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Admissible orders for closed intervals of real numbers not based on the extremes of the intervals

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ABSTRACT

Due to its reasonable properties, the Kulisch and Miranker binary relation on the family of all closed and bounded real intervals has attracted the attention of many researchers, especially in the field of Computation. However, it is not total, so there are intervals that are not comparable. To face this problem, Bustince et al. introduced the notion of *admissible order*, which is coherent to the Kulisch and Miranker binary relation. Due to its technical construction, most of the examples of admissible orders are defined by only employing the extremes of such intervals. In this paper we introduce a non-countable family of admissible orders in the set of all closed and bounded subintervals contained in a concrete closed and bounded real interval. The approach is novel in two senses: on the one hand, due to the mathematical objects that are involved (a dense sequence and a family of continuous functions); and, on the other hand, we do not handle the intervals through their extremes, but only by their interior points.

1. Introduction

Autonomous systems are characterized by the ability of making decisions in any context. However, teaching them to make decisions is a very complicated task, especially when the circumstances in which they may find themselves are very diverse (consider, for example, the case of autonomous driving). Usually, these systems have sensors that provide a finite number of input parameters, and the system makes a decision based on the values it receives. In the decision-making process, one of the main tasks is to compare the real values that are handled, and for this the usual order of real numbers is used. One of the current problems that arise in Computing

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is the comparison of quantities whose order is not clearly determined at the human level, that is, different people would give rise to distinct orderings. These comparisons are crucial in the field of Decision Making, but they are not always according to human intuition. One of the families in which this difficulty can be most clearly seen is the class of all real intervals.

An *interval* is a non-empty continuum set of real numbers, that is, a set enjoying the following property: if it contains two distinct numbers, then it also contains all the real numbers that are placed in the real line between them. When it is bounded, two real numbers (its infimum and its supremum) almost determine the interval. If such values additionally belong to the interval, the continuum set is called a *closed and bounded interval*. These real subsets are completely characterized by their two extremes: minimum and maximum.

Closed and bounded intervals have a dual nature (see [1, Preface]): on the one hand, they can be seen as sets, so they can be compared sometimes through the binary relation given by the *inclusion* of sets; on the other hand, they can be interpreted as numbers (for instance, points of the complex plane), and, in this case, they can be compared through the natural order of their endpoints (this procedure has received several names in the literature, e.g. Kulisch-Miranker order, *component-wise* or *product order*, see [2]). Both comparison methodologies inevitably carry the same drawback: its is easy to find examples of intervals that are incomparable. As we have commented before, when two mathematical objects are incomparable, maybe a decision cannot be made, and this could be a problem in several contexts.

The previous considerations show that it is desirable to work with an order in which all elements are comparable to each other. This property is known as *totality*, or even *linearity* (since it makes it possible to imagine that the elements of the set are arranged as if in a straight line). Some studies making use of total orders are [3–9].

The Kulisch-Miranker order satisfies good properties when the intervals to be ordered are indeed comparable. However, it is not total. In the family $L([0,1])$ of all closed subintervals of $[0,1]$, Bustince et al. introduced in [10] the notion of *admissible order* as that order that, inheriting the good properties of the underlying order (considered when the intervals are comparable), also verifies the property of totality. Admissible orders are, so to speak, total orders that fill in the gaps left by the orders that the researcher wishes to consider. Some applications of this class to real-world problems can be found on [2,11–14].

Although admissible orders solve the problem of the absence of totality, in practice there are currently not enough examples to apply them in concrete contexts. Moreover, the known cases of admissible orders make decisive use of the extremes of the intervals, i.e. they are defined in terms of such extremes. The main aim of this paper is to face these problems: on the one hand, we introduce a large (in fact, a uncountable) class of admissible orders on an arbitrary set $L([a,b])$ based on a dense sequence and on an appropriate family of continuous and surjective functions; on the other hand, considering intervals from the perspective of sets, we show that such a class is not based on the aggregation of the extremes of the intervals, i.e. we can work on this set by avoiding the extremes of the intervals.

This article is organized as follows. In Section 2, some basic notions about binary relations, intervals and functions are reviewed. The third section is devoted to the introduction of the announced class of admissible orders on $L([a,b])$. Later, we describe a concrete family of continuous and surjective functions, which gives way to an uncountable family of examples of admissible orders in the fourth section. Finally, in the last section, we give some concluding remarks and comment on some prospective work.

2. Preliminaries

Let $\mathbb{N}$ be the set of all positive integers and let $\mathbb{R}$ be the set of all real numbers. This set (and each of its subsets) will always be endowed with its usual topology (generated by the Euclidean metric $d(t,s)=|t-s|$ for all $t,s\in\mathbb{R}$). For general notions related to topology (like closedness, boundedness, convergence, etc.), see [15–17]. For instance, a set $X\subseteq\mathbb{R}$ is *closed* if it contains the limit of any convergent sequence whose terms belong to X.

2.1. Binary relations

Henceforth let X be a non-empty set. A *binary relation on* X is a non-empty subset $\mathcal{R}\subseteq X\times X$. For simplicity, if $(x,y)\in\mathcal{R}$, we denote it by $x\preceq y$, and we will say that "$\preceq$" is the binary relation. In particular, we highlight that we denote by "$\preceq$" to an arbitrary binary relation (with no additional properties). We will say that x and y are *$\preceq$-comparable* (or *$\preceq$-related*) if $x\preceq y$ or $y\preceq x$ (or both). A binary relation $\preceq$ on X is:

- *reflexive* if $x\preceq x$ for all $x\in X$;
- *antisymmetric* if, given $x,y\in X$, we can deduce $x=y$ from $x\preceq y$ and $y\preceq x$;
- *transitive* if, given $x,y,z\in X$, we can deduce $x\preceq z$ from $x\preceq y$ and $y\preceq z$;
- *total* if $x\preceq y$ or $y\preceq x$ for all $x,y\in X$ (each two points of X are $\preceq$-comparable).

Notice that each total binary relation is necessarily reflexive. A binary relation is an *order* if it is reflexive, antisymmetric and transitive (see [18–22]).

2.2. Intervals

A real *interval* is a non-empty subset $I\subseteq\mathbb{R}$ verifying the following property: given $t,s\in I$ with $t<s$, if $r\in\mathbb{R}$ satisfies $t<r$ and $r<s$, then $r\in I$, where $<$ denotes the usual order in the real line. In other words, I is an interval when it contains all the real numbers that are placed between any two of its elements. Among all types of intervals, throughout this manuscript, we will only consider *closed and bounded intervals*, which are of the form $[a,b]=\{t\in\mathbb{R} : a\le t\le b\}$, where $a,b\in\mathbb{R}$ satisfy $a\le b$. The numbers a and b are called the (*lower* and *upper*) *extremes* (or *endpoints*) of the interval. A *singleton* is an interval reduced to a single point, that is, $[a,a]=\{a\}$.

Given an interval $I\subseteq\mathbb{R}$, let $L(I)$ denote the family of all closed and bounded intervals of $\mathbb{R}$ contained on I. To fix the notation, throughout this manuscript, we will always denote by A to an interval $[a,b]$, where $a,b\in\mathbb{R}$ satisfy $a<b$ (in particular, A is not a singleton). Hence:

$$L(A)=\{\, X\subset\mathbb{R} : X \text{ is a closed and bounded interval and } X\subseteq A\,\}.$$

In this family, Kulisch and Miranker [2,23] introduced the binary relation $\le_L$ on $L(A)$ (that can be similarly considered on $L(\mathbb{R})$) given, for $X=[\,\underline{x},\overline{x}\,], Y=[\,\underline{y},\overline{y}\,]\in L(A)$,

$$X\le_L Y \quad \text{when} \quad \underline{x}\le\underline{y} \text{ and } \overline{x}\le\overline{y}. \tag{1}$$

The binary relation $\le_L$ is an order on $L(A)$, known as the *point order* or the *pointwise order*. However, it is not total on $L(A)$.

