Omega 132 (2025) 103233

Contents lists available at ScienceDirect

Omega

journal homepage: www.elsevier.com/locate/omega

Revenue distribution in streaming[☆,☆☆]

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ARTICLE INFO

Keywords:
Revenue distribution
Proportional rule
Equal division rule
Axiom
Streaming

ABSTRACT

The streaming industry has experienced exponential growth over the past decade. Streaming platforms provide subscribers with unlimited access to a diverse range of services, including movies, TV shows, and music, in exchange for a subscription fee. We take an axiomatic approach to the problem of how to share the overall revenue obtained from subscription sales among services or content producers. In doing so, we provide normative justifications for several distribution rules. We formulate several axioms that convey ethical and operational principles. In the first group, we consider properties that guarantee equal and impartial treatment of services and subscribers. In the second group, we introduce requirements designed to safeguard allocation schemes from inconvenient alterations, namely, changes in the units of measurement of inputs, subscription sharing, or group decomposition. Our analysis reveals that different combinations of these axioms define two classes of rules that strike a balance between three focal schemes, each representing distinct perspectives on the egalitarian and proportional principles. To illustrate the practical implications of our theoretical model, we explore its potential application by assessing how various types of content impact the revenues of some of the most well-known Twitch streamers.

1. Introduction

In 2022, the Big Four firms in the over-the-top broadcasting market–Netflix, Amazon Prime Video, Disney+, and HBO Max–collectively reported an annual revenue of $103.5 billion. These companies provide viewers with a comprehensive media library comprising movies, series, and documentaries, accessible through a monthly subscription fee for unlimited content access. Some companies emphasize quality, featuring a select few shows that enjoy widespread viewer consumption, while others boast extensive media libraries to appeal to a broader consumer base. Regardless of the approach, it is evident that not all titles are equally in demand, with certain movies or series playing a more pivotal role in attracting subscribers and, consequently, increasing revenues. Given the varying relevance of content, the question arises: How should the revenue derived from selling subscription fees be allocated among content producers? It is noteworthy that this scenario extends beyond the over-the-top broadcasting market and can be applied to other industries and markets, such as Spotify, Twitch, YouTube, and other web-based services, where bundling products into packages proves more lucrative than individual sales (see [1]).

In our model, a *platform* is described by four elements: the set of *services* the platform provides (series, movies, artists, streamers, books, etc.), the set of *subscribers* with unlimited access to those services (viewers, users, readers, etc.), the *profile of subscription prices* indicating the fee paid by each subscriber, and the *consumption matrix* specifying the quantity of each service consumed by each subscriber. We define a *rule* as a mechanism for distributing the earned revenue (the sum of subscription prices) among the services operating on the platform. Three focal rules form the basis of our main findings: the *equal division rule*, which evenly divides the revenue among services; the *proportional rule*, which allocates revenue proportionally to the aggregate consumption

☆ Area: Decision Analysis and Preference Modeling. This manuscript was processed by Associate Editor J. Chen.

☆☆ The authors thanks to the Associate Editor and two anonymous reviewers for their detailed remarks and comments which have been very helpful to improve the paper. Juan Carlos Gonçalves-Dosantos acknowledges the grant PID2021-12403030NB-C31 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe/EU". Ricardo Martínez acknowledges the R&D&I project grant PID2023-147391NB-I00 funded by MCIU/AEI/10.13039/501100011033 and by the FSE+. Joaquín Sánchez-Soriano acknowledges the grant PID2022-137211NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe/EU" and from the Generalitat Valenciana, Spain under project PROMETEO/2021/063.

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https://doi.org/10.1016/j.omega.2024.103233
Received 24 June 2024; Accepted 15 November 2024
Available online 28 November 2024

of services; and the *subscriber-proportional rule*, which first assigns each subscriber's fee individually and proportionally based on their specific consumption before summing across subscribers. The latter two rules implement the *pro-rata* and *user-centric* principles, which are commonly applied in calculating remunerations in the music streaming industry.[1]

For our analysis, we follow the axiomatic approach, which is associated with a longstanding tradition in the economics literature that can be traced back to [4,5]. Rather than selecting from the previous or other rules directly, we propose to do so based on the axioms (or properties) that they satisfy. We consider two groups of properties. The first group encompasses three axioms reflecting fairness principles: *Equal treatment of equals* requires that services that have been equally consumed must obtain equal allocations; *order preservation* states that services that are consumed more must get higher awards; and the last property in this group, *neutrality*, is a standard notion of impartiality stating that subscribers' names must not play a role in revenue distribution. The second group of axioms implements several principles of stability: *Scale invariance* requires that the allocation is independent of the units of measurement of the inputs (namely, price and consumption); *non-advantageous transfer within subscriber* requires that the award of a service is not affected by an exchange in the consumption of two other services; *composition* states that the allocation is additive with respect to separable sets of subscribers; and *sharing proofness* states that revenue distribution among the services is immune to a well-known type of strategic behavior where subscribers share a subscription fee so as to pay less individually and yet have equal access to all the platform's services.

In our main findings, we characterize two families of rules. We identify the unique class of rules that satisfies equal treatment of equals, neutrality, scale invariance, non-advantageous transfer within subscriber, and composition. This class consists of linear compromises between the equal division and the subscriber-proportional rule (Theorem 1). Moreover, if we replace equal treatment of equals by order preservation, the family shrinks and we keep the convex combinations between these two focal rules (Corollary 1). We also find out that if, in addition to equal treatment of equals, neutrality, scale invariance, and non-advantageous transfer within subscriber, we require sharing proofness instead of composition, we characterize a class of rules that are specific linear compromises between the equal division and the proportional rules (Theorem 2). Once again, replacing equal treatment of equals by order preservation yields to the convex combinations between these two rules (Corollary 3). In addition, we have also provide normative foundations for the pro-rata and user-centric mechanisms (Corollaries 2 and 4). These results are obtained by adding the additional requirement of *null service* (non-consumed services are excluded from the distribution of revenue) to the families of rules characterized in Theorems 1 and 2.

Prior literature has also addressed the question of how to measure the contribution of single elements to global or social success. Singal et al. [6] provide an axiomatic justification of the *counterfactual adjusted Shapley value*, which measures the contribution of individual advertiser actions (e.g., emails, display ads, search ads, etc.) to eventual customer acquisition. Lopez-Navarrete et al. [7,8] explore the allocation of revenues in a video-sharing platform, such as YouTube, by considering user navigation patterns within the platform. They employ dynamic games associated with the problem and propose various allocation schemes grounded in the structure of the Shapley value. Schlicher et al. [9] explore conditions under which content creators have no incentives to leave the platform and thus stability can be preserved from the point of view of game theory. In contrast, our approach takes a different route; instead of delving into cooperative games, we adopt an axiomatic perspective to identify suitable rules.

The discourse surrounding the distribution of revenue from subscriptions to music streaming platforms among artists has gained notable significance in recent years. Within this context, the pro-rata and user-centric schemes have emerged as the predominant and widely applied methods, as evidenced by works such as [2,3,10–13]. Alaei et al. [14] explore the strategic implications of the pro-rata and user-centric schemes in a model featuring a two-sided streaming service platform that generates revenue by charging users a subscription fee for unlimited content access. Lei [15] develops an endogenous model that considers both schemes, enabling artists to strategically choose streaming times to maximize their earnings. Instead of the strategic considerations, our contribution to this discourse lies in establishing normative foundations for broader classes of rules that encompass the pro-rata and user-centric schemes. Bergantiños and Moreno-Ternero [16,17] also delve into revenue sharing in the streaming industry, focusing specifically on music streaming services such as Spotify. They formulate and describe allocation mechanisms based on principles such as pro-rata and user-centric approaches, among others. The axioms we propose in this paper differ from those presented in these other works. They consider axioms that reflect the impact of additional users on the platform, as well as properties that propose reasonable minimum payments. In addition, they characterize individual rules, which differ from the family of rules that we obtain in our main results.

In the context of attribution problems, various authors have explored analogous inquiries. Ginsburgh and Zang [18], Bergantiños and Moreno-Ternero [19], and [20] scrutinize the museum pass problem. This problem models how to share revenues obtained by a consortium of museums when visitors can buy a limited-time subscription or access pass allowing unlimited usage of their museums.[2] These problems differ from our model in two relevant aspects. One, in the museum pass problem the consumption is dichotomous, that is, the pass holder visits or not the museum. In contrast, in our setting we do not impose this limitation as we allow for different intensities in the consumption of services. Besides, in the museum pass problem the subscription price is the same for all subscribes, while our model permits different prices. Consequently, our problem represents a broader framework than the museum pass problem. As expected, certain characterization results from the latter problem (e.g. [19,20]) do not hold in our model.

Our proposal also shares some similarities with the bipartite rationing problem [26,27], in which a group of agents collectively shares a single resource that comes in different "types". Each agent has a claim over only a subset of these resource types, and their claims can overlap in arbitrary ways. The objective is to fairly allocate the various resource types among the claimants. A bipartite rationing problem is described by a set of agents (and their claim), a set of types (and their capacity) and a bipartite (unweighted) graph that indicates whether each agent can consume each type. In our scenario, services and subscribers correspond to the agents and types in the bipartite rationing problem. The capacity can be associated with the subscription price, and the challenge lies in distributing this capacity among the services (or agents). However, there are notable differences between the two models. First, determining the services' claims is not straightforward. Various possibilities exist, such as overall revenue, average claims, or subscription prices of subscribers who consume the services. It remains unclear which approach should be adopted. Second, in the bipartite rationing problem, the consumption matrix does not play a role. This model could be extended to consider weighted graphs, where the edge weights represent service consumption by each subscriber. Finally, in the setting we propose any agent (service) may potentially receive supply from any type (subscriber), even when the service has not been consumed by the agent, which contrast with the nature of the bipartite graph to restrict some transfers from agents to types.

[1] See [2,3] for a detailed discussion on these principles.

[2] The museum pass problem can be interpreted to fit with many other applications, see, for instance, [21–24], and [25].

Regarding allocation mechanisms, while both approaches implement proportional schemes, they differ in formulation and information requirements. Notably, the consumption matrix, which is absent in the bipartite rationing problem, plays a crucial role in defining the two proportional rules we propose.

We conclude our analysis by providing an empirical illustration. We acknowledge that the direct application of the rules we propose and characterize in this paper requires access to private information held by streaming platforms such as Amazon, Netflix, and Spotify, yet unfortunately this data is not accessible to third parties. Nevertheless, the theoretical model we formulate is adaptable to Twitch, an online platform where users engage with content provided by streamers. This content typically involves gaming, chatting, or broadcasting sporting events, among other activities. Certain Twitch streamers are rising to celebrity status due to their engaging streams. However, not all types of content carry equal weight in determining their revenues. Through the application of the rules we characterize in this paper, we can assess the impact of each content type on the revenues of individual streamers. This offers a valuable tool for identifying the most lucrative topics, contributing to a deeper understanding of the factors influencing profitability in the Twitch ecosystem.

The rest of the paper is organized as follows. In Section 2, we present the model and three rules to distribute the revenue among the services of a streaming platform. We introduce, in Section 3, several axioms adapted to the streaming context. In Section 4, we characterize two families of rules. In Section 5, we apply the theoretical model to measure the economic impact of different types of content on the Twitch streaming platform. Finally, Section 6 provides concluding remarks.

