real eigenvalues $\lambda_1 < \lambda_2 \leq \ldots$ with $\lambda_k \to \infty$ ([10]). Strong ψ-stability is equivalent to the first eigenvalue being nonnegative.

We show that vertical and horizontal hyperplanes, upper hemispheres and upper half-cylinders are examples of α-singular minimal hypersurfaces.

1. Vertical hyperplanes. The value of H_ψ is 0 and they are strongly ψ-stable because $Q_\psi[u] = \int_\Sigma |\nabla u|^2\, d\mathcal{A}_\psi \geq 0$.
2. Horizontal hyperplanes. If Σ is the horizontal hyperplane of equation $x_{n+1} = a > 0$, then for $N = e_{n+1}$, the weighted mean curvature is $H_\psi = -\alpha/(an)$. Since $A = 0$ and $N_{n+1}^2 = 1$, then
$$Q_\psi[u] = \int_\Sigma \left(|\nabla u|^2 - \frac{\alpha}{a^2}u^2 \right) d\mathcal{A}_\psi,$$
hence Σ is strongly ψ-stable if $\alpha < 0$.
3. Hemispheres $\mathbb{S}_+^n(c,r)$ centered at $c \in P_0$ and of radius $r > 0$. If $N(x) = -(x-c)/r$, then $H_\psi = (n-\alpha)/(rn)$ and
$$Q_\psi[u] = \int_\Sigma \left(|\nabla u|^2 - \left(\frac{n+\alpha}{r^2} \right) u^2 \right) d\mathcal{A}_\psi.$$
Then $\mathbb{S}_+^n(c,r)$ is strongly ψ-stable if $\alpha \leq -n$.

4. Upper half-cylinders $\mathbb{S}^1_+(c,r)\times\mathbb{R}^{n-1}$. Now $H_\psi=(1+\alpha)/(nr)$ and

$$Q_\psi[u]=\int_\Sigma\left(|\nabla u|^2-\left(\frac{1+\alpha}{r^2}\right)u^2\right)\,d\mathcal{A}_\psi.$$

Then $\mathbb{S}^1_+(c,r)\times\mathbb{R}^{n-1}$ is strongly ψ-stable if $\alpha\leq -1$.

A last observation is that there are no closed hypersurface whose weighted mean curvature H_ψ is constant ([15]).

We finish this section with the description of α-singular minimal surfaces of cylindrical type. This will be the class of surfaces for which we will investigate the Plateau-Rayleigh phenomenon. Let Σ be an α-singular minimal surface of cylindrical type. If Σ is not a vertical plane (which is strong ψ-stable), we know from the Introduction that Σ must be a graph on P_0. If Σ is $X(s,t)=(s,t,h(s))$, then Equation (1) is equivalent to

$$\frac{h''}{1+h'^2}=\frac{\alpha}{h}. \tag{5}$$

After a rotation about e_3 and a horizontal translation, $X(s,t)=(s,t,h(s))$, where $h\colon(-m,m)\to\mathbb{R}$ and with the symmetric property $h(-s)=h(s)$ for all $s\in(-m,m)$. The geometric properties of the solutions of (5) depend on α ([15]). See Figure 2. If $\alpha<0$, then $m<\infty$ and the graph of h is concave and intersects orthogonally the s-axis at two points; if $\alpha\in(0,1]$, then h is an entire convex graph on the s-axis; and if $\alpha>1$, then $m<\infty$ and the graph of h is convex and asymptotic to the vertical lines $s=\pm m$. In particular, Σ is a graph on the entire plane P_0 ($\alpha\in(0,1]$) or a graph on a strip of P_0 ($\alpha\not\in(0,1]$). Multiplying (5) by h', we can integrate obtaining $c(1+h'^2)=h^{2\alpha}$ for some positive constant $c>0$. After a translation along the x_1-direction and a dilation with respect to a point of P_0, we can assume $c=1$, or equivalently, $h(0)=1$ and $h'(0)=0$. Thus

$$\frac{h'}{\sqrt{h^{2\alpha}-1}}=\pm 1. \tag{6}$$

This equation can be integrated by quadratures for a few values of α. Some examples are the following.

1. If $\alpha=-1$, then $h(s)=\sqrt{1-s^2}$ is a hemicircle centered at the s-axis.
2. If $\alpha=1$, then $h(s)=\cosh(s)$ is a catenary.
3. If $\alpha=1/2$, then $h(s)=s^2/4+1$ is a parabola

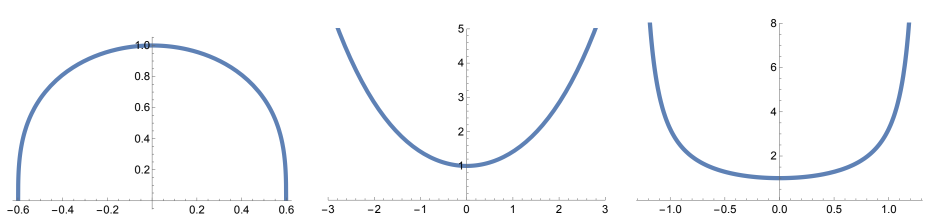

FIGURE 2. Solutions of (5) for values $\alpha=-2$ (right), $\alpha=0.8$ (middle) and $\alpha=2$ (right).