Closed and bounded subsets of $\mathbb{R}$ are *compact*, and this property guarantees the following result.

Proposition 2.1. *The following properties hold.*

1. *Every closed and bounded interval has absolute minimum and absolute maximum.*
2. *If $\{t_n\}_{n\in\mathbb{N}}$ is a sequence contained on a bounded and closed interval, then $\{t_n\}_{n\in\mathbb{N}}$ has a convergent subsequence, and its limit also belongs to such an interval.*

2.3. Real functions of real variable

Let $D,C\subseteq\mathbb{R}$ be non-empty subsets of $\mathbb{R}$ and let $f : D\to C$ be a function. The *image of* f is the set $f(D)=\{f(t) : t\in D\}\subseteq\mathbb{R}$. Any function $f : D\to C$ is surjective if and only if its image $f(D)$ and its codomain C coincide, $f(D)=C$. Analogously, a function $f : X\to Y$ is *bijective* if it is surjective and injective.

Let $I\subseteq\mathbb{R}$ be an interval and let $f : I\to\mathbb{R}$ be a function. We say that f is *increasing* (respectively, *strictly increasing*, *decreasing*, *strictly decreasing*) *on* I if $f(t)\le f(s)$ (respectively, $f(t)<f(s)$, $f(t)\ge f(s)$, $f(t)>f(s)$) for all $t,s\in I$ such that $t<s$. The function f is *monotone* if it is increasing or decreasing. Given two intervals $X,Y\in L(\mathbb{R})$, a function $f : X\to Y$ is an *homeomorphism* (*from* X *onto* Y) if it is continuous and bijective from X onto Y, and its inverse function $f^{-1} : Y\to X$ is also continuous. An *automorphism of* X is an homeomorphism from X onto X. It is easy to show that any two closed and bounded intervals that are not singletons are homeomorphic, and similarly any two singletons of $\mathbb{R}$ are so.

The following is a famous result in Analysis which states that continuous functions preserve the closed and bounded character of intervals.

Theorem 2.2. ***(Intermediate value theorem)*** *If $f : A\to\mathbb{R}$ is a continuous function in the interval $A=[a,b]$, then its image $f(A)$ is also a closed and bounded interval of $\mathbb{R}$ (including $f(a)$ and $f(b)$).*

In other words, if $A\in L(\mathbb{R})$, then $f(A)\in L(\mathbb{R})$.

A function $f : I\to\mathbb{R}$ is *affine* if there are $m,n\in\mathbb{R}$ such that $f(t)=mt+n$ for all $t\in I$, and f is *constant* when f is affine and $m=0$. Affine functions defined on intervals satisfy several well-known properties.

Proposition 2.3. *If $f : A\to\mathbb{R}$ is a function on the interval $A=[a,b]$, then f is affine if and only if*

$$f(t)=\frac{f(b)}{b-a}(t-a)+\frac{f(a)}{b-a}(b-t) \quad \text{for all } t\in A.$$

Immediately we deduce the following consequence.

Proposition 2.4. *If $f,g : A\to\mathbb{R}$ are affine functions on the interval $A=[a,b]$, then $f\le g$ if and only if $f(a)\le g(a)$ and $f(b)\le g(b)$.*

When the function is affine, Theorem 2.2 can gain new properties as stated in the next result.

Corollary 2.5. *If $f : A\to\mathbb{R}$ is an affine function, then f is monotone, continuous and differentiable on A. In fact, f is constant (respectively, strictly increasing, strictly decreasing) on A if and only if $f(a)=f(b)$ (respectively, $f(a)<f(b)$, $f(a)>f(b)$). Furthermore, its image $f(A)$ is the set of all $t\in\mathbb{R}$ verifying $t\ge\min\{f(a),f(b)\}$ and $t\le\max\{f(a),f(b)\}$ (in other words, the closed and bounded interval $[min\{f(a),f(b)\},max\{f(a),f(b)\}]$).*

$$
\begin{array}{cccccccccc}
\frac{0}{1} & & \frac{0}{2} & \longrightarrow & \frac{0}{3} & & \frac{0}{4} & \longrightarrow & \frac{0}{5} & \cdots \\
\downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & \\
\frac{1}{1} & \longrightarrow & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & \cdots \\
 & & & & \downarrow & & \uparrow & & \downarrow & \\
\frac{-1}{1} & \longleftarrow & \frac{-1}{2} & \longleftarrow & \frac{-1}{3} & & \frac{-1}{4} & & \frac{-1}{5} & \cdots \\
\downarrow & & & & & & \uparrow & & \downarrow & \\
\frac{2}{1} & \longrightarrow & \frac{2}{2} & \longrightarrow & \frac{2}{3} & \longrightarrow & \frac{2}{4} & & \frac{2}{5} & \cdots \\
 & & & & & & & & \downarrow & \\
\frac{-2}{1} & \longleftarrow & \frac{-2}{2} & \longleftarrow & \frac{-2}{3} & \longleftarrow & \frac{-2}{4} & \longleftarrow & \frac{-2}{5} & \cdots \\
\downarrow & & & & & & & & & \\
\frac{3}{1} & \longrightarrow & \frac{3}{2} & \longrightarrow & \frac{3}{3} & \longrightarrow & \frac{3}{4} & \longrightarrow & \frac{3}{5} & \cdots \\
 & & & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

Fig. 1. Enumeration of rational numbers.

2.4. Dense sequences on intervals

Given $X \in L(\mathbb{R})$, a sequence $\{s_n\}$ of real numbers is *dense on* X if $\{s_n : n \in \mathbb{N}\} \subseteq X$ and, for all $x \in X$ and all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\left|x - s_{n_0}\right| < \varepsilon$.

Example 2.6. The set $\mathbb{Q}$ of all rational numbers can be enumerated following the algorithm detailed in Fig. 1, giving place to a bijective function $\varrho : \mathbb{N} \to \mathbb{Q}$.

As a result, the sequence $\{s_n\}$ defined by $s_n = \max(\min(\varrho(n), b), a)$ for all $n \in \mathbb{R}$ is a dense sequence on A. In particular, all real intervals admit a dense sequence.

Example 2.7. In [24], two examples of upper dense sequences on $(0, 1]$ were introduced. Adding the value 0, we would find two dense sequences on $[0, 1]$.

Proposition 2.8. *If $\{s_n\}$ is dense on A and $f : A \to \mathbb{R}$ is continuous, then $\{f(s_n)\}$ is dense on $f(A) \in L(\mathbb{R})$.*

Proof. By Theorem 2.2, $f(A) \in L(\mathbb{R})$. Let $y_0 \in f(A)$ and $\varepsilon > 0$ be arbitrary. Let $x_0 \in A$ be such that $f(x_0) = y_0$. Since f is continuous at x_0 and $\varepsilon > 0$, there is $\delta > 0$ such that if $x \in A$ and $\left|x_0 - x\right| < \delta$, then $\left|f(x_0) - f(x)\right| < \varepsilon$. As $\{s_n\}$ is dense on A, there is $n_0 \in \mathbb{N}$ such that $\left|x_0 - s_{n_0}\right| < \delta$. In particular, $f(s_{n_0})$ is a term of the sequence $\{f(s_n)\}$ such that $\left|y_0 - f(s_{n_0})\right| = \left|f(x_0) - f(s_{n_0})\right| < \varepsilon$. $\square$

3. Admissible orders on $L(A)$

The binary relation $\leq_L$ on $L(A)$ described in (1) is explicitly defined by using the extremes of the intervals involved. However, one of the main aims of this manuscript is to show that we can work on $L(A)$ without using the extremes of the intervals. Consequently, given $X \in L(A)$, we will prefer to talk about its infimum and its supremum (whose existence is guaranteed because X is bounded on $\mathbb{R}$) rather than its minimum and its maximum (which are its extremes). For this we will make a double assumption: (1) on the one hand, given $t, s \in \mathbb{R}$, we will always know whether $t < s$, $t = s$ or $t > s$; and (2) on the other hand, for any $t \in A$ and any $X \in L(A)$, we will always know whether $t \in X$ or $t \notin X$. From this point of view, we can work with intervals by comparing points. A first result in this line is the following characterization of the binary relation $\leq_L$ on $L(A)$.

