

Testing LRD in the spectral domain for functional time series in manifolds

María D. Ruiz–Medina¹  and Rosa M. Crujeiras² 

¹*Department of Statistics and Operation Research, University of Granada, e-mail: mruiz@ugr.es*

²*Department of Statistics, Mathematical Analysis and Optimization, University of Santiago de Compostela, e-mail: rosa.crujeiras@usc.es*

Abstract: A statistical hypothesis test for long range dependence (LRD) is formulated in the spectral domain for functional time series in manifolds. The elements of the spectral density operator family are assumed to be invariant with respect to the group of isometries of the manifold. The proposed test statistic is based on the weighted periodogram operator. A Central Limit Theorem is derived to obtain the asymptotic Gaussian distribution of the proposed test statistic operator under the null hypothesis. The rate of convergence to zero, in the Hilbert–Schmidt operator norm, of the bias of the integrated empirical second and fourth order cumulant spectral density operators is obtained under the alternative hypothesis. The consistency of the test follows from the consistency of the integrated weighted periodogram operator under LRD. Practical implementation of our testing approach is based on the random projection methodology. A simulation study illustrates, in the context of spherical functional time series, the asymptotic normality of the test statistic under the null hypothesis, and its consistency under the alternative. The empirical size and power properties are also computed for different functional sample sizes, and under different scenarios.

MSC2020 subject classifications: Primary 60G10, 60G12, 60G18, 60G20, 60G22; secondary 60G60.

Keywords and phrases: Asymptotic normality, bias, compact manifolds, consistency, empirical cumulant spectral density operator, functional time series, integrated weighted periodogram operator, long-range dependence, spectral density operator.

Received February 2025.

1. Introduction

Spherical functional time series analysis helps in understanding the dynamics and spatiotemporal patterns of data that are embedded into the sphere, providing valuable insights for prediction, monitoring, and decision-making. Time series analysis of global temperature data distribution among other climate variables, usually arising in Climate Science and Meteorology, can be performed in a more efficient way by adopting a functional time series framework (see [38]). That is the case of ocean currents, and other marine functional time series to be analyzed in Oceanography studies (see, e.g., [41]; [43]). Other areas demanding this type of techniques are Geophysics, Astronomy and Astrophysics. In the last few decades, the cosmic microwave background radiation variation analysis over time has gained special attention (see [13]; [22]; [23]). In a more general manifold setting, functional time series analysis is often applied in Medical Imaging, Computer Vision and Graphics (see [42]; [45]; [44], among

others). This paper focuses on the spectral analysis of functional time series in manifolds, with special attention to LRD analysis.

The spectral analysis of functional time series has mainly been developed under Short Range Dependence (SRD). In this context, based on the weighted periodogram operator, a nonparametric framework is adopted in [28]. Particularly, the asymptotic normality of the functional discrete Fourier transform (fDFT), and the weighted periodogram operator of the curve data are proved under suitable summability conditions on the L^2 norm of the cumulant spectral density operators. The consistency of the weighted periodogram operator, in the Hilbert–Schmidt operator norm, is derived under SRD. In [29], a harmonic principal component analysis of functional time series in the temporal functional spectral domain is also obtained, based on a Karhunen–Loève–like decomposition, the so–called Cramér–Karhunen–Loève representation. In the context of functional regression, some applications are presented in [30], [34] and [39]. Hypothesis testing for detecting modelling differences in functional time series dynamics is addressed in [40] in the functional spectral domain.

In LRD analysis of functional time series several problems still remain open. One of the key approaches in the current literature is presented in [19], where the eigendecomposition of the long-run covariance operator is considered, under an asymptotic semiparametric functional principal component framework. The consistent estimation of the dimension and the orthonormal functions spanning the dominant subspace, where the projected curve process displays the largest dependence range is derived. Fractionally integrated functional autoregressive moving averages processes constitute an interesting example of this modelling framework.

A first attempt to characterize LRD in functional time series in the spectral domain can be found in [36], adopting the theoretical framework of operator–valued random fields, including fractional Brownian motion with operator–valued Hurst coefficient (see, e.g., [12]; [24]; [31] and [32]). The eigenvalues of the LRD operator are parameterized. These eigenvalues induce different levels of singularity at zero frequency, corresponding to different levels of temporal persistence of the process projected into different eigenspaces of the Laplace Beltrami operator. Under this LRD scenario, the integrated periodogram operator is proved to be asymptotically unbiased in the Hilbert–Schmidt operator norm. Minimum contrast estimation of the LRD operator is achieved in the spectral domain in a weak–consistent way under a Gaussian scenario. Interesting examples of this setting are analyzed in [26], where the spectral analysis of multifractionally integrated functional time series in manifolds is considered. In particular, multifractionally integrated spherical functional ARMA models (i.e., multifractionally integrated SPHARMA models) are analyzed through simulations. In this modelling framework, SRD and LRD can coexist at different spherical scales.

Up to our knowledge, no further developments have been achieved in the spectral analysis of LRD functional time series. Alternative contributions for stationary LRD functional sequences are based on the diagonalization of the heavy tail autocovariance kernel of the time–varying functional error term (see, e.g., [3]). However, under this modelling framework, functional spectral analysis can not be achieved in the time domain due to the assumed independence between the random components of the error term (see also [2]). Similar assertions hold for the two sample problem analyzed in [4]. In [20], a special family of LRD linear functional time series is analyzed with scalar coefficients displaying slow decay. Under stationarity, this LRD scenario constitutes a particular case of our framework when the elements of the spectral density operator family have degenerated pure point spectra corresponding to one

infinite-dimensional eigenspace. The Cramér–Karhunen–Loève representation above referred constitutes a powerful tool in the functional spectral analysis of weak-dependent functional time series (see [35]). In this paper, this representation is extended to our LRD stationary functional time series context, assuming the invariance of the cross-covariance kernels under the group of isometries of the manifold, given by a connected and compact two points homogeneous space.

In this paper we perform a weighted periodogram operator based analysis, requiring the asymptotic analysis of the bias of the integrated empirical fourth-order cumulant spectral density operators, to prove consistency of the integrated weighted periodogram operator under LRD. Its application to spectral statistical hypothesis testing of LRD in $L^2(\mathbb{M}_d, d\nu, \mathbb{R})$ -valued correlated sequences constitutes one of the main goals of this work. Here, $L^2(\mathbb{M}_d, d\nu, \mathbb{R})$ denotes the space of real-valued square integrable functions on a Riemannian manifold \mathbb{M}_d , embedded into \mathbb{R}^{d+1} , given by a connected and compact two-point homogeneous space. The topological dimension of \mathbb{M}_d is d , and $d\nu$ denotes the normalized Riemannian measure on \mathbb{M}_d . In what follows, we will consider $X = \{X_t, t \in \mathbb{Z}\}$ to be a functional sequence such that $\mathcal{P}(X_t \in L^2(\mathbb{M}_d, d\nu, \mathbb{R})) = 1$, for every $t \in \mathbb{Z}$, with \mathcal{P} denoting the probability measure defined on the basic probability space $(\Omega, \mathcal{Q}, \mathcal{P})$, i.e., for every $t \in \mathbb{Z}$,

$$X_t : (\Omega, \mathcal{Q}, \mathcal{P}) \longrightarrow L^2(\mathbb{M}_d, d\nu, \mathbb{R}) \quad (1)$$

is a measurable mapping.

The invariance of the elements of the spectral density operator family of X under the group of isometries of the manifold \mathbb{M}_d is assumed along the paper. A frequency-varying eigenvalue sequence then characterizes the pure point spectra of the elements of the spectral density operator family, with respect to the orthonormal basis of eigenfunctions of the Laplace–Beltrami operator. This invariance assumption is exploited in the derived Central Limit Theorem that characterizes the asymptotic distribution of the proposed test statistic operator under the null hypothesis, which states that X displays SRD. In our formulation of the alternative hypothesis on LRD, we adopt a semiparametric framework in terms of a functional parameter given by the LRD operator. In contrast with the approach presented in [36], here we do not assume a parameterization of the eigenvalues of the LRD operator. Under this scenario, the rate of convergence to zero, in the corresponding L^2 norm, of the bias of the integrated empirical second and fourth order cumulant spectral density operators is respectively obtained in Lemmas 3.1 and 3.3 under suitable conditions. Proposition 4.1 shows the divergence, in the Hilbert–Schmidt operator norm, of the mean of the test statistic operator under the alternative hypothesis. Theorem 4.2 derives suitable conditions, in particular, on the nonparametric functional spectral factor, to ensure consistency of the integrated weighted periodogram operator in the Hilbert–Schmidt operator norm under LRD. Theorem 4.5 then provides the almost surely divergence of the test statistic in the Hilbert–Schmidt operator norm under the alternative, yielding the consistency of the test.

Theorem 4.5 also plays a crucial role in the implementation in practice of the proposed testing procedure, based on rejecting the null hypothesis when the random Fourier coefficients of the test operator statistic, suitably standardized according to Theorem 2.2, cross an upper or lower tail standard normal critical value. The orthonormal basis involved in the computation of these coefficients is constructed by the tensor product of the eigenfunctions of the Laplace–Beltrami operator. The random projection methodology (see Theorem 4.1 in [8]) can be

implemented here to alleviate the dimensionality problem. Specifically, when the moments of our test statistic operator satisfy the Carleman condition under the null hypothesis, our testing procedure is equivalent to rejecting the null hypothesis when the absolute value of a random projection of the test statistic is larger than an upper tail standard normal critical value. In the implementation of the random projection methodology, the Karhunen–Loève expansion in Lemma 4.6 below can be considered for generation of the involved Gaussian random directions. In the simulation study undertaken, robust empirical sizes, and competitive values of the empirical power are displayed by our testing approach (see Section 5.4).

The outline of the paper is as follows. In Section 1.1, the functional spectral background material is introduced. Our hypothesis testing procedure is formulated in a functional semi-parametric spectral framework in Section 1.2. The asymptotic Gaussian distribution of the test statistic operator under the null hypothesis H_0 is obtained in Theorem 2.2 in Section 2. Asymptotics of the bias of the integrated empirical second and fourth order cumulant spectral density operators under LRD are derived in Section 3. Section 4 provides the preliminary results required for consistency of the test, which is derived in Theorem 4.5 of this section. Practical implementation is also discussed in Section 4. In Section 5.1, a simulation study is undertaken to illustrate the asymptotic Gaussian distribution of the proposed test statistic operator under the null hypothesis, in the context of SRD spherical functional time series. The consistency of the test is also illustrated in Section 5.2, in the framework of multifractionally integrated spherical functional time series. This numerical analysis is extended in Section 5.3 to a wider family of LRD operators allowing stronger persistency in time, displayed by the projected process in the dominant subspace. Section 5.4 analyzes empirical size and power properties of the test. Section 6 summarizes conclusions of the simulation study, focusing on the large functional sample size properties of our test statistic operator under different bandwidth parameter scenarios, from additional numerical results. The proofs of the results of this paper can be found in the Appendix.

1.1. Background

Along this work we will assume that $X = \{X_t, t \in \mathbb{Z}\}$ in (1) is a stationary zero–mean functional sequence, with nuclear covariance operator family $\{\mathcal{R}_\tau, \tau \in \mathbb{Z}\}$ satisfying $\mathcal{R}_\tau = E[X_s \otimes X_{s+\tau}] = E[X_{s+\tau} \otimes X_s]$, for every $s, \tau \in \mathbb{Z}$. The elements of this family are characterized in the spectral domain by the spectral density operator family $\{\mathcal{F}_\omega, \omega \in [-\pi, \pi]\}$. The assumed invariance of the elements of these families with respect to the group of isometries of \mathbb{M}_d yields to their diagonal series expansion in terms of $\{S_{n,j}^d \otimes \overline{S_{n,j}^d}, j = 1, \dots, \Gamma(n, d), n \in \mathbb{N}_0\}$, with $\{S_{n,j}^d, j = 1, \dots, \Gamma(n, d), n \in \mathbb{N}_0\}$ being the orthonormal basis of eigenfunctions of the Laplace–Beltrami operator Δ_d on $L^2(\mathbb{M}_d, d\nu, \mathbb{C})$ (see, e.g., [15]; [17]). In particular,

$$\begin{aligned} \mathcal{F}_\omega & \underset{S(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))}{=} \frac{1}{2\pi} \sum_{\tau \in \mathbb{Z}} \exp(-i\omega\tau) \mathcal{R}_\tau \\ & \underset{S(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))}{=} \sum_{n \in \mathbb{N}_0} f_n(\omega) \sum_{j=1}^{\Gamma(n, d)} S_{n,j}^d \otimes \overline{S_{n,j}^d}, \quad \omega \in [-\pi, \pi], \end{aligned} \tag{2}$$

where, for every $n \in \mathbb{N}_0$, $\Gamma(n, d)$ denotes the dimension of the eigenspace \mathcal{H}_n associated with the eigenvalue $\lambda_n(\Delta_d)$ of the Laplace Beltrami operator Δ_d (see, e.g., Section 2.1 in

[21]). The equality $\| \cdot \|_{S(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))} = \| \cdot \|_{L^2(\mathbb{M}_d, d\nu; \mathbb{C})}$ means identity in the norm of the space of Hilbert–Schmidt operators on $L^2(\mathbb{M}_d, d\nu; \mathbb{C})$, the space of complex-valued square integrable functions on \mathbb{M}_d . Specifically, the equality in (2) means that

$$\int_{\mathbb{M}_d \times \mathbb{M}_d} \left| f_\omega(x, y) - \sum_{n=0}^\infty f_n(\omega) \sum_{j=1}^{\Gamma(n,d)} S_{n,j}^d(x) \overline{S_{n,j}^d(y)} \right|^2 d\nu(x) d\nu(y) = 0,$$

where f_ω is the kernel of the integral operator \mathcal{F}_ω , for every $\omega \in [-\pi, \pi]$.

Let $\{X_t, t = 0, \dots, T - 1\}$ be a functional sample of size $T \geq 2$ of X . The fDFT $\tilde{X}_\omega^{(T)}$ is defined as

$$\tilde{X}_\omega^{(T)}(x) = \frac{1}{\sqrt{2\pi T}} \sum_{t=0}^{T-1} X_t(x) \exp(-i\omega t), \quad x \in \mathbb{M}_d, \quad \omega \in [-\pi, \pi]. \tag{3}$$

The kernel $p_\omega^{(T)}$ of the periodogram operator $\mathcal{P}_\omega^{(T)} = \tilde{X}_\omega^{(T)} \otimes \tilde{X}_\omega^{(T)}$ satisfies, for each $\omega \in [-\pi, \pi]$,

$$p_\omega^{(T)}(x, y) = \frac{1}{2\pi T} \sum_{t=0}^{T-1} \sum_{s=0}^{T-1} X_t(x) X_s(y) \exp(-i\omega(t - s)), \quad \forall x, y \in \mathbb{M}_d. \tag{4}$$

We will denote by $f_\omega^{(T)}(x, y) = \text{cum}(\tilde{X}_\omega^{(T)}(x), \tilde{X}_\omega^{(T)}(y)) = E \left[p_\omega^{(T)}(x, y) \right]$, $x, y \in \mathbb{M}_d$, the kernel of the cumulant operator $\mathcal{F}_\omega^{(T)}$ of order 2 of the fDFT $\tilde{X}_\omega^{(T)}$ over the diagonal $\omega \in [-\pi, \pi]$. Note that, for $\omega \in [-\pi, \pi]$, and $T \geq 2$, the Féjer kernel is given by

$$F_T(\omega) = \frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T \exp(-i(t - s)\omega) = \frac{1}{T} \left[\frac{\sin(T\omega/2)}{\sin(\omega/2)} \right]^2. \tag{5}$$

The weighted periodogram operator, denoted as $\hat{\mathcal{F}}_\omega^{(T)}$, has kernel $\hat{f}_\omega^{(T)}(x, y)$ given, for each $\omega \in [-\pi, \pi]$, by

$$\hat{f}_\omega^{(T)}(x, y) = \left[\frac{2\pi}{T} \right] \sum_{s=1}^{T-1} W^{(T)} \left(\omega - \frac{2\pi s}{T} \right) p_{\frac{2\pi s}{T}}^{(T)}(x, y), \quad x, y \in \mathbb{M}_d, \tag{6}$$

where $W^{(T)}$ is a weight function satisfying

$$W^{(T)}(x) = \sum_{j \in \mathbb{Z}} \frac{1}{B_T} W \left(\frac{x + 2\pi j}{B_T} \right), \tag{7}$$

with B_T being the positive bandwidth parameter. Function W is a real-valued function defined on \mathbb{R} such that W is positive, even, and bounded in variation; $W(x) = 0$, if $|x| \geq 1$; $\int_{\mathbb{R}} |W(x)|^2 dx < \infty$; $\int_{\mathbb{R}} W(x) dx = 1$.

1.2. Hypothesis testing

The SRD and LRD scenarios respectively tested under the null H_0 and the alternative H_1 hypotheses are introduced in this section. The proposed test statistic operator based on the weighted periodogram operator is then formulated.

Stationary SRD functional time series are characterized by the summability of the series of trace norms of the elements of the family of covariance operators $\{\mathcal{R}_\tau, \tau \in \mathbb{Z}\}$ (see, e.g., [28]). That is, X displays SRD if and only if $\sum_{\tau \in \mathbb{Z}} \|\mathcal{R}_\tau\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{R}))} < \infty$, where $L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{R}))$ denotes the space of trace operators on $L^2(\mathbb{M}_d, d\nu, \mathbb{R})$. In our setting, this condition can be formulated as follows:

$$\sum_{\tau \in \mathbb{Z}} \|\mathcal{R}_\tau\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{R}))} = \sum_{\tau \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \Gamma(n, d) \left| \int_{-\pi}^\pi \exp(i\omega\tau) f_n(\omega) d\omega \right| < \infty. \tag{8}$$

When (8) fails, X is said to display LRD. In what follows, we will adopt the LRD scenario introduced in [36], given by

$$\mathcal{F}_\omega = \mathcal{M}_\omega |\omega|^{-\mathcal{A}}, \quad \omega \in [-\pi, \pi], \tag{9}$$

where the invariant positive self–adjoint operators \mathcal{M}_ω and $|\omega|^{-\mathcal{A}}$ are composed yielding the definition of \mathcal{F}_ω . Specifically, \mathcal{A} denotes the LRD operator on $L^2(\mathbb{M}_d, d\nu; \mathbb{C})$. Operator $|\omega|^{-\mathcal{A}}$ in (9) is interpreted as in [7], [31] and [32], where \mathcal{A} plays the role of operator–valued Hurst coefficient in the setting of fractional Brownian motion introduced in this framework. Moreover, \mathcal{M}_ω is the regular spectral operator reflecting markovianess when the null space of \mathcal{A} coincides with $L^2(\mathbb{M}_d, d\nu; \mathbb{C})$. Specifically, \mathcal{M}_ω satisfies

$$\sum_{\tau \in \mathbb{Z}} \left\| \int_{[-\pi, \pi]} \exp(i\omega\tau) \mathcal{M}_\omega d\omega \right\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{R}))} < \infty, \tag{10}$$

where the operator integrals are understood as improper operator Stieltjes integrals which converge strongly (see, e.g., Section 8.2.1 in [33]).

We will apply the spectral theory of self–adjoint operators (see, e.g., [11]) in terms of the common spectral kernel

$$Y(x, y) = \sum_{n \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} S_{n, j}^d(x) \overline{S_{n, j}^d(y)}, \quad x, y \in \mathbb{M}_d,$$

under the assumed invariance property with respect to the group of isometries of \mathbb{M}_d .

