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Specificity Measures and Referential Success

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Abstract: In this work we propose a technique for defining specificity measures in terms of measures of referential success. This research completes the study of the relationships between both families of measures opened in a previous proposal for defining referential success in terms of specificity. Our study allows to define new measures of specificity for any application, and it is particularly relevant in the applied and highly topical area of linguistic description of data by means of data-to-text systems.

1 Introduction

Fuzzy sets interpreted as possibility measures can be employed for representing the available information about the actual value of a certain variable. When the fuzzy set is a crisp singleton, there is no uncertainty about the value of the variable. In any other case, there is uncertainty, the amount of which can be assessed by using measures of specificity [DP87; L.G+03; L.G+06; MG82; Yag82; Yag90; Yag92b].

The specificity is related to the difficulty of determining the actual value of the variable given the information provided by the fuzzy set; hence, specificity measures are very useful in many different scenarios.

One particular application of specificity measures that has been recently proposed in [MRGS16b] is the definition of measures of referential success of referring expressions in the setting of the linguistic description of data with fuzzy properties [Gat+16; MS16]. A referring expression is a noun phrase intended to univocally identify a given object among a collection of objects. The referential success of a referring expression is the extent to which the (fuzzy) set of objects that comply with the referring expression is a crisp singleton containing the given object. The proposal in [MRGS16b] is based on a combination, by means of a t-norm, of the accuracy of the referring expression for the object intended to be referred and the specificity of the set of objects actually referred.

In this work the inverse problem is studied, that is, how to define specificity measures in terms of measures of referential success. A technique for solving this problem is proposed, showing that measures defined using the introduced technique satisfy all the properties required for specificity measures. As measures of referential success exist that have not been defined in terms of specificity, see [Gat+16; MRGS16a], our study opens the possibility to use such measures to propose new families of specificity measures. In particular, in this work the case of the two measures of referential success proposed in [Gat+16; MRGS16a] is studied, together with the form and properties of the corresponding specificity measures. In addition, it is studied whether, given a referential success measure $\overrightarrow{RS}_t(Sp)$ defined in terms of a specificity measure Sp and a t-norm t, the specificity defined in terms of this measure is again Sp. It is shown that, though this is true for several families of specificity measures, this is not always the case.

The work is organized as follows: some preliminaries related to definitions of specificity, referring expressions and referential success, and methods for defining measures of referential success from specificity are recalled in Section 2. In Section 3 we introduce our proposal for defining specificity from referential success, with some examples. Section 4 is devoted to introducing the notions of duality and reversibility, and some results regarding relations between these measures. An application example and a reference to another one are detailed in Section 5. Finally, Section 6 contains our conclusions and future research avenues.

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2 Preliminaries

In this section we recall the basics of specificity measures as introduced by Yager, as well as the notions of referring expression, measures of referential success, and how the latter can be obtained through the use of specificity measures.

2.1 Specificity

Let $[0,1]^O$ be the fuzzy power set of a set of objects $O = \{o_1, \ldots, o_n\}$, i.e., the set of all membership functions of the form $A : O \to [0,1]$. A measure of specificity Sp is a mapping $[0,1]^O \to [0,1]$ satisfying the following properties for all $A, B \in [0,1]^O$ [Yag82; Yag92b]:

Property 2.1
$Sp(A) = 1$ iff A is a crisp singleton $A = \{o_j\}$, $o_j \in O$.

Property 2.2
$Sp(\emptyset) = 0$.

Property 2.3
If $A, B \in [0,1]^O$ are normal fuzzy sets (that is, there is $o, o' \in O$ such that $A(o) = B(o') = 1$) with $A \subseteq B$ then it is $Sp(A) \geq Sp(B)$.

where fuzzy set inclusion is defined as usual:

$$A \subseteq B \text{ iff } A(o) \leq B(o) \quad \forall o \in O \tag{1}$$

These three properties are very intuitive according to the intended meaning of any specificity measure. The first property is natural since, when the possibility distribution is comprised of a single object with degree 1, then it is clear that the value of the variable is that object, and hence we have the maximum value for specificity, meaning that the possibility distribution is maximally informative with respect to the value of the variable.

The opposite happens when the possibility is represented by the empty set, since in that case it is clear that no object in O can be the value of the variable. In such case, we have no information, and hence the specificity attains its lower value 0. A different case is that of the possibility measure being represented by O, in which case we have more information (the value of the variable is one of the objects in O). In such case, the value of the specificity is not restricted by the properties (except in that it will be less than 1 unless O is a crisp singleton, according to the first property).

The last property holds for normal fuzzy sets and is also very intuitive: as the membership of objects increase, there are more possible objects and/or objects with more possibility, and hence the distribution is less informative, leading the specificity to diminish.

In the literature it is possible to find different measures of specificity for possibility distributions with finite support [DP87; L.G+03; L.G+06; MG82; Yag82; Yag90; Yag92b]. Some of them are defined individually, whilst some others are grouped into families. We can highlight the following measures and families as the most employed:

- Yager's cardinality-based measure, introduced in [Yag82]. According to this measure, the specificity of a possibility measure defined by a fuzzy set $A \neq \emptyset$ is

  $$Sp^c(A) = \sum_{\alpha_i \in \Lambda_A} \frac{\alpha_i - \alpha_{i+1}}{|A_{\alpha_i}|} \tag{2}$$

  where A_α is the α-cut of A, $\Lambda_A = \{A(o) \mid o \in Supp(A)\} = \{\alpha_1, \ldots, \alpha_m\}$ is the level set of A, with $\alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_{m+1} = 0$, being $m \leq n = |O|$, and $Supp(A)$ the support of A. When $A = \emptyset$ it is $Sp^c(A) = 0$.

  The semantics of this measure is that of relating specificity to cardinality, as proposed in [Yag92b]. More specifically, it follows the idea that specificity can be defined in terms of the inverse of cardinality (when the cardinality is not 0). Dubois and Prade proposed in [DP85] a probabilistic interpretation for this measure, that can be easily understood if we realize that, when A is crisp, $Sp^c(A)$ can be interpreted as the probability that the actual value of the variable is found if we randomly take an object of A.

- The *linear* family of measures [Yag92a; Yag98; Yag90] is defined by

$$Sp_{\vec{w}}(A) = a_1 - \sum_{i=2}^{n} w_i a_i \tag{3}$$

where a_i is the i-th greatest membership value of A, and $\vec{w}$ is a collection of weights with $w_1 = 1$, $\sum_{i=2}^{n} w_i = 1$, $1 \geq w_i \geq w_{i+1} \geq 0$, defining the different members of this family. Notice the difference between a_i and α_i: $a_i \geq a_{i+1}$ whilst $\alpha_i > \alpha_{i+1}$.

Different vectors of weights $\vec{w}$ yield different measures in this family. In particular, it can be shown that the measure Sp^c of Eq. (2) is a member of this family [L.G+03].