Lemma 3.1. *Two intervals $X, Y \in L(A)$ satisfy $X \leq_L Y$ if and only if the following two properties hold:*

(L_1) for all $x \in X$, there is $y \in Y$ such that $x \leq y$;
(L_2) for all $y \in Y$, there is $x \in X$ such that $x \leq y$.

3.1. Admissible orders on $L(A)$

The order $\leq_L$ verifies many properties that human intuition would suggest when ordering real quantities. However, it is not total, and this is its big problem: if we cannot decide whether $X \leq_L Y$ or $Y \leq_L X$, an important decision (related to the order) might not be made. To deal with this problem, we introduce the following notion.

Definition 3.2. (Bustince et al. [10]) An order $\preceq$ on $L(A)$ is *admissible* when it is total and it refines $\leq_L$ (that is, it can be deduced that $X \preceq Y$ for all $X, Y \in L(A)$ such that $X \leq_L Y$).

It is not easy to give examples of admissible orders on $L(A)$. At least, almost all of them were introduced from a theoretical point of view (see [10]). Furthermore, such constructions often made use of the extremes of the intervals through fusion functions. The main aim of this manuscript is to show that, in fact, $L(A)$ can indeed be endowed with a large (more precisely, a non-countable) family of admissible orders. Such a class will depend on both a dense sequence on A and a family of suitable functions, so we introduce the second component in the next subsection.

3.2. Some families of functions

In this subsection we describe one of the main tools we will be using in this manuscript. From now on, suppose that for each $X \in L(A)$ a function $h_X : A \to X$ is known, and let the family of such functions be denoted by $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$. Before studying $\mathcal{F}$, we introduce a result in which the closedness of the intervals is crucial.

Proposition 3.3. *Given $X, Y \in L(A)$, if there is a sequence which is dense on both X and Y, then $X = Y$.*

Proof. Let $\{s_n\}$ be a sequence which is dense on both X and Y. By definition, $\{s_n : n \in \mathbb{N}\} \subseteq X \cap Y$. Reasoning by contradiction, suppose that there is $x_0 \in X \setminus Y$. Let consider the function $h : Y \to \mathbb{R}$ defined by $h(y) = |y - x_0|$ for all $y \in Y$. This function is continuous and it is defined on $Y \in L(A)$. By Theorem 2.2, $h(Y) \in L(\mathbb{R})$. But as $h(y) > 0$ for each $y \in Y$ and $h(Y)$ has absolute minimum, there is $\varepsilon_0 > 0$ such that $|y - x_0| > \varepsilon_0$ for all $y \in Y$. Since $\{s_n\}$ is dense on X, there is $n_0 \in \mathbb{N}$ such that $\left|x_0 - s_{n_0}\right| < \varepsilon_0$. However, as $s_{n_0} \in \{s_n : n \in \mathbb{N}\} \subseteq Y$, the term $s_{n_0} \in Y$ satisfies $\left|x_0 - s_{n_0}\right| < \varepsilon_0$, which contradicts that $|y - x_0| > \varepsilon_0$ for all $y \in Y$. □

Remark 3.4. From the contrapositive of Proposition 3.3, we have that if $X \neq Y$, then there is no common dense sequence. A direct consequence of this is that for each $X, Y \in L(A)$ the set of dense sequences on X and Y, respectively, are either equal or disjoint.

Although the previous statement seems to be irrelevant, it is the key to proving that, under the continuity of functions, a set of interest is not empty.

Lemma 3.5. *Let $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$ be a family of continuous and surjective functions and let $\{s_n\}$ be a dense sequence on A. Then, given two different $X, Y \in L(A)$, the set $\{\, n \in \mathbb{N} : h_X(s_n) \neq h_Y(s_n) \,\}$ is not empty.*

Proof. By contradiction, suppose that the above mentioned set is empty, i.e. $h_X(s_n) = h_Y(s_n)$ for all $n \in \mathbb{N}$. So the sequences $\{h_X(s_n)\}$ and $\{h_Y(s_n)\}$ are equal. By Proposition 2.8, since $h_X : A \to X$ is continuous and surjective, the sequence $\{h_X(s_n)\}$ is dense on $h_X(A) = X$, and similarly, $\{h_Y(s_n)\}$ is dense on $h_Y(A) = Y$. So there is a common sequence which is dense on X and on Y. Proposition 3.3 guarantees that $X = Y$, which contradicts that X and Y are different. □

Since the set $\{\, n \in \mathbb{N} : h_X(s_n) \neq h_Y(s_n) \,\}$ is not empty, it has an absolute minimum. For convenience, given two different $X, Y \in L(A)$, we denote it by:

$$j_0(X, Y) = \min\left(\{\, n \in \mathbb{N} : h_X(s_n) \neq h_Y(s_n) \,\}\right).$$

Its existence is guaranteed by Lemma 3.5 if $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$ is a family of continuous and surjective functions. Note that the value of $j_0(X, Y)$ depends directly on both the family $\mathcal{F}$ and the dense sequence $\{s_n\}$, but we avoid complicating the notation. Also by definition,

$$\begin{cases} \bullet\ h_X(s_n) = h_Y(s_n) \text{ for all } n \in \{1, 2, \dots, j_0(X, Y) - 1\}, \quad \text{and} \\ \bullet\ h_X(s_{j_0(X,Y)}) \neq h_Y(s_{j_0(X,Y)}). \end{cases} \tag{2}$$

3.3. The binary relation $\preceq_S^{\mathcal{F}}$ on $L(A)$

In the following definition, we introduce the binary relation $\preceq_S^{\mathcal{F}}$ on $L(A)$.

Definition 3.6. Let $S = \{s_n\}$ be a dense sequence on A and let $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$ be a family of continuous and surjective functions. Given $X, Y \in L(A)$, we will write $X \preceq_S^{\mathcal{F}} Y$ if either $X = Y$ or $h_X(s_{j_0(X,Y)}) < h_Y(s_{j_0(X,Y)})$.

Notice that, when $X \preceq_S^{\mathcal{F}} Y$, using (2), in general,

$$h_X(s_n) \leq h_Y(s_n) \quad \text{for all } n \in \{1, 2, \dots, j_0(X, Y)\}. \tag{3}$$

Firstly we check that the binary relation $\preceq_S^{\mathcal{F}}$ is a total order.