The point spectrum of \mathcal{A} is given by $\{\alpha(n), n \in \mathbb{N}_0\}$, with $l_\alpha \leq \alpha(n) \leq L_\alpha$, for every $n \in \mathbb{N}_0$, and $l_\alpha, L_\alpha \in (0, 1/2)$. It is assumed that LRD operator \mathcal{A} has kernel $\mathcal{K}_\mathcal{A}$ admitting the following series expansion in the weak–sense:

$$\mathcal{K}_\mathcal{A}(x, y) = \sum_{n \in \mathbb{N}_0} \alpha(n) \sum_{j=1}^{\Gamma(n, d)} S_{n, j}^d \otimes \overline{S_{n, j}^d}(x, y). \tag{11}$$

Specifically, identity (11) is understood as

$$\mathcal{A}(f)(g) = \int_{\mathbb{M}_d \times \mathbb{M}_d} f(x)g(y) \sum_{n \in \mathbb{N}_0} \alpha(n) \sum_{j=1}^{\Gamma(n, d)} S_{n, j}^d(x) \overline{S_{n, j}^d}(y) d\nu(x) d\nu(y), \tag{12}$$

for all $f, g \in C^\infty(\mathbb{M}_d)$, where $C^\infty(\mathbb{M}_d)$ denotes the space of infinitely differentiable functions with compact support contained in \mathbb{M}_d . Note that, under the conditions assumed, \mathcal{A} and \mathcal{A}^{-1} are bounded, and $\|\mathcal{A}\|_{\mathcal{L}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} < 1/2$, with $\|\cdot\|_{\mathcal{L}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}$ denoting the norm in the space $\mathcal{L}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$ of bounded linear operators on $L^2(\mathbb{M}_d, d\nu, \mathbb{C})$.

In a similar way, operator $|\omega|^{-\mathcal{A}}$ is interpreted as

$$|\omega|^{-\mathcal{A}}(f)(g) = \int_{\mathbb{M}_d \times \mathbb{M}_d} f(x)g(y) \sum_{n \in \mathbb{N}_0} \frac{1}{|\omega|^{\alpha(n)}} \sum_{j=1}^{\Gamma(n,d)} S_{n,j}^d \otimes \overline{S_{n,j}^d}(x, y), dv(x)dv(y), \tag{13}$$

for every $f, g \in C^\infty(\mathbb{M}_d)$ and $\omega \in [-\pi, \pi] \setminus \{0\}$.

Operator \mathcal{M}_ω in (9) is a Hilbert–Schmidt operator on $L^2(\mathbb{M}_d, dv; \mathbb{C})$, whose kernel $\mathcal{K}_{\mathcal{M}_\omega}(x, y)$ admits the following series expansion in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, dv; \mathbb{C}))$:

$$\mathcal{K}_{\mathcal{M}_\omega}(x, y) = \sum_{n \in \mathbb{N}_0} M_n(\omega) \sum_{j=1}^{\Gamma(n,d)} S_{n,j}^d \otimes \overline{S_{n,j}^d}(x, y), \quad x, y \in \mathbb{M}_d, \omega \in [-\pi, \pi], \tag{14}$$

where $\{M_n(\omega), n \in \mathbb{N}_0\}$ denotes the sequence of positive eigenvalues. For each $n \in \mathbb{N}_0$, $M_n(\omega), \omega \in [-\pi, \pi]$, is a continuous positive slowly varying function at $\omega = 0$ in the Zygmund’s sense (see Definition 6.6 in [1], and Assumption IV in [36]). Equation (10) can be equivalently expressed, in terms of $\{M_n(\omega), n \in \mathbb{N}_0, \omega \in [-\pi, \pi]\}$, as

$$\sum_{\tau \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \Gamma(n, d) \left| \int_{-\pi}^{\pi} \exp(i\omega\tau) M_n(\omega) d\omega \right| < \infty. \tag{15}$$

As before, equation (15) implies that X displays SRD, when $\alpha(n) = 0$, for every $n \in \mathbb{N}_0$. Under (15), $\{\mathcal{M}_\omega, \omega \in [-\pi, \pi]\}$ is also included in the trace class.

Under the above setting of conditions,

$$\int_{-\pi}^{\pi} \|\mathcal{F}_\omega\|_{\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))}^2 d\omega < \infty, \tag{16}$$

i.e., $\|\mathcal{F}_\omega\|_{\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))} \in L^2([-\pi, \pi])$, with $L^2([-\pi, \pi])$ being the space of square integrable functions on the interval $[-\pi, \pi]$. Condition (16) plays a crucial role in the derivation of the results of this paper under LRD.

From equations (9)–(14), the elements of the positive frequency varying eigenvalue sequence $\{f_n(\omega), n \in \mathbb{N}_0\}$ in (2) admit the following expression:

$$f_n(\omega) = \frac{M_n(\omega)}{|\omega|^{\alpha(n)}}, \quad \omega \in [-\pi, \pi], n \in \mathbb{N}_0. \tag{17}$$

Note that, since $\sin(\omega) \sim \omega, \omega \rightarrow 0$,

$$|1 - \exp(-i\omega)|^{-\mathcal{A}} = [4 \sin^2(\omega/2)]^{-\mathcal{A}/2} \sim |\omega|^{-\mathcal{A}}, \quad \omega \rightarrow 0. \tag{18}$$

Sequence (17) is involved in the formulation of the alternative hypothesis H_1 stating that X displays LRD against H_0 where SRD is assumed. Specifically,

$$H_0 : f_n(\omega) = M_n(\omega), \omega \in [-\pi, \pi], \forall n \in \mathbb{N}_0 \tag{19}$$

$$H_1 : f_n(\omega) = M_n(\omega) |\omega|^{-\alpha(n)}, \omega \in [-\pi, \pi], \forall n \in \mathbb{N}_0. \tag{20}$$

In our context, the formulation of the test statistic must capture the singularities at zero frequency for different manifold resolution levels under H_1 . The proposed test statistic operator \mathcal{S}_{B_T} is then given by

$$\mathcal{S}_{B_T} = \sqrt{B_T T} \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \widehat{\mathcal{F}}_\omega^{(T)} \frac{d\omega}{\sqrt{B_T}}, \tag{21}$$

where the kernel of the integral operator $\widehat{\mathcal{F}}_\omega^{(T)}$ has been introduced in equation (6), with, as before, B_T being the bandwidth parameter. The indicator function on the interval $[-\sqrt{B_T}/2, \sqrt{B_T}/2]$ is denoted by $\mathbb{I}_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]}(\omega)$, for $\omega \in [-\pi, \pi]$. Note that, as $T \rightarrow \infty$,

$$\int_{-\pi}^{\pi} \frac{\mathbb{I}_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]}(\omega)}{\sqrt{B_T}} h(\omega) d\omega \rightarrow \int_{-\pi}^{\pi} \delta(0 - \omega) h(\omega) d\omega = h(0),$$

for every $h \in L^2([-\pi, \pi])$ (see [14] for the usual notion of convergence in the sense of generalized functions or distributions). Here, $\delta(0 - \omega)$ denotes the Dirac Delta distribution at zero frequency. Hence, in what follows we adopt the notation $\delta_T(0 - \omega) = \frac{\mathbb{I}_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]}(\omega)}{\sqrt{B_T}}$ representing a truncated Dirac Delta distribution.

2. Preliminary results under SRD

The following lemma will be applied in the proof of the main result of this section, Theorem 2.2, deriving the asymptotic Gaussian distribution of \mathcal{S}_{B_T} in (21) under H_0 . Specifically, Lemma 2.1 provides the asymptotic Gaussian distribution of the weighted periodogram operator $\widehat{\mathcal{F}}_\omega^{(T)}$ under H_0 . Its proof can be obtained in the same way as in [28], where this result is formulated for the separable Hilbert space $H = L^2([0, 1], \mathbb{C})$.

Lemma 2.1. *Assume that $E\|X_0\|^k < \infty$, for all $k \geq 2$, and*

- (i) $\sum_{t_1, \dots, t_{k-1} \in \mathbb{Z}} \|\text{cum}(X_{t_1}, \dots, X_{t_{k-1}}, X_0)\|_{L^2(\mathbb{M}_d^k, \otimes_{i=1}^k d\nu(x_i), \mathbb{R})} < \infty, k \geq 2$
- (i') $\sum_{t_1, \dots, t_{k-1} \in \mathbb{Z}} (1 + |t_j|) \|\text{cum}(X_{t_1}, \dots, X_{t_{k-1}}, X_0)\|_{L^2(\mathbb{M}_d^k, \otimes_{i=1}^k d\nu(x_i), \mathbb{R})} < \infty, \text{ for } k \in \{2, 4\}, j < k$
- (ii) $\sum_{t \in \mathbb{Z}} (1 + |t|) \|\mathcal{R}_t\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{R}))} < \infty$
- (iii) $\sum_{t_1, t_2, t_3 \in \mathbb{Z}} \|\mathcal{R}_{t_1, t_2, t_3}\|_{L^1(L^2(\mathbb{M}_d^3, \otimes_{i=1}^3 d\nu(x_i), \mathbb{R}))} < \infty.$

Then, for every frequencies $\omega_j, j = 1, \dots, J$, with $J < \infty$,

$$\sqrt{B_T T} (\widehat{f}_{\omega_j}^{(T)} - E[\widehat{f}_{\omega_j}^{(T)}]) \rightarrow_D \widehat{f}_{\omega_j}, j = 1, \dots, J \tag{22}$$

where \rightarrow_D denotes the convergence in distribution. Here, $\widehat{f}_{\omega_j}, j = 1, \dots, J$, are jointly zero–mean complex Gaussian elements in $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C})) = L^2(\mathbb{M}_d^2, d\nu \otimes d\nu, \mathbb{C})$, with covariance kernel:

$$\begin{aligned} \text{cov}(\widehat{f}_{\omega_i}(x_1, y_1), \widehat{f}_{\omega_j}(x_2, y_2)) &= 2\pi \|W\|_{L^2(\mathbb{R})}^2 \{ \eta(\omega_i - \omega_j) f_{\omega_i}(x_1, x_2) f_{-\omega_i}(y_1, y_2) \\ &+ \eta(\omega_i + \omega_j) f_{\omega_i}(x_1, y_2) f_{-\omega_i}(y_1, x_2) \}, \quad (x_i, y_i) \in \mathbb{M}_d^2, i = 1, 2, \end{aligned} \tag{23}$$

with $\eta(\omega) = 1$, for $\omega \in 2\pi\mathbb{Z}$, and $\eta(\omega) = 0$, otherwise. Thus, \widehat{f}_{ω_i} and \widehat{f}_{ω_j} are independent for $\omega_i + \omega_j \neq 0, \text{ mod } 2\pi$ and $\omega_i - \omega_j \neq 0, \text{ mod } 2\pi$. For zero frequency modulus 2π the limit Gaussian random element is in $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{R})) = L^2(\mathbb{M}_d^2, d\nu \otimes d\nu, \mathbb{R})$.

Proof. See Theorem 3.7 in [28].

The next result provides the asymptotic Gaussian distribution of the test statistic operator \mathcal{S}_{B_T} under H_0 . The convergence to a Gaussian random element in the norm of the

space $\mathcal{L}^2_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}(\Omega, \mathcal{A}, P)$ is also obtained. Here, $\mathcal{L}_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}(\Omega, \mathcal{A}, P)$ denotes the space of zero–mean second–order $\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))$ –valued random variables with the norm $\sqrt{E\|\cdot\|^2_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))}}$.

Theorem 2.2. *Under H_0 , assume that the conditions of Lemma 2.1 hold. Then,*

$$\mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}] \rightarrow_D Y_0^{(\infty)}, \quad T \rightarrow \infty, \tag{24}$$

where \mathcal{S}_{B_T} has been introduced in (21), and $Y_0^{(\infty)}$ is a zero–mean Gaussian random element in $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{R}))$, with autocovariance operator $\mathcal{R}_{Y_0^{(\infty)}} = E\left[Y_0^{(\infty)} \otimes Y_0^{(\infty)}\right]$ having kernel introduced in equation (23) in Lemma 2.1, with $\omega_i = \omega_j = 0$.

Proof. See Appendix A.1.

3. Second and fourth order bias asymptotics under LRD

This section provides new results on the bias asymptotics in the Hilbert-Schmidt operator norm of the integrated empirical second and fourth order cumulant spectral density operators of X under H_1 . These results are applied in the derivation of Theorem 4.2 and Corollary 4.4, providing the consistency of the integrated weighted periodogram operator under H_1 . In what follows, we assume $B_T \rightarrow 0$ and $B_T T \rightarrow \infty$ as $T \rightarrow \infty$.

The rate of convergence to zero of the norm of the bias in the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$ of the integrated periodogram operator is obtained under LRD in the next lemma. The following well-known identity will be applied:

$$\mathcal{F}_\omega^{(T)} = E\left[\mathcal{P}_\omega^{(T)}\right] = [F_T * \mathcal{F}_\bullet](\omega) = \int_{-\pi}^\pi F_T(\omega - \xi)\mathcal{F}_\xi d\xi, \quad T \geq 2, \tag{25}$$

for $\omega \in [-\pi, \pi] \setminus \{0\}$, where $F_T(\omega)$ denotes the Féjer kernel introduced in equation (5) of Section 1.1.

Lemma 3.1. *Under H_1 , as $T \rightarrow \infty$,*

$$\int_{-\pi}^\pi \mathcal{F}_\omega^{(T)} d\omega = \int_{-\pi}^\pi E_{H_1}\left[\mathcal{P}_\omega^{(T)}\right] d\omega \underset{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}{=} \int_{-\pi}^\pi \mathcal{F}_\omega d\omega + O(T^{-1}), \tag{26}$$

where E_{H_1} denotes expectation under the alternative H_1 , and, as before, $\underset{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}{=}$ denotes the equality in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$.

Proof. See Appendix B.1.

The following corollary is obtained from Lemma 3.1, and provides the rate of convergence to zero of the bias of the integrated weighted periodogram operator, in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$ under H_1 .

Corollary 3.2. Under H_1 , as $T \rightarrow \infty$,

$$\int_{-\pi}^{\pi} E_{H_1}[\widehat{\mathcal{F}}_{\omega}^{(T)}]d\omega \underset{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}{=} \int_{-\pi}^{\pi} \int_{\mathbb{R}} W(\xi)\mathcal{F}_{\omega-\xi} B_T d\xi d\omega + O(B_T^{-1}T^{-1}) + O(T^{-1}). \tag{27}$$

Proof. See Appendix B.2.

The rate of convergence to zero, in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C})) \equiv L^2(\mathbb{M}_d^4, \otimes_{i=1}^4 \nu(dx_i), \mathbb{C})$, of the bias of the integrated empirical fourth–order cumulant spectral density operators of X under LRD is obtained in Lemma 3.3 below. The following assumption is required:

Assumption I. For every $t_1, t_2, t_3 \in \mathbb{Z}$, $\text{cum}(X_{t_1}, X_{t_2}, X_{t_3}, X_0)$ defines an isotropic kernel in $L^2(\mathbb{M}_d^4, \otimes_{i=1}^4 \nu(dx_i), \mathbb{R})$, and the following convergence holds:

$$\sum_{t_1, t_2, t_3 \in \mathbb{Z}} \|\text{cum}(X_{t_1}, X_{t_2}, X_{t_3}, X_0)\|_{L^2(\mathbb{M}_d^4, \otimes_{i=1}^4 \nu(dx_i), \mathbb{R})}^2 < \infty, \tag{28}$$

where

$$\begin{aligned} & \|\text{cum}(X_{t_1}, X_{t_2}, X_{t_3}, X_0)\|_{L^2(\mathbb{M}_d^4, \otimes_{i=1}^4 \nu(dx_i), \mathbb{R})}^2 \\ &= \int_{\mathbb{M}_d^4} |\text{cum}(X_{t_1}(x), X_{t_2}(y), X_{t_3}(z), X_0(v))|^2 d\nu(x)d\nu(y)d\nu(z)d\nu(v). \end{aligned}$$

Lemma 3.3. Under H_1 and Assumption I, uniformly in $\omega_4 \in [-\pi, \pi]$,

$$\begin{aligned} & \int_{[-\pi, \pi]^3} T \text{cum}(\widetilde{X}_{\omega_1}^{(T)}(\tau_1), \widetilde{X}_{\omega_2}^{(T)}(\tau_2), \widetilde{X}_{\omega_3}^{(T)}(\tau_3), \widetilde{X}_{\omega_4}^{(T)}(\tau_4)) d\omega_1 d\omega_2 d\omega_3 \\ & \underset{\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C}))}{=} 2\pi \int_{[-\pi, \pi]^3} \mathcal{F}_{\omega_1, \omega_2, \omega_3}(\tau_1, \tau_2, \tau_3, \tau_4) d\omega_1 d\omega_2 d\omega_3 + O(T^{-1}), \end{aligned} \tag{29}$$

where, for $\omega_i \in [-\pi, \pi]$, $i = 1, 2, 3$,

$$\begin{aligned} \mathcal{F}_{\omega_1, \omega_2, \omega_3} \underset{\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C}))}{=} & \frac{1}{(2\pi)^3} \sum_{t_1, t_2, t_3 = -\infty}^{\infty} \exp\left(\sum_{j=1}^3 \omega_j t_j\right) \\ & \times \text{cum}(X_{t_1}, X_{t_2}, X_{t_3}, X_0) \end{aligned} \tag{30}$$

denotes the cumulant spectral density operator of order 4 of X , and, as before,

$$\underset{\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C}))}{=}$$

means the identity in the norm of the space

$$\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C})) \equiv L^2(\mathbb{M}_d^4, \otimes_{i=1}^4 \nu(dx_i), \mathbb{C}).$$

Proof. See Appendix B.3.

4. A test for LRD in functional time series on \mathbb{M}_d

Consistency of the test based on \mathcal{S}_{B_T} is derived in this section. Specifically, Theorem 4.5 provides the almost surely divergence in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$ of \mathcal{S}_{B_T} under H_1 . The proof of this result follows from Proposition 4.1, showing the divergence in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$ of the centering operator of \mathcal{S}_{B_T} , and from Theorem 4.2 and Corollary 4.4, establishing the consistency of the integrated weighted periodogram operator under H_1 . The implementation of the testing procedure in practice is also discussed.

Proposition 4.1. *Under H_1 , as $T \rightarrow \infty$,*

$$\begin{aligned} \left\| E_{H_1} \left[\frac{\mathcal{S}_{B_T}}{\sqrt{T}B_T} \right] \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} &= \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} [\widehat{\mathcal{F}}_\omega^{(T)}] \frac{d\omega}{\sqrt{B_T}} \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\ &\geq g(T) = O(B_T^{-1\alpha-1/2}). \end{aligned} \tag{31}$$

Proof. See Appendix C.1.

Theorem 4.2. *Under H_1 , Assumption I, and*

$$\int_{[-\pi, \pi]} \|\mathcal{M}_\omega\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 |\omega|^{-2L\alpha} d\omega < \infty, \tag{32}$$

as $T \rightarrow \infty$,

$$\int_{-\pi}^{\pi} E_{H_1} \left\| \widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} [\widehat{\mathcal{F}}_\omega^{(T)}] \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 d\omega \leq h(T) = O(B_T^{-1}T^{-1}), \tag{33}$$

where, as before, $\|\cdot\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}$ denotes the norm in the space $L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$ of nuclear operators on $L^2(\mathbb{M}_d, d\nu, \mathbb{C})$.

Remark 4.3. *Note that condition (32) is satisfied, for instance, when the family $\{\mathcal{M}_\omega, \omega \in [-\pi, \pi]\}$ lies in a ball of radius $R > 0$ of the space $L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$.*

Proof. See Appendix C.2.

Theorem 4.2 implies the weak consistency of the integrated weighted periodogram operator under H_1 in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$.

