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Solitons of the mean curvature flow in $\mathbb{S}^2 \times \mathbb{R}$

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ABSTRACT

A soliton of the mean curvature flow in the product space $\mathbb{S}^2 \times \mathbb{R}$ is a surface whose mean curvature H satisfies the equation $H = \langle N, X\rangle$, where N is the unit normal of the surface and X is a Killing vector field of $\mathbb{S}^2 \times \mathbb{R}$. In this paper we consider the cases that X is the vector field tangent to the second factor and the vector field associated to rotations about an axis of $\mathbb{S}^2$, respectively. We give a classification of the solitons with respect to these vector fields assuming that the surface is invariant under a one-parameter group of vertical translations or rotations of $\mathbb{S}^2$.

1. Introduction

Let $\psi\colon \Sigma \to \mathbb{R}^3$ be an immersion of a surface Σ in Euclidean space $\mathbb{R}^3$. A variation $\{\psi_t\colon \Sigma \to \mathbb{R}^3 : t \in [0, T)\}$, $T > 0$, $\psi_0 = \psi$, evolves by the mean curvature flow (MCF in short) if $\dfrac{\partial \psi_t}{\partial t} = H(\psi_t)N(\psi_t)$, where $H(\psi_t)$ is the mean curvature of ψ_t and $N(\psi_t)$ is its unit normal. The surface Σ is called a soliton of MCF if the evolution of Σ under a one-parameter family of dilations or isometries remains constant. An important type of solitons are the translators whose shape is invariant by translations along a direction $\vec{v} \in \mathbb{R}^3$. Translators are characterized by the equation $H = \langle N, \vec{v}\rangle$, where H and N are the mean curvature and unit normal of Σ respectively. Translators play a special role in the theory of MCF because they are, after rescaling, a type of singularities of the MCF according to Huisken and Sinestrari [6]. In the meantime, the development of the theory of solitons of the MCF in other ambient spaces has been developed. Without

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to be complete, we refer: a general product space $M^2 \times \mathbb{R}$ [10]; hyperbolic space [3,4,9,12]; the product $\mathbb{H}^2 \times \mathbb{R}$ [1,2,5,8]; the Sol space [15]; the Heisenberg group [16]; the special linear group [11].

In this paper, we focus on the MCF in the product space $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ is the unit sphere of $\mathbb{R}^3$. We consider surfaces evolving under MCF by isometries of the ambient space in $\mathbb{S}^2 \times \mathbb{R}$. We give the following definition.

Definition 1.1. Let $X \in \mathfrak{X}(\mathbb{S}^2 \times \mathbb{R})$ be a Killing vector field. A surface Σ in $\mathbb{S}^2 \times \mathbb{R}$ is said to be a X-soliton if its mean curvature H and unit normal vector N satisfy

$$H = \langle N, X \rangle. \tag{1}$$

In our paper we adopt the convention that H represents the sum of the principal curvatures of the surface. Let (x, y, z, t) denote the global coordinates in $\mathbb{R}^3 \times \mathbb{R}$, where $\mathbb{S}^2 \times \mathbb{R}$ is embedded. Recall that the dimension of the space of Killing vector fields in $\mathbb{S}^2 \times \mathbb{R}$ is 4. One relevant Killing vector field is $V = \partial_t$, which is tangent to the fibers of the natural submersion $\mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2$. Other Killing vector fields arise from the rotations of $\mathbb{S}^2$. Upon coordinates renaming, we consider the vector field $R = -y\partial_x + x\partial_y$, which is the infinitesimal generator of the rotations, with unit angular speed, about the z-axis of $\mathbb{S}^2$.

Examples of solitons are the following.

(1) Vertical cylinders over geodesic of $\mathbb{S}^2$ are V-solitons. Indeed, let $\Sigma = C \times \mathbb{R} \subset \mathbb{S}^2 \times \mathbb{R}$ be a surface constructed as a cylinder over a curve $C \subset \mathbb{S}^2$. Then the mean curvature of Σ is $H = \kappa$, where κ the curvature of C. Since the unit normal vector N of Σ is orthogonal to ∂_t, then $\langle N, V \rangle = 0$. Thus Σ is a V-soliton if and only if $\kappa = 0$, that is, if C is a geodesic of $\mathbb{S}^2$.
(2) Slices $\mathbb{S}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$, are R-solitons. Notice that $H = 0$ because a slice is totally geodesic. Since $N = \partial_t$, then $\langle N, R \rangle = 0$, proving that $H = \langle N, R \rangle$.

In this article, we are interested in examples of V-solitons and R-solitons that are invariant by a one-parameter group of isometries of $\mathbb{S}^2 \times \mathbb{R}$. Here we consider two types of such surfaces. First, surfaces invariant by vertical translations in the t-coordinate (vertical surfaces). Second, rotational surfaces, which are invariant by a group of rotations about an axis of $\mathbb{S}^2$. Under these geometric conditions, we give a full classification of V-solitons (Sect. 3) and R-solitons (Sect. 4). In Theorem 3.2 we prove that vertical V-solitons are trivial in the sense that they are vertical cylinders over geodesics of $\mathbb{S}^2$. Similarly, rotational R-solitons are slices $\mathbb{S}^2 \times \{t_0\}$ or rotational minimal surfaces (Theorem 4.4). The most interesting cases of solitons are rotational V-solitons and vertical R-solitons. In Theorems 3.4 and 4.2 we show the main properties of these solitons. In particular, we prove that they are not embedded and they are asymptotic to the cylinder $(\mathbb{S}^1 \times \{0\}) \times \mathbb{R}$ at infinity.