2. Model and allocation rules

Let $\mathbb{N}$ represent the set of natural numbers, and let $\mathcal{N}$ be the set of all finite and non-empty subsets of $\mathbb{N}$. A **platform** is described by a 4-tuple (N, S, p, C), where $N = \{1, \ldots, |N|\} \in \mathcal{N}$ ($|N| \geq 3$) is the set of **services** provided by the platform, $S = \{1, \ldots, |S|\} \in \mathcal{N}$ is the set of **subscribers** who have unlimited access to the services in N, $p = (p_1, \ldots, p_{|S|}) \in \mathbb{R}^{|S|}_{++}$ represents the **subscription prices** paid by each subscriber, and $C \in \mathbb{R}^{|N|\times|S|}_{+}$ is the **consumption matrix**, each of whose entries $C_{is} \in \mathbb{R}_+$ indicates the quantity of service i consumed by subscriber s. The quantity of service can be measured in terms of time, the number of streams viewed, or other metrics. Furthermore, these quantities reflect actual consumption by the subscribers, not their willingness to consume. Finally, we denote by $\mathcal{D}^N$ the set of all platforms with service set N, and by $\mathcal{D}$ the set of all platforms.

We denote by $C_{i\cdot}$ the ith row of C, which represents the consumption of service i. We also denote by $C_{\cdot s}$ the sth column of C, which represents the consumption made by subscriber s. Therefore, $\|C_{i\cdot}\| = \sum_{s\in S} C_{is}$ and $\|C_{\cdot s}\| = \sum_{i\in N} C_{is}$ are the total consumption of service $i \in N$ and subscriber $s \in S$, respectively. We restrict ourselves to consumption matrices such that $\|C_{\cdot s}\| > 0$ for all $s \in S$, that is, we assume that any subscriber has paid the subscription fee actually consumes part of the content offered by the platform. Given a set of services $N' \subseteq N$, or a set of subscribers $S' \subseteq S$, $C_{N\setminus N'}$ and $C_{S\setminus S'}$ denote the matrices resulting from removing the rows in N' and the columns in S', respectively. For any two vectors $x, y \in \mathbb{R}^{|N|}$, we denote the vector $x + y = \left(x_1 + y_1, x_2 + y_2, \ldots, x_{|N|} + y_{|N|}\right)$. Given two matrices $C \in \mathbb{R}^{|N|\times|S|}_{+}$ and $C' \in \mathbb{R}^{|N|\times|S'|}_{+}$ for any two disjoint set of subscribers S and S', (C, C') is the matrix resulting from concatenating (by columns) C and C', that is, $(C, C')_{\cdot s} = C_{\cdot s}$ if $s \in S$ or $(C, C')_{\cdot s} = C'_{\cdot s}$ if $s \in S'$.

A *rule* is a way to distribute among the services the aggregate revenue obtained by the platform ($\|p\| = \sum_{s\in S} p_s$), that is, it is a mapping $R : \mathcal{D}^N \longrightarrow \mathbb{R}^{|N|}_{+}$ such that

$$\sum_{i\in N} R_i(N, S, p, C) = \|p\|,$$

where $R_i(N, S, p, C)$ indicates the award of service $i \in N$.

Although it is possible to generate many rules for revenue allocation, we now propose three rules that emerge as natural mechanisms. The first one is straightforward, it equally splits the revenue among all services, as follows:

Equal division rule. For each $(N, S, p, C) \in \mathcal{D}^N$ and each $i \in N$,

$$R^{ED}_i(N, S, p, C) = \frac{\|p\|}{|N|}.$$

The next two rules are inherently intuitive and both adhere to the principle of proportionality, albeit from different perspectives. The *proportional rule* operates from a macro perspective, distributing revenue and allocating the aggregate income among services based on their overall consumption. On the other hand, the *subscriber-proportional rule* adopts a micro perspective; in particular, it proportionally divides the price paid by each individual subscriber among the services she has consumed. This process is applied to all subscribers, and the resulting allocation is simply the aggregate sum across them. The latter two rules mirror the underlying principles of two prominent remuneration methods used in the domain of music streaming: the *pro-rata mechanism* (for the *proportional rule*) and the *user-centric rule* (for the *subscriber-proportional rule*).[3]

Proportional rule. For each $(N, S, p, C) \in \mathcal{D}^N$ and each $i \in N$,

$$R^{P}_i(N, S, p, C) = \frac{\|C_{i\cdot}\|}{\sum_{j\in N} \|C_{j\cdot}\|}\|p\|.$$

Subscriber-proportional rule. For each $(N, S, p, C) \in \mathcal{D}^N$ and each $i \in N$,

$$R^{SP}_i(N, S, p, C) = \sum_{s\in S} \frac{C_{is}}{\|C_{\cdot s}\|} p_s.$$

The next example illustrates the functioning of the equal division, proportional, and subscriber-proportional rules.

Example 1. Consider the platform where $N = \{1, 2, 3\}$, $S = \{1, 2, 3, 4, 5, 6\}$, $p = (2, 3, 1, 2, 1, 3)$, and C is given by

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The next table shows the distribution of the revenue ($\|p\| = 12$) among the services according to the equal division, proportional, and subscriber-proportional rules.

	Services		
Rule	1	2	3
R^{ED}	4	4	4
R^{P}	$\frac{20}{7}$	$\frac{64}{7}$	0
R^{SP}	$\frac{141}{28}$	$\frac{195}{28}$	0

3. Properties of fairness and stability

No single criterion exists for determining the preference of one rule over another. In this regard, we propose an axiomatic approach by introducing a set of axioms tailored to the streaming context. By selecting the axioms that are most suitable for the specific circumstances of the problem, it is possible to make a decision regarding which rule to use. The first axiom is a minimum requirement of impartiality, that is, if two services are equally consumed by all viewers, then they should be equally awarded.

[3] For a detailed formulation of these principles and a discussion of their implications in the case of the music streaming industry, see [2,3].

Equal treatment of equals. For each $(N,S,p,C) \in \mathcal{D}^N$, and each pair $\{i,j\} \subseteq N$, if $C_{i\cdot} = C_{j\cdot}$,

$$R_i(N,S,p,C) = R_j(N,S,p,C).$$

A slightly stronger version of the previous principle requires that services that are consumed more frequently must get higher allocations.

Order preservation. For each $(N,S,p,C) \in \mathcal{D}^N$, and each $\{i,j\} \subseteq N$, if $C_{is} \geq C_{js}$ for all $s \in S$,

$$R_i(N,S,p,C) \geq R_j(N,S,p,C).$$

The next axiom also represents a standard notion of fairness. The distribution of revenue should depend only on the consumption and the subscription prices, but not on the subscribers' names. Therefore, if we permute the subscribers, the allotment should remain unaltered. A permutation of a finite set S is a bijection $\sigma : S \longrightarrow S$. We denote by Π_S the set of all permutations of S.[4]

Neutrality. For each $(N,S,p,C) \in \mathcal{D}^N$, each $\sigma \in \Pi_S$, and each $i \in N$,

$$R_i(N,S,p,C) = R_i(N,S^\sigma,p^\sigma,C^\sigma),$$

where $S^\sigma_s = S_{\sigma(s)}$, $p^\sigma_s = p_{\sigma(s)}$, and $C^\sigma_{is} = C_{i\sigma(s)}$ for all $i \in N$ and $s \in S$.

Scale invariance states that the units of measurement of the inputs do not affect the allocation. That is to say, for example, that the distribution of revenue does not change if consumption is measured in thousands or millions of viewers. Similarly, if subscription prices are converted from dollars to euros (and, therefore, the currency of the revenue is also converted), then the distribution changes accordingly.

Scale invariance. For each $(N,S,p,C) \in \mathcal{D}^N$, and each $(\lambda_1,\lambda_2) \in \mathbb{R}^2_{++}$,

(i) $R(N,S,\lambda_1 p,C) = \lambda_1 R(N,S,p,C)$. [Scale invariance in prices].
(ii) $R(N,S,p,\lambda_2 C) = R(N,S,p,C)$. [Scale invariance in consumption].

Consider the case of a subscriber who exchanges α units of consumption of service i for the same α units of consumption of service j. The next property requires that such a conversion does not alter the allocation of other services different from i and j.

Non-advantageous transfer within subscriber. For each $(N,S,p,C) \in \mathcal{D}^N$, if $\{i,j\} \subseteq N$, $s \in S$, and $\alpha \in \mathbb{R}_+$ are such that $C_{is} - \alpha \geq 0$, then for each $k \in N \backslash \{i,j\}$,

$$R_k(N,S,p,C) = R_k(N,S,p,C^\alpha),$$

where $C^\alpha_{is} = C_{is} - \alpha$, $C^\alpha_{js} = C_{js} + \alpha$, $C^\alpha_{ks} = C_{ks}$ for all $k \in N\backslash\{i,j\}$, and $C^\alpha_{ks'} = C_{ks'}$ for all $k \in N$ and $s' \in S\backslash\{s\}$.

Suppose that subscribers are split into two classes (for example, premium and regular) and we solve each problem separately. By *composition*, the sum of those two allocations must equal the allotment obtained by applying the rule to the whole set of subscribers.[5]

Composition. For each $(N,S,p,C),(N,S',p',C') \in \mathcal{D}^N$ such that $S \cap S' = \emptyset$, it holds that

$$R(N,S\cup S',(p,p'),(C,C')) = R(N,S,p,C) + R(N,S',p',C'),$$

where (C,C') is the matrix resulting from concatenating (by columns) C and C', and (p,p') is the vector resulting from concatenating p and p'.

[4] This property should not be confused with the standard notion of *anonymity*. While neutrality refers to subscribers (their labeling is not relevant for the distribution of the revenue), *anonymity* states what is not relevant is the label of the agents, in our case the services.

[5] Relating the outcome of a group of services with the outcome of subgroups is a typical approach in the literature. *Composition* is akin to those namesakes used in the classical literature on income inequality measurement (e.g., [28]), poverty measurement (e.g., [29]), income mobility measurement (e.g., [30]), voting (e.g., [31]), and allocation problems (e.g., [24,32,33], and [25]).

It is common for streaming platform subscribers to share their subscriptions with friends and relatives. When two subscribers merge, if the platform receives the same revenue as when they were separate, and also their joint consumption equals the sum of their individual consumption; then, it seems reasonable that the allocation of services consumed by both subscribers should be the same in both individual and joint cases. *Sharing proofness* requires that the allocation is not altered by this type of situation.

Sharing proofness. For each $(N,S,p,C) \in \mathcal{D}^N$, each non-empty $S' \subseteq S$, and each $s \in S'$, if $p'_s = \sum_{t\in S'} p_t$ and $C'_{is} = \sum_{t\in S'} C_{it}$ for any $i \in N$, then

$$R_i(N,S,p,C) = R_i\left(N,\{s\}\cup S\backslash S',(p'_s,p_{S\backslash S'}),(C'_{\cdot s},C_{S\backslash S'})\right).$$

The following axiom states that if a service is not consumed by any subscriber, then its allocation should be zero.

Null service. For each $(N,S,p,C) \in \mathcal{D}^N$, and each $i \in N$, if $C_{is} = 0$ for any $s \in S$, then

$$R_i(N,S,p,C) = 0.$$

4. Two families of parametrized rules

In this section, we present our main characterization results. First, let us observe that the non-advantageous transfer within subscriber property implies that, if we reallocate the consumption from a set of services, then the aggregate allocation of all of them remains unchanged.