Theorem 3.7. *If $S = \{s_n\}$ is a dense sequence on A and $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$ is a family of continuous and surjective functions, then the binary relation $\preceq_S^{\mathcal{F}}$ is a total order on $L(A)$.*

Proof. Reflexivity is included in the definition. The antisymmetry and the totality follows from the fact that if $X \neq Y$, then $h_X(s_{j_0(X,Y)}) \neq h_Y(s_{j_0(X,Y)})$, and it necessarily holds that either $h_X(s_{j_0(X,Y)}) < h_Y(s_{j_0(X,Y)})$ or $h_X(s_{j_0(X,Y)}) > h_Y(s_{j_0(X,Y)})$. Let us prove the transitivity. Let $X, Y, Z \in L(A)$ be such that $X \preceq_S^{\mathcal{F}} Y$ and $Y \preceq_S^{\mathcal{F}} Z$. If $X = Y$ or $Y = Z$, then $X \preceq_S^{\mathcal{F}} Z$. Suppose that $X \neq Y$ and $Y \neq Z$. Then:

$$\begin{cases} \bullet\ h_X(s_n) = h_Y(s_n) \text{ for all } n \in \{1,2,\dots,j_0(X,Y)-1\}, \quad \text{and} \quad h_X(s_{j_0(X,Y)}) < h_Y(s_{j_0(X,Y)}); \\ \bullet\ h_Y(s_n) = h_Z(s_n) \text{ for all } n \in \{1,2,\dots,j_0(Y,Z)-1\}, \quad \text{and} \quad h_Y(s_{j_0(Y,Z)}) < h_Z(s_{j_0(Y,Z)}). \end{cases}$$

In particular, using (3),

$$\begin{cases} \bullet\ h_X(s_n) \leq h_Y(s_n) \quad \text{for all } n \in \{1,2,\dots,j_0(X,Y)\}; \\ \bullet\ h_Y(s_n) \leq h_Z(s_n) \quad \text{for all } n \in \{1,2,\dots,j_0(Y,Z)\}. \end{cases} \tag{4}$$

Let $j_0 = \min\{j_0(X,Y), j_0(Y,Z)\}$. If $j_0 = j_0(X,Y) \leq j_0(Y,Z)$, then, using (4),

$$h_X(s_n) = h_Y(s_n) = h_Z(s_n) \quad \text{for all } n \in \{1,2,\dots,j_0-1\}, \quad \text{and}$$
$$h_X(s_{j_0}) = h_X(s_{j_0(X,Y)}) < h_Y(s_{j_0(X,Y)}) \leq h_Z(s_{j_0(X,Y)}) = h_Z(s_{j_0}).$$

Hence, $h_X(s_{j_0}) < h_Z(s_{j_0})$, so $X \preceq_S^{\mathcal{F}} Z$. On the contrary case, if $j_0 = j_0(Y,Z) < j_0(X,Y)$, then

$$h_X(s_n) = h_Y(s_n) = h_Z(s_n) \quad \text{for all } n \in \{1,2,\dots,j_0-1\}, \quad \text{and}$$
$$h_X(s_{j_0}) = h_X(s_{j_0(Y,Z)}) = h_Y(s_{j_0(Y,Z)}) < h_Z(s_{j_0(Y,Z)}) = h_Z(s_{j_0}).$$

So we also deduce that $X \preceq_S^{\mathcal{F}} Z$. □

An additional property on the functions of the family completes the admissibility.

Theorem 3.8. *Let $S = \{s_n\}$ be a dense sequence on A and let $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$ be a family of continuous and surjective functions. Suppose that*

$$h_X \leq h_Y \quad \textit{for all } X, Y \in L(A) \textit{ such that } X \leq_L Y. \tag{5}$$

Then, the binary relation $\preceq_S^{\mathcal{F}}$ is an admissible order on $L(A)$.

Proof. Theorem 3.7 guarantees that $\preceq_S^{\mathcal{F}}$ is a total order on $L(A)$. To prove that $\preceq_S^{\mathcal{F}}$ refines $\leq_L$, let $X, Y \in L(A)$ be such that $X \leq_L Y$. By hypothesis, $h_X \leq h_Y$. Since, when $X \neq Y$, $h_X(s_{j_0(X,Y)}) \neq h_Y(s_{j_0(X,Y)})$ by definition of $j_0(X,Y)$, then it necessarily holds $h_X(s_{j_0(X,Y)}) < h_Y(s_{j_0(X,Y)})$. Hence $X \preceq_S^{\mathcal{F}} Y$, so $\preceq_S^{\mathcal{F}}$ is an admissible order on $L(A)$. □

The next corollary shows that each automorphism of A generates a new family of functions, which can give rise to new families of admissible orders on $L(A)$.

Corollary 3.9. *Let $S = \{s_n\}$ be a dense sequence on A and let $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$ be a family of continuous and surjective functions fulfilling (5). Let $\varrho : A \to A$ be an increasing automorphism of A. Then $\mathcal{F}^\varrho = \{h_X \circ \varrho : A \to X\}_{X \in L(A)}$ is a new family of continuous and surjective functions satisfying (5) and the binary relation $\preceq_S^{\mathcal{F}^\varrho}$ is an admissible order on $L(A)$.*

4. A notable family of admissible orders on $L(A)$

Theorem 3.8 gives a large family of admissible orders on $L(A)$ if $S = \{s_n\}$ is a dense sequence on A and $\mathcal{F} = \{h_X : A \to X\}_{X \in L(A)}$ is a family of continuous and surjective functions satisfying the property (5). In this section we show a concrete family, verifying all hypotheses and avoiding the use of the extremes of the intervals.

The first step in this construction is the following result.

Theorem 4.1. *Given $X \in L(A)$, with $A = [a,b]$, let $M_X = \{(m,k) \in [0,1] \times X : k + m(b-a) \in X\}$. Then, the following properties hold.*

1. *The set M_X is not empty, and $M_X \subseteq M_A$.*
2. *The set $M_X^1 = \{m \in [0,1]$: there is $k \in X$ with $(m,k) \in M_X\}$ is an interval of $L([0,1])$. In particular, $\sup M_X^1 \in M_X^1$.*
3. *For each $m \in M_X^1$, the set $K_{X,m} = \{k \in X : (m,k) \in M_X\}$ is an interval of $L(X)$ and verifies the following property:*

$$\textit{if } k \in K_{X,m} \textit{ and } k' \in X \textit{ are such that } k' \leq k, \textit{ then } k' \in K_{X,m}. \tag{6}$$

In particular, the infimum of $K_{X,m}$ is the infimum of X.

4. *If $m_X = \sup M_X^1$, then the interval K_{X,m_X} is the singleton containing the infimum of X.*

Proof. Item (1) The set M_X is not empty because if $k_0 \in X$, then $(0, k_0) \in M_X$. Furthermore, since $X \subseteq A$, then it is straightforward that $M_X \subseteq M_A$.

Item (2) The first item shows that $0 \in M_X^1$, so M_X^1 is not empty (and 0 is, in fact, its infimum). Let us prove that M_X^1 is an interval. Let $m_1, m_2 \in M_X^1$ be two values such that $m_1 < m_2$, and let $m_0 \in [0,1]$ be a number such that $m_1 < m_0$ and $m_0 < m_2$. We claim that $m_0 \in M_X^1$. To prove it, let $k_1, k_2 \in X$ be such that $k_1 + m_1(b-a) \in X$ and $k_2 + m_2(b-a) \in X$. Let $k_0 = \min\{k_1, k_2\}$. Clearly $k_0 \in X$. Moreover, $k_0 \leq k_0 + m_0(b-a) \leq k_2 + m_2(b-a)$. As $k_0 \in X$, $k_2 + m_2(b-a) \in X$ and X is an interval, then $k_0 + m_0(b-a) \in X$. Hence $(m_0, k_0) \in M_X$, which means that $m_0 \in M_X^1$ and proves that M_X^1 is an interval.