2. Preliminaries

In this section, we compute each one of the terms of Eq. (1) for vertical and rotational surfaces. The isometry group of $\mathbb{S}^2 \times \mathbb{R}$ is isomorphic to $\mathrm{Isom}(\mathbb{S}^2) \times \mathrm{Isom}(\mathbb{R})$. The group $\mathrm{Isom}(\mathbb{S}^2)$ is generated by the identity, the antipodal map, rotations and reflections. The group $\mathrm{Isom}(\mathbb{R})$ contains the identity, translations, and reflections. Therefore there are two important one-parameter groups of isometries in $\mathbb{S}^2 \times \mathbb{R}$: vertical translations in the factor $\mathbb{R}$ and rotations in the factor $\mathbb{S}^2$. This leads to two types of invariant surfaces.

(1) *Vertical surfaces.* A vertical translation is a map of type $T_\lambda \colon \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}$ defined by $T_\lambda(p, t) = (p, t + \lambda)$, where λ is fixed. This defines a one-parameter group of vertical translations $\mathcal{T} = \{T_\lambda : \lambda \in \mathbb{R}\}$. A vertical surface is a surface Σ invariant by the group $\mathcal{T}$, that is, $T_\lambda(\Sigma) \subset \Sigma$ for all $\lambda \in \mathbb{R}$. The generating curve of Σ is a curve $\alpha \colon I \subset \mathbb{R} \to \mathbb{S}^2$ in the unit sphere $\mathbb{S}^2$. Let us write this curve as

$$\alpha(s) = (\cos u(s) \cos v(s), \cos u(s) \sin v(s), \sin u(s)), \tag{2}$$

for some smooth functions $u = u(s)$ and $v = v(s)$. Then a parametrization of Σ is

$$\Psi(s,t) = (\cos u(s) \cos v(s), \cos u(s) \sin v(s), \sin u(s), t), \quad s \in I, t \in \mathbb{R}. \tag{3}$$

In what follows, we parametrize the curve $\beta(s) = (u(s), v(s))$ to have

$$u'(s) = \cos u(s) \cos \theta(s), \quad v'(s) = \sin \theta(s).$$

(2) *Rotational surfaces.* After a choice of coordinates on $\mathbb{S}^2$, a rotation in $\mathbb{S}^2 \times \mathbb{R}$ about the z-axis is a map $\mathcal{R}_\varphi \colon \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}$, given by

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The set $\mathcal{R} = \{\mathcal{R}_\varphi : \varphi \in \mathbb{R}\}$ of all $\mathcal{R}_\varphi$, is a one-parameter group of rotations, that is SO(2). A rotational surface is a surface Σ invariant by the group $\mathcal{R}$, namely, $\mathcal{R}_\varphi(\Sigma) \subset \Sigma$ for all $\varphi \in \mathbb{R}$. The generating curve of Σ is a curve $\alpha \colon I \subset \mathbb{R} \to \mathbb{S}^2 \times \mathbb{R}$ contained in the xzt-hyperplane which we suppose parametrized by

$$\alpha(s) = (\cos u(s), 0, \sin u(s), v(s)), \quad s \in I \subset \mathbb{R}, \tag{4}$$

where $u = u(s)$ and $v = v(s)$ are smooth functions. Then a parametrization of Σ is

$$\Psi(s,\varphi) = (\cos u(s) \cos\varphi, \cos u(s) \sin\varphi, \sin u(s), v(s)), \quad s \in I, \varphi \in \mathbb{R}. \tag{5}$$

From now on, we suppose that the curve $\beta(s) = (u(s), v(s))$ is parametrized by the Euclidean arc length, that is,

$$u'(s) = \cos\theta(s), \quad v'(s) = \sin\theta(s),$$

for some smooth function $\theta = \theta(s)$. Notice that θ' is the curvature of β as a planar curve of $\mathbb{R}^2$.

We now compute the mean curvature H and the unit normal vector N for vertical surfaces and for rotational surfaces.

Proposition 2.1. *Suppose that Σ is a vertical surface parametrized by (3). Then the unit normal vector N is expressed as*

$$N = (\cos\theta \sin v - \sin\theta \sin u \cos v, -\cos\theta \cos v - \sin\theta \sin u \sin v, \sin\theta \cos u, 0), \tag{6}$$

and the mean curvature H is given by

$$H = \frac{\sin u \sin\theta - \theta'}{\cos u}. \tag{7}$$

Proof. Suppose that Σ is parametrized by (3). Then the tangent plane at each point of Σ is spanned by $\{\Psi_s, \Psi_t\}$, where

$$\begin{aligned}
&\Psi_s = (-\cos u(\cos\theta\sin u\cos v + \sin\theta\sin v), \cos u(\sin\theta\cos v - \cos\theta\sin u\sin v), \cos\theta\cos^2 u, 0),\\
&\Psi_t = (0, 0, 0, 1).
\end{aligned} \tag{8}$$

A straightforward computation yields the expression for the unit normal vector as stated in equation (6).