- The *product* family [Yag98] is given by the general expression

$$Sp_k(A) = a_1 \prod_{i=2}^{n} (ka_i + (1 - a_i)) \tag{4}$$

where $k \in [0, 1)$ is a parameter that yields the specific measure.

- The *fractional* specificity of A [Yag12] is

$$Sp^f(A) = \frac{a_1^2}{\sum_{i=1}^{n} a_i} \tag{5}$$

2.2 Referring Expressions and Referential Success

As mentioned in the introduction, one of the application domains where measures based on specificity can be used is that of identifying objects in a given set of objects. An example is the identification of objects in a visual scene containing different objects. Noun phrases are the conventional way to carry out such identification by means of natural language; these phrases are commonly known in the literature as referring expressions.

The automatic generation of such expressions is a major problem in the field of natural language generation, where it is a challenge of growing interest. As they are noun phrases aimed at identifying, referring expressions usually include adjectives and adverbs either in simple form (*The round object* in a scene, for instance) or by using relative clauses (*The object that is over the table*). Since the concepts behind many adjectives and adverbs do not have a precise application in real-world scenes, fuzzy logic is an appropriate tool in order to provide models to handle the semantics of a referring expression [MS16].

One of the problems to be solved when generating referring expressions by means of computers is their evaluation. Paying attention to its objective, a referring expression should identify an object (the one intended to be referred) within a set. Therefore, the expression must describe the object under consideration and no other [RD92]. Thus, in order to evaluate the quality of an expression, we need to measure both its validity as descriptor for the desired object and its ability to distinguish it among the other objects in the universe under study.

In its most basic and usual form, a referring expression can be managed as a set of properties. Let $re = \{p_1, \ldots, p_n\}$ be such a set. If the object under consideration in the scene is o, the expression is accurate for object o when [Gat+16]:

$$\forall p_i \in re, o \in [\![p_i]\!]$$

where $[\![p_i]\!]$ is the set of objects that satisfy p_i.

If, as previously mentioned, properties in the expression have fuzzy semantics, the latter calculus should be appropriately adapted. Let $p_i(o)$ stand for the accomplishment degree of property p_i for object o. The accuracy of the referring expression is then calculated as [Gat+16]:

Sp	$\overrightarrow{RS}_t(Sp)$
Yager's measure, Eq. (2)	$$\overrightarrow{RS}_t(Sp_c)(re,o_i)=\begin{cases} t\left(O_{re}(o_i), \sum_{\alpha_i\in\Lambda_{O^*_{re}}} \frac{\alpha_i-\alpha_{i+1}}{\lvert (O^*_{re})_{\alpha_i}\rvert}\right) & \text{if condmax} \\ 0 & \text{otherwise.}\end{cases} \quad (6)$$ where *condmax* is the following condition: $$\max_{o\in O} O_{re}(o)=O_{re}(o_i)>0$$ Also,$(O^*_{re})_\alpha$ is the α-cut of O^*_{re}, and $\Lambda_{O^*_{re}}=\{O^*_{re}(o)\mid o\in Supp(O^*_{re})\}$ is the set of membership degrees of objects to the support of O^*_{re} (level set of O^*_{re}), with $\Lambda_{O^*_{re}}=\{\alpha_1,\dots,\alpha_m\}$, and $\alpha_i>\alpha_{i+1}\ \forall i$.
Linear family of measures, Eq. (3)	$$\overrightarrow{RS}_t(Sp_{\vec{w}})(re,o_i)=\begin{cases} t(O_{re}(o_i),(w_1 O^*_{re}(o_i)-\sum_{i=2}^n w_i a_i)) & \text{if condmax}\\ 0 & \text{otherwise,}\end{cases} \quad (7)$$ where a_i is the i-th greatest value among the memberships of O^*_{re}, and $\vec{w}$ is a collection of weights with $w_1=1$, $\sum_{i=2}^n w_i=1$, $1\ge w_i\ge w_{i+1}\ge 0$.
Product family of measures, Eq. (4)	$$\overrightarrow{RS}_t(Sp_k)(re,o_i)=\begin{cases} t(O_{re}(o_i),(O^*_{re}(o_i)\prod_{i=2}^n(ka_i+(1-a_i)))) & \text{if condmax}\\ 0 & \text{otherwise,}\end{cases} \quad (8)$$ where a_i is the i-th greatest value among the memberships of O^*_{re}, and $k\in[0,1]$.
Fractional measure of specificity, Eq. (5)	$$\overrightarrow{RS}_t(Sp_f)(re,o_i)=\begin{cases} t\left(O_{re}(o_i),\frac{O^*_{re}(o_i)^2}{\sum_{i=1}^n a_i}\right) & \text{if condmax}\\ 0 & \text{otherwise,}\end{cases} \quad (9)$$ where a_i is the i-th greatest value among the memberships of O^*_{re}.

Table 1. Example of referential success measures obtained from specificity

$$acc_{re}(o)=\bigotimes_{i=1}^{n} p_i(o) \tag{10}$$

where $\otimes$ is a t-norm.

In addition to correctly describe the desired object, the expression must distinguish it from other objects in the context, in which case it is said, in the field of natural language generation, that the expression *has referential success*.

When the properties used in the expression are crisply defined, the accomplishment of the referential success requires that the set of objects that meet the expression is a singleton. That is:

$$\bigcap_{p_i\in re} [\![\, p_i \,]\!] = \{o\} \tag{11}$$

Conversely, if the properties are fuzzily defined, graduality in their compliance with the objects in the universe under study should be considered. In this case, if $O=\{o_1,\dots,o_n\}$ is the universe of objects, the set O_{re} of objects referred by the expression re is defined as follows:

$$O_{re}(o)=acc_{re}(o), \forall o\in O \tag{12}$$

For each object $o \in O$, $O_{re}(o)$ represents the possibility that object o is the referred object. The referential success of the expression re is in line with the information that the former possibility distribution O_{re} brings about which is the referred object among those in O.

In [MRGS16b], a minimal set of properties that a referential success measure $rs(re, o)$ must fulfill in relation to the induced O_{re} is presented:

Property 2.4
$rs(re, o_i) = 1$ iff $O_{re} = \{o_i\}$.

Property 2.5
If $O_{re}(o_i) = 0$ then $rs(re, o_i) = 0$.

Property 2.6
If $O_{re}(o) \leq O_{re'}(o)\ \forall o \in O \backslash \{o_i\}$ and $O_{re}(o_i) \geq O_{re'}(o_i)$ then $rs(re, o_i) \geq rs(re', o_i)$.

That is, the referential success is maximum when O_{re} is the singleton comprised of the object to be referred and is minimum when the object to be referred in no way fulfills the expression (the degree of membership is 0). The third property states that, as membership of unwanted objects increase, the referential success decreases, just the opposite that is observed when the membership of the desired object grows.