Corollary 4.4. *Under the conditions of Theorem 4.2, as $T \rightarrow \infty$,*

$$\begin{aligned} \left\| \int_{-\pi}^{\pi} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} - \int_{-\pi}^{\pi} W(\xi) \mathcal{F}_{\omega-B_T\xi} d\xi \right] d\omega \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\ \leq \widetilde{g}(T) = O(T^{-1/2}B_T^{-1/2}). \end{aligned}$$

Proof. See Appendix C.3.

Under the conditions assumed in the following result, consistency of the test follows.

Theorem 4.5. Under H_1 , assume that $l_\alpha > 1/4$, and that the bandwidth parameter $B_T = T^{-\beta}$ for $\beta \in (0, 1)$. If conditions of Theorem 4.2 hold, then, as $T \rightarrow \infty$,

$$\|S_{B_T}\|_{\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))} \rightarrow_{a.s.} \infty,$$

where $\rightarrow_{a.s.} \infty$ denotes almost surely divergence.

Proof. See Appendix C.4.

4.1. Practical implementation

The practical implementation of the proposed statistical testing procedure, in terms of the Gaussian random projection methodology (see Theorem 4.1 in [8]), is now briefly discussed. The Karhunen–Loève expansion in Lemma 4.6 below can be applied in such an implementation. Let us consider the random Fourier coefficients

$$Y_{n,j,h,l}(\omega) = \frac{(\sqrt{2\pi}\|W\|_{L^2(\mathbb{R})})^{-1}}{\sqrt{f_n(\omega)f_h(\omega)}} \int_{\mathbb{M}_d^2} \widehat{f}_\omega(\tau, \sigma) \overline{S_{n,j}^d(\tau)} S_{h,l}^d(\sigma) dv(\sigma) dv(\tau),$$

$$j = 1, \dots, \Gamma(n, d), \quad l = 1, \dots, \Gamma(h, d), \quad n, h \in \mathbb{N}_0, \quad \omega \in [-\pi, \pi] \setminus \{0\},$$
(34)

where integration is understood in the mean–square sense, and \widehat{f}_ω is the limit Gaussian random element in $\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))$ introduced in Lemma 2.1.

Lemma 4.6. Let \widehat{f}_ω be defined, as before, satisfying equation (23) in Lemma 2.1. Then, the following series expansion holds in the mean–square sense: For every $(\tau, \sigma) \in \mathbb{M}_d^2$,

$$\frac{1}{\sqrt{2\pi}\|W\|_{L^2(\mathbb{R})}} \widehat{f}_\omega(\tau, \sigma) \underset{\mathcal{L}^2_{\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))}(\Omega, \mathcal{A}, \mathcal{P})}}{=} \sum_{n,h \in \mathbb{N}_0}^{\Gamma(n,d)} \sum_{j=1}^{\Gamma(h,d)} \sum_{l=1}^{\Gamma(h,d)} \sqrt{f_n(\omega)f_h(\omega)} \times Y_{n,j,h,l}(\omega) \overline{S_{n,j}^d(\tau)} S_{h,l}^d(\sigma), \quad \omega \in [-\pi, \pi] \setminus \{0\},$$
(35)

where, as before, $\mathcal{L}^2_{\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))}(\Omega, \mathcal{A}, \mathcal{P})$ denotes the space of zero–mean second–order $\mathcal{S}(L^2(\mathbb{M}_d, dv; \mathbb{C}))$ –valued random variables with the norm $\sqrt{E\|\cdot\|_{\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))}}^2$. The random Fourier coefficients

$$\{Y_{n,j,h,l}(\omega), \quad j = 1, \dots, \Gamma(n, d), \quad l = 1, \dots, \Gamma(h, d), \quad n, h \in \mathbb{N}_0\},$$

for $\omega \in [-\pi, \pi] \setminus \{0\}$, have been introduced in equation (34). They are independent and identically distributed complex–valued standard Gaussian random variables.

Proof. See Appendix C.5.

Theorem 4.5 motivates the methodology to be adopted in practice. Specifically, as illustrated in the simulation study undertaken in the next section, a consistent test for LRD is obtained by rejecting H_0 , when, for every $j = 1, \dots, \Gamma(n, d), l = 1, \dots, \Gamma(h, d), n, h \in \mathbb{N}_0$,

$$\frac{|[S_{B_T} - E[S_{B_T}]](\overline{S_{h,l}^d})(S_{n,j}^d)|}{\sqrt{\text{Var}\left(S_{B_T}(\overline{S_{h,l}^d})(S_{n,j}^d)\right)}} \tag{36}$$

is larger than an upper tail standard normal critical value. Note that for T sufficiently large, and for $n, h \in \mathbb{N}_0$,

$$\begin{aligned}
 & [\mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}]] (\overline{S_{h,l}^d})(S_{n,j}^d) = \int_{\mathbb{M}_d^2} [\mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}]](\tau, \sigma) \overline{S_{n,j}^d}(\tau) S_{h,l}^d(\sigma) d\nu(\sigma) d\nu(\tau) \\
 & \text{Var} \left(\mathcal{S}_{B_T} (\overline{S_{h,l}^d})(S_{n,j}^d) \right) = \text{Var} \left(\int_{\mathbb{M}_d^2} \mathcal{S}_{B_T}(\tau, \sigma) \overline{S_{n,j}^d}(\tau) S_{h,l}^d(\sigma) d\nu(\sigma) d\nu(\tau) \right) \\
 & = 2\pi \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]^2 \times [-\pi, \pi]} W \left(\frac{\omega - \alpha}{B_T} \right) \left[W \left(\frac{\xi - \alpha}{B_T} \right) + W \left(\frac{\xi + \alpha}{B_T} \right) \right. \\
 & \quad \left. \times \left\langle S_{n,j}^d, \overline{S_{h,l}^d} \right\rangle_{L^2(\mathbb{M}_d, d\nu, \mathbb{C})} \left\langle \overline{S_{n,j}^d}, S_{h,l}^d \right\rangle_{L^2(\mathbb{M}_d, d\nu, \mathbb{C})} \right] f_n(\alpha) f_h(\alpha) \frac{d\alpha d\omega d\xi}{B_T^2} \\
 & + O(B_T^{-2} T^{-2}) + O(T^{-1}), \quad j = 1, \dots, \Gamma(n, d), \quad l = 1, \dots, \Gamma(h, d).
 \end{aligned} \tag{37}$$

The associated dimensionality problem can be substantially alleviated if we restrict our attention to the case where all moments of \mathcal{S}_{B_T} are finite and satisfy the Carleman condition. In that case, Theorem 4.1 in [8] leads to the following test statistic, evaluated conditionally to the observed functional value \mathbf{k} of a non-degenerated functional Gaussian random variable in the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))$, whose probability measure on $\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))$ is denoted as μ . Specifically, consider

$$\mathcal{T}_{B_T}^{\mathbf{k}} = \frac{\left| \left\langle \mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}], \mathbf{k} \right\rangle_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))} \right|}{\sqrt{\text{Var} \left(\left\langle \mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}], \mathbf{k} \right\rangle_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))} \right)}}. \tag{38}$$

Then, $H_0^{\mathbf{k}}$ will be rejected if the observed value of $\mathcal{T}_{B_T}^{\mathbf{k}}$ is larger than an upper tail standard normal critical value. Note that, if H_0 holds then $H_0^{\mathbf{k}}$ also holds, and if H_0 fails then $H_0^{\mathbf{k}}$ also fails μ -a.s. Thus, with probability one, we will generate a realization of random direction \mathbf{k} in $\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))$ for which $H_0^{\mathbf{k}}$ fails (see also [9]).

In the spirit of the Gaussian random degree- l spherical harmonics introduced in [25] ($l \geq 0$), we will consider a truncated Karhunen-Loève expansion in the generation of a non-degenerated Gaussian measure characterizing the random direction \mathbf{k} , where our test statistics is projected (see equation (38)). Specifically, we consider a zero-mean Gaussian random element \mathbf{k} in $\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))$ with trace covariance operator $C_{\mathbf{k}}$ having kernel

$$\begin{aligned}
 C_{\mathbf{k}}(\tau_1, \sigma_1, \tau_2, \sigma_2) &= E [\mathbf{k}(\tau_1, \sigma_1) \mathbf{k}(\tau_2, \sigma_2)] \\
 &= \sum_{n \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n,d)} \sum_{h \in \mathbb{N}_0} \sum_{l=1}^{\Gamma(h,d)} \lambda_{n,h} S_{n,j}^d(\tau_1) \overline{S_{h,l}^d(\sigma_1) S_{n,j}^d(\tau_2) S_{h,l}^d(\sigma_2)},
 \end{aligned} \tag{39}$$

for every $(\tau_1, \sigma_1), (\tau_2, \sigma_2) \in \mathbb{M}_d^2$. The trace property of $C_{\mathbf{k}}$ can be equivalently expressed as $\sum_{n \in \mathbb{N}_0} \sum_{h \in \mathbb{N}_0} \Gamma(n, d) \Gamma(h, d) \lambda_{n,h} < \infty$. Therefore, \mathbf{k} admits a series expansion, Karhunen-Loève expansion (see, e.g., Lemma 4.6), whose truncated version is implemented. Particularly, one can construct centered isotropic Gaussian random fields by finite-dimensional projection

$$f_{n,h}(\tau, \sigma) = \frac{1}{\Gamma(n, d) \Gamma(h, d)} \sum_{j=1}^{\Gamma(n,d)} \sum_{l=1}^{\Gamma(h,d)} Y_{n,j,h,l} S_{n,j}^d \otimes \overline{S_{h,l}^d}(\tau, \sigma), \tag{40}$$

for $(\tau, \sigma) \in \mathbb{M}_d^2$, and $n, h \in \mathbb{N}_0$, involving the Gaussian random directions in $\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))$

$$\mathbf{k}_{n,j,h,l}(\tau, \sigma) = Y_{n,j,h,l}[S_{n,j}^d \otimes \overline{S_{h,l}^d}(\tau, \sigma)], \quad (\tau, \sigma) \in \mathbb{M}_d^2, \tag{41}$$

where $Y_{n,j,h,l}$ denotes a zero–mean Gaussian random variable with variance $\lambda_{n,h}$, for $j = 1, \dots, \Gamma(n, d), l = 1, \dots, \Gamma(h, d), n, h \in \mathbb{N}_0$. Kernel $f_{n,h}(\tau, \sigma)$ plays a key role in our approach when \mathcal{H}_n and/or \mathcal{H}_h are dominant eigenspaces of the Laplace Beltrami operator on \mathbb{M}_d .

5. Simulation study

Our simulations will be set on $\mathbb{M}_d = \mathbb{S}_d \subset \mathbb{R}^{d+1}$. An alternative generation algorithm to the ones considered in [26] and [27] is implemented reducing computational burden allowing for the consideration of large functional sample sizes. Theorem 2.2 is illustrated in the context of SPHARMA(p,q) processes, and the illustration of Theorem 4.5 is carried out in the context of multifractionally integrated SPHARMA(p,q) processes. These numerical results are respectively reported in Sections 5.1 and 5.2 for $\beta = 1/4$, i.e., $B_T = T^{-1/4}$ (see also Section 6 for a wider analysis for different β values). When $L_\alpha > 1/2$, Section 5.3 opens new research lines beyond condition (16), providing empirical evidence of a faster a.s. divergence rate of \mathcal{S}_{B_T} in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$. Finally, Section 5.4 shows empirical size and power properties of the testing procedure.

5.1. Asymptotic Gaussian distribution of \mathcal{S}_{B_T} under H_0

Let us consider that the elements of the family of spectral density operators of X have frequency varying eigenvalues, with respect to the system of eigenfunctions of the Laplace–Beltrami operator, obeying the following equation under H_0 (see [36]):

$$f_n(\omega) = \frac{\lambda_n(\mathcal{R}_0^\eta)}{2\pi} \left| \frac{\Psi_{q,n}(\exp(-i\omega))}{\Phi_{p,n}(\exp(-i\omega))} \right|^2, \quad n \in \mathbb{N}_0, \quad \omega \in [-\pi, \pi], \tag{42}$$

where $\{\lambda_n(\mathcal{R}_0^\eta), n \in \mathbb{N}_0\}$ is the system of eigenvalues of the autocovariance operator \mathcal{R}_0^η of the innovation process $\eta = \{\eta_t, t \in \mathbb{Z}\}$, with respect to the system of eigenfunctions of the Laplace–Beltrami operator. Process η is assumed to be strong–white noise in $L^2(\mathbb{S}_d, d\nu, \mathbb{R})$. That is, η is assumed to be a sequence of independent and identically distributed $L^2(\mathbb{S}_d, d\nu, \mathbb{R})$ –valued random variables such that $E[\eta_t] = 0$, and $E[\eta_t \otimes \eta_s] = \delta_{t,s} \mathcal{R}_0^\eta$, with $\mathcal{R}_0^\eta \in L^1(L^2(\mathbb{S}_d, d\nu, \mathbb{R}))$, and $\delta_{t,s} = 0$, for $t \neq s$, and $\delta_{t,s} = 1$, for $t = s$. For $n \in \mathbb{N}_0$, $\Phi_{p,n}(z) = 1 - \sum_{j=1}^p \lambda_n(\varphi_j) z^j$ and $\Psi_{q,n}(z) = \sum_{j=1}^q \lambda_n(\psi_j) z^j$, with $\{\lambda_n(\varphi_j), n \in \mathbb{N}_0\}$ and $\{\lambda_n(\psi_l), n \in \mathbb{N}_0\}$ denoting the sequences of eigenvalues, with respect to the system of eigenfunctions of the Laplace–Beltrami operator, of the self–adjoint invariant integral operators φ_j and ψ_l , for $j = 1, \dots, p$, and $l = 1, \dots, q$, respectively. These operators satisfy the following equations:

$$\Phi_p(B) = 1 - \sum_{j=1}^p \varphi_j B^j, \quad \Psi_q(B) = \sum_{j=1}^q \psi_j B^j,$$

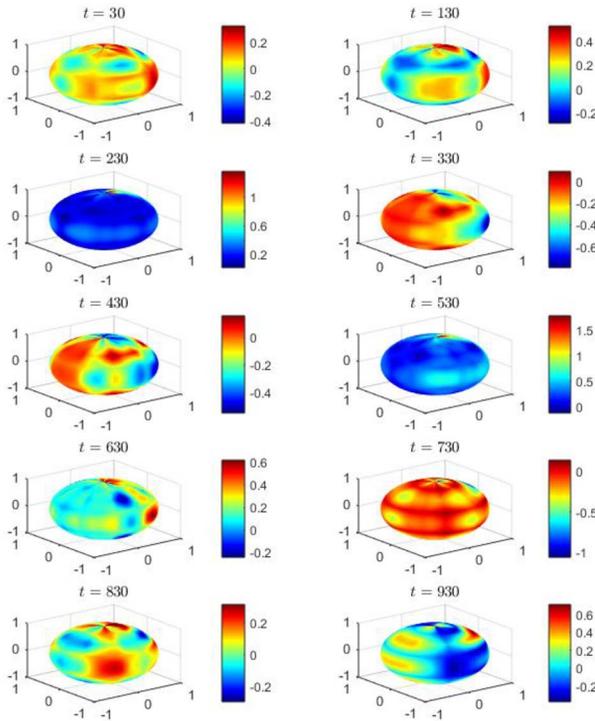


Fig 1. One realization at times $t = 30, 130, 230, 330, 430, 530, 630, 730, 830, 930$ of SPHARMA(1,1) process $\left(\lambda_n(\varphi_1) = 0.7 \left(\frac{n+1}{n}\right)^{-3/2}, \text{ and } \lambda_n(\psi_1) = (0.4) \left(\frac{n+1}{n}\right)^{-5/1.95}, n = 1, 2, 3, 4, 5, 6, 7, 8\right)$, projected into the direct sum $\bigoplus_{n=1}^8 \mathcal{H}_n$ of eigenspaces $\mathcal{H}_n, n = 1, \dots, 8$, of Δ_2 .

where B is a difference operator such that

$$E\|B^j X_t - X_{t-j}\|_{L^2(\mathbb{S}_d, d\nu, \mathbb{R})}^2 = 0, \quad \forall t, j \in \mathbb{Z}. \tag{43}$$

Here, Φ_p and Ψ_q are the so-called autoregressive and moving average operators, respectively. Also, for each $n \in \mathbb{N}_0$, $\Phi_{p,n}(z) = 1 - \sum_{j=1}^p \lambda_n(\varphi_j)z^j$ and $\Psi_{q,n}(z) = \sum_{j=1}^q \lambda_n(\psi_j)z^j$ have not common roots, and their roots are outside of the unit circle (see also Corollary 6.17 in [1]). Thus, X satisfies an SPHARMA(p,q) equation (see also [5]; [6]).

In the simulations we have generated an SPHARMA(1,1) process, i.e., $p = q = 1$, with $H = L^2(\mathbb{S}_2, d\nu, \mathbb{R})$, and $\lambda_n(\varphi_1) = 0.7 \left(\frac{n+1}{n}\right)^{-3/2}$ and $\lambda_n(\psi_1) = (0.4) \left(\frac{n+1}{n}\right)^{-5/1.95}$, $n \in \mathbb{N}_0$. Figure 1 displays one realization of the generated SPHARMA(1,1) process projected into $\bigoplus_{n=1}^8 \mathcal{H}_n$, at times $t = 30, 130, 230, 330, 430, 530, 630, 730, 830, 930$.

For each $n = 1, \dots, 8$, the empirical distribution of the centered and standardized projection into $\mathcal{H}_n \otimes \mathcal{H}_n$ of \mathcal{S}_{B_T} is displayed in Figure 2 for functional sample size $T = 1000$ and $R = 3000$ repetitions, in Figure 3 for functional sample size $T = 2000$ and $R = 3000$ repetitions, and in Figure 4 for functional sample size $T = 3000$ and $R = 3000$ repetitions. These empirical distributions approximate the support and shape of a standard Gaussian probability density. The empirical standardization displays a decreasing pattern over the spherical scale n , meaning that the respective limit one-dimensional Gaussian probability measures of these projections have decreasing support. According to Theorem 1.2.1 in [10], this property is satisfied by the infinite product Gaussian measure on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$, whose restriction to $L^2(\mathbb{S}_d, d\nu, \mathbb{C})$ is

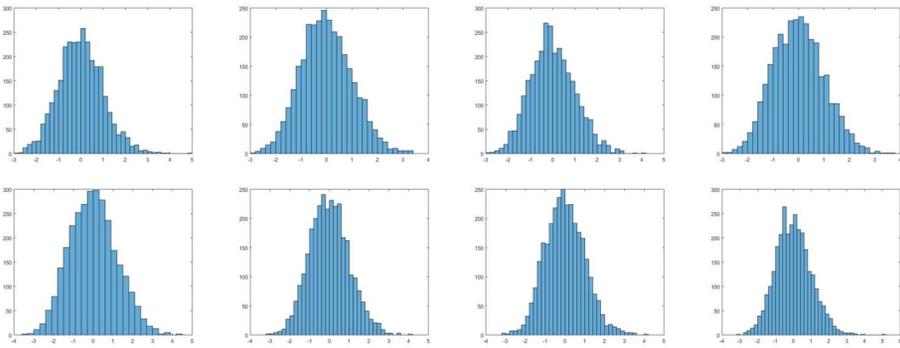


Fig 2. Empirical projections of the probability measure of standardized S_{B_T} , $B_T = T^{-1/4}$, under H_0 , into the eigenspaces $\mathcal{H}_n \otimes \mathcal{H}_n$, for $n = 1, 2, 3, 4, 5, 6, 7, 8$, respectively displayed from the left to the right, and from the top to the bottom, for functional samples size $T = 1000$ and $R = 3000$ repetitions.