As usual, denote by g_{ij} the coefficients of the first fundamental form of Ψ, where

$$g_{11} = \langle\Psi_s, \Psi_s\rangle, \quad g_{12} = \langle\Psi_s, \Psi_t\rangle, \quad g_{22} = \langle\Psi_t, \Psi_t\rangle.$$

The formula of H is

$$H = \frac{g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}}{g_{11}g_{22} - {g_{12}}^2},$$

where b_{ij} are the coefficients of the second fundamental form, namely,

$$b_{11} = -\langle N_s, \Psi_s\rangle, \quad b_{12} = -\langle N_s, \Psi_t\rangle, \quad b_{22} = -\langle N_t, \Psi_t\rangle.$$

A computation of g_{ij} gives $g_{11} = (\cos u)^2$, $g_{22} = 1$ and $g_{12} = 0$. In particular, $\cos u(s) \neq 0$ for all $s \in I$. Then $g_{11}g_{22} - g_{12}^2 = (\cos u)^2$. For the coefficients of the second fundamental, we have $b_{12} = b_{22} = 0$ and

$$b_{11} = \cos u(\sin\theta\sin u - \theta'). \tag{9}$$

Then the mean curvature H is (7). □

Proposition 2.2. *Suppose that Σ is a rotational surface parametrized by (5). Then the unit normal vector N is defined by*

$$N(s, \varphi) = (\sin\theta\sin u\cos\varphi, \sin\theta\sin u\sin\varphi, -\sin\theta\cos u, \cos\theta), \tag{10}$$

and the mean curvature H is expressed as

$$H = \theta' - \sin\theta\tan u. \tag{11}$$

Proof. From (5), the basis $\{\Psi_s, \Psi_t\}$ at each tangent plane of Σ is

$$\begin{aligned}
&\Psi_s(s, \varphi) = (-u'\sin u\cos\varphi, -u'\sin u\sin\varphi, u'\cos u, v'),\\
&\Psi_\varphi(s, \varphi) = (-\sin\varphi\cos u, \cos\varphi\cos u, 0, 0).
\end{aligned} \tag{12}$$

Thus $g_{11} = 1$, $g_{12} = 0$ and $g_{22} = \cos^2 u$; in particular, $\cos u \neq 0$. As a consequence, the unit normal vector N is (10). The computation of the coefficients of the second fundamental form gives

$$\begin{aligned}
&b_{11} = \theta'\\
&b_{12} = 0\\
&b_{22} = -\sin\theta\sin u\cos u.
\end{aligned} \tag{13}$$

Hence we deduce the expression of H given in (11). □

3. The class of V-solitons

Let V be the vector field given by

$$V = \partial_t. \tag{14}$$

The fact that V is tangent to the fibers of the submersion $\mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2$ confers special properties to V. For example, V-solitons of $\mathbb{S}^2 \times \mathbb{R}$ can be viewed as weighted minimal surfaces in a space with density: see [10, Sect. 2] in a general context of product spaces. So, let $e^t dA$ and $e^t dV$ the area and volume of $\mathbb{S}^2 \times \mathbb{R}$ with a weight e^t, where t is the last coordinate of the space. Considering the energy functional $\Omega \mapsto E(\Omega) = \int_\Omega e^t dA$ defined for compact subdomains $\Omega \subset \Sigma$, a critical point of this functional, also called a weighted minimal surface, is a surface characterized by the equation $H - \langle N, \nabla t\rangle = 0$, where ∇ is the gradient in $\mathbb{S}^2 \times \mathbb{R}$. Since $\nabla t = \partial_t = V$, we have proved that a weighted minimal surface in $(\mathbb{S}^2 \times \mathbb{R}, e^t\langle,\rangle)$ is a V-soliton. One property of weighted minimal surfaces is that they satisfy a principle of tangency as a consequence of the Hopf maximum principle for elliptic equations of divergence type. In our context, the tangency principle asserts that if two V-solitons Σ_1 and Σ_2 touch at some interior point $p \in \Sigma_1 \cap \Sigma_2$ and one surface is in one side of the other around p, then Σ_1 and Σ_2 coincide in a neighborhood of p. The following result proves that slices are the only closed V-solitons.

Theorem 3.1. *There are no closed (compact without boundary) V-solitons in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. Let $\psi\colon \Sigma \to \mathbb{S}^2 \times \mathbb{R}$ be a closed V-soliton. Define on Σ the height function $h\colon \Sigma \to \mathbb{R}$ by $h(q) = \langle \psi(q), \partial_t\rangle$. It is known that for any surface of $\mathbb{S}^2 \times \mathbb{R}$, the Laplacian of h is $\Delta h = H\langle N, \partial_t\rangle$ [18].

Using that Σ is a V-soliton, then $\Delta h = \langle N, \partial_t\rangle^2 = \langle N, V\rangle^2$. Integrating on Σ, the divergence theorem yields $\int_\Sigma \langle N, V\rangle^2 \, d\Sigma = 0$. Thus $\langle N, V\rangle = 0$ on Σ and $H = 0$. In particular, $\Delta h = 0$. By the maximum principle, h is a constant function, namely $h(q) = t_0$, for some $t_0 \in \mathbb{R}$. This proves that $\Sigma \subset \mathbb{S}^2 \times \{t_0\}$ and hence, $\Sigma = \mathbb{S}^2 \times \{t_0\}$. However, a slice $\mathbb{S}^2 \times \{t_0\}$ is not a V-soliton. □

We begin with the study of V-solitons invariant by the group $\mathcal{T}$. We prove that any vertical V-soliton is trivial in the sense that it is a minimal surface. Even more, we prove that it is a cylinder of type $\mathbb{S}^1 \times \mathbb{R} \subset \mathbb{S}^2 \times \mathbb{R}$.

Theorem 3.2. *Suppose that Σ is a vertical surface. Then Σ is a V-soliton if and only if its generating curve is a geodesic of $\mathbb{S}^2$ and Σ is a vertical surface on a geodesic of $\mathbb{S}^2$.*

Proof. Let Σ be a vertical surface. Since the vertical lines are fibers of the submersion $\mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2$, the mean curvature H of Σ is $H = \kappa$, where κ is the curvature of α. Moreover, the unit normal vector is horizontal, hence $\langle N, V\rangle = 0$. This proves the result. □

We now study V-solitons of rotational type. As we have indicated in the previous section, we can assume that the rotation axis is the z-axis. Thus a rotational surface Σ can be parametrized by (5).