Since $M_X^1 \subseteq [0,1]$, then it is bounded from above. We check that is supremum $m_X = \sup M_X^1 \in [0,1]$ also belongs to M_X^1. If $M_X^1 = \{0\}$, this property is obvious. In other case, there is an strictly increasing sequence $\{m_n\} \subseteq M_X^1$ converging to m_X. Hence $m_n < m_{n+1} \leq m_X$ for all $n \in \mathbb{N}$ and $\{m_n\} \to m_X$. By definition, there is a sequence $\{k_n\}$ such that $k_n \in X$ and $k_n + m_n(b-a) \in X$ for all $n \in \mathbb{N}$. Since $\{k_n\} \subset X$ and X is closed and bounded, item 2 of Proposition 2.1 guarantees the existence of a partial subsequence $\{k_{\sigma(n)}\}$ converging to a point $k' \in X$. Therefore, $\{k_{\sigma(n)} + m_{\sigma(n)}(b-a)\} \subset X$ is a sequence of points of X convergent to

$$\lim_{n\to+\infty} \left(k_{\sigma(n)} + m_{\sigma(n)}(b-a)\right) = k' + m_X(b-a).$$

As X is closed, then $k' + m_X(b-a) \in X$, so $(m_X, k') \in M_X$, which concludes that $m_X \in M_X^1$.

Item (3) Let $m \in M_X^1$ be arbitrary. By definition, there is $k \in X$ such that $(m,k) \in M_X$. Hence $k \in K_{X,m}$, which proves that $K_{X,m}$ is not empty. Notice that $K_{X,m} \subseteq X$ because if $k \in M_{X,m}$, then $(m,k) \in M_X$, so $k \in X$. Hence $K_{X,m}$ is bounded. To prove that $K_{X,m}$ is an interval, let $k_1, k_2 \in K_{X,m}$ be such that $k_1 < k_2$, and let $k_0 \in X$ such that $k_1 < k_0$ and $k_0 < k_2$. Since $k_1, k_2 \in K_{X,m}$, then $(m, k_1) \in M_X$ and $(m, k_2) \in M_X$, so $k_1, k_2 \in X$, $k_1 + m(b-a) \in X$ and $k_2 + m(b-a) \in X$. Having in mind that X is an interval and $k_1, k_2 \in X$, the intermediate value $k_0 \in X$. In fact, $k_0 \leq k_0 + m(b-a) \leq k_2 + m(b-a)$. Since $k_0 \in X$, $k_2 + m(b-a) \in X$ and X is an interval, then $k_0 + m(b-a) \in X$. Hence $k_0 \in K_{X,m}$, which concludes that $K_{X,m}$ is an interval. Property (6) is apparent because if $k \in K_{X,m}$ and $k' \in X$ are such that $k' \leq k$, then $k' \leq k' + m(b-a) \leq k + m(b-a) \in X$, so $k' + m(b-a)$ and consequently $k' \in K_{X,m}$. This property guarantees that $K_{X,m}$ has an infimum, and that infimum is the infimum of X. To conclude that $K_{X,m} \in L(X)$, it only remains to show that the supremum $k_X = \sup K_{X,m}$ also belongs to $K_{X,m}$. If $K_{X,m}$ is a singleton, this property is obvious. Suppose that $K_{X,m}$ is not a singleton. Hence there is a strictly increasing sequence $\{k_n\} \subset K_{X,m}$ converging to k_X. As $\{k_n\} \subset K_{X,m} \subseteq X$ and X is closed, its limit $k_X \in X$. Furthermore, as $k_n + m(b-a) \in X$ for all $n \in \mathbb{N}$, then its limit $k_X + m(b-a) \in X$. Therefore $k_X \in K_{X,m}$, and such set is closed.

Item (4) Suppose, by contradiction, that there are $k_1, k_2 \in K_{X,m_X}$ such that $k_1 < k_2$. Hence $k_2 + m_X(b-a) \in X$. Let define $m' = m_X + (k_2 - k_1)/(b-a)$. Since $m' > m_X$ and m_X is the supremum of M_X^1, then $m' \notin M_X^1$. However, $k_1 \in X$ and

$$k_1 + m'(b-a) = k_1 + \left(m_X + \frac{k_2 - k_1}{b-a}\right)(b-a) = k_1 + m_X(b-a) + k_2 - k_1 = k_2 + m_X(b-a) \in X.$$

Therefore, $m' \in M_X^1$, which contradicts the fact that $m' \notin M_X^1$. As item 3 guarantees that the infimum of X belongs to K_{X,m_X}, then such singleton is $\{\inf X\}$. $\square$

Remark 4.2. If we know the endpoints of X, for example $X = [c,d] \subseteq [a,b]$, then M_X^1 is the interval $[0, \frac{d-c}{b-a}]$ and therefore $m_X = \frac{d-c}{b-a}$. However, since we do not treat the intervals by their endpoints, but only by their interior points, this way of obtaining m_X does not follow this principle. On the other hand, considering the dense sequence $S' = \{s'_n\}$ of X, generated from the sequence S of A, the function ϱ in Example 2.6 and $M'^1_X = \cup_{j\in\mathbb{N}}\{\varrho(i) : s'_j + \varrho(i)(b-a) \in X\}$, then $m_X = \sup M'^1_X$ which would be an infinitary computation of m_X in the sense of [25]. The infinitary computation of k_X is analogous.

The previous result is the basis for considering the following family of functions. Given $(m,k) \in M_X$, let consider the function $\iota_{m,k} : A \to \mathbb{R}$ defined as

$$\iota_{m,k}(t) = k + m(t-a) \quad \text{for all } t \in A. \tag{7}$$

Clearly, the function $\iota_{m,k}$ is an increasing affine function. As a consequence, Corollary 2.5 guarantees that its image $\iota_{m,k}(A) \in L(\mathbb{R})$ is the closed and bounded interval of $\mathbb{R}$ that includes all the real numbers between $\iota_{m,k}(a) = k$ and $\iota_{m,k}(b) = k + m(b-a)$. Since $\iota_{m,k}(a) = k \in X$, $\iota_{m,k}(b) = k + m(b-a) \in X$ and X is an interval, then

$$\iota_{m,k}(A) = [a, k + m(b-a)] \subseteq X.$$

Thus, for each $X \in L(A)$, we have a bivariate family of continuous surjective functions given by

$$\left\{ \iota_{m,k} : A \to [a, k + m(b-a)] : (m,k) \in M_X \right\}.$$

From now, we will use $\iota_{m,k} : A \to \iota_{m,k}(A)$ instead of $\iota_{m,k} : A \to [a, k + m(b-a)]$. When $m = 0$, $\iota_{0,k}(A)$ is the singleton $\{k\}$. Then, any singleton $\{k\} \subset X$ can be obtained as $\iota_{0,k}(A)$. Next, suppose that $m > 0$. In this case, $\iota_{m,k}$ is strictly increasing (its first derivative is

$m > 0$). Then $\iota_{m,k} : A \to \iota_{m,k}(A)$ is continuous and bijective. In fact, as $\iota_{m,k}$ is a polynomial of degree exactly one, then $\iota_{m,k}^{-1} : \iota_{m,k}(A) \to A$ is also a polynomial of degree one, which is a new affine function. It is therefore continuous, so we have deduced the following statement.

Proposition 4.3. *Given $X \in L(A)$ and $(m,k) \in M_X$, the image $\iota_{m,k}(A) \in L(X)$ and the function $\iota_{m,k} : A \to \iota_{m,k}(A)$ is continuous. Furthermore, if $m = 0$, then $\iota_{m,k}(A)$ is a singleton and $\iota_{m,k} : A \to \iota_{m,k}(A)$ is constant, and if $m > 0$, the function $\iota_{m,k} : A \to \iota_{m,k}(A)$ is an strictly increasing homeomorphism from A onto $\iota_{m,k}(A)$.*

In a particular case, the interval $\iota_{m,k}(A)$ coincides with X.