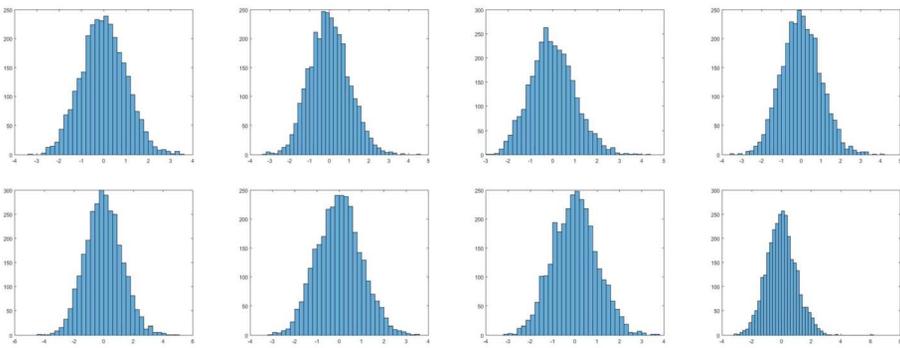


Fig 3. Empirical projections of the probability measure of standardized S_{B_T} , $B_T = T^{-1/4}$, under H_0 , into the eigenspaces $\mathcal{H}_n \otimes \mathcal{H}_n$, for $n = 1, 2, 3, 4, 5, 6, 7, 8$, respectively displayed from the left to the right, and from the top to the bottom, for functional samples size $T = 2000$ and $R = 3000$ repetitions.

identified in the ℓ^2 -sense with the probability measure of the limit Gaussian random element in Theorem 2.2.

5.2. Consistency of the test

Under H_1 , for each $\omega \in [-\pi, \pi]$, the eigenvalues $\{f_n(\omega), n \in \mathbb{N}_0\}$ satisfy (see equations (17), (18) and (20))

$$f_n(\omega) = \frac{\lambda_n(\mathcal{R}_0^\eta)}{2\pi} \left| \frac{\Psi_{q,n}(\exp(-i\omega))}{\Phi_{p,n}(\exp(-i\omega))} \right|^2 |1 - \exp(-i\omega)|^{-\alpha(n)}, \quad n \in \mathbb{N}_0. \quad (44)$$

Again, we consider the projection of X into $\bigoplus_{n=1}^8 \mathcal{H}_n$. Three multifractional integration operators, applied to SPHARMA(1,1) process generated in Section 5.1, are considered in Sections 5.2.1–5.2.3. In Example 1, $\alpha(n)$ is decreasing over n , in Example 2 $\alpha(n)$ is increasing over n , and non-monotone in Example 3. Note that, under the generated Gaussian scenario, condition (16) implies that Assumption I holds. Furthermore, condition (32) also holds since φ_1 lies in the unit ball of the space $\mathcal{L}(L^2(\mathbb{S}_2, d\nu, \mathbb{C}))$, and ψ_1 belongs to the trace class.

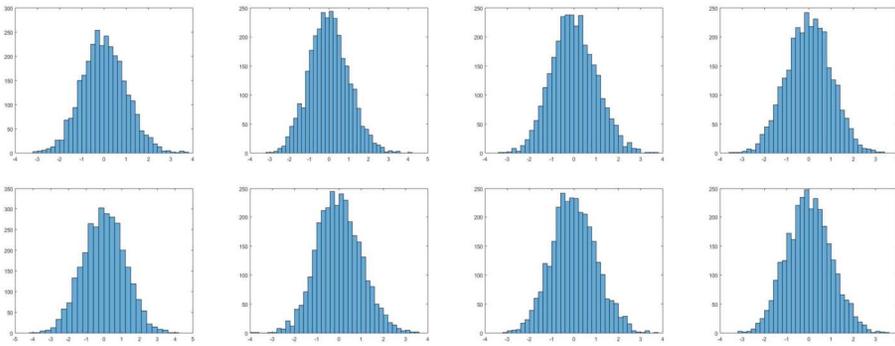


Fig 4. Empirical projections of the probability measure of standardized \mathcal{S}_{B_T} , $B_T = T^{-1/4}$, under H_0 , into the eigenspaces $\mathcal{H}_n \otimes \mathcal{H}_n$, for $n = 1, 2, 3, 4, 5, 6, 7, 8$, respectively displayed from the left to the right, and from the top to the bottom, for functional samples size $T = 3000$ and $R = 3000$ repetitions.

5.2.1. Example 1

Theorem 4.5 is now illustrated in the case where the largest dependence range is displayed by the process projected into the eigenspace \mathcal{H}_1 . Figure 5 displays a sample realization of the corresponding multifractionally integrated SPHARMA (1,1) process X projected into $\bigoplus_{i=1}^8 \mathcal{H}_i$.

In this example, $L_\alpha = 0.4733$, $l_\alpha = 0.2678$, and $\alpha(n) = l_\alpha = 0.2678$, $n \geq 9$ (see plot at the left-hand side of Figure 6).

The a.s. divergence of \mathcal{S}_{B_T} , for $B_T = T^{-1/4}$, in the Hilbert–Schmidt operator norm (see Table 1) is also reflected in the observed increasing sample values of each one of its projections into $\mathcal{H}_n \otimes \mathcal{H}_n$, $n = 1, \dots, 8$, for the increasing functional samples sizes $T = 1000, 10000, 30000$ (see the three plots at the right-hand side of Figure 6).

5.2.2. Example 2

The dominant subspace in this example, where the projected process displays the largest dependence range, is eigenspace \mathcal{H}_8 . One sample realization of the generated multifractionally integrated SPHARMA (1,1) process, projected into $\bigoplus_{n=1}^8 \mathcal{H}_n$, is displayed in Figure 7.

The LRD operator eigenvalues $\alpha(n)$, $n = 1, 2, 3, 4, 5, 6, 7, 8$, are given in the plot at the left-hand side of Figure 8, where $L_\alpha = 0.3327$, $l_\alpha = 0.2550$, and $\alpha(n) = l_\alpha = 0.2550$, $n \geq 9$. The a.s. divergence of our test statistic operator in the Hilbert–Schmidt operator norm (see also Table 1) is illustrated in the three plots at the right-hand side of such Figure 8, in terms of the sample values of each projection of \mathcal{S}_{B_T} , $B_T = T^{-1/4}$, into $\mathcal{H}_n \otimes \mathcal{H}_n$, $n = 1, \dots, 8$, for increasing functional samples sizes $T = 1000, 10000, 30000$.

5.2.3. Example 3

In this third example, the dominant subspace is eigenspace \mathcal{H}_5 of the Laplace–Beltrami operator Δ_2 . One sample realization of the generated multifractionally integrated SPHARMA (1,1) process, projected into $\bigoplus_{n=1}^8 \mathcal{H}_n$, is displayed in Figure 9.

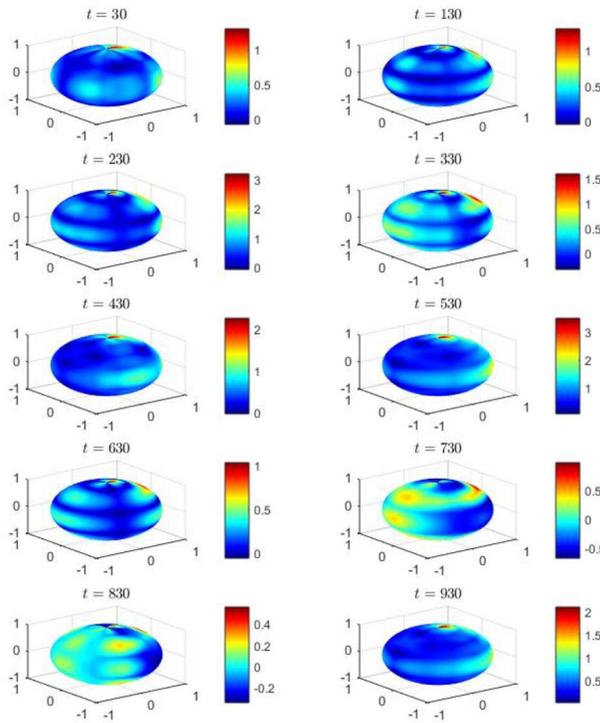


Fig 5. Example 1. One sample realization at times $t = 30, 130, 230, 330, 430, 530, 630, 730, 830, 930$ of multifractionally integrated SPHARMA(1,1) process projected into $\bigoplus_{n=1}^8 \mathcal{H}_n$.

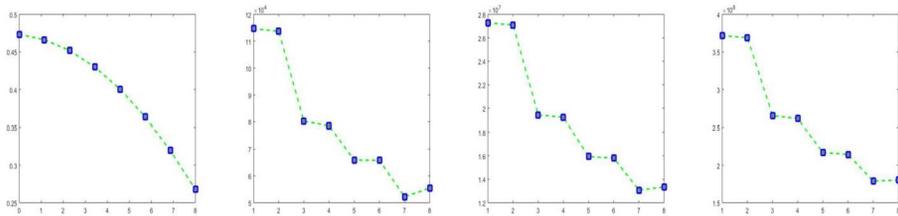


Fig 6. Example 1. Eigenvalues $\alpha(n)$, $n = 1, 2, 3, 4, 5, 6, 7, 8$, of LRD operator \mathcal{A} , $L_\alpha = 0.4733$, and $l_\alpha = 0.2678$ (plot at the left–hand side). Sample values of the test operator statistic \mathcal{S}_{B_T} , $B_T = T^{-1/4}$, projected into $\mathcal{H}_n \otimes \mathcal{H}_n$, $n = 1, \dots, 8$ for functional sample sizes $T = 1000, 10000, 30000$ (three plots at the right–hand side).

The eigenvalues $\alpha(n)$, $n = 1, 2, 3, 4, 5, 6, 7, 8$, of LRD operator \mathcal{A} are showed in the plot at the left–hand side of Figure 10 with $L_\alpha = 0.4000$, and $l_\alpha = 0.2753 = \alpha(n)$, $n \geq 9$. The sample values of the projections of \mathcal{S}_{B_T} , $B_T = T^{-1/4}$, into $\mathcal{H}_n \otimes \mathcal{H}_n$, $n = 1, \dots, 8$, for functional samples sizes $T = 1000, 10000, 30000$, can be found in the three plots at the right–hand side of Figure 10 (see also Table 1).

5.2.4. Almost surely divergence of \mathcal{S}_{B_T} in $\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))$ norm under H_1

The observed values of the Hilbert-Schmidt operator norm of \mathcal{S}_{B_T} , for $B_T = T^{-1/4}$, projected into $\bigoplus_{n=1}^8 \mathcal{H}_n \otimes \mathcal{H}_n$, is displayed in Table 1, for the three numerical examples generated, and for the functional sample sizes $T = 1000, 5000, 10000, 30000, 50000, 100000$.

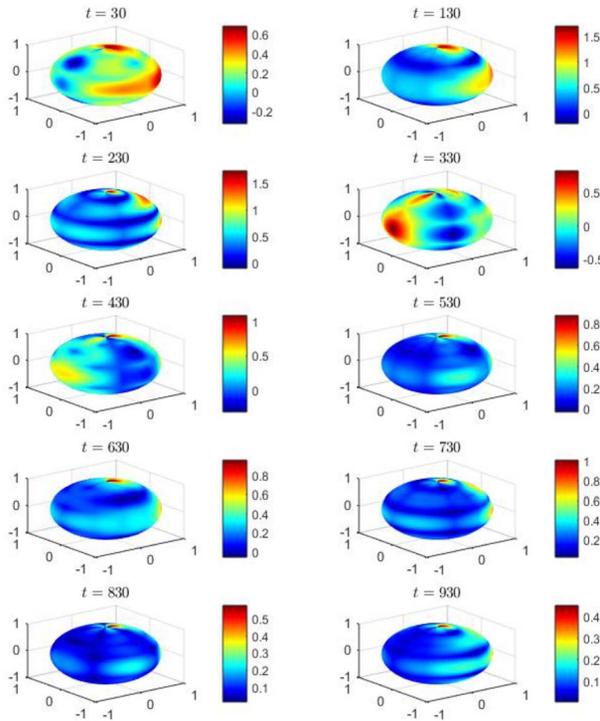


Fig 7. Example 2. One sample realization at times $t = 30, 130, 230, 330, 430, 530, 630, 730, 830, 930$ of multifractionally integrated SPHARMA(1,1) process projected into $\bigoplus_{n=1}^8 \mathcal{H}_n$.

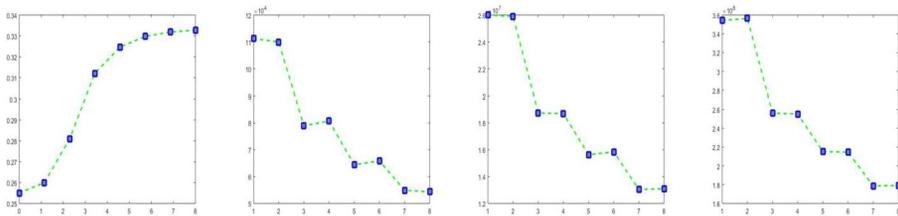


Fig 8. Example 2. Eigenvalues $\alpha(n)$, $n = 1, 2, 3, 4, 5, 6, 7, 8$, of LRD operator \mathcal{A} , $L_\alpha = 0.3327$, and $l_\alpha = 0.2550$ (plot at the left-hand side). Sample values of the kernel of the test operator statistic \mathcal{S}_{B_T} , $B_T = T^{-1/4}$, projected into $\mathcal{H}_n \otimes \mathcal{H}_n$, $n = 1, \dots, 8$, for functional sample sizes $T = 1000, 10000, 30000$ (three plots at the right-hand side).

One can observe in Table 1 the increasing sample values of the Hilbert–Schmidt operator norm of the projected \mathcal{S}_{B_T} as T increases, in all the examples under $l_\alpha > 1/4$, with $B_T = T^{-\beta}$, $\beta = 1/4$, satisfying $TB_T \rightarrow \infty$, $T \rightarrow \infty$. Spherical sample patterns and scales induced by the multifractional integration operator (see Figures 5, 7 and 9) have no significant effect (see Table 1), when the condition $l_\alpha > 1/4$ is satisfied under the bandwidth parameter modelling $B_T = T^{-\beta}$, $\beta \in (0, 1)$. This fact is also reflected in Figures 6, 8 and 10, respectively, where decreasing patterns, and almost the same divergence rates are displayed by the sample values of \mathcal{S}_{B_T} projected into $\mathcal{H}_n \otimes \mathcal{H}_n$, for $n = 1, 2, 3, 4, 5, 6, 7, 8$, in all the examples. However, the scenario under which $\alpha(n)$ crosses the threshold $1/2$ at some spherical scale n requires a separated analysis, as briefly discussed in Example 4 in the next section (see Figure 12).

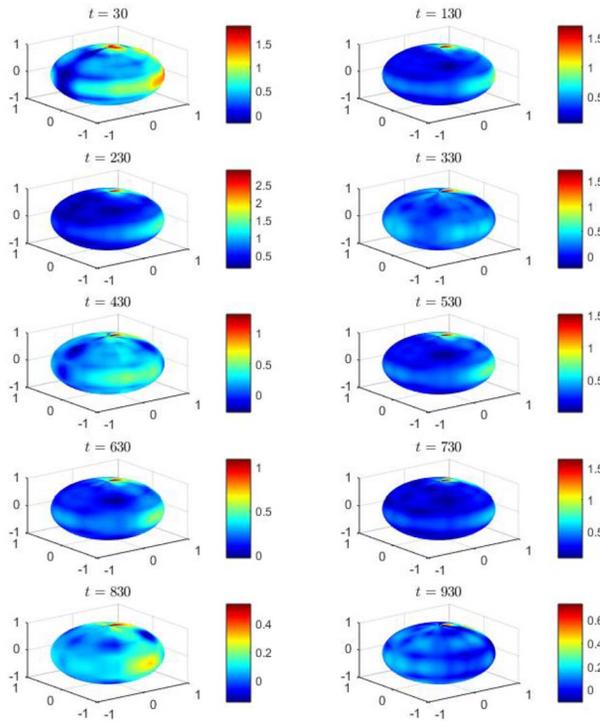


Fig 9. Example 3. One sample realization at times $t = 30, 130, 230, 330, 430, 530, 630, 730, 830, 930$ of multifractionally integrated SPHARMA(1,1) process projected into $\bigoplus_{n=1}^8 \mathcal{H}_n$.

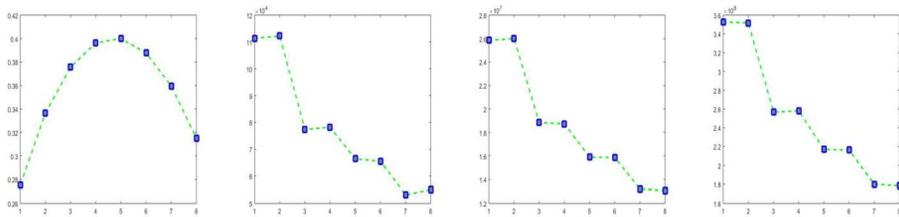


Fig 10. Example 3. Eigenvalues $\alpha(n)$, $n = 1, 2, 3, 4, 5, 6, 7, 8$, of the LRD operator \mathcal{A} , $L_\alpha = 0.4000$, and $l_\alpha = 0.2753$ (plot at the left–hand side). Sample values of the kernel of the test operator statistic \mathcal{S}_{B_T} , $B_T = T^{-1/4}$, projected into $\mathcal{H}_n \otimes \mathcal{H}_n$, $n = 1, \dots, 8$, for functional sample sizes $T = 1000, 10000, 30000$ (three plots at the right–hand side).

Table 1
Hilbert-Schmidt operator norm of projected \mathcal{S}_{B_T} , $\beta = 1/4$.

T	Example 1	Example 2	Example 3
1000	2.3036e+05	2.2595e+05	1.9934e+05
5000	1.0612e+07	1.0223e+07	9.0697e+06
10000	5.5172e+07	5.3770e+07	4.7303e+07
30000	7.5695e+08	7.3377e+08	6.4742e+08
50000	2.5516e+09	2.4764e+09	2.1844e+09
100000	1.3256e+10	1.2892e+10	1.2906e+10

5.3. Example 4

Our numerical analysis is extended here beyond the restriction $L_\alpha < 1/2$. Specifically, this section shows some preliminary numerical results regarding the effect of higher levels of singularity at zero frequency when $L_\alpha > 1/2$, i.e., $\|\mathcal{A}\|_{\mathcal{L}(L^2(\mathbb{S}_2, d\nu, \mathbb{C}))} > 1/2$, corresponding to a stronger persistency in time of the projected process into the dominant subspace (see Figure 11).

Under this scenario, conditions (16), and (32) in Theorem 4.2, are not satisfied. Indeed, we are out of the scenario where the summability in time of the square of the Hilbert–Schmidt operator norms of the elements of the covariance operator family holds. Then, new technical tools are required to address the asymptotic analysis in the spectral domain of this family of manifold supported functional time series displaying stronger levels of persistency in time. Let us again consider \mathcal{S}_{B_T} , for $B_T = T^{-1/4}$, projected into $\bigoplus_{n=1}^8 \mathcal{H}_n \otimes \mathcal{H}_n$. In this example, the multifractional integration of SPHARMA(1,1) process generated in Section 5.1 has been achieved in terms of LRD operator \mathcal{A} having eigenvalues displayed at the left–hand side of Figure 12, with $L_\alpha = 0.9982$ and $l_\alpha = 0.3041$, and \mathcal{H}_8 being the dominant subspace. The same functional sample sizes as in Examples 1–3 have been considered. One can observe, in the three plots displayed at the right–hand side of Figure 12, that the decreasing patterns over $n = 1, \dots, 8$, displayed in Figures 6, 8 and 10 do not hold in this example. Table 2 also illustrates a faster increasing than in Examples 1–3 of $\|\mathcal{S}_{B_T}\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}$, for functional sample sizes ranging from 1000 to 100000, under $l_\alpha > 1/4$, and $B_T = T^{-1/4}$.