An immediate example of rotational V-soliton is the cylinder $\mathcal{C} = (\mathbb{S}^1 \times \{0\}) \times \mathbb{R}$. This surface corresponds with the curve $(u(s), v(s)) = (0, s)$, $s \in \mathbb{R}$, in (4). Thus $\alpha(s) = (1, 0, 0, s)$ is the vertical line through the point $(1, 0, 0) \in \mathbb{S}^2$. The unit normal N is orthogonal to V. Since the generating curve is a geodesic of $\mathbb{S}^2$, the surface is minimal, proving that $\mathcal{C}$ is a V-soliton. This surface is also a vertical R-soliton (Theorem 4.2). We now characterize rotational V-solitons in terms of its generating curve α.

Proposition 3.3. *Let Σ be a rotational surface in $\mathbb{S}^2 \times \mathbb{R}$. If Σ is parametrized by (5), then Σ is a V-soliton if and only if the generating curve α satisfies*

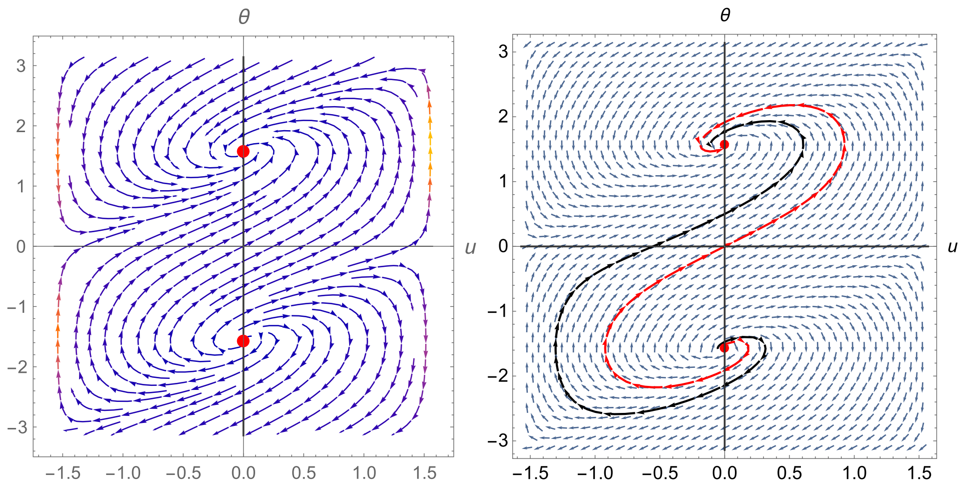

Fig. 1. The (u,θ)-phase plane of (16). Left: the red points are the equilibrium points $(0,\pm\frac{\pi}{2})$, where the surface is the cylinder $\mathcal{C}$. Right: two trajectories in the phase plane thought the points $(0,0.5)$ (black) and $(0,0)$ (red). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\begin{cases} u' = \cos\theta \\ v' = \sin\theta \\ \theta' = \sin\theta\tan u + \cos\theta \end{cases} \tag{15}$$

Proof. This is an immediate consequence of Proposition 2.2. Indeed, from the expression of N in (10), we have

$$\langle N, V\rangle = \cos\theta.$$

Using (11), then Eq. (1) is $\theta' = \sin\theta\tan u + \cos\theta$. $\square$

We now study the solutions of (15), describing their main geometric properties. Recall that $\cos u \neq 0$ by regularity of the surface (Proposition 2.2). Since the last equation of (15) does not depend on v, we can study the solutions α of (15) projecting in the (u,θ)-plane, which, in turn, gives rise to the following autonomous planar ordinary system:

$$\begin{cases} u' = \cos\theta \\ \theta' = \sin\theta\tan u + \cos\theta. \end{cases} \tag{16}$$

The phase plane of (16) is depicted in Fig. 1, left. By regularity of the surface, $u(s) \in (-\pi/2, \pi/2)$. Thus the phase plane of (16) is the set

$$A = \{(u,\theta)\colon u \in (\frac{\pi}{2}, \frac{\pi}{2}), \theta \in \mathbb{R}\}.$$

The trajectories of (16) are the solutions $\gamma(s) = (u(s), \theta(s))$ of (16) when regarded in A and once initial conditions $(u_0, \theta_0) \in A$ have been fixed. These trajectories foliate A as a consequence of the existence and uniqueness of the Cauchy problem of (16).

The equilibrium points of (16) are $(u,\theta) = (0, \frac{\pi}{2})$ and $(u,\theta) = (0, -\frac{\pi}{2})$. The rest of equilibrium points can be obtained by translations by multiples of π along the u and θ-coordinates. If $(u,\theta) = (0, \frac{\pi}{2})$, then $u(s) = 0$, $v(s) = s$. For this trajectory, the generating curve α is the vertical fiber at $(1,0,0) \in \mathbb{S}^2$ parametrized with increasing variable s, $v(s) = s$. Consequently, the corresponding surface is the vertical right cylinder $\mathcal{C}$ and this solution is already known. If $(u, \theta) = (0, -\frac{\pi}{2})$, then $v(s) = -s$ and the generating curve is again the above vertical line but parametrized by decreasing variable s. The surface is the cylinder $\mathcal{C}$ again.