2.3 From Specificity to Referential Success

Measures of referential success can be obtained by using the specificity measures mentioned in the previous section. In [MRGS16b], it is shown that this can be done through a t-norm based combination of, first, the accomplishment degree between the expression and the object to be referred and, second, the specificity of the set of objects actually referred:

Definition 2.1 [MRGS16b]
Let t be a t-norm and Sp be a specificity measure. The referential success measure associated to Sp and t is defined as follows:

$$\overrightarrow{RS}_t(Sp)(re, o_i) = \begin{cases} t(O_{re}(o_i), Sp(O^*_{re})) & if condmax \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

where

$$O^*_{re}(o_j) = \frac{O_{re}(o_j)}{\max_{o \in O} O_{re}(o)} \tag{14}$$

and *condmax* stands for the following condition:

$$\max_{o \in O} O_{re}(o) = O_{re}(o_i) > 0.$$

In [MRGS16b] it is shown that Eq. (13) satisfies properties 2.4-2.6. This definition allows the generation of a measure of referential success for each specificity measure and t-norm considered.

As an example, Table 1 shows the measures of referential success obtained when using some well known families of specifity measures [MRGS16b].

3 From Referential Success to Specificity

In the previous section, it has been shown that it is possible to define measures of referential success using as basis specificity measures. In this section, the study of the relationship between both families of measures is complemented by analyzing the inverse path, that is, we show that specificity measures can also be obtained from measures of referential success.

Before presenting our proposal, let us introduce the following property that holds for every measure of referential success:

Proposition 3.1
Let re be a referring expression, and let $O_{re}(o_i) \geq O_{re}(o_j)$. Then $rs(re, o_i) \geq rs(re, o_j)$.

Proof: Let re' be a referring expression such that: $O_{re'}(o_i) = O_{re}(o_j)$, $O_{re'}(o_j) = O_{re}(o_i)$, and $O_{re'}(o_k) = O_{re}(o_k)$ $\forall k \neq i, j$. Then it is evident that $rs(re, o_j) = rs(re', o_i)$. Also it is $O_{re}(o_i) \geq O_{re'}(o_i) = O_{re}(o_j)$ and $O_{re}(o_j) \leq O_{re'}(o_j) = O_{re}(o_i)$. Hence it is $O_{re}(o) \leq O_{re'}(o)$ $\forall o \neq o_i$ and $O_{re}(o_i) \geq O_{re'}(o_i)$, and by property RS3 it is $rs(re, o_i) \geq rs(re', o_i) = rs(re, o_j)$ ∎

Corollary 3.1
Let re be a referring expression, and let $\max_{o \in O} O_{re}(o) = O_{re}(o_i)$. Then $rs(re, o_i) \geq rs(re, o_j)$ $\forall j \neq i$.

Now, let us introduce our proposal. As any fuzzy set A represents a property that can be noted also A for simplicity, $\{A\}$ is a referring expression and $A = O_{\{A\}}$ is the possibility distribution induced by the referring expression on O. Hence, in order to build a specificity measure from a measure of referential success, it is possible to use the following proposition:

Definition 3.1
Let rs be a measure of referential success and let A be a fuzzy subset of $O = \{o_1, \dots, o_n\}$. Let $max_{o \in O} A(o) = A(o_i)$. The specificity measure associated to rs is defined as follows:

$$\overrightarrow{SP}(rs)(A) = \max_{o \in O} rs(\{A\}, o) = rs(\{A\}, o_i) \tag{15}$$

Proposition 3.2
Eq. (15) satisfies properties S1-S3.

Proof:

S1: $\overrightarrow{SP}(rs)(A) = 1$ iff $\max_{o \in O} rs(\{A\}, o) = 1$ iff $\exists o_i \in O$ such that $rs(\{A\}, o) = 1$ iff $A = \{o_i\}$, by property RS1.

S2: $\overrightarrow{SP}(rs)(\emptyset) = \max_{o \in O} rs(\emptyset, o) = 0$ by property RS2.

S3: Let $A, B \in [0,1]^O$ be normal fuzzy sets and $A \subseteq B$. Then $\exists o_i \in O$ such that $A(o_i) = B(o_i) = 1$ and, by corollary 3.1, it is $\max_{o \in O} rs(\{A\}, o) = rs(\{A\}, o_i)$ and $\max_{o \in O} rs(\{B\}, o) = rs(\{B\}, o_i)$. Since $A(o_i) \geq B(o_i)$ and $A(o) \leq B(o)$ $\forall o \neq o_i$, by property RS3 it is $rs(\{A\}, o_i) \geq rs(\{B\}, o_i)$ and hence $\overrightarrow{SP}(rs)(A) \geq \overrightarrow{SP}(rs)(B)$.

∎

That is, as the referential success is a measure focused on a particular object of the fuzzy set under study, one way to compute a measure of specificity of the whole fuzzy set is to consider that this specificity coincides with the maximum referential success among the elements in the set. This proposition brings to the front a semantics for specificity in relation with the act of referring: *the specificity of a fuzzy set can be interpreted as the extent to which there is one element in its support that can be successfully referred to by the property represented by the fuzzy set.*

3.1 Some examples

Let us show some specificity measures that can be obtained from measures of referential success according to definition 3.1:

3.1.1 From the measure of referential success proposed in [Gat+16]

Consider for instance the following family of measures introduced in [Gat+16]:

$$rs_{(\otimes,\neg)}(re, o_i) = O_{re}(o_i) \otimes \left(\bigotimes_{o_j \in O \wedge j \neq i} \neg(O_{re}(o_j)) \right) \tag{16}$$

where $\otimes$ is a t-norm and $\neg$ is a fuzzy negation. The corresponding family of specificity measures is obtained as follows:

$$Sp_{(\otimes,\neg)}(A) = \overrightarrow{SP}(rs_{(\otimes,\neg)})(A) =$$
$$= \max_{o\in O}\left(A(o) \otimes \left(\bigotimes_{o\neq o_j\in O} \neg(A(o_j))\right)\right) \tag{17}$$

where $\otimes$ is a t-norm and $\neg$ is a fuzzy negation.

It is possible to show that one particular member of this family coincides with a member of the product family, defined in Eq. (4):

Proposition 3.3
The measure $\overrightarrow{SP}(rs)$ of Eq. (17) with $\otimes$ being the product and $\neg$ the standard negation is the measure of specificity of the product family defined in Eq. (4) with $k = 0$.

