Banach spaces with the Daugavet property

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To our families

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Abstract

The aim of this manuscript is to study Banach spaces with the *Daugavet property*: Banach spaces X satisfying that the norm equality

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

(known as the *Daugavet equation*, (DE) for short) holds for every bounded linear operator $T\colon X \to X$ of rank one. Its starting point is a review of I. K. Daugavet's result from 1963 showing that (DE) holds for compact linear operators on $C[0,1]$ and of related results which were established in the XX Century. Next, a chapter on those results from Banach space theory and topology that are used in the book is included. The core part of the text deals with the "geometrical" treatment of the subject developed in the XXI Century using slices, narrow operators, and slicely countably determined sets. It presents the main consequences, the main examples, and some generalisations such as Daugavet centres, the almost Daugavet property, and a Lipschitz version of (DE). Finally, some geometric properties related to the Daugavet property are commented on: other possible norm equalities for operators, the so-called big slice phenomena, the alternative Daugavet property, and alternatively convex or smooth spaces.

Each chapter ends with some notes, remarks and open questions.

We acknowledge the use of the free version of ©Google Gemini (https://gemini.google.com/) that produced raw `tikz` code for preliminary versions of some of the figures.

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Basic Notation

Throughout the book the letters $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{E}$ are used for Banach spaces, $\boldsymbol{S_X}$ and $\boldsymbol{B_X}$ are the unit sphere and the closed unit ball of X, respectively. Unless stated otherwise, we will consider real and complex Banach spaces at the same time, writing $\mathbb{K}$ to denote the base field (which could be $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), and using the notation $\mathbf{Re}(\boldsymbol{z})$ for the real part of $z \in \mathbb{C}$; in the case when the space is real, this just means $\mathrm{Re}(z) = z$ for every $z \in \mathbb{R}$. The symbol $\mathbb{T}$ will denote the set of modulus one scalars, that is, $\mathbb{T} := \{-1, 1\}$ in the real case and $\mathbb{T}$ being the unit circle of $\mathbb{C}$ in the complex case. We will use the notations (x_n), $(x_n)_{n\in\mathbb{N}}$, and $(x_n)_{n=1}^{\infty}$ for sequences, and we also use the notation $(x_n) \in X^{\mathbb{N}}$.

The words "operator" and "functional", unless the contrary is stated, are used for linear continuous operators and functionals, $\boldsymbol{L(X, Y)}$ is the space of linear continuous operators between the Banach spaces X and Y, $\boldsymbol{L(X)}$ is the short notation for $L(X, X)$, and $\boldsymbol{X^*}$ stands for $L(X, \mathbb{K})$, the Banach space of continuous linear functionals on X. Mostly, our notation agrees with the standard Banach space terminology from the classical Lindenstrauss–Tzafriri book [**206**].

Let us include here for easy reference a (short) list of usual notation which we will use all along.

$\mathbb{1}$: constant function with value 1.
$\mathbb{1}_{\boldsymbol{A}}$: characteristic function of the set A.
$\overline{\mathbf{1}, \boldsymbol{n}}$: positive integers between 1 and n, $\overline{1, n} := \{1, \ldots, n\}$.
$\oplus_{\boldsymbol{E}}$ / $\oplus_{\boldsymbol{p}}$: absolute E-sum / ℓ_p-sum of spaces (see page 2.9.1).
$\left(\bigoplus_{i\in I} X_n\right)_E$: (absolute) E-sum of the spaces X_i's (see page 45).
$\otimes$: algebraic tensor product (see page 52).
$\widehat{\otimes}_{\boldsymbol{\pi}}$: projective tensor product (see page 53).
$\widehat{\otimes}_{\boldsymbol{\varepsilon}}$: injective tensor product of X and Y (see page 55).
aconv$(\boldsymbol{A})$: absolutely convex hull of A.
AP: Approximation property (see page 56).
BAP: Bounded approximation property (see page 56).
Bil$(\boldsymbol{X} \times \boldsymbol{Y})$: (bounded linear) bilinear forms on $X \times Y$.
$\boldsymbol{C(K)}$ / $\boldsymbol{C(K, X)}$: space of scalar-valued / X-valued continuous functions on the compact space K.
$\boldsymbol{C_0(L)}$ / $\boldsymbol{C_0(L, X)}$: space of scalar-valued / X-valued continuous functions on the locally compact space L vanishing at infinity.
$\boldsymbol{C_b(\Omega)}$ / $\boldsymbol{C_b(\Omega, X)}$: space of scalar-valued / X-valued bounded continuous functions on the completely regular topological space Ω.
codim$_{\boldsymbol{X}}(\boldsymbol{Y})$: codimension of the subspace Y in the space X.
conv$(\boldsymbol{A})$: convex hull of A.
$\overline{\mathbf{conv}}(\boldsymbol{A})$: closed convex hull of A.

δ_x: Dirac measure or functional (i.e., $f \mapsto f(x)$) on a function space.
dens(X): density character of the Banach space X (i.e., minimal cardinality of a dense subset)
dent(A): denting points of A (see page 38).
ext(C): extreme points of C (see page 32).
FIN(Γ): family of all finite subsets of a set Γ.
FinRan(X, Y): finite-rank (continuous linear) operators from X to Y. In the case when $X = Y$, we just write $\mathrm{FinRan}(X)$.
$K(X, Y)$: compact (linear) operators from X to Y. In the case when $X = Y$, we just write $K(X)$.
$L_1(\lambda)$ / $L_1(\lambda, X)$: space of (classes of) scalar-valued / X-valued integrable / Bochner integrable functions with respect to λ (see page 36).
$L_\infty(\lambda)$ / $L_\infty(\lambda, X)$: space of (classes of) scalar-valued / X-valued (strongly) λ-measurable and essentially bounded functions (see page 38).
$L(X, Y)$: bounded (linear) operators from X to Y. In the case when $X = Y$, we just write $L(X)$.
lin(A): linear span of A.
ℓ_p: space of sequences for which the usual p-norm is finite, $1 \leqslant p \leqslant \infty$.
$\ell_p^{(n)}$: n-dimensional ℓ_p-space, $1 \leqslant p \leqslant \infty$
MAP: Metric approximation property (see page 56).
$M(K)$: finite regular Borel signed measures on a compact Hausdorff topological space K, i.e., $M(K) = C(K)^*$.
$\mathcal{OP}(X)$: semigroup of operators on a Banach space X identified by the equivalent relation $\sim$ (see page 150).
Pol$(^m X, Y)$: m-homogeneous (continuous) polynomials from X to Y (see page 114).
Pol(X, Y): (continuous) polynomials from X to Y (see page 114).
RNP: Radon-Nikodým property (see page 37).
Σ^+: relative to a measure space (Ω, Σ, μ), the collection of those $A \in \Sigma$ for which $0 < \mu(A) < \infty$ (see page 140).
Slice(A, f, α): slice of A determined by the functional f and the positive number α (see page 29).
stexp(A): strongly exposed points of A (see page 38).
supp(f): support of the function/family/sequence f.
Var(f): variation of the function f.
$W(X, Y)$: weakly compact (linear) operators from X to Y. In the case when $X = Y$, we just write $W(X)$.
w^*-stexp(A): weak-star strongly exposed points of A (see page 38).

CHAPTER 1

The history of the subject

The first chapter will be devoted to a potpourri of results around the Daugavet equation for various classes of operators on classical Banach spaces. (We recall that by *operator* we shall mean *bounded linear operator*.) Also, our results will be formulated for *real* Banach spaces in this chapter.

1.1. Daugavet's theorem

In 1963, the mathematician Igor K. Daugavet published the following result in one of his first-ever papers [88].

THEOREM 1.1.1. *If* $T\colon C[a,b] \to C[a,b]$ *is a compact linear operator, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

Daugavet, in his later career, made a name for himself in approximation theory (see Figure 1.1), and also Theorem 1.1.1 connects to this field, as we shall comment shortly.

The present volume sets out to study consequences and ramifications of results like Theorem 1.1.1. Daugavet's eponymous equation for an operator T, i.e.,

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

will be called the *Daugavet equation*; thus Theorem 1.1.1 can be rephrased by saying that compact operators on $C[a,b]$ satisfy the Daugavet equation.

In this introductory chapter we shall gather a number of results on this subject that were proved by various techniques. Our main emphasis in later chapters, however, will be on the interplay of the geometry of slices of the unit ball of X and the validity of the Daugavet equation for certain classes of operators on X.

Let us now present Daugavet's proof of Theorem 1.1.1. He describes this theorem as "almost obvious, but at the same time unexpected"; here is his argument.

One first observes that it is enough to consider a finite-rank operator T since these operators are dense in the space of compact operators on $C[a,b]$. Such an operator has the form

$$Tx = \sum_{k=1}^{n} \varphi_k(x) z_k \tag{1.1.1}$$

with $z_k \in C[a,b]$ and continuous linear functionals $\varphi_k \in C[a,b]^*$ that can be represented by Riemann-Stieltjes integrals

$$\varphi_k(x) = \int_a^b x(t)\, d\sigma_k(t),$$

where σ_k is a function of bounded variation. Denote

$$\max_k \|z_k\| = M. \tag{1.1.2}$$

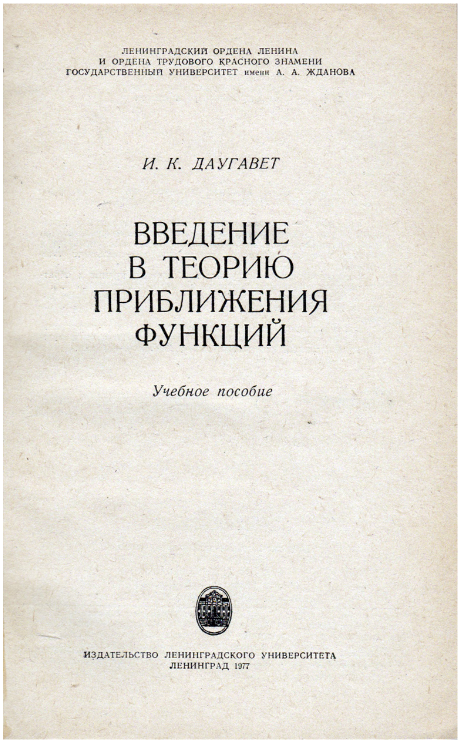

FIGURE 1.1. Daugavet's *Introduction to the theory of approximation of functions*

Let $\varepsilon > 0$. Pick $x_0 \in C[a,b]$ such that $\|x_0\| = 1$ and $\|Tx_0\| > \|T\| - \varepsilon/2$. Put $y_0 = Tx_0$ and let $\Delta \subset [a,b]$ be a subinterval on which $|y_0(t)| > \|T\| - \varepsilon/2$. Replacing x_0 with $-x_0$ if necessary we can even assume that $y_0(t) > \|T\| - \varepsilon/2$ on Δ. Further pick a subinterval $I = [t_0 - \delta, t_0 + \delta] \subset \Delta$ such that

$$\operatorname{Var}(\sigma_k|_I) \leqslant \frac{\varepsilon}{4nM} \tag{1.1.3}$$

for $k = 1, \dots, n$. Indeed, if Δ is written as a union of m non-overlapping closed intervals $I_1, \dots, I_m$, then one of the I_l will work provided that $m \geqslant (8nM/\varepsilon) \max_k \operatorname{Var}(\sigma_k|_\Delta)$.

Now, let $x_1 \in C[a,b]$ be the function that coincides with x_0 off the set I, $x_1(t_0) = 1$, and x_1 is affine-linear on $[t_0 - \delta, t_0]$ and on $[t_0, t_0 + \delta]$; put $y_1 = Tx_1$. Obviously $\|x_1\| = 1$, and it follows from (1.1.1), (1.1.2) and (1.1.3), that

$$\|y_1 - y_0\| \leqslant \frac{\varepsilon}{2}. \tag{1.1.4}$$

Indeed, by (1.1.1)

$$y_1 - y_0 = Tx_1 - Tx_0 = \sum_{k=1}^{n} (\varphi_k(x_1) - \varphi_k(x_0)) z_k$$

and, since $\|x_1 - x_0\| \leqslant 2$,

$$|\varphi_k(x_1) - \varphi_k(x_0)| = \left| \int_{t_0-\delta}^{t_0+\delta} (x_1(t) - x_0(t))\, d\sigma_k(t) \right| \leqslant 2 \operatorname{Var}(\sigma_k|_I) \leqslant \frac{\varepsilon}{2nM}$$

by (1.1.3), which implies (1.1.4) by (1.1.2).

One now has

$$\|\mathrm{Id} + T\| \geqslant \|x_1 + Tx_1\| \geqslant x_1(t_0) + y_1(t_0) = 1 + y_0(t_0) - [y_0(t_0) - y_1(t_0)].$$

But $y_0(t_0) \geqslant \|T\| - \varepsilon/2$ and $y_0(t_0) - y_1(t_0) \leqslant \|y_0 - y_1\| \leqslant \varepsilon/2$ by (1.1.4). Hence

$$\|\mathrm{Id} + T\| \geqslant 1 + \|T\| - \varepsilon,$$

and the theorem is proved. □

We may mention that later we will show that proving the validity of the Daugavet equation for rank-one operators is enough to get its validity for all compact operators (and more!), a fact which will make life easier. To understand why Theorem 1.1.1 is of interest in approximation theory, consider a sequence of bounded finite-rank norm-one linear operators $L_n\colon C[0,1] \to C[0,1]$ such that $L_n f \to f$ for each $f \in C[0,1]$; e.g., operators of interpolation. One then has the uniform estimate $\|L_n f - f\|_\infty \leqslant \|L_n - \mathrm{Id}\| \|f\|_\infty$. By Daugavet's theorem $\|L_n - \mathrm{Id}\| = 2$; hence the estimate $\|L_n f - f\|_\infty \leqslant 2\|f\|_\infty$ is optimal and cannot be replaced by $[\dots] \leqslant \|f\|_\infty$.

It might be tempting to surmise that Theorem 1.1.1 extends to all $C(K)$-spaces; but there is one caveat. Namely, if K has an isolated point, then the Daugavet equation will fail for certain compact, indeed rank-one operators. To wit, if $t_0 \in K$ is an isolated point and $Tf = -f(t_0)\mathbb{1}_{\{t_0\}}$, then $\|T\| = 1 = \|\mathrm{Id} + T\|$. However, if K is perfect (= without isolated points) one can translate the above proof using the language of neighbourhoods (instead of intervals) and signed measures (instead of functions of bounded variation) to show:

Theorem 1.1.2. *If K is a perfect compact Hausdorff space and $T\colon C(K) \to C(K)$ is compact, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

See Theorem 3.3.1 for the "modern" proof.

We also note that Daugavet's argument literally translates for weak*-weak*-continuous finite-rank operators on $L_\infty[0,1]$, i.e., for adjoints of such operators on $L_1[0,1]$. Since $\|T\| = \|T^*\|$ for all Banach space operators, one can deduce that compact operators on $L_1[0,1]$ satisfy the Daugavet equation.

1.2. The space L_1

We have just seen an argument for the following counterpart of Daugavet's theorem.

Theorem 1.2.1. *If $T\colon L_1[0,1] \to L_1[0,1]$ is compact, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

This result is originally due to Grigoriĭ Ya. Lozanovskiĭ in a paper submitted December 1, 1964 and published in 1966 [**212**]. He obtained it as a by-product of his theory of almost integral operators on KB-spaces. To indicate his proof, which was later streamlined by Abramovich, Aliprantis, and Burkinshaw in [**14**], we will freely use the language of Banach lattices, as outlined for instance in the monographs [**22**], [**233**], [**282**]; see also Subsection 2.9.5 on Banach lattices.

Although Lozanovskiĭ phrased his proof for L_1-spaces over nonatomic measure spaces, we will stick to the classical space $L_1[0,1]$. We remark that, in analogy to the $C(K)$-case, the presence of atoms will, in general, invalidate the Daugavet equation; see Theorem 3.3.2.

The key point of Lozanovskiĭ's argument is that the ordered space of operators $L(L_1[0,1])$ is in fact a Dedekind complete vector lattice (also known as a Riesz space), and it is a Banach lattice at that. Thus, every operator $T\colon L_1[0,1] \to L_1[0,1]$ has a modulus, and

$$\|T\| = \|\,|T|\,\|. \tag{1.2.1}$$

Now, two elements x, y in a vector lattice are said to be *disjoint* if $|x| \wedge |y| = 0$, and for disjoint elements one has

$$|x+y| = |x| + |y| \tag{1.2.2}$$

(this is a little tricky to verify from first principles; see [**304**, p. 19]).

The proof of Theorem 1.2.1 now takes two steps: (1) The Daugavet equation holds for T if T and Id are disjoint in the Banach lattice $L(L_1[0,1])$; (2) if T is compact, then T and Id are disjoint.

To check (1), note that by (1.2.1) and (1.2.2)

$$\|\mathrm{Id} + T\| = \|\,|\mathrm{Id} + T|\,\| = \|\,\mathrm{Id} + |T|\,\|,$$

and using that $\mathrm{Id} + |T|$ is a positive operator, one has

$$\|\,\mathrm{Id} + |T|\,\| = \sup\{\|f + |T|f\|_1\colon f \geqslant 0,\ \|f\|_1 = 1\}$$

and, since $\|f+g\|_1 = \|f\|_1 + \|g\|_1$ for positive functions,

$$\begin{aligned}\|\,\mathrm{Id} + |T|\,\| &= \sup\{1 + \|\,|T|f\,\|\colon f \geqslant 0,\ \|f\|_1 = 1\}\\ &= 1 + \|\,|T|\,\| = 1 + \|T\|.\end{aligned}$$

This completes the proof of (1).

As for (2), let $T\colon L_1[0,1] \to L_1[0,1]$ be compact and consider $S = \mathrm{Id} \wedge |T|$. Then $0 \leqslant S \leqslant \mathrm{Id}$ so that S is in the centre of $L_1[0,1]$; hence S is a multiplication operator, $Sf = M_\varphi f = \varphi f$ for some $\varphi \in L_\infty[0,1]$, $0 \leqslant \varphi \leqslant \mathbb{1}$ (cf. [**233**, p. 149]). Further, $|T|$ is compact ([**12**, Theorem 3.14]), and we shall argue that $S = 0$. If not, take a set E of positive measure and some $a > 0$ such that $\varphi \geqslant a$ on E. Pick a sequence (f_n) of pairwise disjoint, positive, normalised functions supported on E; since $|T|$ is compact we can assume in addition that $(|T|f_n)$ converges, say to g, and that $v_n = |T|f_n - g$ has norm $\leqslant 2^{-n}$. Then, the series $v := \sum_n |v_n|$ converges and

$$0 \leqslant af_n \leqslant \varphi f_n \leqslant |T|f_n \leqslant g + v$$

on E. But $af_n \to 0$ a.e., so $\|f_n\|_1 \to 0$ by the dominated convergence theorem (the f_n are dominated by $g + v$), in contradiction to $\|f_n\|_1 = 1$. □

A more pedestrian argument for Theorem 1.2.1 was given by Babenko and Pichugov [**30**]; we shall provide other proofs in Corollary 1.3.6 and Corollary 1.4.4.

It should be noted that the above proof made use of the nonatomic nature of the Lebesgue measure when we chose the functions f_n. Using more advanced machinery of operator theory in Banach lattices, Abramovich, Aliprantis and Burkinshaw show in [**14**] that even weakly compact operators on $L_1[0,1]$ satisfy the Daugavet equation, which will turn out to be a general feature of the theory described in this monograph. (For the case of L_1 cf. Corollary 1.3.6; for the general case see Theorem 3.2.6.)

1.3. Almost diffuse operators

In their 1965 paper [**111**], Ciprian Foiaş and Ivan Singer approached the Daugavet equation for operators on $C(K)$-spaces using a new idea. Actually, they even deal with vector-valued spaces, but we'll only present the scalar-valued case. They attribute a number of improvements to Aleksander Pełczyński, so he should be considered as one of the originators of the results in this section as well.

Here is the basic definition.

Definition 1.3.1. Let K be a compact Hausdorff space, X be a Banach space and $T\colon C(K) \to X$ be a bounded linear operator.

(a) A point $s_0 \in K$ is called a *point of diffusion* for T if for every $\varepsilon > 0$ there exists a neighbourhood $U(s_0)$ such that

$$g \in C(K),\ \|g\|_\infty \leqslant 1,\ g(s) = 0 \text{ for all } s \notin U(s_0) \quad \Longrightarrow \quad \|Tg\| < \varepsilon.$$

(b) T is called *almost diffuse* if the set of points of diffusion is dense in K.

We shall revisit this idea later in Definition 8.3.6.

The interest in almost diffuse operators in our context stems from the following theorem.

Theorem 1.3.2. *If $T\colon C(K) \to C(K)$ is almost diffuse, then T satisfies the Daugavet equation:*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

Proof. Let $\varepsilon > 0$, w.l.o.g. $\varepsilon < \|T\|$. Pick a function $f \in C(K)$ of norm 1 such that $\|Tf\|_\infty > \|T\| - \varepsilon$, that is, $|(Tf)(t_0)| > \|T\| - \varepsilon$ for some $t_0 \in K$. Therefore, the inequality $|(Tf)(t)| > \|T\| - \varepsilon$ holds true in a certain neighbourhood $V(t_0)$. Since T is almost diffuse, $V(t_0)$ contains a point of diffusion, say s_0. By definition, there exists a neighbourhood $U(s_0) \subset V(t_0)$ of s_0 such that

$$g \in C(K),\ \|g\|_\infty \leqslant 1,\ g(s) = 0 \text{ for all } s \notin U(s_0) \quad \Longrightarrow \quad \|Tg\| < \varepsilon.$$

Let $s \in U(s_0)$. Pick some $h \in C(K)$ such that

$$\|h\|_\infty \leqslant 1,\ h = f \text{ on } K \setminus U(s_0),\ h(s) = \frac{(Tf)(s)}{|(Tf)(s)|}.$$

(Since $s \in V(t_0)$, the denominator here is nonzero.) It follows $\|h - f\|_\infty \leqslant 2$, $h - f = 0$ on $K \setminus U(s_0)$ and, therefore,

$$\|T(h - f)\| < 2\varepsilon.$$

Consequently,

$$\begin{aligned}
\|\mathrm{Id} + T\| &\geqslant \|h + Th\|_\infty \geqslant \|h + Tf\|_\infty - 2\varepsilon \\
&\geqslant |h(s) + (Tf)(s)| - 2\varepsilon = \big|\,|(Tf)(s)| + 1\,\big| - 2\varepsilon \\
&= 1 + |(Tf)(s)| - 2\varepsilon \geqslant 1 + \|T\| - 3\varepsilon.
\end{aligned}$$

In the last step we used $s \in V(t_0)$, and in the last but second, the definition of $h(s)$. Since $\varepsilon > 0$ was arbitrary, the Daugavet equation follows. $\square$

We shall argue that compact operators are almost diffuse if K is perfect; in fact, weakly compact operators are. The way forward to see this is the integral representation theorem of Bartle, Dunford and Schwartz ([**99**, Th. VI.2.5], [**104**, p. 493]): Let $T\colon C(K) \to X$ be a weakly compact operator. Then there is a countably additive vector measure G of bounded semivariation such that

$$Tf = \int_K f\,dG \qquad (f \in C(K)).$$

Let us show that in this situation, a point $s_0 \in K$ is not a point of diffusion if $x_0 := G(\{s_0\}) \neq 0$. We shall freely use the elementary theory of vector measures. Indeed, is s_0 were a point of diffusion, we could find, for $\varepsilon = \frac{1}{2}\|x_0\|$, a neighbourhood $U(s_0)$ as spelt out in Definition 1.3.1. But then for $g \in C(K)$, $\|g\|_\infty \leqslant 1$, $g = 0$ on $K \setminus U(s_0)$ with $g(s_0) = 1$

$$\|Tg\| = \left\| \int_{\{s \neq s_0\}} g\,dG \right\| + \|x_0\| \geqslant 2\varepsilon$$

refuting the condition of Definition 1.3.1. (The converse statement is also valid, but we won't need it.)

Now, since G is countably additive, there can only be countably many s_0 with $G(\{s_0\}) \neq 0$; for otherwise there would be a sequence $s_1, s_2, \dots$ and some $\alpha > 0$ such that $\|G(\{s_n\})\| > \alpha$ for all n, and $\sum_n G(\{s_n\})$ would be divergent.

It follows, for a weakly compact T, that the points of diffusion form the complement of an at most countable set. Here is the punchline.

PROPOSITION 1.3.3. *If K is perfect, then every weakly compact operator is almost diffuse.*

PROOF. This follows from the Baire category theorem, because in a perfect compact space, a countable set is of the first category. □

COROLLARY 1.3.4. *If K is perfect, then every weakly compact operator $T\colon C(K) \to C(K)$ satisfies the Daugavet equation.*

Let us take the chance and rederive and extend the results of Section 1.2. We note a simple lemma.

LEMMA 1.3.5. *Let $T\colon X \to X$. Then T satisfies the Daugavet equation if and only if T^* does.*

PROOF. This is clear since the norm of an operator and that of its adjoint coincide. □

Now, let (Ω, Σ, μ) be a σ-finite atomless measure space. Then $L_1(\mu)^*$ is isometric to $L_\infty(\mu)$ (for this it would have sufficed that μ is localisable, see [**112**, Theorem 243G]). It follows from the Gelfand-Naimark theorem that $L_\infty(\mu)$, being a commutative C^*-algebra with unit, is a $C(K)$-space, and K does not have isolated points since μ is atomless. Now, suppose that $T\colon L_1(\mu) \to L_1(\mu)$ is weakly compact; then so is $T^*\colon L_1(\mu)^* \to L_1(\mu)^*$, see [**104**, p. 485]. Corollary 1.3.4 implies that T^* satisfies the Daugavet equation; hence we have from Lemma 1.3.5:

COROLLARY 1.3.6. *If (Ω, Σ, μ) is atomless and σ-finite and $T\colon L_1(\mu) \to L_1(\mu)$ is weakly compact, then T satisfies the Daugavet equation.*

1.4. The kernel approach

The basic idea of this section, following [**298**], is to represent an operator $T\colon X \to C(K)$ by its kernel, that is, the weak* continuous function $\tau\colon K \to X^*$ defined by $\tau(s) = T^*(\delta_s)$. It is quickly seen that τ efficiently encodes the properties of T in that

$$\|\tau\|_\infty := \sup_{s\in K} \|\tau(s)\| = \|T\|,$$

and τ is continuous for the weak topology of X^* if and only if T is weakly compact, τ is continuous for the norm topology if and only if T is compact. (See [**104**, p. 490] for details.)

We shall investigate the validity of the Daugavet equation for an operator $T\colon C(K) \to C(K)$ by means of its kernel. In the sequel we shall denote $\mu_s = T^*(\delta_s) \in M(K)$ so that

$$\int_K f\, d\mu_s = \langle f, \mu_s\rangle = \langle Tf, \delta_s\rangle = (Tf)(s).$$

We now formulate a technical condition that will allow us to prove the Daugavet equation for weakly compact operators and for operators that factor through c_0. Note that the identity operator is represented by the kernel $(\delta_s)_{s\in K}$.

LEMMA 1.4.1. *Let K be a compact Hausdorff space, and let $T\colon C(K) \to C(K)$ be a bounded linear operator with representing kernel $(\mu_s)_{s\in K}$. If the kernel satisfies*

$$\sup_{s\in U} \mu_s(\{s\}) \geqslant 0 \text{ for all nonvoid open sets } U \subset K, \tag{1.4.1}$$

then

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

In fact, a necessary and sufficient condition for this to hold is

$$\sup_{\{s\colon\ \|\mu_s\| > \|T\| - \varepsilon\}} \mu_s(\{s\}) \geqslant 0 \qquad \text{for all } \varepsilon > 0. \tag{1.4.2}$$

PROOF. We have

$$\|\mathrm{Id} + T\| = \sup_{s\in K} \|\delta_s + \mu_s\| = \sup_{s\in K} \bigl(|1 + \mu_s(\{s\})| + |\mu_s|(K \setminus \{s\})\bigr)$$

and

$$1 + \|T\| = \sup_{s\in K} \bigl(1 + \|\mu_s\|\bigr) = \sup_{s\in K} \bigl(1 + |\mu_s(\{s\})| + |\mu_s|(K \setminus \{s\})\bigr);$$

so problems with showing the Daugavet equation can only arise in case some of the $\mu_s(\{s\})$ are negative.

Given $\varepsilon > 0$, we now apply (1.4.1) to the open set

$$U = \{s \in K\colon \|\mu_s\| > \|T\| - \varepsilon\}$$

(that is, we apply (1.4.2)) and obtain

$$\begin{aligned}
\|\mathrm{Id} + T\| &\geqslant \sup_{s\in U} \|\delta_s + \mu_s\| = \sup_{s\in U} \bigl(|1 + \mu_s(\{s\})| + |\mu_s|(K \setminus \{s\})\bigr) \\
&\geqslant \sup_{s\in U,\ \mu_s(\{s\}) \geqslant -\varepsilon} \bigl(1 + \|\mu_s\| + \mu_s(\{s\}) - |\mu_s(\{s\})|\bigr) \\
&\geqslant 1 + \|T\| - \varepsilon + \sup_{s\in U,\ \mu_s(\{s\}) \geqslant -\varepsilon} \bigl(\mu_s(\{s\}) - |\mu_s(\{s\})|\bigr) \\
&\geqslant 1 + \|T\| - 3\varepsilon;
\end{aligned}$$

hence T satisfies the Daugavet equation. A similar calculation shows that (1.4.2) is not only sufficient, but also necessary. $\square$

Next, we deal with weakly compact operators.