The qualitative behavior of the trajectories near the equilibrium points is analyzed, as usually, by the linearized system (see [14, Ch. 1] as a general reference). At the point $(u, \theta) = (0, \frac{\pi}{2})$, we find

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

as the matrix of the linearized system. The eigenvalues of this matrix are the two conjugate complex numbers $\frac{1}{2}(-1 \pm i\sqrt{3})$. Since the real parts are negative, then the point $(0, \frac{\pi}{2})$ is a stable spiral. Thus all the trajectories will move in towards the equilibrium point as s increases. Similarly, for the point $(u, \theta) = (0, -\frac{\pi}{2})$, the matrix of the corresponding linearized system is

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are $\frac{1}{2}(1 \pm i\sqrt{3})$ and the point $(0, -\frac{\pi}{2})$ is an unstable spiral. In Fig. 1, right we show as the trajectories start in the unstable spiral $(0, -\frac{\pi}{2})$ and end in the stable spiral $(0, \frac{\pi}{2})$.

In order to give initial conditions at $s = 0$, notice that if we do a vertical translation in $\mathbb{R}^4$ of the generating curve α, the surface is a translated from the original. This vertical translation is simply adding a constant to the last coordinate function $v = v(s)$. Thus, at the initial time $s = 0$, we can assume $v(0) = 0$. On the other hand, the fact that the trajectories go from $(0, -\frac{\pi}{2})$ towards $(0, \frac{\pi}{2})$ implies that the function θ attains the value 0. Therefore we can consider that the function θ at the initial point $s = 0$ takes the value 0, $\theta(0) = 0$. So, let

$$u(0) = v(0) = \theta(0) = 0. \tag{17}$$

It is immediate from (15) that

$$\bar{u}(s) = -u(-s), \quad \bar{v}(s) = v(-s), \quad \bar{\theta}(s) = -\theta(-s)$$

is also a solution of (15) with the same initial conditions (17). Thus the graphic of $\beta(s) = (u(s), v(s))$ is symmetric about the v-axis.

Given initial conditions (17), we know that $(u(s), \theta(s))$ goes to the stable spiral $(0, \frac{\pi}{2})$. Then the right hand-sides of (16) (also in (15)) are bounded functions, proving that the domain of solutions is $\mathbb{R}$. Since $|v'(s)| \to 1$, then $\lim_{s\to\pm\infty} v(s) = \infty$ by symmetry of β. Thus $\lim_{s\to\pm\infty} \beta(s) = (0, \infty)$, that is β is asymptotic to the v-axis at infinity. The projection of α on the factor $\mathbb{S}^2$ converges to $(1, 0, 0)$. This implies that Σ is asymptotic to the cylinder $\mathcal{C}$.

Because $(0, \pi/2)$ is a stable spiral, the function $\theta(s)$ converges to $\pi/2$ oscillating around this value, and the same occurs for the function $u(s)$ around $u = 0$. In particular, the graphic of β intersects infinitely many times the v-axis. By the symmetry of β with respect to the v-axis, we deduce that β has (infinitely many) self-intersections.

We claim that the coordinate function $v(s)$ of β has no critical points except $s = 0$. We know

$$v''(s) = \theta'(s)\cos\theta(s) = \sin\theta(s)\cos\theta(s)\tan u(s) + \cos^2\theta(s).$$

If $v'(s) = 0$ at $s = s_0$, then $\sin\theta(s_0) = 0$, hence $v''(s_0) = 1$. Thus all critical points are local minimum deducing that $s = 0$ is the only minimum. Once we have proved that $v' \neq 0$ for all $s \neq 0$, then each branch

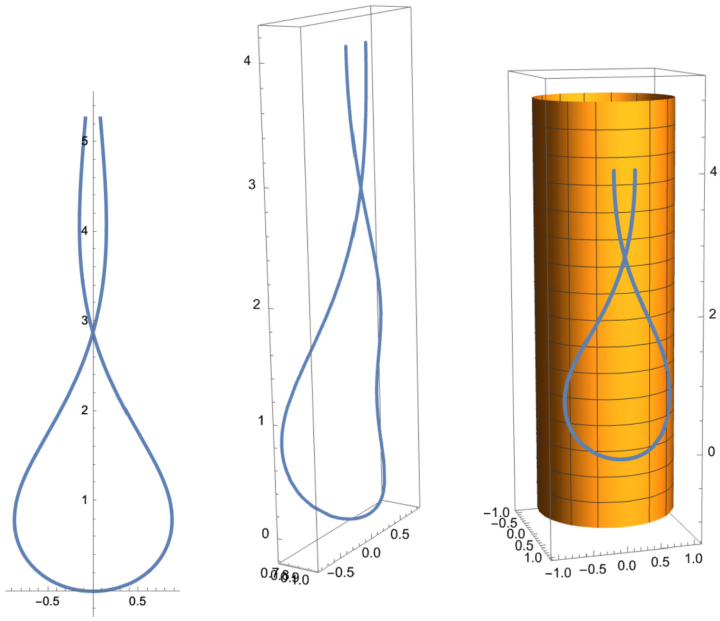

Fig. 2. Generating curves of rotational V-solitons with initial conditions (17). Left: the curve β. Middle and right: projection of the generating curve α on the xzt-space (middle) and as a subset of the cylinder $\mathbb{S}^1 \times \mathbb{R}$ (right).

of β, that is, $\beta(0,\infty)$ and $\beta(-\infty,0)$, are graphs on the v-axis. This proves that β is a bi-graph on the v-axis. See Fig. 2.

If $u(0) = u_0 \neq 0$ and $\theta(0) = 0$, then we know that $(u(s), \theta(s))$ converges towards the point $(0, \frac{\pi}{2})$. Since this point is a stable spiral, the curve β meets again the v-axis being asymptotic to this axis.

We summarize the above arguments.