Lemma 4.4. *Using the notation of Theorem 4.1, given $X \in L(A)$, let $m_X = \sup M_X^1$ and let k_X be the unique element of K_{X,m_X}. Then $\iota_{m_X,k_X}(A) = X$.*

Proof. Proposition 4.3 guarantees that $\iota_{m_X,k_X}(A) \in L(X)$, so $\iota_{m_X,k_X}(A) \subseteq X$. We prove the contrary inclusion by contradiction, assuming that there is $x_0 \in X \setminus \iota_{m_X,k_X}(A)$. Theorem 4.1 states that $m_X = \sup M_X^1 \in M_X^1$ and $K_{X,m_X} = \{k_X\}$. Hence $\iota_{m_X,k_X}(a) = k_X \in X$ and $\iota_{m_X,k_X}(b) = k_X + m_X(b-a) \in X$. If $k_X \le x_0$ and $x_0 \le k_X + m_X(b-a)$, as $\iota_{m_X,k_X}(A)$ is an interval containing k_X and $k_X + m_X(b-a)$, then $x_0 \in \iota_{m_X,k_X}(A)$, which contradicts the fact that $x_0 \notin \iota_{m_X,k_X}(A)$. Hence either $x_0 < k_X$ or $x_0 > k_X + m_X(b-a)$.

- If $x_0 < k_X$, property (6) guarantees that $x_0 \in K_{X,m_X} = \{k_X\}$, which is false.
- Suppose that $x_0 > k_X + m_X(b-a)$. Let define $m' = (x_0 - k_X)/(b-a) > 0$. Notice that $k_X \in X$ and

$$k_X + m'(b-a) = k_X + \frac{x_0 - k_X}{b-a}(b-a) = k_X + x_0 - k_X = x_0 \in X.$$

  Hence $(m, k_X) \in M_X$. Therefore $m \in M_X^1$, so $m \le \sup M_X^1 = m_X$. However,

$$x_0 > k_X + m_X(b-a) \quad \Rightarrow \quad m = \frac{x_0 - k_X}{b-a} > m_X.$$

This contradiction concludes that $\iota_{m_X,k_X}(A) = X$. □

In the next result, we show that any interval of $L(A)$ is of the type $\iota_{m,k}(A)$.

Corollary 4.5. $L(A) = \left\{\iota_{m,k}(A) : (m,k) \in M_A\right\}$.

Proof. If $(m,k) \in M_A$, Proposition 4.3 guarantees that $\iota_{m,k}(A) \in L(A)$. Conversely, given an arbitrary $X \in L(A)$, there is $(m_X, k_X) \in M_X$ such that $\iota_{m_X,k_X}(A) = X$. Since $M_X \subseteq M_A$, the equality is proven. □

Given $X \in L(A)$, the function $\iota_{m_X,k_X} : A \to \iota_{m_X,k_X}(A) = X$ will be of particular importance, so we will denote it in a special way. From now on, given $X \in L(A)$, we denote by $\iota_X : A \to X$ the function ι_{m_X,k_X}, where $m_X = \sup M_X^1$ and k_X is the unique element of K_{X,m_X}.

Proposition 4.6. *The function $\iota_A : A \to A$ is the identity mapping on A.*

Proof. Since $A = [a,b]$ and $a + 1 \cdot (b-a) = b$, then $(1,a) \in M_A$. Hence $m_A = \sup M_A^1 = 1$ and $K_{A,m_A} = \{a\}$. Therefore, for all $t \in A$, $\iota_A(t) = a + (t-a) = t$. □

Theorem 4.7. *Given $X \in L(A)$, the function $\iota_X : A \to X$ is affine and continuous. If X is a singleton, then ι_X is constant, and if X is not a singleton, $\iota_X : A \to X$ is an strictly increasing homeomorphism from A onto $\iota_X(A) = X$. In any case, $\iota_X : A \to X$ is surjective.*

Furthermore, given $X, Y \in L(A)$ it holds that $X \le_L Y$ if and only if $\iota_X \le \iota_Y$.

Proof. The first part follows from Proposition 4.3 and Lemma 4.4. The second part follows from Proposition 2.4, since the values $\iota_X(a)$ and $\iota_X(b)$ are the extremes of $\iota_X(A) = X$. □

From Theorems 3.8 and 4.7 we get one of the aims of the manuscript.

Corollary 4.8. *If $S = \{s_n\}$ is a dense sequence on A and $\mathcal{F}'$ is the family $\{\iota_X : A \to X\}_{X \in L(A)}$, then the binary relation $\preceq_S^{\mathcal{F}'}$ is an admissible order on $L(A)$.*

Next, we show that suitably distinct dense sequences give rise to distinct admissible orders.

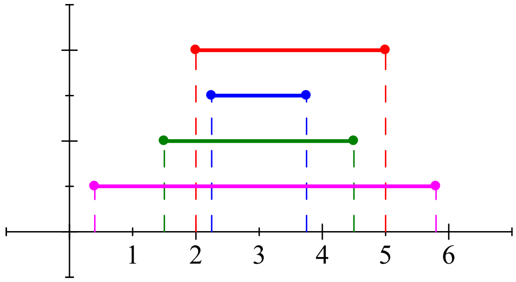

Fig. 2. Graphic representation of intervals $X_\alpha = [3-\alpha, 3+\alpha]$ and $Y = [2,5]$ of Example 4.11. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Theorem 4.9. *Let $S = \{s_n\}$ and $S' = \{s'_n\}$ be dense sequences on A such that $s_1 \neq s'_1$, and let X be the singleton $\{(s_1 + s'_1)/2\}$. Then $A \preceq_S^{F'} X$, $X \preceq_{S'}^{F'} A$ and $X \neq A$. In particular, the admissible orders $\preceq_S^{F'}$ and $\preceq_{S'}^{F'}$ are distinct.*

Proof. Let denote $x_0 = (s_1 + s'_1)/2$ and $X = \{x_0\}$. Clearly $s_1 < x_0 < s'_1$ and $X \neq A$. Since $S \subset A$ and $S' \subset A$, then $s_1 \in A$ and $s'_1 \in A$. As A is an interval, then $x_0 \in A$, so $X \in L(A)$. As X is a singleton, $\iota_X : A \to X$ is constant, so

$$\iota_X(t) = x_0 \quad \text{for all } t \in A.$$

Proposition 4.6 says that $\iota_A : A \to A$ is the identity mapping on A. Hence

$$\iota_A(s_1) = s_1 < x_0 = \iota_X(s_1).$$

By definition, $A \preceq_S^{F'} X$. However, as

$$\iota_X(s'_1) = x_0 < s'_1 = \iota_A(s'_1),$$

then $X \preceq_{S'}^{F'} A$. Therefore, $\preceq_S^{F'}$ and $\preceq_{S'}^{F'}$ are distinct admissible orders because they are antisymmetric (if $\preceq_S^{F'}$ and $\preceq_{S'}^{F'}$ would be equal, from $A \preceq_S^{F'} X$ and $X \preceq_S^{F'} A$ we could deduce that $X = A$, which is false). $\square$

The previous result finally proves the existence of a non-countable family of admissible orders of the type $\preceq_S^{F'}$ taking into account the following property.

Proposition 4.10. *Given a dense sequence $S = \{s_n\}$ on A and a point $t \in A$, let $S_t = \{s_n^t\}$ be the sequence defined, for all $n \in \mathbb{N}$, by:*

$$s_n^t = \begin{cases} t, & \text{if } n = 1, \\ s_{n-1}, & \text{if } n > 1. \end{cases}$$

Then S_t is also dense in A. Furthermore, if $t_1, t_2 \in A$ are distinct, then the first terms of S_{t_1} and S_{t_2} are distinct.