3. **Stability of α-singular minimal graphs.** The first result of stability concerns to graphs with respect to e_{n+1}, that is, graphs on the hyperplane P_0. Let us observe that the Question in Introduction is posed for α-singular minimal surfaces of cylindrical type, which are all graphs on P_0. The stability of graphs has been studied in other weighted manifolds, proving that graphs are strongly ψ-stable. In all them, the arguments use known properties of diagonalization of elliptic operators and the proof is as follows. If Σ is a graph over the hyperplane $P = \{x \in \mathbb{R}^{n+1} : \langle x, \vec{v}\rangle = 0\}$, the function $u = \langle N, \vec{v}\rangle$ has sign on Σ. The key now is that u is an eigenfunction of the ψ-Jacobi operator for the eigenvalue $\lambda = 0$. Standard theory asserts that 0 is the first eigenvalue of L_ψ, $\lambda_1 = 0$, hence the graph is strongly ψ-stable.

For α-singular minimal hypersurfaces, the hyperplane with respect to which the hypersurface is a graph is not arbitrary, and it has a relation with Equation (1). So, $\vec{v}$ must be e_{n+1} because $N_{n+1} = \langle N, e_{n+1}\rangle$. In contrast to the situation in other weighted manifolds, N_{n+1} is not an eigenfunction of the ψ-Jacobi operator. However, $L_\psi[N_{n+1}]$ is well controlled which will be enough for our purposes when $\alpha < 0$. First, we need to compute $L_\psi[N_{n+1}]$.

Lemma 3.1. *Let Σ be an α-singular minimal hypersurface with constant weighted mean curvature $H_\psi = \mu$. Then the function N_{n+1} satisfies*

$$L_\psi[N_{n+1}] = \frac{\alpha}{x_{n+1}^2} N_{n+1}. \tag{7}$$

Proof. It is known that the Gauss map N in any hypersurface satisfies

$$\Delta N + |A|^2 N + \nabla(nH) = 0. \tag{8}$$

If we now assume that $H_\psi = \mu$, then $nH = \alpha N_{n+1}/x_{n+1} + n\mu$, so

$$\begin{aligned}\langle \nabla(nH), e_{n+1}\rangle &= \frac{\alpha}{x_{n+1}}\langle \nabla N_{n+1}, e_{n+1}\rangle - \frac{\alpha N_{n+1}}{x_{n+1}^2}\langle \nabla x_{n+1}, e_{n+1}\rangle\\ &= \frac{\alpha}{x_{n+1}}\langle \nabla N_{n+1}, e_{n+1}\rangle - \frac{\alpha N_{n+1}}{x_{n+1}^2}(1 - N_{n+1}^2)\end{aligned}$$

because $\nabla x_{n+1} = e_{n+1} - N_{n+1}N$. From (8),

$$\Delta N_{n+1} + |A|^2 N_{n+1} + \frac{\alpha}{x_{n+1}}\langle \nabla N_{n+1}, e_{n+1}\rangle - \frac{\alpha N_{n+1}}{x_{n+1}^2}(1 - N_{n+1}^2) = 0.$$

Inserting this expression in (4), we obtain (7).

□

The proof of the following theorem follows the same ideas of Fischer-Colbrie and Schoen for minimal surfaces ([11]).

Theorem 3.2. *Let Σ be an α-singular minimal hypersurface with constant weighted mean curvature H_ψ. Suppose Σ is a graph over P_0. If $\alpha < 0$, then Σ is strongly ψ-stable.*

Proof. Since Σ is a graph with respect to e_{n+1}, the function N_{n+1} has sign on Σ. Without loss of generality, we suppose $N_{n+1} > 0$. Let $u \in C_0^\infty(\Sigma)$ arbitrary and set $v = u/N_{n+1}$, which is well defined and belongs to $C_0^\infty(\Sigma)$. For $u = vN_{n+1}$, we have

$$\begin{aligned}\Delta u &= v\Delta N_{n+1} + N_{n+1}\Delta v + 2\langle \nabla v, \nabla N_{n+1}\rangle,\\ \langle \nabla u, e_{n+1}\rangle &= N_{n+1}\langle \nabla v, e_{n+1}\rangle + v\langle \nabla N_{n+1}, e_{n+1}\rangle.\end{aligned}$$

Using (7),

$$\begin{aligned}L_\psi[u] &= v\cdot L[N_{n+1}] + N_{n+1}\Delta v + 2\langle\nabla v,\nabla N_{n+1}\rangle + \frac{\alpha N_{n+1}}{x_{n+1}}\langle \nabla v, e_{n+1}\rangle\\ &= v\cdot L_\psi[N_{n+1}] + N_{n+1}\Delta_\psi v + 2\langle\nabla v,\nabla N_{n+1}\rangle\\ &= \frac{\alpha}{x_{n+1}^2} vN_{n+1} + N_{n+1}\Delta_\psi v + 2\langle\nabla v,\nabla N_{n+1}\rangle.\end{aligned}$$

Then

$$\begin{aligned}Q_\psi[u] = &-\int_\Sigma \frac{\alpha N_{n+1}^2}{x_{n+1}^2} v^2\, d\mathcal{A}_\psi - \int_\Sigma vN_{n+1}^2\Delta_\psi v\, d\mathcal{A}_\psi\\ &- 2\int_\Sigma vN_{n+1}\langle\nabla v,\nabla N_{n+1}\rangle\, d\mathcal{A}_\psi.\end{aligned} \tag{9}$$