Theorem 3.4. *Let Σ be a rotational V-soliton. Then Σ is the cylinder $\mathcal{C}$ or Σ is parametrized by (5) with the following properties:*

(1) *The curve $\beta(s) = (u(s), v(s))$ has self-intersections and it is asymptotic to the v-axis at infinity. In case that β satisfies the initial conditions (17), then β is a symmetric bi-graph with respect to the v-axis.*
(2) *The surface Σ is not embedded with infinitely many intersection points with the z-axis.*
(3) *The surface Σ is asymptotic to the cylinder $\mathcal{C}$ at infinity.*

In Fig. 3 we plot the surface Σ after the stereographic projection p_r of the first factor $\mathbb{S}^2$ into $\mathbb{R}^2$, $p_r \colon \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$, $p_r(x,y,z,t) = (\frac{x}{1-z}, \frac{y}{1-z}, t)$.

4. The class of R-solitons

In this section we study R-solitons, where the vector field R is

$$R = -y\partial_x + x\partial_y. \tag{18}$$

Notice that R is a Killing vector field whose infinitesimal isometries are rotations about the z-axis. Following our scheme, we will classify R-solitons that are vertical surfaces and next, rotational surfaces.

First, suppose that Σ is a vertical R-soliton. We know that Σ is parametrized by (3) and that the generating curve α is contained in $\mathbb{S}^2$, see (2). By (3) it is immediate that the curve α is a R-soliton to the curve shortening flow in $\mathbb{S}^2$. Such curves were originally studied in Section 2.3 of [7]. More recently,

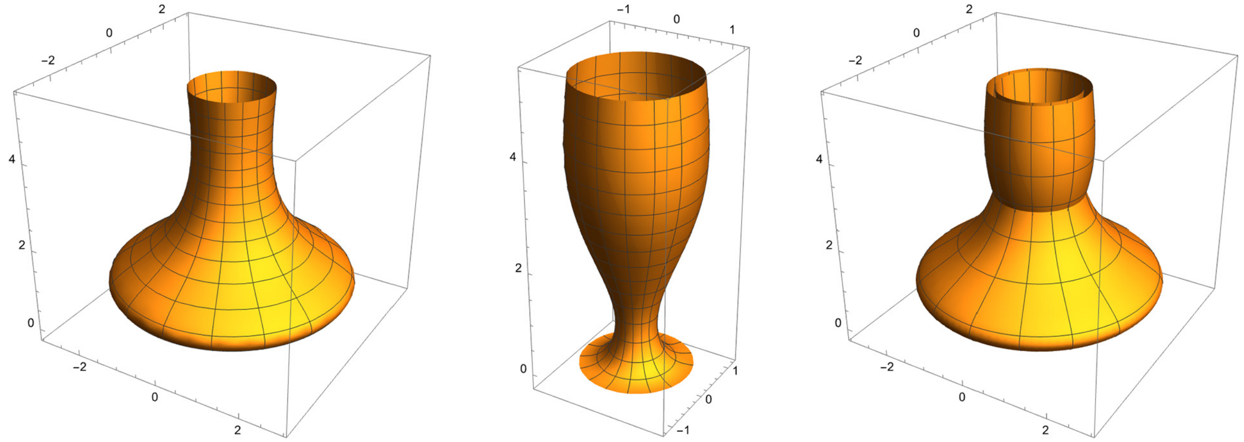

Fig. 3. A rotational V-soliton after the stereographic projection p_r. The surface after rotating β in the interval $[0, \infty)$ (left) and in the interval $(-\infty, 0]$ (middle). Right: the full surface.

properties of these curves were obtained in [17] by using the Frenet frame of α. In order to be self-contained, we restate some these properties in Theorem 4.2 by presenting a different proof thanks to an autonomous system of ordinary differential equations. The behavior of the trajectories of this system in Proposition 4.1 leads to the desired properties.

A first observation is that a vertical cylinder over the geodesic $\mathbb{S}^1 \times \{0\}$ of $\mathbb{S}^2$ is an example of vertical R-soliton. To be precise, let $(u(s), v(s)) = (0, s)$. Then $\alpha(s) = (\cos s, \sin s, 0)$ in (2) and the surface is the vertical cylinder over α which we have denoted by $\mathcal{C}$ in the previous section. This surface is minimal and it is immediate that N is orthogonal to R. Thus $\mathcal{C}$ is a R-soliton. Recall that $\mathcal{C}$ is also a rotational V-soliton.

Proposition 4.1. *Suppose that Σ is a vertical surface parametrized by (3). Then Σ is a R-soliton if and only if the generating curve α satisfies*

$$\begin{cases} u' = \cos u \cos\theta \\ v' = \sin\theta \\ \theta' = \sin\theta \sin u + (\cos u)^2 \cos\theta. \end{cases} \tag{19}$$

Proof. The expression of the unit normal N is given in (3). Thus

$$\langle N, R\rangle = -\cos u \cos\theta.$$

Since the expression of H is given in (7), then Eq. (1) becomes $\theta' = \sin\theta \sin u + (\cos u)^2 \cos\theta$, proving the result. □

As in the previous section, we project the solutions of (19) on the (u, θ)-plane, obtaining the autonomous system of differential equations

$$\begin{cases} u' = \cos u \cos\theta \\ \theta' = \sin\theta \sin u + (\cos u)^2 \cos\theta. \end{cases} \tag{20}$$

The equilibrium points are $(u, \theta) = (0, \pm\pi/2)$ together the points $(u, \theta) = (\pm\frac{\pi}{2}, 0)$ and translations of length π of these points in the θ variable. The equilibrium points $(u, \theta) = (0, \pm\pi/2)$ corresponds with the solution $u(s) = 0$ and $v(s) = \pm s$. In this case, and by the observation pointed out before this theorem, the surface Σ is the vertical cylinder $\mathcal{C}$. In contrast, the equilibrium points $(u, \theta) = (\pm\frac{\pi}{2}, 0)$ do not correspond with surfaces