LEMMA 1.4.2. *If K is a compact Hausdorff space without isolated points and $T\colon C(K) \to C(K)$ is weakly compact, then T fulfills (1.4.1) of Lemma 1.4.1.*

PROOF. To prove this lemma we argue by contradiction. Suppose there are a nonvoid open set $U \subset K$ and some $\beta > 0$ such that

$$\mu_s(\{s\}) < -2\beta \qquad \text{for all } s \in U.$$

At this stage we note that, for each $t \in K$, the function $s \mapsto \mu_s(\{t\})$ is continuous, since T is weakly compact. For, $\mu \mapsto \mu(\{t\})$ is in $M(K)^*$ and, as noted in the introduction of this section, $s \mapsto \mu_s$ is weakly continuous.

Returning to our argument, we pick some $s_0 \in U$ and consider the set

$$U_1 = \{s \in U\colon |\mu_s(\{s_0\}) - \mu_{s_0}(\{s_0\})| < \beta\}$$

which – as we have just observed – is an open neighbourhood of s_0. Since s_0 is not isolated, there is some $s_1 \in U_1$, $s_1 \neq s_0$. We thus have

$$\mu_{s_1}(\{s_1\}) < -2\beta,$$

because $s_1 \in U$, and

$$\mu_{s_1}(\{s_0\}) < \mu_{s_0}(\{s_0\}) + \beta < -2\beta + \beta = -\beta.$$

In the next step we let

$$U_2 = \{s \in U_1\colon |\mu_s(\{s_1\}) - \mu_{s_1}(\{s_1\})| < \beta\} \ (\subset U).$$

Likewise, this is an open neighbourhood of s_1, hence there is some $s_2 \in U_2$, $s_2 \neq s_1$, $s_2 \neq s_0$. We conclude, using that $s_2 \in U$, $s_2 \in U_2$ and $s_2 \in U_1$,

$$\begin{aligned} \mu_{s_2}(\{s_2\}) &< -2\beta, \\ \mu_{s_2}(\{s_1\}) &< -\beta, \\ \mu_{s_2}(\{s_0\}) &< -\beta. \end{aligned}$$

Thus we inductively define a descending sequence of open sets $U_n \subset U$ and distinct points $s_n \in U$ by

$$\begin{aligned} &U_{n+1} = \{s \in U_n\colon |\mu_s(\{s_n\}) - \mu_{s_n}(\{s_n\})| < \beta\}, \\ &s_{n+1} \in U_{n+1} \setminus \{s_0, \dots, s_n\} \end{aligned}$$

yielding

$$\mu_{s_n}(\{s_j\}) < -\beta \qquad \text{for all } j = 0, \dots, n-1.$$

Consequently,

$$\|T\| \geqslant \|\mu_{s_n}\| \geqslant |\mu_{s_n}|(\{s_0, \dots, s_{n-1}\}) \geqslant n\beta \qquad \text{for all } n \in \mathbb{N},$$

which furnishes a contradiction. $\square$

Lemmas 1.4.1 and 1.4.2 immediately yield the first main result of this section.

THEOREM 1.4.3. *Suppose K is a compact Hausdorff space without isolated points. If $T\colon C(K) \to C(K)$ is weakly compact, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

COROLLARY 1.4.4. *If μ is an atomless σ-finite (or just localisable) measure and $T\colon L_1(\mu) \to L_1(\mu)$ is weakly compact, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

PROOF. The argument is the same as the one leading to Corollary 1.3.6. □

REMARKS 1.4.5. (1) *If T is compact, the proof of Lemma 1.4.2 can considerably be simplified. In fact, if $\mu_s(\{s\}) < -2\beta < 0$ on an open nonvoid set U, let us pick some $s \in U$ and consider the set*

$$U_1 = \{t \in U\colon \|\mu_s - \mu_t\| < \beta\}.$$

Since T is compact, this is an open neighbourhood of s, and for each $t \in U_1$ we deduce that

$$\mu_s(\{t\}) \leqslant \mu_t(\{t\}) + |\mu_t(\{t\}) - \mu_s(\{t\})| < -2\beta + \|\mu_t - \mu_s\| < -\beta.$$

Since s is not isolated, there are infinitely many distinct points $t_1, t_2, \ldots \in U_1$, and we obtain $|\mu_s|(\{t_1, t_2, \ldots\}) = \infty$, a contradiction.

(2) *The proof of Theorem 1.4.3 shows that weakly compact operators on $C_0(L)$, L locally compact without isolated points, satisfy the Daugavet equation.*

(3) *We also see immediately that positive operators on $C(K)$-spaces (and likewise on (AL)- and (AM)-spaces) satisfy the Daugavet equation.*

(4) *For weakly compact operators T on $C(K)$, represented by $(\mu_s)_{s\in K}$, the functions $\varphi_A\colon s \mapsto \mu_s(A)$, $A \subset S$ a Borel set, are continuous; in fact, weakly compact operators are characterised by this property* [**104**, p. 493]. *In Lemma 1.4.2 it is even enough to assume that only the functions $\varphi_{\{t\}}$, $t \in K$, are continuous, provided K has no isolated points. Hence also such operators satisfy the Daugavet equation. A special case of this situation (a trivial one, though) occurs if $\mu_s(\{t\}) = 0$ for all $s, t \in K$; see also the following remark.*

(5) *A particular class of operators for which (1.4.1) of Lemma 1.4.1 is valid are those for which*

$$\{t \in K\colon \mu_s(\{t\}) = 0\ \forall s \in K\} \text{ is dense in } K. \tag{1.4.3}$$

Since this class is seen to contain the almost diffuse operators of Foiaş and Singer (Definition 1.3.1), we have obtained their result that almost diffuse operators satisfy the Daugavet equation; see Theorem 1.3.2.

We shall apply the last remark in the next result. Recall that an operator $T\colon X \to Y$ factors through the space Z if there are operators $T_1\colon X \to Z$ and $T_2\colon Z \to Y$ such that $T = T_2T_1$.

The following result is originally due to Holub [**139**] for $[0, 1]$ and Ansari [**24**] for general K.

THEOREM 1.4.6. *If K is a compact Hausdorff space without isolated points and $T\colon C(K) \to C(K)$ factors through c_0, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

PROOF. Let $(\mu_s)_{s\in K}$ be the representing kernel of T. By Remark 1.4.5(5) it is enough to show that

$$K' := \{t \in K\colon \mu_s(\{t\}) = 0\ \forall s \in K\}$$

is dense in K. Let us write $T = T_2 T_1$ with bounded linear operators $T_1\colon C(K) \to c_0$, $T_2\colon c_0 \to C(K)$. We have

$$\begin{aligned}(T_1 f)(n) &= \int_K f\, d\rho_n \qquad \text{for all } n \in \mathbb{N},\\ \big(T_2(a_n)\big)(s) &= \sum_{n=1}^{\infty} \nu_s(n) a_n \qquad \text{for all } s \in K\end{aligned}$$

for a sequence of measures ρ_n and a family $\big(\nu_s(n)\big)_n$ of sequences in ℓ_1. Consequently,

$$\mu_s = \sum_{n=1}^{\infty} \nu_s(n) \rho_n.$$

Now, $K' \supset \bigcap_n \{t \in S\colon \rho_n(\{t\}) = 0\}$, which is a set whose complement is at most countable. Since no point in K is isolated, countable sets are of the first category, and Baire's theorem implies that K' is dense. □

REMARKS 1.4.7. (6) The same proof applies to operators that factor through a $C(\kappa)$-space where κ is a countable compact space, since on such spaces all regular Borel measures are discrete. We recall that there are countable compact spaces κ such that $C(\kappa)$ is not isomorphic to c_0 [**52**].

(7) The Baire argument in Theorem 1.4.6 implies a very simple proof of Theorem 1.4.3 if in addition K is supposed to be separable. In fact, let us show that then (1.4.3) of Remark 1.4.5(5) holds. The complement of the set spelt out there is $\{t \in K\colon \exists s \in K\ \mu_s(\{t\}) \neq 0\}$. Since $s \mapsto \mu_s(\{t\})$ is continuous, this is, with $\{s_1, s_2, \dots\}$ denoting a countable dense subset of K, $\bigcup_n \{t \in K\colon \mu_{s_n}(\{t\}) \neq 0\}$ and hence a countable union of countable sets, i.e., of the first category. Again, $\{t \in S\colon \mu_s(\{t\}) = 0\ \forall s \in K\}$ must be dense.

We finally mention a property related to the Daugavet equation that is valid for all operators on all $C(K)$- or $L_1(\mu)$-spaces. It was first discovered by J. Duncan, C. McGregor, J. Pryce, and A. White in 1970 [**103**] (with a different proof).

PROPOSITION 1.4.8. *If K is a compact Hausdorff space and $T\colon C(K) \to C(K)$ is a bounded linear operator, then*

$$\max\{\|\mathrm{Id} + T\|, \|\mathrm{Id} - T\|\} = 1 + \|T\|.$$

PROOF. Let $(\mu_s)_{s\in K}$ be the representing kernel of T. Then

$$\begin{aligned}\max_{\pm} \|\mathrm{Id} \pm T\| &= \max_{\pm} \sup_{s\in K} \|\delta_s \pm \mu_s\|\\ &= \sup_{s\in K} \max_{\pm} \big(|\delta_s \pm \mu_s|(\{s\}) + |\delta_s \pm \mu_s|(K \setminus \{s\})\big)\\ &= \sup_{s\in K} \max_{\pm} \big(|1 \pm \mu_s(\{s\})| + |\mu_s|(K \setminus \{s\})\big)\\ &= \sup_{s\in K} \big(1 + |\mu_s(\{s\})| + |\mu_s|(K \setminus \{s\})\big)\\ &= \sup_{s\in K} \big(1 + \|\mu_s\|\big) = 1 + \|T\|,\end{aligned}$$

as claimed. □

COROLLARY 1.4.9. *If E is an (AL)-space or an (AM)-space and $T\colon E \to E$ is a bounded linear operator, then*

$$\max\{\|\mathrm{Id} + T\|, \|\mathrm{Id} - T\|\} = 1 + \|T\|.$$

PROOF. An (AL)-space E is representable as $L_1(\mu)$ for some localisable measure μ, hence E^* is representable as $L_\infty(\mu) \cong C(K)$. So the assertion follows from Proposition 1.4.8 by passing to T^*. If E is an (AM)-space, then E^* is an (AL)-space, and again we obtain the assertion by considering the adjoint operator. □

REMARK 1.4.10. *We finally wish to comment on the case of complex scalars. All the results and proofs in this chapter remain valid – mutatis mutandis – in the setting of complex Banach spaces. In Proposition 1.4.8 the proper formulation of the conclusion is*

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega T\| = 1 + \|T\|,$$

and (1.4.1) *in Lemma 1.4.1 should be replaced by*

$$\sup_{s \in U}\bigl(|1 + \mu_s(\{s\})| - (1 + |\mu_s(\{s\})|)\bigr) \geqslant 0 \text{ for all nonvoid open sets } U \subset K.$$

1.5. Notes and remarks

This chapter has described some early results on the Daugavet equation using a variety of methods and techniques. In the final section we would like to take the chance to discuss a couple of more theorems.

Using the approach via narrow operators (to be discussed in Chapter 6) the following extension of Corollary 1.3.4 was proved in [**175**]; incidentally, using the approach of Section 1.4, the same theorem was proved in [**297**].

We say that an operator $T\colon X \to Y$ *does not fix a copy of a Banach space E* (or is *E-singular*) if there is no subspace $Z \subset X$ isomorphic to E on which T is bounded below, i.e., acts as an isomorphism.

THEOREM 1.5.1. *If K is a perfect compact Hausdorff space and $T\colon C(K) \to C(K)$ does not fix a copy of $C[0,1]$, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

The corresponding result in the context of L_1-spaces reads as follows.

THEOREM 1.5.2. *If μ is atomless and $T\colon L_1(\mu) \to L_1(\mu)$ does not fix a copy of $L_1[0,1]$, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

This theorem was first proved by Plichko and Popov in [**252**, Theorem 8, Section 9] using narrow operators. Actually, it can also be traced back to a famous representation theorem due to Nigel Kalton [**180**, Theorem 3.1]. It asserts that an operator $T\colon L_1(\mu) \to L_1(\mu)$ can be represented by a random measure; namely, if Ω is a compact metric space equipped with its Borel-σ-algebra and a probability measure μ and $T\colon L_1(\mu) \to L_1(\mu)$ is an operator, then there is a weak* measurable map $\omega \mapsto \mu_\omega$ into $M(\Omega)$ such that

$$(Tf)(\omega) = \int_\Omega f(s)\, d\mu_\omega(s) \qquad \mu\text{-a.e.}$$

Godefroy, Kalton and Li [**123**, p. 266] deduce Theorem 1.5.2 from this and Kalton's [**180**, Theorem 5.5].

As yet, we haven't said much (if anything) about operators on other spaces apart from $C(K)$ and L_1. The first decisive step in this direction was taken in [**173**] where it was shown by an adaption of Daugavet's argument that the Daugavet equation holds for compact operators on finite-codimensional subspaces of $C[0,1]$. Also, the case of uniform algebras was discussed in [**175**] and [**299**] (in the former reference rather implicitly), and Abramovich showed in [**11**] that $L_1[0,1] \oplus_\infty L_1[0,1]$ and $C[0,1] \oplus_1 C[0,1]$ are Banach spaces (in fact, Banach lattices) on which the weakly compact operators satisfy the Daugavet equation. This was generalised to infinite sums of general "Daugavet prone" spaces (that we shall soon call "Banach spaces with the Daugavet property") by Wojtaszczyk [**302**].

We finally provide a list of papers from the last millennium establishing the Daugavet equation in a number of cases (the millennium threshold seems to be aptly chosen since the subject appeared in a different light after the publication of the papers [**178**] and [**285**] (see also the announcement [**177**]); some of them have already been mentioned: Y. Abramovich [**10**], [**11**]; Y. Abramovich and C. D. Aliprantis [**12**, Chap. 11], [**13**, Chap. 11]; Y. Abramovich, C. D. Aliprantis and O. Burkinshaw [**14**]; S. I. Ansari [**24**]; V. F. Babenko and S. A. Pichugov [**30**]; P. Chauveheid [**78**]; I. K. Daugavet [**88**]; U. U. Diallo and P. P. Zabrejko [**95**]; C. Foiaş and I. Singer [**111**]; J. R. Holub [**137**], [**138**], [**139**]; V. M. Kadets [**173**], [**174**]; V. M. Kadets and M. M. Popov [**175**]; H. Kamowitz [**185**]; R. Khalil [**187**]; C.-S. Lin [**204**]; G. Ya. Lozanovskiĭ [**212**]; A. M. Plichko and M. M. Popov [**252**]; K. D. Schmidt [**283**]; L. Weis [**296**]; L. Weis and D. Werner [**297**]; D. Werner [**298**], [**299**]; P. Wojtaszczyk [**302**].

On the topic of Proposition 1.4.8 and Corollary 1.4.9 let us point out that there is a close relation with the theory of numerical ranges, numerical radius, and the numerical index that we will discuss in Section 12.3.

CHAPTER 2

Some results from Banach space theory and topology that are used in the book

In this chapter we briefly recall terminology and results that go beyond a standard Functional Analysis course. Doing this we don't pretend completeness: several other results will be mentioned later in other chapters at the moment when we first need them. An expert in Banach spaces could probably skip this chapter (or, at least, most of it), but it is included here also to fix the terminology and notation. Let us mention that our bedside books on Banach space theory are those of Albiac–Kalton [**108**], Benyamini–Lindenstrauss [**51**], Bourgin [**65**], Carothers [**74**], Deville–Godefroy–Zizler [**94**], Diestel [**97**], Fabian–Habala–Hájek–Montesinos–Zizler [**109**], Kadets [**156**], and Lindenstrauss–Tzafriri [**206**]. Most of the results in this chapter can be found there and most of the references in the chapter are to these books.

2.1. Duality in Banach spaces

The concepts and results listed in this section can be found in advanced textbooks in Functional Analysis like [**156**, Chapters 16–17] or in introductory parts of Banach space theory books like [**109**, Chapter 3].

Let V, W be linear spaces over the same field $\mathbb{K}$. A mapping that assigns to each pair of elements $(x, y) \in V \times W$ a number $\langle x, y\rangle \in \mathbb{K}$ is called a *duality* if: (1) $(x, y) \mapsto \langle x, y\rangle$ is a bilinear form:

$$\langle a_1 x_1 + a_2 x_2, y\rangle = a_1\langle x_1, y\rangle + a_2\langle x_2, y\rangle;$$
$$\langle x, a_1 y_1 + a_2 y_2\rangle = a_1\langle x, y_1\rangle + a_2\langle x, y_2\rangle;$$

and (2) it satisfies the non-degeneracy condition

- for every $x \in V \setminus \{0\}$ there exists some $y \in W$ such that $\langle x, y\rangle \neq 0$, and
- for every $y \in W \setminus \{0\}$ there exists some $x \in V$ such that $\langle x, y\rangle \neq 0$.

A pair of spaces V, W with a given duality on them is called a *pair of spaces in duality* or a *duality pair*.

Let (V, W) be a pair of spaces in duality. For each element $y \in W$ we define its action on the elements of the space V by the rule $y(x) = \langle x, y\rangle$. With this definition every element $y \in W$ becomes a linear functional on V. The *weak topology* $\sigma(V, W)$ on V is defined as the weakest topology on V in which all elements $y \in W$ define continuous functionals on V. A neighbourhood basis of zero in the topology $\sigma(V, W)$ is given by the family of sets

$$U_{\varepsilon,G} = \left\{ x \in V\colon \max_{y\in G} |\langle x, y\rangle| < \varepsilon \right\},$$

where $\varepsilon > 0$ and G runs over all finite subsets of the space W.

The roles of the spaces V, W in the above definitions may be exchanged, which leads to the topology $\sigma(W, V)$ on W.

The *polar* of the $A \subset V$ is the set $A^\circ \subset W$, defined by the following rule: $y \in A^\circ$ if $|\langle x, y\rangle| \leqslant 1$ for all $x \in A$. The polar $A^\circ \subset V$ of a set $A \subset W$ is defined by symmetry.

For $A \subset V$ the set $(A^\circ)^\circ \subset V$ is called the *bipolar* of A and is denoted by $A^{\circ\circ}$.

The bipolar theorem. Let (V, W) be a pair of spaces in duality. The bipolar $A^{\circ\circ}$ of a set $A \subset V$ is equal to the $\sigma(V, W)$-closed absolutely convex hull of the set A. □

Let X be a Banach space. We denote the action of a functional $x^* \in X^*$ on $x \in X$ either by $x^*(x)$ or by $\langle x^*, x\rangle$. We identify the elements of X as usual with the functionals on X^* generated by them: for a given $x \in X$, the corresponding functional $\hat{x}\colon X^* \to \mathbb{K}$ acts by the rule $\hat{x}(x^*) = x^*(x)$. If needed, we will use the notation $J_X\colon X \to X^{**}$ for the isometric embedding $x \mapsto \hat{x}$. The Hahn-Banach theorem implies that $\|\hat{x}\| = \|x\|$, and accordingly, we consider X as a subspace of X^{**}.

A subset $A \subset B_{X^*}$ is said to be *one-norming* if for every $x \in X$

$$\sup\{|\langle x^*, x\rangle| : x^* \in A\} = \|x\|.$$

$A \subset X^*$ is said to be *total* (over X) if for every $x \in X$ with $x \neq 0$ there is $x^* \in A$ with $x^*(x) \neq 0$. A subspace $Y \subset X^*$ is said to be a *one-norming subspace* if its unit sphere S_Y is a one-norming subset.

A subset $A \subset S_{X^*}$ is said to be a *boundary* (or *James-boundary*) for X if for every $x \in X$ there is $x^* \in A$, such that $|x^*(x)| = \|x\|$. Each boundary is evidently a one-norming set, but the opposite is false (just consider the open unit ball of X^*).

A pair (X, Y) where X is a Banach space and $Y \subset X^*$ is a total linear subspace is a good example of a duality pair.

Let us remark an easy but important property of one-norming subspaces.

Proposition 2.1.1. *Let X be a Banach space and $Y \subset X^*$ be a one-norming subspace. Then B_X is $\sigma(X, Y)$-closed. Consequently, every ball $x_0 + rB_X$ is $\sigma(X, Y)$-closed as well.*

Proof. Let us demonstrate that $X \backslash B_X$ is $\sigma(X, Y)$-open. For every $x \in X \backslash B_X$ we have

$$\sup\{|x^*(x)| : x^* \in S_Y\} = \|x\| > 1.$$

So, there is $x^* \in S_Y$ with $|x^*(x)| > 1$. On the other hand, $|x^*(y)| \leqslant 1$ for all $y \in B_X$. So, the set $\{y \in X\colon |x^*(y)| > 1\} \subset X \setminus B_X$ is the desired $\sigma(X, Y)$-open neighbourhood of x. □

In the special case of the duality pair (X, X^*) the corresponding topology $\sigma(X, X^*)$ is simply called *the weak topology of* X and is denoted by w. The duality pair (X^*, X) leads to the topology $\sigma(X^*, X)$, which is called the *weak-star topology* of X^* and is denoted by w^*. Remark that Proposition 2.1.1 is applicable to both w and w^*.

The weak topology of X has many close connections to the norm topology, in particular the following theorem is true.

Theorem 2.1.2 (S. Mazur). *Suppose the sequence (x_n) of elements of the Banach space X converges weakly to an element $x \in X$. Then x lies in the strong closure of the convex hull of the sequence (x_n). Moreover, there exists a sequence*

(y_n) of convex combinations of the elements x_n that converges strongly to x, such that $y_n \in \operatorname{conv}(\{x_k\colon k \geqslant n\})$, $n = 1, 2, \ldots$.

The first statement in Theorem 2.1.3 below is evident, the remaining ones are standard consequences of the bipolar theorem.

THEOREM 2.1.3.

1. A subspace $Y \subset X^*$ is total over X if and only if the pairing $(x, y) \mapsto \langle x, y\rangle$ on (X, Y) satisfies the non-degeneracy condition. In this case, the topology $\sigma(X, Y)$ is Hausdorff.

2. A subspace $Y \subset X^*$ is total if and only if it is w^*-dense in X^*.

3. A subset $A \subset S_{X^*}$ is one-norming if and only if its absolute convex hull $\operatorname{aconv}(A)$ *is w^*-dense in B_{X^*}.*

4. A subspace $Y \subset X^*$ is one-norming if and only if B_Y is weak dense in B_{X^*}.*

5. For an infinite-dimensional Banach space X, a subspace $Y \subset X^*$ is one-norming if and only if S_Y is weak dense in B_{X^*}.*

Due to the classical Banach-Alaoglu theorem, the unit ball of X^* is compact in the w^*-topology. At the same time, for some spaces X the unit ball B_{X^*} contains sequences that do not have w^*-convergent subsequences. In other words, (B_{X^*}, w^*) is not necessarily sequentially compact. Such effects are not something "exotic". For example, the sequence of coordinate functionals $(e_k^*)_{k=1}^{\infty}$ on ℓ_∞, $e_k^*(x_1, x_2, \ldots) := x_k$, does not contain $\sigma((\ell_\infty)^*, \ell_\infty)$-convergent subsequences, although in the topological sense it has incredibly many (more than continuum many) $\sigma((\ell_\infty)^*, \ell_\infty)$-cluster points.

This means that the sequential language does not describe the w^*-topology adequately. This is one of the reasons why we need the language of filters and ultrafilters.

A highly non-trivial sequential property of the weak topology, called "countable tightness", is described in the following theorem due to Kaplansky [**191**, page 312] (we only present the particular case that we need).

THEOREM 2.1.4. *Let A be a subset of a Banach space X, and let $x \in X$ belong to the weak closure of A. Then there is a countable subset $\tilde{A} \subset A$ such that x belongs to the weak closure of $\tilde{A}$.*

2.2. Filters, ultrafilters, ultrapowers

The results about filters and ultrafilters mentioned below can be found in [**156**, Chapter 16]; for an introduction to ultraproducts and ultrapowers we refer to [**134**].

A family of subsets $\mathfrak{F}$ of a set $\Gamma \neq \emptyset$ is said to be a *filter* on Γ if it possesses the following properties:

(i) $\Gamma \in \mathfrak{F}$;

(ii) $\emptyset \notin \mathfrak{F}$;

(iii) if $A, B \in \mathfrak{F}$, then $A \cap B \in \mathfrak{F}$;

(iv) if $A \in \mathfrak{F}$ and $A \subset B \subset \Gamma$, then $B \in \mathfrak{F}$.

The definition implies the stability of $\mathfrak{F}$ under finite intersections, so a finite intersection of the form $\bigcap_{k=1}^n A_k$ is not empty if all the A_k are elements of $\mathfrak{F}$. A good example of a filter is the system of neighbourhoods of a given point in a topological space. An important example of a filter on $\mathbb{N}$ is the *Fréchet filter*, whose elements are all sets of the form $\mathbb{N} \setminus A$ where A is finite.

A non-empty family $\mathfrak{D}$ of subsets of Γ is a *filter base* if
(a) $\emptyset \notin \mathfrak{D}$ and
(b) for all $A, B \in \mathfrak{D}$ there is $C \in \mathfrak{D}$ such that $C \subset A \cap B$.

Let $\mathfrak{D}$ be a filter base. *The filter generated by the base* $\mathfrak{D}$ is the family $\mathrm{Filt}(\mathfrak{D})$ of all those sets which contain an element of $\mathfrak{D}$ as a subset: $A \in \mathrm{Filt}(\mathfrak{D})$ if and only if there exists $B \in \mathfrak{D}$ such that $B \subset A$.

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be filters on the same set Γ. $\mathfrak{F}_1$ is said to *dominate* $\mathfrak{F}_2$ if $\mathfrak{F}_1 \supset \mathfrak{F}_2$; in other words, if each element of $\mathfrak{F}_2$ is at the same time an element of $\mathfrak{F}_1$.

A filter on Γ that is maximal by inclusion is called an *ultrafilter*. Zorn's lemma implies that for every filter $\mathfrak{F}$ on Γ there is an ultrafilter that dominates $\mathfrak{F}$. The only explicit examples of ultrafilters are the *trivial* ones, that is, the filters of the form $\mathrm{Filt}(\{\{x\}: x \in \Gamma\})$. The remaining nontrivial ultrafilters are called *free*. All elements of a free ultrafilter are infinite sets.

Let Γ be a set, X a Hausdorff topological space, $\mathfrak{F}$ a filter on Γ, and $f: \Gamma \to X$ a function. A point $x \in X$ is said to be the *limit of f by* (or *along*) $\mathfrak{F}$ (the corresponding notation is $x = \lim_{\mathfrak{F}} f$ or $x = \mathfrak{F}\text{-}\lim_t f(t)$) if for every neighbourhood U of x there is $A \in \mathfrak{F}$ such that $f(A) \subset U$.

$x \in X$ is called a *limiting point of f by* $\mathfrak{F}$ if each neighbourhood of x intersects all elements of the form $f(A)$, $A \in \mathfrak{F}$.

A large class of filters comes from directed sets.

DEFINITION 2.2.1. A set Γ endowed with a binary relation $\succ$ is called a *directed set* if the following axioms are satisfied:

(a) $g \succ g$ for all $g \in \Gamma$;
(b) if $g_2 \succ g_1$ and $g_3 \succ g_2$, then $g_3 \succ g_1$;
(c) for any two elements $g_1, g_2 \in \Gamma$ there exists an element $g_3 \in \Gamma$ such that $g_3 \succ g_1$ and $g_3 \succ g_2$.

The relation $\succ$ is called a *(non-strict) ordering*. A *strict ordering* is a relation on Γ that satisfies (b) and (c), but (a) is substituted by $(g_1 \succ g_2) \Rightarrow (g_2 \not\succ g_1)$. For every strict ordering $\succ$ there is an induced non-strict ordering (that is usually denoted by the same symbol if it does not lead to confusion): $\gamma_1 \succ \gamma_2$ non-strictly if $\gamma_1 \succ \gamma_2$ or $\gamma_1 = \gamma_2$. Remark that a non-strict ordering induced by a strict ordering possesses the following additional property: $(g_1 \succ g_2) \wedge (g_2 \succ g_1) \Rightarrow (g_1 = g_2)$.

DEFINITION 2.2.2. Let $(\Gamma, \succ)$ be a directed set. The *section filter* on Γ is the filter $\mathfrak{F}_\succ$ a basis of which consists of all sets of the form $\{x \in \Gamma: x \succ a\}$ with $a \in \Gamma$.

DEFINITION 2.2.3. Let $(\Gamma, \succ)$ be a directed set, X a topological space, and let x_g, $g \in \Gamma$, be elements of X. The function $g \mapsto x_g$ is called a *net*. The standard notation for a net is $(x_g, g \in \Gamma)$. A limit (respectively, a limiting point) of a net are defined to be a limit (respectively, a limiting point) of the function $g \mapsto x_g$ with respect to the corresponding section filter. In other words, $x \in X$ is a limit of $(x_g, g \in \Gamma)$ if for every neighbourhood U of x there is $g \in \Gamma$ such that $f(h) \in U$ for all $h \succ g$. A point $x \in X$ is a limiting point of $(x_g, g \in \Gamma)$ (another name is "cluster point") if for each neighbourhood of x and each $g \in \Gamma$ there is $h \succ g$ such that $f(h) \in U$.

The following theorem is a good substitute for the Bolzano-Weierstrass theorem, which is especially useful in non-metrisable spaces where the sequential language does not work well.