Using the ψ-divergence theorem in weighted manifolds,

$$\begin{aligned}\int_\Sigma vN_{n+1}^2\Delta_\psi v\, d\mathcal{A}_\psi &= -\int_\Sigma \langle \nabla(vN_{n+1}^2),\nabla v\rangle\, d\mathcal{A}_\psi\\ &= -\int_\Sigma N_{n+1}^2|\nabla v|^2\, d\mathcal{A}_\psi - 2\int_\Sigma vN_{n+1}\langle\nabla N_{n+1},\nabla v\rangle\, d\mathcal{A}_\psi.\end{aligned}$$

Putting into (9),

$$Q_\psi[u] = -\int_\Sigma \frac{\alpha N_{n+1}^2}{x_{n+1}^2} v^2\, d\mathcal{A}_\psi + \int_\Sigma N_{n+1}^2|\nabla v|^2\, d\mathcal{A}_\psi \geq 0$$

because $\alpha < 0$. □

By the proof, notice that if $Q_\psi[u] = 0$, then $v = 0$ on Σ, so if $u \neq 0$, then $Q_\psi[u] > 0$.

Corollary 1. *Cylindrical α-singular minimal surfaces are strongly ψ-stable if $\alpha < 0$.*

In the next two sections, we will see that for each $\alpha > 0$, there are α-singular minimal graphs that are not strongly ψ-stable neither ψ-stable.

4. **Stability of singular minimal catenaries.**

We study the stability of singular minimal surfaces of cylindrical type. From Section 2, we know that the generating curve is a catenary, hence we call these surfaces *singular minimal catenaries*. The structure of this section is firstly the computation of the ψ-Jacobi operator L_ψ of a cylindrical α-singular minimal surface for arbitrary α. Next, we will consider the technique of separation of variables for the normal component of the variation, or equivalently, for the input functions u of the ψ-Jacobi operator L_ψ. Finally, we consider the case $\alpha = 1$ and the study will consist in varying both the value of the amplitude as well as the length of the surface.

Following Section 2, consider $X(s,t) = (s,t,h(s))$ the parametrization of a cylindrical α-singular minimal surface. The coefficients of the first fundamental form are $E = 1 + h'^2$, $G = 1$ and $F = 0$. Thus the Laplacian Δ of a function u is

$$\Delta u = \frac{1}{E}u_{ss} + \frac{1}{\sqrt{E}}\left(\frac{1}{\sqrt{E}}\right)_s u_s + u_{tt}.$$

For the computation of $\langle\nabla u, \nabla\psi\rangle$, notice that $\psi = \alpha \log x_3 = \alpha \log h$ does not depend on the variable t. Thus

$$\langle\nabla u, \nabla\psi\rangle = \frac{1}{E}\langle\nabla u, \partial_s X\rangle\langle\nabla\psi, \partial_s X\rangle = \frac{\psi_s}{E}u_s = \alpha\frac{h'}{hE}u_s.$$

Since the Gaussian curvature on a cylindrical surface is 0, we have

$$|A|^2 = 4H^2 = (2H)^2 = \alpha^2\frac{N_3^2}{x_3^2} = \frac{\alpha^2}{h^2E}.$$

Finally,

$$\overline{\nabla}^2\psi(N,N) = -\alpha\frac{N_3^2}{x_3^2} = -\frac{\alpha}{h^2E}.$$

Then

$$L_\psi[u] = \frac{1}{E}u_{ss} + \left(\frac{1}{\sqrt{E}}\left(\frac{1}{\sqrt{E}}\right)_s + \alpha\frac{h'}{hE}\right)u_s + u_{tt} + \frac{\alpha+\alpha^2}{h^2E}u.$$

Using (5), the computation of the coefficient of u_s leads to

$$\frac{1}{\sqrt{E}}\left(\frac{1}{\sqrt{E}}\right)_s + \alpha\frac{h'}{hE} = -\frac{h'h''}{(1+h'^2)^2} + \frac{\alpha h'}{h(1+h'^2)} = 0.$$

We can be a little more precise. From (6), $E = 1 + h'^2 = h^{2\alpha}$, so

$$L_\psi[u] = h^{-2\alpha}u_{ss} + u_{tt} + \frac{\alpha+\alpha^2}{h^{2+2\alpha}}u. \tag{10}$$

We now consider the rectangle $[-a,a]\times[0,L]$, $a < m$, to be the domain of $X(s,t)$ and we denote by $\Sigma(a,L)$ this piece of surface. Let us observe that $\Sigma(a,L)$ is symmetric about the vertical plane of equation $x_1 = 0$. In order to perform a stability analysis of these surfaces, we consider separation of variables for the function u, writing $u(s,t) = f(s)g(t)$ for some functions f and g, $s \in (-m,m)$, $t \in \mathbb{R}$. In other words, we are doing perturbations of the surface along the s-direction and then along the t-direction. According to (10),

$$L_\psi[u] = h^{-2\alpha}f''g + fg'' + \frac{\alpha+\alpha^2}{h^{2+2\alpha}}fg.$$

On the other hand, the area element is

$$d\mathcal{A}_\psi = e^\psi d\mathcal{A} = x_3^\alpha\sqrt{E}\,dsdt = h^{2\alpha}\,dsdt.$$

Since $u(s,t) = f(s)g(t)$ vanishes on the boundary of $[-a,a]\times[0,L]$, the boundary conditions are $f(\pm a) = 0$ and $g(0) = g(L) = 0$. For the function g, we can choose trigonometric functions such as $g(t) = \sin(\pi t/L)$ or $g(t) = \sin(2\pi t/L)$. Let us observe that the integral of u is

$$\int_\Sigma u\,d\mathcal{A}_\psi = \int_0^L g(t)\ dt\int_{-a}^a f(s)h(s)^{2\alpha}\ ds.$$

Let us notice that if $g(t) = \sin(2\pi t/L)$, then $\int_0^L g(t)\ dt = 0$, so u can be utilized as a test function for Q_ψ in the ψ-stability problem. In conclusion, the above choice of g can treat both situations of strong ψ-stability and ψ-stability.