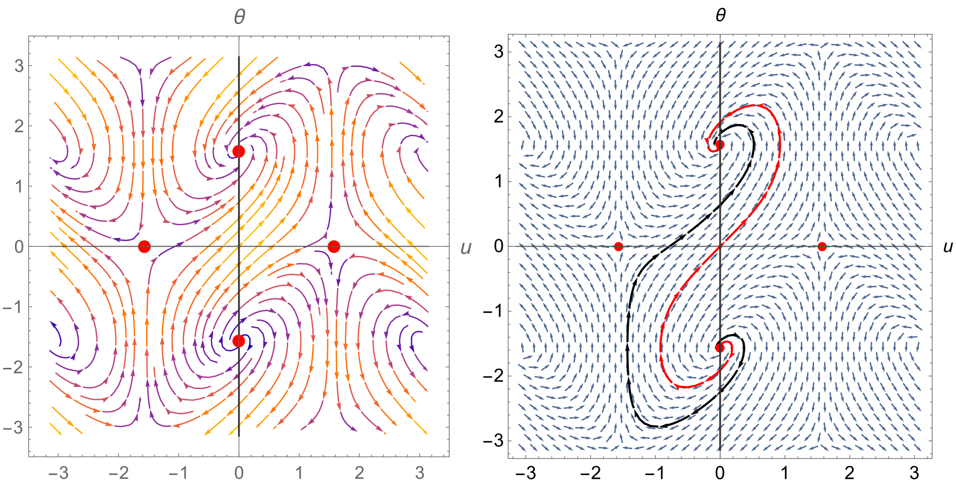

Fig. 4. Left: the (u, θ)-phase plane of (20). The red points are the equilibrium points $(0, \pm\frac{\pi}{2})$ and $(\pm\frac{\pi}{2}, 0)$. Right: two trajectories in the (u, θ)-phase plane.

because regularity is lost. In fact, coming back to the parametrization (3), the map Ψ is the parametrization of the vertical fiber at $(0, 0, 1) \in \mathbb{S}^2$.

The phase plane of (20) is the set $A = (-\pi, \pi) \times (-\pi, \pi)$ in the (u, θ)-plane by the periodicity of θ. If we now compute the linearized system at the points $(u, \theta) = (0, \pm\frac{\pi}{2})$, we find that they have the same character that the ones of the system (15). Thus we have that $(u, \theta) = (0, \frac{\pi}{2})$ is a stable spiral and $(u, \theta) = (0, -\frac{\pi}{2})$ is an unstable spiral. See Fig. 4.

For the points $(\frac{\pi}{2}, 0)$ and $(-\frac{\pi}{2}, 0)$, the linearized systems are

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

respectively. Since the eigenvalues are two real numbers with opposite signs, then both equilibrium points are saddle points ([14, Ch. 1]). See Fig. 4. A solution α of (19) for given initial conditions corresponds with a trajectory in the (u, θ)-phase plane. In Fig. 4, right, we show two trajectories. The red one acrosses the point $(0, 0)$ which corresponds with the initial conditions $(u(0), v(0)) = (0, 0)$ and $\theta(0) = 0$ in (19). From this trajectory, we deduce that the u-coordinate of α is bounded. On the other hand, the angle coordinate θ goes from $-\pi/2$ to $\pi/2$, that is, from an unstable spiral point to a stable one. In the limit, $\theta = \pi/2$, the angle θ of the curve α acrosses several times the value $\pi/$: see Fig. 5, left. Since the arguments now are similar as in the proof of Theorem 3.4, we omit the details. Fig. 5 shows generating curves of vertical R-solitons: see also Figs. 5 and 6 in [7].

Theorem 4.2. *Let Σ be a vertical R-soliton. Then Σ is the cylinder $\mathcal{C}$ or Σ is parametrized by (3) with the following properties:*

(1) *The curve $\beta(s) = (u(s), v(s))$ has self-intersections and it is asymptotic to the v-axis at infinity. In the case when β satisfies the initial conditions (17), then β is a symmetric bi-graph with respect to the v-axis.*
(2) *The surface Σ is not embedded with infinitely many intersection points with the z-axis.*
(3) *The surface Σ is asymptotic to the cylinder $\mathcal{C}$ at infinity.*

Remark 4.3. The vector field R in (18) corresponds with the evolution of the mean curvature flow where the angular speed is unitary. Suppose now that instead R, we consider λR, where $\lambda > 0$ represents the