Proof: Let a_i be the i-th greatest value among the membership of objects to A. Let $a_1 = A(o_i)$ with $o_i \in O$ and let $k = 0$. By corollary 3.1 it is

$$\begin{aligned}
\overrightarrow{SP}(rs_{(\otimes,\neg)})(A) &= \max_{o\in O}\left(A(o) \otimes \left(\bigotimes_{o\neq o_j\in O} \neg(A(o_j))\right)\right) = \\
&= A(o_i) \otimes \left(\bigotimes_{o_j\in O\wedge j\neq i} \neg(A(o_j))\right) = \\
&= A(o_i)\prod_{j\neq i}(1 - A(o_j)) = a_1\prod_{l=2}^{n}(1-a_l) = \\
&= a_1\prod_{l=2}^{n}(0a_l + (1-a_l)) = Sp_0(A)
\end{aligned}$$

■

3.1.2 From the measure of referential success proposed in [MRGS16a]

Let O be a set of objects, re be a referring expression defined on O, and let $V^o_{re} \subset [0,1]$ be

$$V^o_{re} = \left\{\alpha \mid (O_{re})_\alpha = \{o\}\right\} \tag{18}$$

where $(O_{re})_\alpha$ is the α-cut of O_{re}. V^o_{re} is the set of levels where the expression has referential success in a crisp sense [MRGS16a]. Let

$$\begin{aligned}
\alpha_1 &= \sup(V^o_{re}) && (19)\\
\alpha_2 &= \inf(V^o_{re}) && (20)
\end{aligned}$$

Then

$$rs_\Lambda(re,o) = \begin{cases} \alpha_1(\alpha_1-\alpha_2), & V^o_{re}\neq\emptyset \\ 0, & \text{otherwise}\end{cases} \tag{21}$$

The corresponding specificity measure is the following:

This measure has been also proposed under a different setting in [SPY16].

Proposition 3.4
The specificity measure corresponding to the measure of referential success rs of Eq. (21) is

$$Sp_{\Lambda}(A) = \overrightarrow{SP}(rs_{\Lambda})(A) = \max_{o \in O} rs_{\Lambda}(re, o) = a_1(a_1 - a_2) \tag{22}$$

where a_i is the i-th greatest membership value of A.

Proof: Let $re = \{A\}$. As shown in [MRGS16a], $rs_{\Lambda}(re, o) > 0$ iff $O(o) = a_1 > a_2$ and in such case $\alpha_1 = a_1$ and $\alpha_2 = a_2$. Hence, $\max_{o \in O} rs_{\Lambda}(re, o) = a_1(a_1 - a_2)$. ∎

To the best of our knowledge, this specificity measure has not been previously studied in the literature.

3.1.3 From a new measure of referential success

Let us introduce the following new measure of referential success:

Proposition 3.5
The expression

$$rs_{\sum}(re, o_i) = \frac{O_{re}(o_i)}{1 + \sum_{j \neq i} O_{re}(o_j)} \tag{23}$$

defines a measure of referential success.

Proof:

1. $rs_{\sum}(re, o_i) = 1$ iff $O_{re}(o_i) = 1$ and $O_{re}(o_j) = 0$ $\forall i \neq j$ iff $O_{re} = \{o_i\}$.
2. Trivial.
3. If $O_{re}(o) \leq O_{re'}(o)$ $\forall o \in O \backslash \{o_i\}$ then $1 + \sum_{j \neq i} O_{re}(o_j) \leq 1 + \sum_{j \neq i} O_{re'}(o_j)$. Since by the conditions it is $O_{re}(o_i) \geq O_{re'}(o_i)$, then $rs_{\sum}(re, o_i) \geq rs_{\sum}(re', o_i)$.

∎

Proposition 3.6
The corresponding specificity measure for the referential success measure rs of Eq. (23) is

$$Sp_{\sum}(A) = \overrightarrow{SP}(rs_{\sum})(A) = \frac{a_1}{1 + \sum_{i=2}^{n} a_i} \tag{24}$$

where a_i is the i-th greatest membership value of A.

Proof: The maximum value of Eq. (23) is attained clearly when $O_{re}(o_i) = a_1$, since in that case $O_{re}(o_i)$ attains its larger value and at the same time $\sum_{j \neq i} O_{re}(o_j)$ attains its lower value. ∎

Again, to the best of our knowledge, this specificity measure has not been previously studied in the literature.

4 Duality and reversibility of specificity and referential success

According to the results in previous sections, it is clear that there is a close association between specificity and referential success. In this section we go further into the study of the relationship between these types of measures, by studying their behaviour when going back and forth from ones to the others through the techniques previously proposed.

4.1 Duality

The concept of duality is intended to formalize the relationship existing between measures of specificity and referential success when they are obtained one from the other on the basis of the results in previous sections.

Definition 4.1
Let Sp be a specificity measure, rs a measure of referential success, and t a t-norm. We say that Sp and rs are *dual through* t, or t-dual for short, in $\mathcal{A} \subseteq [0,1]^O$, if and only if they satisfy the following two conditions for every $A \in \mathcal{A}$:

$$\overrightarrow{SP}(rs)(A) = Sp(A) \tag{25}$$
$$\overrightarrow{RS}_t(Sp)(\{A\},o) = rs(\{A\},o) \quad \forall o \in O \tag{26}$$

When $\mathcal{A} = [0,1]^O$ we will simply say that Sp and rs are t-dual.
In the following section we shall see several examples.

4.2 Reversibility

The concept of duality is tightly related to the concept of reversibility, which reflects the fact that a measure remains the same after going back and forth using the techniques proposed in previous sections.
Let us introduce the following definitions:

Definition 4.2
Let t be a t-norm and Sp be a specificity measure. The specificity measure associated to Sp and t via referential success is defined as follows:

$$\overleftrightarrow{SP}_t(Sp) = \overrightarrow{SP}(\overrightarrow{RS}_t(Sp)) \tag{27}$$

By the definition of the operators $\overrightarrow{SP}()$ and $\overrightarrow{RS}()$, it is clear that $\overleftrightarrow{SP}()$ yields a specificity measure.

Definition 4.3
Let t be a t-norm and rs be a measure of referential success. The measure of referential success associated to rs and t via specificity is defined as follows:

$$\overleftrightarrow{RS}_t(rs) = \overrightarrow{RS}_t(\overrightarrow{SP}(rs)) \tag{28}$$

Similarly to the previous definition, it is clear that $\overleftrightarrow{RS}()$ defines a measure of referential success.
Based on these two definitions, let us introduce the following two concepts of reversibility:

Definition 4.4
Let rs be a measure of referential success and let t be a t-norm. We say that rs is *reversible* through t, or t-*reversible* for short, in $\mathcal{A} \subseteq [0,1]^O$, if and only if $\forall A \in \mathcal{A}$ it is

$$\overleftrightarrow{RS}_t(rs)(\{A\},o) = rs(\{A\},o) \quad \forall o \in O \tag{29}$$

When $\mathcal{A} = [0,1]^O$ we will simply say that rs is t-reversible.