We begin with the study of the strong ψ-stability and let $g(t)=\sin(\pi t/L)$. Then $g''=-\frac{\pi^2}{L^2}g$, and this gives

$$\begin{aligned} Q_\psi[u] &= -\int_0^L g(t)^2\,dt\int_{-a}^{a}\left\{ff''h^{-2\alpha}+\left(\frac{\alpha+\alpha^2}{h^{2+2\alpha}}-\frac{\pi^2}{L^2}\right)f^2\right\}h^{2\alpha}\,ds\\ &= -\int_0^L g(t)^2\,dt\int_{-a}^{a}\left\{ff''+\left(\frac{\alpha+\alpha^2}{h^2}-\frac{\pi^2}{L^2}h^{2\alpha}\right)f^2\right\}ds \qquad (11)\\ &= -\int_0^L g(t)^2\,dt\int_{-a}^{a}\left\{\left(\frac{\alpha+\alpha^2}{h^2}-\frac{\pi^2}{L^2}h^{2\alpha}\right)f^2-f'^2\right\}ds \end{aligned}$$

where in the last identity we have integrated by parts. Since we are looking for a type of Plateau-Rayleigh instability, our objective is to find a function u such that $Q_\psi[u]<0$, or equivalently, a function f such that the integral

$$I(a,L)=\int_{-a}^{a}\left\{\left(\frac{\alpha+\alpha^2}{h^2}-\frac{\pi^2}{L^2}h^{2\alpha}\right)f^2-f'^2\right\}ds \qquad (12)$$

is positive. Since $f(\pm a)=0$, and thanks to the symmetry of h, we will choose f to be, up to a constant, the height function on $\Sigma(a,L)$. Let

$$f(s)=h(s)-h(a),$$

which will be the function to test the instability of the surface $\Sigma(a,L)$. We now restrict to the case $\alpha=1$. Because $h(s)=\cosh(s)$, we have from (12),

$$I(a,L)=\int_{-a}^{a}\left\{\left(\frac{2}{\cosh(s)^2}-\frac{\pi^2\cosh(s)^2}{L^2}\right)(\cosh(s)-\cosh(a))^2-\sinh(s)^2\right\}ds. \qquad (13)$$

We will now play with the two variables a and L in this integral to find that $I(a,L)$ is positive for some values of a and L. We give our Plateau-Rayleigh instability criterion.

Theorem 4.1. *There is a value $a_0>0$, $a_0\approx 1.2391$, such that for all $a>a_0$, there is a critical length $L_0>0$, depending only on the amplitude a, such that the singular minimal catenaries $\Sigma(a,L)$ are not strongly ψ-stable for all $L>L_0$.*

Proof. The integral (13) can be solved by quadratures, obtaining

$$I(a,L)=I_1(a)+I_2(a,L),$$

where

$$I_1(a)=5a+\cosh(a)\left(3\sinh(a)-16\tan^{-1}\left(\tanh\left(\frac{a}{2}\right)\right)\right),$$

$$I_2(a,L)=-\frac{\pi^2}{48L^2}(60a-44\sinh(2a)+\sinh(4a)+24a\cosh(2a)).$$

The behaviour of the function $I_1(a)$ appears in Figure 3. In particular, there is a unique number a_0 such that $I_1(a)<0$ if $a<a_0$, $I_1(a_0)=0$ and $I_1(a)>0$ if $a>a_0$. The value a_0 is $a_0\approx 1.2391$. On the other hand, the parenthesis in the expression of $I_2(a,L)$ is always positive. If we see $I_2(a,L)$ as a function on the variable L, then

$$\lim_{L\to 0^+} I_2(a,L)=-\infty,\qquad \lim_{L\to\infty} I_2(a,L)=0 \qquad (14)$$

and the function $L\mapsto I_2(a,L)$ is strictly increasing on L. See Figure 3.

It is clear that if $a<a_0$, then $I(a,L)<0$ because it is the sum of two negative functions. Let now $a>a_0$, where we know that $I_1(a)>0$. Letting $L\to\infty$, from

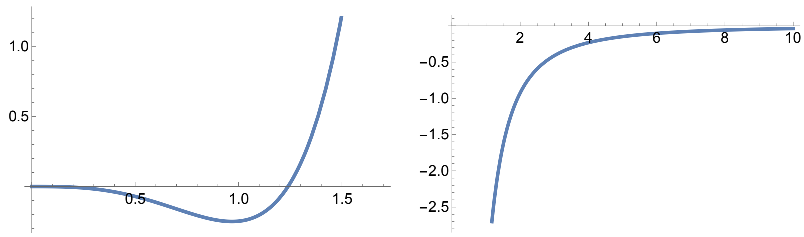

FIGURE 3. Case $\alpha = 1$. Left: the function $I_1(a)$. Right: the function $L \mapsto I_2(a, L)$ (here $a = 1$).