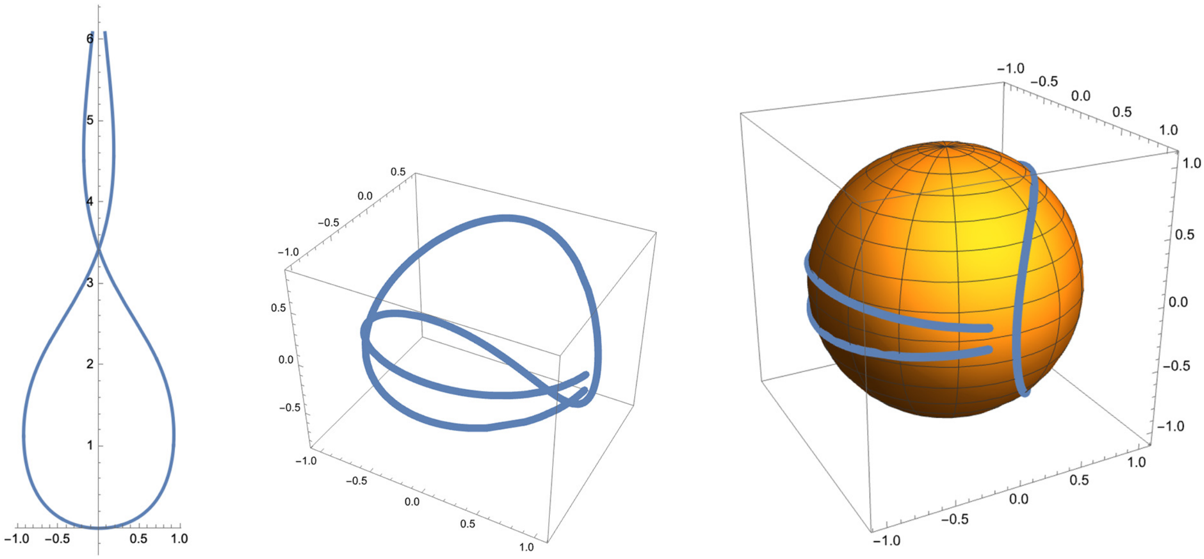

Fig. 5. Generating curves of vertical R-solitons with initial conditions (17). Left: solution curve $\beta(s) = (u(s), v(s))$. Middle: the generating curve α. Right: the generating curve α contained in the unit sphere $\mathbb{S}^2$.

angular speed. If Σ is a vertical surface which it is also a λ-soliton, the equation that satisfies the generating curve is (19) with the difference that the third equation is now $\theta' = \sin\theta\sin u + \lambda(\cos u)^2\cos\theta$. However, the presence of λ does not affect neither the equilibrium points, which are the same, nor their nature. In conclusion, we can assure that the qualitative properties of the vertical solitons for the vector λR are the same as those described in Theorem 4.2.

The second type of R-solitons of our study is those surfaces that are invariant by a one-parameter group of rotations of the first factor $\mathbb{S}^2$. Since we have defined in (18) the vector field R as the rotation about the direction $(0, 0, 1) \in \mathbb{S}^2$, we cannot *a priori* prescribe the rotational axis of the surface.

Theorem 4.4. *The only rotational R-solitons are:*

(1) *Slices $\mathbb{S}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$, viewed as rotational surfaces with respect to any axis of $\mathbb{S}^2$ and;*
(2) *Rotational minimal surfaces about the z-axis.*

Proof. Let Σ be a rotational R-soliton. In order to have manageable computations of the mean curvature H and the unit normal N of Σ, we will assume in this proof that the rotation axis of Σ is the z-axis. In particular, the surface is parametrized by (5). In consequence, the vector field R is now arbitrary (not necessarily given by (18)) because there is no *a priori* relation with the z-axis. The vector field R is determined by an orthonormal basis $B = \{E_1, E_2, E_3\}$ of $\mathbb{R}^3$. Let $p = (x_1, x_2, x_3, t) \in \Sigma \in \mathbb{S}^2 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R}$, where (x_1, x_2, x_3) are coordinates of $\mathbb{R}^3$ with respect to B. Then the vector field R can be expressed by

$$R(p) = -x_2E_1 + x_1E_2 = -\langle p, E_2\rangle E_1 + \langle p, E_1\rangle E_2.$$

We now write E_1 and E_2 with respect to the canonical basis of $\mathbb{R}^3$,

$$E_i = (\cos m_i \cos n_i, \cos m_i \sin n_i, \sin m_i), \quad i = 1, 2,$$

where $m_i, n_i \in \mathbb{R}$. The unit normal N and the mean curvature H of Σ were computed in (10) and (11), respectively. Then

$$\begin{aligned}\langle N,R\rangle &= -\langle\Psi,E_2\rangle\langle N,E_1\rangle+\langle\Psi,E_1\rangle\langle N,E_2\rangle\\ &=(\sin m_1\cos m_2\sin n_2-\cos m_1\sin n_1\sin m_2)\sin\theta\sin\varphi\\ &\quad+(\sin m_1\cos m_2\cos n_2-\cos m_1\cos n_1\sin m_2)\sin\theta\cos\varphi.\end{aligned}\tag{21}$$

Looking now at the soliton equation (1), we have that the right hand-side of (1), that is, $\langle N,R\rangle$, the variable φ does appear because of (21). However in the left hand-side of (1), the mean curvature H, formula (11), does not depend on φ. This implies that the coefficients of $\sin\varphi$ and $\cos\varphi$ in (21) must vanish. Both coefficients contain the factor $\sin\theta$. This gives the following discussion of cases.

(1) Case $\sin\theta(s)=0$ for all s. Then $u(s)=s$ and $v(s)$ is a constant function, $v(s)=t_0$, $t_0\in\mathbb{R}$. This proves that Σ is a slice $\mathbb{S}^2\times\{t_0\}$.
(2) Case $\sin\theta(s_0)\neq 0$ at some s_0. Then in an interval around $s=s_0$, we deduce

$$\sin m_1\cos m_2\sin n_2-\cos m_1\sin n_1\sin m_2=0,$$
$$\sin m_1\cos m_2\cos n_2-\cos m_1\cos n_1\sin m_2=0.$$

Both identities imply $E_1\times E_2=(0,0,1)$. Thus R coincides with the vector field defined in (18) and the rotation axis is the z-axis. Moreover, the right hand-side of (1) is 0, proving that the surface is minimal. This proves the result. □

Remark 4.5. Minimal surfaces in $\mathbb{S}^2\times\mathbb{R}$ of rotational type with respect to an axis in the first factor $\mathbb{S}^2$ were classified by Pedrosa and Ritoré [13].

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Data availability

No data was used for the research described in the article.

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