5.4. Empirical size and power analysis

The empirical size and power properties of the testing approach presented are now illustrated. We have applied the random projection methodology. Tables 3 and 4 display the numerical results for 8 random functional directions (see equation (38)). Model SPHARMA(1,1) generated in Section 5.1 has been considered in the computation of the empirical size of the test. Multifractionally integrated SPHARMA(1,1) model, generated in Section 5.2.1, defines the scenario under the alternative to compute the empirical power. For each one of the eight random directions tested, we have analyzed the functional samples sizes $T = 50, 100, 500, 1000$, and, for each functional sample size, we have considered $R = 500, 1000, 3000$ repetitions.

The empirical size properties of the proposed testing procedure are quite robust, as one can observe in the numerical results displayed in Table 3. Specifically, since we are working with finite sample sizes ($T = 50, 100, 500, 1000$), despite the statistical distance to the normal distribution which holds asymptotically, small deviations are observed from the theoretical α value for all number of repetitions R considered. One can also observe, in Table 4, the increasing patterns displayed by the empirical power with respect to the functional sample sizes tested in all random directions. Note that these empirical power values are in the interval $[0.776, 1]$. In particular, since the threshold $T = 1000$, the empirical power is almost 1 for any of the three values of R studied.

6. Final comments. Reliable inference from our approach

The simulation study illustrates six key aspects of our approach, briefly summarized in points (i)–(vi) below:

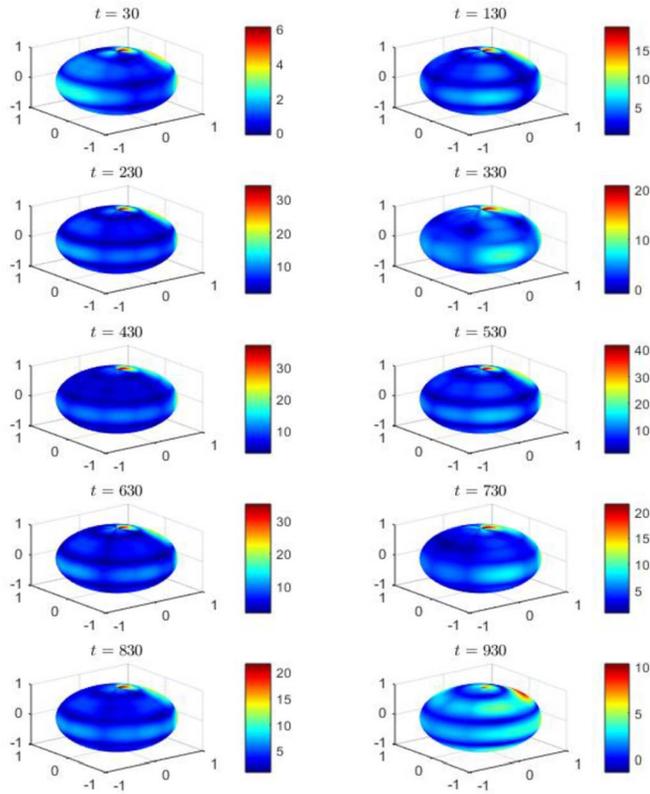


Fig 11. Example 4. One sample realization at times $t = 30, 130, 230, 330, 430, 530, 630, 730, 830, 930$ of multi-fractionally integrated SPHARMA(1,1) process projected into $\bigoplus_{n=1}^8 \mathcal{H}_n$.

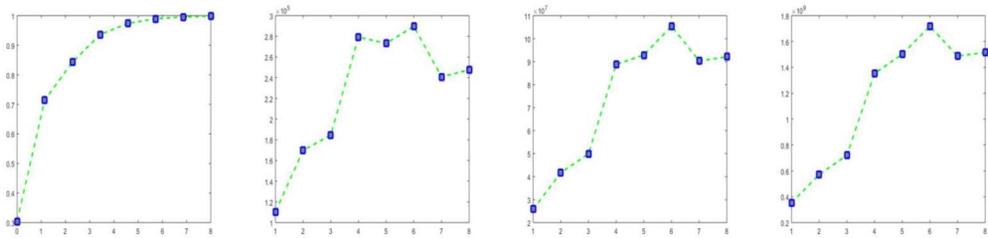


Fig 12. Example 4. Eigenvalues $\alpha(n), n = 1, 2, 3, 4, 5, 6, 7, 8$, of LRD operator \mathcal{A} , $L_\alpha = 0.9982$ and $l_\alpha = 0.3041$ (plot at the left–hand side). Sample values of the kernel of the test operator statistic S_{B_T} , for $B_T = T^{-1/4}$, projected into $\mathcal{H}_n \otimes \mathcal{H}_n, n = 1, \dots, 8$, for the functional sample sizes $T = 1000, 10000, 30000$ (three plots at the right–hand side).

Table 2

Example 4. $\|S_{B_T}\|_{\mathcal{S}(L^2(\mathbb{M}_d, dv, \mathbb{C}))}$, of S_{B_T} projected into $\bigoplus_{n=1}^8 \mathcal{H}_n \otimes \mathcal{H}_n$ ($\beta = 1/4, L_\alpha = 0.9982$ and $l_\alpha = 0.3041$).

Sample Size	1000	5000	10000	30000	50000	100000
	6.5651e+05	3.8623e+07	2.2172e+08	3.5383e+09	1.2688e+10	7.2258e+10

Table 3
Empirical size ($\beta = 1/4, \mathbf{k}_{n,j,h,l}, n = h = 1, 2, 3, \alpha = 0.05$).

R	T=50							
500	0.0280	0.0560	0.0360	0.0480	0.0600	0.0600	0.0360	0.0520
1000	0.0480	0.0420	0.0320	0.0380	0.0420	0.0440	0.0320	0.0300
3000	0.0420	0.0447	0.0453	0.0353	0.0413	0.0400	0.0440	0.0507
R	T=100							
500	0.0360	0.0680	0.0440	0.0720	0.0520	0.0240	0.0360	0.0400
1000	0.0280	0.0380	0.0380	0.0500	0.0380	0.0740	0.0340	0.0360
3000	0.0373	0.0507	0.0407	0.0460	0.0440	0.0360	0.0600	0.0480
R	T=500							
500	0.0440	0.0520	0.0320	0.0640	0.0320	0.0480	0.0320	0.0480
1000	0.0420	0.0400	0.0500	0.0460	0.0420	0.0380	0.0540	0.0240
3000	0.0453	0.0393	0.0447	0.0407	0.0427	0.0507	0.0453	0.0553
R	T=1000							
500	0.0520	0.0360	0.0400	0.0560	0.0600	0.0640	0.0480	0.0520
1000	0.0440	0.0380	0.0400	0.0580	0.0500	0.0360	0.0520	0.0400
3000	0.0573	0.0480	0.0507	0.0467	0.0440	0.0453	0.0487	0.0447

Table 4
Empirical power ($\beta = 1/4, \mathbf{k}_{n,j,h,l}, n = h = 1, 2, 3, \alpha = 0.05$).

R	T=50							
500	0.9200	0.9240	0.8640	0.8880	0.8480	0.8160	0.7760	0.7800
1000	0.9000	0.9000	0.8860	0.8980	0.8000	0.8320	0.7840	0.7840
3000	0.9247	0.9253	0.8713	0.8760	0.8247	0.8273	0.7980	0.8013
R	T=100							
500	0.9920	0.9920	0.9920	0.9880	0.9840	0.9800	0.9880	0.9720
1000	0.9880	0.9880	0.9860	0.9840	0.9800	0.9720	0.9740	0.9840
3000	0.9893	0.9893	0.9920	0.9827	0.9820	0.9773	0.9767	0.9747
R	T=500							
500	1.0000	1.0000	1.0000	0.9960	1.0000	0.9960	1.0000	1.0000
1000	1.0000	1.0000	1.0000	1.0000	0.9980	1.0000	0.9980	1.0000
3000	1.0000	0.9987	1.0000	1.0000	0.9993	1.0000	0.9993	0.9993
R	T=1000							
500	1	1	1	1	1	1	1	1
1000	1	1	1	1	1	1	1	1
3000	1	1	1	1	1	1	1	1

(i) The tight property under H_0 of the random projection sequence

$$\left\langle \sqrt{B_T T} (\widehat{\mathcal{F}}_\omega^{(T)} - E[\widehat{\mathcal{F}}_\omega^{(T)}]), S_{n,j}^d \otimes \overline{S_{h,l}^d} \right\rangle_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}$$

$$j = 1, \dots, \Gamma(n, d), l = 1, \dots, \Gamma(h, d), n, h \in \mathbb{N}_0,$$

allows the application of Prokhorov Theorem to prove the convergence, as $T \rightarrow \infty$, of $\mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}]$ to $\widehat{\mathcal{F}}_0$, in the space $\mathcal{L}^2_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}(\Omega, \mathcal{A}, P)$. In particular, the asymptotic Gaussian distribution of \mathcal{S}_{B_T} under H_0 follows from this result. This result is illustrated in Section 5.1 from Theorem 1.2.1 in [10].

(ii) The crucial role played by the design of the test statistic operator \mathcal{S}_{B_T} in the derivation of the conditions assumed to obtain consistency of the test (see Proposition 4.1

Table 5

$$\left\| \frac{S_{B_T}}{(TB_T)^{1/2}} \right\|_{S(L^2(\mathbb{M}_d, dv, \mathbb{C}))}, S_{B_T} \text{ projected into } \bigoplus_{n=1}^8 \mathcal{H}_n \otimes \mathcal{H}_n.$$

$B_T = T^{-\beta}$	T	Example 1	Example 2	Example 3
$\beta = 0.2$	1000	1.6885(1.0e+04)	1.6384(1.0e+04)	1.6438(1.0e+04)
	50000	4.3549(1.0e+07)	4.2251(1.0e+07)	4.2290(1.0e+07)
	100000	1.7493(1.0e+08)	1.6944(1.0e+08)	1.6989(1.0e+08)
$\beta = 0.55$	1000	1.8067 (1.0e+04)	1.7733(1.0e+04)	1.7693(1.0e+04)
	50000	4.4789(1.0e+07)	4.3747(1.0e+07)	4.3510(1.0e+07)
	100000	1.7984(1.0e+08)	1.7470(1.0e+08)	1.7476(1.0e+08)
$\beta = 0.9$	1000	2.0296(1.0e+04)	2.0109(1.0e+04)	1.9993(1.0e+04)
	50000	4.5133(1.0e+07)	4.4271(1.0e+07)	4.4138(1.0e+07)
	100000	1.8040 (1.0e+08)	1.7518(1.0e+08)	1.7624(1.0e+08)

and Theorem 4.2). The simulation study also reveals that the additional conditions assumed in Theorem 4.5 lead to universal a.s. divergence rates. These rates are not affected by the localization of the dominant eigenspace, or the value of the parameter $\beta \in (0, 1)$ chosen, under the bandwidth parameter scenario $B_T = T^{-\beta}$ when $l_\alpha > 1/4$. Table 5 visualizes this fact for parameter values $\beta = 0.2, 0.55, 0.9$. Specifically, in the three examples analyzed, the sample values of $\left\| \frac{S_{B_T}}{(TB_T)^{1/2}} \right\|_{S(L^2(\mathbb{M}_d, dv, \mathbb{C}))}$, projected into $\bigoplus_{n=1}^8 \mathcal{H}_n \otimes \mathcal{H}_n$, are displayed under these three bandwidth parameter scenarios, for functional sample sizes $T = 1000, 50000, 100000$. Note that, although, as expected, the sample values of $\left\| \frac{S_{B_T}}{(TB_T)^{1/2}} \right\|_{S(L^2(\mathbb{M}_d, dv, \mathbb{C}))}$ slightly increase when β increases (see Examples 1–3), no significant differences are observed in the sample divergence rate of $\left\| \frac{S_{B_T}}{(TB_T)^{1/2}} \right\|_{S(L^2(\mathbb{M}_d, dv, \mathbb{C}))}$, between the three values of parameter β analyzed. Furthermore, under condition $l_\alpha > 1/4$, when $B_T = T^{-\beta}$, $\beta \in (0, 1)$, one can observe the invariance of the sample divergence rate against the location of the dominant eigenspace. Keeping in mind that, as illustrated in [26], the regular spectral factor \mathcal{M}_ω , which here is represented by the SPHARMA(p,q) functional spectrum, has not impact in the asymptotic analysis, one can conclude the findings in Table 5 are representative.

- (iii) As expected, when higher orders of singularity are displayed at zero frequency, beyond the restriction $L_\alpha < 1/2$, a faster divergence of the sample values of the Hilbert–Schmidt operator norm of S_{B_T} , and of its diagonal projections, is observed.
- (iv) The testing approach adopted shows good empirical size and power properties for finite functional samples, as reported in Tables 3 and 4. Namely, Table 3, in the simulation study undertaken, shows empirical test sizes very close to the theoretical value $\alpha = 0.05$ for the minimum sample size $T = 50$ considered, and for the number of repetitions $R = 500, 1000, 3000$. Table 4 displays, for $T = 50$, empirical powers in the interval $(0.7760, 0.9253)$ at the eight random directions tested, and for the number of repetitions $R = 500, 1000, 3000$. Summarizing, as illustrated in Section 5.4, for relatively small functional sample sizes, reliable inference based on our functional spectral nonparametric approach is possible. On the other hand, asymptotic properties like consistency of the test are verified to hold, as given in Table 1, displaying the increasing order of magnitude of the Hilbert–Schmidt operator norm of our test statistics, which is around 10^5 ,

for the smallest functional sample size $T = 1000$ in all examples. By the same reasons explained in (ii), i.e., the invariance of the results displayed in Tables 3 and 4 against the location of the dominant eigenspace (\mathcal{H}_1 , \mathcal{H}_8 and \mathcal{H}_5 respectively in Examples 1,2,3), and the absence of asymptotic impact of the choice of the SPHARMA(p,q) functional spectrum, one can conclude the representativeness of the numerical results reflected in Tables 3 and 4.

- (v) We remark that all computations involved in the simulation study undertaken, in particular, in the implementation of the proposed inference tools and testing approach, have been achieved in terms of a unique orthonormal basis, given by the eigenfunctions of the Laplace Beltrami operator. Thus, we have worked under the scenario where the eigenfunctions of the elements of the covariance and spectral density operator families are known. This fact constitutes an important advantage of the analyzed setting, avoiding the use of empirical eigenfunction bases. We have also worked under the context of fully observed functional data. The case of sparse discretely observed and contaminated functional data can be addressed from the nonparametric series least-squares estimation of our functional data set, and plug-in implementation of our test statistics (see, e.g., [46] and [37]). Specifically, under suitable restrictions on the local Hölder regularity of our functional data set, and on the supremum norm of the sieve basis elements, as well as on the pure point spectral properties of the autocovariance matrix of the random sieve basis (involved in the nonparametric series least-squares estimation of the discretely observed functional data set), one can derive similar asymptotic results. To this aim, a suitable manifold random uniform sampling design must be considered. The time-varying manifold sampling frequency must display a faster divergence rate than the time-varying sieve basis dimension, but slower than the functional sample size. Under these conditions, similar results on asymptotic L^2 bias analysis can be obtained under H_1 , depending on the almost surely uniform convergence rates to the theoretical values of our functional data, involved in their nonparametric series least-squares estimation.
- (vi) As commented in the Introduction, the presented approach allows the statistical inference from spatiotemporal data sets embedded into the sphere. Thus, the non-euclidean spatial statistical analysis of such data sets can be performed enhancing geometrical interpretability, avoiding the usual transformations of longitudes and latitudes of the data to work in a cartesian reference coordinate system. Under the invariance properties assumed in our setting, the case of discretely observed functional data can be addressed considering the sieve basis constructed from the eigenfunctions of the Laplace Beltrami operator. This sieve basis allows an easier geometrical implementation (see, e.g., [18], and [46] for alternative sieve bases in nonparametric series least-squares ridge regression in an euclidean spatial setting). An important dimension reduction is obtained in terms of this sieve basis, which is crucial in the reconstruction of high-dimensional data sets. This fact constitutes another remarkable feature of our approach that reduces computational burden, allowing the implementation of resampling techniques in real data applications. Note that temporal information can be incorporated under our functional time series framework, extending recent developments in the purely spatial statistical euclidean context (see, e.g., [18]). Just to mention a motivating data example for implementation of our approach, [18] analyze a georeferenced spatiotemporal discretely observed population data set to predict nighttime population in Tokyo. They implement

series spatial ridge regression estimation after monthly averaging the data, ignoring time information that is crucial in this prediction problem. Our functional spectral non-parametric approach allows time information to be processed in an efficient way in a non–euclidean setting as commented in (iv) and (v).

Appendix. Proof of the results

Appendix A: Proofs of the results in Section 2

A.1. Proof of Theorem 2.2

Proof. From Lemma 2.1,

$$\sqrt{B_T T}(\widehat{f}_{\omega_j}^{(T)} - E[\widehat{f}_{\omega_j}^{(T)}]) \rightarrow_D \widehat{f}_{\omega_j}, \quad j = 1, \dots, J, \tag{45}$$

where \rightarrow_D denotes the convergence in distribution. Here, $\widehat{f}_{\omega_j}, j = 1, \dots, J$, are jointly zero–mean complex Gaussian elements in $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C})) \equiv L^2(\mathbb{M}_d^2, d\nu \otimes d\nu, \mathbb{C})$, with covariance kernel (23).