(14) and the monotonicity of $L \mapsto I_2(a, L)$, we deduce that there is a unique value L_0 depending on a such that $I(a, L_0) = 0$ and $I(a, L) > 0$ for all $L > L_0$. This proves the result. The determination of the value L_0 is obtained by solving the equation $I_1(a) + I_2(a, L) = 0$, whose solution is

$$L_0 = \frac{\pi\sqrt{60a - 44\sinh(2a) + \sinh(4a) + 24a\cosh(2a)}}{4\sqrt{15a + 9\cosh(a)\left(\sinh(a) - 6\tan^{-1}\left(\tanh\left(\frac{a}{2}\right)\right)\right)}}. \tag{15}$$

□

Remark 1. The classical Plateau-Rayleigh instability is a particular (and easier) case of the above arguments. Now $\alpha = 0$, but we assume $H_\psi = \mu$ is constant. In particular, $4H^2 = 4\mu^2$. Because Σ is a circular cylinder of radius r, $4H^2 = 1/r^2$. So, the ψ-Jacobi operator is $L_\psi = u_{ss} + u_{tt} + u/r^2$ and the eigenvalues can be explicitly obtained using cylindrical coordinates. In such a case, for strong stability, $L_0 = \pi r$ and for stability, $L_0 = 2\pi r$ ([17]).

Compared with the classical Plateau-Rayleigh criterion for constant mean curvature surfaces, Theorem 4.1 only gives sufficient conditions to ensure strong ψ-instability of $\Sigma(a, L)$ because we have only utilized the functions $h(s) - h(a)$ as inputs for Q_ψ. For example, we do not know if the pieces $\Sigma(a, L)$ with $a < a_0$ are not stable for some values of L or, on the contrary, they are always stable. On the other hand, and thinking in the classical scenario of circular cylinders, one expects that if the amplitude a increases, then one would need large lengths L to ensure the instability. However, the function $L_0 = L_0(a)$ is not increasing: there is a value a_1, with $a_1 \approx 1.7964$, such that $L_0(a)$ is decreasing until $a = a_1$, and then increases for $a > a_1$. See Figure 4.

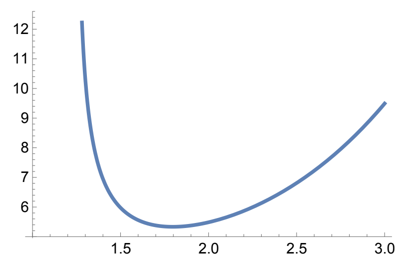

FIGURE 4. The function $L_0 = L_0(a)$ given in (15).

With a similar argument than Theorem 4.1, we give a Plateau-Rayleigh criterion of ψ-stability.

Theorem 4.2. *For the same value $a_0 \approx 1.2391$, we have that for all $a > a_0$, there is a critical length $\widetilde{L}_0 > 0$, depending only on a, such that singular minimal catenaries $\Sigma(a, L)$ are not ψ-stable for all $L > \widetilde{L}_0$.*

Proof. The difference with Theorem 4.1 is that the test functions u must have zero integral on the surface. For this purpose, it is enough to choose $g(t) = \sin(2\pi t/L)$. The term π^2/L^2 in the expression (13) is now replaced by $4\pi^2/L^2$, obtaining $I(a, L) = I_1(a) + 4I_2(a, L)$. The arguments follow the same steps than Theorem 4.1. $\square$

The value $\widetilde{L}_0$ is

$$\widetilde{L}_0 = \frac{\pi\sqrt{60a - 44\sinh(2a) + \sinh(4a) + 24a\cosh(2a)}}{2\sqrt{15a + 3\cosh(a)\left(3\sinh(a) - 16\tan^{-1}\left(\tanh\left(\frac{a}{2}\right)\right)\right)}}.$$

Comparing with the critical length L_0 of Theorem 4.1, we have $L_0 < \widetilde{L}_0$ for all a.

In the above two theorems, the singular minimal surfaces are pinched along the intersection with planes Π_l of equation $x_2 = l$, $l \in \{0, L\}$, this intersection consisting of two copies of the catenary. But we can extend the results to other mixed boundary conditions. To be precise, we pose the same variational problem as in Section 2 where now $\text{int}(\Sigma)$ lies included in the strip $0 < x_2 < L$ and $\partial\Sigma \subset \Pi_0 \cup \Pi_L$. Now the admissible variations of Σ allow that the curves $\Gamma_0 = \partial\Sigma \cap \Pi_0$ and $\Gamma_L = \partial\Sigma \cap \Pi_L$ can move freely on Π_0 and Π_L, respectively. This situation for surfaces with constant mean curvature is related with capillarity problems, where the boundary of the surface freely moves on the support plane. In the determination of the critical points Σ of $\mathcal{A}_\psi$, the only difference is that the intersection of Σ with the planes $\Pi_0 \cup \Pi_L$ must be orthogonal ([4]). For the expression of $Q_\psi[u]$ in (3), we need to add the extra boundary term

$$\int_{\Gamma_0 \cup \Gamma_L} \sigma(N, N) u^2 \, d\mathcal{L}_\psi,$$

where σ is the second fundamental form of $\Pi_0 \cup \Pi_L$ and $\mathcal{L}_\psi$ is the weighted length element on $\Gamma_0 \cup \Gamma_L$. Since $\sigma = 0$ because Π_0 and Π_L are planes, the quadratic form $Q_\psi[u]$ has the same expression (3). Returning to the case of cylindrical α-singular minimal surfaces $\Sigma(a, L)$, the boundary of $\Sigma(a, L)$ is formed by four arcs, namely, Γ_0, Γ_L and the two segments $C_{\pm a} = \{X(\pm a, t) : 0 \leq t \leq L\}$. The test functions u must satisfy the mixed Dirichlet and Neumann conditions

$$u = 0 \text{ along } C_{\pm a}, \quad D_n u = 0 \text{ along } \Gamma_0 \cup \Gamma_L,$$

where $D_n u$ is the derivative of u along an orthogonal direction to $\Pi_0 \cup \Pi_L$. In case of ψ-stability, we add the condition $\int_\Sigma u \, d\mathcal{A}_\psi = 0$.