Lemma 1. *If a rule satisfies non-advantageous transfer within subscriber, then it has the following property: Let $(N,S,p,C) \in \mathcal{D}^N$, and let $M \subseteq N$ and $\overline{C} \in \mathbb{R}^{|N|\times|S|}_+$ such that*

(i) $\sum_{j\in M} \overline{C}_{j\cdot} = \sum_{j\in M} C_{j\cdot}$, and
(ii) $\overline{C}_{j\cdot} = C_{j\cdot}$ for any $j \in N\backslash M$.

It holds that

$$\sum_{j\in M} R_j(N,S,p,C) = \sum_{j\in M} R_j(N,S,p,\overline{C}).$$

Proof. Let $(N,S,p,C) \in \mathcal{D}^N$ and consider $M \subseteq N$, if $\overline{C} \in \mathbb{R}^{|N|\times|S|}_+$ is such that $\sum_{j\in M} \overline{C}_{j\cdot} = \sum_{j\in M} C_{j\cdot}$, and $\overline{C}_{j\cdot} = C_{j\cdot}$ for any $j \in N\backslash M$. We can pass from C to $\overline{C}$ in a finite sequence of applications of *non-advantageous transfer within subscriber*. Notice that *non-advantageous transfer within subscriber* requires that in all those steps, the distribution among the services out of M is not altered. For this reason, $R_k(N,S,p,C) = R_k(N,S,p,\overline{C})$ for any $k \in N\backslash M$; and by definition of rule, we know that $\sum_{j\in N} R_j(N,S,p,C) = \sum_{j\in N} R_j(N,S,p,\overline{C})$. Therefore,

$$\begin{aligned}\sum_{j\in M} R_j(N,S,p,C) &= \sum_{j\in N} R_j(N,S,p,C) - \sum_{j\in N\backslash M} R_j(N,S,p,C)\\ &= \sum_{j\in N} R_j(N,S,p,\overline{C}) - \sum_{j\in N\backslash M} R_j(N,S,p,\overline{C})\\ &= \sum_{j\in M} R_j(N,S,p,\overline{C}). \quad \square\end{aligned}$$

In order to mitigate repetitive arguments in the proofs the characterization results, we introduce the following two technical lemmas.

Lemma 2. *For each rule R and each parameter $\beta \in \left[0,\frac{|N|}{|N|-1}\right]$, the following rule is well-defined*

$$T = \beta R^{ED} + (1-\beta)R.$$

Proof. First, we have that

$$\sum_{i\in N} T_i(N,S,p,C) = \beta \sum_{i\in N} R^{ED}_i(N,S,p,C) + (1-\beta)\sum_{i\in N} R_i(N,S,p,C)$$

$$= \beta\|p\| + (1-\beta)\|p\| = \|p\| \cdot$$

Now, for any $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$, we have that $T_i(N,S,p,C) \geq 0$ for all $i \in N$. Indeed, when $\beta \in [0,1]$, the result is trivially true since $R_i^{ED}(N,S,p,C) \geq 0$ and $R_i(N,S,p,C) \geq 0$, for all $i \in N$. Next, we check the cases for $\beta > 1$. In these cases, we have that

$$\begin{aligned} T_i(N,S,p,C) &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i(N,S,p,C) \\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)R_i(N,S,p,C) \\ &\geq \beta\frac{\|p\|}{|N|} + (1-\beta)\|p\| = \|p\| - \|p\|\beta\left(\frac{|N|-1}{|N|}\right). \end{aligned}$$

Therefore, if $\beta \leq \frac{|N|}{|N|-1}$, then $\|p\| - \|p\|\beta\left(\frac{|N|-1}{|N|}\right) \geq 0$. □

Lemma 3. *If a rule R satisfies equal treatment of equals, scale invariance and non-advantageous transfer within subscriber, then, for each* $(N,\{s\},p^{(s)},(C_{1s}^{(s)},0,\dots,0)^T) \in \mathcal{D}^N$*, there exists* $\hat{\mu}^s \in \mathbb{R}_+$*, such that*

$$R_j(N,\{s\},p^{(s)},(C_{1s}^{(s)},0,\dots,0)^T) = p^{(s)}\hat{\mu}^s,$$

for all $j \in N\backslash\{1\}$.

Proof. Let us define $\mu^s(p^{(s)}) = R_2(N,\{s\},p^{(s)},(1,0,\dots,0)^T)$. In particular, $\mu^s(1) = R_2(N,\{s\},1,(1,0,\dots,0)^T)$. To simplify the notation, we define $\hat{\mu}^s = \mu^s(1)$. *Equal treatment of equals* implies that $R_j(N,\{s\},1,(1,0,\dots,0)^T) = \hat{\mu}^s$ for any $j \in N\backslash\{1\}$. In application of *scale invariance*, it follows that $\hat{\mu}^s$ is independent of the consumption. That is, $\hat{\mu}^s = R_2(N,\{s\},1,(C_{1s}^{(s)},0,\dots,0)^T)$ for any $C_{1s}^{(s)} \in \mathbb{R}_+$. Besides, $\hat{\mu}^s$ is also independent of what the *non-null service* is. Indeed, let $i \in N\backslash\{1\}$ and let $\hat{C}^{(s)} \in \mathbb{R}_+^{|N|}$ be such that $\hat{C}_{is}^{(s)} = 1$ and $\hat{C}_{js}^{(s)} = 0$ for any $j \in N\backslash\{i\}$. *Non-advantageous transfer within subscriber* implies that $R_j(N,\{s\},1,(1,0,\dots,0)^T) = R_j(N,\{s\},1,\hat{C}^{(s)}) = \hat{\mu}^s$ for any $j \in N\backslash\{1,i\}$. By *equal treatment of equals*, we conclude that $R_j(N,\{s\},p^{(s)},\hat{C}^{(s)}) = \hat{\mu}^s$ for any $j \in N\backslash\{i\}$. *Scale invariance*, in this case with respect to the prices, also implies that $\mu^s(p^{(s)}) = R_2(N,\{s\},p^{(s)},(1,0,\dots,0)^T) = p^{(s)}R_2(N,\{s\},1,(1,0,\dots,0)^T) = p^{(s)}\hat{\mu}^s$. □

The first theorem states that any rule that fulfills equal treatment of equals, neutrality, scale invariance, non-advantageous transfer within subscriber, and composition must be a combination between the *equal division* and the *susbcriber-proportional* rules.

Theorem 1. *A rule R satisfies equal treatment of equals, neutrality, scale invariance, non-advantageous transfer within subscriber, and composition if and only if there exists* $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$ *such that, for each* $(N,S,p,C) \in \mathcal{D}^N$,

$$R(N,S,p,C) = \beta R^{ED}(N,S,p,C) + (1-\beta)R^{SP}(N,S,p,C).$$

Proof. First, by Lemma 2 the previous family is well-defined. Now, we prove that any of the members of the family satisfies the axioms in the statement. Let $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$.

- Equal treatment of equals. Let $(N,S,p,C) \in \mathcal{D}^N$ and $\{i,j\} \subseteq N$ such that $C_{i\cdot} = C_{j\cdot}$. It follows that

$$\begin{aligned} R_i(N,S,p,C) &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^{SP}(N,S,p,C) \\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{C_{is}}{\|C_{\cdot s}\|}p_s = \beta\frac{\|p\|}{|N|} \\ &+ (1-\beta)\sum_{s\in S}\frac{C_{js}}{\|C_{\cdot s}\|}p_s \\ &= \beta R_j^{ED}(N,S,p,C) + (1-\beta)R_j^{SP}(N,S,p,C) \\ &= R_j(N,S,p,C). \end{aligned}$$

- Neutrality. Let $(N,S,p,C) \in \mathcal{D}^N$, $\sigma \in \Pi_S$, and $i \in N$. It follows that

$$\begin{aligned} R_i(N,S,p,C) &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^{SP}(N,S,p,C) \\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{C_{is}}{\|C_{\cdot s}\|}p_s \\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{C_{i\sigma(s)}}{\|C_{\cdot\sigma(s)}\|}p_\sigma(s) \\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{C_{is}^\sigma}{\|C_{\cdot s}^\sigma\|}p_s^\sigma \\ &= \beta R_i^{ED}(N,S^\sigma,p^\sigma,C^\sigma) + (1-\beta)R_i^{SP}(N,S^\sigma,p^\sigma,C^\sigma) \\ &= R_i(N,S^\sigma,p^\sigma,C^\sigma). \end{aligned}$$

- Scale invariance. Let $(N,S,p,C) \in \mathcal{D}^N$, $i \in N$, and $(\lambda_1,\lambda_2) \in \mathbb{R}_{++}^2$. It follows that

$$\begin{aligned} R_i(N,S,\lambda_1 p,C) &= \beta R_i^{ED}(N,S,\lambda_1 p,C) + (1-\beta)R_i^{SP}(N,S,\lambda_1 p,C) \\ &= \beta\frac{\lambda_1\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{C_{is}}{\|C_{\cdot s}\|}\lambda_1 p_s \\ &= \lambda_1\left(\beta\frac{\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{C_{is}}{\|C_{\cdot s}\|}p_s\right) \\ &= \lambda_1\left(\beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^{SP}(N,S,p,C)\right) \\ &= \lambda_1 R_i(N,S,p,C); \end{aligned}$$

$$\begin{aligned} R_i(N,S,p,\lambda_2 C) &= \beta R_i^{ED}(N,S,p,\lambda_2 C) + (1-\beta)R_i^{SP}(N,S,p,\lambda_2 C) \\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{\lambda_2 C_{is}}{\|\lambda_2 C_{\cdot s}\|}p_s \\ &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^{SP}(N,S,p,C) \\ &= R_i(N,S,p,C). \end{aligned}$$

- Non-advantageous transfer within subscriber. Let $(N,S,p,C) \in \mathcal{D}^N$, $i,j \in N$, $s \in S$, $\alpha \in \mathbb{R}_+$ such that $C_{is} - \alpha \geq 0$, and C^α such that $C_{is}^\alpha = C_{is} - \alpha$, $C_{js}^\alpha = C_{js} + \alpha$, and $C_{zs}^\alpha = C_{zs}$ for all $z \in N\backslash\{i,j\}$. It follows that

$$\begin{aligned} R_z(N,S,p,C) &= \beta R_z^{ED}(N,S,p,C) + (1-\beta)R_z^{SP}(N,S,p,C) \\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\sum_{s\in S}\frac{C_{zs}}{\|C_{\cdot s}\|}p_s = \beta\frac{\|p\|}{|N|} \\ &+ (1-\beta)\sum_{s\in S}\frac{C_{zs}^\alpha}{\|C_{\cdot s}^\alpha\|}p_s \\ &= \beta R_z^{ED}(N,S,p,C^\alpha) + (1-\beta)R_z^{SP}(N,S,p,C^\alpha) \\ &= R_z(N,S,p,C^\alpha). \end{aligned}$$

- Composition. Let $(N,S,p,C),(N,S',p',C') \in \mathcal{D}^N$ such that $S \cap S' = \emptyset$. Let $i \in N$. It follows that

$$\begin{aligned} R_i(N,S\cup S',(p,p'),(C,C')) &= \beta R_i^{ED}(N,S\cup S',(p,p'),(C,C')) \\ &+ (1-\beta)R_i^{SP}(N,S\cup S',(p,p'),(C,C')) \\ &= \beta\frac{\sum_{s\in S\cup S'}(p,p')_s}{|N|} \\ &+ (1-\beta)\sum_{s\in S\cup S'}\frac{(C,C')_{is}}{\|(C,C')_{\cdot s}\|}(p,p')_s \\ &= \beta\left(\frac{\sum_{s\in S}p_s}{|N|} + \frac{\sum_{s\in S'}p_s}{|N|}\right) \\ &+ (1-\beta)\left(\sum_{s\in S}\frac{C_{is}}{\|C_{\cdot s}\|}p_s + \sum_{s\in S'}\frac{C'_{is}}{\|C'_{\cdot s}\|}p'_s\right) \\ &= \beta\left(R_i^{ED}(N,S,p,C) + R_i^{ED}(N,S',p',C')\right) \\ &+ (1-\beta)\left(R_i^{SP}(N,S,p,C)\right. \\ &+ \left.R_i^{SP}(N,S',p',C')\right) \\ &= R_i(N,S,p,C) + R_i(N,S',p',C'). \end{aligned}$$

Note that by arguments analogous to those used previously, it can be concluded that the equal division and subscriber-proportional rules also satisfy *composition*.