Definition 4.5
Let Sp be a specificity measure and let t be a t-norm. We say that Sp is *reversible* through t, or t-*reversible* for short, in $\mathcal{A} \subseteq [0,1]^O$, if and only if $\forall A \in \mathcal{A}$ it is

$$\overleftrightarrow{SP}_t(Sp)(A) = Sp(A) \tag{30}$$

Again, when $\mathcal{A} = [0,1]^O$ we will simply say that Sp is t-reversible.
The following propositions state the relationship between duality and reversibility:

Proposition 4.1
Let Sp be a specificity measure, rs a measure of referential success, t a t-norm, and $\mathcal{A} \subseteq [0,1]^O$. The following holds:

1. Sp is t-reversible in $\mathcal{A}$ iff Sp and $\overrightarrow{RS}_t(Sp)$ are t-dual in $\mathcal{A}$.
2. rs is t-reversible in $\mathcal{A}$ iff rs and $\overrightarrow{SP}(rs)$ are t-dual in $\mathcal{A}$.

Proof: Follows directly from definitions 4.2-4.5. ∎

Is is possible to show that, for every specificity measure Sp, t-reversibility holds in the set of all normal fuzzy subsets of O for any t-norm:

Proposition 4.2
Let Sp be a specificity measure and t a t-norm. Let $\mathcal{N} \subset [0,1]^O$ be the set of all normal fuzzy subsets of O. Then, Sp is t-reversible in $\mathcal{N}$.

Proof: Since A is normal then $A^* = A$ and $\exists o_i \in O$ such that $A(o_i) = 1$. Hence

$$\overleftarrow{SP}_t(Sp)(A) = \overrightarrow{SP}(\overrightarrow{RS}_t(Sp))(A) = \max_{o \in O} \overrightarrow{RS}_t(Sp)(\{A\}, o) =$$
$$= t(A(o_i), Sp(A^*)) = t(1, Sp(A)) = Sp(A)$$

∎

In contrast, the reversibility property for $\mathcal{N}$ does not hold for all the measures of referential success. As a counterexample, consider for instance the measure $rs_{\sum}$ in Eq. (23). Let t be the minimum and let $A = 1/o_1 + 0.3/o_2$. Then

- $rs_{\sum}(A, o_2) = 0.3/1.3 > 0$
- $\overleftarrow{RS}_t(rs_{\sum})(\{A\}, o_2) = \overrightarrow{RS}_t(\overrightarrow{SP}(rs_{\sum}))(\{A\}, o_2) = 0$ since condmax does not hold.

Hence, $rs_{\sum}(A, o_2) \neq \overleftarrow{RS}_t(rs_{\sum})(\{A\}, o_2)$ and $rs_{\sum}$ is not min-reversible in $\mathcal{N}$.
However, there are measures of referential success for which reversibility for $\mathcal{N}$ holds, as the following proposition shows:

Proposition 4.3
All measures in the family $rs_{(\otimes,\neg)}$ of Eq. (16) are t-reversible in $\mathcal{N}$ for every t-norm t.

Proof: Let $A \in \mathcal{N}$, hence $A^* = A$ and $\exists o_j \in O$ such that $A(o_j) = 1$. Let us consider two cases:

1. Let $o_i \in O$ such that $A(o_i) = 1$. Then, since condmax is satisfied
$$\overleftarrow{RS}_t(rs_{(\otimes,\neg)})(\{A\}, o_i) = \overrightarrow{RS}_t(\overrightarrow{SP}(rs_{(\otimes,\neg)}))(\{A\}, o_i) =$$
$$= t(A(o_i), \max_{o \in O}(A^*(o) \otimes (\bigotimes_{o' \neq o} \neg A^*(o'))) =$$
$$= t(A(o_i), \max_{o \in O}(A(o) \otimes (\bigotimes_{o' \neq o} \neg A(o'))) =$$
$$= t(A(o_i), A(o_i) \otimes (\bigotimes_{o' \neq o_i} \neg A(o'))) =$$
$$= t(1, A(o_i) \otimes (\bigotimes_{o' \neq o_i} \neg A(o')) =$$
$$= (A(o_i) \otimes (\bigotimes_{o' \neq o_i} \neg A(o'))) = rs_{(\otimes,\neg)}(\{A\}, o_i).$$
2. Let $o_i \in O$ such that $A(o_i) < 1$. Then, since condmax is not satisfied
$$\overleftarrow{RS}_t(rs_{(\otimes,\neg)})(\{A\}, o_i) = \overrightarrow{RS}_t(\overrightarrow{SP}(rs_{(\otimes,\neg)}))(\{A\}, o_i) = 0$$
And
$$rs_{(\otimes,\neg)}(\{A\}, o_i) = A(o_i) \otimes (\bigotimes_{o \neq o_i} \neg A(o)) = A(o_i) \otimes 0 = 0$$
since $\neg A(o_j) = 0$ and $o_j \neq o_i$. Hence $\overleftarrow{RS}_t(rs_{(\otimes,\neg)})(\{A\}, o_i) = rs_{(\otimes,\neg)}(\{A\}, o_i)$.

∎

4.3 Duality and reversibility of usual specificity measures

In this section we shall show some results about duality and reversibility of linear, product and fractional specificity measures.

Proposition 4.4
Let $Sp_{\vec{w}}$ be a specificity measure in Yager's linear family (Eq. (3)). Let $\times$ be the product t-norm. Then, $Sp_{\vec{w}}$ is $\times$-reversible.

Proof: When $A = \emptyset$ the proposition holds trivially. Now, consider $A \neq \emptyset$. By definition 2.1 we have

$$\overrightarrow{RS}_{\times}(Sp_{\vec{w}})(\{A\}, o) = \begin{cases} A(o) \times Sp_{\vec{w}}(A^*) & \text{if condmax} \\ 0 & \text{otherwise} \end{cases}$$

Let $o_i \in O$ such that $A(o_i) = \max_{o \in O} A(o) = a_1$. Then it is $A^*(o) = A(o)/a_1 \ \forall o \in O$, and

$$\overleftrightarrow{SP}_{\times}(Sp_{\vec{w}})(A) = \max_{o \in O} \overrightarrow{RS}_{\times}(Sp_{\vec{w}})(\{A\}, o) = A(o_i) \times Sp_{\vec{w}}(A^*) =$$

$$= a_1 \times \left[\frac{a_1}{a_1} - \sum_{i=2}^{n} w_i \frac{a_i}{a_1} \right] = a_1 - \sum_{i=2}^{n} w_i a_i = Sp_{\vec{w}}(A)$$

∎

Proposition 4.5
Let Sp_f be the fractional specificity measure of Eq. (5). Let $\times$ be the product t-norm. Then, Sp_f is $\times$-reversible.