The previous result shows a family $\{S_t\}_{t \in A}$ of dense sequences on A whose first terms are distinct for different indices. Then $\{\preceq_{S_t}^{F'}\}_{t \in A}$ is a non-countable family of admissible orders on $L(A)$.

Let us show next how the admissible order $\preceq_S^{F}$ works when it is applied to some intervals. So far, we have been dealing with intervals without having to name them by their endpoints. However, in the following examples, for convenience, we will denote them by their endpoints.

Example 4.11. Let us consider the universe interval $A = [0,6]$. Given $\alpha \in [0,3]$, let X_α be the interval $[3-\alpha, 3+\alpha]$, and let call $Y = [2,5]$. Fig. 2 represents the interval Y in red color, and the interval X_α in blue color for $\alpha \in [0,1]$, in green color for $\alpha \in [1,2]$, and in magenta color for $\alpha \in [2,3]$.

If $\alpha \in [1,2]$, then $3-\alpha \leq 2$ and $3+\alpha \leq 5$. Then $X_\alpha \leq_L Y$. In this case, the admissibility of the order (recall Theorem 3.8) guarantees that $X_\alpha \preceq_S^{F} Y$ whatever the sequence S and the family F (provided that $S = \{s_n\}$ is a dense sequence on $[0,6]$ and $F = \{h_X : [0,6] \to X\}_{X \in L([0,6])}$ is a family of continuous and surjective functions satisfying (5)). When $\alpha \in [0,1) \cup (2,3]$, the intervals X_α and Y are not $\leq_L$-comparable, so this order cannot decide what interval is the lesser one. The great advantage of the admissible order $\preceq_S^{F}$ is that it will always be able to decide what interval is the lesser one (because they are distinct). The keys of the criterion will be the dense sequence S and the family F.

Suppose that F is the family $F' = \{\iota_X : [0,6] \to X\}_{X \in L([0,6])}$. Associated to $Y = [2,5]$, it can be checked that $m_Y = 1/2$ and $k_Y = 2$. Therefore $\iota_Y : [0,6] \to Y$ is the function given, for each $t \in [0,6]$, by:

$$\iota_Y(t) = \iota_{m_Y, k_Y}(t) = k_Y + m_Y(t-a) = 2 + \frac{1}{2}(t-0) = \frac{t+4}{2}.$$

Given $\alpha \in [0,3]$, we similarly can compute that $\iota_{X_\alpha}(t) = (9 - 3\alpha + \alpha t)/3$ for all $t \in [0,6]$. When $\alpha \neq 1.5$, the unique point where $\iota_Y(t) = \iota_{X_\alpha}(t)$ is $t = 6(1-\alpha)/(3-2\alpha)$.

Suppose that $\alpha \in [0,1) \cup (2,3]$, and let $\mathcal{S} = \{s_1, s_2, s_3, \dots\}$ be a dense sequence in $[0,6]$ such that $s_1 = 6(1-\alpha)/(3-2\alpha)$ and $s_2 \neq s_1$. Then $s_1 \in (0,2] \cup [4,6) \subset [0,6] = A$. In this case, note that the first point of the sequence is not enough to $\preceq_{\mathcal{S}}^{\mathcal{F}'}$-compare the intervals X_α and Y because $\iota_{X_\alpha}(s_1) = \iota_Y(s_1)$. Precisely, s_1 is the unique point where this equality holds. But, as $s_2 \neq s_1$, then necessarily either $\iota_{X_\alpha}(s_2) < \iota_Y(s_2)$ or $\iota_{X_\alpha}(s_2) > \iota_Y(s_2)$. In the first case, $X_\alpha \prec_{\mathcal{S}}^{\mathcal{F}'} Y$, and, in the second case, $Y \prec_{\mathcal{S}}^{\mathcal{F}'} X_\alpha$. To decide what is the case, we observe that, given $t \in [0,6]$:

$$\iota_Y(t) < \iota_{X_\alpha}(t) \quad \Leftrightarrow \quad (3-2\alpha)t < 6(1-\alpha).$$

Therefore, the criterion to compare X_α and Y by $\preceq_{\mathcal{S}}^{\mathcal{F}'}$ is the following one:

$$\begin{cases} \bullet \ \text{if } \alpha \in [0,1), & \begin{cases} Y \prec_{\mathcal{S}}^{\mathcal{F}'} X_\alpha & \text{if} \quad s_2 < s_1, \\ X_\alpha \prec_{\mathcal{S}}^{\mathcal{F}'} Y & \text{if} \quad s_2 > s_1; \end{cases} \\ \bullet \ \text{if } \alpha \in (2,3], & \begin{cases} Y \prec_{\mathcal{S}}^{\mathcal{F}'} X_\alpha & \text{if} \quad s_2 > s_1, \\ X_\alpha \prec_{\mathcal{S}}^{\mathcal{F}'} Y & \text{if} \quad s_2 < s_1. \end{cases} \end{cases}$$

In any case, the intervals X_α and Y are $\preceq_{\mathcal{S}}^{\mathcal{F}'}$-comparable.

Example 4.12. In general, the first terms of the sequence $\mathcal{S} = \{s_1, s_2, s_3, \dots\}$ are sufficient to determine the ordering by $\preceq_{\mathcal{S}}^{\mathcal{F}}$ of two intervals X and Y. However, when the functions of the family $\mathcal{F}$ take often the same value, then it could be necessary to appeal to advanced terms of the sequence $\mathcal{S}$. This is the case in this example. Suppose that $A = [0,6]$ and let $\varepsilon_0 \in (0,1)$ be a very small positive real number (for instance, $\varepsilon_0 = 10^{-10}$). Let $\mathcal{F} = \{h_X : [0,6] \to X\}_{X \in L([0,6])}$ be a family of functions such that, for each $X = [c,d]$ verifying $0 \leq c \leq 3 \leq d \leq 6$, the function h_X is defined, for each $t \in [0,6]$, by:

$$h_X(t) = \begin{cases} \dfrac{c\varepsilon_0 + (3-c)t}{\varepsilon_0}, & \text{if} \ \ 0 \leq t < \varepsilon_0, \\ 3, & \text{if} \ \ \varepsilon_0 \leq t \leq 6 - \varepsilon_0, \\ \dfrac{d\varepsilon_0 + (3-d)(6-t)}{\varepsilon_0}, & \text{if} \ \ 6 - \varepsilon_0 < t \leq 6. \end{cases}$$

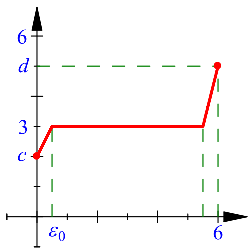

Fig. 3. Definition and graphic representation of some functions h_X of Example 4.12.

The function h_X takes the value 3 in the interval $[\varepsilon_0, 6 - \varepsilon_0]$ (see Fig. 3), which is near to be the whole interval $[0,6]$. Then, for intervals $X = [1,5]$ and $Y = [2,4]$, it is reasonable to find that $h_X(s_n) = h_Y(s_n)$ for a large number of terms $\{s_1, s_2, \dots, s_{n_0}\}$. In this case, it would be necessary to determine the first term s_{m_0} of the sequence $\mathcal{S}$ such that $s_{m_0} \in [0, \varepsilon_0) \cup (6 - \varepsilon_0, 6]$ before comparing the intervals X and Y through the order $\preceq_{\mathcal{S}}^{\mathcal{F}}$.

5. Interval valued orders in brain–computer interface classification

This section offers a brain–computer interface (BCI) example that contrasts the conventional *admissible* order on intervals with the one proposed in this work inside an interval–valued Sugeno (IV–Sugeno) fusion scheme. Although limited in scope, the experiment is intended as an open work that can be extended with additional datasets, orders or fusion functions.