THEOREM 2.2.4. *Let $\mathfrak{F}$ be an ultrafilter on Γ, X be a Hausdorff topological space, and let the range $f(\Gamma)$ of $f\colon \Gamma \to X$ be a subset of a compact $K \subset X$. Then the limit of f by $\mathfrak{F}$ exists, and that limit lies in K.*

If a function f has a compact range, then the limit by an ultrafilter selects one of the limiting points of f. When the ultrafilter is fixed, this selection has the following advantage, which makes it a useful technical tool even for real-valued functions: if X is a Hausdorff topological vector space then the mapping $f \mapsto \lim_{\mathcal{F}} f$ is linear in f on its domain.

DEFINITION 2.2.5. Let X be a Banach space and $\mathfrak{U}$ be a free ultrafilter on $\mathbb{N}$. Denote by $\ell_\infty(X)$ the Banach spaces of all bounded X-valued sequences equipped with the standard sup-norm. The *ultrapower* $X^{\mathfrak{U}}$ of X by $\mathfrak{U}$ is the quotient of $\ell_\infty(X)$ by the subspace of those $x = (x_n)$ for which $\mathfrak{U}\text{-}\lim_n x_n = 0$.

The elements of the ultrapower $X^{\mathfrak{U}}$ are equivalence classes in the space of all bounded sequences (x_n), $x_n \in X$, under the equivalence relation $(x_n) \sim (y_n)$ if $\lim_{\mathfrak{U}} \|x_n - y_n\| = 0$, equipped with the norm $\|[(x_n)]\| = \lim_{\mathfrak{U}} \|x_n\|$. If there is a necessity to stress the fact that $[(x_n)] \in X^{\mathfrak{U}}$, the notation $[(x_n)]_{\mathfrak{U}}$ is used. In particular, such a necessity may appear when one speaks of an ultrapower of a quotient space.

It is convenient to consider (and we sometimes do it this way) the ultrapower $X^{\mathfrak{U}}$ as the space of all bounded sequences $x = (x_n)$, $x_n \in X$, in the norm $\|x\| = \lim_{\mathfrak{U}} \|x_n\|$ under the agreement that $x = (x_n)$ is equal to $y = (y_n)$ in the sense of $X^{\mathfrak{U}}$ if $\lim_{\mathfrak{U}} \|x_n - y_n\| = 0$.

X is identified with a subspace of $X^{\mathfrak{U}}$ in the following canonical way: $x \mapsto [(x, x, \ldots)]$.

DEFINITION 2.2.6. Let $\mathfrak{U}$ be a nontrivial ultrafilter on $\mathbb{N}$ and let T be an operator acting from a Banach space X to a Banach space E. We denote by $T^{\mathfrak{U}}$ the natural operator between the ultrapowers $X^{\mathfrak{U}}$ and $E^{\mathfrak{U}}$ defined by $T^{\mathfrak{U}}[(x_n)] = [(Tx_n)]$.

In infinite-dimensional spaces, where the unit ball is not compact and the ultrafilter limit of a bounded sequence does not always exist, the element $x = (x_n)$ of the ultrapower is a substitute for that non-existent limit. In many instances the ultrapower language is a useful tool that helps to avoid boring ε-δ reasoning, substituting approximate equality in X by exact ones in the ultrapower. An easy example of this is the following proposition.

PROPOSITION 2.2.7. *Under the conditions of Definition 2.2.6, $\|T^{\mathfrak{U}}\| = \|T\|$ and $T^{\mathfrak{U}}$ is norm-attaining. Moreover, if T is unbounded below, then $T^{\mathfrak{U}}$ is non-injective.*

PROOF. For every $n \in \mathbb{N}$ choose $x_n \in S_X$ such that $\|Tx_n\| > \|T\| - \frac{1}{n}$. Then $[(x_n)] \in X^{\mathfrak{U}}$ has norm one and $\|T^{\mathfrak{U}}[(x_n)]\| = \lim_{\mathfrak{U}} \|Tx_n\| = \|T\|$. Analogously, if T is unbounded below, choose $y_n \in S_X$ such that $\|Ty_n\| < \frac{1}{n}$. Then $\|[(y_n)]\| = 1$ and $\|T^{\mathfrak{U}}[(y_n)]\| = \lim_{\mathfrak{U}} \|Ty_n\| = 0$. $\square$

This technique works especially smoothly for those Banach space properties that are inherited by ultrapowers. That was one of the motivations for the extensive study of super-properties (i.e., the properties that are inherited by finite representability and, consequently, by ultrapowers) in the 1970ies. Let us give the corresponding definition.

DEFINITION 2.2.8. Let $\varepsilon > 0$ be a positive real. Two Banach spaces E, F are said to be $(1+\varepsilon)$-*isometric* if there exists an isomorphism $T: E \to F$ with $\|T\|\|T^{-1}\| \leqslant 1+\varepsilon$. A Banach space X is said to be *finitely representable* in a Banach space Y if for every $\varepsilon > 0$ and for every finite-dimensional subspace $E \subset X$ there is a finite-dimensional subspace $F \subset Y$ such that E and F are $(1+\varepsilon)$-isometric.

DEFINITION 2.2.9. Let $\mathbf{P}$ be a Banach space property. A Banach space Y is said to possess the property *super*-$\mathbf{P}$ if for every Banach space X, if X is finitely representable in Y, then X possesses $\mathbf{P}$. In particular, a Banach space Y is said to be *superreflexive* if every Banach space X finitely representable in Y is reflexive.

Key examples of superreflexive spaces are the L_p-spaces for $1 < p < \infty$.

It is easy to see that every ultrapower of a space is finitely representable in that space. Moreover (this takes a little bit more work), if a separable space X is finitely representable in Y, then X is isometric to a subspace of $Y^{\mathfrak{U}}$.

A good example of when superreflexivity plays an important role is the relation between $(X^{\mathfrak{U}})^*$ and $(X^*)^{\mathfrak{U}}$. For every $F = [(f_n)] \in (X^*)^{\mathfrak{U}}$, $f_n \in X^*$, one can define the action on $X^{\mathfrak{U}}$ by the rule $F([(x_n)]) = \lim_{\mathfrak{U}} f_n(x_n)$. This induces the *canonical embedding* $(X^*)^{\mathfrak{U}} \subset (X^{\mathfrak{U}})^*$. But every functional F defined above attains its norm, so the equality $(X^*)^{\mathfrak{U}} = (X^{\mathfrak{U}})^*$ can happen only if $X^{\mathfrak{U}}$ is reflexive (by James's theorem). The reflexivity of $X^{\mathfrak{U}}$ is equivalent to the superreflexivity of X. So, in almost all examples that we are interested in, $(X^*)^{\mathfrak{U}} \subsetneq (X^{\mathfrak{U}})^*$. Nevertheless, it is easy to see that $(X^*)^{\mathfrak{U}}$ is always a one-norming subspace, as we will show in Proposition 2.2.10 below.

One more piece of notation. Let $\Gamma \subset S_{X^*}$. We denote $\Gamma^{\mathfrak{U}}$ the set of the linear functionals $F = [(f_n)]$, $f_n \in \Gamma$, of the form $F[(x_n)] = \lim_{\mathfrak{U}} f_n(x_n)$.

PROPOSITION 2.2.10. *$\Gamma \subset S_{X^*}$ is one-norming if and only if $\Gamma^{\mathfrak{U}}$ is a boundary, and this happens if and only if $\Gamma^{\mathfrak{U}}$ is one-norming. In particular, $(X^*)^{\mathfrak{U}}$ is a one-norming subspace of $(X^{\mathfrak{U}})^*$, moreover, $S_{(X^*)^{\mathfrak{U}}}$ is a boundary.*

PROOF. Let $\Gamma \subset S_{X^*}$ be one-norming. For every $[(x_n)] \in S_{X^{\mathfrak{U}}}$, $x_n \in X$, choose $f_n \in \Gamma$, $n \in \mathbb{N}$, such that $|f_n(x_n)| > \|x_n\| - \frac{1}{n}$. Then $F := [(f_n)] \in \Gamma^{\mathfrak{U}}$ has norm 1 and $|F[(x_n)]| = \lim_{\mathfrak{U}} |f_n(x_n)| = \|[(x_n)]\| = 1$, which demonstrates that $\Gamma^{\mathfrak{U}}$ is a boundary. A boundary is one-norming, so it remains to prove that if $\Gamma^{\mathfrak{U}}$ is one-norming, then Γ is one-norming. To this end, fix $x \in S_X$ and $\varepsilon > 0$. For the element $[(x, x, \ldots)] \in S_{X^{\mathfrak{U}}}$ take some $F := [(f_n)] \in \Gamma^{\mathfrak{U}}$ with $f_n \in \Gamma$ such that $|F([(x, x, \ldots)])| > 1 - \varepsilon$. This means that

$$\lim_{\mathfrak{U}} |f_n(x)| > 1 - \varepsilon,$$

so for some n we have $|f_n(x)| > 1 - \varepsilon$, and the job is done. □

The ultrapower is a particular case of an *ultraproduct*. The ultraproduct of a sequence $(X_n)_{n\in\mathbb{N}}$ of Banach spaces by the ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, denoted $(X_n)_{\mathfrak{U}}$, is the space of all equivalence classes of $x = (x_n)$, $x_n \in X_n$, $n = 1, 2, \ldots$, equipped by the norm $\|x\| = \lim_{\mathfrak{U}} \|x_n\|$, where $x = (x_n)$, $y = (y_n)$ are equivalent if $\lim_{\mathfrak{U}} \|x_n - y_n\| = 0$.

2.3. Bases, basic sequences, and Schauder decompositions

The concepts and results listed in this section can be found, for example, in [**206**], where we refer the interested reader for more information and background.

DEFINITION 2.3.1. A sequence of elements $(e_n)_{n\in\mathbb{N}}$ of a Banach space X is called a *Schauder basis* (or just *basis*) of X if for any element $x \in X$ there exists a unique sequence of scalar coefficients (a_n) such that the series $\sum_{n=1}^{\infty} a_n e_n$ converges to x.

An example of a basis is provided by any orthonormal basis in a separable Hilbert space. Another standard example is the *canonical basis* in each of the sequence spaces ℓ_p $(1 \leqslant p < \infty)$ or c_0. That canonical basis is the system of vectors $(e_n)_{n=1}^{\infty}$, where $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, ... One more example is the trigonometric system $\{1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \dots\}$ in $L_p[0, 2\pi]$ for $1 < p < \infty$.

An example of a basis in $L_1[0, 1]$ that is important for us is the *Haar system* (which, in fact forms a basis in all $L_p[0, 1]$ for $1 \leqslant p < \infty$). This system consists of $h_0 = 1$ and the functions

$$h_{k,j} = \mathbb{1}_{[(2j-2)2^{-k},(2j-1)2^{-k})} - \mathbb{1}_{[(2j-1)2^{-k},2j2^{-k})},$$

where $k \in \mathbb{N}$, $1 \leqslant j \leqslant 2^{k-1}$. In order to make a basis out of these functions, one needs to write them in a sequence in the following natural order:

$$h_0,\ h_1 = h_{1,1},\ h_2 = h_{2,1},\ h_3 = h_{2,2},\ h_4 = h_{3,1},\ \dots$$

see Figure 2.1.

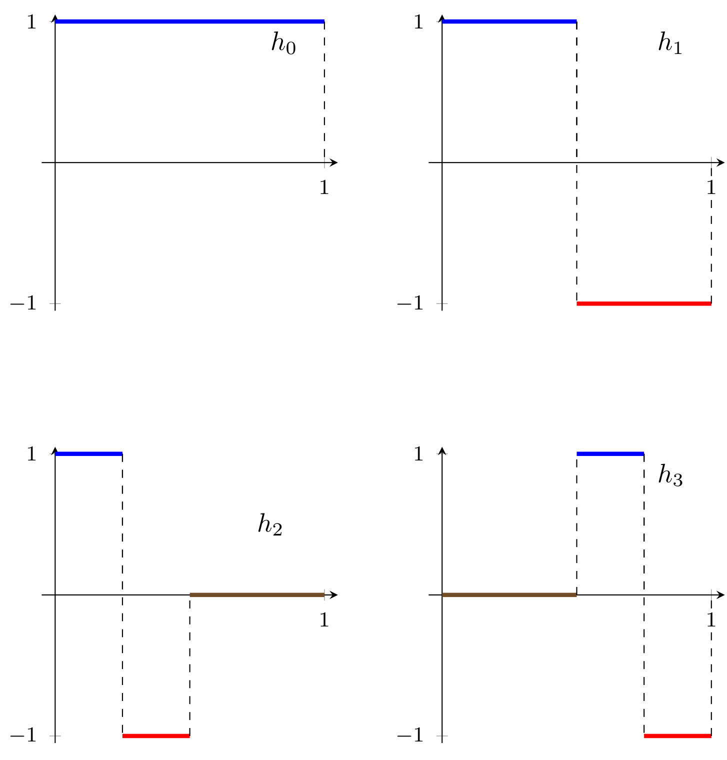

FIGURE 2.1. The Haar system

A generalisation of the Haar system, called *Haar-like systems*, will play a crucial role in Section 5.4.1 below.

It is plain that a basis may exist only in a separable space. Banach's question whether every separable infinite-dimensional Banach space has a basis turned out to be very difficult. A negative answer was provided in 1973 by P. Enflo (see [**206**, Sect. 2.d]).

DEFINITION 2.3.2. Let $(e_n)_{n=1}^\infty$ be a basis of the Banach space X, and let $x \in X$. Denote by $e_n^*(x)$ the coefficients of the decomposition of x in the basis $(e_n)_{n=1}^\infty$, and by $S_n(x)$ the n-th partial sum of the decomposition, i.e., $S_n(x) = \sum_{k=1}^n e_k^*(x)e_k$.

It is easy to see that $e_n^*(x)$ and $S_n(x)$ are linear in the variable x. Their continuity is non-trivial and was first demonstrated by S. Banach:

THEOREM 2.3.3. *Let $(e_n)_{n=1}^\infty$ be a basis of the Banach space X. Then the corresponding partial sum operators S_n and coordinate functionals e_n^* are continuous and* $\sup_n \|S_n\| < \infty$.

We will refer to the partial sum operators S_n as the *basis projections* or the *projections associated to the basis*. The functionals e_n^* are called *coordinate functionals* associated to the basis.

The quantity $C = \sup_{n\in\mathbb{N}} \|S_n\|$ is called the *basis constant* of the basis (e_n).

DEFINITION 2.3.4. A sequence of non-zero subspaces $(X_n)_{n=1}^\infty$ of a Banach space X is called a *Schauder decomposition* of X if for any element $x \in X$ there exists a unique sequence of elements (x_n), $x_n \in X_n$ for every $n \in \mathbb{N}$, such that $x = \sum_{n=1}^\infty x_n$. The mappings $x \mapsto x_n$ are called *coordinate projections*, and $x \mapsto \sum_{k=1}^n x_n$ are called *partial sum projections.*

Taking $X_n = \operatorname{lin} e_n$ for a basis $(e_n)_{n=1}^\infty$ one gets the easiest example of a Schauder decomposition. The continuity and uniform boundedness of partial sums projections extend to general Schauder decompositions.

DEFINITION 2.3.5. A sequence $(e_n)_{n=1}^\infty$ in a Banach space called a *basic sequence* if it forms a basis of its closed linear span $\overline{\operatorname{lin}}\{e_n \colon n \in \mathbb{N}\}$.

According to Mazur's *basic sequence selection principle* every infinite-dimensional Banach space contains a basic sequence, which makes basic sequences an important tool of general Banach space theory. Of special importance for us will be ℓ_1-sequences and, sometimes, c_0-sequences as defined below.

DEFINITION 2.3.6. Let X, Y be Banach spaces, $(x_n) \in X^{\mathbb{N}}$, $(y_n) \in Y^{\mathbb{N}}$ be two sequences. The sequences (x_n), (y_n) are said to be *equivalent* if there exist constants $C > c > 0$ such that

$$c\left\|\sum_{k=1}^n a_k x_k\right\| \leqslant \left\|\sum_{k=1}^n a_k y_k\right\| \leqslant C\left\|\sum_{k=1}^n a_k x_k\right\|$$

for all $n \in \mathbb{N}$ and every collection of scalars $a_1, \dots, a_n$.

In particular, $(x_n) \subset X$ is equivalent to the canonical basis of ℓ_1 if there are $C > c > 0$ such that

$$c\sum_{k=1}^n |a_k| \leqslant \left\|\sum_{k=1}^n a_k x_k\right\| \leqslant C\sum_{k=1}^n |a_k| \tag{2.3.1}$$

for all $n \in \mathbb{N}$ and every collection of scalars $a_1, \dots, a_n$.

A sequence which is equivalent to the canonical basis of ℓ_1 is called *an ℓ_1-sequence* for short. If (2.3.1) holds true with $C = 1$ and $c \leqslant 1$, the sequence is said to be c-equivalent to the canonical basis of ℓ_1.

Analogously, a *c_0-sequence* is a sequence $(x_n) \subset X$ which is equivalent to the canonical basis of c_0. In other words, $(x_n) \subset X$ is a c_0-sequence if there are $C > c > 0$ such that for all $n \in \mathbb{N}$ and every collection of scalars $a_1, \dots, a_n$

$$c \max_{1 \leqslant k \leqslant n} |a_k| \leqslant \left\| \sum_{k=1}^{n} a_k x_k \right\| \leqslant C \max_{1 \leqslant k \leqslant n} |a_k|. \tag{2.3.2}$$

Every disjoint sequence of norm-one elements of $C(K)$ gives an example of a c_0-sequence, and disjoint sequences of norm-one elements of L_1 are ℓ_1-sequences.

For a bounded sequence $(x_n) \subset X$ the right-hand inequality in (2.3.1) follows automatically with $C = \sup_n \|x_n\|$. This simplifies the search for ℓ_1-sequences. A good example of this effect is the following simple lemma.

LEMMA 2.3.7. *Let E, X be Banach spaces, $G \in L(E, X)$ and $(x_n) \subset X$ be an ℓ_1-sequence. Assume that the sequence $(e_n) \subset E$ is bounded and consists of preimages of the corresponding x_n's: $Ge_n = x_n$ for each $n \in \mathbb{N}$. Then (e_n) is an ℓ_1-sequence.*

PROOF. Let $c, C > 0$ be the constants from (2.3.1). For $n \in \mathbb{N}$ and every collection of scalars $a_1, \dots, a_n$, we have

$$\left\| \sum_{k=1}^{n} a_k x_k \right\| = \left\| G \left(\sum_{k=1}^{n} a_k e_k \right) \right\| \leqslant \|G\| \left\| \sum_{k=1}^{n} a_k e_k \right\|.$$

Consequently,

$$\left\| \sum_{k=1}^{n} a_k e_k \right\| \geqslant \frac{1}{\|G\|} \left\| \sum_{k=1}^{n} a_k x_k \right\| \geqslant \frac{c}{\|G\|} \sum_{k=1}^{n} |a_k|,$$

and the job is done (as the other inequality holds automatically by the boundedness of the sequence (e_n)). □

The following corollary of Lemma 2.3.7 is called *the lifting property of ℓ_1*.

THEOREM 2.3.8. *Let X be a quotient space of a Banach space E. If X contains an isomorphic copy of ℓ_1, then E also contains an isomorphic copy of ℓ_1.*

PROOF. The words "contains an isomorphic copy of ℓ_1" mean the same as "contains an ℓ_1-sequence". So, let $X = E/Y$, $q\colon E \to X$ be the corresponding quotient map, $(x_n) \subset X$ be an ℓ_1-sequence. Then (x_n) is bounded. For each n we may select $e_n \in q^{-1}(x_n)$ in such a way that $\|e_n\| < \|x_n\| + 1$. Then $(e_n) \subset X$ is also bounded and it remains to apply Lemma 2.3.7 with $G = q$. □

A possible proof of Mazur's basic sequence selection principle mentioned above uses a simple lemma on ε-orthogonal subspaces (see [**155**, Section 6.3]). Since we are going to use that lemma in a somewhat modified form shortly, we present it below with a detailed proof.

DEFINITION 2.3.9. Let Y, Z be subspaces of a Banach space X and let ε be a positive number. Z is said to be ε-orthogonal to Y if

$$\|y + z\| \geqslant (1 - \varepsilon)\|y\| \tag{2.3.3}$$

for all $y \in Y$, $z \in Z$.

LEMMA 2.3.10. *Let X be a Banach space $F \subset X^*$ be a one-norming subspace, $Y \subset X$ be a finite-dimensional subspace. Then, for every $\varepsilon > 0$, there is a finite collection of $f_k \in S_F$, $k \in \overline{1,n} := \{1, \dots, n\}$, such that the subspace*

$$Z := \bigcap_{k \in \overline{1,n}} \ker f_k$$

is ε-orthogonal to Y.

PROOF. Using the compactness of the unit sphere of the finite-dimensional subspace Y we may select an $\varepsilon/2$-net $\{y_k \colon k \in \overline{1,n}\} \subset S_Y$. After that, for each y_k we choose $f_k \in S_F$ in such a way that $|f_k(y_k)| > 1 - \varepsilon/2$. Let us demonstrate that the collection $\{f_k \colon k \in \overline{1,n}\} \subset S_F$ thus obtained is what we need. Indeed, let $y \in Y$, $z \in Z$. For $y = 0$ the validity of (2.3.3) is evident, so we consider the case of $y \neq 0$. In this case we may find some $k \in \overline{1,n}$ such that $\left\| \frac{y}{\|y\|} - y_k \right\| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
\|y + z\| &\geqslant f_k(y + z) = f_k(y) \\
&= \|y\| \left(f_k \left(\frac{y}{\|y\|} - y_k \right) + f_k(y_k) \right) \\
&\geqslant \|y\| \left(|f_k(y_k)| - \left| f_k \left(\frac{y}{\|y\|} - y_k \right) \right| \right) \\
&\geqslant \|y\| \, (1 - \varepsilon/2 - \varepsilon/2) = (1 - \varepsilon)\|y\|.
\end{aligned}$$

□

2.4. Unconditional convergence of series and unconditional bases

A series $\sum_{n=1}^\infty x_n$ in a Banach space X is said to be *unconditionally convergent* if it converges under every rearrangement of terms. The unconditional convergence of a series is equivalent to the convergence of all series $\sum_{n=1}^\infty \theta_n x_n$ with $\theta_n = \pm 1$. According to Gelfand's theorem, the set $\{\sum_{n=1}^\infty \theta_n x_n \colon \theta_n = \pm 1\}$ is compact and, in particular, is bounded. All these well-known facts can be found, for example, in [**155**, Chapter 1].

In finite-dimensional spaces unconditional convergence of $\sum_{n=1}^\infty x_n$ is equivalent to absolute convergence, i.e., to the condition $\sum_{n=1}^\infty \|x_n\| < \infty$. Due to the famous Dvoretzky-Rogers theorem (see [**155**, Chapter 4, §1]), this characterisation does not extend to any infinite-dimensional Banach space.

Thanks to the equality

$$\max_{(a_k) \subset [-1,1]} \left\| \sum_{k=1}^n a_k x_k \right\| = \max_{\theta_k = \pm 1} \left\| \sum_{k=1}^n \theta_k x_k \right\|$$

(a convex continuous function on a compact convex set attains its maximum at an extreme point), the unconditional convergence of a series $\sum_{n=1}^\infty x_n$ implies the convergence of $\sum_{n=1}^\infty a_n x_n$ for every sequence of real coefficients $a_n \in [-1, 1]$, and

$$\sup_{(a_n) \subset [-1,1]} \left\| \sum_{n=1}^\infty a_n x_n \right\| = \sup_{\theta_n = \pm 1} \left\| \sum_{n=1}^\infty \theta_n x_n \right\| < \infty. \tag{2.4.1}$$

Observe that (2.4.1) implies the following result in which FIN($\mathbb{N}$) denotes the family of finite subsets of $\mathbb{N}$:

PROPOSITION 2.4.1. *Let $\sum_{n=1}^{\infty} x_n$ be an unconditionally convergent series in a Banach space X. Then*

$$\sup \left\{ \left\| \sum_{n\in A} x_n \right\| : A \in \mathrm{FIN}(\mathbb{N}) \right\} < \infty. \tag{2.4.2}$$

DEFINITION 2.4.2. A series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is said to be *weakly unconditionally Cauchy* if for every $x^* \in X^*$

$$\sum_{n=1}^{\infty} |x^*(x_n)| < \infty. \tag{2.4.3}$$

An occasionally used synonym for "weakly unconditionally Cauchy series" is "weakly absolutely convergent series". However, the first name stresses the fact that such a series need not converge.

Since for an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ the corresponding numerical series $\sum_{n=1}^{\infty} x^*(x_n)$ also converges for all rearrangements of terms and hence is absolutely convergent, the unconditional convergence implies the weak absolute one. The converse implication needs an additional condition.

THEOREM 2.4.3 (Bessaga-Pełczyński Theorem, see [**155**, Theorem 6.4.3]). *For a Banach space X, the following assertions are equivalent:*

(i) *every weakly unconditionally Cauchy series in X is unconditionally convergent;*
(ii) *X does not have subspaces isomorphic to c_0.*

Remark that the class of spaces without c_0-subspaces includes in particular all reflexive spaces, all weakly sequentially complete ones (like ℓ_1, L_1 or $C(K)^*$) and all spaces with non-trivial cotype. The spaces $C(K)$ for infinite compact K contain isomorphic copies of c_0.

For the proof of Theorem 8.3.5 below we will need the following result.

PROPOSITION 2.4.4. *Let X be a Banach space, $Y \subset X^*$ be a one-norming subspace and $\sum_{n=1}^{\infty} x_n$ be a series in X. Then the following assertions are equivalent:*

(1) *The series $\sum_{n=1}^{\infty} x_n$ is weakly unconditionally Cauchy.*
(2) *For every $x^* \in Y$ the condition (2.4.3) holds true.*
(3) *There is a constant $C > 0$ such that for every $x^* \in Y$*

$$\sum_{n=1}^{\infty} |x^*(x_n)| \leqslant C\|x^*\|$$

(4) $\sup\limits_{n\in\mathbb{N}} \max\limits_{\theta_k\in\mathbb{T}} \left\| \sum\limits_{k=1}^{n} \theta_k x_k \right\| < \infty.$

PROOF. The implication (1) $\Rightarrow$ (2) is evident. Let us demonstrate (2) $\Rightarrow$ (3). The proof will be a minor modification of that from [**155**, Lemma 6.4.1]. Denote by $(e_n)_{n=1}^{\infty}$ the canonical basis of ℓ_1. Consider the operators $T_n \in L(Y, \ell_1)$ that act by the rule $T_n x^* = \sum_{k=1}^n x^*(x_k)e_k$. This set of operators is pointwise bounded on Y:

$$\sup_{n\in\mathbb{N}} \|T_n x^*\| = \sup_{n\in\mathbb{N}} \sum_{k=1}^n |x^*(x_k)| = \sum_{k=1}^\infty |x^*(x_k)| < \infty.$$

Consequently, by the Uniform Boundedness Principle, $C := \sup_{n\in\mathbb{N}} \|T_n\| < \infty$, and

$$\sum_{n=1}^\infty |x^*(x_n)| = \sup_{n\in\mathbb{N}} \|T_n x^*\| \leqslant C\|x^*\|.$$

Before demonstrating the implication (3) ⇒ (4) let us remark that

$$\begin{aligned}\sup_{x^*\in S_Y} \sum_{k=1}^n |x^*(x_k)| &= \sup_{x^*\in S_Y} \max_{\theta_k\in\mathbb{T}} \left|\sum_{k=1}^n \theta_k x^*(x_k)\right| \\ &= \max_{\theta_k\in\mathbb{T}} \sup_{x^*\in S_Y} \left|x^*\left(\sum_{k=1}^n \theta_k x_k\right)\right| = \max_{\theta_k\in\mathbb{T}} \left\|\sum_{k=1}^n \theta_k x_k\right\|. \qquad (2.4.4)\end{aligned}$$

Now, it remains to estimate

$$\sup_{n\in\mathbb{N}} \max_{\theta_k\in\mathbb{T}} \left\|\sum_{k=1}^n \theta_k x_k\right\| = \sup_{n\in\mathbb{N}} \sup_{x^*\in S_Y} \sum_{k=1}^n |x^*(x_k)| \leqslant C < \infty.$$

The remaining implication (4) ⇒ (1) follows from (2.4.4). Indeed, (2.4.4) has been shown to hold for every one-norming subspace $Y \subset X^*$, so it remains valid for $Y = X^*$. Then, for every $x^* \in S_{X^*}$ we have

$$\sum_{n=1}^\infty |x^*(x_n)| = \sup_{n\in\mathbb{N}} \sum_{k=1}^n |x^*(x_k)| \leqslant \sup_{n\in\mathbb{N}} \max_{\theta_k\in\mathbb{T}} \left\|\sum_{k=1}^n \theta_k x_k\right\| < \infty. \qquad \square$$

DEFINITION 2.4.5. A sequence $(e_n)_{n=1}^\infty$ in a Banach space X is called an *unconditional basis* if it forms a basis of X and for every $x \in X$ its expansion $x = \sum_{n=1}^\infty a_n e_n$ converges unconditionally. A basis which is not unconditional is called *conditional*.

The canonical bases of the sequence spaces ℓ_p, $1 \leqslant p < \infty$, or c_0 are unconditional. The Haar system in $L_p[0,1]$ for $1 < p < \infty$ is unconditional, but the same system (written in the natural order) is a conditional basis in $L_1[0,1]$. Below in Corollary 5.3.2 we demonstrate that spaces with the Daugavet property (in particular $C[0,1]$ and $L_1[0,1]$) do not possess any unconditional basis.