Let us focus now on the converse implication. Let R be a rule that satisfies all the properties in the statement of the theorem. Let

$(N, S, p, C) \in \mathcal{D}^N$. Suppose that S is a singleton (i.e., there is only one subscriber $S = \{s\}$), and let $p^{(s)}$ and $C^{(s)} = \left(C^{(s)}_{1s}, \ldots, C^{(s)}_{|N|s}\right)^T$ be the corresponding subscription price and consumption matrix of this platform, respectively.

Since R satisfies *non-advantageous transfer within subscriber* and *equal treatment of equals*, and taking into account Lemma 1, there exists a function $A^s : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_+$ such that, for each $i \in N$,[6]

$$R_i(N, \{s\}, p^{(s)}, C^{(s)}) = A^s(\|C^{(s)}\|, p^{(s)}) + \frac{C^{(s)}_{is}}{\|C^{(s)}\|}\left[p^{(s)} - \sum_{i \in N} A^s(\|C^{(s)}\|, p^{(s)})\right].$$

Observe that as the term $A^s(\|C^{(s)}\|, p^{(s)})$ only depends on the aggregate consumption $\|C^{(s)}\|$ (but not its distribution across services) and the subscription price $p^{(s)}$, it can be written as

$$A^s(\|C^{(s)}\|, p^{(s)}) = R_2(N, \{s\}, p^{(s)}, (\|C^{(s)}\|, 0, \ldots, 0)^T).$$

Since R satisfies *equal treatment of equals*, *scale invariance* and *non-advantageous transfer within subscriber*, by Lemma 3, $R_2(N, \{s\}, p^{(s)}, (\|C^{(s)}\|, 0, \ldots, 0)^T) = p^{(s)}\hat{\mu}^s$, for some $\hat{\mu}^s \in \mathbb{R}_+$. Therefore, $A^s(\|C^{(s)}\|, p^{(s)}) = p^{(s)}\hat{\mu}^s$, and then, for each $i \in N$,

$$R_i(N, \{s\}, p^{(s)}, C^{(s)}) = p^{(s)}\hat{\mu}^s + \frac{C^{(s)}_{is}}{\|C^{(s)}\|} p^{(s)}\left[1 - |N|\hat{\mu}^s\right].$$

Now, letting $\beta^s = |N|\hat{\mu}^s$, we obtain that, for each $i \in N$,

$$R_i(N, \{s\}, p^{(s)}, C^{(s)}) = \beta^s R^{ED}_i(N, \{s\}, p^{(s)}, C^{(s)}) + (1-\beta^s) R^{SP}_i(N, \{s\}, p^{(s)}, C^{(s)}).$$

Since $\hat{\mu}^s = R_2(N, \{s\}, 1, (1, 0, \ldots, 0)^T)$, it follows that $\hat{\mu}^s \in \left[0, \frac{1}{|N|-1}\right]$. Therefore, $\beta^s \in \left[0, \frac{|N|}{|N|-1}\right]$.

Now, suppose that S is such that $|S| \geq 2$. Notice that

$$p = (p^{(1)}, \ldots, p^{(|S|)}) \text{ and } C = (C^{(1)}, \ldots, C^{(|S|)}),$$

where each pair $p^{(s)}$ and $C^{(s)}$ corresponds to the subscription price and the consumption matrix with only one subscriber, respectively. Since R satisfies *composition*, it follows that, for each $i \in N$,

$$\begin{aligned} R_i(N, S, p, C) &= \sum_{s \in S} R_i(N, S, p^{(s)}, C^{(s)}) \\ &= \sum_{s \in S} \left(\beta^s R^{ED}_i(N, \{s\}, p^{(s)}, C^{(s)}) \right. \\ &\quad \left. + (1-\beta^s) R^{SP}_i(N, \{s\}, p^{(s)}, C^{(s)})\right). \end{aligned}$$

Neutrality implies that $\beta^s = \beta$ for any $s \in S$. Therefore, since R^{ED} and R^{SP} satisfy *composition*, we have that

$$\begin{aligned} R_i(N, S, p, C) &= \beta \sum_{s \in S} R^{ED}_i(N, \{s\}, p^{(s)}, C^{(s)}) \\ &\quad + (1-\beta) \sum_{s \in S} R^{SP}_i(N, \{s\}, p^{(s)}, C^{(s)}) \\ &= \beta R^{ED}_i(N, S, p, C) + (1-\beta) R^{SP}_i(N, S, p, C), \end{aligned}$$

where $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$. $\square$

As the next remark shows, Theorem 1 is tight and all the axioms are necessary for the characterization.

Remark 1. The axioms of Theorem 1 are independent.

(a) The proportional rule satisfies equal treatment of equals, neutrality, scale invariance, and non-advantageous transfer within subscriber, but not composition.

(b) Let R^1 be defined as follows. For each $i \in N$,

$$R^1_i(N, S, p, C) = \begin{cases} \|p\| & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The rule R^1 satisfies neutrality, scale invariance, non-advantageous transfer within subscriber, and composition, but not equal treatment of equals.

(c) Let R^2 be defined as follows. For each $i \in N$,

$$R^2_i(N, S, p, C) = \sum_{s \in S, i \in N_s} \frac{1}{|N_s|} p_s,$$

where $N_s = \{j \in N : C_{js} \neq 0\}$. The rule R^2 satisfies equal treatment of equals, neutrality, scale invariance, and composition, but not non-advantageous transfer within subscriber.

(d) Let R^3 be defined as follows. For each $i \in N$,

$$R^3_i(N, S, p, C) = \begin{cases} R^{ED}_i(N, S, p, C) & \text{if } \|p\| < 5 \\ R^{SP}_i(N, S, p, C) & \text{otherwise.} \end{cases}$$

The rule R^3 satisfies equal treatment of equals, neutrality, non-advantageous transfer within subscriber, and composition, but not scale invariance.

(e) Let R^4 be defined as follows. For each $i \in N$,

$$R^4_i(N, S, p, C) = \begin{cases} R^{ED}_i(N, S, p, C) & \text{if } \|C_{\cdot 1}\| \geq \|C_{\cdot s}\| \forall s \in S \\ R^{SP}_i(N, S, p, C) & \text{otherwise.} \end{cases}$$

The rule R^4 satisfies equal treatment of equals, scale invariance, non-advantageous transfer within subscriber, and composition, but not neutrality.

In the previous theorem, the parameter β varies between 0 and $\frac{|N|}{|N|-1}$. When $\beta = 0$, we get the subscriber-proportional rule, while $\beta = 1$ leads to the equal division rule. However, when $\beta \in \left(1, \frac{|N|}{|N|-1}\right]$, the resulting rule assigns higher awards to less consumed services. This situation is illustrated using the following example, which shows the effect of β on the revenue distribution.

Example 2. As in Example 1, consider the platform where $N = \{1, 2, 3\}$, $S = \{1, 2, 3, 4, 5, 6\}$, $p = (2, 3, 1, 2, 1, 3)$, and C is given by

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The revenue to distribute is $\|p\| = 12$. According to Theorem 1, any rule that satisfies the properties in its statement can be expressed as $\beta R^{ED} + (1-\beta) R^{SP}$, where $\beta \in [0, \frac{3}{2}]$. The next table shows how five of these rules apply to this particular example.

	Services		
$\beta R^{ED} + (1-\beta) R^{SP}$	1	2	3
$\beta = 0$ $[R^{SP}]$	$\frac{141}{28}$	$\frac{195}{28}$	0
$\beta = \frac{1}{2}$	$\frac{253}{56}$	$\frac{307}{56}$	2
$\beta = 1$ $[R^{ED}]$	4	4	4
$\beta = \frac{5}{4}$	$\frac{419}{112}$	$\frac{365}{112}$	5
$\beta = \frac{3}{2}$	$\frac{195}{56}$	$\frac{141}{56}$	6

Fig. 1 plots the evolution of the allocations of services 1, 2, and 3 as β varies from 0 to $\frac{3}{2}$. The solid (red) line indicates the allocations of service 1, the dashed (blue) line refers to service 2, and the dot-dashed (green) line refers to service 3. As we can observe, when $\beta = 0$ the allocations correspond to the subscriber-proportional rule. As β increases from 0 to 1, the role of R^{SP} gradually decreases and the impact of R^{ED} progressively emerges. For $\beta = 1$, the allocations coincide with the equal division rule. When β is larger than 1, the weight of the subscriber-proportional rule $(1-\beta)$ becomes negative, which inverses the distributive behavior of the proportionality, allocating more awards to less consumed services.

[6] Theorem 3 in [34].

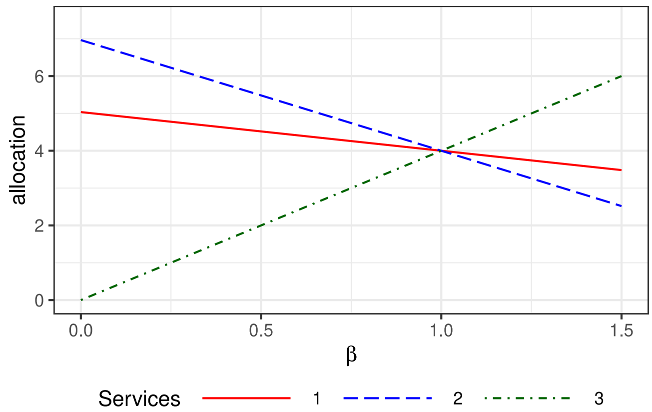

Fig. 1. Representation of $\beta R^{ED} + (1-\beta) R^{SP}$ allocations for any $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$ in Example 2.

Therefore, the combination of equal treatment of equals, neutrality, scale invariance, non-advantageous transfer within subscriber, and composition results in a parametrized family of rules with sufficient degrees of freedom to accommodate several circumstances. If $\beta \in [0,1]$ we obtain the complete range of pure compromises of the equal division and subscriber-proportional rules, which may be suitable in multiple scenarios. The choice of $\beta \in \left(1, \frac{|N|}{|N|-1}\right]$ may be coherent in contexts in which emerging services or agents must be specifically supported. However, the next result shows that if equal treatment of equals is strengthened to order preservation (which guarantees higher rewards to more consumed services) in Theorem 1, then values of $\beta \in \left(1, \frac{|N|}{|N|-1}\right]$ must be dismissed and the unique admissible rules are the convex combinations of the equal division and subscriber-proportional schemes.