Theorem 4.3. *Theorems 4.1 and 4.2 hold for the above mixed variational problem.*

Proof. We consider separation of variables $u(s, t) = f(s)g(t)$. Then the condition $D_n u = 0$ along $\Gamma_0 \cup \Gamma_L$ is equivalent to $g'(0) = g'(L) = 0$. We define $g(t) = \cos(\pi t/L)$ or $g(t) = \cos(2\pi t/L)$ depending if we are considering the problem of strongly ψ-stability or ψ-stability, respectively. For the function f, we will take $f(s) = \cosh(s) - \cosh(a)$ again. Since $g'' = -\pi^2/L^2 g$ or $g'' = -4\pi^2/L^2 g$ depending on the case, the term (12) is the same than Theorems 4.1 and 4.2, respectively, and the result follows immediately. □

We can formulate the one-dimensional version of Theorems 4.1 and 4.2. For this, we see the catenary as a 1-singular minimal curve. The ψ-Jacobi operator (10) is now

$$L_\psi[u] = h^{-2}u'' + 2h^{-4}u = \frac{u''}{\cosh(s)^2} + \frac{2}{\cosh(s)^4}u$$

and

$$Q_\psi[u] = -\int_{-a}^{a}\left(u'^2 - \frac{2}{\cosh(s)^2}u^2\right) ds.$$

Theorem 4.4. *If $a > a_0 \approx 1.2391$, the piece of catenary $y(s) = \cosh(s)$ defined in the interval $[-a, a]$ is not strong ψ-stable.*

Proof. Taking $f(s) = \cosh(s) - \cosh(a)$ in Q_ψ, we have $Q_\psi[f] = -I_1[a]$, obtaining the result. □

We add a last word about the corridors constructed by Gaudí described in Introduction. Although the notion of 'stability' (or solidity) in architecture does not coincide with the one in mathematics, Theorems 4.1 and 4.2 prove that, in a 'perfect' corridor made by copies of catenaries, where loads and tensions act tangentially on the roof, the corridor is not stable if the amplitude of the catenary and the length of the corridor are sufficiently large.

5. **Numerical approximation to the general case.** In this section we will investigate if Theorems 4.1 and 4.2 are valid for all positive values of α. As we indicated in Section 2, the explicit integration of (5) is not possible in general. As a sample of this study, we will consider the case $\alpha = 1/2$, where the solution of (5) is known, and the cases $\alpha = 2$ and $\alpha = 3$. In the latter cases, we use numerical computations to assess the influence of the amplitude and the length of the surface in its stability. The numerical simulations are performed using the software `Mathematica`. We begin with $\alpha = 1/2$, where we know an exact solution of (5).

Theorem 5.1. *For $a_0 \approx 2.0426$, Theorems 4.1 and 4.2 hold for cylindrical $\frac{1}{2}$-singular minimal surfaces.*

Proof. We know that $h(s) = s^2/4 + 1$ and the maximal domain is $\mathbb{R}$, in particular the amplitude a of the surface can take any value. We prove the result for strong ψ-stability, and analogously the reasoning holds for ψ-stability. Computing $I(a, L)$ in (12), we have $I(a, L) = I_1(a) + I_2(a, L)$, where now

$$I_1(a) = \frac{a\left(35\left(9\pi a^3 + 4a^2 + 72\left(6 - \pi a - \frac{6\pi}{a}\right)\right)\right) - 630\left(a^4 - 8a^2 - 48\right)\tan^{-1}\left(\frac{2}{a}\right)}{6720},$$

$$I_2(a, L) = -\frac{\pi^2 a^5(a^2 + 28)}{420L^2}.$$

The behaviour of both functions is similar to the case $\alpha = 1$, where now the value a_0 where $I_1(a) = 0$ is $a_0 \approx 2.0426$. The rest of arguments are equally valid. □

5.1. **Case $\alpha = 2$.** We only study the problem of strong ψ-stability. In contrast with $\alpha = 1/2$ and $\alpha = 1$, if $\alpha = 2$ (also $\alpha = 3$), the domain of h is a bounded interval, so first we need to find the value m of the maximal domain of h. This is obtained with the `NDSolve` command

$$\texttt{k=4;}\alpha = 2\texttt{; sol=}$$

$$\texttt{NDSolve}[\{\texttt{h}''[\texttt{s}] == \alpha\frac{1+\texttt{h}'[\texttt{s}]^2}{\texttt{h}[\texttt{s}]}, \texttt{h}[0] == 1, \texttt{h}'[0] == 0\}, \texttt{h}[\texttt{s}], \{\texttt{s}, -\texttt{k}, \texttt{k}\}]\}$$

and choosing a large value m so the output message of `Mathematica` shows the maximum value of the domain. In the present case, and taking $k = 4$, we receive the following message from `Mathematica`:

$$\{\{h[s] \to \text{InterpolatingFunction}\left[\begin{pmatrix} -1.31103 & 1.31103 \end{pmatrix}, \text{Null} <>\right][s]\}\}$$