The dataset used in the experiment consists of ten stroke volunteers producing forty unseen trials of six–second electroencephalogram (EEG) windows recorded while left– or right–hand grasp movements are made. Every signal was band–pass filtered into four overlapping μ/β ranges (6–10, 8–15, 14–28 and 24–35 Hz). Per band, the first two principal components were retained and summarised by five descriptive statistics (mean, minimum, maximum, 0.15– and 0.85–quantiles), yielding a ten-dimensional feature vector. Three probabilistic models, k–nearest neighbours (k-NN), support–vector machines (SVM) and Gaussian processes (GP) are applied to each vector, producing nine posterior probabilities for the *left* class. The interval $X_b^{\text{left}} = [\min p, \max p]$ was formed per band; the complementary interval X_b^{right} was obtained analogously. Repeating this procedure across the four bands delivered four left intervals and four right intervals for every trial.

5.1. Overview of the IV-Sugeno fusion

The four intervals of a given label are fused by an IV–Sugeno integral $S_{m,H,G}(x_1, x_2, x_3, x_4) = G(H(x_{(1)}, m(A_{(1)}), \ldots, H(x_{(4)}, m(A_{(4)}))$ that employs the uniform fuzzy measure $m(U) = |U|/N$, the arithmetic mean G as outer aggregator and the bivariate mapping $H(a, b) = a^2 b + a(1 - b)$, and where $(x_{(1)}, \ldots, x_{(4)})$ is an increasing reordering of (x_1, x_2, x_3, x_4) and $A_{(i)} = \{(i), \ldots, 4\}$ for every $i \in \{1, \ldots, 4\}$. The fusion requires (i) sorting the four band intervals, (ii) applying a local function f to each ordered pair {interval, measure}, and (iii) aggregating the four outputs with G. Replacing the sorting step with a different total order leaves the rest of the pipeline unchanged.

5.2. Pseudocode of the complete decision rule

Algorithm 1: IV–Sugeno decision rule with interior ordering (single module).

Input: Feature vectors $\mathbf{v}_b$ for the four bands ($b = 1..4$)
Output: Predicted label $c \in \{\text{left}, \text{right}\}$

1 **Step 1: Build band-wise probability intervals**;
2 Initialize two empty lists `L_left`, `L_right`;
3 **for** $b \leftarrow 1$ **to** 4 **do**
4 Obtain nine characteristics $p^{\ell}_{b,1..9}$ for *left* and *right* using trained classifiers (K-NN, SVM and GP);
5 $X^{\ell}_b \leftarrow [\min_i p^{\ell}_{b,i}, \max_i p^{\ell}_{b,i}]$;
6 $X^{r}_b \leftarrow [\min_i (1 - p^{\ell}_{b,i}), \max_i (1 - p^{\ell}_{b,i})]$;
7 Append X^{ℓ}_b to `L_left`; append X^{r}_b to `L_right`;
8 **end**
9 **Step 2: Fuse the four intervals of each label with IV–Sugeno**;
10 **foreach** *label* $\in$ *{L_left, L_right}* **do**
11 Let $k \leftarrow 4$ and rename the current list as $L[1..k]$;
12 **for** $i \leftarrow 1$ **to** $k - 1$ **do**
13 **for** $j \leftarrow 1$ **to** $k - i$ **do**
14 Compare $L[j]$ and $L[j+1]$ by scanning S;
15 Find first t_m with $h_{L[j]}(t_m) \neq h_{L[j+1]}(t_m)$;
16 **if** $h_{L[j]}(t_m) > h_{L[j+1]}(t_m)$ **then**
17 Swap $L[j], L[j+1]$;
18 **end**
19 **end**
20 **end**
21 **for** $i \leftarrow 1$ **to** k **do**
22 $\mu_i \leftarrow (k - i + 1)/k$;
23 $g_i \leftarrow H(L[i], \mu_i)$ where $H(a, b) = a^2 b + a(1 - b)$;
24 **end**
25 $Y^{\text{label}} \leftarrow G(g_1, \ldots, g_k)$; // G = arithmetic mean
26 **end**
27 **Step 3: Final comparison of the two fused intervals**;
28 Scan S until first t_m with $h_{Y^{\ell}}(t_m) \neq h_{Y^{r}}(t_m)$;
29 **if** $h_{Y^{\ell}}(t_m) < h_{Y^{r}}(t_m)$ **then**
30 $c \leftarrow \text{left}$;
31 **end**
32 **else**
33 $c \leftarrow \text{right}$;
34 **end**
35 **return** c;

5.3. Complexity analysis

Building the four band intervals (lines 3–8) is $O(k)$ with $k = 4$. The two bubble sorts (for left and right classes) in lines 13–19 dominate: each needs $k(k-1)/2$ comparisons and every comparison scans the n-point sequence S. The fusion step, therefore, costs $2\,O(k^2 n)$. Computing the four g_i values (lines 22–24) is $O(k)$ and the final interval comparison (lines 28–32) adds $O(n)$.

Putting everything together, a single trial costs

$$T_{\text{trial}}(k, n) = 2\,O(k^2 n) + O(n) + O(k) = O(k^2 n).$$

The order proposed in this work injects only a linear factor in the length of the dense sequence while preserving the overall $O(k^2 n)$ character of the pipeline; it can therefore be adopted without jeopardising throughput in larger-scale or online experiments.

Table 1
Correct classifications per participant (40 trials each).

Participant	admissible order	order without extremes
1	28	22
2	18	16
3	23	23
4	26	24
5	20	18
6	7	12
7	31	28
8	25	23
9	31	30
10	36	12

5.4. Results and discussion

Improvements appear for subject 6, whose probability interval contain noisy extremes: discarding the bounds adds five correct predictions. In the case of subject 3 the results remain equal with both types of order. Conversely, subjects 1, 4 and 7 lose up to six hits each, indicating that for them, the end points of the fused intervals carry meaningful evidence (Table 1).

The example confirms that changing the total order on intervals is a lightweight but influential modification. The order without the extremes adds a linear factor in the sequence length yet remains computationally modest. Although overall accuracy dropped in this dataset, performance gains for specific users underline the value of using the proposed order to individual uncertainty profiles. The current pipeline, therefore, serves as a flexible starting point for future studies that, enlarge the number of information sources, or deploy the method in online BCI sessions.

6. Final remarks and prospect works

Admissible orders, recently introduced by Bustince et al. in [10], are examples of orders that satisfy reasonable properties from a human point of view. At present, however, a sufficiently large family of such orders had not been introduced to work on concrete applications. In this paper, based on a dense sequence and on a family of continuous and surjective functions, we have introduced a large class of admissible orders that generalize and extend the Kulisch and Miranker partial order. Furthermore, we have shown that this family can be considered by avoiding the use of the extremes of the intervals.

This line of research can be developed by using this family of admissible orders on real, closed and bounded intervals in concrete applications. In particular, the application of the proposed admissible order in BCI is important, as it opens a line of research for other applications, such as in social network analysis, as done in [26]. The idea would be to use and compare this IV-Sugeno fusion function with other classes of IV-Sugeno fusion functions, such as [26,27], but based on the admissible order $\preceq_S^{\mathcal{F}}$. We can also investigate other families of admissible orders defined without using interval limits.

CRediT authorship contribution statement

Humberto Bustince: Writing – original draft, Conceptualization, Investigation. **Benjamín Bedregal:** Investigation, Writing – original draft. **Susana Montes:** Investigation, Writing – original draft. **Radko Mesiar:** Investigation, Writing – original draft. **Antonio Francisco Roldán López de Hierro:** Investigation, Writing – original draft. **Graçaliz Pereira Dimuro:** Investigation, Writing – original draft. **Javier Fernández:** Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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