Corollary 1. *A rule R satisfies order preservation, neutrality, scale invariance, non-advantageous transfer within subscriber, and composition if and only if there exists $\beta \in [0,1]$ such that, for each $(N,S,p,C) \in \mathcal{D}^N$,*

$$R(N,S,p,C) = \beta R^{ED}(N,S,p,C) + (1-\beta) R^{SP}(N,S,p,C).$$

Proof. In Theorem 1, we already proved that any of such rules satisfies neutrality, scale invariance, non-advantageous transfer within subscriber, and composition. The fulfillment of order preservation is obvious. We focus on the converse implication. Let $(N,S,p,C^1) \in \mathcal{D}^N$ in which the consumption matrix C^1 is such that all entries in the first row are equal to 1, and all the others entries are equal to zero ($C^1_{1s} = 1$ for all $s \in S$, and $C^1_{is} = 0$ for all $i \in N\backslash\{1\}$ and all $s \in S$). Since *order preservation* implies equal treatment of equals, Theorem 1 guarantees that there exists a $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$ such that

$$R(N,S,p,C^1) = \beta R^{ED}(N,S,p,C^1) + (1-\beta) R^{SP}(N,S,p,C^1).$$

Since $C^1_{1s} > C^1_{2s}$ for all $s \in S$, *order preservation* implies that $R_1(N,S,p,C^1) \geq R_2(N,S,p,C^1)$, which is equivalent to requiring that $(1-\beta)\|p\| \geq (1-\beta)\cdot 0$. That is, $\beta \leq 1$. □

The rules used in Remark 1 can be used to prove the independence of the properties in Corollary 1 by replacing equal treatment of equals with order preservation.

If the properties equal treatment of equals, neutrality and scale invariance in Theorem 1 are replaced by the property of null service, a characterization of the subscriber-proportional rule is obtained, as shown in the following corollary. In addition, this result serves as a normative foundation for the user-centric mechanism used in the music streaming industry.

Corollary 2. *The subscriber-proportional rule is the unique rule that satisfies non-advantageous transfer within subscriber, composition and null service.*

Proof. In Theorem 1, we have already proved that the subscriber-proportional rule satisfies non-advantageous transfer within subscriber, and composition. Let us now prove that the subscriber-proportional rule satisfies the null service axiom.

- Null service. Let $(N,S,p,C) \in \mathcal{D}^N$ and $i \in N$such that $C_{is} = 0$ for all $s \in S$. It follows that
$$R_i^{SP}(N,S,p,C) = \sum_{s\in S} \frac{C_{is}}{\|C_{\cdot s}\|} p_s = 0.$$

Now, let us see the converse implication. Let R be a rule that satisfies the axioms of the statement. Let $(N,S,p,C) \in \mathcal{D}^N$. Suppose that $S = \{s\}$, and let $p^{(s)}$ and $C^{(s)} = \left(C^{(s)}_{1s}, \ldots, C^{(s)}_{ns}\right)^T$ be the corresponding subscription price and consumption matrix of this platform, respectively.

Since R satisfies *non-advantageous transfer within subscriber*, and taking into account Lemma 1, there exists functions $A_i^s : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_+$ for each $i \in N$,[7]

$$R_i(N,\{s\},p^{(s)},C^{(s)}) = A_i^s(\|C^{(s)}\|, p^{(s)}) + \frac{C^{(s)}_{is}}{\|C^{(s)}\|}\left[p^{(s)} - \sum_{i\in N} A_i^s(\|C^{(s)}\|, p^{(s)})\right].$$

Now, let $\bar{C}^{(s)}$ such that $\bar{C}^{(s)}_i = \|C^{(s)}\|$ and $\bar{C}^{(s)}_j = 0$ for any $i \in N$ and all $j \in N\backslash\{i\}$, then

$$R_j(N,\{s\},p^{(s)},\bar{C}^{(s)}) = A_j^s(\|\bar{C}^{(s)}\|, p^{(s)}),$$

and by *null service* we have that

$$R_j(N,\{s\},p^{(s)},\bar{C}^{(s)}) = A_j^s(\|\bar{C}^{(s)}\|, p^{(s)}) = 0.$$

Since $\|\bar{C}^{(s)}\| = \|C^{(s)}\|$ we have that $A_j^s(\|\bar{C}^{(s)}\|, p^{(s)}) = A_j^s(\|C^{(s)}\|, p^{(s)}) = 0$. Therefore,

$$R_i(N,\{s\},p^{(s)},C^{(s)}) = \frac{C^{(s)}_{is}}{\|C^{(s)}\|} p^{(s)},$$

for all $i \in N$. Since R satisfies *composition*, it follows that, for each $i \in N$,

$$R_i(N,S,p,C) = \sum_{s\in S} R_i\left(N,\{s\},p^{(s)},C^{(s)}\right) = \sum_{s\in S} R_i^{SP}(N,\{s\},p^{(s)},C^{(s)}) = R_i^{SP}(N,S,p,C). \quad \square$$

The following remark proves the independence of the properties of Corollary 2.

Remark 2. The axioms of Corollary 2 are independent.

(a) The equal division rule satisfies non-advantageous transfer within subscriber and composition, but not null service.
(b) The proportional rule satisfies non-advantageous transfer within subscriber and null service, but not composition.
(c) Let R^2 be defined as follows. For each $i \in N$,
$$R_i^2(N,S,p,C) = \sum_{s\in S, i\in N_s} \frac{1}{|N_s|} p_s,$$
where $N_s = \{j \in N : C_{js} \neq 0\}$. The rule R^2 satisfies composition and null service, but not non-advantageous transfer within subscriber.

The next result states that any rules that fulfill equal treatment of equals, neutrality, scale invariance, non-advantageous transfer within subscriber, and sharing proofness must be a compromise between the equal division and the proportional rules. In comparison with Theorem 1, Theorem 2 shows that if we replace composition by sharing

[7] Theorem 3 in [34].

proofness, we must also replace the subscriber-proportional rule with the proportional rule in the family that we characterize.

Theorem 2. *A rule R satisfies equal treatment of equals, neutrality, scale invariance, non-advantageous transfer within subscriber, and sharing proofness if and only if there exists* $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$ *such that, for each* $(N,S,p,C) \in \mathcal{D}^N$,

$$R(N,S,p,C) = \beta R^{ED}(N,S,p,C) + (1-\beta)R^P(N,S,p,C).$$

Proof. First, by Lemma 2 the previous family is well-defined. Let us now prove that any of the members of the family satisfies the axioms in the statement. Let $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$.

- Equal treatment of equals. Let $(N,S,p,C) \in \mathcal{D}^N$ and $i,j \in N$ such that $C_{i\cdot} = C_{j\cdot}$. It follows that
$$\begin{aligned}R_i(N,S,p,C) &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^P(N,S,p,C)\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|C_{i\cdot}\|}{\sum_{z\in N}\|C_{z\cdot}\|}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|C_{j\cdot}\|}{\sum_{z\in N}\|C_{z\cdot}\|}\|p\|\\ &= \beta R_j^{ED}(N,S,p,C) + (1-\beta)R_j^P(N,S,p,C)\\ &= R_j(N,S,p,C).\end{aligned}$$

- Neutrality. Let $(N,S,p,C) \in \mathcal{D}^N$, $\sigma \in \Pi_S$, and $i \in N$. It follows that
$$\begin{aligned}R_i(N,S,p,C) &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^P(N,S,p,C)\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|C_{i\cdot}\|}{\sum_{j\in N}\|C_{j\cdot}\|}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|C^\sigma_{i\cdot}\|}{\sum_{j\in N}\|C^\sigma_{j\cdot}\|}\|p^\sigma\|\\ &= \beta R_i^{ED}(N,S^\sigma,p^\sigma,C^\sigma) + (1-\beta)R_i^P(N,S^\sigma,p^\sigma,C^\sigma)\\ &= R_i(N,S^\sigma,p^\sigma,C^\sigma).\end{aligned}$$

- Scale invariance. Let $(N,S,p,C) \in \mathcal{D}^N$, $i \in N$, and $(\lambda_1,\lambda_2) \in \mathbb{R}^2_{++}$. It follows that
$$\begin{aligned}R_i(N,S,\lambda_1 p,C) &= \beta R_i^{ED}(N,S,\lambda_1 p,C) + (1-\beta)R_i^P(N,S,\lambda_1 p,C)\\ &= \beta\frac{\lambda_1\|p\|}{|N|} + (1-\beta)\frac{\|C_{i\cdot}\|}{\sum_{j\in N}\|C_{j\cdot}\|}\|\lambda_1 p\|\\ &= \lambda_1\left(\beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|C_{i\cdot}\|}{\sum_{j\in N}\|C_{j\cdot}\|}\|p\|\right)\\ &= \lambda_1\left(\beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^P(N,S,p,C)\right)\\ &= \lambda_1 R_i(N,S,p,C);\end{aligned}$$
$$\begin{aligned}R_i(N,S,p,\lambda_2 C) &= \beta R_i^{ED}(N,S,p,\lambda_2 C) + (1-\beta)R_i^{SP}(N,S,p,\lambda_2 C)\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|\lambda_2 C_{i\cdot}\|}{\sum_{j\in N}\|\lambda_2 C_{j\cdot}\|}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\lambda_2\|C_{i\cdot}\|}{\sum_{j\in N}\lambda_2\|C_{j\cdot}\|}\|p\|\\ &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^P(N,S,p,C)\\ &= R_i(N,S,p,C).\end{aligned}$$

- Non-advantageous transfer within subscriber. Let $(N,S,p,C) \in \mathcal{D}^N$, $i,j \in N$, $s \in S$, $\alpha \in \mathbb{R}_+$ such that $C_{is} - \alpha \geq 0$, and C^α such that $C^\alpha_{is} = C_{is} - \alpha$, $C^\alpha_{js} = C_{js} + \alpha$ and $C^\alpha_{zs} = C_{zs}$ for all $z \in N\backslash\{i,j\}$. It follows that
$$\begin{aligned}R_z(N,S,p,C) &= \beta R_z^{ED}(N,S,p,C) + (1-\beta)R_z^P(N,S,p,C)\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|C_{z\cdot}\|}{\sum_{l\in N}\|C_{l\cdot}\|}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|C^\alpha_{z\cdot}\|}{\sum_{l\in N}\|C^\alpha_{l\cdot}\|}\|p\|\\ &= \beta R_z^{ED}(N,S,p,C^\alpha) + (1-\beta)R_z^P(N,S,p,C^\alpha)\\ &= R_z(N,S,p,C^\alpha).\end{aligned}$$

- Sharing proofness. Let $(N,S,p,C) \in \mathcal{D}^N$, and let $S' \subseteq S$ and $s \in S'$ such that $p'_s = \sum_{t\in S'} p_t$ and $C'_{is} = \sum_{t\in S'} C_{it}$ for any $i \in N$. It follows that
$$\begin{aligned}&R_i\left(N,\{s\}\cup S\backslash S',(p'_s,p_{S\backslash S'}),(C'_{\cdot s},C_{S\backslash S'})\right)\\ &= \beta R_i^{ED}\left(N,\{s\}\cup S\backslash S',(p'_s,p_{S\backslash S'}),(C'_{\cdot s},C_{S\backslash S'})\right)\\ &+ (1-\beta)R_i^P\left(N,\{s\}\cup S\backslash S',(p'_s,p_{S\backslash S'}),(C'_{\cdot s},C_{S\backslash S'})\right)\\ &= \beta\frac{\|(p'_s,p_{S\backslash S'})\|}{|N|} + (1-\beta)\frac{\|(C'_{\cdot s},C_{S\backslash S'})_{i\cdot}\|}{\sum_{j\in N}\|(C'_{\cdot s},C_{S\backslash S'})_{j\cdot}\|}\|(p'_s,p_{S\backslash S'})\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\|(C'_{\cdot s},C_{S\backslash S'})_{i\cdot}\|}{\sum_{j\in N}\|(C'_{\cdot s},C_{S\backslash S'})_{j\cdot}\|}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\sum_{t\in\{s\}\cup S\backslash S'}(C'_{\cdot s},C_{S\backslash S'})_{it}}{\sum_{j\in N}\sum_{t\in\{s\}\cup S\backslash S'}(C'_{\cdot s},C_{S\backslash S'})_{jt}}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{C'_{is}+\sum_{t\in S\backslash S'}C_{it}}{\sum_{j\in N}\left(C'_{js}+\sum_{t\in S\backslash S'}C_{jt}\right)}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\sum_{t\in S'}C_{it}+\sum_{t\in S\backslash S'}C_{it}}{\sum_{j\in N}\left(\sum_{t\in S'}C_{jt}+\sum_{t\in S\backslash S'}C_{jt}\right)}\|p\|\\ &= \beta\frac{\|p\|}{|N|} + (1-\beta)\frac{\sum_{t\in S}C_{it}}{\sum_{j\in N}\sum_{t\in S}C_{jt}}\|p\|\\ &= \beta R_i^{ED}(N,S,p,C) + (1-\beta)R_i^P(N,S,p,C)\\ &= R_i(N,S,p,C).\end{aligned}$$

Note that by similar arguments, it can be concluded that the equal division and proportional rules also satisfy *sharing proofness*.