The c_0- and ℓ_1-subspaces of spaces with an unconditional basis determine many properties of the space and of the basis. In particular, according to the classical James theorem [**206**, Theorem 1.c.12(a)], a Banach space with an unconditional basis is reflexive if and only if X does not contain isomorphic copies of neither c_0 nor ℓ_1. More results in this vein are formulated in the following two theorems that in a more extended form are contained in [**206**, Sections 1.b and 1.c].

THEOREM 2.4.6. *Let X be a Banach space with an unconditional basis (e_n). Then the following assertions are equivalent:*

(1) *X does not contain a copy of c_0.*
(2) *(e_n) is boundedly complete, that is, for every sequence (a_n) of scalars such that $\sup_{n\in\mathbb{N}} \|\sum_{k=1}^n a_k e_k\| < \infty$, the series $\sum_{k=1}^\infty a_k e_k$ converges.*
(3) *X is isomorphic to a dual space.*

(4) *X is weakly sequentially complete.*

THEOREM 2.4.7. *Let X be a Banach space with an unconditional basis (e_n). Then the following assertions are equivalent:*

(1) *X does not contain a copy of ℓ_1.*
(2) *(e_n) is shrinking, that is, the coordinate functionals form a basis in X^*.*
(3) *X^* is separable.*

2.5. Spaces without ℓ_1-subspaces. The Schur property

Detailed expositions of the results listed in this section can be found in [**97**] and [**292**].

The space ℓ_1 possesses a number of special properties, which makes a difference compared to other classical spaces. It is separable but its dual is nonseparable, it is weakly sequentially complete and the weak convergence of sequences in ℓ_1 is the same as the strong one (Schur's lemma – actually a theorem). Consequently, none of the elements $x^{**} \in (\ell_1)^{**} \setminus \ell_1$ can be represented as a limit of $\sigma((\ell_1)^{**}, (\ell_1)^*)$-convergent sequences $(x_n) \subset \ell_1$. In this section we recall the statements of two deep and extremely important Banach space theory results which demonstrate that the ℓ_1-subspaces (i.e., isomorphic copies of ℓ_1 in the space X) are "responsible" for possible bad sequential properties of the topology $\sigma(X^{**}, X^*)$.

DEFINITION 2.5.1. A sequence (x_n) in X is *weakly Cauchy* if the limit $\lim_{n\to\infty} x^*(x_n)$ exists for every $x^* \in X^*$.

It is easy to see from the uniform boundedness principle that $(x_n) \in X^{\mathbb{N}}$ is a weak Cauchy sequence if and only if it is $\sigma(X^{**}, X^*)$-convergent in X^{**} to some $x^{**} \in X^{**}$.

THEOREM 2.5.2 (Rosenthal's Alternative or Rosenthal's ℓ_1 Theorem [**261**]). *Every bounded sequence in a Banach spaces contains a weak Cauchy subsequence or an ℓ_1-subsequence.*

The next result is a part of [**292**, Theorem 4.1]. In a little weaker form it can be found in the original paper [**239**] by Odell and Rosenthal.

THEOREM 2.5.3. *Let X be a separable Banach space that does not have subspaces isomorphic to ℓ_1, and let $A \subset X$ be bounded. Then every $\sigma(X^{**}, X^*)$-cluster point $x^{**} \in X^{**}$ of A is a $\sigma(X^{**}, X^*)$-limit of a sequence $(x_n) \subset A$. In other words, the $\sigma(X^{**}, X^*)$-closure of A in X^{**} is the same as its $\sigma(X^{**}, X^*)$-sequential closure.*

REMARK 2.5.4. *The Kaplansky Theorem 2.1.4 implies that, if in the above theorem $x^{**} \in X$, then the condition of separability of X can be waived.* Indeed, in this case there is a countable subset $\tilde{A} \subset A$ such that x^{**} belongs to the weak closure of $\tilde{A}$, and it remains to apply the above theorem to $\tilde{A}$ as a subset of the separable space $\overline{\text{lin}}\,\tilde{A}$.

In fact, we shall need a generalisation of this result and first provide a lemma.

LEMMA 2.5.5. *Let X be a separable Banach space without subspaces isomorphic to ℓ_1, and let $(x_{n,m})_{n,m\in\mathbb{N}} \subset X$ be a bounded double sequence. Let $x^{**} \in X^{**}$ be a $\sigma(X^{**}, X^*)$-limit point of every column $(x_{n,m})_{n\in\mathbb{N}}$ of $(x_{n,m})_{n,m\in\mathbb{N}}$. Then there are strictly increasing sequences $(n(k))$, $(m(k))$ of indices such that $x_{n(k),m(k)} \to x^{**}$*

in $\sigma(X^{**}, X^*)$. Moreover, if $x^{**} \in X$, then the condition of separability of X may be omitted.

PROOF. Consider an auxiliary space $Y = X \times \mathbb{R}$ and an auxiliary matrix $(y_{n,m})_{n,m\in\mathbb{N}} \subset Y$, $y_{n,m} = (x_{n,m}, 1/n + 1/m)$. Since Y contains no copies of ℓ_1 either and since $(x^{**}, 0)$ is a $\sigma(Y^{**}, Y^*)$-limit point of $(y_{n,m})_{n,m\in\mathbb{N}}$, there is, according to Theorem 2.5.3 (combined with Remark 2.5.4 in the case of $x^{**} \in X$ for the "moreover" part without the separability assumption), a sequence of the form $(y_{n(k),m(k)})_{k\in\mathbb{N}}$ which converges to $(x^{**}, 0)$ in $\sigma(Y^{**}, Y^*)$. This means, in particular, that $x_{n(k),m(k)} \to x^{**}$ in $\sigma(X^{**}, X^*)$ and $1/n + 1/m \to 0$. So $(n(k))$ and $(m(k))$ both tend to ∞ which, after passing to a subsequence, provides the desired sequence. □

The next result is a direct generalisation of Theorem 2.5.3. It is taken from [**179**, Theorem 4.3].

THEOREM 2.5.6. *Let X be a separable Banach space without ℓ_1-subspaces, $(\Gamma, \prec)$ be a directed set, and let $F\colon \Gamma \to X$ be a bounded function. Then for every $\sigma(X^{**}, X^*)$-limit point $x^{**} \in X^{**}$ of the function F, there is a strictly increasing sequence $\gamma(1) \prec \gamma(2) \prec \dots$ in Γ such that $\big(F(\gamma(n))\big)$ converges to x^{**} in $\sigma(X^{**}, X^*)$. Moreover, if $x^{**} \in X$, then the condition of separability of X may be omitted.*

PROOF. Using inductively Theorem 2.5.3 (combined with Remark 2.5.4 for the "moreover" part) to the sets of the form

$$\{F(\gamma)\colon \gamma \succ \gamma_j\}$$

we can select column-by-column a doubly indexed sequence $(\gamma_{n,m})_{n,m\in\mathbb{N}}$ in Γ with the following properties:

(1) for every $m \in \mathbb{N}$, $x^{**} \in X^{**}$ is a $\sigma(X^{**}, X^*)$-limit point of every column $(F(\gamma_{n,m}))_{n\in\mathbb{N}}$;
(2) for every $m, n, k, l \in \mathbb{N}$, if $\max\{k, l\} < m$, then $\gamma_{k,l} \prec \gamma_{n,m}$.

Applying Lemma 2.5.5 and passing to a subsequence if necessary, we obtain strictly increasing sequences $(n(k))$, $(m(k))$ such that $\max_{k<j}\{n(k), m(k)\} < m(j)$ and $\big(F(\gamma_{n(k),m(k)}\big)$ converges to x^{**} in $\sigma(X^{**}, X^*)$. To finish the proof, put $\gamma(k) = \gamma_{n(k),m(k)}$. □

The above theorem is going to find its application in Section 6.3. There are other applications which are not related to the main subject of the book. As an example let us prove the following selection theorem which was earlier established by E. Behrends [**49**] under the more restrictive condition of separability of X^*.

THEOREM 2.5.7 ([**179**, Theorem 4.14]). *Let X be a Banach space without ℓ_1-subspaces and $A_n \subset X$ be bounded subsets with $0 \in \overline{\mathrm{conv}}(A_n)$ for each $n \in \mathbb{N}$. Then there exists a sequence (a_n) in X with $a_n \in A_n$ for every n such that $0 \in \overline{\mathrm{conv}}\big(\{a_1, a_2, \dots\}\big)$.*

PROOF. In each A_n there is a separable subset whose closed convex hull contains 0. So, passing to the linear span of these separable subsets we may assume that X is separable. Introduce a directed set $(\Gamma, \prec)$ as follows: the elements of Γ are of the form

$$\gamma = \big(n, m, (a_k)_{k=n}^m, (\lambda_k)_{k=n}^m\big),$$

where $n, m \in \mathbb{N}$, $n < m$, $a_k \in A_k$, $\lambda_k > 0$, $\sum_{k=n}^{m} \lambda_k = 1$. Define $\prec$ as follows: let $\gamma_1 = (n_1, m_1, (a_k)_{k=n_1}^{m_1}, (\lambda_k)_{k=n_1}^{m_1})$, $\gamma_2 = (n_2, m_2, (b_k)_{k=n_2}^{m_2}, (\mu_k)_{k=n_2}^{m_2})$; then $\gamma_1 \prec \gamma_2$ if $m_1 < n_2$ or $\gamma_1 = \gamma_2$. Define $F\colon \Gamma \to X$ by the formula $F(\gamma) = \sum_{k=n}^{m} \lambda_k a_k$. Now, 0 is a weak limit point of F; see the proof of [**49**, Th. 4.3]. So, by Theorem 2.5.6 there is a sequence of elements

$$\gamma_j = (n_j, m_j, (a_k)_{k=n_j}^{m_j}, (\lambda_k)_{k=n_j}^{m_j})$$

such that $n_1 < m_1 < n_2 < m_2 < n_3 < \ldots$ and $\sum_{k=n_j}^{m_j} \lambda_k a_k$ tends weakly to zero. To finish the proof one just needs to apply Mazur's Theorem 2.1.2. $\square$

Definition 2.5.8. A Banach space E possesses the *Schur property* if every weakly convergent sequence in E converges strongly.

The above definition is motivated by the Schur theorem: in ℓ_1 the weak and the strong convergence of sequences are the same. Clearly, apart from ℓ_1, there are other spaces with the Schur property, for example all finite-dimensional spaces and all subspaces of ℓ_1. On the other hand, Rosenthal's alternative implies that each infinite-dimensional Banach space E with the Schur property is ℓ_1*-saturated* in the following sense: every infinite-dimensional subspace $Z \subset E$ in its turn contains a subspace isomorphic to ℓ_1. In particular, a space with the Schur property has no infinite-dimensional reflexive subspaces.

Another very commonly used property of ℓ_1 is its quotient universality (see the proof of [**108**, Theorem 2.3.1] for this version).

Theorem 2.5.9. *Let X be a separable Banach space. Then, there is an operator $T\colon \ell_1 \to X$ that maps the open unit ball of ℓ_1 onto the open unit ball of X:*

$$T\Big(\overset{\circ}{B}_{\ell_1}\Big) = \overset{\circ}{B}_X.$$

In particular, X is isometrically isomorphic to a quotient of ℓ_1.

2.6. Quasi-codirected vectors, extreme points, and slices

For vectors $x, y \in X$ in a normed space one has the following evident statement: if x and y are codirected (meaning that $y = \lambda x$ or $x = \lambda y$ with $\lambda \geqslant 0$), then

$$\|x + y\| = \|x\| + \|y\|. \tag{2.6.1}$$

The converse statement does not hold true in general, which motivates the following definition.

Definition 2.6.1. The elements $x, y \in X$ are said to be *quasi-codirected* if they satisfy (2.6.1).

Thanks to the triangle inequality, the equality (2.6.1) is equivalent to the inequality $\|x + y\| \geqslant \|x\| + \|y\|$.

Geometrically, for $x, y \in S_X$ the property of being quasi-codirected means that the (straight line) segment that connects x and y lies in the unit sphere. (Recall that a *segment* in a Banach space is the convex hull of two (distinct) points; for $x, y \in X$, we write $[x, y] := \{tx + (1 - t)y\colon t \in [0, 1]\}$, which is the segment that connects x and y.)

REMARK 2.6.2. *For $a, b \geqslant 0$ and any quasi-codirected $x, y \in X$, the vectors ax, by are quasi-codirected as well. Indeed, without loss of generality we may assume $a \geqslant b$. Then*

$$\begin{aligned}\|ax + by\| &= \|a(x+y) - (a-b)y\| \\ &\geqslant a\|(x+y)\| - (a-b)\|y\| = a\|x\| + b\|y\|\end{aligned}$$

which shows that

$$\|ax + by\| = a\|x\| + b\|y\|.$$

In the sequel we sometimes need the following "small perturbation" of the above concept.

DEFINITION 2.6.3. Let X be a normed space, $\varepsilon > 0$ and $x, y \in X$. The elements x, y are said to be *ε-quasi-codirected* if $\|x + y\| > \|x\| + \|y\| - \varepsilon$.

LEMMA 2.6.4. *Let $x, y \in X$ be ε-quasi-codirected. Then for every $a, b > 0$ the elements ax, by are $(\varepsilon \max\{a, b\})$-quasi-codirected.*

PROOF. Without loss of generality we may assume $a \geqslant b$. Then $a = \max\{a, b\}$ and

$$\begin{aligned}\|ax + by\| &= \|a(x+y) - (a-b)y\| \geqslant a\|x+y\| - (a-b)\|y\| \\ &> a(\|x\| + \|y\| - \varepsilon) - (a-b)\|y\| = a\|x\| + b\|y\| - a\varepsilon.\end{aligned}$$ □

Let A be a bounded subset of a topological vector space X (usually we consider a Banach space X). A *slice* of A is a non-empty part S of A that is cut out by a closed real hyperplane (see Figure 2.2).

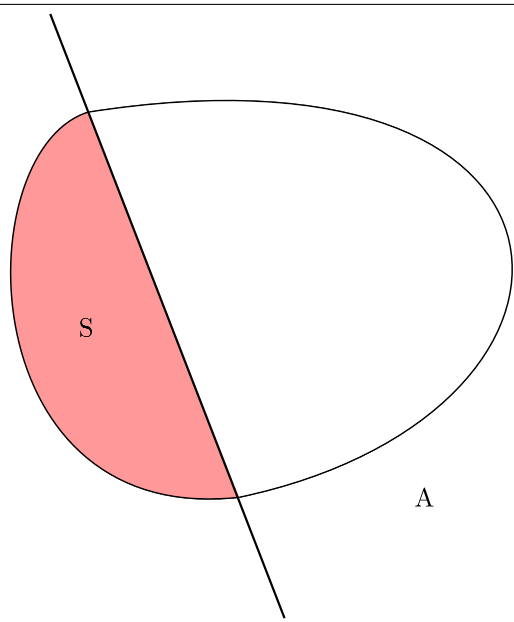

FIGURE 2.2. A picture of a slice

Given $x^* \in X^*$ and $\varepsilon > 0$, denote the corresponding slice as

$$\mathrm{Slice}(A, x^*, \varepsilon) := \{x \in A\colon \operatorname{Re} x^*(x) > \sup \operatorname{Re} x^*(A) - \varepsilon\}.$$

If $A \subset X^*$ and the functional defining the slice is taken in the predual, i.e., this functional is some $z \in X \subset X^{**}$, then $\mathrm{Slice}(A, z, \alpha)$ is called a *weak-star slice* (or *w^*-slice*) of A.

Taking into account that $\sup \operatorname{Re} x^*(B_X) = \|x^*\|$, in the case of $A = B_X$ and $x^* \in S_{X^*}$ the definition of slice simplifies to

$$\mathrm{Slice}(B_X, x^*, \varepsilon) = \{x \in B_X\colon \operatorname{Re} x^*(x) > 1 - \varepsilon\}.$$

Remark that a slice of A is a non-empty relatively weakly open subset of A, and that every **non-empty** set of the form $\{x \in A\colon \operatorname{Re} x^*(x) > \alpha\}$ is a slice of A.

As a rule, we consider slices of convex sets, most often slices of the unit ball of a Banach space, but sometimes we need slices of nonconvex sets like the unit sphere or the set of extreme points of a convex compact set. Observe that if A is convex, given a slice S of A, $A \setminus S$ is also convex.

PROPOSITION 2.6.5 (Slice preimage remark). *Let X, Y be normed spaces, $T \in L(X, Y)$, $A \subset X$. Then for every slice S of $\overline{T(A)}$ the set $T^{-1}(S) \cap A$ is a slice of A. The same is true for slices of $T(A)$.*

PROOF. Let $y^* \in Y^*$ and $\alpha > 0$ be such that

$$S = \{y \in T(A)\colon \operatorname{Re} y^*(y) > \alpha\}.$$

Consider $x^* := T^* y^*$ and denote by $\tilde{S}$ the following slice of A:

$$\tilde{S} = \{x \in A\colon \operatorname{Re} x^*(x) > \alpha\}.$$

First, $T^{-1}(S) \cap A \neq \emptyset$ because S, being a relatively open subset of $\overline{T(A)}$, intersects $T(A)$. Next, for $x \in X$ we have the following equivalences:

$$\begin{aligned}(x \in T^{-1}(S) \cap A) &\iff ((x \in A) \wedge (Tx \in S)) \iff ((x \in A) \wedge (\operatorname{Re} y^*(Tx) > \alpha)) \\ &\iff ((x \in A) \wedge (\operatorname{Re} x^*(x) > \alpha)) \iff (x \in \tilde{S}),\end{aligned}$$

which proves that $T^{-1}(S) \cap A = \tilde{S}$ and so it is a slice of A. The same argument works for slices of $T(A)$. □

Remark that *the image of a slice does not necessarily form a slice of the image.* This becomes clear from the following example in $\mathbb{R}^2$ (see Figure 2.3). Let

$$A = \big\{(x, y)\colon 0 \leqslant x \leqslant 1,\ 1 - x \leqslant y \leqslant x - 1\big\},$$

x^* be the first coordinate functional (i.e., $x^*((a, b)) := a$), and $T \in L(\mathbb{R}^2)$ be the projection onto the vertical axis: $T((a, b)) := (0, b)$. Then, for the slice

$$S := \mathrm{Slice}(A, x^*, 1/2) = \left\{x \in A\colon \operatorname{Re} x^*(x) > \frac{1}{2}\right\},$$

we have $T(S) = \{(0, b)\colon -1/2 < b < 1/2\}$, which is not a slice in $T(A) = \{(0, b)\colon -1 \leqslant b \leqslant 1\}$.

One more easy observation that we will use repeatedly:

PROPOSITION 2.6.6. *Let X be a locally convex topological vector space and $A \subset X$ be a bounded subset. Then every slice of $\overline{\mathrm{conv}}(A)$ intersects A.*

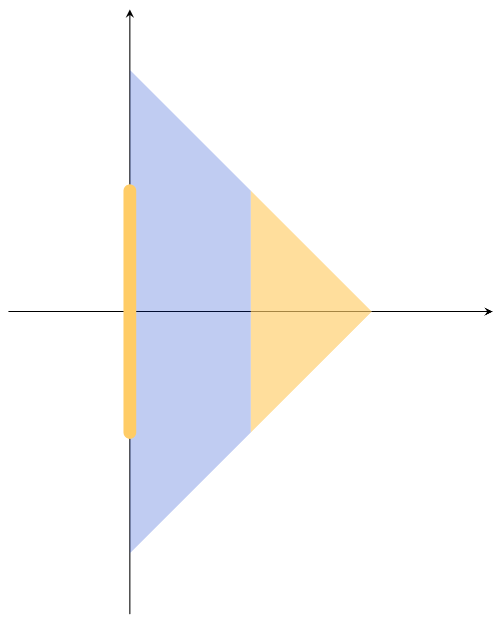

FIGURE 2.3. An image of a slice that is not a slice

PROOF. Let $S = \mathrm{Slice}(\overline{\mathrm{conv}}(A), x^*, \varepsilon)$ be an arbitrary slice. If $S \cap A$ is empty, then $A \subset (\overline{\mathrm{conv}}(A)) \setminus S$, which is a closed convex set and, therefore, $\overline{\mathrm{conv}}(A) \subset (\overline{\mathrm{conv}}(A)) \setminus S$. This means that $(\overline{\mathrm{conv}}(A)) \cap S = \emptyset$. This contradiction completes the proof. $\square$

The following useful observation is a consequence of the Hahn-Banach separation theorem.

LEMMA 2.6.7. *Let X be a locally convex topological vector space, A, B be bounded subsets of X, and suppose that B intersects all the slices of A. Then $\overline{\mathrm{conv}}(B) \supset A$. In particular, if $A \subset X$ is bounded, closed and convex, then for $B \subset A$ the following assertions are equivalent:*

(i) *B intersects all slices of A;*
(ii) *$\overline{\mathrm{conv}}(B) = A$.*

PROOF. Assume to the contrary the existence of $x_0 \in A \setminus \overline{\mathrm{conv}}(B)$. By the Hahn-Banach separation theorem there are a continuous linear functional x^* and $\alpha \in \mathbb{R}$ such that $\mathrm{Re}\, x^*(b) \leqslant \alpha$ for $b \in \overline{\mathrm{conv}}(B)$ and $\mathrm{Re}\, x^*(x_0) > \alpha$. Then $S = \{x \in A\colon \mathrm{Re}\, x^*(x) > \alpha\}$ is a slice of A and $S \cap B = \emptyset$. $\square$

There is a relationship between ε-quasi-codirectedness and slices containing the vectors in question.

LEMMA 2.6.8. *Let X be a Banach space and let $\varepsilon \in (0, 1)$.*

(a) *If $x, y \in S_X$ are ε-quasi-codirected, then there is $x^* \in S_{X^*}$ such that both x and y belong to the slice $\mathrm{Slice}(B_X, x^*, \varepsilon)$.*

(b) *If $x^* \in S_{X^*}$ and $x, y \in \mathrm{Slice}(B_X, x^*, \varepsilon)$, then x, y are 2ε-quasi-codirected.*

Proof. (a) According to Lemma 2.6.4, for every $t \in [0,1]$ we have

$$\|tx + (1-t)y\| > 1 - \varepsilon,$$

that is, the segment $[x, y] := \{tx + (1-t)y \colon t \in [0,1]\}$ does not intersect $(1-\varepsilon)B_X$. By the Hahn-Banach separation theorem, there are $x^* \in S_{X^*}$ and $\alpha \in \mathbb{R}$ such that $\sup \mathrm{Re}\, x^*((1-\varepsilon)B_X) \leqslant \alpha$ and $\inf \mathrm{Re}\, x^*([x,y]) > \alpha$. From the first condition we deduce that $\alpha > 1-\varepsilon$, and the second condition implies that $x, y \in \mathrm{Slice}(B_X, x^*, \varepsilon)$.

(b) Conversely, for $x, y \in \mathrm{Slice}(B_X, x^*, \varepsilon)$ we have

$$\|x + y\| \geqslant \mathrm{Re}\, x^*(x+y) > 2 - 2\varepsilon \geqslant \|x\| + \|y\| - 2\varepsilon. \qquad \square$$

Remark 2.6.9. Let X be a Banach space and $Y \subset X^*$ be a one-norming subspace. Then, according to Proposition 2.1.1, B_X is $\sigma(X,Y)$-closed. Consequently, in the proof of the statement (a) of the above lemma we may separate the $\sigma(X,Y)$-compact segment $[x, y]$ from $(1-\varepsilon)B_X$ by $x^* \in S_Y$. Consequently, *$x^* \in S_{X^*}$ in the statement (a) of Lemma 2.6.8 can be selected from any given one-norming subspace of X^*.*

Some kind of stability of ε-quasi-codirectedness is highlighted by the next proposition.

Proposition 2.6.10. *Let X be a Banach space, $Y \subset X^*$ be a one-norming subspace, $x, y \in X$, $\alpha > 0$ be such that $\|x + y\| > \alpha$. Then, for every $\sigma(X,Y)$-neighbourhood U of x there is another $\sigma(X,Y)$-neighbourhood $V \subset U$ of x such that the inequality $\|z + y\| > \alpha$ is true for every $z \in V$. In particular, for $x, y \in S_X$, $\alpha = 2 - \varepsilon$, intersecting weak neighbourhoods with the unit ball, we obtain that for every $\tilde{\varepsilon} < \varepsilon$, and every pair of $\tilde{\varepsilon}$-quasi-codirected vectors $x, y \in S_X$ and every relative $\sigma(X,Y)$-neighbourhood U of x in B_X, there is another relative $\sigma(X,Y)$-neighbourhood $V \subset U$ of x in B_X that contains only vectors that are ε-quasi-codirected with y.*

Proof. The set $W := \{z \in X \colon \|z + y\| \leqslant 2 - \varepsilon\}$ is equal to the ball of radius α centred in $-y$. By Proposition 2.1.1, W is $\sigma(X,Y)$-closed. So, we can just take $V := U \setminus W$. $\square$

The concept of quasi-codirectedness from Definition 2.6.1 easily extends to n-tuples of vectors.

Definition 2.6.11. The elements $x_1, \ldots, x_n$ of a normed space are said to form a *quasi-codirected n-tuple* if

$$\|x_1 + \cdots + x_n\| = \|x_1\| + \cdots + \|x_n\|.$$

In the sequel we shall simply say "quasi-codirected vectors" for "quasi-codirected n-tuple".

We will need a simple lemma.

Lemma 2.6.12. *Suppose that $x_1, \ldots, x_n$ are quasi-codirected. Then*

(a) *$\|a_1 x_1 + \cdots + a_n x_n\| = a_1\|x_1\| + \cdots + a_n\|x_n\|$ for all nonnegative coefficients a_k.*

(b) *If x_{n+1} is quasi-codirected to $x_1 + \cdots + x_n$, then all the vectors $x_1, \ldots, x_{n+1}$ are quasi-codirected.*

PROOF. (a) The function $F\colon \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$F(a_1, \dots, a_n) = \|a_1x_1 + \dots + a_nx_n\| - (a_1\|x_1\| + \dots + a_n\|x_n\|)$$

is convex, takes values $\leqslant 0$ and $F(1, \dots, 1) = 0$. Hence $F = 0$.

(b) follows from (a) and Remark 2.6.2:

$$\begin{aligned}\|x_1 + \dots + x_n + x_{n+1}\| &= \left\|n\frac{x_1 + \dots + x_n}{n} + x_{n+1}\right\| \\ &= n\left\|\frac{x_1 + \dots + x_n}{n}\right\| + \|x_{n+1}\| \\ &= \|x_1\| + \dots + \|x_n\| + \|x_{n+1}\|.\end{aligned}$$

□

LEMMA 2.6.13 (Diminishing of slices lemma, [**144**]). *Let X be a Banach space, $x^* \in S_{X^*}$, and $\varepsilon > 0$. Then, for every $x \in \mathrm{Slice}(B_X, x^*, \varepsilon) \cap S_X$ and every $\delta \in (0, \varepsilon)$, there is $y^* \in S_{X^*}$ such that*

$$x \in \mathrm{Slice}(B_X, y^*, \delta) \subset \mathrm{Slice}(B_X, x^*, \varepsilon).$$

PROOF. In order to simplify the notation, we deal with real-linear functionals. Without loss of generality, we may assume that $\varepsilon \in (0, 2]$. Indeed, if $\varepsilon > 2$, then the corresponding slice is the whole B_X, and there is nothing to do. Also notice that the case of $x^*(x) = 1$ is plain: we can simply take $y^* = x^*$. So, below we assume $-1 < x^*(x) < 1$. Fix a *supporting functional* f_x of x, that is, $\|f_x\| = 1$ and $f_x(x) = 1$. Remark that our additional assumptions imply that $f_x \neq \pm x^*$, so f_x and x^* are linearly independent. Let $\alpha_0 > 0$ be a root of the equation

$$\frac{1 + \alpha(1 - \varepsilon)}{\|f_x + \alpha x^*\|} = 1 - \delta. \tag{2.6.2}$$

Such a root exists because the left hand side of (2.6.2), call it $F(\alpha)$, is continuous in α, $F(0) = 1 > 1 - \delta$ and $\lim_{\alpha\to\infty} F(\alpha) = 1 - \varepsilon < 1 - \delta$. Put

$$y^* = \frac{f_x + \alpha_0 x^*}{\|f_x + \alpha_0 x^*\|}.$$

Then

$$y^*(x) = \frac{1 + \alpha_0 x^*(x)}{\|f_x + \alpha_0 x^*\|} > \frac{1 + \alpha_0(1 - \varepsilon)}{\|f_x + \alpha_0 x^*\|} = 1 - \delta,$$

i.e., $x \in \mathrm{Slice}(S_X, y^*, \delta)$. To prove the inclusion $\mathrm{Slice}(B_X, y^*, \delta) \subset \mathrm{Slice}(B_X, x^*, \varepsilon)$ take an arbitrary $y \in \mathrm{Slice}(B_X, y^*, \delta)$. Then

$$1 + \alpha_0 x^*(y) \geqslant f_x(y) + \alpha_0 x^*(y) > (1 - \delta)\|f_x + \alpha_0 x^*\| = 1 + \alpha_0(1 - \varepsilon).$$

So, $x^*(y) > 1 - \varepsilon$, which means that $y \in \mathrm{Slice}(B_X, x^*, \varepsilon)$. □

Let C be a convex subset of a vector space X. A point $x \in C$ is called *extreme* for C if it is not the midpoint of any non-trivial segment whose endpoints belong to C. We denote the set of all extreme points of C by $\mathrm{ext}(C)$. In detail: $x \in \mathrm{ext}(C)$ if and only if for every $x_1, x_2 \in C$, if $\frac{x_1 + x_2}{2} = x$ then $x_1 = x_2$ (and, consequently, x_1, x_2 are equal to x).