Let us consider $\{S_{n,j}^d, j = 1, \dots, \Gamma(n, d), n \in \mathbb{N}_0\}$, the orthonormal basis of eigenfunctions of the Laplace–Beltrami operator Δ_d on $L^2(\mathbb{M}_d, d\nu, \mathbb{C})$. From equation (23), applying invariance property leading to

$$f_{\omega}(\tau, \sigma) \underset{\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))}{=} \sum_{n \in \mathbb{N}_0} f_n(\omega) \sum_{j=1}^{\Gamma(n,d)} S_{n,j}^d \otimes \overline{S_{n,j}^d}(\tau, \sigma), \quad (\tau, \sigma) \in \mathbb{M}_d^2, \tag{46}$$

we obtain, for $\tau_1, \sigma_1, \tau_2, \sigma_2 \in \mathbb{M}_d$,

$$\begin{aligned} \text{cov}(\widehat{f}_{\omega}(\tau_1, \sigma_1), \widehat{f}_{\omega}(\tau_2, \sigma_2)) &= 2\pi \|W\|_{L^2(\mathbb{R})}^2 \\ &\times \left[\sum_{n,h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n,d)} \sum_{l=1}^{\Gamma(h,d)} f_n(\omega) f_h(\omega) S_{n,j}^d(\tau_1) \overline{S_{h,l}^d(\sigma_1) S_{n,j}^d(\tau_2) S_{h,l}^d(\sigma_2)} \right. \\ &\left. + \eta(2\omega) \sum_{n,h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n,d)} \sum_{l=1}^{\Gamma(h,d)} f_n(\omega) f_h(\omega) S_{n,j}^d(\tau_1) \overline{S_{h,l}^d(\sigma_1) S_{h,l}^d(\tau_2) S_{n,j}^d(\sigma_2)} \right]. \end{aligned} \tag{47}$$

Under H_0 , from Theorem D2 in the Supplementary Material of [28], keeping in mind (46),

$$\begin{aligned} &\text{cov}(\widehat{f}_{\omega}^{(T)}(\tau_1, \sigma_1), \widehat{f}_{\omega}^{(T)}(\tau_2, \sigma_2)) \\ &= \frac{2\pi}{T} \int_{-\pi}^{\pi} W^{(T)}(\omega - \alpha) W^{(T)}(\omega - \alpha) f_{\alpha}(\tau_1, \tau_2) \overline{f_{\alpha}(\sigma_1, \sigma_2)} d\alpha \\ &+ \frac{2\pi}{T} \int_{-\pi}^{\pi} W^{(T)}(\omega - \alpha) W^{(T)}(\omega + \alpha) f_{\alpha}(\tau_1, \sigma_2) \overline{f_{\alpha}(\sigma_1, \tau_2)} d\alpha \\ &+ O(B_T^{-2} T^{-2}) + O(T^{-1}) \end{aligned}$$

$$\begin{aligned}
 &= \frac{2\pi}{T} \sum_{n,h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n,d)} \sum_{l=1}^{\Gamma(h,d)} \int_{-\pi}^{\pi} W^{(T)}(\omega - \alpha) W^{(T)}(\omega - \alpha) f_n(\alpha) f_h(\alpha) d\alpha \\
 &\quad \times S_{n,j}^d(\tau_1) \overline{S_{h,l}^d(\sigma_1)} S_{n,j}^d(\tau_2) \overline{S_{h,l}^d(\sigma_2)} \\
 &\quad + \frac{2\pi}{T} \int_{-\pi}^{\pi} W^{(T)}(\omega - \alpha) W^{(T)}(\omega + \alpha) f_n(\alpha) f_h(\alpha) d\alpha \\
 &\times S_{n,j}^d(\tau_1) \overline{S_{h,l}^d(\sigma_1)} \overline{S_{h,l}^d(\tau_2)} S_{n,j}^d(\sigma_2) + O(B_T^{-2} T^{-2}) + O(T^{-1}). \tag{48}
 \end{aligned}$$

From (48), applying Cauchy–Schwartz inequality, and the orthonormality of the basis of eigenfunctions of the Laplace Beltrami operator, the following inequalities hold: For every $j = 1, \dots, \Gamma(n, d)$, $n \in \mathbb{N}_0$, and $l = 1, \dots, \Gamma(h, d)$, $h \in \mathbb{N}_0$,

$$\begin{aligned}
 &E \left[\left| \left\langle \sqrt{B_T T} (\widehat{\mathcal{F}}_{\omega}^{(T)} - E[\widehat{\mathcal{F}}_{\omega}^{(T)}]), S_{n,j}^d \otimes \overline{S_{h,l}^d} \right\rangle_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \right|^2 \right] \\
 &= 2\pi \int_{-\pi}^{\pi} W \left(\frac{\omega - \alpha}{B_T} \right) \left[W \left(\frac{\omega - \alpha}{B_T} \right) + W \left(\frac{\omega + \alpha}{B_T} \right) \right] \left\langle S_{n,j}^d, \overline{S_{h,l}^d} \right\rangle_{L^2(\mathbb{M}_d, d\nu, \mathbb{C})} \\
 &\quad \times \left\langle \overline{S_{n,j}^d}, S_{h,l}^d \right\rangle_{L^2(\mathbb{M}_d, d\nu, \mathbb{C})} \left[f_n(\alpha) f_h(\alpha) \frac{d\alpha}{B_T} + O(B_T^{-2} T^{-2}) + O(T^{-1}) \right] \\
 &\leq 2\pi \int_{-\pi}^{\pi} W \left(\frac{\omega - \alpha}{B_T} \right) \left[W \left(\frac{\omega - \alpha}{B_T} \right) + W \left(\frac{\omega + \alpha}{B_T} \right) \right] f_n(\alpha) f_h(\alpha) \frac{d\alpha}{B_T} \\
 &\quad + O(B_T^{-2} T^{-2}) + O(T^{-1}) \leq \mathcal{N}_1 \left[\sum_{\tau \in \mathbb{Z}} \|\mathcal{R}_{\tau}\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \right]^2 + \varepsilon(T) < \infty, \tag{49}
 \end{aligned}$$

under SRD, and for certain $\mathcal{N}_1 > 0$, and $\varepsilon(T) > 0$, with $\varepsilon(T) \rightarrow 0$, as $T \rightarrow \infty$. As before, $\widehat{\mathcal{F}}_{\omega}^{(T)}$ denotes the weighted periodogram operator with kernel $\widehat{f}_{\omega}^{(T)}$. In equation (49), we have considered T sufficiently large to apply the identity $W(x) = 1/B_T W(x/B_T)$, for $B_T < 1$, and $x \in [-\pi, \pi]$ (see Lemma F11 in the Supplementary Material of [28]). Thus, under H_0 , assuming the conditions in Lemma 2.1, the sequence $\sqrt{B_T T} (\widehat{\mathcal{F}}_{\omega}^{(T)} - E[\widehat{\mathcal{F}}_{\omega}^{(T)}])$ is tight. Hence, the convergence, as $T \rightarrow \infty$, of $\sqrt{B_T T} (\widehat{\mathcal{F}}_{\omega}^{(T)} - E[\widehat{\mathcal{F}}_{\omega}^{(T)}])$ to the Gaussian random operator $\widehat{\mathcal{F}}_{\omega}$ with kernel \widehat{f}_{ω} (see equation (45)), in the norm of the space $\mathcal{L}_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2(\Omega, \mathcal{A}, P)$, follows from Prokhorov Theorem. Here, $\mathcal{L}_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2(\Omega, \mathcal{A}, P)$ denotes the space of zero–mean second–order $\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))$ –valued random variables with the norm $\sqrt{E\|\cdot\|^2}_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))}$.

Let us consider

$$\begin{aligned}
 &C_{n,j,h,l}(\omega, \xi) \\
 &= \text{Cov} \left(\sqrt{TB_T} \widehat{\mathcal{F}}_{\omega}^{(T)} - \widehat{\mathcal{F}}_0, \sqrt{TB_T} \widehat{\mathcal{F}}_{\xi}^{(T)} - \widehat{\mathcal{F}}_0 \right) \left(S_{n,j}^d \otimes \overline{S_{h,l}^d} \right) \left(S_{n,j}^d \otimes \overline{S_{h,l}^d} \right),
 \end{aligned}$$

for $j = 1, \dots, \Gamma(n, d)$, and $l = 1, \dots, \Gamma(h, d)$, $n, h \in \mathbb{N}_0$, where for a bounded linear operator \mathcal{A} on a separable Hilbert space H , $\mathcal{A}(\varphi)(\phi) = \langle \mathcal{A}(\varphi), \phi \rangle_H$, for every $\varphi, \phi \in \text{Dom}(\mathcal{A})$. Here, as before, $\widehat{\mathcal{F}}_0$ is the Gaussian random element with random kernel \widehat{f}_0 introduced in equation (45) for $\omega_j = 0$. Then, applying Cauchy–Schwartz inequality in $\mathcal{L}^2(\Omega, \mathcal{A}, P)$, the space of complex–valued zero–mean second–order random variables on (Ω, \mathcal{A}, P) , and Jensen’s

inequality, we obtain

$$\begin{aligned}
 & E \left\| (\mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}]) - \widehat{\mathcal{F}}_0 \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\
 &= \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \int_{[-\pi, \pi]^2} \delta_T(0 - \omega) \delta_T(0 - \xi) C_{n, j, h, l}(\omega, \xi) d\omega d\xi \\
 &\leq \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \int_{[-\pi, \pi]^2} \delta_T(0 - \omega) \delta_T(0 - \xi) \sqrt{C_{n, j, h, l}(\omega, \omega) C_{n, j, h, l}(\xi, \xi)} d\omega d\xi \\
 &\leq \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \left[\int_{[-\pi, \pi]^2} \delta_T(0 - \omega) \delta_T(0 - \xi) C_{n, j, h, l}(\omega, \omega) C_{n, j, h, l}(\xi, \xi) d\omega d\xi \right]^{1/2} \\
 &= \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \int_{[-\pi, \pi]} \delta_T(0 - \omega) C_{n, j, h, l}(\omega, \omega) d\omega \\
 &= \int_{[-\pi, \pi]} \delta_T(0 - \omega) E \left\| \sqrt{TB_T} \left(\widehat{\mathcal{F}}_\omega^{(T)} - E \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right) - \widehat{\mathcal{F}}_0 \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 d\omega \\
 &\leq \mathcal{N}_2 \left[\left(\sum_{\tau \in \mathbb{Z}} \|\mathcal{R}_\tau\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \right)^2 + \left(\sum_{\tau \in \mathbb{Z}} \|\mathcal{R}_\tau\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \right)^4 \right] < \infty,
 \end{aligned} \tag{50}$$

under H_0 , for certain positive constant \mathcal{N}_2 , where the last inequality follows from equations (47) and (48), for T sufficiently large.

Applying Dominated Convergence Theorem, we then obtain

$$\begin{aligned}
 & \lim_{T \rightarrow \infty} E \left\| (\mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}]) - \widehat{\mathcal{F}}_0 \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\
 &= \lim_{T \rightarrow \infty} \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \int_{[-\pi, \pi]^2} \delta_T(0 - \omega) \delta_T(0 - \xi) \\
 &\times \text{Cov} \left(\sqrt{TB_T} \widehat{\mathcal{F}}_\omega^{(T)} - \widehat{\mathcal{F}}_0, \sqrt{TB_T} \widehat{\mathcal{F}}_\xi^{(T)} - \widehat{\mathcal{F}}_0 \right) \left(S_{n, j}^d \otimes \overline{S_{h, l}^d} \right) \left(S_{n, j}^d \otimes \overline{S_{h, l}^d} \right) d\omega d\xi \\
 &= \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \lim_{T \rightarrow \infty} \int_{[-\pi, \pi]^2} \delta_T(0 - \omega) \delta_T(0 - \xi) \\
 &\times \text{Cov} \left(\sqrt{TB_T} \widehat{\mathcal{F}}_\omega^{(T)} - \widehat{\mathcal{F}}_0, \sqrt{TB_T} \widehat{\mathcal{F}}_\xi^{(T)} - \widehat{\mathcal{F}}_0 \right) \left(S_{n, j}^d \otimes \overline{S_{h, l}^d} \right) \left(S_{n, j}^d \otimes \overline{S_{h, l}^d} \right) d\omega d\xi = 0,
 \end{aligned} \tag{51}$$

in view of the convergence in $\mathcal{L}_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2(\Omega, \mathcal{A}, P)$ of $\sqrt{TB_T} \left(\widehat{\mathcal{F}}_0^{(T)} - E \left[\widehat{\mathcal{F}}_0^{(T)} \right] \right)$ to $\widehat{\mathcal{F}}_0$. Thus, the convergence in distribution of $\mathcal{S}_{B_T} - E[\mathcal{S}_{B_T}]$ to $\widehat{\mathcal{F}}_0$ holds. \square

Appendix B: Proofs of the results in Section 3

B.1. Proof of Lemma 3.1

Proof. Let us consider

$$\begin{aligned} & \left\| \int_{-\pi}^{\pi} \left[\mathcal{F}_{\omega} - \mathcal{F}_{\omega}^{(T)} \right] d\omega \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\ &= \sum_{n \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n,d)} \int_{[-\pi, \pi]^2} \left[f_n(\omega) - f_n^{(T)}(\omega) \right] \overline{f_n(\xi)} d\xi d\omega \\ &+ \int_{[-\pi, \pi]^2} \left[f_n^{(T)}(\omega) - f_n(\omega) \right] \overline{f_n^{(T)}(\xi)} d\xi d\omega, \end{aligned} \tag{52}$$

where the sequence of functions

$$f_n^{(T)}(\omega) = \int_{-\pi}^{\pi} F_T(\omega - \xi) f_n(\xi) d\xi, \quad \forall \omega \in [-\pi, \pi], n \in \mathbb{N}_0,$$

defines the frequency-varying pure point spectra of the operator family

$$\left\{ \mathcal{F}_{\omega}^{(T)} = E_{H_1} \left[\mathcal{P}_{\omega}^{(T)} \right], \omega \in [-\pi, \pi] \right\},$$

for every $T \geq 2$, with $\mathcal{P}_{\omega}^{(T)}$ denoting the periodogram operator (see equation (4)), and E_{H_1} denoting the expectation under the alternative H_1 .

Under H_1 , for every $n \in \mathbb{N}_0$, $f_n(\cdot) \in L^1([-\pi, \pi])$. Applying well-known properties of Féjer kernel, we then obtain, as $T \rightarrow \infty$,

$$f_n^{(T)}(\omega) \rightarrow f_n(\omega), \quad \forall \omega \in [-\pi, \pi] \setminus \Lambda_0, \text{ with } \int_{\Lambda_0} d\omega = 0. \tag{53}$$

From Young convolution inequality in $L^2([-\pi, \pi])$, for each $n \in \mathbb{N}_0$, and $T \geq 2$,

$$\int_{[-\pi, \pi]} \left| f_n^{(T)}(\omega) \right|^2 d\omega \leq \int_{[-\pi, \pi]} |f_n(\omega)|^2 d\omega < \infty, \tag{54}$$

under H_1 , since $l_{\alpha}, L_{\alpha} \in (0, 1/2)$. Thus, $f_n^{(T)} \in L^2([-\pi, \pi])$, for every $n \in \mathbb{N}_0$, and T .

To apply Dominated Convergence Theorem in (52), the following additional inequalities are considered, obtained from triangle inequality, Young convolution inequality for functions in $L^1([-\pi, \pi])$, and Jensen’s inequality, keeping in mind that $f_n(\omega) \geq 0$, a.s. in $\omega \in [-\pi, \pi]$, for every $n \in \mathbb{N}_0$,

$$\begin{aligned} & \left| \int_{[-\pi, \pi]^2} \overline{f_n^{(T)}(\xi)} f_n^{(T)}(\omega) d\xi d\omega \right| \leq \int_{[-\pi, \pi]^2} \left| \overline{f_n^{(T)}(\xi)} f_n^{(T)}(\omega) \right| d\xi d\omega \\ & \leq \int_{[-\pi, \pi]^2} |f_n(\xi) f_n(\omega)| d\xi d\omega = \left[\int_{[-\pi, \pi]} f_n(\omega) d\omega \right]^2 \leq \int_{[-\pi, \pi]} |f_n(\omega)|^2 d\omega. \end{aligned} \tag{55}$$

Also, in a similar way,

$$\left| \int_{[-\pi, \pi]^2} \overline{f_n^{(T)}(\xi)} f_n(\omega) d\xi d\omega \right| \leq \int_{[-\pi, \pi]} |f_n(\omega)|^2 d\omega$$

$$\begin{aligned} \left| \int_{[-\pi, \pi]^2} \overline{f_n(\xi)} f_n^{(T)}(\omega) d\xi d\omega \right| &\leq \int_{[-\pi, \pi]} |f_n(\omega)|^2 d\omega \\ \left| \int_{[-\pi, \pi]^2} f_n(\xi) f_n(\omega) d\xi d\omega \right| &\leq \int_{[-\pi, \pi]} |f_n(\omega)|^2 d\omega. \end{aligned} \tag{56}$$

Under H_1 , from equation (16),

$$\sum_{n \in \mathbb{N}_0} \Gamma(n, d) \int_{[-\pi, \pi]} |f_n(\omega)|^2 d\omega = \int_{[-\pi, \pi]} \|\mathcal{F}_\omega\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 d\omega < \infty. \tag{57}$$

From equations (53)–(57), one can apply Dominated Convergence Theorem in equation (52), obtaining

$$\begin{aligned} \lim_{T \rightarrow \infty} \left\| \int_{-\pi}^{\pi} \left[\mathcal{F}_\omega - \mathcal{F}_\omega^{(T)} \right] d\omega \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\ = \sum_{n \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \int_{[-\pi, \pi]^2} \lim_{T \rightarrow \infty} \overline{f_n^{(T)}(\xi)} \left[f_n^{(T)}(\omega) - f_n(\omega) \right] d\xi d\omega \\ + \int_{[-\pi, \pi]^2} \lim_{T \rightarrow \infty} \overline{f_n(\xi)} \left[f_n(\omega) - f_n^{(T)}(\omega) \right] d\xi d\omega = 0. \end{aligned} \tag{58}$$

The rate of convergence to zero of the bias is now obtained in the time domain. Let B_n be defined as

$$B_n(t) = \int_{-\pi}^{\pi} \exp(it\omega) f_n(\omega) d\omega, \quad t \in \mathbb{Z}, n \in \mathbb{N}_0. \tag{59}$$

The function sequence

$$\left\{ \mathbb{I}_{[-(T-1), T-1]}(t) \frac{T - |t|}{T} B_n(t), t \in \mathbb{Z}, n \in \mathbb{N}_0 \right\}_{T \geq 2}$$

pointwise converges, as $T \rightarrow \infty$, to $B_n(t)$ with rate of convergence T^{-1} , and satisfies, for every $T \geq 2$,

$$\left| \mathbb{I}_{[-(T-1), T-1]}(t) \frac{T - |t|}{T} B_n(t) \right|^2 \leq |B_n(t)|^2. \tag{60}$$

From (57) and Parseval identity (see equation (59)),

$$\begin{aligned} \sum_{t \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \Gamma(n, d) |B_n(t)|^2 &= \sum_{t \in \mathbb{Z}} \|\mathcal{R}_t\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\ &= \int_{-\pi}^{\pi} \|\mathcal{F}_\omega\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 d\omega < \infty. \end{aligned} \tag{61}$$

From equations (60) and (61), Dominated Convergence Theorem then leads to

$$\begin{aligned} \lim_{T \rightarrow \infty} \sum_{t \in \mathbb{Z}} \left\| \mathcal{R}_t - \mathbb{I}_{[-(T-1), T-1]}(t) \frac{T - |t|}{T} \mathcal{R}_t \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\ = \sum_{t \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \Gamma(n, d) \lim_{T \rightarrow \infty} \left| B_n(t) - \mathbb{I}_{[-(T-1), T-1]}(t) \frac{T - |t|}{T} B_n(t) \right|^2 = 0, \end{aligned} \tag{62}$$

and $\sum_{t \in \mathbb{Z}} \left\| \mathcal{R}_t - \mathbb{I}_{[-(T-1), T-1]}(t) \frac{T-|t|}{T} \mathcal{R}_t \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 = O(T^{-2})$. Hence, the desired result follows from Parseval identity. \square

B.2. Proof of Corollary 3.2

Proof. Applying Lemma 3.1, and Lemmas F10 and F12 of Appendix F in the Supplementary Material of [28],

$$\begin{aligned} & \int_{-\pi}^{\pi} E_{H_1} [\widehat{\mathcal{F}}_{\omega}^{(T)}] d\omega \\ &= S(L^2(\mathbb{M}_d, d\nu, \mathbb{C})) \int_{\mathbb{R}} W(\xi) \int_{-\pi}^{\pi} [\mathcal{F}_{\omega-\xi B_T} + O(T^{-1})] d\omega d\xi + O(B_T^{-1} T^{-1}) \\ &= S(L^2(\mathbb{M}_d, d\nu, \mathbb{C})) \int_{-\pi}^{\pi} \int_{\mathbb{R}} W(\xi) \mathcal{F}_{\omega-\xi B_T} d\xi d\omega + O(T^{-1}) + O(B_T^{-1} T^{-1}), \end{aligned} \tag{63}$$

as we wanted to prove. \square

B.3. Proof of Lemma 3.3

Proof. Under Assumption I, there exists an orthonormal basis $\{\phi_n, n \in \mathbb{N}\}$ of

$$L^2 \left(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{R} \right)$$

such that (see [17])

$$\begin{aligned} & \int_{\mathbb{M}_d} \text{cum} (X_{u_1}, X_{u_2}, X_{u_3}, X_0) (\tau_1, \tau_2, \tau_3, \tau_4) \phi_n(\tau_3, \tau_4) \nu(d\tau_3) \nu(d\tau_4) \\ &= B_n(u_1, u_2, u_3) \phi_n(\tau_1, \tau_2), \quad \forall (\tau_1, \tau_2) \in \mathbb{M}_d \times \mathbb{M}_d, u_1, u_2, u_3 \in \mathbb{Z}, n \geq 1. \end{aligned} \tag{64}$$