Proof: When $A = \emptyset$ the proposition holds trivially. Now, consider $A \neq \emptyset$. By definition 2.1 we have

$$\overrightarrow{RS}_{\times}(Sp_f)(\{A\}, o) = \begin{cases} A(o) \times Sp_f(A^*) & \text{if condmax} \\ 0 & \text{otherwise} \end{cases}$$

Let $o_i \in O$ such that $A(o_i) = \max_{o \in O} A(o) = a_1$. Then it is $A^*(o) = A(o)/a_1 \ \forall o \in O$ and, since $(ka_i + (1 - a_i)) = 1$ when $a_i = 0$, it is

$$\overleftrightarrow{SP}_{\times}(Sp_f)(A) = \max_{o \in O} \overrightarrow{RS}_{\times}(Sp_f)(\{A\}, o) = A(o_i) \times Sp_f(A^*) =$$

$$= a_1 \times \left[\frac{\left(\frac{a_1}{a_1}\right)^2}{\sum_{i=1}^{n} \frac{a_i}{a_1}} \right] = a_1 \times \left[\frac{1}{\frac{1}{a_1} \sum_{i=1}^{n} a_i} \right] = a_1 \times \left[\frac{a_1}{\sum_{i=1}^{n} a_i} \right] =$$

$$= \frac{a_1^2}{\sum_{i=1}^{n} a_i} = Sp_f(A)$$

∎

Finally, t-reversibility does not hold for the product family in general. As a counterexample, let us consider the specificity measure of the product family with $k = 0.5$, let t be the minimum, and let $A = 0.9/o_1 + 0.3/o_2$. In this case $A^* = 1/o_1 + (1/3)/o_2$, and

- $Sp_{0.5}(A) = a_1 \times (0.5 \times a_2 + (1 - a_2)) = 0.9 \times (0.5 \times 0.3 + 0.7) = 0.765$
- $Sp_{0.5}(A^*) = a_1^* \times (0.5 \times a_2^* + (1 - a_2^*)) = 1 \times (0.5 \times 1/3 + 2/3) = 5/6$
- $\overleftrightarrow{SP}_t(Sp_{0.5})(A) = \max_{o \in O} \overrightarrow{RS}_t(Sp_{0.5})(\{A\}, o) = \overrightarrow{RS}_t(Sp_{0.5})(\{A\}, o_1) = \min(0.9, 5/6) = 5/6 \neq 0.765$

4.4 Isomorphisms

In previous sections we have seen that there is a strong relation between specificity and referential success. Regarding the extent of this relation, the question arises about whether some kind of isomorphism exists between both types of measures on the basis of Definitions 2.1 and 3.1.

Let O be any finite set with $|O| > 1$, $\mathbb{SP}$ be the set of all specificity measures with domain $[0,1]^O$, and $\mathbb{RS}$ the set of all measures of referential success with domain $[0,1]^O$. Let t be a t-norm, and let $f_t : \mathbb{SP} \to \mathbb{RS}$ and $g : \mathbb{RS} \to \mathbb{SP}$ be defined as follows:

$$f_t(Sp) = \overrightarrow{RS}_t(Sp) \tag{31}$$

$$g(rs) = \overrightarrow{SP}(rs) \tag{32}$$

$\forall Sp \in \mathbb{SP}, rs \in \mathbb{RS}$. It is possible to show that there is no isomorphism between the sets of measures $\mathbb{SP}$ and $\mathbb{RS}$ on the basis of neither f_t nor g, for every O and for every t-norm t.

Let us consider first the case of f_t:

Proposition 4.6
The expression $Sp(A) = a_1^2(1-a_2)$ defines a specificity measure.

Proof: We can prove the three required properties:

1. A is a crisp singleton iff $a_1 = 1$ and $a_2 = 0$ iff $Sp(A) = 1$.
2. Let $A = \emptyset$. Then $a_1 = a_2 = 0$ and $Sp(A) = 0$.
3. Let A be a normal fuzzy set. Then $a_1 = 1$ and it is $Sp(A) = a_1^2(1-a_2) = a_1(a_1 - a_2)$. That is, for normal fuzzy subsets, Sp coincides with the specificity measure of Eq. (22), and hence it satisfies property 3.

■

Proposition 4.7
Let $Sp_1(A) = a_1^2(1-a_2)$ and let $Sp_2(A) = Sp_\Lambda = a_1(a_1 - a_2)$. For every t-norm t it is $\overrightarrow{RS}_t(Sp_1) = \overrightarrow{RS}_t(Sp_2)$.

Proof: Let us consider a certain element o. If o does not satisfy *condmax*, then $\overrightarrow{RS}_t(Sp_1)(\{A\}, o) = \overrightarrow{RS}_t(Sp_2)(\{A\}, o) = 0$. Otherwise, it is

$$\overrightarrow{RS}_t(Sp_1) = t(A(o), Sp_1(A^*)) = t(A(o), 1^2(1 - a_2/a_1)) =$$
$$= t(A(o), 1(1 - a_2/a_1)) = t(A(o), Sp_2(A^*)) = \overrightarrow{RS}_t(Sp_2)$$

since we have shown in the proof of Proposition 4.6 that when applied to normal fuzzy sets, Sp_1 and Sp_2 yield the same result (and A^* is a normal fuzzy set). ■

Theorem 4.1
The sets $\mathbb{SP}$ and $\mathbb{RS}$ are not isomorphic via f_t for any t-norm t.

Proof: The specificity measures Sp_1 and Sp_2 in Proposition 4.7 are different, that is, $Sp_1(A) \neq Sp_2(A)$ in general. As an example, consider $A = 0.7/o_1 + 0.3/o_2$. Then it is $Sp_1(A) = 0.343$ whilst $Sp_2(A) = 0.28$. However, as shown in Proposition 4.7, it is $f_t(Sp_1) = \overrightarrow{RS}_t(Sp_1) = \overrightarrow{RS}_t(Sp_2) = f_t(Sp_2)$ for every t-norm t. Hence, f_t is not a bijection for any t-norm t. ■

Now, let us consider the case of g. Let

$$rs_1(\{A\}, o) = \min(A(o), \min_{o' \neq o}\{1 - A(o')\}) \tag{33}$$

be a measure of referential success of the family defined in Eq. (16) using the minimum for both t-norms and the standard negation.

Proposition 4.8
The expression

$$rs_2(\{A\}, o) = \begin{cases} rs_1(\{A\}, o) & \text{if } condmax \\ 0 & \text{otherwise} \end{cases} \tag{34}$$

defines another measure of referential success.

Proof: We can prove the three required properties:

1. $rs_2(\{A\}, o) = 1$ iff $rs_1(\{A\}, o) = 1$ iff $A = o$ since rs_1 is a measure of referential success.
2. Trivial.
3. Let A, A' be such that $A(o_i) \leq A'(o_i)\ \forall o_i \in O \backslash \{o\}$ and $A(o) \geq A'(o)$. Let us consider two cases:
   a. Let $A'(o) = 0$. Then $rs_2(\{A\}, o) \geq rs_2(\{A'\}, o) = 0$.
   b. Let $A'(o) > 0$. Then $rs_2(\{A'\}, o) = rs_1(\{A'\}, o)$ and $rs_2(\{A\}, o) = rs_1(\{A\}, o)$ and, since rs_1 is a measure of referential success, it is $rs_2(\{A\}, o) \geq rs_2(\{A'\}, o)$.