THEOREM 2.6.14 (Krein-Milman theorem).

(a) *Every convex compact subset K of a Hausdorff locally convex space E is equal to the closure of the convex hull of* $\mathrm{ext}(K)$. (See [**156**, Section 18.1.2, Theorem 3]). *Moreover, (see* [**156**, Section 18.1.2, Theorem 2]*), every real-valued continuous linear functional $f\colon E \to \mathbb{R}$ attains its maximum on K at some extreme point of K.*

(b) *In particular, for a Banach space X the unit ball of X^* is equal to the w^*-closure of* $\operatorname{conv}\big(\operatorname{ext}(B_{X^*})\big)$*, and for every $x \in X$*

$$\|x\| = \max\{\operatorname{Re} x^*(x) \colon x^* \in \operatorname{ext}(B_{X^*})\}.$$

In the particular case of finite-dimensional spaces, the Krein-Milman theorem has a nicer form, known as Carathéodory's theorem, see [**109**, Proposition 3.70], for instance.

THEOREM 2.6.15 (Carathéodory's theorem). *Let K be a compact convex subset of a finite-dimensional Banach space X with $\dim_{\mathbb{R}}(X) = n$. Then, every element of K can be written as a convex combination of at most $n+1$ extreme points of K.*

The following result [**84**, Proposition 25.13] will be used mostly for the weak-star topology of X^* or of X^{**}.

THEOREM 2.6.16 (Choquet lemma). *For every extreme point x_0 of a compact convex set K in a Hausdorff locally convex space E, the collection of those slices of K that contain x_0 forms a base of neighbourhoods of x_0 in K.*

Let us present an easy argument to get this nice and powerful result.

SKETCH OF THE PROOF. First, we observe that the compactness allows us to suppose that the topology of E restricted to K is just the weak topology $\sigma(E, E^*)$. Then, we only have to deal with neighbourhoods of x_0 of the form $V = S_1 \cap \cdots \cap S_m$ for suitable open slices $S_1, \ldots, S_m$ of K. Now,

$$x_0 \notin K \setminus V = \bigcup_{j=1}^{m} K \setminus S_j.$$

As x_0 is extreme, we actually have that

$$x_0 \notin \operatorname{conv}\left(\bigcup_{j=1}^{m} K \setminus S_j\right) = \overline{\operatorname{conv}}\left(\bigcup_{j=1}^{m} K \setminus S_j\right)$$

where the equality holds since all the sets $K \setminus S_j$ are convex and compact. Finally, a call to the Hahn-Banach separation theorem allows to separate the point x_0 from $\overline{\operatorname{conv}}(K \setminus V)$, that is, to produce a slice containing x_0 which is contained in V. $\square$

Figure 2.4 contains three pictures. The first one is a scheme of the geometric idea of the proof of the Choquet lemma. The second shows that the result is not valid when x_2 is not an extreme point of the compact convex set K_2, and the third picture shows that the result is not valid when K_3 is not compact, even though x_3 is an extreme point.

We will occasionally use some more properties of the set of extreme points.

LEMMA 2.6.17. *Let X be a Banach space and let $A \subset X^*$ be convex and weak-star compact.*

(a) *(Milman's Theorem) If $D \subset A$ satisfies that $\overline{\operatorname{conv}}^{w^*}(D) \supset A$, then $\overline{D}^{w^*} \supset$* $\operatorname{ext}(A)$*.*

(b) *The topological space* $(\operatorname{ext}(A), w^*)$ *is a Baire space, that is, the intersection of every sequence of G_δ dense subsets of* $\operatorname{ext}(A)$ *is again dense (and, of course, a G_δ), (see* [**84**, p. 146, Theorem 27.9]*).*

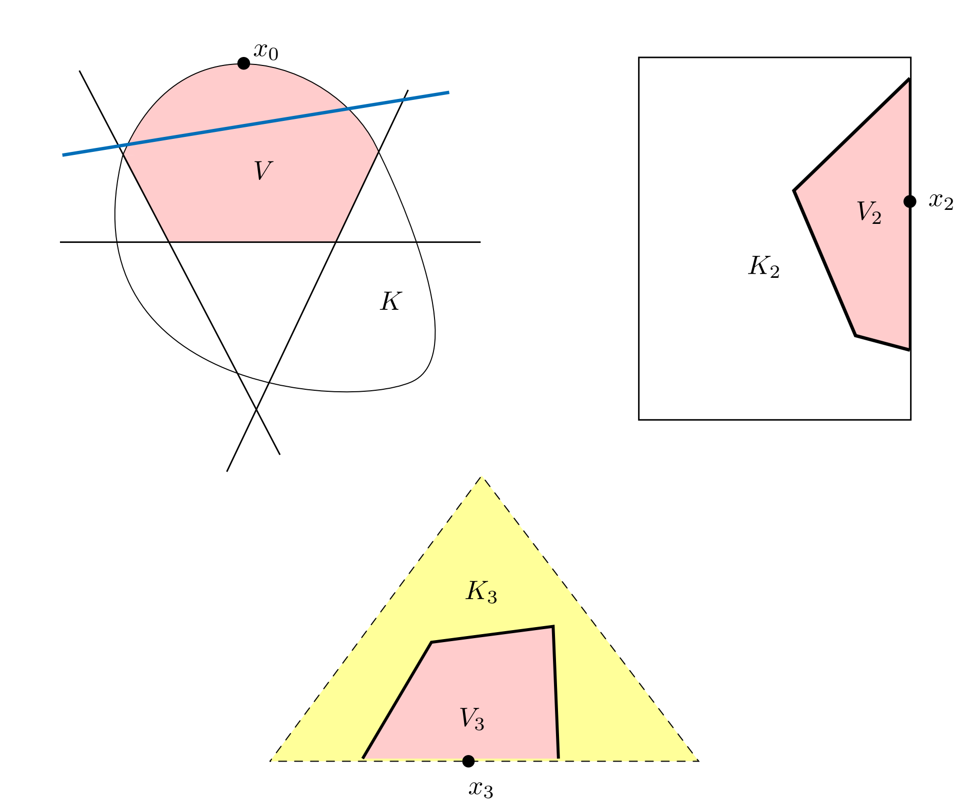

FIGURE 2.4. On the Choquet lemma

Nowadays, Milman's theorem is usually deduced from the Choquet lemma with the help of a Hahn-Banach separation argument.

A last result related to extreme points is the following sufficient condition for a *real* Banach space to contain a copy of c_0 and ℓ_1 which was obtained in the 1999 paper [**208**].

PROPOSITION 2.6.18 ([**208**, Proposition 2]). *Let X be a* real *Banach space, and assume that there is an infinite set $A \subset S_X$ such that $|x^*(a)| = 1$ for every $a \in A$ and $x^* \in \operatorname{ext}(B_{X^*})$. Then X contains (an isomorphic copy of) c_0 or ℓ_1.*

We include a short sketch of the proof which is slightly different from the original one from [**208**] and which appeared in the PhD dissertation [**217**].

PROOF. If X does not contain ℓ_1, Rosenthal's ℓ_1-theorem (Theorem 2.5.2) provides a sequence (a_n) of distinct elements of A which is weakly Cauchy. By the hypothesis and using that $\operatorname{ext}(B_{X^*})$ separates the points of X (it is actually one-norming!), it follows that $\|a_{n+1} - a_n\| = 2$ for every $n \in \mathbb{N}$. Now, the Bessaga-Pełczyński theorem (Theorem 2.4.3) gives us that X contains c_0 if the sequence $(a_{n+1} - a_n)_{n\in\mathbb{N}}$ is weakly unconditionally Cauchy, that is, if we show that

$$\sum_{n=1}^{\infty} \big|x^*\big(a_{n+1} - a_n\big)\big| < \infty \qquad (2.6.3)$$

for every $x^* \in X^*$. But a result by Elton [**106**] says that it is enough to check that (2.6.3) holds just for $x^* \in \mathrm{ext}(B_{X^*})$. This latter fact is immediate, as the sequence $\big(x^*(a_{n+1} - a_n)\big)_{n\in\mathbb{N}}$ is eventually null for every $x^* \in \mathrm{ext}(B_{X^*})$. □

We now would like to introduce and give some useful results on convex combinations of slices. As as the name suggests, a *convex combination of slices* of a set A is a subset of the form $\sum_{k=1}^m \lambda_k S_k$, where $\lambda_k > 0$, $\sum_{k=1}^m \lambda_k = 1$, and S_k are slices of A. The basic result here is *Bourgain's lemma* (see [**62**, Lemma 5.3] or [**292**, Lemma 7.3]):

LEMMA 2.6.19 (Bourgain's lemma). *Let A be a bounded closed convex subset of a locally convex space X. Then each relatively weakly open subset of A contains a convex combination of slices of A.*

REMARK 2.6.20. *The condition of closedness in the previous Lemma can be omitted.* Indeed, let $A \subset X$ be bounded and convex, and let $U \subset A$ be a relatively weakly open subset. Denote by V a relatively weakly open subset of $\overline{A}$ such that $V \cap A = U$. According to Bourgain's lemma, there are slices $S_1, S_2, \ldots, S_n$ of $\overline{A}$ and convex combination coefficients $\lambda_k > 0$ such that $\sum_1^n \lambda_k S_k \subset V$. Then, $S_k \cap A$ are slices of A, and $\sum_1^n \lambda_k (S_k \cap A) \subset V \cap A = U$.

One may wonder whether, given a relative weakly open subset W of a bounded closed convex subset A of a Banach space X and a point $x \in W$, Bourgain's lemma gives a convex combination of slices C of A contained in W *and containing* x. The answer is negative in general, but it is positive is x is a *preserved extreme point* of A (that is, $J_X(x)$ is an extreme point of $\overline{J_X(A)}^{w^*}$) or, even, if x is a convex combination of preserved extreme points of A, a result which can be found in [**224**, Lemma 2.2 and Remark 2.3], where Bourgain's lemma is proved just using Choquet's lemma and the Krein-Milman theorem on the convex and weak-star compact set $\overline{J_X(A)}^{w^*}$.

The next result is a small improvement of Bourgain's lemma above which we will use in Chapter 6.

LEMMA 2.6.21. *Let $U \subset B_X$ be a relatively weakly open subset, $x \in U$. Then for every $\varepsilon > 0$ there is a convex combination of slices of the unit sphere W_ε, $W_\varepsilon \subset U$, such that* $\mathrm{dist}(x, W_\varepsilon) < \varepsilon$.

PROOF. Without loss of generality, we may assume that U is convex: otherwise we substitute U by a smaller convex relatively weakly open neighbourhood of x. Denote by $\mathcal{W}$ the collection of all those $W \subset U$ that are convex combinations of slices of the unit sphere. Further, denote by $\mathbf{W}$ the union of all $W \in \mathcal{W}$. Since $\mathcal{W}$ is stable under convex combinations of its elements, the set $\mathbf{W}$ is convex. Bourgain's Lemma 2.6.19 implies that $\mathbf{W}$ is weakly dense in U, but, for a convex set, the closure in norm is the same as the weak closure, so $\mathbf{W}$ is dense in U in norm topology. Consequently, for every $\varepsilon > 0$ there is $y \in \mathbf{W}$ with $\|x - y\| < \varepsilon$. By the definition of $\mathbf{W}$, there is $W \in \mathcal{W}$ such that $y \in W$. Take this W as W_ε, and the job is done. □

Remark that in the statement above we speak about slices of the unit sphere because, at least formally, this version is stronger than that for slices of the ball: for a convex relatively weakly open neighbourhood U of x, apply the lemma in order to get the convex combination of slices W_ε, $W_\varepsilon \subset U$, of the sphere and then substitute the corresponding slices of the sphere by their convex hulls, which are slices of the ball.

2.7. Vector-valued integration and the Radon-Nikodým property

For a detailed exposition of the concepts and results listed below we refer to the books [**99**] and [**51**, Chapter 5].

Let $(\Omega, \Sigma, \lambda)$ be a measure space (with finite or σ-finite measure λ).

DEFINITION 2.7.1. Let X be a Banach space, $f\colon \Omega \to X$ be a function. The function f is said to be *measurable* if $f^{-1}(U) \in \Sigma$ for every Borel subset $U \subset X$. f is said to be *scalarly measurable* if the composition of f with every $x^* \in X^*$ is a measurable scalar function. A function $f\colon \Omega \to X$ is called *simple* if it is of the form $f = \sum_{n=1}^{m} x_n \mathbb{1}_{A_n}$ with $x_n \in X$, and $A_n \in \Sigma$.

In the scalar case, every measurable function can be approximated by simple functions. In the general situation one needs a separability condition.

DEFINITION 2.7.2. A function $f\colon \Omega \to X$ is said to be *strongly measurable* if there is a sequence of simple functions converging to f a.e. A function f is *almost separably valued* if there is a set $A \in \Sigma$ of zero measure for which $f(\Omega \setminus A)$ is separable.

THEOREM 2.7.3. *A function $f\colon \Omega \to X$ is strongly measurable if and only if it is scalarly measurable and almost separably valued.* $\square$

DEFINITION 2.7.4. Let $f\colon \Omega \to \mathbb{R}^+$ be a function. We call the value

$$\overline{\int_\Omega} f\, d\lambda = \inf\left\{\int_\Omega g\, d\lambda : g \geqslant f,\ g \in L_1(\Omega, \Sigma, \lambda)\right\}$$

upper Lebesgue integral of f. In particular, $\overline{\int_\Omega} f\, d\lambda = +\infty$ if f has no Lebesgue integrable majorant.

If the function is measurable and $\overline{\int_\Omega} f\, d\lambda < +\infty$, then f is Lebesgue integrable. If the function is Lebesgue integrable, its upper Lebesgue integral coincides with the usual Lebesgue integral.

Using the notion of the upper Lebesgue integral we introduce the *upper-L_1 space* $\overline{L}_1$.

DEFINITION 2.7.5. The space $\overline{L}_1(\Omega, \Sigma, \lambda, X)$ is defined as the space of all functions $f\colon \Omega \to X$ such that $\overline{\int_\Omega} \|f(t)\|\, d\lambda(t) < +\infty$. The norm on this space is given by $\|f\| = \overline{\int_\Omega} \|f(t)\|\, d\lambda(t)$.

Under the usual agreement that a.e. equal functions are the same element of $\overline{L}_1$, it is easy to see that $\overline{L}_1$ is a Banach space.

Denote by $L_1^s(\Omega, \Sigma, \lambda, X)$ the subspace of $\overline{L}_1(\Omega, \Sigma, \lambda, X)$ formed by simple functions and by $L_1(\Omega, \Sigma, \lambda, X)$ (or by $L_1(\lambda, X)$ for short) the closure of $L_1^s(\Omega, \Sigma, \lambda, X)$ in $\overline{L}_1(\Omega, \Sigma, \lambda, X)$. It is obvious how to define the integral for simple functions: if $f = \sum x_i \mathbb{1}_{A_i}$, then $\int_\Omega f\, d\lambda = \sum x_i \lambda(A_i)$. The integral assigns to every function $f \in L_1^s(\Omega, \Sigma, \lambda, X)$ an element of X, and this correspondence is continuous. So there is a natural way to define an integral for elements of $L_1(\lambda, X)$: if $f \in L_1(\lambda, X)$, select a sequence $(f_n) \subset L_1^s(\Omega, \Sigma, \lambda, X)$ such that $\|f_n - f\| \to 0$ and put

$$\int_\Omega f\, d\lambda = \lim_{n\to\infty} \int_\Omega f_n\, d\lambda.$$

Elements of $L_1(\lambda, X)$ are called *Bochner integrable* functions and the integral defined above is called the *Bochner integral.*

THEOREM 2.7.6. *A function $f\colon \Omega \to X$ is Bochner integrable if and only if it is strongly measurable and the mapping $t \mapsto \|f(t)\|$ is a Lebesgue integrable function.*

The properties of the Bochner integral are very close to those of the Lebesgue integral, which makes the Bochner integral commonly used in the theory and applications of vector integration. In particular, if $f \in L_1([0,1], X)$, then its "antiderivative" or "primitive function" $F(t) = \int_0^t f\,d\lambda$ is differentiable almost everywhere and $F' = f$.

Let $\mu\colon \Sigma \to X$ be a *vector measure*, i.e., a countably additive function on the σ-algebra Σ; and let $A \in \Sigma$.

DEFINITION 2.7.7. The quantity

$$|\mu|(A) = \sup\left\{\sum_{k=1}^{n} \|\mu(A_k)\|\right\},$$

where supremum is taken over all finite measurable partitions of A, is said to be the *variation of μ on A.* μ is said to be *a measure of bounded variation* if $|\mu|(\Omega) < \infty$.

A typical example of a vector measure of bounded variation comes from integration theory: for every $f \in L_1(\lambda, X)$

$$\lambda_f(A) = \int_A f\,d\lambda$$

gives such an example. In this example the vector measure λ_f is *absolutely continuous* with respect to λ, that is, if $\lambda(A) = 0$, then $\lambda_f(A) = 0$, too. For a Bochner integrable f the variation of λ_f can be calculated as follows:

$$|\lambda_f|(A) = \int_A \|f(t)\|\,d\lambda(t) \tag{2.7.1}$$

It follows easily from (2.7.1) that if $\lambda_f = 0$ and $f \in L_1(\lambda, X)$, then $f = 0$. Hence, if $f_1, f_2 \in L_1(\lambda, X)$ and $\lambda_{f_1} = \lambda_{f_2}$, then $f_1 = f_2$.

DEFINITION 2.7.8. A vector measure μ is said to be *representable* if $\mu = \lambda_f$ for some $f \in L_1(\lambda, X)$. Then f is said to be the *Radon-Nikodým derivative* of μ with respect to λ (as in the scalar case). A space X is said to have the *Radon-Nikodým property* ($X \in$ RNP) if for every $(\Omega, \Sigma, \lambda)$ every X-valued measure of bounded variation which is absolutely continuous with respect to λ is representable.

More generally, a subset $D \subset X$ is said to have the *Radon-Nikodým property* ($D \in$ RNP) if for every bounded closed convex subset $A \subset D$ and for every $(\Omega, \Sigma, \lambda)$, every X-valued measure of bounded variation $\mu\colon \Sigma \to X$ that satisfies the condition

$$\mu(\Delta) \in \lambda(\Delta)A \qquad \text{for all } \Delta \in \Sigma$$

is representable.

THEOREM 2.7.9. *If μ is representable, then its range $\mu(\Sigma)$ is precompact.*

For an easy example of a space without the RNP consider $L_1[0,1]$. Take as $(\Omega, \Sigma, \lambda)$ the segment $[0,1]$ with the standard Lebesgue measure. The measure $\mu\colon \Sigma \to L_1[0,1]$ of bounded variation defined by $\mu(A) = \mathbb{1}_A$ is absolutely continuous with respect to λ, but its range is not precompact, so μ is not representable.

The Radon-Nikodým property of X^* is related to the description of the dual space of $L_1(\lambda, X)$. Namely, for every $F \in L_1(\lambda, X)^*$ and every $\Delta \in \Sigma$ one can define the functional $\mu(\Delta) \in X^*$ by the rule $\langle \mu(\Delta), x\rangle := F(x \cdot \mathbb{1}_\Delta)$, $x \in X$. This $\mu\colon \Sigma \to X^*$ is a finitely-additive vector measure that satisfies the condition

$$\mu(\Delta) \in \lambda(\Delta)\|F\|B_{X^*} \qquad \text{for all } \Delta \in \Sigma,$$

and for every $\Delta \in \Sigma$ of finite measure and every disjoint partition $\Delta = \bigsqcup_{n\in\mathbb{N}} \Delta_n$ we have the equality $\mu(\Delta) = \sum_{n\in\mathbb{N}} \mu(\Delta_n)$ in the sense of w^*-convergence of functionals. If X^* does not contain copies of c_0 (this happens in particular when $X^* \in$ RNP), the above condition implies the countable additivity in the sense of the norm-convergence (Bessaga-Pełczyński theorem, Theorem 2.4.3), and if $X^* \in$ RNP this gives a bounded strongly measurable function $g\colon \Omega \to X^*$ such that $\mu(\Delta) = \int_\Delta g\, d\lambda$. With this g we obtain the representation of the functional $F \in L_1(\lambda, X)^*$ as an integration functional

$$F(f) = \int_\Omega \langle g(t), f(t)\rangle\, d\lambda(t),$$

which leads to the identification of $L_1(\lambda, X)^*$ with the space $L_\infty(\lambda, X^*)$. Recall that the space $L_\infty(\lambda, X)$ consists of all equivalence classes (with respect to the λ-almost everywhere equality) of bounded strongly measurable functions $f\colon \Omega \to X$ equipped with the norm

$$\|f\| = \|f\|_{L_\infty} := \min\{c > 0\colon \|f(t)\| \leqslant c \text{ a.e.}\}.$$

Theorem 2.7.10 ([**99**, Theorem 1 on page 98] or [**141**, Theorem 1.3.10]). *If $X^* \in$ RNP, then $L_1(\lambda, X)^* = L_\infty(\lambda, X^*)$.*

There are many characterisations of the RNP (in [**99**, Section VII.6] there is a long list of them). Among the purely geometrical characterisations of the RNP, we will mostly use the ones defined by strongly exposed points and denting points. First we need some definitions and notation.

Definition 2.7.11. Let $A \subset X$. An element $e \in A$ is said to be a *strongly exposed point* of A if there is $x^* \in X^*$ (called a *strongly exposing functional*) such that $\operatorname{diam}\{x \in A\colon \operatorname{Re} x^*(x) > \operatorname{Re} x^*(e) - \varepsilon\}$ tends to 0 as $\varepsilon \to 0$ (equivalently, $\operatorname{Re} x^*(e) = \max \operatorname{Re} x^*(A)$ and $\operatorname{diam} \operatorname{Slice}(A, \operatorname{Re} x^*, \varepsilon) \to 0$ as $\varepsilon \to 0$). We write $\operatorname{stexp}(A)$ for the set of strongly exposed points of A. If A is a subset of a dual Banach space $X = Z^*$, we write w^*-$\operatorname{stexp}(A)$ to denote those points which are strongly exposed by weak-star continuous functionals (i.e., functionals from Z), which are called weak-star strongly exposed points. An element $y \in A$ is said to be a *denting point* of A if for every $\varepsilon > 0$ there is a slice S of A such that $y \in S$ and $\operatorname{diam} S < \varepsilon$. We write $\operatorname{dent}(A)$ to denote the set of denting points of A. If A is a subset of a dual Banach space $X = Z^*$, $y^* \in A$ is a weak-star denting point of A if it belongs to weak-star open slices of A of arbitrarily small diameter. We say that the set A is *dentable* if $A = \overline{\operatorname{conv}}\big(\operatorname{dent}(A)\big)$ (or, equivalently, if every slice A contains slices of A or arbitrarily small diameter, see [**117**, Proposition III.3]).

Evidently, every strongly exposed point is denting, every denting point is extreme, but the reverse implications do not hold in general. Indeed, in Figure 2.5 below, x_1 and x_2 are strongly exposed while y is denting but not strongly exposed.

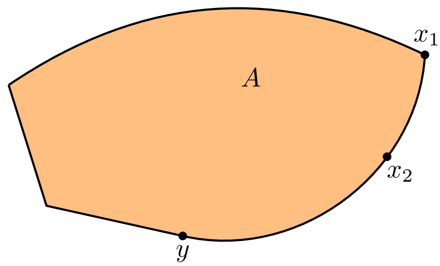

FIGURE 2.5. Strongly exposed vs denting

On the other hand, extreme points of norm compact sets are denting (use, for instance, Choquet's lemma given in Theorem 2.6.16). However, the constant function $\mathbb{1}$ in $C[0,1]$ is an extreme point of the unit ball which is not denting.

THEOREM 2.7.12. *For a Banach space X and a closed convex subset $D \subset X$ the following assertions are equivalent:*

(i) $D \in \mathrm{RNP}$*;*
(ii) *every closed convex bounded subset $A \subset D$ has a strongly exposed point;*
(iii) *every closed convex bounded subset $A \subset D$ has a denting point;*
(iv) *every closed convex bounded subset $A \subset D$ is dentable.*

In particular, every space with the RNP has the *Krein-Milman Property* (*KMP* in short): every closed convex bounded subset $A \subset X$ has an extreme point. It is still an open problem whether the KMP is equivalent to the RNP. A strong partial result was proved by Schachermayer [**279**]: for a Banach space isomorphic to its square, the Radon-Nikodým property and the Krein-Milman property are equivalent. This is also so for dual Banach spaces [**99**, Corollary VII.8, p. 198].

Since the closed unit balls of the spaces $L_1[0,1]$ and c_0 do not have any extreme points, these spaces fail to have the KMP, and hence, $L_1[0,1]$ and c_0 fail to have the RNP. On the other hand, it is known that weakly compact convex sets possess the RNP [**65**, Theorem 3.6.1].

For dual spaces there is a nice topological characterisation of the RNP: if X is separable, then $X^* \in \mathrm{RNP}$ if and only if X^* is separable. As a corollary one can deduce that both $L_1[0,1]$ and c_0 are not isomorphic to a dual space.

There is another characterisation of sets with the RNP, formulated in terms of vector-valued martingales, which easily implies the stability of the RNP with respect to direct sums – a fact that we are going to use later.

Let $\mathcal{T} \subset \Sigma$ be a finite algebra generated by a partition $\Omega = \bigsqcup_{k=1}^n A_k$, with $A_k \in \Sigma$, $\lambda(A_k) > 0$ for $k = 1, \dots, n$. The *conditional expectation* of the function $f \in L_1(\Omega, \Sigma, \lambda, X)$ with respect to $\mathcal{T}$ is defined as

$$E_{\mathcal{T}}^X(f) = \sum_{k=1}^n \left(\frac{1}{\lambda(A_k)} \int_{A_k} f \, d\lambda \right) \mathbb{1}_{A_k}.$$

In other words, we obtain $E_{\mathcal{T}}^X(f)$ by substituting on each A_k the original function f by its average on A_k. The conditional expectation operator $E_{\mathcal{T}}^X \colon L_1(\Omega, \Sigma, \lambda, X) \to$

$L_1(\Omega, \Sigma, \lambda, X)$ is a linear projection of the whole $L_1(\Omega, \Sigma, \lambda, X)$ onto the subspace $L_1(\Omega, \mathcal{T}, \lambda, X)$. Moreover, a direct calculation demonstrates that $\|E_{\mathcal{T}}\| = 1$.

Let $\mathcal{T}_j \subset \Sigma$, $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots$ be an increasing sequence of finite algebras, $g_j \in L_1(\Omega, \mathcal{T}_j, \lambda, X)$, $j = 1, 2, \ldots$, be a sequence of X-valued simple functions. The sequence of pairs $(g_j, \mathcal{T}_j)_{j\in\mathbb{N}}$ is called an *X-valued martingale, subordinate to the sequence of algebras $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \cdots$* if $E^X_{\mathcal{T}_j}(g_{j+1}) = g_j$ for all $j \in \mathbb{N}$.

THEOREM 2.7.13. *For a closed convex bounded subset $W \subset X$ the following assertions are equivalent:*
(i) $W \in \mathrm{RNP}$*;*
(ii) *for every increasing sequence of finite algebras $(\mathcal{T}_j)_{j\in\mathbb{N}} \subset \Sigma$ and every X-valued martingale $(g_j, \mathcal{T}_j)_{j\in\mathbb{N}}$ such that $g_j(t) \in W$ for all $j \in \mathbb{N}$ and $t \in \Omega$, the sequence (g_j) converges almost everywhere.*

COROLLARY 2.7.14. *Let X_1, X_2 be subspaces of a Banach space X such that $X = X_1 \oplus X_2$, $W_k \subset X_k$ be closed convex bounded subsets with the RNP, $k = 1, 2$. Then $W := W_1 + W_2 \in$* RNP.

PROOF. Let $\mathcal{T}_j \subset \Sigma$, $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots$ be an increasing sequence of finite algebras and $(g_j, \mathcal{T}_j)_{j\in\mathbb{N}}$ be a W-valued martingale. Denote P_k: $X \to X_k$ the natural projections; then $(P_k g_j, \mathcal{T}_j)_{j\in\mathbb{N}}$ is a W_k-valued martingale, $k = 1, 2$. By the RNP of W_k, the sequence $(P_k g_j)$ converges almost everywhere as $j \to \infty$, hence $(g_j) = (P_1 g_j + P_2 g_j)$ converges almost everywhere as well. $\square$

Every "smallness" property of sets induces in a natural way a property of operators: one just demands that the closure of the image of the unit ball has the corresponding property. This way one defines the classical concepts of compact operators and weakly compact operators. One may do the same with the Radon-Nikodým property.

DEFINITION 2.7.15. Let X, Y be Banach spaces. An operator $T \in L(X, Y)$ is called a *strong Radon-Nikodým operator* if $\overline{T(B_X)} \in \mathrm{RNP}$.

Weakly compact operators are strong Radon-Nikodým operators.

2.8. ℓ_1-type sequences

In this section we will study how to obtain, given a Banach space X, an element $u \in X^{**} \setminus \{0\}$ so that the equality

$$\|x + u\| = 1 + \|x\|$$

holds for every $x \in X$. These elements will play an important role in many characterisations of the Daugavet property, Daugavet centers, and narrow operators in Banach spaces with small density character. Let us start with the formal definitions.