Hence $m \approx 1.31103$. We now need to redefine the function $h(s)$ in order to numerically manage it. So let us write in `Mathematica`

$$\texttt{h2}[\texttt{s\_}] = \texttt{h}[\texttt{s}]/.\texttt{sol}$$

Using separation of variables and for the question of strong ψ-stability, we take $g(t) = \sin(\pi t/L)$ and $f(s) = h(s) - h(a)$, where $a < m$. The term $I(a, L)$ given in (12) is transformed into a numerical integration thanks to `NIntegrate` command:

$$\texttt{I}[\texttt{a\_}, \texttt{L\_}] = \texttt{NIntegrate}[-\texttt{f}'[\texttt{s}]^2 + (\frac{\alpha+\alpha^2}{\texttt{h2}[\texttt{s}]^2} - \frac{\texttt{Pi}^2}{\texttt{L}^2}\texttt{h2}[\texttt{s}]^{2\alpha})\texttt{f}[\texttt{s}]^2, \{\texttt{s}, -\texttt{a}, \texttt{a}\}]$$

Such as it occurs for the case $\alpha = 1$, for values of a close to 0, $I(a, L)$ is negative for any L. Table 1 shows the computation of $I(a, L)$ for different values of a and L, where we highlight the first positive value of $I(a, L)$ for each a.

TABLE 1. Values of $I(a, L)$ for $\alpha = 2$.

L	1	10	20	30	40	50
a						
0.2	-0.02389	-0.02031	-0.02028	-0.02028	-0.02028	-0.02028
0.3	-0.09371	-0.06398	-0.06375	-0.06371	-0.06370	-0.06369
0.4	-0.27824	-0.13500	-0.13392	-0.13372	-0.13365	-0.13362
0.5	-0.74222	-0.21609	-0.21210	-0.21137	-0.21111	-0.21099
0.6	-1.93468	-0.253826	-0.24109	-0.23873	-0.23790	-0.237527
0.7	-5.20226	-0.12268	-0.08420	-0.07708	-0.07458	-0.07343
0.8	-15.15280	0.45027	0.5684	0.5903	0.59802	0.60157
0.9	-50.9705	2.0161	2.41751	2.49185	2.51786	2.52991

So, until $a \approx 0.7$, the value of $I(a, L)$ is negative for all values of L. However, for $a = 0.8$ and $a = 0.9$ there appear positive values, which it is our target. We then need to compute $I(a, L)$ for values of a close to 0.8 and 0.9, as well as, values of L less than 10. As the reader can see in Table 1, the value of $I(a, L)$ barely changes when L is large but there are major changes from $L = 1$ to $L = 10$, which should warn us against errors of `Mathematica`. For this reason, we compute $I(a, L)$ around the value $a = 0.7$. Taking the values $a = 0.70$, $a = 0.71$ and $a = 0.72$, we see that it is for $a = 0.72$ when we find the first positive values of $I(a, L)$ for $L_0 \approx 30$. See Table 2.

TABLE 2. Case $\alpha = 2$. Values of $I(0.72, L)$.

L	22	24	26	28	30	32
	-0.0053	-0.0032	-0.0015	-0.0002	0.0007	0.0016

Next we compute $I(a, L)$ for values a with $a > 0.72$ as shown in Table 3. The numerical computations of $I(a, L)$ prove that there is a value a_0, $a_0 \approx 0.72$, such that for $a > a_0$, there is a first value $L_0 = L_0(a)$ such that $I(a, L) > 0$ for all $L > L_0$. We can also see that, as in the cases $\alpha = 1/2$ and $\alpha = 1$, the function $L_0 = L_0(a)$ has a minimum at a certain value a_1 (here $a_1 \approx 0.90$).

TABLE 3. Case $\alpha = 2$. Values of $I(a, L)$ when $a > 0.72$.

L	2	3	4	5	6	7	8	9
a								
0.75	-2.061	-0.820	-0.386	-0.185	-0.076	-0.010	0.032	0.061
0.80	-3.332	-1.143	-0.377	-0.022	0.170	0.286	0.361	0.413
0.90	-10.829	-3.395	-0.793	0.410	1.064	1.459	1.715	1.890
1.00	-49.743	-17.942	-6.812	-1.660	1.137	2.824	3.920	4.670
1.05	-126.723	-49.655	-22.681	-10.196	-3.414	0.674	3.328	5.148

5.2. **Case $\alpha = 3$.** As a final example, we discuss the case of $\alpha = 3$. The procedure is the same as in case $\alpha = 2$. The value m of the maximal domain is $m \approx 0.7010$. After a preliminary study of the values a such that $I(a, L)$ is positive, we deduce $a_0 \approx 0.53$. We show in Table 4 the value of $I(a, L)$ for some values of a with $a > a_0$.

TABLE 4. Case $\alpha = 3$. Values of $I(a, L)$ when $a > 0.53$.