Let us now focus on the converse implication. Let R be a rule that satisfies all the properties in the statement of the theorem. Let $(N,S,p,C) \in \mathcal{D}^N$. *Sharing proofness* requires that

$$R(N,S,p,C) = R\left(N,\{s\},p^{(s)},C^{(s)}\right),$$

where $p^{(s)}$ and $C^{(s)} = \left(C^{(s)}_{1s},\dots,C^{(s)}_{|N|s}\right)^T$ are the corresponding subscription price and consumption matrix, respectively, with $p^{(s)} = \|p\|$ and $\|C_{is}\| = \sum_{s'\in S} C_{is'}$.

Since R satisfies *non-advantageous transfer within subscriber* and *equal treatment of equals*, applying a reasoning similar to the proof of Theorem 1, there is a function $A^s : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_+$ such that, for each $i \in N$,

$$R_i(N,\{s\},p^{(s)},C^{(s)}) = A^s(\|C^{(s)}\|,p^{(s)}) + \frac{C^{(s)}_{is}}{\|C^{(s)}\|}\left[p^{(s)} - \sum_{i\in N} A^s(\|C^{(s)}\|,p^{(s)})\right].$$

Observe that as the term $A^s(\|C^{(s)}\|,p^{(s)})$ only depends on the aggregate consumption $\|C^{(s)}\|$ (but not its distribution across services) and the subscription price $p^{(s)}$, it can be written as

$$A^s(\|C^{(s)}\|,p^{(s)}) = R_2(N,\{s\},p^{(s)},(\|C^{(s)}\|,0,\dots,0)^T).$$

Since R satisfies *equal treatment of equals*, *scale invariance* and *non-advantageous transfer within subscriber*, by Lemma 3, $R_2(N,\{s\},p^{(s)},(\|C^{(s)}\|,0,\dots,0)^T) = p^{(s)}\hat{\mu}^s$, for some $\hat{\mu}^s \in \mathbb{R}_+$. Therefore, $A^s(\|C^{(s)}\|,p^{(s)}) = p^{(s)}\hat{\mu}^s$, and then, for each $i \in N$,

$$R_i(N,\{s\},p^{(s)},C^{(s)}) = p^{(s)}\hat{\mu}^s + \frac{C^{(s)}_{is}}{\|C^{(s)}\|}p^{(s)}\left[1-|N|\hat{\mu}^s\right].$$

Now, letting $\beta^s = |N|\hat{\mu}^s$, we obtain that, for each $i \in N$,

$$R_i(N,\{s\},p^{(s)},C^{(s)}) = \beta^s R_i^{ED}(N,\{s\},p^{(s)},C^{(s)}) + (1-\beta^s)R_i^P(N,\{s\},p^{(s)},C^{(s)}).$$

Since $\hat{\mu}^s = R_2(N, \{s\}, 1, (1, 0, \ldots, 0)^T)$, it follows that $\hat{\mu}^s \in \left[0, \frac{1}{|N|-1}\right]$. Therefore, $\beta^s \in \left[0, \frac{|N|}{|N|-1}\right]$.

Now, since R satisfies *sharing proofness*, it follows that, for each $i \in N$,

$$R_i(N, S, p, C) = R_i(N, \{s\}, p^{(s)}, C^{(s)}) \\ = \beta^s R_i^{ED}(N, \{s\}, p^{(s)}, C^{(s)}) + (1 - \beta^s) R_i^P(N, \{s\}, p^{(s)}, C^{(s)}).$$

Neutrality implies that $\beta^s = \beta$ for any $s \in S$. Therefore, since R^{ED} and R^P satisfy *sharing proofness*, we have that

$$R_i(N, S, p, C) = \beta R_i^{ED}(N, S, p, C) + (1 - \beta) R_i^P(N, S, p, C),$$

where $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$. □

The following remark shows that the axioms of Theorem 2 are necessary for the characterization.

Remark 3. The axioms of Theorem 2 are independent.

(a) The subscriber-proportional rule satisfies equal treatment of equals, neutrality, scale invariance, and non-advantageous transfer within subscriber, but not sharing proofness.

(b) Let R^1 be defined as follows. For each $i \in N$,

$$R_i^1(N, S, p, C) = \begin{cases} \|p\| & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The rule R^1 satisfies neutrality, scale invariance, non-advantageous transfer within subscriber, and sharing proofness, but not equal treatment of equals.

(c) Let R^2 be defined as follows. For each $i \in N$,

$$R_i^2(N, S, p, C) = \begin{cases} \frac{1}{|\hat{N}|}\|p\| & \text{if } i \in \hat{N} \\ 0 & \text{otherwise,} \end{cases}$$

where $\hat{N} = \{j \in N : \|C_{j\cdot}\| \neq 0\}$.
The rule R^2 satisfies equal treatment of equals, neutrality, scale invariance, and sharing proofness, but not non-advantageous transfer within subscriber.

(d) Let R^3 be defined as follows. For each $i \in N$,

$$R_i^3(N, S, p, C) = \begin{cases} R_i^{ED}(N, S, p, C) & \text{if } \|p\| < 5 \\ R_i^P(N, S, p, C) & \text{otherwise.} \end{cases}$$

The rule R^3 satisfies equal treatment of equals, neutrality, non-advantageous transfer within subscriber, and sharing proofness, but not scale invariance.

(e) Let R^4 be defined as follows. For each $i \in N$,

$$R_i^4(N, S, p, C) = \begin{cases} R_i^{ED}(N, S, p, C) & \text{if } \|C_{\cdot 1}\| \geq \|C_{\cdot s}\| \forall s \in S \\ R_i^P(N, S, p, C) & \text{otherwise.} \end{cases}$$

The rule R^4 satisfies equal treatment of equals, scale invariance, non-advantageous transfer within subscriber, and sharing proofness, but not neutrality.

Once again, the parameter β in Theorem 2 may vary between 0 and $\frac{|N|}{|N|-1}$. As Example 3 illustrates, when $\beta \in \left(1, \frac{|N|}{|N|-1}\right]$, the rule assigns higher allocations to less consumed services.

Example 3. As in Examples 1 and 2, consider the platform where $N = \{1, 2, 3\}$, $S = \{1, 2, 3, 4, 5, 6\}$, $p = (2, 3, 1, 2, 1, 3)$, and C is given by

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 2 & 6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

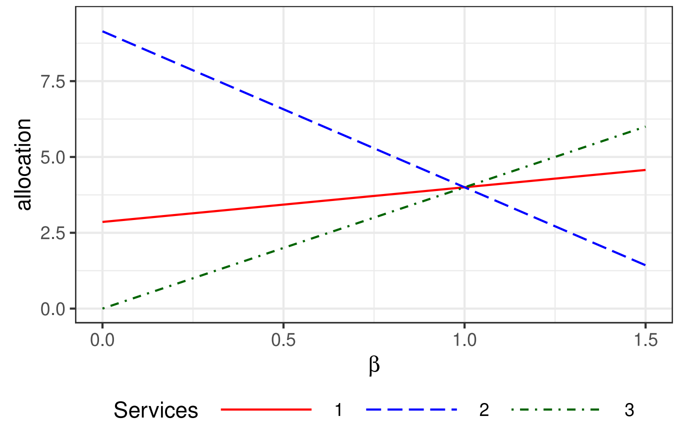

Fig. 2. Representation of $\beta R^{ED} + (1 - \beta) R^P$ allocations for any $\beta \in \left[0, \frac{|N|}{|N|-1}\right]$ in Example 3.

The revenue to distribute is $\|p\| = 12$, and the overall consumption levels of the three services are

$$\begin{pmatrix} \|C_{1\cdot}\| \\ \|C_{2\cdot}\| \\ \|C_{3\cdot}\| \end{pmatrix} = \begin{pmatrix} 5 \\ 16 \\ 0 \end{pmatrix}.$$

According to Theorem 2, any rule that satisfies the properties in its statement if and only if there exists $\beta \in [0, \frac{3}{2}]$ such that the rule is expressed as $\beta R^{ED} + (1 - \beta) R^P$. In this example, it implies that $R(N, S, p, C) = \beta(4, 4, 4) + (1 - \beta)\left(\frac{5}{21} \cdot 12, \frac{16}{21} \cdot 12, 0\right)$, where $\beta \in [0, \frac{3}{2}]$. The next table shows the application of five of these rules.

	Services		
$\beta R^{ED} + (1 - \beta) R^P$	1	2	3
$\beta = 0$ $[R^P]$	$\frac{20}{7}$	$\frac{64}{7}$	0
$\beta = \frac{1}{2}$	$\frac{24}{7}$	$\frac{46}{7}$	2
$\beta = 1$ $[R^{ED}]$	4	4	4
$\beta = \frac{5}{4}$	$\frac{30}{7}$	$\frac{19}{7}$	5
$\beta = \frac{3}{2}$	$\frac{32}{7}$	$\frac{10}{7}$	6

Fig. 2 plots the evolution of the allocations of services 1, 2, and 3 as β varies from 0 to $\frac{3}{2}$. The solid (red) line indicates the allocations of service 1, the dashed (blue) line refers to service 2, and the dot-dashed (green) line refers to service 3. The behavior is similar to Example 2; as β moves from 0, the equal allocation gains relevance to the detriment of the proportional allocation. When $\beta \in \left(1, \frac{3}{2}\right]$ the weight of the proportional rule $(1 - \beta)$ becomes negative, and thus the effect of the proportionality is inverse, providing higher awards to less consumed services.

If we replace equal treatment of equals by order preservation in Theorem 2, we characterize the family of convex combinations of the equal division and proportional rules. We omit the proof because it is analogous to Corollary 1.

Corollary 3. *A rule R satisfies order preservation, neutrality, scale invariance, non-advantageous transfer within subscriber, and sharing proofness if and only if there exists* $\beta \in [0, 1]$ *such that, for each* $(N, S, p, C) \in \mathcal{D}^N$,

$$R(N, S, p, C) = \beta R^{ED}(N, S, p, C) + (1 - \beta) R^P(N, S, p, C).$$

The independence of the properties in Corollary 3 can be proved, as shown in Remark 3, by replacing equal treatment of equals with order preservation.