DEFINITION 2.8.1. Let X be a Banach space.
(a) A norm-one element $u \in X^{**}$ is said to be *L-orthogonal* if the equality

$$\|x + u\| = 1 + \|x\|$$

holds for every $x \in X$.
(b) A sequence $(x_n) \subset B_X$ is an *ℓ_1-type sequence* if

$$\|x + x_n\| \to 1 + \|x\|$$

for every $x \in X$.

This nomenclature is derived from the language of Krivine and Maurey [192] who call a function of the form

$$\tau(x) = \mathfrak{U}\text{-}\lim_n \|x + x_n\|$$

a *type* on X; here (x_n) is a bounded sequence and $\mathfrak{U}$ is a free ultrafilter.

REMARK 2.8.2. From Lemma 2.6.4 it is easy to deduce that *in order to verify the condition $\|x+x_n\| \to 1+\|x\|$ from (b) of Definition 2.8.1 it is sufficient to check it only for $x \in S_X$*. Indeed, if $x \in S_X$ and $\alpha > 0$, then the condition $\|x + x_n\| \to 2$ means that for every $\varepsilon > 0$ and sufficiently large n, the vectors x and x_n are ε-quasi-codirected. But then, for every $\alpha > 0$ the vectors αx and x_n are $(\varepsilon \max\{\alpha, 1\})$-quasi-codirected, that is, $\|\alpha x + x_n\| \geqslant 1 + \|\alpha x\| - \varepsilon \max\{\alpha, 1\}$ for large n, which implies that $\|\alpha x + x_n\| \to 1 + \|\alpha x\|$.

There is a general procedure that allows one to construct ℓ_1-type sequences in several spaces, which will be of much use in our book.

LEMMA 2.8.3. *Let X be a separable subspace of an infinite-dimensional Banach space Z, $A \subset S_Z$ be a subset with the following property: for every finite-dimensional subspace $E \subset X$ and every $\varepsilon > 0$ there is $a \in A$ which is ε-quasi-codirected to all elements of S_E. Then, A contains a sequence (a_n) such that for every $x \in X$*

$$\|x + a_n\| \xrightarrow[n\to\infty]{} 1 + \|x\|.$$

In particular, if Z is a separable infinite-dimensional Banach space and the subset $A \subset S_Z$ has the above property, then A contains an ℓ_1-type sequence.

PROOF. Fix a sequence $(E_n)_{n\in\mathbb{N}}$ of finite-dimensional subspaces of X so that

$$E_1 \subset E_2 \subset \dots \quad \text{and} \quad \overline{\bigcup_{n\in\mathbb{N}} E_n} = X. \tag{2.8.1}$$

Also select a sequence $(\varepsilon_n) \subset \mathbb{R}^+$ such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. By our assumption, for every $n \in \mathbb{N}$ there is $a_n \in A$ such that $\|e + a_n\| \geqslant 2 - \varepsilon_n$ for every $e \in S_{E_n}$. Let us show that this (a_n) is the requested sequence. Fix $x \in S_X$ and a $\delta > 0$. According to (2.8.1), there are $n \in \mathbb{N}$ and $e \in S_{E_n}$ such that $\|x - e\| < \delta$. Then, for every $m \geqslant n$ we still have $e \in S_{E_m}$, so

$$\|x + a_m\| = \|(x - e) + (e + a_m)\| > \|e + a_m\| - \delta \geqslant 2 - \varepsilon_m - \delta.$$

Consequently, $\liminf_{m\to\infty} \|x + a_m\| \geqslant 2 - \delta$, which by arbitrariness of δ gives us

$$\lim_{m\to\infty} \|x + a_m\| = 2 = 1 + \|x\|. \qquad \square$$

In general, we will show a procedure to obtain, from a given ℓ_1-type sequence (x_n) in a Banach space X, a subsequence (y_n) of (x_n) so that every w^*-cluster point of (y_n) is an L-orthogonal element. Let us observe, however, that we cannot avoid the procedure of finding appropriate subsequences in the sense that there are ℓ_1-type sequences satisfying that not all its w^*-cluster points are L-orthogonal elements, as the following example shows.

EXAMPLE 2.8.4. There exists an ℓ_1-type sequence (x_n) in ℓ_1 such that $0 \in \overline{\{x_n\colon n \in \mathbb{N}\}}^{w}$. In particular, there exists an ultrafilter $\mathfrak{U}$ such that $w\text{-}\lim_{\mathfrak{U}} x_n = 0$.

Proof. Define, for every $n \in \mathbb{N}$, A_n to be the set of elements $x \in B_{\ell_1}$ such that $\|x\| \geqslant 1 - \frac{1}{2^n}$ with coordinates $x(i) \in \{\frac{z}{4^n} \colon z \in \mathbb{Z}\}$ if $2^n < i \leqslant 2^{n+1}$ and $x(i) = 0$ otherwise. Notice that A_n is finite for every $n \in \mathbb{N}$, so $A := \bigcup_{n\in\mathbb{N}} A_n$ is a countable subset of S_{ℓ_1}.

Note that if we consider A as a sequence in S_{ℓ_1}, then A is an ℓ_1-type sequence. This easily follows from the density of finitely-supported sequences in ℓ_1 and from the fact that, for every $k \in \mathbb{N}$, the set $\{x \in A \colon \operatorname{supp}(x) \cap \{1, \ldots, k\} \neq \emptyset\}$ is finite for every $k \in \mathbb{N}$.

Let us prove that 0 is a weak cluster point of A. To this end, pick a basic weak neighbourhood $O := \{x \in B_{\ell_1} \colon |x_i^*(x)| < \varepsilon \text{ for every } 1 \leqslant i \leqslant p\}$ of 0, with $x_1^*, \ldots, x_p^* \in S_{\ell_\infty}$ and $\varepsilon > 0$, and let us prove that $O \cap A \neq \emptyset$. For this, pick n large enough so that $2^n > p$ and $\frac{1}{2^n} < \varepsilon$. By a dimension argument we can find an element $y \in S_{\ell_1}$ which is supported on the coordinates $\{2^n + 1, \ldots, 2^{n+1}\}$ and such that $x_i^*(y) = 0$ holds for $1 \leqslant i \leqslant p$. Let $x \in B_{\ell_1}$ be supported on the coordinates $\{2^n + 1, \ldots, 2^{n+1}\}$ such that $x(i) = \frac{k}{4^n}$ and $|x(i) - y(i)| < \frac{1}{4^n}$. Note that hence $\|x - y\| < \frac{1}{2^n}$. This implies, on the one hand, that $\|x\| > 1 - \frac{1}{2^n}$, from where $x \in A_n \subset A$. On the other hand, $x_i^*(x) = 0$ for every $i \leqslant p$, which proves that $x \in O$ as desired. $\square$

Let us now show how to get L-orthogonal elements from an ℓ_1-type sequence. Let us begin with the following auxiliary lemma.

Definition 2.8.5. Let F be subspace of a Banach space X and $\varepsilon > 0$. An element $e \in B_X$ is said to be $(\varepsilon, 1)$-*orthogonal* to F if for every $x \in F$ and $t \in \mathbb{K}$

$$\|x + te\| \geqslant (1 - \varepsilon)(\|x\| + |t|). \tag{2.8.2}$$

Remark that the above condition implies that $\|e\| \geqslant 1-\varepsilon$ and that every $y \in S_F$ is α-quasi-codirected with e for every $\alpha > 2\varepsilon$. Indeed, we get the first condition substituting $x = 0$ and $t = 1$. The second one is also plain: $\|y + e\| \geqslant (1 - \varepsilon)2 = 2 - 2\varepsilon > \|y\| + \|e\| - \alpha$. The next lemma is in some sense a converse to this remark.

Lemma 2.8.6. *Let X be a Banach space, $F \subset X$ be a subspace, $\varepsilon, \delta > 0$ and $e \in B_X$ with $\|e\| > 1-\delta$. Then in order to demonstrate that e is $(\varepsilon+\delta, 1)$-orthogonal to F it is sufficient to show that e is ε-quasi-codirected with each of $y \in S_F$.*

Proof. Since F is a linear subspace, it is sufficient to demonstrate (2.8.2) for $t = 1$. We know that $y := x/\|x\|$ is ε-quasi-codirected with e. According to Lemma 2.6.4, x and e are $(\varepsilon \max\{\|x\|, 1\})$-quasi-codirected, that is

$$\begin{aligned}
\|x + e\| &> \|x\| + \|e\| - \varepsilon \max\{\|x\|, 1\} \\
&\geqslant \|x\| + 1 - \delta - \varepsilon \max\{\|x\|, 1\} \\
&> \|x\| + 1 - \delta - \varepsilon(\|x\| + 1) \\
&\geqslant (1 - (\varepsilon + \delta))(\|x\| + 1) \\
&\geqslant (1 - (\varepsilon + \delta))(\|x\| + \|e\|).
\end{aligned}$$

$\square$

Lemma 2.8.7. *Let X be a Banach space, $F \subset X$ be a finite-dimensional subspace, $\{y_1, \ldots, y_N\} \subset S_F$ be an ε-net of S_F, and let $e \in B_X$ with $\|e\| > 1 - \varepsilon$ be ε-quasi-codirected with each of y_k. Then e is $(3\varepsilon, 1)$-orthogonal to F.*

Proof. Let us apply the previous lemma. For $y \in S_F$ there is $k \in \{1, \ldots, N\}$ such that $\|y - y_k\| < \varepsilon$. Then $\|y + e\| \geqslant \|y_k + e\| - \varepsilon > 1 + \|e\| - 2\varepsilon$, that is, e is 2ε-quasi-codirected with y. $\square$

LEMMA 2.8.8. *Let X be a Banach space with an ℓ_1-type sequence (x_n). For every finite-dimensional subspace F of X and every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ so that for every $n \geqslant m$ the element x_n is $(\varepsilon, 1)$-orthogonal to F.*

PROOF. Since S_F is compact, we can find a finite $\frac{\varepsilon}{3}$-net $\{y_1, \ldots, y_N\} \subset S_F$ for it. Since (x_n) is an ℓ_1-type sequence, find $m \in \mathbb{N}$ so that $n \geqslant m$ implies that $\|x_n\| > 1 - \varepsilon/3$ and

$$\|y_k + x_n\| \geqslant 2 - \frac{\varepsilon}{3} \quad \text{for every } k \in \{1, \ldots, N\}.$$

It remains to apply Lemma 2.8.7. □

With the previous lemma in mind, we are able to prove the following result which says that, if we have an ℓ_1-type sequence, we can get a subsequence so that, roughly speaking, all the convex combinations of terms of the subsequence are orthogonal elements. More precisely, we have the following result.

LEMMA 2.8.9. *Let X be a Banach space with an ℓ_1-type sequence (x_n). Take a separable subspace Z of X and a sequence of positive real numbers (ε_n). Write $Z = \overline{\bigcup_{n\in\mathbb{N}} F_n}$, where $(F_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite dimensional subspaces of X. Then, there exists a subsequence (y_n) of (x_n) such that the inequality*

$$(1 - \varepsilon_n)(1 + \|x\|) \leqslant \left\| x + \sum_{j=n+1}^{\infty} \alpha_j y_j \right\|$$

holds for every $n \in \mathbb{N}$, $x \in F_n$ and every $(\alpha_j) \subset \mathbb{R}^+$ with $\sum_{j=n+1}^{\infty} \alpha_j = 1$.

PROOF. Pick a sequence (δ_n) of positive scalars so that

$$(1 - \varepsilon_n) < \prod_{j=n+1}^{\infty} (1 - \delta_j).$$

By induction, we can construct a subsequence (y_n) of (x_n) so that

$$\|y + \lambda y_n\| \geqslant (1 - \delta_n)(\|y\| + |\lambda|)$$

holds for every $y \in \operatorname{lin}(F_n \cup \{y_1, \ldots, y_{n-1}\})$ and every $\lambda \in \mathbb{R}$. Let us prove that the sequence (y_n) satisfies our requirements. To this end, pick $n \in \mathbb{N}$, $x \in F_n$ and a sequence of positive scalars $(\alpha_j)_{j=n+1}^{\infty}$ with $\sum_{j=n+1}^{\infty} \alpha_j = 1$. Then, for every $k > n$ it follows that

$$\begin{aligned}
\left\| x + \sum_{j=n+1}^{k} \alpha_j y_j \right\| &\geqslant (1 - \delta_k) \left(\left\| x + \sum_{j=n+1}^{k-1} \alpha_j y_j \right\| + \alpha_k \right) \\
&\geqslant (1 - \delta_k) \left((1 - \delta_{k-1}) \left(\left\| x + \sum_{j=n+1}^{k-2} \alpha_j y_j \right\| + \alpha_{k-1} \right) + \alpha_k \right) \\
&\geqslant (1 - \delta_k)(1 - \delta_{k-1}) \left(\left\| x + \sum_{j=n+1}^{k-2} \alpha_j y_j \right\| + \alpha_{k-1} + \alpha_k \right).
\end{aligned}$$

Continuing in this fashion, we get that

$$\begin{aligned}\left\| x + \sum_{j=n+1}^{k} \alpha_j y_j \right\| &\geqslant \prod_{i=n+1}^{k} (1-\delta_i) \left(\|x\| + \sum_{j=n+1}^{k} \alpha_j \right) \\ &\geqslant \prod_{i=n+1}^{\infty} (1-\delta_i) \left(\|x\| + \sum_{j=n+1}^{k} \alpha_j \right) \\ &\geqslant (1-\varepsilon_n) \left(\|x\| + \sum_{j=n+1}^{k} \alpha_j \right).\end{aligned}$$

Note that $(x + \sum_{j=n+1}^{k} \alpha_j y_j)_k$ converges in norm to $x + \sum_{j=n+1}^{\infty} \alpha_j y_j$. Since the previous inequality holds for every $n \in \mathbb{N}$ we get that

$$\left\| x + \sum_{j=n+1}^{\infty} \alpha_j y_j \right\| \geqslant (1-\varepsilon_n) \left(\|x\| + \sum_{j=n+1}^{\infty} \alpha_j \right),$$

as desired. $\square$

Putting everything together, we are now able to describe how to obtain L-orthogonal elements for separable Banach spaces.

THEOREM 2.8.10. *Let X be a Banach space. Let Z be a separable subspace of X and let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite-dimensional subspaces of X so that $\bigcup_{n\in\mathbb{N}} F_n$ is dense in Z. Assume that there are a sequence (y_n) in B_X and a sequence of positive scalars (ε_n) converging to zero satisfying that the inequality*

$$(1-\varepsilon_n)(1+\|x\|) \leqslant \left\| x + \sum_{j=n+1}^{\infty} \alpha_j y_j \right\|$$

holds for every $n \in \mathbb{N}$, $x \in F_n$ and every $(\alpha_j) \subset \mathbb{R}^+$ with $\sum_{j=n+1}^{\infty} \alpha_j = 1$.

Then, any $u \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}}^{w^*}(\{y_j\colon j > n\}) \subset B_{X^{**}}$ (in particular, any weak*-cluster point of the sequence) satisfies that

$$\|x+u\| = 1 + \|x\|$$

for every $x \in Z$.

In particular, if X is a separable Banach space with an ℓ_1-type sequence, then there are L-orthogonal elements among the weak* cluster points of (x_n).

PROOF. Given any $x \in Z$ we can assume, up to a density argument, that there exists $m \in \mathbb{N}$ so that $x \in F_n$ holds for every $n \geqslant m$. Given $n \geqslant m$, notice that the condition on the sequence (y_n) implies that

$$(x + \operatorname{conv}(\{y_j\colon j > n\})) \cap (1-2\varepsilon_n)(1+\|x\|)B_X = \emptyset.$$

By the Hahn-Banach separation theorem (see [**109**, Proposition 2.13], for instance), there exist $f_n \in S_{X^*}$ such that the inequality

$$\operatorname{Re} f_n(x+z) \geqslant (1-2\varepsilon_n)(1+\|x\|)$$

holds for every $z \in \operatorname{conv}(\{y_j\colon j > n\})$. Note that the condition on f_n also implies that

$$\left(x + \overline{\operatorname{conv}(\{y_j\colon j > n\})}^{w^*}\right) \cap (1-2\varepsilon_n)(1+\|x\|)B_{X^{**}} = \emptyset. \tag{2.8.3}$$

Finally, if $u \in \bigcap_{n\in\mathbb{N}} \overline{\operatorname{conv}(\{y_j : j > n\})}^{w^*}$ then, for every $n > m$, (2.8.3) implies that $\|x + u\| \geqslant (1 - 2\varepsilon_n)(1 + \|x\|)$.

The final part is a direct consequence of Lemma 2.8.9. □

2.9. Summands and ideals in Banach spaces

In this section we discuss several special classes of subspaces of a Banach spaces which will be used throughout the book.

2.9.1. Absolute sums of Banach spaces. Let $E = (\mathbb{R}^n, \|\cdot\|_E)$ be a normed space, and denote e_k, $k = 1, \dots, n$, the elements of the canonical basis: the k-th coordinate of e_k is equal to 1, and the remaining ones are zero. The norm $\|\cdot\|_E$ is called *absolute* if it satisfies the following conditions:

(i) $\|e_k\|_E = 1$, $k = 1, \dots, n$;
(ii) for every $a = (a_1, \dots, a_n)$ the vector $|a| := (|a_1|, \dots, |a_n|)$ has the same norm as a:

$$\|(a_1, \dots, a_n)\|_E = \|(|a_1|, \dots, |a_n|)\|_E.$$

The above properties imply that the norm is *monotone* in the following sense: if $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ satisfy $0 \leqslant a_k \leqslant b_k$, $k = 1, \dots, n$, then $\|a\|_E \leqslant \|b\|_E$.

We identify the dual E^* of E in the standard way with $(\mathbb{R}^n, \|\cdot\|_{E^*})$, where a functional $b = (b_1, \dots, b_n) \in E^*$ acts on $a = (a_1, \dots, a_n) \in E$ by the formula $b(a) = b_1 a_1 + \dots + b_n a_n$. Remark that the norm $\|\cdot\|_{E^*}$ is also absolute.

Let $X_1, \dots, X_n$ be normed spaces, and $E = (\mathbb{R}^n, \|\cdot\|_E)$ be a space with an absolute norm. The E-sum of the spaces X_k is the vector space $\left(\bigoplus_{i=1}^n X_i\right)_E$ of all n-tuples $x = (x_1, \dots, x_n)$, $x_k \in X_k$, $k = 1, \dots, n$, equipped with the norm

$$\|x\| = \big\|(\|x_1\|, \dots, \|x_n\|)\big\|_E. \tag{2.9.1}$$

Recall that condition (ii) from the definition of an absolute norm guarantees that the expression (2.9.1) satisfies the triangle inequality. It also gives the following natural property: if $X_k = \mathbb{R}$, $k = 1, \dots, n$, then $\left(\bigoplus_{i=1}^n X_i\right)_E = E$. The condition (i) is not so essential because it can easily be achieved by rescaling, but it is usually assumed for the sake of convenience. In order to shorten the notation, for $x = (x_1, \dots, x_n) \in \left(\bigoplus_{i=1}^n X_i\right)_E$ we denote $N(x) = (\|x_1\|, \dots, \|x_n\|)$. In this notation, $\|x\| = \|N(x)\|_E$. The dual space of $\left(\bigoplus_{i=1}^n X_i\right)_E$ is $\left(\bigoplus_{i=1}^n X_i^*\right)_{E^*}$, where $f = (f_1, \dots, f_n) \in \left(\bigoplus_{i=1}^n X_i^*\right)_{E^*}$ acts on $x = (x_1, \dots, x_n) \in \left(\bigoplus_{i=1}^n X_i\right)_E$ by the rule $f(x) = f_1(x_1) + \dots + f_n(x_n)$.

In the case of two terms we simplify our notation to $X_1 \oplus_E X_2$.

The above construction generalises to infinite sums in a natural way. Given a sequence $\big((X_n, \|\cdot\|_n)\big)_{n\in\mathbb{N}}$ of Banach spaces, and a Banach space E of sequences such that $((t_i) \in E) \Leftrightarrow ((|t_i|) \in E)$ and whose norm satisfies

$$\|(t_i)\|_E = \|(|t_i|)\|_E \qquad \big((t_i) \in E\big),$$

we denote by $\left(\bigoplus_{n\in\mathbb{N}} X_n\right)_E$ the Banach space of all sequences $(x_n) \in \prod_{n=1}^\infty X_n$ so that $(\|x_n\|_n) \in E$ and equip it with the norm

$$\|(x_n)\| = \big\|(\|x_n\|_n)\big\|_E.$$

In particular, E can be any sequence space whose unit vector basis is 1-unconditional.

The most important cases for us are $E = \ell_1$ and $E = \ell_\infty$. In this case even uncountable sums make sense, which we explain below on the example of ℓ_1-sums.

Let I be an index set, and X_i, $i \in I$, be a fixed collection of Banach spaces. We denote by $\left(\bigoplus_{i\in I} X_i\right)_{\ell_1}$ the ℓ_1-sum of the X_i. This means that $\left(\bigoplus_{i\in I} X_i\right)_{\ell_1}$ is the set of all points $z = (z_i)_{i\in I}$, where $z_i \in X_i$ for all $i \in I$, with at most countable support $\operatorname{supp}(z) := \{i\colon z_i \neq 0\}$ and such that $\sum_{i\in I} \|z_i\|_{X_i} < \infty$. The space $\left(\bigoplus_{i\in I} X_i\right)_{\ell_1}$ is equipped with the natural norm

$$\|z\| = \|(z_i)_{i\in I}\| = \sum_{i\in I} \|z_i\|_{X_i}. \tag{2.9.2}$$

If $I = \{1, \dots, n\}$, we may use the self-explanatory notation $(\bigoplus_{i=1}^n X_i)_{\ell_1}$ or $X_1 \oplus_1 \cdots \oplus_1 X_n$.

If I is infinite (countable or uncountable), the corresponding sum in (2.9.2) reduces to an ordinary at most countable sum $\sum_{i\in \operatorname{supp}(z)} \|z_i\|_{X_i}$, which does not depend on the order of its terms, so there is no need to introduce an ordering on I and to appeal to any kind of definition for uncountable sum.

In the sequel we will regard each X_j as a subspace of $\left(\bigoplus_{i\in I} X_i\right)_{\ell_1}$ in the following natural way: $X_j = \left\{z \in \left(\bigoplus_{i\in I} X_i\right)_{\ell_1} : \operatorname{supp}(z) \subset \{j\}\right\}$.

2.9.2. *L*-summands and *M*-ideals. We compile in this section the results of the theory of M-ideals, M-summands, and L-summands spaces that we will use in the book. They are the most natural and most useful cases of absolute sums. For a subspace J of a Banach space X, we write $J^\perp = \{x^* \in X^*\colon x^*|_J = 0\} \subset X^*$. The basic reference here is the Lecture Notes volume of P. Harmand, D. Werner, and W. Werner [**133**].

Definition 2.9.1. Let X be a real or complex Banach space and let $P \in L(X)$ be a *projection* (i.e. $P^2 = P$).

(a) P is an *M-projection* if $\|x\| = \max\{\|Px\|, \|x - Px\|\}$ for all $x \in X$.
(b) P is an *L-projection* if $\|x\| = \|Px\| + \|x - Px\|$ for all $x \in X$.
(c) The range of an M-projection is called an *M-summand*; the range of an L-projection is called an *L-summand*.
(d) A subspace J of X is an *M-ideal* of X if $J^\perp$ is an L-summand in X^*.

Examples of M-ideals which are not M-summands are easy to get in $C_0(L)$ spaces.

Example 2.9.2. Let L be a locally compact Hausdorff topological space. Then, a subspace J of $C_0(L)$ is an M-ideal of $C_0(L)$ if and only if there is a closed subset $D \subset L$ such that

$$J = \{f \in C_0(L)\colon f|_D = 0\}.$$

Moreover, such a J is an M-summand if and only if D is clopen in L.

The following result on L- and M- projections will be needed.

Proposition 2.9.3 ([**133**, Theorem I.1.10]). *Let X be a Banach space.*

(a) *Two L- (respectively, M-) projections on X commute.*
(b) *The set of all L-projections on X forms a complete Boolean algebra under the operations*

$$P \wedge Q = PQ,\ P \vee Q = P + Q - PQ,\ P^c = \mathrm{Id} - P.$$

The following easy result on L-projections follows routinely from the above proposition. We include a proof for the sake of completeness.

LEMMA 2.9.4. *Let Z be a Banach space and $z_1, \ldots, z_n \in S_Z$ be pairwise linearly independent elements such that each $\mathbb{K}z_k$ is an L-summand of Z for $k = 1, \ldots, n$. For each $k \in \{1, \ldots, n\}$, write P_k for the L-projection with range $\mathbb{K}z_k$, so $Z = \mathbb{K}z_k \oplus_1 \ker P_k$. Then, $P_k P_j = 0$ when $k \neq j$, $P := P_1 + \cdots + P_n$ is an L-projection with kernel $\bigcap_{k=1}^n \ker P_k$, and $P(Z) \cong \ell_1^{(n)}$ with $B_{P(Z)} = \operatorname{aconv}(\{z_1, \ldots, z_n\})$. In particular, the points $z_1, \ldots, z_n$ are linearly independent.*

PROOF. First, fix k, j with $k \neq j$ and use that $P_k P_j = P_j P_k$ by Proposition 2.9.3 to get that $P_k P_j(Z) \subset P_k(Z) \cap P_j(Z) = (\mathbb{K}z_k) \cap (\mathbb{K}z_j)$. As z_k and z_j are linearly independent, we get that $P_k P_j(Z) = \{0\}$, that is, $P_k P_j = 0$. Now, it follows also from Proposition 2.9.3 that $P = P_1 + \cdots + P_n$ is an L-projection. It is straightforward to show that $\ker P = \bigcap_{k=1}^n \ker P_k$ using that the projections are orthogonal. Finally, it is also immediate that $P(Z) \cong \ell_1^{(n)}$ and that $B_{P(Z)} = \operatorname{aconv}(\{z_1, \ldots, z_n\})$. □

If J is an M-ideal in a Banach space X then, by definition, $X^* = V \oplus_1 J^\perp$ for some closed subspace V of X^*. Then $\{x^*|_J \colon x^* \in V\}$ is linearly isometric to J^*, and we shall write

$$X^* = J^* \oplus_1 J^\perp \tag{2.9.3}$$

(see [**133**, Remark I.1.13]). This formula allows us to consider the $\sigma(X, J^*)$-topology on X. An application of the Hahn-Banach theorem gives the following result which can be found in [**133**, Remark I.1.13].

LEMMA 2.9.5. *Let J be an M-ideal in a Banach space X. Then B_J is $\sigma(X, J^*)$-dense in B_X.*

Another property of M-ideals is the following one, known as the restricted 3-ball property of M-ideals (see [**133**, Theorem I.2.2]).

LEMMA 2.9.6. *Let J be a closed subspace of a Banach space X. Then, J is an M-ideal in X if and only if for all $y_1, y_2, y_3 \in B_J$, $x \in B_X$, and $\varepsilon > 0$, there is $y \in J$ satisfying that*

$$\|x + y_i - y\| \leqslant 1 + \varepsilon$$

for $i = 1, 2, 3$.

In the following definition, X is considered as canonically embedded into its bidual X^{**} via J_X.

DEFINITION 2.9.7. Let X be a Banach space.

(a) X is said to be *L-embedded* if $X^{**} = J_X(X) \oplus_1 X_s$ for some closed subspace X_s of X^{**}.

(b) X is said to be *M-embedded* if X is an M-ideal in X^{**}, that is, $X^{***} = \big(J_{X^*}(X^*)\big)^\perp \oplus_1 Z$ for some subspace Z of X^{***}. It is known that, in this case, $Z = J_{X^*}(X^*)$ [**133**, Proposition III.1.2].

A list of examples of L- and M-embedded spaces is given below.

EXAMPLE 2.9.8 ([**133**, Example IV.1.1]). The following classes of spaces are L-embedded:

(a) $L_1(\mu)$ spaces,
(b) preduals of von Neumann algebras,
(c) the Hardy space H_0^1, the dual of the disk algebra $\mathbb{A}^*$, and $L_1[0,1]/H_0^1$ (the predual of H^∞).

EXAMPLE 2.9.9 ([**133**, Example III.1.4]). The following classes of spaces are M-embedded:
(a) c_0,
(b) $K(\ell_p)$ for $1 < p < \infty$,
(c) the space $C(\mathbb{T})/\mathbb{A}$,
(d) closed subspaces and quotients of the above spaces.

We next give an easy result on the behaviour of extreme points with respect to L- and M-summand which will be useful later on.

REMARK 2.9.10. *Let X be a Banach space and let Z, W be closed subspaces of X.*

(a) *If $X = Z \oplus_1 W$, then*
$$\operatorname{ext}(B_X) = \{(z,0)\colon z \in \operatorname{ext}(B_Z)\} \cup \{(0,w)\colon w \in \operatorname{ext}(B_W)\}.$$
(b) *If $X = Z \oplus_\infty W$, then*
$$\operatorname{ext}(B_X) = \{(z,w)\colon z \in \operatorname{ext}(B_Z),\ w \in \operatorname{ext}(B_W)\}.$$

2.9.3. Almost isometric ideals.

DEFINITION 2.9.11. Let Z be a subspace of a Banach space X. We say that Z is an *almost isometric ideal* (*ai-ideal*) in X if X is locally complemented in Z by almost isometries. This means that for each $\varepsilon > 0$ and for each finite-dimensional subspace $E \subset X$ there exists a linear operator $T\colon E \to Z$ satisfying

(1) $T(e) = e$ for each $e \in E \cap Z$, and
(2) $(1-\varepsilon)\|e\| \leqslant \|T(e)\| \leqslant (1+\varepsilon)\|e\|$ for each $e \in E$,

i.e., T is a $\frac{1+\varepsilon}{1-\varepsilon}$-isometry fixing the elements of E. If T satisfies only (1) and the right-hand side of (2), we get the well-known concept of Z being an *ideal* in X.