Furthermore,

$$\begin{aligned} & \int_{[-\pi, \pi]^3} \text{cum} \left(\widetilde{X}_{\omega_1}^{(T)}(\tau_1), \widetilde{X}_{\omega_2}^{(T)}(\tau_2), \widetilde{X}_{\omega_3}^{(T)}(\tau_3), \widetilde{X}_{\omega_4}^{(T)}(\tau_4) \right) d\omega_1 d\omega_2 d\omega_3 \\ &= S(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C})) \frac{1}{(2\pi T)^2} \int_{[-\pi, \pi]^3} \sum_{t_1, t_2, t_3, t_4=0}^{T-1} \exp \left(-i \sum_{j=1}^3 (t_j - t_4) \omega_j \right) \\ & \times \exp \left(-it_4 \sum_{j=1}^4 \omega_j \right) \text{cum} (X_{t_1-t_4}(\tau_1), X_{t_2-t_4}(\tau_2), X_{t_3-t_4}(\tau_3), X_0(\tau_4)) \prod_{j=1}^3 d\omega_j \\ &= S(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C})) \int_{[-\pi, \pi]^3} \frac{1}{(2\pi T)^2} \sum_{u_1, u_2, u_3=-(T-1)}^{T-1} \exp \left(-i \sum_{j=1}^3 u_j \omega_j \right) \\ & \times \text{cum} (X_{u_1}(\tau_1), X_{u_2}(\tau_2), X_{u_3}(\tau_3), X_0(\tau_4)) \\ & \times \sum_{t \in \mathbb{Z}} h^{(T)}(u_1+t) h^{(T)}(u_2+t) h^{(T)}(u_3+t) h^{(T)}(t) \exp \left(-it \left(\sum_{j=1}^4 \omega_j \right) \right) \prod_{j=1}^3 d\omega_j, \end{aligned} \tag{65}$$

with $h(t) = 1, 0 \leq t \leq T$, and $h(t) = 0$, otherwise. In (65), we have considered the change of variable $u_j = t_j - t_4, j = 1, 2, 3$, and $t = t_4$. Denote, for every $n \geq 1$, and $(\omega_1, \omega_2, \omega_3) \in [-\pi, \pi]^3$,

$$f_n(\omega_1, \omega_2, \omega_3) = \frac{1}{(2\pi)^3} \sum_{u_1, u_2, u_3 \in \mathbb{Z}} \exp\left(-i \sum_{j=1}^3 \omega_j u_j\right) B_n(u_1, u_2, u_3),$$

where, for $u_1, u_2, u_3 \in \mathbb{Z}, \{B_n(u_1, u_2, u_3), n \geq 1\}$ satisfies (64). From equations (64)–(65), applying Fourier transform inversion formula, for each $n \geq 1$,

$$\begin{aligned} & T \int_{[-\pi, \pi]^3 \times \mathbb{M}_d^4} \text{cum}\left(\tilde{X}_{\omega_1}^{(T)}(\tau_1), \tilde{X}_{\omega_2}^{(T)}(\tau_2), \tilde{X}_{\omega_3}^{(T)}(\tau_3), \tilde{X}_{\omega_4}^{(T)}(\tau_4)\right) \\ & \quad \times \phi_n(\tau_1, \tau_2) \phi_n(\tau_3, \tau_4) \prod_{j=1}^4 d\tau_j \prod_{i=1}^3 d\omega_i \\ &= \frac{(2\pi)^3}{(2\pi)^{2T}} \int_{[-\pi, \pi]^6} \sum_{u_1, u_2, u_3 = -(T-1)}^{T-1} \exp\left(-i \sum_{j=1}^3 u_j(\omega_j - \xi_j)\right) \\ & \quad \times \sum_{t \in \mathbb{Z}} h^{(T)}(u_1 + t) h^{(T)}(u_2 + t) h^{(T)}(u_3 + t) h^{(T)}(t) \exp\left(-it \left(\sum_{j=1}^4 \omega_j\right)\right) \\ & \quad \times f_n(\xi_1, \xi_2, \xi_3) \prod_{j=1}^3 d\xi_j \prod_{i=1}^3 d\omega_i \\ &= \frac{2\pi}{T} \int_{[-\pi, \pi]^6} \left[\sum_{u_1, u_2, u_3 = -(T-1)}^{T-1} \exp\left(-i \sum_{j=1}^3 u_j(\omega_j - \xi_j)\right) \right. \\ & \quad \left. \times \sum_{t \in \mathbb{Z}} \exp\left(-it \left(\sum_{j=1}^4 \omega_j\right)\right) h\left(t + \max_{j=1,2,3} |u_j|\right) \right] f_n(\xi_1, \xi_2, \xi_3) \prod_{j=1}^3 d\xi_j \prod_{i=1}^3 d\omega_i. \quad (66) \end{aligned}$$

As $T \rightarrow \infty$, uniformly in $\omega_4 \in [-\pi, \pi]$,

$$\begin{aligned} & \frac{1}{T} \left[\sum_{u_1, u_2, u_3 = -(T-1)}^{T-1} \exp\left(-i \sum_{j=1}^3 u_j(\omega_j - \xi_j)\right) \right. \\ & \quad \left. \times \sum_{t \in \mathbb{Z}} \exp\left(-it \left(\sum_{j=1}^4 \omega_j\right)\right) h\left(t + \max_{j=1,2,3} |u_j|\right) \right] \rightarrow \delta(\omega - \xi), \quad (67) \end{aligned}$$

where $\delta(\omega - \xi) = \prod_{j=1}^3 \delta(\omega_j - \xi_j)$ denotes the Dirac Delta distribution, defining the kernel of the identity operator on $L^2([-\pi, \pi]^3)$. Using the notation

$$\begin{aligned} \delta_T(\omega - \xi) &:= \frac{1}{T} \left[\sum_{u_1, u_2, u_3 = -(T-1)}^{T-1} \exp\left(-i \sum_{j=1}^3 u_j(\omega_j - \xi_j)\right) \right. \\ & \quad \left. \times \sum_{t \in \mathbb{Z}} \exp\left(-it \left(\sum_{j=1}^4 \omega_j\right)\right) h\left(t + \max_{j=1,2,3} |u_j|\right) \right], \quad (68) \end{aligned}$$

equation (66) can be rewritten as

$$\begin{aligned}
 & T \int_{[-\pi, \pi]^3 \times \mathbb{M}_d^4} \text{cum} \left(\tilde{X}_{\omega_1}^{(T)}(\tau_1), \tilde{X}_{\omega_2}^{(T)}(\tau_2), \tilde{X}_{\omega_3}^{(T)}(\tau_3), \tilde{X}_{\omega_4}^{(T)}(\tau_4) \right) \\
 & \quad \times \phi_n(\tau_1, \tau_2) \phi_n(\tau_3, \tau_4) \prod_{j=1}^4 d\tau_j \prod_{i=1}^3 d\omega_i \\
 & = 2\pi \int_{[-\pi, \pi]^6} \delta_T(\omega - \xi) f_n(\xi) d\xi d\omega, \quad n \geq 1.
 \end{aligned} \tag{69}$$

Note that, for $T \geq T_0$, with T_0 sufficiently large,

$$|\delta_T(\omega - \xi) f_n(\xi)| \leq |f_n(\xi)|, \quad \omega \neq \xi, \tag{70}$$

since $\delta_T(\omega - \xi) \rightarrow 0$, $T \rightarrow \infty$, for every $(\omega, \xi) \in [-\pi, \pi]^6 \setminus \Lambda$, with $\Lambda = \{(\omega, \xi) \in [-\pi, \pi]^6; \omega = \xi\} \subset [-\pi, \pi]^6$. Under Assumption I, applying Parseval identity,

$$\begin{aligned}
 & \sum_{n \geq 1} \int_{[-\pi, \pi]^6} |f_n(\omega_1, \omega_2, \omega_3)| \prod_{j=1}^6 d\omega_j \\
 & \leq (2\pi)^3 \sum_{n \geq 1} \int_{[-\pi, \pi]^3} |f_n(\omega_1, \omega_2, \omega_3)|^2 \prod_{j=1}^3 d\omega_j \\
 & = (2\pi)^3 \int_{[-\pi, \pi]^3} \|\mathcal{F}_{\omega_1, \omega_2, \omega_3}\|_{\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C}))}^2 \prod_{j=1}^3 d\omega_j \\
 & = (2\pi)^3 \sum_{t_1, t_2, t_3 \in \mathbb{Z}} \|\text{cum}(X_{t_1}, X_{t_2}, X_{t_3}, X_0)\|_{L^2(\mathbb{M}_d^4, \otimes_{i=1}^4 \nu(x_i), \mathbb{R})}^2 < \infty.
 \end{aligned} \tag{71}$$

Hence, from (70)–(71), applying Dominated Convergence Theorem,

$$\begin{aligned}
 & \lim_{T \rightarrow \infty} \int_{[-\pi, \pi]^6} \delta_T(\omega - \xi) f_n(\xi) d\xi d\omega \\
 & = \int_{[-\pi, \pi]^6} \lim_{T \rightarrow \infty} \delta_T(\omega - \xi) f_n(\xi) d\xi d\omega \\
 & = \int_{[-\pi, \pi]^3} f_n(\omega) d\omega, \quad n \geq 1.
 \end{aligned} \tag{72}$$

and, as $T \rightarrow \infty$,

$$\left| \int_{[-\pi, \pi]^6} \delta_T(\omega - \xi) f_n(\xi) d\xi d\omega - \int_{[-\pi, \pi]^3} f_n(\omega) d\omega \right| = O(T^{-1}). \tag{73}$$

Therefore, from (69), (71) and (73), uniformly in $\omega_4 \in [-\pi, \pi]$,

$$\begin{aligned}
 & \lim_{T \rightarrow \infty} \sum_n \left| T \int_{[-\pi, \pi]^3 \times \mathbb{M}_d^4} \text{cum} \left(\tilde{X}_{\omega_1}^{(T)}(\tau_1), \tilde{X}_{\omega_2}^{(T)}(\tau_2), \tilde{X}_{\omega_3}^{(T)}(\tau_3), \tilde{X}_{\omega_4}^{(T)}(\tau_4) \right) \right. \\
 & \quad \times \phi_n(\tau_1, \tau_2) \phi_n(\tau_3, \tau_4) \prod_{j=1}^4 d\tau_j - 2\pi f_n(\omega_1, \omega_2, \omega_3) \prod_{i=1}^3 d\omega_i \left. \right| = 0.
 \end{aligned}$$

It then follows that, as $T \rightarrow \infty$, the norm

$$\left\| \int_{[-\pi, \pi]^3} \left[T \text{cum} \left(\tilde{X}_{\omega_1}^{(T)}, \tilde{X}_{\omega_2}^{(T)}, \tilde{X}_{\omega_3}^{(T)}, \tilde{X}_{\omega_4}^{(T)} \right) - 2\pi \mathcal{F}_{\omega_1, \omega_2, \omega_3} \right] \prod_{j=1}^3 d\omega_j \right\|_{\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C}))} \quad (74)$$

goes to zero, with

$$\begin{aligned} T \int_{[-\pi, \pi]^3} \text{cum} \left(\tilde{X}_{\omega_1}^{(T)}, \tilde{X}_{\omega_2}^{(T)}, \tilde{X}_{\omega_3}^{(T)}, \tilde{X}_{\omega_4}^{(T)} \right) \prod_{i=1}^3 d\omega_i \\ = 2\pi \int_{[-\pi, \pi]^3} \mathcal{F}_{\omega_1, \omega_2, \omega_3} \prod_{j=1}^3 d\omega_j + O(T^{-1}), \end{aligned}$$

in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d^2, \otimes_{i=1}^2 \nu(dx_i), \mathbb{C}))$, where $\mathcal{F}_{\omega_1, \omega_2, \omega_3}$ denotes the cumulant spectral density operator of order 4 of X under H_1 , introduced in equation (30). \square

Appendix C: Proofs of the results in Section 4

C.1. Proof of Proposition 4.1

Proof. Under H_1 , we have $0 < l_\alpha \leq \alpha(n) \leq L_\alpha < 1/2$, for every $n \in \mathbb{N}_0$. From Lemma 3.1, considering T sufficiently large,

$$\begin{aligned} & \left| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} [\hat{f}_n^{(T)}(\omega)] \frac{d\omega}{\sqrt{B_T}} \right| \\ &= \int_{-\pi}^{\pi} \frac{1}{B_T} \left[\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} W \left(\frac{\omega - \alpha}{B_T} \right) \frac{d\omega}{\sqrt{B_T}} \right] f_n^{(T)}(\alpha) d\alpha \\ &+ O(B_T^{-1} T^{-1}) \\ &\approx \frac{1}{\sqrt{B_T}} \int_{-\pi}^{\pi} \frac{1}{\sqrt{B_T}} W \left(\frac{-\alpha}{B_T} \right) f_n(\alpha) d\alpha + O(B_T^{-1} T^{-1}) + O(T^{-1}) \\ &\geq g(T) = O(B_T^{-1/2 - l_\alpha}), \quad \forall n \in \mathbb{N}_0, \end{aligned} \quad (75)$$

where $\{\hat{f}_n^{(T)}(\omega), n \in \mathbb{N}_0\}$ and $\{f_n^{(T)}(\omega), n \in \mathbb{N}_0\}$ respectively denote the frequency varying eigenvalues of the weighted periodogram operator $\hat{\mathcal{F}}_\omega^{(T)}$ and the mean operator $\mathcal{F}_\omega^{(T)} = E[\mathcal{P}_\omega^{(T)}]$. Here, $a_T \simeq b_T$ means that the two sequences $\{a_T, T > 0\}$ and $\{b_T, T > 0\}$ have the same limit as $T \rightarrow \infty$. From (75),

$$\begin{aligned} & \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} [\hat{\mathcal{F}}_\omega^{(T)}] \frac{d\omega}{\sqrt{B_T}} \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\ &\geq \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} [\hat{\mathcal{F}}_\omega^{(T)}] \frac{d\omega}{\sqrt{B_T}} \right\|_{\mathcal{L}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\ &= \sup_{n \in \mathbb{N}_0} \left| \int_{-\pi}^{\pi} \frac{1}{B_T} \left[\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} W \left(\frac{\omega - \alpha}{B_T} \right) \frac{d\omega}{\sqrt{B_T}} \right] f_n^{(T)}(\alpha) d\alpha + O(B_T^{-1} T^{-1}) \right| \end{aligned}$$

$$\geq g(T) = O(B_T^{-1/2-l\alpha}), \quad T \rightarrow \infty, \tag{76}$$

where $\mathcal{L}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$ denotes the space of bounded linear operators on $L^2(\mathbb{M}_d, d\nu, \mathbb{C})$. □

C.2. Proof of Theorem 4.2

Proof. From Lemmas 3.1 and 3.3, applying trace formula, for T sufficiently large,

$$\begin{aligned} & \int_{-\pi}^{\pi} E_{H_1} \left\| \widehat{\mathcal{F}}_{\omega}^{(T)} - E_{H_1} [\widehat{\mathcal{F}}_{\omega}^{(T)}] \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 d\omega \\ & \leq \frac{2\pi}{TB_T} \int_{-\pi}^{\pi} \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \left| \int_{-\pi}^{\pi} \frac{1}{\sqrt{B_T}} W\left(\frac{\omega - \alpha}{B_T}\right) \frac{1}{\sqrt{B_T}} W\left(\frac{\omega - \alpha}{B_T}\right) f_n(\alpha) f_h(\alpha) d\alpha \right| d\omega \\ & + \frac{2\pi}{TB_T} \int_{-\pi}^{\pi} \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \left| \int_{-\pi}^{\pi} \frac{1}{\sqrt{B_T}} W\left(\frac{\omega - \alpha}{B_T}\right) \frac{1}{\sqrt{B_T}} W\left(\frac{\omega + \alpha}{B_T}\right) f_n(\alpha) f_h(\alpha) d\alpha \right| d\omega \\ & + O(B_T^{-2}T^{-2}) + O(T^{-1}) \\ & \simeq \frac{2\pi}{TB_T} \int_{-\pi}^{\pi} \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \left| \int_{-\pi}^{\pi} \delta_T(\omega - \alpha) f_n(\alpha) f_h(\alpha) d\alpha \right| d\omega \\ & + \frac{2\pi}{TB_T} \int_{-\pi}^{\pi} \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \left| \int_{-\pi}^{\pi} \delta_T(\omega - \alpha) f_n(\alpha) f_h(\alpha) d\alpha \right| d\omega \\ & + O(B_T^{-2}T^{-2}) + O(T^{-1}) \\ & \simeq \frac{2\pi}{TB_T} \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} 2 \int_{[-\pi, \pi]} f_n(\omega) f_h(\omega) d\omega + O(B_T^{-2}T^{-2}) + O(T^{-1}) \\ & = h(T) = O(B_T^{-1}T^{-1}), \quad T \rightarrow \infty, \tag{77} \end{aligned}$$

where, as before, $a_T \simeq b_T$ means that the two sequences $\{a_T, T > 0\}$ and $\{b_T, T > 0\}$ have the same limit as $T \rightarrow \infty$.

Note that, under H_1 , equation (77) follows from condition

$$\int_{[-\pi, \pi]} \|\mathcal{M}_{\omega}\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 |\omega|^{-2L\alpha} d\omega < \infty,$$

since

$$\begin{aligned} & \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \int_{[-\pi, \pi]} f_n(\omega) f_h(\omega) d\omega \\ & \leq \sum_{n, h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n, d)} \sum_{l=1}^{\Gamma(h, d)} \int_{[-\pi, \pi]} M_n(\omega) M_h(\omega) |\omega|^{-2L\alpha} d\omega \\ & = \int_{[-\pi, \pi]} \|\mathcal{M}_{\omega}\|_{L^1(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 |\omega|^{-2L\alpha} d\omega = O(1). \end{aligned}$$

□

C.3. Proof of Corollary 4.4

Proof. Applying triangle and Jensen inequalities, we obtain from Corollary 3.2 and Theorem 4.2,

$$\begin{aligned}
 & \left\| \int_{-\pi}^{\pi} E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} - \int_{-\pi}^{\pi} W(\xi) \mathcal{F}_{\omega-B_T\xi} d\xi \right] d\omega \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\
 & \leq \left\| \int_{-\pi}^{\pi} E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \right] d\omega \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\
 & \quad + \left\| \int_{-\pi}^{\pi} \left[E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] - \int_{-\pi}^{\pi} W(\xi) \mathcal{F}_{\omega-B_T\xi} d\xi \right] d\omega \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\
 & \leq \left[\int_{-\pi}^{\pi} E_{H_1} \left\| \widehat{\mathcal{F}}_{\omega}^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 d\omega \right]^{1/2} \\
 & \quad + \left\| \int_{-\pi}^{\pi} \left[E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] - \int_{-\pi}^{\pi} W(\xi) \mathcal{F}_{\omega-B_T\xi} d\xi \right] d\omega \right\|_{\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\
 & = O(T^{-1/2} B_T^{-1/2}), \quad T \rightarrow \infty.
 \end{aligned} \tag{78}$$

□

C.4. Proof of Theorem 4.5

Proof. The proof of this result shares some ideas with the proof of Theorem 2 of [16], formulated in the time domain for real-valued time series. Specifically, the test statistic operator \mathcal{S}_{B_T} is rewritten as

$$\begin{aligned}
 \mathcal{S}_{B_T} &= \sqrt{B_T T} \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \\
 & \circ \left[\mathbb{I}_{L^2(\mathbb{M}_d, d\nu, \mathbb{C})} + \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \left(\widehat{\mathcal{F}}_{\omega}^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \right) \frac{d\omega}{\sqrt{B_T}} \right] \\
 & \circ \left[\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1},
 \end{aligned} \tag{79}$$

where \circ means the composition of operators, $\mathbb{I}_{L^2(\mathbb{M}_d, d\nu, \mathbb{C})}$ denotes the identity operator on the space $L^2(\mathbb{M}_d, d\nu, \mathbb{C})$, and $\left[\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1}$ is the inverse of operator $\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \frac{d\omega}{\sqrt{B_T}}$.