In this case, if the properties equal treatment of equals, neutrality and scale invariance in Theorem 2 are replaced by the property of null service, a characterization of the proportional rule is obtained. This result can also be interpreted as a normative justification of the pro-rata mechanism.

Corollary 4. *The proportional rule is the unique rule that satisfies non-advantageous transfer within subscriber, sharing proofness and null service.*

Proof. We have already proved that the proportional rule satisfies non-advantageous transfer within subscriber, and composition in Theorem 2. Let us now prove that the proportional rule satisfies the null service axiom.

- Null service. Let $(N,S,p,C) \in \mathcal{D}^N$ and $i \in N$such that $C_{is} = 0$ for all $s \in S$. It follows that

$$R_i^P(N,S,p,C) = \frac{\|C_{i\cdot}\|}{\sum_{j\in N}\|C_{j\cdot}\|}\|p\| = 0.$$

Now, let us see the converse implication. Let R be a rule that satisfies the axioms of the statement. *Sharing proofness* requires that

$$R(N,S,p,C) = R\left(N,s,p^{(s)},C^{(s)}\right),$$

where $p^{(s)}$ and $C^{(s)} = \left(C^{(s)}_{1s},\ldots,C^{(s)}_{|N|s}\right)^T$ are the corresponding subscription price and consumption matrix, respectively, with $p^{(s)} = \|p\|$ and $\|C_{is}\| = \sum_{s'\in S} C_{is'}$.

Since R satisfies *non-advantageous transfer within subscriber*, and taking into account Lemma 1, there exists functions $A_i^s : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_+$ for each $i \in N$,[8]

$$R_i(N,\{s\},p^{(s)},C^{(s)}) = A_i^s(\|C^{(s)}\|,p^{(s)}) + \frac{C_{is}^{(s)}}{\|C^{(s)}\|}\left[p^{(s)} - \sum_{i\in N} A_i^s(\|C^{(s)}\|,p^{(s)})\right].$$

Now, let $\bar{C}^{(s)}$ such that $\bar{C}_i^{(s)} = \|C^{(s)}\|$ and $\bar{C}_j^{(s)} = 0$ for any $i \in N$ and all $j \in N\backslash\{i\}$, then

$$R_j(N,\{s\},p^{(s)},\bar{C}^{(s)}) = A_j^s(\|\bar{C}^{(s)}\|,p^{(s)}),$$

and by *null service* we have that

$$R_j(N,\{s\},p^{(s)},\bar{C}^{(s)}) = A_j^s(\|\bar{C}^{(s)}\|,p^{(s)}) = 0.$$

Since $\|\bar{C}^{(s)}\| = \|C^{(s)}\|$ we have that $A_j^s(\|\bar{C}^{(s)}\|,p^{(s)}) = A_j^s(\|C^{(s)}\|,p^{(s)}) = 0$. Therefore,

$$R_i(N,\{s\},p^{(s)},C^{(s)}) = \frac{C_{is}^{(s)}}{\|C^{(s)}\|}p^{(s)},$$

for all $i \in N$. Since R satisfies *sharing proofness*, it follows that, for each $i \in N$,

$$\begin{aligned} R_i(N,S,p,C) &= R_i(N,\{s\},p^{(s)},C^{(s)}) \\ &= R_i^P(N,\{s\},p^{(s)},C^{(s)}) = R_i^P(N,S,p,C). \quad \square \end{aligned}$$

The independence of the properties in Corollary 4 is proved in the following remark.

Remark 4. The axioms of Corollary 4 are independent.

(a) The equal division rule satisfies non-advantageous transfer within subscriber and sharing proofness, but not null service.
(b) The subscriber-proportional rule satisfies non-advantageous transfer within subscriber and null service, but not sharing proofness.
(c) Let R^2 be defined as follows. For each $i \in N$,

$$R_i^2(N,S,p,C) = \begin{cases} \frac{1}{|\hat{N}|}\|p\| & \text{if } i \in \hat{N} \\ 0 & \text{otherwise,} \end{cases}$$

where $\hat{N} = \{j \in N : \|C_{j\cdot}\| \neq 0\}$.

The rule R^2 satisfies sharing proofness and null service, but not non-advantageous transfer within subscriber.

[8] Theorem 3 in [34].

Table 1
Users and exclusive users.

Streamer	Category	Users		Exclusive users	
		Number	%	Number	%
Auronplay					
	Just Chatting	455 214	29.91	161 123	42.00
	Fall Guys	195 836	12.87	31 508	8.21
	Valorant	222 128	14.59	47 188	12.30
	GTA V	218 511	14.36	46 351	12.08
	The Forest	244 093	16.04	56 510	14.73
	Minecraft	186 272	12.24	40 944	10.67
elXokas					
	Just Chatting	146 880	46.28	64 777	42.30
	Escape from Tarkov (EFT)	170 461	53.72	88 358	57.70
Fextralife					
	Overwatch 2	180 595	41.82	89 797	59.84
	Death Stranding	98 162	22.73	27 004	18.00
	Pokemon	61 571	14.26	16 479	10.98
	Eversoul	29 038	6.72	4635	3.09
	League of Legends (LoL)	26 022	6.02	2988	1.99
	Elden Ring	36 479	8.45	9144	6.09

5. An illustrative application: Content and revenues on Twitch

In practice, it is clear that computing the allocations of any of the rules presented in the previous sections requires the use of private information, such as data regarding the consumption of each individual subscriber for each service. This implies that only the streaming platforms themselves, which have direct access to the data, can perform such computations. In this section, we use the families of rules characterized in Section 4 to determine the revenue provenance of different types of content, grouped by categories, for a number of streamers on Twitch.[9] This information is important for both streamers and Twitch, as it allows both parties to determine the profitability of the different content offered.

We analyzed three popular streamers during a three-week period from 23 December 2022 to 15 January 2023. We collected data on the users (identified through their nicknames) who watched each streamer for each hour of the broadcast, in addition to the category streamed. According to the theoretical model presented in Section 2, each streamer is considered to be an individual platform, in the sense each streamer provides different types of content. Therefore, the set of services, N, represent the categories of the content (i.e., the name of the played game, such as Minecraft, or Just Chatting) that each streamer broadcast. The subscribers S are the users, and the consumption C_{is} is the time each user spent watching the content provided by the streamer in each category. The subscription price is assumed to be fixed (and normalized to 1 monetary unit).

For the timeframe examined, Table 1 shows, for each streamer, the categories of the content that they streamed, the number of users during the three-week period, the number of exclusive users (i.e., users who consumed only one category), and the percentage they represent out of the total amount of users and exclusive users. Hence, for instance, the streamer Auronplay (or, in our theoretical terminology, the platform Auronplay) has six categories (or services): Just Chatting, Fall Guys, Valorant, GTA V, The Forest, and Minecraft. During the three-week period, 455,214 users watched content provided in the category Just Chatting, representing 29.91% of the overall users that consumed Auronplay's content. In this application, a rule is a mechanism to distribute among the different categories the revenue obtained by a streamer from her viewers. This can also be interpreted as a measure of the profitability of the different contents the streamer produces.

Table 2
Profitability of categories for Auronplay according to different allocation rules.

Auronplay	$\beta = 0$		$\beta = \frac{1}{3}$		$\beta = \frac{2}{3}$		$\beta = 1$	
Categories	Value	%	Value	%	Value	%	Value	%
	$\beta R^{ED} + (1-\beta)R^{P}$ [Corollary 3]							
Just Chatting	95 340.37	12.98	104 351.4	14.21	113 362.5	15.44	122 373.5	16.67
Fall Guys	144 665.56	19.70	137 234.9	18.69	129 804.2	17.68	122 373.5	16.67
Valorant	109 543.28	14.92	113 820.0	15.50	118 096.8	16.08	122 373.5	16.67
GTA V	133 901.58	18.28	130 058.9	17.71	126 216.2	17.19	122 373.5	16.67
The Forest	120 928.61	16.47	121 410.2	16.54	121 891.9	16.60	122 373.5	16.67
Minecraft	129 861.59	17.69	127 365.6	17.35	124 869.5	17.01	122 373.5	16.67
	$\beta R^{ED} + (1-\beta)R^{SP}$ [Corollary 1]							
Just Chatting	211 344.88	28.78	181 687.8	24.74	152 030.6	20.71	122 373.5	16.67
Fall Guys	97 576.39	13.29	105 842.1	14.42	114 107.8	15.54	122 373.5	16.67
Valorant	99 719.75	13.58	107 271.0	14.61	114 822.2	15.64	122 373.5	16.67
GTA V	110 896.25	15.10	114 722.0	15.62	118 547.7	16.15	122 373.5	16.67
The Forest	117 945.33	16.06	119 421.4	16.26	120 897.4	16.47	122 373.5	16.67
Minecraft	96 758.41	13.18	105 296.8	14.34	113 835.1	15.50	122 373.5	16.67

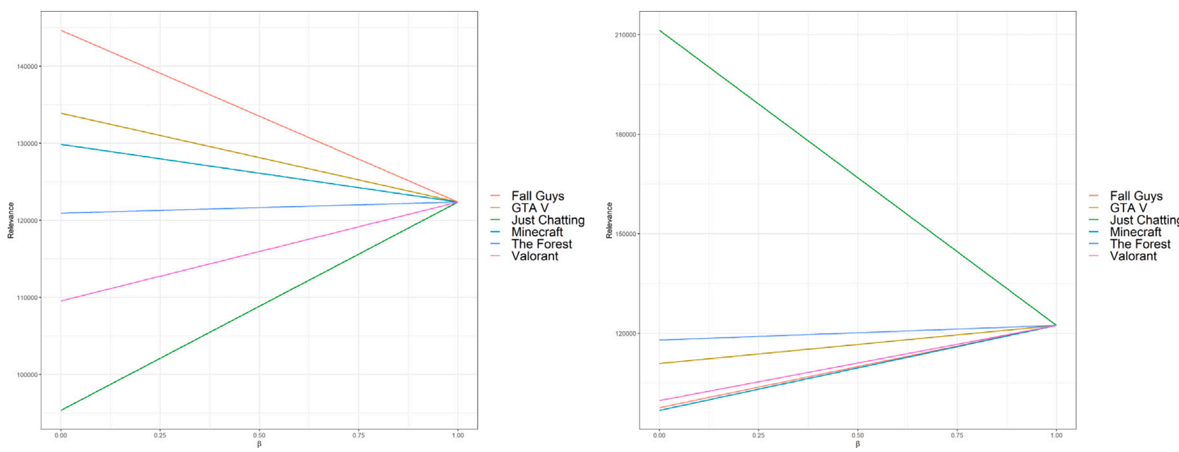

Fig. 3. Distribution of Auronplay's revenues across the categories using the family $\beta R^{ED} + (1-\beta)R^{P}$ (left) and the family $\beta R^{ED} + (1-\beta)R^{SP}$ (right).