Almost isometric ideals were introduced in [**8**] as examples of a certain class of subspaces which inherit diameter two properties (cf. Section 12.2) and the Daugavet property, to be introduced shortly; see also [**1**]. It should be mentioned that, despite their name, M-ideals need not be ai-ideals.

Note that the Principle of Local Reflexivity means that X is an ai-ideal in X^{**} for every Banach space X.

Throughout the text we will make use of the following two results, which we include here for easy reference.

THEOREM 2.9.12. [**8**, Theorem 1.4] *Let X be a Banach space and let Z be an almost isometric ideal in X. Then there is a linear isometry $\varphi\colon Z^* \to X^*$ such that*
$$\varphi(z^*)(z) = z^*(z)$$
holds for every $z \in Z$ and $z^* \in Z^*$ and satisfying that, for every $\varepsilon > 0$, every finite-dimensional subspace E of X and every finite-dimensional subspace F of Z^*, we can find an operator $T\colon E \to Z$ satisfying

(1) *$T(e) = e$ for every $e \in E \cap Z$,*

(2) $(1-\varepsilon)\|e\| \leqslant \|T(e)\| \leqslant (1+\varepsilon)\|e\|$ *holds for every* $e \in E$, *and;*
(3) $f(T(e)) = \varphi(f)(e)$ *holds for every* $e \in E$ *and every* $f \in F$.

Following [1], we will refer to such an operator φ as *an almost-isometric Hahn-Banach extension operator*. Notice that if $\varphi\colon Z^* \to X^*$ is an almost isometric Hahn-Banach extension operator, then $\varphi^*\colon X^{**} \to Z^{**}$ is a norm-one projection (see e.g. [181, Theorem 3.5]).

Another central result for our main theorems will be the following, coming from [1, Theorem 1.5]

THEOREM 2.9.13. *Let X be a Banach space, let Y be a separable subspace of X and let $W \subset X^*$ be a separable subspace. Then there exists a separable almost isometric ideal Z in X containing Y and an almost isometric Hahn-Banach extension operator $\varphi\colon Z^* \to X^*$ such that $\varphi(Z^*) \supset W$.*

2.9.4. u-summands and u-ideals. We recall that according to [124], given a Banach space X and a subspace Y, Y is a *u-summand* in X if there exists a subspace Z of X such that $X = Y \oplus Z$ and such that the projection $P\colon X \to X$ along Z such that $P(X) \subset Y$ satisfies that $\|\mathrm{Id} - 2P\| \leqslant 1$ (in this case we say that P is a *u-projection*). We say that Y is a *u-ideal* in X if there exists a u-projection $P\colon X^* \to Y^*$ such that $\ker(P) = Y^\perp$, and we say that Y is a *strict u-ideal* in X if $P^{**}(X^{***})$ is norming in X^{***}.

2.9.5. Banach lattices. At some places we will have occasion to deal with Banach lattices; see for instance Section 1.2. Here is a short review of the basic definitions.

Let E be a real vector space equipped with a partial order $\leqslant$. The pair $(E, \leqslant)$ is called an *ordered vector space* if the order and the algebraic operations are compatible:

$$\begin{aligned} x \leqslant y,\ z \in E \quad &\Longrightarrow \quad x+z \leqslant y+z \\ x \leqslant y,\ a \in \mathbb{R},\ a \geqslant 0 \quad &\Longrightarrow \quad ax \leqslant ay \end{aligned}$$

An ordered vector space is called a *Riesz space* (or *vector lattice*) if, given $x, y \in E$, the supremum (= least upper bound) $x \vee y$ and the infimum (= greatest lower bound) $x \wedge y$ exist. It is called *Dedekind complete* if the supremum of any nonvoid order bounded subset exists (then so does the infimum). In a vector lattice one can form, for $x \in E$, the elements $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$; then $x = x^+ - x^-$. The vector $|x| = x^+ + x^-$ is called the *absolute value* of x.

Finally, suppose E is a Riesz space equipped with a norm satisfying $\|x\| = \|\,|x|\,\|$ for all x; then E is called a *normed Riesz space*. If this normed space is complete, it is called a *Banach lattice*.

The spaces $L_p(\mu)$ are among the best known examples of Banach lattices; for a function $f \in L_p(\mu)$, the above notions f^+, f^- and $|f|$ have their traditional meaning from real analysis. More general examples are the Köthe function spaces, in particular rearrangement invariant spaces like Orlicz spaces (more on this in Section 4.4).

Suppose E is a Banach space with a 1-unconditional Schauder basis (e_n). Let $x = \sum_{k=1}^\infty a_k e_k$ be the expansion of $x \in E$ in this basis. Then defining

$$x \geqslant 0 \quad \Longleftrightarrow \quad \text{all } a_k \geqslant 0$$

provides a partial order (of course $x \leqslant y$ iff $y - x \geqslant 0$) that makes E a Banach lattice.

Standard references on Banach lattices include the monographs [**22**], [**233**], [**282**].

2.10. Geometric properties of norms and corresponding renormings

Recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space X are said to be *equivalent* if there are $a, b \in (0, +\infty)$ such that for every $x \in X$

$$a\|x\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1.$$

In this short section we collect some results on equivalent renormings that we will use in our book.

THEOREM 2.10.1. *Let X be a normed space, $Y \subset X$ be a subspace. Then, every equivalent norm $\|\cdot\|_1$ on Y can be extended to an equivalent norm defined on the whole of X.*

PROOF. Denote $B_1 = \{y \in Y\colon \|y\|_1 < 1\}$ the open unit ball of $\|\cdot\|_1$ in Y and let U_X be the open unit ball of the original norm $\|\cdot\|$ in X. Let $r > 0$ be such that $rU_X \cap Y \subset B_1$. Denote $W = rU_X \cup B_1$ (geometrically W looks like the standard picture of Saturn rU_X with the disk B_1 as one of its rings, see Figure 2.6). For the required equivalent norm on X we may take the norm whose open unit ball is equal to $\overline{\text{conv}}(W)$. □

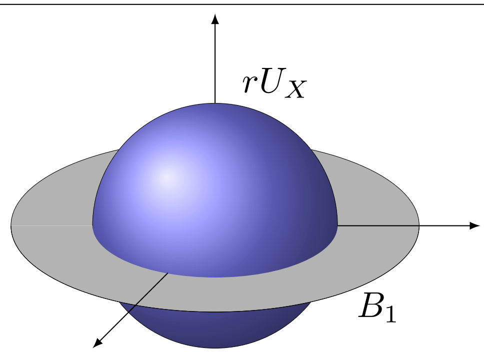

FIGURE 2.6. Renorming a space keeping the norm on a subspace

DEFINITION 2.10.2. The norm $\|\cdot\|$ on a Banach space X is said to be *LUR* at $x_0 \in S_X$ (another name is x_0 *is a LUR point of* S_X) if $\lim_{n\to\infty} \|x_n - x_0\| = 0$ whenever $(x_n)_{n\in\mathbb{N}} \subset B_X$ is such that $\lim_{n\to\infty} \|x_n + x_0\| = 2$. If the norm is LUR at each point of S_X, we say that X is LUR ($X \in \text{LUR}$ for short).

The acronym LUR above is the abbreviation for *Locally Uniformly Rotund*. Another standard name for the same property is *Locally Uniformly Convex*.

PROPOSITION 2.10.3. *Let X be a Banach space and $x_0 \in S_X$. If the norm is LUR at x_0, then x_0 is a strongly exposed point of B_X; so, in particular, it is denting. Consequently, if $X \in$ LUR then every point in S_X is denting and so B_X is dentable.*

PROOF. Let $x_0^* \in S_{X^*}$ be a supporting functional at x_0, that is, $x_0^*(x_0) = 1$. Our goal is to demonstrate that x_0^* is the strongly exposing functional that we need, i.e., that $\operatorname{diam}\operatorname{Slice}(B_X, x_0^*, \varepsilon)$ tends to 0 as $\varepsilon \to 0$ (see Definition 2.7.11). Let us take $x_n \in \operatorname{Slice}(B_X, x_0^*, 1/n)$ with

$$\|x_n - x_0\| > \frac{1}{4}\operatorname{diam}\operatorname{Slice}(B_X, x_0^*, 1/n). \tag{2.10.1}$$

Then $2 \geqslant \|x_n + x_0\| \geqslant x_0^*(x_n + x_0) > 2 - \frac{1}{n}$, so $\lim_{n\to\infty} \|x_n + x_0\| = 2$. By the assumption that the norm is LUR at x_0, we have that $\lim_{n\to\infty} \|x_n - x_0\| = 0$. Applying (2.10.1) we obtain that $\lim_{n\to\infty} \operatorname{diam}\operatorname{Slice}(B_X, x_0^*, 1/n) = 0$, which, by the monotonicity of $\operatorname{diam}\operatorname{Slice}(B_X, x_0^*, \varepsilon)$ in ε, gives what we need. $\square$

THEOREM 2.10.4 (M. I. Kadets [**154**], see also [**94**, Theorem II.2.6]). *Every separable Banach space X admits an equivalent LUR renorming, i.e., there is an equivalent norm p on X such that $(X, p) \in \mathrm{LUR}$. This norm p may be selected arbitrarily close to the original norm $\|\cdot\|$, that is, for given $\varepsilon > 0$ the corresponding p may be selected to satisfy the inequalities $(1-\varepsilon)\|x\| \leqslant p(x) \leqslant \|x\|$ for all $x \in X$.*

We now deal with differentiability of the norm of a Banach space. Recall that the norm of a Banach space X is *Gâteaux differentiable* at $x \in X \setminus \{0\}$ or that $x \neq 0$ is a point of *Gâteaux differentiability* (respectively, *Fréchet differentiable* or a point of *Fréchet differentiability*) if

$$\lim_{t\to 0} \frac{\|x + th\| + \|x - th\| - 2\|x\|}{t} = 0 \tag{2.10.2}$$

for every $h \in S_X$ (respectively, uniformly on $h \in S_X$). A very well-known and useful criterion for differentiability of the norm is given by Shmulyan's test. We refer to [**94**, Theorem I.1.4] or to [**109**, Theorems 7.15 and 7.17] for a proof.

LEMMA 2.10.5 (Shmulyan's test). *Let X be a Banach space and let $x \in X \setminus \{0\}$.*

(a) *x is a point of Gâteaux differentiability of X if there exists a unique functional $f \in S_{X^*}$ such that $f(x) = \|x\|$ (we usually say that x is a* smooth point *or that the norm is* smooth at *x);*
(b) *x is a point of Fréchet differentiability of X if there exists a functional $f \in S_{X^*}$ such that for every sequence (f_n) in S_{X^*} with $f_n(x) \to \|x\|$ one has that $f_n \to f$ in norm, equivalently, if there is a strongly exposed point of B_{X^*} attaining its norm at $x/\|x\|$.*

If $X = Z^*$ is a dual space and $f \in Z^* \setminus \{0\}$, then

(c) *f is a point of Fréchet differentiability of Z^* if there exists $z_0 \in S_{X^*}$ such that for every sequence (z_n) in S_Z with $f(z_n) \to \|f\|$, one has that $z_n \to z_0$ in norm, equivalently, if f attains its norm at a strongly exposed point $z_0 \in S_Z$.*

A couple of consequences of the previous result deserve to be mentioned.

(1) If X is a Banach space such that X^* is separable, then X admits an equivalent norm which is Fréchet differentiable at every non zero element. Indeed, X admits a norm $|\cdot|$ whose dual norm is LUR (this is an improvement of Theorem 2.10.4 in the case of a dual space, see [**94**, Theorem II.2.6]). By Proposition 2.10.3, every point of $S_{(X,|\cdot|)^*}$ is strongly exposed, hence by item (b) of Shmulyan's test, the norm $|\cdot|$ is Fréchet differentiable at every non zero element.

(2) If the norm of the dual of a Banach space X is Fréchet differentiable at every non zero element, then X is reflexive. Indeed, by item (c) of Shmulyan's test, every element of X^* attains its norm (at a strongly exposed point, but this is not important now). Hence, James's theorem proves that X is reflexive.

The complete opposite of Fréchet differentiability for a Banach space X is given by replacing the zero (uniform) limit in (2.10.2) by the existence of a constant $\rho > 0$ such that the following inequality holds

$$\limsup_{\|h\|\to 0} \frac{\|x+h\| + \|x-h\| - 2\|x\|}{\|h\|} \geqslant \rho. \tag{2.10.3}$$

In such a case, we say that (the norm of) X is *rough* (or *ρ-rough* to be more precise). In the extreme case that $\rho = 2$ works, we say that (the norm of) X is *extremely rough*. It is easy to see that infinite-dimensional L_1-spaces are always extremely rough. Obviously, a rough norm has no point of Fréchet differentiability. We refer the reader to [**94**] for more information and background. For instance, the following result can be found in [**94**, Proposition I.1.11].

LEMMA 2.10.6 (See [**94**, Proposition I.1.11]). *Let X be a Banach space and let $\rho > 0$. Then, the norm of X is ρ-rough if and only if all weak-star slices of B_{X^*} have diameter greater than or equal to ρ.*

Finally, recall that the *modulus of uniform convexity* of a Banach space X is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X,\ \|x-y\| \geqslant \varepsilon\right\}.$$

The space X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A notion dual to this is the one of uniform smoothness. The *modulus of smoothness* of a Banach space X is defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau h\| + \|x+\tau h\| - 2}{2} : \|x\| = \|h\| = 1\right\} \qquad (\tau > 0).$$

We say that X is *uniformly smooth* if $\lim\limits_{\tau\downarrow 0} \dfrac{\rho_X(\tau)}{\tau} = 0$. These two notions are in duality as X is uniformly convex if and only if X^* is uniformly smooth, and X is uniformly smooth if and only if X^* is uniformly convex. $L_p(\mu)$ spaces for $1 < p < \infty$ are both uniformly convex and uniformly smooth. Besides, a classical result by Milman and Pettis shows that uniformly convex spaces are reflexive. Moreover, a Banach space is superreflexive if and only if it admits an equivalent norm which makes it uniformly convex.

2.11. Tensor product spaces

In this section we will provide an introduction to tensor product spaces. A more detailed treatment of tensor products and proofs of the results of this sections can be found in the books [**91, 274**].

Given two Banach spaces, we denote by $X \otimes Y$ the *algebraic tensor product* of X and Y. Given an element $z \in X \otimes Y$, there are $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$ so that

$$z = \sum_{i=1}^{n} x_i \otimes y_i. \tag{2.11.1}$$

Notice that the previous expression is not unique. Our next aim is to define norms in $X \otimes Y$ in two different ways: the largest reasonable one (projective norm) and the smallest reasonable one (injective norm).

2.11.1. Projective tensor product. Now, we want to define a norm on $X \otimes Y$. What should we require from such a norm? It is natural to require that, given x and y,

$$\|x \otimes y\| \leqslant \|x\|\|y\|.$$

Now, the triangle inequality implies that $\|z\| \leqslant \sum_{i=1}^n \|x_i\|\|y_i\|$. Since this must hold for every representation of z, we get that

$$\|z\| \leqslant \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| \colon z = \sum_{i=1}^n x_i \otimes y_i\right\},$$

where the previous infimum runs over all the possible representations of z described by (2.11.1). Now, if we take the biggest possible norm satisfying the previous condition, we obtain the definition of the projective tensor product.

Definition 2.11.1. Given two Banach spaces X and Y, we define the *projective norm* of an element $z \in X \otimes Y$ as

$$\|z\|_\pi := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| \colon z = \sum_{i=1}^n x_i \otimes y_i\right\},$$

where the previous infimum runs over all the possible representations of z. Now, the *projective tensor product of X and Y*, denoted by $X \widehat{\otimes}_\pi Y$, is defined as the completion of the normed space $(X \otimes Y, \|\cdot\|_\pi)$.

In the following result we will summarise the first properties of the projective tensor product.

Proposition 2.11.2. *Let X and Y be two Banach spaces.*

(1) $\|x \otimes y\|_\pi = \|x\|\|y\|$ *holds for every $x \in X$ and every $y \in Y$.*
(2) $B_{X\widehat{\otimes}_\pi Y} = \overline{\mathrm{conv}}(B_X \otimes B_Y)$.
(3) *If V, W are Banach spaces and $T \colon X \to V$ and $S \colon Y \to W$ are operators, then there is a unique operator $T \otimes S \colon X \widehat{\otimes}_\pi Y \to V \widehat{\otimes}_\pi W$ so that $(T \otimes S)(x \otimes y) = T(x) \otimes S(y)$ holds for every $x \in X$ and $y \in Y$. Moreover, $\|T \otimes S\| = \|T\|\|S\|$. In addition, if T and S are quotient operators, $T \otimes S$ is also a quotient operator.*

The last assertion in the previous proposition reveals why this space is known as the "projective" tensor product. In contrast with what happens with quotient spaces, projective tensor products do not behave well with respect to subspaces. To be more precise, consider two Banach spaces X and Y and two subspaces $V \subset X$ and $W \subset Y$. Consider $\|\cdot\|_{X,Y}$, the norm of $X \widehat{\otimes}_\pi Y$, and $\|\cdot\|_{V,W}$, the one of $V \widehat{\otimes}_\pi W$. If we consider $z \in V \otimes W$ it is clear that, if we view $z \in X \widehat{\otimes}_\pi Y$, there are more representations of z in $X \otimes Y$ than in $V \otimes W$. Because of the very definition of the projective norms,

$$\|z\|_{X,Y} \leqslant \|z\|_{V,W},$$

and this inequality may be strict (see [**274**, Proposition 2.11]). There is, however, equality when the subspaces satisfy extra assumptions. Indeed, let us show the following result, which slightly generalises the well-known result that projective tensor products respect 1-complemented subspaces isometrically by making use of the concept of ideal introduced in Definition 2.9.11. We include a proof for the sake of completeness.

PROPOSITION 2.11.3. *Let X and Y be two Banach spaces and let $V \subset X$ and $W \subset Y$ be two subspaces which are ideals. Then, $V \widehat{\otimes}_\pi W$ is an isometric subspace of $X \widehat{\otimes}_\pi Y$.*

PROOF. Take $z = \sum_{i=1}^k v_i \otimes w_i \in V \otimes W$, and let us prove that $\|z\|_{V,W} = \|z\|_{X,Y}$. Of course, the inequality $\geqslant$ is clear. For the reverse one, pick $\varepsilon > 0$ and take a representation $z = \sum_{i=1}^n x_i \otimes y_i$ so that $\sum_{i=1}^n \|x_i\|\|y_i\| < \|z\|_{X,Y} + \varepsilon$. Define $E := \operatorname{lin}\{v_1, \ldots, v_k, x_1, \ldots, x_n\} \subset X$ and $F := \operatorname{lin}\{w_1, \ldots, w_k, y_1, \ldots, y_n\}$. Let T: $E \to V$ (respectively S: $F \to W$) be an operator of norm $\leqslant 1 + \varepsilon$ and so that $T(x) = x$ for all $x \in E \cap V$ (respectively $S(y) = y$ for every $y \in F \cap W$). Define $T \otimes S$: $E \widehat{\otimes}_\pi F \to V \widehat{\otimes}_\pi W$ to be the operator described in Proposition 2.11.2. It is clear that $\|T \otimes S\| \leqslant (1+\varepsilon)^2$. Moreover $z = \sum_{i=1}^k v_i \otimes w_i = \sum_{i=1}^n x_i \otimes y_i$. Now,

$$\begin{aligned}
[T \otimes S](z) &= [T \otimes S]\left(\sum_{i=1}^k v_i \otimes w_i\right) \\
&= \sum_{i=1}^k T(v_i) \otimes T(w_i) = \sum_{i=1}^k v_i \otimes w_i = z
\end{aligned}$$

since T (respectively, S) fixes the elements of $E \cap V$ (respectively, $F \cap Y$). Then, using the norm of the operator $T \otimes S$, we get

$$\begin{aligned}
\|z\|_{V,W} &= \left\|\sum_{i=1}^k v_i \otimes w_i\right\|_{V,W} \\
&= \left\|(T \otimes S)\left(\sum_{i=1}^k v_i \otimes w_i\right)\right\|_{V,W} \leqslant (1+\varepsilon)^2 \left\|\sum_{i=1}^k v_i \otimes w_i\right\|_{E\widehat{\otimes}_\pi F}.
\end{aligned}$$

By the very definition of the projective norm and, using that $\sum_{i=1}^k v_i \otimes w_i = \sum_{i=1}^n x_i \otimes y_i$ in $E \otimes F$, we get that

$$\left\|\sum_{i=1}^k v_i \otimes w_i\right\|_{E\widehat{\otimes}_\pi F} = \left\|\sum_{i=1}^n x_i \otimes y_i\right\|_{E\widehat{\otimes}_\pi F} \leqslant \sum_{i=1}^n \|x_i\|\|y_i\| \leqslant \|z\|_{X,Y} + \varepsilon.$$

Consequently, $\|z\|_{V,W} \leqslant (1+\varepsilon)^2(\|z\|_{X,Y} + \varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we conclude the desired equality of norms. □

Let us now describe the dual of a projective tensor product. Given two Banach spaces X and Y, the space $(X \widehat{\otimes}_\pi Y)^*$ is linearly isometric to $L(X, Y^*)$. In order to exhibit this identification, we associate to $\Phi \in (X \widehat{\otimes}_\pi Y)^*$ an operator $T \in L(X, Y^*)$ by means of

$$T(x)(y) = \Phi(x \otimes y).$$

This provides an isometric isomorphism between $(X \widehat{\otimes}_\pi Y)^*$ and $L(X, Y^*)$. One can likewise identify $(X \widehat{\otimes}_\pi Y)^*$ and $L(Y, X^*)$ via $S(y)(x) = \Phi(x \otimes y)$ or $(X \widehat{\otimes}_\pi Y)^* \cong \operatorname{Bil}(X \times Y)$, the space of bounded bilinear forms, via $\beta(x, y) = \Phi(x \otimes y)$. This justifies another construction of the projective tensor product as a free object which linearises continuous bilinear mappings.

2.11.2. Injective tensor product. In this subsection we will define another tensor product space. Consider two Banach spaces X and Y, and take $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$. Note that z can be seen as a finite-rank operator $z\colon X^* \to Y$ acting as

$$z(x^*) := \sum_{i=1}^n x^*(x_i) y_i.$$

Thus, from this point of view, we can visualise $X \otimes Y = \mathrm{FinRan}(X^*, Y) \subset L(X^*, Y)$. Now, the *injective norm on* $X \otimes Y$ will be the operator norm inherited by the previous inclusion, denoted by $\|\cdot\|_\varepsilon$. Finally, the *injective tensor product*, denoted by $X \widehat{\otimes}_\varepsilon Y$, will be the completion of the normed space $(X \otimes Y, \|\cdot\|_\varepsilon)$.

By the very definition of the injective norm, given an element $z = \sum_{i=1}^n x_i \otimes y_i$, we get that

$$\begin{aligned}
\|z\|_\varepsilon &= \sup\left\{ \left\| \sum_{i=1}^n x^*(x_i) y_i \right\| : x^* \in B_{X^*} \right\} \\
&= \sup\left\{ \left| \sum_{i=1}^n x^*(x_i) y^*(y_i) \right| : x^* \in B_{X^*},\ y^* \in B_{Y^*} \right\} \\
&= \sup\left\{ \left\| \sum_{i=1}^n y^*(y_i) x_i \right\| : y^* \in B_{Y^*} \right\}
\end{aligned}$$

It is clear that, in the previous equalities, we can replace B_{X^*} (respectively B_{Y^*}) with any norming subset $A \subset B_{X^*}$ (respectively $B \subset B_{Y^*}$).

It is not difficult to prove that, in contrast to what happens with the projective tensor product, the injective tensor product always respects subspaces in the following sense: if we consider a subspace V of X and a subspace W of Y, then $V \widehat{\otimes}_\varepsilon W$ is an isometric subspace of $X \widehat{\otimes}_\varepsilon Y$.

This phenomenom allows us to describe the dual of an injective tensor product. Given Banach spaces X and Y, there are linear isometric embeddings $X \hookrightarrow C(B_{X^*})$ and $Y \hookrightarrow C(B_{Y^*})$. Consequently,

$$X \widehat{\otimes}_\varepsilon Y \hookrightarrow C(B_{X^*}) \widehat{\otimes}_\varepsilon C(B_{Y^*}) = C(B_{X^*} \times B_{Y^*})$$

[**274**, Section 3.2]. This implies that the adjoint is a surjective mapping, in fact a quotient mapping, $C(B_{X^*} \times B_{Y^*})^* \to (X \widehat{\otimes}_\varepsilon Y)^*$. Taking into account the description of the dual of $C(B_{X^*} \times B_{Y^*})$ we get the following known result.

Theorem 2.11.4. *Let X and Y be two Banach spaces. A bilinear mapping $B \in \mathrm{Bil}(X \times Y)$ is a continuous linear functional of $X \widehat{\otimes}_\varepsilon Y$ if, and only if, there exists a regular Borel measure μ on $B_{X^*} \times B_{Y^*}$ so that*

$$B(x, y) := \int_{B_{X^*} \times B_{Y^*}} \varphi(x) \phi(y) \, d\mu(\varphi, \psi)$$

for every $x \in X$ and every $y \in Y$. Moreover, we have that

$$\|B\|_{(X \widehat{\otimes}_\varepsilon Y)^*} = \inf \|\mu\|,$$

where the previous infimum runs over all the measures representing B.

For this reason the bilinear forms appearing in Theorem 2.11.4 are called *integral bilinear forms.*

One central topic in tensor product theory is the study of approximation properties. Recall that a Banach space X is said to have the *approximation property* (AP) if there exists a net (T_α) of finite-rank operators on X such that $T_\alpha(x) \to x$ uniformly on compact subsets of X. If the previous net satisfies $\sup_\alpha \|T_\alpha\| < \infty$, we say that X has the *bounded approximation property* (BAP). More precisely, if there exists $\lambda \geqslant 1$ so that $\|T_\alpha\| \leqslant \lambda$ holds for every α, we will say that X has the λ-*bounded approximation property* (λ-BAP). Finally, we will say that X has the *metric approximation property* (MAP) if X has the 1-BAP.

One application of the approximation property is the following result, which is due to Grothendieck.

Proposition 2.11.5. *Let X and Y be two Banach spaces. Assume that X^* or Y has the approximation property. Then, $K(X,Y) = X^* \widehat{\otimes}_\varepsilon Y$, i.e., every compact operator from X to Y can be approximated by finite-rank operators with respect to the operator norm.*

Another application of the AP for our purposes is based on the following result.

Theorem 2.11.6. *Let X and Y be two Banach spaces. Assume that X^* or Y^* has the RNP. If X^* or Y^* has the AP, then $(X \widehat{\otimes}_\varepsilon Y)^* = X^* \widehat{\otimes}_\pi Y^*$.*

One of the reasons why the previous result is interesting is that, under the assumptions of Theorem 2.11.6, we can completely describe $(X \widehat{\otimes}_\varepsilon Y)^{**}$. Indeed,

$$(X \widehat{\otimes}_\varepsilon Y)^{**} = (X^* \widehat{\otimes}_\pi Y^*)^* = L(X^*, Y^{**}).$$

2.12. Notes and remarks

Most of the results in this chapter belong to the core of modern Banach space theory, and specific references can be found in the text above.

The results of Section 2.8 are more recent. They are presented in the same way as in [**29**] (in that paper, the ℓ_1-type sequences went by the name of L-orthogonal sequences). Example 2.8.4 is extended in [**29**, Theorem 7.4], where it is proved that the following assertion

- There exists a Banach space X and an ℓ_1-type sequence (x_n) so that no w^*-cluster point is L-orthogonal.

is consistent with ZFC theory.

The techniques behind Theorem 2.8.10 and its preliminary results are based on the results of [**229**, Section 1], where more general versions of them are obtained as a tool to prove the following celebrated result: a separable Banach space X contains an isomorphic copy of ℓ_1 if, and only if, there exists an element $u \in X^{**} \setminus \{0\}$ so that

$$\|x + u\| = \|x - u\|$$

holds for every $x \in X$.

Let us finally observe that non-separable versions of Theorem 2.8.10 have been obtained in [**29**]. For instance, it is proved there that if X is a Banach space with an ℓ_1-type sequence (x_n) and whose density character $\text{dens}(X)$ is at most the *pseudointersection number* $\mathfrak{p}$, then there are L-orthogonal elements u in $\{x_n\colon n \in \mathbb{N}\}'$, the set of weak* cluster points of (x_n) [**29**, Theorem 4.1]. Moreover, it is proved in [**29**, Theorem 5.3] that the assertion

- Given a Banach space X and an ℓ_1-type sequence (x_n) in X there exists $u \in \{x_n\colon n \in \mathbb{N}\}'$ which is an L-orthogonal element.

is consistent with ZFC theory (for instance, this occurs under the Continuum Hypothesis).

We further remark that, taking into account the description of the dual of a projective tensor product, Proposition 2.11.3 can be deduced from [**181**, Proposition 4.2]. A stronger version appeared in [**258**, Theorem 1], from where it can even be obtained that $X \widehat{\otimes}_\pi Z$ is an ideal in $X \widehat{\otimes}_\pi Y$ if Z is an ideal in Y.