L	4	5	6	7	8	9	10	11	12
a									
0.54	-0.394	-0.216	-0.119	-0.060	-0.022	0.003	0.021	0.035	0.046
0.56	-0.531	-0.247	-0.093	0.000	0.059	0.100	0.130	0.152	0.168
0.58	-0.906	-0.433	-0.176	-0.021	0.079	0.148	0.197	0.234	0.262
0.60	-1.928	-1.079	-0.618	-0.340	-0.160	-0.036	0.051	0.117	0.167

As a conclusion of the stability analysis of the cases $\alpha = 2$ and $\alpha = 3$, and supported by the above numerical simulations if $\alpha > 0$, the Plateau-Rayleigh mechanism for cylindrical α-singular minimal surfaces is similar than $\alpha = 1$. So, we have the next conclusion:

Theorem 5.2 (numerical). *Let $\alpha > 0$. There is a value a_0 of the amplitude, a_0 depending on α, such that for all $a > a_0$, cylindrical α-minimal surfaces $\Sigma(a, L)$ are not strong ψ-stable if the length L is bigger than a certain value L_0, L_0 depending on a.*

6. **Appendix: the conformal metric $x_{n+1}^{2\alpha/n}\langle,\rangle$.** We have pointed out that α-singular minimal hypersurfaces are minimal hypersurfaces in $\mathbb{R}^{n+1}$ after a conformal change of metric. We point this out since in weighted manifolds the area and the volume element change with the same factor, which does not occur under a conformal change of the metric. Consider a conformal metric $\widetilde{g}=e^{2\varphi}\langle,\rangle$ in $\mathbb{R}^{n+1}_+$. The area element is $d\widetilde{\mathcal{A}}=e^{n\varphi}d\mathcal{A}$ and the volume element is $d\widetilde{V}=e^{(n+1)\varphi}d\mathcal{V}$. If we want $d\widetilde{A}$ and $d\mathcal{A}_\psi$ to coincide, then $\varphi=\psi/n$ and the conformal metric is

$$\widetilde{g}=x_{n+1}^{2\alpha/n}\langle,\rangle.$$

We compute the mean curvature $\widetilde{H}$ of Σ as a hypersurface isometrically immersed in $(\mathbb{R}^{n+1}_+,\widetilde{g})$. Because the metrics $\widetilde{g}$ and $\langle,\rangle$ are conformal, the relation between the second fundamental forms $\widetilde{A}$ and A of $\widetilde{g}$ and $\langle,\rangle$, respectively, is

$$\widetilde{A}(X,Y)=e^{\varphi}\left(A(X,Y)-\langle\overline{\nabla}\varphi,N\rangle\langle X,Y\rangle\right).$$

See [2]. Taking traces,

$$n\widetilde{H}=e^{-\varphi}(nH-n\langle\overline{\nabla}\varphi,N\rangle).$$

Thus

$$\widetilde{H}=e^{-\varphi}(H-\langle\overline{\nabla}\varphi,N\rangle)=x_{n+1}^{-\alpha/n}\left(H-\frac{\alpha}{n}\frac{N_{n+1}}{x_{n+1}}\right)=x_{n+1}^{-\alpha/n}H_\psi.$$

In particular, the mean curvature $\widetilde{H}$ does not coincide with the weighted mean curvature H_ψ. However, $\widetilde{H}=0$ if and only $H_\psi=0$. Definitively, *α-singular minimal hypersurfaces are minimal hypersurfaces in the Riemannian manifold $(\mathbb{R}^{n+1}_+,\widetilde{g})$ and vice-versa*. Let us notice that, except for horizontal hyperplanes, if H_ψ is a nonzero constant, then $\widetilde{H}$ is not constant.

A last observation about the space $(\mathbb{R}^3,\widetilde{g})$ is that sectional curvatures $\widetilde{K}_{ij}$ are not bounded. The calculation is motivated by the works of Schoen ([22]) because one needs boundedness of the sectional curvatures to ensure convergence of singular minimal surfaces viewed as minimal surfaces in the 3-manifold $(\mathbb{R}^3_+,\widetilde{g})$. First, we calculate the Christoffel symbols of the metric $\widetilde{g}$:

$$\widetilde{\Gamma}_{ij}^k=\frac{1}{2}\left(\frac{\partial}{\partial x_i}\widetilde{g}_{jm}+\frac{\partial}{\partial x_j}\widetilde{g}_{im}-\frac{\partial}{\partial x_m}\widetilde{g}_{ij}\right)\widetilde{g}^{mk}=\frac{1}{2}\frac{\alpha}{x_3}\left(\delta_{i3}\delta_{jk}+\delta_{j3}\delta ik-\delta_{ij}\delta_{3k}\right).$$

Hence $\widetilde{\Gamma}_{ij}^k=0$ for $1\leq i,j,k\leq 2$ and $\widetilde{\Gamma}_{12}^3=\widetilde{\Gamma}_{13}^3=\widetilde{\Gamma}_{23}^3=0$. The other Christoffel symbols are

$$\widetilde{\Gamma}_{11}^3=\widetilde{\Gamma}_{22}^3=-\widetilde{\Gamma}_{33}^3=-\widetilde{\Gamma}_{13}^1=-\widetilde{\Gamma}_{23}^2=-\frac{\alpha}{2x_3}.$$

The coefficients of the curvature are

$$\widetilde{R}_{1212}=-\frac{\alpha^2}{4x_3^2},\quad \widetilde{R}_{1313}=\widetilde{R}_{2323}=\frac{\alpha}{2x_3^2}.$$

Finally, the sectional curvatures are

$$\widetilde{K}_{12}=-\frac{\alpha^2}{4x_3^{2+\alpha}},\quad \widetilde{K}_{13}=\widetilde{K}_{23}=\frac{\alpha}{2x_3^{2+\alpha}}.$$

If $\alpha>0$, then $\widetilde{K}_{ij}$ have different signs, and if $\alpha<0$, then $\widetilde{K}_{ij}<0$ (the case $\alpha=-2$, corresponding to the hyperbolic metric, gives $\widetilde{K}_{ij}=-1$, such as it was expectable). In particular, the sectional curvatures are not bounded by letting $x_3\searrow 0$ if $\alpha>-2$ or $x_3\nearrow\infty$ if $\alpha<-2$.

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