Table 2 shows the application of the families of rules in Corollaries 1 and 3, used to distribute the revenue obtained Auronplay by among the categories he offers. For each family of rules and for some selected $\beta \in [0,1]$, we indicate both the value and the percentage of revenues that is attributable to each category. The equal division rule assigns equal revenues to all categories, as expected. The proportional rule takes into account the time that each category has been viewed when assigning revenues. Among the categories, Fall Guys is the most watched and, therefore, the most profitable for streamer Auronplay, according to this rule. Despite being one of the categories with the fewest users and the lowest number of exclusive users, it is noteworthy that this category was the most consumed by users during Auronplay's streaming sessions. The subscriber-proportional rule distributes the revenues of each viewer individually among the categories from which they consumed content; therefore, the revenue of exclusive users is directly assigned to the category they consume. Under this rule, the most profitable category is Just Chatting. This category has a significant proportion of exclusive users and, therefore, receives the full subscription fee from these subscribers. This, in addition to its high number of users, makes Just Chatting the streamer's most profitable category. Fig. 3 shows the allocation of revenues to each category according to each family of rules, according to β. Comparing the families of rules characterized in Corollaries 1 and 3, the Just Chatting category has a significantly greater economic impact according to the latter than to the former. It even switches from being the least profitable category to the most profitable category. In fact, as we can observe, the order in the profitability of the type of content is nearly entirely reversed, depending on the family of rules we consider.

In Table 3, we apply the same allocation rules for streamer elXokas. His most viewed category is Just Chatting, and thus the one with the highest profitability according to the proportional rule. On the other hand, Escape from Tarkov is the category with the highest number of both users and exclusive users. This implies that, using the subscriber-proportional rule, it is now the most profitable category for the streamer. However, even with 7% more users and 15% more exclusive users, this category does not obtain more than 2% of revenues compared to Just Chatting. This is due to, among other factors, the fact that elXokas is a streamer known for his controversial Just Chatting. Therefore, although Escape from Tarkov achieves 15% more revenues from the exclusive users, this is compensated by a greater number of views for Just Chatting from common users in both categories. Fig. 4 illustrates these results. As in the case of Auronplay, we can observe a reversal in the profitability of the categories when comparing the families in Corollaries 1 and 3, although it is not highly significant.

Finally, Table 4 presents the results for streamer Fextralife. Based on these results, we emphasize that the six categories have effectively the same number of total views. In other words, the allocations of each category according to the proportional rule do not exhibit significant differences. The most viewed category is Elden Ring with 18.85% of revenues, while the least viewed is Death Stranding with 15.34%, representing a difference of 3.51%. Nevertheless, significant differences emerge when taking into account the subscriber-proportional rule. Overwatch 2 leads the categories with the highest profitability, at

[9] https://www.twitch.tv/

Table 3
Profitability of categories for elXokas according to different allocation rules.

elXokas	$\beta = 0$		$\beta = \frac{1}{3}$		$\beta = \frac{2}{3}$		$\beta = 1$	
Categories	Value	%	Value	%	Value	%	Value	%
				$\beta R^{ED} + (1-\beta)R^{P}$ [Corollary 3]				
Just Chatting	132 732.2	56.42	127 694.5	54.28	122 656.7	52.14	117 619	50
EFT	102 505.8	43.58	107 543.5	45.72	112 581.3	47.86	117 619	50
				$\beta R^{ED} + (1-\beta)R^{SP}$ [Corollary 1]				
Just Chatting	115 255.9	49	116 043.6	49.33	116 831.3	49.67	117 619	50
EFT	119 982.1	51	119 194.4	50.67	118 406.7	50.33	117 619	50

Table 4
Profitability of categories for Fextralife according to different allocation rules.

Fextralife	$\beta = 0$		$\beta = \frac{1}{3}$		$\beta = \frac{2}{3}$		$\beta = 1$	
Categories	Value	%	Value	%	Value	%	Value	%
				$\beta R^{ED} + (1-\beta)R^{P}$ [Corollary 3]				
Overwatch 2	40 547.82	16.31	40 839.71	16.43	41 131.61	16.55	41 423.5	16.67
Death Stranding	38 118.57	15.34	39 220.21	15.78	40 321.86	16.22	41 423.5	16.67
Pokemon	41 700.27	16.78	41 608.01	16.74	41 515.76	16.70	41 423.5	16.67
Eversoul	41 321.06	16.63	41 355.21	16.64	41 389.35	16.65	41 423.5	16.67
LoL	40 002.49	16.09	40 476.16	16.29	40 949.83	16.48	41 423.5	16.67
Elden Ring	46 850.80	18.85	45 041.70	18.12	43 232.60	17.39	41 423.5	16.67
				$\beta R^{ED} + (1-\beta)R^{SP}$ [Corollary 1]				
Overwatch 2	109 931.08	44.23	87 095.22	35.04	64 259.36	25.85	41 423.5	16.67
Death Stranding	49 983.14	20.11	47 129.93	18.96	44 276.71	17.81	41 423.5	16.67
Pokemon	36 081.79	14.52	37 862.36	15.23	39 642.93	15.95	41 423.5	16.67
Eversoul	15 755.45	6.34	24 311.47	9.78	32 867.48	13.22	41 423.5	16.67
LoL	14 397.92	5.79	23 406.44	9.42	32 414.97	13.04	41 423.5	16.67
Elden Ring	22 391.61	9.01	28 735.58	11.56	35 079.54	14.11	41 423.5	16.67

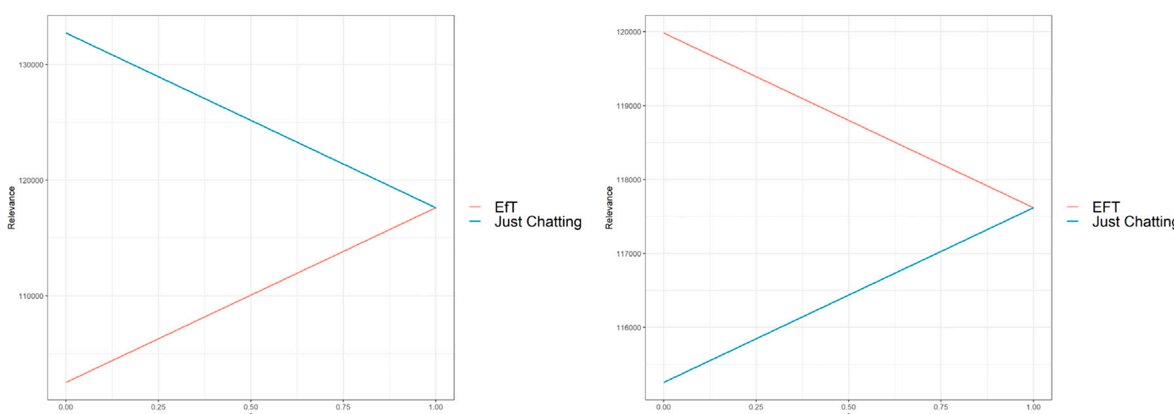

Fig. 4. Distribution of elXokas's revenues across the categories using the family $\beta R^{ED} + (1-\beta)R^{P}$ (left), and family $\beta R^{ED} + (1-\beta)R^{SP}$ (right).

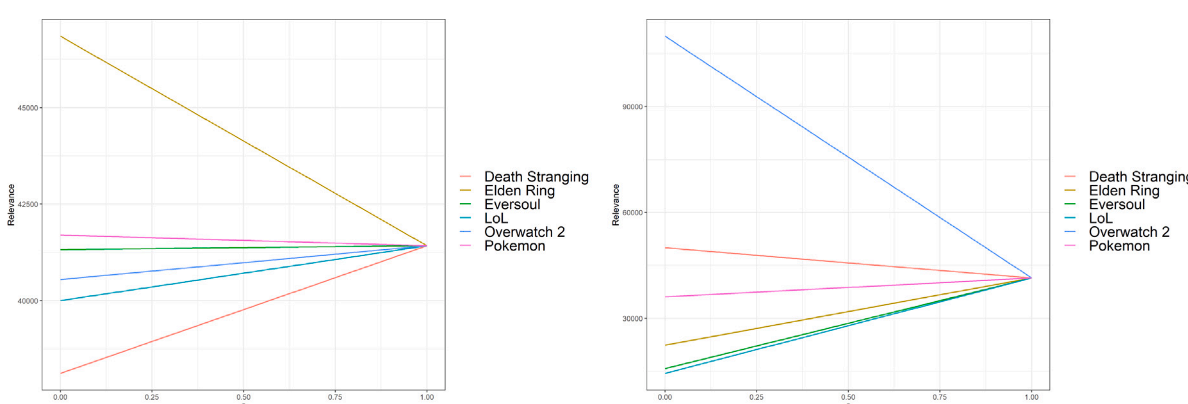

Fig. 5. Distribution of Fextralife's revenues across the categories using the family $\beta R^{ED} + (1-\beta)R^{P}$ (left), and family $\beta R^{ED} + (1-\beta)R^{SP}$ (right).

44.23%, followed by Death Stranding, at 20.11%. These results align more closely with the total number of users and exclusive users. Fig. 5 shows the distribution assigned with each of the two families to each category for the streamer Fextralife.

6. Concluding remarks

The sharing of revenues on streaming platforms is currently a topic of interest. Since not all services provided by a streaming platform hold the same level of interest for subscribers, allocations could be adjusted accordingly. In this paper, we introduce a model for revenue distribution on streaming platforms. The model incorporates three allocation schemes: the equal division, the proportional, and the subscriber-proportional rules. The two latter rules implement the so-called *pro-rata* and *user-centric* principles applied in the music streaming industry. Beyond a mere mathematical description, we apply axiomatic analysis to provide normative foundations for two families of award methods, which represent a compromise between the equal division and subscriber-proportional rules (Theorem 1 and Corollary 1) and the equal division and proportional rules (Theorem 2 and Corollary 3). In addition, we have also presented a normative justification of the *pro-rata* and *user-centric* mechanisms (Corollaries 3 and 4). These results are obtained by adding the additional requirement of *null service* to the families of rules characterized in Theorems 1 and 2.

With regard to the characterizations, it is worth noting that although Remarks 1 and 3 show that all the properties in Theorems 1 and 2 are necessary, both results can be generalized by removing certain axioms. That would lead to a class of rules that are technically more general but, in practice, less appealing. For instance, if we do not require scale invariance in consumption, then β is dependent on the aggregate consumption of each subscriber or the total consumption. Similarly, if we do not consider scale invariance in prices, then β depends on the subscription price of each individual subscriber or the total revenue generated by the platform. Analogous generalizations can be made by removing neutrality.

Finally, we have applied the families of rules that we characterized to identify the impact of specific content on the revenues of some of the most popular Twitch streamers. Over three weeks, we collected hourly data on users who were watching these Twitch streamers, as well as the streaming category. With this information, we obtained those categories that lead to greater profitability on the platform. We found that the share of the revenue assigned to each category may significantly vary depending on the applied family of rules (Corollaries 1 or 3). From a normative perspective, these two classes of rules simply differ in one axiom. The combinations of the equal division and the subscriber-proportional rules (Theorem 1) satisfy composition but violate sharing proofness, while combinations of the equal division and the proportional rules (Theorem 2) satisfy sharing proofness but violate composition. The choice of the axiom to require crucially determines the relevance of the different categories in the revenues of streamers. In general, we obtained that the subscriber-proportional rule has a more direct relationship with the number of users and exclusive users. The proportional rule, however, is related to a more significant degree with viewing time. As such, categories that are viewed for a longer period of time, even if they receive a smaller flow of users, are more profitable for streamers.

CRediT authorship contribution statement

Juan Carlos Gonçalves-Dosantos: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Ricardo Martínez:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Joaquín Sánchez-Soriano:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors state that there is no conflict of interest.

Acknowledgment

All authors approved the final version of the manuscript to be published' to Acknowledgment section.

Data availability

The authors do not have permission to share data.

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