■

Proposition 4.9
Let rs_1 and rs_2 be those of Equations (33) and (34), respectively. Then it is $\overrightarrow{SP}(rs_1) = \overrightarrow{SP}(rs_2)$.

Proof: Immediate by Eq. (15) since when *condmax* is satisfied for an object o, it is $\overrightarrow{SP}(rs_2)(A) = rs_2(\{A\}, o) = rs_1(\{A\}, o) = \overrightarrow{SP}(rs_1)(A)$. ■

Theorem 4.2
The sets $\mathbb{SP}$ and $\mathbb{RS}$ are not isomorphic via g.

Proof: The measures of referential success rs_1 and rs_2 in Proposition 4.9 are different, that is, $rs_1(\{A\}, o) \neq rs_2(\{A\}, o)$ in general. As an example, consider $A = 0.7/o_1 + 0.3/o_2$. Then it is $rs_1(\{A\}, o_2) = 0.3$ whilst $rs_2(\{A\}, o_2) = 0$. However, as shown in Proposition 4.9, it is $g(rs_1) = \overrightarrow{SP}(rs_1) = \overrightarrow{SP}(rs_2) = g(rs_2)$. Hence, g is not a bijection. ■

5 Application example

In the previous sections, we have seen how to obtain measures of specificity from measures of referential success and vice versa, and we have analyzed the relationships between the two families of measures. The fields of application of this type of measures are varied. In the case of measures of referential success, the most widespread use is in the field of referring expression generation within the area of natural language generation [Gat+16; MRGS16a], although they have been used without that name in other problems as in the field of data modelling in Business Intelligence [SPY16]. Measures of specificity have been widely used for a long time in determining the informativeness of possibility distributions, among other uses.

The semantics for specificity in relation to referential success that is explored in this work opens new opportunities for the application of this type of measures. As a particular case, it permits to solve problems such as "Finding those sets X in $\mathcal{X} \subseteq \{0,1\}^O$ where a given restriction P can be used to identify a single object $o \in X$", where P is represented by a fuzzy set $O_P \in [0,1]^O$. Referring expressions defined by the conjunction of several fuzzy sets representing properties are particular cases of P.

For example, let $X = \{Img_1, ..., Img_n\}$ be a set of images, each image Img_i containing a set of objects O^i. Let $O = \bigcup O^i$. Given a certain restriction P, our problem would be to locate those images Img_i in X where it is clear enough that there is a single object satisfying P. The resolution of this type of queries is useful in the context of

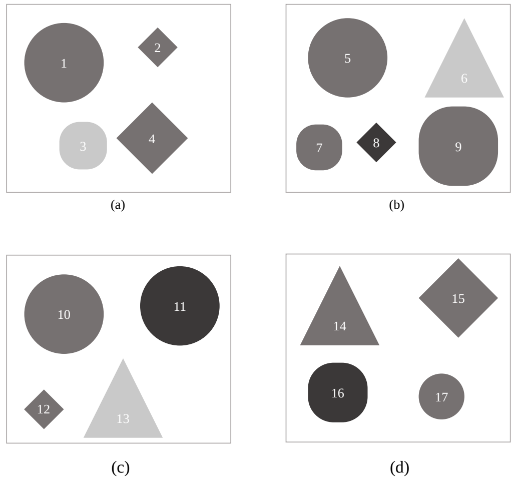

Figure 1. Four images containing objects. (a) Img_1. (b) Img_2. (c) Img_3. (d) Img_4.

image retrieval systems because it complements the usual query capability based on the inclusion in the image of *at least* one object that fulfills P.

As P is a fuzzy constraint, the query is flexible and the membership of a given image Img_i to the query solution is a matter of degree. This degree can be calculated through a measure of specificity as $Sp(O_P^i)$.

Although any measure of specificity could be valid to solve the query, those derived from a measure of referential success allow us to link the semantics of the query to the underlying object identification problem. Given an image Img_i consisting of a set of objects $\{o_1, ..., o_m\}$, the degree to which the property P uniquely identifies the object o_j within the image can be obtained by means of a suitable measure of referential success rs as $rs(O_P^i, o_j)$. Then, if we intend to locate images where it is easy to identify a single object satisfying P, it seems reasonable to use a specificity measure derived from the selected measure of referential success.

Let $X = \{Img_1, Img_2, Img_3, Img_4\}$ be the set of four images in Figure 1, where numbers are attached to objects with the only purpose that the reader can easily identify them. We consider the properties “circular” and “large” as fuzzy properties represented by suitable fuzzy sets. The fulfilment degree of each object regarding both properties, as well as the degree to which each object is “circular and large” are shown in Table 2 (the minimum has been used as t-norm for combining both properties).

Table 3 shows the values obtained when computing the specificity of the example images by means of several measures with respect to the property “circular and large”:

- In the first image it is clearly observed that there is only one object that fits the constraint. All measures, without exception, show this fact by yielding high values.

- A different case is that of images two and three, where two objects highly fit with the description presented. It can be observed a wide variability in the values produced by the different measures of specificity, which gives idea of the variability in their semantics.

- Finally, the fourth image raises a case in which no object fits significantly with the description. In this case, it is remarkable the low value produced by the $Sp_\wedge$ measure, while the others move in the range [0.35-0.5].

The three new measures (last lines in the table) offer different behaviors since they are obtained from measures of referential success with different semantics:

- $Sp_{\sum}$ is derived from a measure of referential success penalized by cardinality, using in this case the sigma count. For that reason, it delivers values which are similar to those provided by Sp_c.

- $Sp_{(\otimes,\neg)}$ comes from a logical interpretation of the referential success which, translated to specificity, corresponds to a restrictive version of the semantics associated with the product family, as shown in Proposition 3.3.

Table 2. Fulfilment degree of “circular”, “large”, and “circular and large” by objects in the different images in Figure 1.

Image	Object	circular	large	circular & large
Img_1 (a)	1	1	1	1
	2	0	0	0
	3	0,8	0,2	0,2
	4	0	0,8	0
Img_2 (b)	5	1	1	1
	6	0	1	0
	7	0,8	0,15	0,15
	8	0	0	0
	9	0,8	1	0,8
Img_3 (c)	10	1	1	1
	11	1	1	1
	12	0	0	0
	13	0	1	0
Img_4 (d)	14	0	0	0
	15	0	1	0
	16	0,8	0,5	0,5
	17	1	0,15	0,15

· Finally, Sp_Λ comes from an interpretation based on the analysis of the reference power by levels, according to which the referential success is directly proportional to both the highest level and the size of the range of levels where the reference is successful. As illustrated in the example, this measure is very restrictive in the presence of several objects with high membership degree, as well as when the referred object has a low membership degree.