Our strategy in the proof of this result consists of first proving, under H_1 , the divergence, in the norm of the space $\mathcal{S}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$, of operator

$$\sqrt{B_T T} \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_{\omega}^{(T)} \right] \frac{d\omega}{\sqrt{B_T}}.$$

Then, under the conditions of Theorem 4.2, we derive the convergence to zero, as $T \rightarrow \infty$, of

random operator

$$\left[\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right] \circ \left[\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1}, \tag{80}$$

in the space $\mathcal{L}^2_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}(\Omega, \mathcal{A}, P)$, which holds with a suitable rate under $l_\alpha > 1/4$ and $B_T = T^{-\beta}$, $\beta \in (0, 1)$, allowing the application of Borell Cantelli Lemma to ensure almost surely convergence. Specifically, from Proposition 4.1, as $T \rightarrow \infty$,

$$\begin{aligned} & \left\| \sqrt{B_T T} \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \\ & \geq g(T) = O\left(T^{1/2} B_T^{-l_\alpha}\right). \end{aligned} \tag{81}$$

For the random operator in (80), the following inequality holds:

$$\begin{aligned} & E_{H_1} \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right. \\ & \quad \left. \circ \left[\int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\ & \leq \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right\|_{\mathcal{L}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^{-1} \right\|^2 \\ & \quad \times E_{H_1} \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \end{aligned} \tag{82}$$

From equation (76), as $T \rightarrow \infty$,

$$\left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right\|_{\mathcal{L}(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^{-1} \right\|^2 \leq b(T) = O\left(B_T^{2l_\alpha+1}\right), \tag{83}$$

and, from Theorem 4.2,

$$\begin{aligned} & \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \text{Var}_{H_1} \left(\widehat{\mathcal{F}}_\omega^{(T)} \right) \frac{d\omega}{\sqrt{B_T}} \\ & \leq \int_{[-\pi, \pi]} \text{Var}_{H_1} \left(\widehat{\mathcal{F}}_\omega^{(T)} \right) \frac{d\omega}{\sqrt{B_T}} \leq u(T) = O\left(T^{-1} B_T^{-1-1/2}\right), \quad T \rightarrow \infty. \end{aligned} \tag{84}$$

For each $T \geq 2$, applying Jensen inequality, in terms of the uniform probability measure on the interval $[-\sqrt{B_T}/2, \sqrt{B_T}/2]$,

$$\begin{aligned} & E_{H_1} \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\ & = \left\| \int_{[-\sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right\|_{\mathcal{L}^2_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}(\Omega, \mathcal{A}, P)}^2 \end{aligned}$$

$$\begin{aligned}
 &= \varphi_{H_1} \left(E_{\mathcal{U}([- \sqrt{B_T}/2, \sqrt{B_T}/2])} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \right) \\
 &\leq E_{\mathcal{U}([- \sqrt{B_T}/2, \sqrt{B_T}/2])} \left[\varphi_{H_1} \left(\left\| \widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right\| \right) \right] \\
 &= \int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left\| \widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \frac{d\omega}{\sqrt{B_T}}, \tag{85}
 \end{aligned}$$

where $E_{\mathcal{U}([- \sqrt{B_T}/2, \sqrt{B_T}/2])}$ denotes expectation under the uniform probability measure on the interval $[- \sqrt{B_T}/2, \sqrt{B_T}/2]$, and $\varphi_{H_1}(\cdot) = \|\cdot\|_{\mathcal{L}^2_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}(\Omega, \mathcal{A}, P)}^2 = E_{H_1} \|\cdot\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2$ is a convex function. Thus, from equations (82)–(85),

$$\begin{aligned}
 &E_{H_1} \left\| \int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right. \\
 &\quad \left. \circ \left[\int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 \\
 &\leq h(T) = O(T^{-1} B_T^{2l_\alpha - 1/2}), \quad T \rightarrow \infty. \tag{86}
 \end{aligned}$$

From equation (86), applying Chebyshev’s inequality,

$$\begin{aligned}
 &P \left[\left\| \int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right. \right. \\
 &\quad \left. \left. \circ \left[\int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} > \varepsilon \right] \\
 &\leq E_{H_1} \left\| \int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right. \\
 &\quad \left. \circ \left[\int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}^2 / \varepsilon^2 \\
 &\leq h(T) / \varepsilon^2 = O(T^{-1} B_T^{2l_\alpha - 1/2}). \tag{87}
 \end{aligned}$$

Since $l_\alpha > 1/4$, hence, $2l_\alpha - 1/2 = \rho > 0$, and, for $B_T = T^{-\beta}$, $T^{-1} B_T^{2l_\alpha - 1/2} = T^{-1-\beta\rho}$, with $\beta \in (0, 1)$, and $\rho \in (0, 1/2)$. From equation (87), Borel–Cantelli lemma then leads to

$$\begin{aligned}
 &\left\| \int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} \left[\widehat{\mathcal{F}}_\omega^{(T)} - E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \right] \frac{d\omega}{\sqrt{B_T}} \right. \\
 &\quad \left. \circ \left[\int_{[- \sqrt{B_T}/2, \sqrt{B_T}/2]} E_{H_1} \left[\widehat{\mathcal{F}}_\omega^{(T)} \right] \frac{d\omega}{\sqrt{B_T}} \right]^{-1} \right\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))} \rightarrow_{a.s.} 0. \tag{88}
 \end{aligned}$$

as $T \rightarrow \infty$. The a.s. divergence of $\|\mathcal{S}_{B_T}\|_{S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))}$, as $T \rightarrow \infty$, follows from equations (79), (81) and (88). □

C.5. Proof of Lemma 4.6

Proof. For $\omega \in (-\pi, \pi) \setminus \{0\}$, consider a Gaussian random element $\widehat{\mathcal{F}}_\omega$ in the space $S(L^2(\mathbb{M}_d, d\nu, \mathbb{C}))$, with kernel \widehat{f}_ω satisfying (23). Hence, from (47),

$$\begin{aligned} & \frac{1}{2\pi\|W\|_{L^2(\mathbb{R})}^2} E \left[\widehat{f}_\omega \otimes \widehat{f}_\omega \right] (\tau_1, \sigma_1, \tau_2, \sigma_2) \\ &= \sum_{n,h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n,d)} \sum_{l=1}^{\Gamma(h,d)} f_n(\omega) f_h(\omega) S_{n,j}^d(\tau_1) \overline{S_{h,l}^d(\sigma_1)} S_{n,j}^d(\tau_2) \overline{S_{h,l}^d(\sigma_2)}, \end{aligned} \quad (89)$$

for every $(\tau_i, \sigma_i) \in \mathbb{M}_d^2$, $i = 1, 2$, $\omega \in (-\pi, \pi) \setminus \{0\}$. Thus, the diagonal coefficients $\{\lambda_{n,h}(\omega), n, h \in \mathbb{N}_0\} = \{f_n(\omega) f_h(\omega), n, h \in \mathbb{N}_0\}$ define the eigenvalues of the autocovariance operator (89). Since \mathbb{M}_d^2 is a compact set, and $\mathcal{R}_{\widehat{\mathcal{F}}_\omega, \widehat{\mathcal{F}}_\omega}$ is a trace positive semidefinite self-adjoint operator under H_0 , the orthogonal expansion

$$\frac{1}{\sqrt{2\pi}\|W\|_{L^2(\mathbb{R})}} \widehat{f}_\omega(\tau, \sigma) = \sum_{n,h \in \mathbb{N}_0} \sum_{j=1}^{\Gamma(n,d)} \sum_{l=1}^{\Gamma(h,d)} \sqrt{f_n(\omega) f_h(\omega)} Y_{n,j,h,l}(\omega) S_{n,j}^d(\tau) \overline{S_{h,l}^d(\sigma)} \quad (90)$$

holds in the space $\mathcal{L}_{S(L^2(\mathbb{M}_d, d\nu; \mathbb{C}))}^2(\Omega, \mathcal{A}, \mathcal{P})$. The random Fourier coefficients are given by

$$\begin{aligned} Y_{n,j,h,l}(\omega) &= \frac{(\sqrt{2\pi}\|W\|_{L^2(\mathbb{R})})^{-1}}{\sqrt{f_n(\omega) f_h(\omega)}} \int_{\mathbb{M}_d^2} \widehat{f}_\omega(\tau, \sigma) \overline{S_{n,j}^d(\tau)} S_{h,l}^d(\sigma) d\nu(\tau) d\nu(\sigma), \\ j &= 1, \dots, \Gamma(n, d), \quad l = 1, \dots, \Gamma(h, d), \quad n, h \in \mathbb{N}_0, \quad \omega \in [-\pi, \pi] \setminus \{0\}. \quad \square \end{aligned} \quad (91)$$

Funding

This work has been supported in part by projects PID2022–142900NB-I00 and PID2020–116587GB-I00, financed by MCIU/AEI/10.13039/501100011033 and by FEDER UE, and CEX2020–001105–M MCIN/AEI/10.13039/501100011033), as well as supported by grant ED431C 2021/24 (Grupos competitivos) financed by Xunta de Galicia through European Regional Development Funds (ERDF).

References

- [1] BERAN, J. (2017). *Mathematical Foundations of Time Series Analysis*. Springer, Switzerland.
- [2] BERAN, J. and LIU, H. (2014). On estimation of mean and covariance functions in repeated time series with long–memory errors. *Lithuanian Mathematical Journal* **54** 8–34. [MR3189134](#)
- [3] BERAN, J. and LIU, H. (2016). Estimation of eigenvalues, eigenvectors and scores in FDA models with dependent errors. *Journal of Multivariate Analysis* **147** 218–233. [MR3484179](#)
- [4] BERAN, J., LIU, H. and TELKMANN, K. (2016). On two sample inference for eigenspaces in functional data analysis with dependent errors. *Journal of Statistical Planning and Inference* **174** 20–37. [MR3477698](#)

- [5] CAPONERA, A. (2021). SPHARMA approximations for stationary functional time series in the sphere. *Stat. Infer. Stoch. Proc.* **24** 609–634. [MR4321852](#)
- [6] CAPONERA, A. and MARINUCCI, D. (2021). Asymptotics for spherical functional autoregressions. *Ann. Stat.* **49** 346–369. [MR4206681](#)
- [7] CHARACIEJUS, V. and RÄCKAUSKAS, A. (2014). Operator self-similar processes and functional central limit theorems. *Stochastic Process Appl.* **124** 2605–2627. [MR3200727](#)
- [8] CUESTA-ALBERTOS, J.A., FRAIMAN, R. and RANSFORD, T. (2007). A sharp form of the Cramér–Wold theorem. *J. Theoret. Probab.* **20** 201–209. [MR2324526](#)
- [9] CUESTA-ALBERTOS, J. A., GARCÍA-PORTUGUÉS, E., FEBRERO-BANDE, M. and GONZÁLEZ-MANTEIGA, W. (2019). Goodness-of-fit tests for the functional linear model based on randomly projected empirical processes. *The Annals of Statistics* **47** 439–467. [MR3909938](#)
- [10] DA PRATO, G. and ZABCZYK, J. (2002). *Second Order Partial Differential Equations in Hilbert Spaces*. Cambridge, London. [MR1985790](#)
- [11] DAUTRAY, R. and LIONS, J.L. (1985). *Mathematical Analysis and Numerical Methods for Science and Technology*. Volume 3. *Spectral Theory and Applications*. Springer, New York. [MR0918560](#)
- [12] DÜKER, M. (2018). Limit theorems for Hilbert space-valued linear processes under long range dependence. *Stochastic Processes and Their Applications* **128** 1439–1465. [MR3780685](#)
- [13] DUQUE, J. C., CARONES, A., MARINUCCI, D., MIGLIACCIO, M. and VITTORIO, N. (2024). Minkowski functionals in SO(3) for the spin-2 CMB polarisation field. *Journal of Cosmology and Astroparticle Physics*. <https://doi.org/10.1088/1475-7516/2024/01/039>. [MR4723629](#)
- [14] GELFAND, I. M. and VILENKIN, N. Y. (1964). *Generalized Functions* **4** Academic Press, New York. [MR0173945](#)
- [15] GINÉ, M. (1975). The addition formula for the eigenfunctions of the Laplacian. *Advances in Mathematics* **18** 102–107. [MR0377997](#)
- [16] HARRIS, D., MCCABE, B. and LEYBOURNE, S. (2008). Testing for long memory. *Econometric Theory* **24** 143–175. [MR2408862](#)
- [17] HELGASON, S. (1959). Differential operators on homogeneous spaces. *Acta Math.* **102** 239–299. [MR0117681](#)
- [18] KURISU, D. and MATSUDA, Y. (2025). Series ridge regression for spatial data on \mathbb{R}^d . Forthcoming papers of Bernoulli journal (<https://www.bernoullisociety.org/publications/bernoulli-journal/bernoulli-journal-papers>).
- [19] LI, D., ROBINSON, P.M. and SHANG, H.L. (2019). Long-range dependent curve time series. *Journal of the American Statistical Association* **115** 957–971. [MR4107692](#)
- [20] LI, D., ROBINSON, P.M. and SHANG, H.L. (2021). Local whittle estimation of long-range dependence for functional time series. *J. Time Series Anal.* **42** 685–695. [MR4325670](#)
- [21] MA, C. and MALYARENKO, A. (2020). Time varying isotropic vector random fields on compact two points homogeneous spaces. *J. Theor. Probab.* **33** 319–339. [MR4064303](#)
- [22] MARINUCCI, D. (2004). Testing for non-gaussianity on cosmic microwave background radiation: A review. *Statistical Science* **19** 294–307. [MR2140543](#)

- [23] MARINUCCI, D. and PECCATI, G. (2011). *Random fields on the Sphere. Representation, Limit Theorems and Cosmological Applications*. Mathematical Society Lecture Note Series **389**. Cambridge University Press, London. [MR2840154](#)
- [24] MARINUCCI, D., ROSSI, M. and VIDOTTO, A. (2020). Non-universal fluctuations of the empirical measure for isotropic stationary fields on $\mathbb{S}^2 \times \mathbb{R}$. *Ann. Appl. Probab.* **31** 2311–2349. [MR4332698](#)
- [25] MARINUCCI, D., ROSSI, M. and WIGMAN, I. (2020). The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 374–390. [MR4058991](#)
- [26] OVALLE–MUÑOZ, D. P. and RUIZ–MEDINA, M. D. (2024a). LRD spectral analysis of multifractional functional time series on manifolds. *TEST* **33** 564–588. [MR4756648](#)
- [27] OVALLE–MUÑOZ, D. P. and RUIZ–MEDINA, M. D. (2024b). Climate change analysis from LRD manifold functional regression. *Stochastic Environmental Research and Risk Assessment* **39** 1555–1580.
- [28] PANARETOS, V. M. and TAVAKOLI, S. (2013a). Fourier analysis of stationary time series in function space. *Ann. Statist.* **41** 568–603. [MR3099114](#)
- [29] PANARETOS, V. M. and TAVAKOLI, S. (2013b). Cramér–Karhunen–Loève representation and harmonic principal component analysis of functional time series. *Stochastic Process and their Applications* **123** 2779–2807. [MR3054545](#)
- [30] PHAM, T. and PANARETOS, V. (2018). Methodology and convergence rates for functional time series regression. *Statistica Sinica* **28** 2521–2539. [MR3839872](#)
- [31] RACKAUSKAS, A. and SUQUET, CH. (2010). On limit theorems for Banach-space-valued linear processes. *Lithuanian Mathematical J.* **50** 71–87. [MR2607680](#)
- [32] RACKAUSKAS, A. and SUQUET, CH. (2011). Operator fractional brownian motion as limit of polygonal lines processes in Hilbert space. *Stochastics and Dynamics* **11** 49–70. [MR2771341](#)
- [33] RAMM, A.G. (2005). *Random Fields Estimation*. Longman Scientific & Technical, England.
- [34] RUBÍN, T. and PANARETOS, V. M. (2020a). Functional lagged regression with sparse noisy observations. *Journal of Time Series Analysis* **41** 858–882. [MR4176363](#)
- [35] RUBÍN, T. and PANARETOS, V. M. (2020b). Spectral simulation of functional time series. *arXiv preprint* [arXiv:2007.08458](#).
- [36] RUIZ–MEDINA, M. D. (2022). Spectral analysis of long range dependence functional time series. *Fractional Calculus and Applied Analysis* **25** 1426–1458. [MR4468520](#)
- [37] RUIZ–MEDINA, M.D. and CRUJEIRAS, R.M. (2025). An LRD spectral test for irregularly discretely observed functional time series in manifolds. *New Trends in Functional Statistics and Related Fields*. Chapter 56, pp. 467–474.
- [38] SHAH, I., MUBASSIR, P., ALI, S. and ALBALAWI, O. (2024). A functional autoregressive approach for modeling and forecasting short-term air temperature. *Environ. Sci.* **12** <https://doi.org/10.3389/fenvs.2024.1411237>.
- [39] TAVAKOLI, S. (2014). *Fourier Analysis of Functional Time Series with Applications to DNA Dynamics*. Ph.D. dissertation, EPFL. Available at <http://dx.doi.org/10.5075/epfl-thesis-6320>.
- [40] TAVAKOLI, S. and PANARETOS, V. M. (2016). Detecting and localizing differences in functional time series dynamics: a case study in molecular biophysics. *Journal of the American Statistical Association* **111** 1020–1035. [MR3561926](#)

- [41] TIRELLI, I., MENDEZ, M. A., IANIRO, A. and DISCETTI, S. (2024). A meshless method to compute the proper orthogonal decomposition and its variants from scattered data. [arXiv:2407.03173](https://arxiv.org/abs/2407.03173). [MR4926739](#)
- [42] WU, Y., HUANG, CH. and SRIVASTAVA, A. (2024). Shape based functional data analysis. *TEST* **33** 1–47. [MR4727775](#)
- [43] YARGER, D., STOEV, S. and HSING, T. (2022). A functional-data approach to the Argo data. *Ann. Appl. Stat.* **16** 216–246. [MR4400508](#)
- [44] ZHU, H., CHEN, Y., IBRAHIM, J. G., LI, Y., HALL, C. and LIN, W. (2009). Intrinsic regression models for positive-definite matrices with applications to diffusion tensor imaging. *J. Am. Stat. Assoc.* **104** 1203–1212. [MR2750245](#)
- [45] ZHOU, H., LI, L. and ZHU, H. (2013). Tensor regression with applications in neuroimaging data analysis. *J. Am. Stat. Assoc.* **108** 540–552. [MR3174640](#)
- [46] ZHOU, H., WEI, D. and YAO, F. (2024). Theory of functional principal component analysis for discretely observed data. [arXiv:2209.08768v4](https://arxiv.org/abs/2209.08768v4).