6 Conclusion

Specificity measures and referential success are tightly connected and the study of their relationship is of high interest in many areas of Computer Science, as in the field of referring expression generation within Natural Language Generation and Business Intelligence, among others.

The specificity of a fuzzy set induced by a given property or restriction can be interpreted as the extent to which there is one element in its support that can be successfully referred to by the property, that is, as the degree to which there is an element that can be distinguished from the rest according to that property. This interpretation provides a very useful tool to define and/or to consider the semantics of specificity measures in terms of the semantics of the corresponding measures of referential success.

Given a couple of measures, one of specificity and one of referential success, their degree of connection is explored through the concepts of duality and reversibility introduced in this paper. Despite the fact that it is not possible to define an isomorphism between both sets of measures in the most general case using our proposals, there are many such connections between large families of measures, particularly when the fuzzy sets involved are normal. A further study of these relations remains open for future research, in order to advance in the development of both types of measures.

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Table 3. Different specificity measures for the fulfillment of property "circular and large" for the four images in Figure 1. The first 9 lines correspond to measures of the product family with different k values. Next 4 lines correspond to measures of the linear family with different weight vectors. Fractional, and cardinality based measure of Yager follow. The three last measures are those obtained from referential success in Section 3.1. $Sp_{(\otimes,\neg)}$ has been calculated using the product as t-norm and the standard negation.

Measure	Img_1	Img_2	Img_3	Img_4
$Sp_{0.1}$	0.82	0.24	0.10	0.43
$Sp_{0.2}$	0.84	0.32	0.20	0.44
$Sp_{0.3}$	0.86	0.39	0.30	0.45
$Sp_{0.4}$	0.88	0.47	0.40	0.46
$Sp_{0.5}$	0.90	0.56	0.50	0.46
$Sp_{0.6}$	0.92	0.64	0.60	0.47
$Sp_{0.7}$	0.94	0.73	0.70	0.48
$Sp_{0.8}$	0.96	0.81	0.80	0.49
$Sp_{0.9}$	0.98	0.91	0.90	0.49
$Sp_{(1,1,0,\ldots)}$	0.80	0.20	0.00	0.35
$Sp_{(1,0.5,0.5,0,\ldots)}$	0.90	0.53	0.50	0.43
$Sp_{(1,0.334,0.333,0.333,0,\ldots)}$	0.93	0.68	0.67	0.45
$Sp_{(1,0.2,0.2,0.2,0.2,0.2,0,\ldots)}$	0.96	0.81	0.80	0.47
Sp_f	0.83	0.51	0.50	0.38
Sp_c	0.90	0.57	0.50	0.43
$Sp_{(\otimes,\neg)}$	0.80	0.17	0.00	0.43
Sp_{Λ}	0.80	0.20	0.00	0.17
$Sp_{\sum}$	0.83	0.51	0.50	0.43

References

[DP85] D. Dubois and H. Prade. "A note on measures of specificity for fuzzy sets". In: *Internat. J. Gen. Systems* 10 (1985), pp. 279–283.

[DP87] D. Dubois and H. Prade. "Properties of measures of information in evidence and possibility theories". In: *Fuzzy Sets and Systems* 24 (1987), pp. 161–182.

[Gat+16] A. Gatt, N. Marín, F. Portet, and D. Sánchez. "The role of graduality for referring expression generation in visual scenes". In: *Proceedings IPMU 2016, Volume 610 of the series Communications in Computer and Information Science*. Ed. by J.P. Carvalho et al. Springer, 2016, pp. 191–203.

[L.G+03] L.Garmendia, R.Yager, E.Trillas, and A.Salvador. "On t-norms based specificity measures". In: *Fuzzy Sets and Systems* 133.2 (2003), pp. 237–248.

[L.G+06] L.Garmendia, R.Yager, E.Trillas, and A.Salvador. "Measures of specificity of fuzzy sets under t-indistinguishabilities". In: *IEEE Transactions on Fuzzy Systems* 14.4 (2006), pp. 568–572.

[MRGS16a] N. Marín, G. Rivas-Gervilla, and D. Sánchez. "A measure of referential success based on alpha-cuts". In: *Proceedings SUM 2016, LNAI 9858*. Ed. by Steven Schockaert and Pierre Senellart. Springer, 2016, pp. 345–351.

[MRGS16b] N. Marín, G. Rivas-Gervilla, and D. Sánchez. "Using Specificity to Measure Referential Success in Referring Expressions with Fuzzy Properties". In: *Proceedings FUZZ-IEEE 2016*. 2016, pp. 563–570.

[MS16] Nicolás Marín and Daniel Sánchez. "On generating linguistic descriptions of time series". In: *Fuzzy Sets and Systems* 285 (2016), pp. 6–30.

[MG82] M.Higashi and G.J.Klir. "Measures of uncertainty and information based on possibility distributions". In: *Int.J.Gen.Syst* 8 (1982), pp. 43–58.

[RD92] Ehud Reiter and Robert Dale. “A Fast Algorithm for the Generation of Referring Expressions”. In: *COLING'92*. 1992, pp. 232–238.

[SPY16] G. Smits, O. Pivert, and R. Yager. “A Soft Computing Approach to Agile Business Intelligence”. In: *Proceedings FUZZ-IEEE 2016*. 2016, pp. 1850–1857.

[Yag92a] Ronald Yager. “Default knowledge and measures of specificity”. In: *Information Sciences* 61.1–2 (1992), pp. 1–44.

[Yag98] Ronald Yager. “Measures of Specificity”. In: *Computational Intelligence: Soft Computing and Fuzzy-Neuro Integration with Applications*. Ed. by Okyay Kaynak, Lotfi A. Zadeh, Burhan Türkşen, and Imre J. Rudas. Berlin, Heidelberg: Springer Berlin Heidelberg, 1998, pp. 94–113.

[Yag82] R.R. Yager. “Measuring tranquility and anxiety in decision-making: An application of fuzzy sets”. In: *Internat. J. Gen. Systems* 8 (1982), pp. 139–146.

[Yag90] R.R. Yager. “Ordinal measures of specificity”. In: *Internat. J. Gen. Systems* 17 (1990), pp. 57–72.

[Yag92b] R.R. Yager. “On the specificity of a possibility distribution”. In: *Fuzzy Sets and Systems* 50 (1992), pp. 279–292.

[Yag12] R.R. Yager. “Expansible measures of specificity”. In: *Internat. J. Gen. Systems* 41.3 (2012), pp. 247–263.

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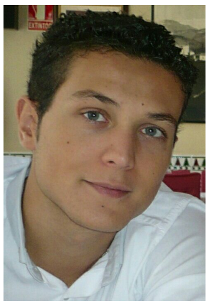

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