CHAPTER 3

The Daugavet property

We now formally introduce the Daugavet property of a Banach space by requiring that all rank-one operators satisfy the Daugavet equation. The Daugavet property will be characterised in terms of slices of the unit ball; this leads to transfer theorems that assert the validity of the Daugavet equation for much larger classes of operators, e.g., the weakly compact ones. In this way we obtain a new approach to the results of Chapter 1, and we discuss many new examples. Also, we obtain structural properties of Banach spaces with the Daugavet property: they contain subspaces isomorphic to ℓ_1, but fail the RNP.

3.1. Definition and basic reformulations in terms of slices

Recall that the rank of a linear operator is the dimension of its range. An operator $T \in L(X, E)$ is of rank one if and only if there are $e \in E$ and $x^* \in X^*$, both nonzero, such that $Tx = x^*(x)e$ for all $x \in X$. The standard abbreviation for the above operator is $T = x^* \otimes e$. Remark that $\|x^* \otimes e\| = \|x^*\| \cdot \|e\|$.

DEFINITION 3.1.1. A Banach space X has *the Daugavet property* ($X \in \mathrm{DPr}$ for short) if for every rank-one operator $T \in L(X)$ the following *Daugavet equation* holds true:

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{3.1.1}$$

REMARK 3.1.2. *The Daugavet equation means that* Id *and* T *are quasi-codirected elements of* $L(X)$*, so all properties of quasi-codirected elements listed at the beginning of Section 2.6 are applicable. In particular, if* T *satisfies* (3.1.1)*, then the same equation holds true for* aT *with* $a > 0$*. Consequently, it is sufficient to consider in the above Definition 3.1.1 only operators of norm 1. So,* $X \in \mathrm{DPr}$ *if and only if every* $T \in L(X)$ *of rank one with* $\|T\| = 1$ *satisfies the identity*

$$\|\mathrm{Id} + T\| = 2. \tag{3.1.2}$$

Finally, the condition $\|\mathrm{Id} + T\| \leqslant 2$ *follows from the triangle inequality; consequently, in order to verify* (3.1.2) *it is enough to check that* $\|\mathrm{Id} + T\| \geqslant 2$.

The following lemma allows us to reformulate the Daugavet equation and some of its future generalisations in terms of the geometry of slices.

LEMMA 3.1.3. *Let* Z, E *be Banach spaces,* $G \in S_{L(Z,E)}$*, and consider elements* $z^* \in S_{Z^*}$ *and* $e \in S_E$*. Then the following assertions are equivalent:*

(i) $\|G + z^* \otimes e\| = 2$.

(ii) *For all* $\delta, \varepsilon > 0$ *there is* $y \in \mathrm{Slice}(B_Z, z^*, \delta)$ *such that*

$$\|Gy + e\| \geqslant 2 - \varepsilon. \tag{3.1.3}$$

Proof. (i) $\Rightarrow$ (ii). Let $\varepsilon' = \min\{\delta, \varepsilon\}$. The definition of the operator norm gives us the existence of $y_0 \in S_Z$ such that $\|(G + z^* \otimes e)y_0\| > 2 - \varepsilon'/2$, i.e.,

$$\|Gy_0 + z^*(y_0)e\| > 2 - \varepsilon'/2. \tag{3.1.4}$$

Since $\|Gy_0\| \leqslant 1$ and $\|e\| = 1$, we deduce from (3.1.4) that $|z^*(y_0)| > 1 - \varepsilon'/2$. Choose $\theta \in \mathbb{K}$ with $|\theta| = 1$ in such a way that $z^*(\theta y_0) = |z^*(y_0)|$, and put $y := \theta y_0$. Then, $\operatorname{Re} z^*(y) = z^*(y) = |z^*(y_0)| > 1-\varepsilon'/2 > 1-\delta$. In particular, $|1-z^*(y)| < \varepsilon'/2$ and

$$\|Gy + e\| = \|Gy + z^*(y)e + (1 - z^*(y))e\| > \|Gy + z^*(y)e\| - \varepsilon'/2 > 2 - \varepsilon$$

(in the last step we used (3.1.4) once more).

(ii) $\Rightarrow$ (i). Due to (ii), for every $\varepsilon > 0$ there is $y \in \operatorname{Slice}(B_Z, z^*, \varepsilon)$ satisfying (3.1.3). Then $|z^*(y)| \leqslant \|y\| \leqslant 1$ and $\operatorname{Re} z^*(y) \geqslant 1 - \varepsilon$, consequently we have $|1 - z^*(y)| \leqslant \sqrt{2\varepsilon}$. Finally,

$$\|G + z^* \otimes e\| \geqslant \|Gy + z^*(y)e\| \geqslant 2 - \varepsilon - |1 - z^*(y)| \geqslant 2 - \varepsilon - \sqrt{2\varepsilon},$$

which, by the arbitrariness of ε, means that $\|G + z^* \otimes e\| = 2$. □

The proof shows that it is sufficient to consider $\delta = \varepsilon$ in (ii).

Remark 3.1.4. In the proof of (i) $\Rightarrow$ (ii) above we obtained some $y \in \operatorname{Slice}(B_Z, z^*, \delta)$ that, in addition to the properties listed in (ii), satisfies $\|y\| = 1$. This gives us one more equivalent reformulation (the point is that y is in the unit sphere):

(iii) *For all $\delta, \varepsilon > 0$ there is $y \in \operatorname{Slice}(S_Z, z^*, \delta)$ such that (3.1.3) holds true.*

The next reformulation formalises the connection of the Daugavet property with the geometry of slices of the unit ball, which is the cornerstone of the whole theory. (Below it is understood that $\|x^*\| = 1$ and $\delta > 0$.)

Theorem 3.1.5. *For a Banach space X the following assertions are equivalent:*

(i) *X has the Daugavet property.*

(ii) *For every $x \in S_X$, for every $\varepsilon > 0$, and for every slice $S = \operatorname{Slice}(B_X, x^*, \delta)$ of B_X there is some $y \in S$ such that*

$$\|x + y\| > 2 - \varepsilon. \tag{3.1.5}$$

(iii) *For every $x \in S_X$, for every $\varepsilon > 0$, and for every slice $S = \operatorname{Slice}(B_X, x^*, \delta)$ of B_X there is some $y \in S$ such that*

$$\|x - y\| > 2 - \varepsilon. \tag{3.1.6}$$

Proof. For (i) $\Rightarrow$ (ii) it is sufficient to remark that $T := x^* \otimes x$ is a rank-one operator, so the Daugavet property of X gives us $\|\mathrm{Id} + x^* \otimes x\| = 2$. It remains to apply Lemma 3.1.3 with $Z = E = X$, $G = \mathrm{Id}$, $z^* = x^*$, and $e = x$.

For the converse implication (ii) $\Rightarrow$ (i), according to Remark 3.1.2, for every rank-one $T \in L(X)$ with $\|T\| = 1$ we have to verify (3.1.2). This T is of the form $T = x^* \otimes x$, where $x \in X$, $x^* \in X^*$. Since $\|T\| = 1$, the representation $T = x^* \otimes x$ can be taken in such a way that $x \in S_X$ and $x^* \in S_{X^*}$. So, again it remains to apply Lemma 3.1.3.

That (ii) and (iii) are equivalent can be seen by replacing x with $-x$. □

We remark that (iii) implies that in a Banach space with the Daugavet property, every slice of the unit ball has diameter 2 (just take $x \in S$); see Theorem 3.2.1 and Section 12.2 for more on this circle of ideas.

Recall that every normed space X over the field $\mathbb{C}$ is at the same time a normed space over the field $\mathbb{R}$, which is denoted $X_{\mathbb{R}}$. There is a bijective correspondence between $(X^*)_{\mathbb{R}}$ and $(X_{\mathbb{R}})^*$: each complex linear functional corresponds to its real part [**156**, Subsection 9.1.1]. Since the definition of slices deals with real parts of linear functionals, that is, with real continuous functionals, Theorem 3.1.5 leads to the following corollary.

Corollary 3.1.6. *A complex Banach space X has the Daugavet property if and only if $X_{\mathbb{R}} \in$ DPr.*

The geometric property of an individual point x that appears in (ii) of Theorem 3.1.5 is extracted for future reference in the definition below.

Definition 3.1.7. Let X be a Banach space. An element $x \in S_X$ is said to be a *Daugavet point* if for every $\varepsilon > 0$ and every slice S of B_X there is some $y \in S$ such that $\|x + y\| \geqslant 2 - \varepsilon$.

In view of Lemma 3.1.3, $x \in S_X$ is a Daugavet point if and only if

$$\|\mathrm{Id} + x^* \otimes x\| = 2$$

for every $x^* \in S_{X^*}$.

In this terminology Theorem 3.1.5 can be reformulated as follows:

Proposition 3.1.8. *$X \in$ DPr if and only if every point of S_X is a Daugavet point.*

Let us list some easy but useful reformulations; see Definition 2.6.3 for the notion of ε-quasi-codirected vectors. We introduce the following notation which will be used all over the section: given $x \in S_X$ and $\varepsilon > 0$, $Q(x, \varepsilon)$ is the set of those $y \in S_X$ that are ε-quasi-codirected with x, that is,

$$Q(x, \varepsilon) = \{y \in S_X \colon \|x + y\| > 2 - \varepsilon\}.$$

Lemma 3.1.9. *Let X be a Banach space. For an element $x \in S_X$ the following assertions are equivalent:*

(i) *x is a Daugavet point;*
(ii) *for every $\varepsilon > 0$ and every slice S of B_X, there is $y \in S$ such that $\|x+y\| \geqslant 2-\varepsilon$ and $\|y\| = 1$;*
(iii) *for every $\varepsilon > 0$ the set $Q(x, \varepsilon)$ intersects all slices of B_X;*
(iv) *$\overline{\mathrm{conv}}\big(Q(x, \varepsilon)\big) = B_X$ for every $\varepsilon > 0$;*
(v) *for every $\varepsilon > 0$ and every slice S of B_X, there is $y \in S$ such that $\|x-y\| \geqslant 2-\varepsilon$ and $\|y\| = 1$;*
(vi) *for every $\varepsilon \in (0, 1)$ the set $S_X \setminus (x + (2 - \varepsilon)B_X)$ intersects all slices of B_X;*
(vii) *$\overline{\mathrm{conv}}\big(S_X \setminus (x + (2 - \varepsilon)B_X)\big) = B_X$ for every $\varepsilon \in (0, 1)$.*

Proof. (i) $\Leftrightarrow$ (ii) follows from the symmetry (just change x^* to $-x^*$ in the definition). (iii) $\Rightarrow$ (ii) is evident, and the converse implication (ii) $\Rightarrow$ (iii) is contained in Remark 3.1.4.

(iv) is just a rephrasing of (iii); and the equivalence (iv) $\Leftrightarrow$ (v) follows from the Hahn-Banach separation theorem, which was already remarked in a more general setting in Lemma 2.6.7.

Finally, (vi), (vii) and (viii) are symmetric reformulations of (iii), (iv) and (v), respectively (sometimes it is more convenient to think about elements at almost maximal possible distance from x than about ε-quasi-codirected ones). $\square$

The following lemma allows one to iterate Theorem 3.1.5: there is not only one point y satisfying (3.1.5), but this is satisfied on a whole subslice.

LEMMA 3.1.10. *Let $x \in S_X$ be a Daugavet point. Then, for every slice S of B_X and every $\delta \in (0,1)$ there is another slice $\tilde{S}$ of B_X such that $\tilde{S} \subset S$ and $\|x+z\| > 2-\delta$ for all $z \in \tilde{S}$.*

PROOF. Without loss of generality we may assume that S is of the form $S = \mathrm{Slice}(B_X, x_0^*, \varepsilon_0)$ with $x_0^* \in S_{X^*}$ and $\varepsilon_0 \in (0,1)$. Consider $\varepsilon \in (0, \min\{\delta/3, \varepsilon_0/2\})$. Applying (iii) of the above Lemma to the smaller slice $\mathrm{Slice}(B_X, x_0^*, \varepsilon)$, we obtain $x_0 \in S_X$ with $\mathrm{Re}\, x_0^*(x_0) > 1-\varepsilon$ and $\|x_0 + x\| > 2-\varepsilon$. Then x_0 and x are ε-quasi-codirected, so Lemma 2.6.8 gives us a functional $x^* \in S_{X^*}$ such that both x_0 and x belong to $\mathrm{Slice}(B_X, x^*, \varepsilon)$. Take as $\tilde{S}$ the slice

$$\tilde{S} = \{z \in B_X \colon \mathrm{Re}(x_0^*(z) + x^*(z)) > 2 - 2\varepsilon\}.$$

Since $\tilde{S}$ is an intersection of a half-space with the unit ball and is not empty (as $x_0 \in \tilde{S}$), we didn't cheat the reader saying that $\tilde{S}$ is a slice. Let us check that $\tilde{S} \subset S$. Indeed, for every $z \in \tilde{S}$

$$\mathrm{Re}\, x_0^*(z) > 2 - 2\varepsilon - \mathrm{Re}\, x^*(z) > 1 - 2\varepsilon > 1 - \varepsilon_0.$$

It remains to check that all $z \in \tilde{S}$ satisfy $\|x+z\| \geqslant 2-\delta$. Indeed, for $z \in \tilde{S}$

$$\mathrm{Re}\, x^*(z) = \mathrm{Re}(x_0^*(z) + x^*(z)) - \mathrm{Re}(x_0^*(z)) > 2 - 2\varepsilon - 1 = 1 - 2\varepsilon,$$

and consequently $\|x+z\| \geqslant \mathrm{Re}\, x^*(x+z) > 2 - 3\varepsilon \geqslant 2-\delta$. $\square$

Putting together the above results we obtain the following list of characterisations of the Daugavet property. Only (ii)* and (x)* are new. The first assertion is clearly equivalent to the fact that $\|\mathrm{Id}_{X^*} + x \otimes x^*\|_{X^*} = 2$ for all $x \in S_X$ and $x^* \in S_{X^*}$. To prove (x)*, we may repeat the proof of Lemma 3.1.10 using weak*-slices and replacing Lemma 3.1.9(iii) for the corresponding weak-star assertion using (ii)*.

THEOREM 3.1.11. *For a Banach space X the following assertions are equivalent:*

(i) $X \in \mathrm{DPr}$.

(ii) *For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon \in (0,1)$, there is $y \in B_X$ such that $\mathrm{Re}\, x^*(y) \geqslant 1-\varepsilon$ and $\|x+y\| \geqslant 2-\varepsilon$.*

(ii)* *For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon \in (0,1)$, there is $y^* \in B_{X^*}$ such that $\mathrm{Re}\, y^*(x) \geqslant 1-\varepsilon$ and $\|x^*+y^*\| \geqslant 2-\varepsilon$.*

(iii) *Every point of S_X is a Daugavet point.*

(iv) *For every $x \in S_X$, every $\varepsilon > 0$, and every slice S of B_X, there is $y \in S$ such that $\|x+y\| \geqslant 2-\varepsilon$.*

(v) *For every $x \in S_X$, every $\varepsilon > 0$, and every slice S of B_X, there is $y \in S$ such that $\|x+y\| \geqslant 2-\varepsilon$ and $\|y\| = 1$.*

(vi) *For every $x \in S_X$ and every $\varepsilon > 0$, the set $Q(x,\varepsilon)$ of those $y \in S_X$ that are ε-quasi-codirected with x intersects all slices of B_X.*

(vii) *For every* $x \in S_X$ *and every* $\varepsilon > 0$*, the condition* $\overline{\mathrm{conv}}\big(Q(x,\varepsilon)\big) = B_X$ *holds true.*

(viii) *For every* $x \in S_X$ *and every* $\varepsilon > 0$*, the set* $Q(-x,\varepsilon) = S_X \setminus (x + (2-\varepsilon)B_X)$ *intersects all slices of* B_X*.*

(ix) *For every* $x \in S_X$ *and every* $\varepsilon > 0$*, the condition*

$$\overline{\mathrm{conv}}\big(S_X \setminus (x + (2-\varepsilon)B_X)\big) = B_X$$

holds true.

(x) *For every* $x \in S_X$*, every slice* S *of* B_X *and every* $\delta \in (0,1)$*, there is another slice* $\widetilde{S}$ *of* B_X *such that* $\widetilde{S} \subset S$ *and* $\|x+z\| \geqslant 2-\delta$ *for all* $z \in \widetilde{S}$*.*

(x)* *For every* $x^* \in S_{X^*}$*, every weak-star slice* S^* *of* B_{X^*}*, and every* $\delta \in (0,1)$*, there is another weak-star slice* $\widetilde{S}^*$ *of* B_{X^*} *such that* $\widetilde{S}^* \subset S^*$ *and* $\|x^*+z^*\| \geqslant 2-\delta$ *for all* $z^* \in \widetilde{S}^*$*.*

Item (x)* will be extended in Section 9.1 to the Daugavet property with respect to a one-norming subspace of its dual.

Remark 3.1.12. *In (ii) of Theorem 3.1.11 the condition* $y \in B_X$ *may be weakened to* $\|y\| < 1+\varepsilon$. Indeed, in this case the condition $\operatorname{Re} x^*(y) \geqslant 1-\varepsilon$ implies that $|\|y\|-1| < \varepsilon$, and the auxiliary element $v = y/\|y\| \in S_X$ will satisfy $\|y-v\| < \varepsilon$. Then $\operatorname{Re} x^*(v) \geqslant 1-2\varepsilon$ and $\|v+x\| \geqslant 2-2\varepsilon$, i.e., v fits into (ii) of Theorem 3.1.11 with 2ε instead of ε, which is fine for us.

Analogously, item (vii) may be substituted by the following weaker condition: for every $x \in S_X$ *and* $\varepsilon > 0$ *the closed convex hull of the set* $l^+(x,\varepsilon) := \{y \in X\colon \|y\| \leqslant 1+\varepsilon,\ \|x+y\| > 2-\varepsilon\}$ *contains* B_X*.*

The following immediate consequence of the above theorem shows the relation of the Daugavet property with the roughness of norms.

Corollary 3.1.13. *Let* X *be a Banach space with the Daugavet property. Then, every weak-star slice of* B_{X^*} *has diameter two. Equivalently, the norm of* X *is extremely rough. In particular, the norm of* X *is not Fréchet differentiable at any point.*

The equivalent reformulation at the end of the corollary is well known (see Lemma 2.10.6).

The next lemma will be the main ingredient in the proof of existence of copies of ℓ_1 in every $X \in \mathrm{DPr}$.

Recall that according to Definition 2.8.5, for a subspace E of a Banach space X and $\varepsilon > 0$, an element $x \in B_X$ is $(\varepsilon,1)$-orthogonal to E if for every $y \in E$ and $t \in \mathbb{K}$

$$\|y+tx\| \geqslant (1-\varepsilon)(\|y\|+|t|). \tag{3.1.7}$$

Lemma 3.1.14. *If* $X \in \mathrm{DPr}$ *then, for every finite-dimensional subspace* $X_0 \subset X$ *and for every* $\varepsilon > 0$*, each slice* S *of* B_X *contains a smaller slice* $\tilde{S} \subset S$ *of* B_X *such that every* $x \in \tilde{S} \cap S_X$ *is* $(\varepsilon,1)$*-orthogonal to* X_0*.*

Proof. For $\delta = \varepsilon/3$ we may select a finite δ-net $\{v_1,\dots,v_n\} \subset S_{X_0}$ of the unit sphere S_{X_0}. Applying repeatedly item (x) of Theorem 3.1.11, we obtain a decreasing chain $S = S_0 \supset S_1 \supset \ldots \supset S_n$ of slices of B_X such that

$$\|v_k+z\| \geqslant 2-\delta \tag{3.1.8}$$

for all $z \in S_k$. Take $\tilde{S} := S_n$; then (3.1.8) will hold true for all $x \in \tilde{S}$ and all $k = 1, \dots, n$. It remains to apply Lemma 2.8.7. $\square$

The following lemma will be used a number of times in this text, as it allows us to shift our efforts from slices to weakly compact sets, via the technique of convex combinations of slices.

LEMMA 3.1.15 (Shvydkoy's lemma [**285**, Lemma 2.2]). *Let $X \in \mathrm{DPr}$, $U \subset B_X$ be a subset that contains a convex combination of slices of the ball, $x_0 \in S_X$ and $\varepsilon > 0$. Then there is $z \in U$ such that $\|x_0 + z\| > 2 - \varepsilon$. In particular, by Bourgain's lemma 2.6.19, the result is applicable for every relatively weakly open $\emptyset \neq U \subset B_X$.*

PROOF. Let $\sum_{k=1}^m \lambda_k S_k \subset U$ be a convex combination of slices of B_X that exists according to the conditions of the Lemma, where $\lambda_k > 0$, $\sum_{k=1}^m \lambda_k = 1$ and S_k are slices of B_X. Fix $\tilde{\varepsilon} > 0$ small enough in order to have $2(1 - \tilde{\varepsilon})^m > 2 - \varepsilon$. Applying Lemma 3.1.14 to $X_0 := \mathrm{lin}\{x_0\}$ and the slice S_1, we obtain $\tilde{S}_1 \subset S_1$ such that the inequality (3.1.7) holds true for all $y \in X_0$, $x \in \tilde{S}_1 \cap S_X$ and $t \in \mathbb{K}$. Take an arbitrary $x_1 \in \tilde{S}_1$ and denote $X_1 := \mathrm{lin}\{x_0, x_1\}$. Now, we can apply Lemma 3.1.14 once more, this time to X_1 and the slice S_2; we obtain $\tilde{S}_2 \subset S_2$ such that (3.1.7) holds for all $y \in X_1$, $x \in \tilde{S}_2 \cap S_X$ and $t \in \mathbb{K}$. Doing this step-by-step, we obtain a collection of slices $\tilde{S}_k \subset S_k$ and elements $x_k \in \tilde{S}_k$, $k = 1, \dots, m$, such that (3.1.7) holds for all $y \in X_{k-1} = \mathrm{lin}\{x_0, \dots, x_{k-1}\}$, $x \in \tilde{S}_k \cap S_X$ and $t \in \mathbb{K}$.

Let us show that $z = \sum_{k=1}^m \lambda_k x_k \in \sum_{k=1}^m \lambda_k S_k \subset U$ is the promised element. Indeed, by construction, for every $j = 1, 2, \dots, m$ we have

$$\begin{aligned}\left\| x_0 + \sum_{k=1}^{j} \lambda_k x_k \right\| &= \left\| x_0 + \sum_{k=1}^{j-1} \lambda_k x_k + \lambda_j x_j \right\| \\ &\geqslant (1 - \tilde{\varepsilon}) \left(\left\| x_0 + \sum_{k=1}^{j-1} \lambda_k x_k \right\| + |\lambda_j| \right).\end{aligned}$$

Putting these m inequalities in one chain we get the desired one:

$$\|x_0 + z\| \geqslant (1 - \tilde{\varepsilon})^m \left(1 + \sum_{k=1}^{m} \lambda_k \right) = 2(1 - \tilde{\varepsilon})^m > 2 - \varepsilon. \qquad \square$$

An immediate consequence of Shvydkoy's lemma is the weak denseness of $Q(x_0, \varepsilon)$, strengthening assertion (vii) of Theorem 3.1.11.

COROLLARY 3.1.16. *Let $X \in \mathrm{DPr}$. Then, for every $x_0 \in S_X$ and every $\varepsilon > 0$, the set*

$$Q(x_0, \varepsilon) = \{y \in S_X \colon \|x_0 + y\| > 2 - \varepsilon\}$$

is weakly dense in B_X.

Going to the bidual space, as a consequence of the Baire category theorem, we may get the following two interesting corollaries.

COROLLARY 3.1.17. *Let X be a Banach space with the Daugavet property. Then, for every $x_0 \in S_X$ and every $\varepsilon > 0$, the set*

$$Q^{**}(x_0, \varepsilon) := \{x^{**} \in B_{X^{**}} \colon \|x_0 + x^{**}\| > 2 - \varepsilon\}$$

is weak-star relatively open and weak-star dense in $B_{X^{**}}$. As a consequence, given $x_0 \in S_X$, the set

$$\{x^{**} \in B_{X^{**}} \colon \|x_0 + x^{**}\| = 2\}$$

is weak-star G_δ dense.

PROOF. The weak-star openness of the sets $Q^{**}(x, \varepsilon)$ is consequence of the weak-star lower semicontinuity of the norm of X^{**}. The weak-star denseness is a consequence of Corollary 3.1.16 and Goldstine's theorem. Now, the Baire category theorem implies that

$$\{x^{**} \in B_{X^{**}}\colon \|x_0 + x^{**}\| = 2\} = \bigcap_{n\in\mathbb{N}} Q^{**}(x_0, 1/n)$$

is weak-star dense in $B_{X^{**}}$, and it is clearly a G_δ subset. □

For separable Banach spaces X, the result can even be strengthened.

COROLLARY 3.1.18. *Let X be a separable Banach space with the Daugavet property. Then, there is a weak-star G_δ dense subset $\mathcal{A}$ of $B_{X^{**}}$ such that*

$$\|x + x^{**}\| = \|x\| + 1$$

for every $x \in X$ and every $x^{**} \in \mathcal{A}$.

PROOF. By separability, consider a set $\{x_n\colon n \in \mathbb{N}\}$ which is norm dense in S_X. For each $n \in \mathbb{N}$, the set

$$\mathcal{A}_n := \{x^{**} \in B_{X^{**}}\colon \|x_n + x^{**}\| = 2\}$$

is weak-star G_δ dense in $B_{X^{**}}$ by Corollary 3.1.17. Applying the Baire category theorem once again, we get that $\mathcal{A} := \bigcap_{n\in\mathbb{N}} \mathcal{A}_n$ is also weak-star dense in $B_{X^{**}}$, and it is clearly G_δ. But it readily follows that $\|x + x^{**}\| = 2$ for every $x \in S_X$ and every $x^{**} \in \mathcal{A}$. □

We will provide a sharper version of this result in Section 4.3.

It is immediate from the weak-star density of B_X in $B_{X^{**}}$ and the weak-star lower semicontinuity of the bidual norm that Corollary 3.1.17 actually characterises the Daugavet property. Actually, the following result provides a weaker property which is sufficient to get Daugavet points.

LEMMA 3.1.19. *Let X be a Banach space and let $x \in S_X$. Then, x is a Daugavet point provided that for every $y \in B_X$ and every $\varepsilon > 0$, there is a net $(x^{**}_\lambda)_{\lambda\in\Lambda}$ in $B_{X^{**}}$ weak-star converging to y and satisfying that*

$$\limsup \|x + x^{**}_\lambda\| \geqslant 2 - \varepsilon.$$

3.2. The very first circle of surprising results

We first note that, although the Daugavet property is of isometric nature, it has strong isomorphic consequences.

THEOREM 3.2.1. *Let $X \in$ DPr, then every convex combination of slices of the unit ball of X has diameter 2. In particular, every relatively weak open subset of B_X and every slice of B_X has diameter 2.*

Consequently, X fails the Radon-Nikodým property; indeed, X is not even strongly regular and so fails the CPCP as well.

Strong regularity and the convex point of continuity property CPCP are generalisations of the RNP; one can find the definitions on page 283 below.

PROOF. Let C be a convex combination of slices of B_X. A first application of Lemma 3.1.15 shows that C contains elements of norm arbitrarily closed to 1. Fix $\varepsilon > 0$ and consider $x \in C$ with $\|x\| > 1 - \varepsilon$. Applying Lemma 3.1.15 again, there is $u \in C$ such that $\|x - u\| > 2 - \varepsilon$, hence the diameter of C is greater than $2 - \varepsilon$. The arbitrariness of $\varepsilon > 0$ provides the result. As every slice is also a convex combination of slices and every relatively weakly open subset contains a convex combination of slices by Bourgain's lemma (see 2.6.19), the two particular cases also follow. □

Most particularly, this theorem implies that every space with the Daugavet property is infinite-dimensional and that reflexive spaces do not possess the Daugavet property.

The next theorem is the very first sample result about the relationship between the Daugavet property and its connection with ℓ_1-subspaces. It implies, in particular, that the dual space of a separable space $X \in \mathrm{DPr}$ must be non-separable. We refer to Definition 2.8.1 for the notion of an ℓ_1-type sequence.

THEOREM 3.2.2. *Every separable $X \in \mathrm{DPr}$ contains an ℓ_1-type sequence, so it has a subspace isomorphic to ℓ_1.*

PROOF. Lemma 3.1.14 implies that $Z = X$ and $A = S_X$ satisfy the conditions of Lemma 2.8.3. Consequently, S_X contains an ℓ_1-type sequence. Then, Lemma 2.8.9 gives us a subsequence equivalent to the canonical basis of ℓ_1. □

Let us remark that it will be demonstrated in Theorem 4.1.7 that every $X \in \mathrm{DPr}$ contains a separable subspace Y with the Daugavet property. This Y has ℓ_1-subspaces, which means that the presence of ℓ_1-subspaces in all spaces with the Daugavet property remains true without the separability assumption:

THEOREM 3.2.3. *Every $X \in \mathrm{DPr}$ has a subspace isomorphic to ℓ_1.*

Let us mention that this result can be proved directly, without relying on the separable determination of the Daugavet property (as it is done in [**178**, Theorem 2.9]). Using Lemma 3.1.14 inductively, it is easy to construct a sequence of vectors $(e_n)_{n\in\mathbb{N}}$ and a sequence of slices $S_n := \mathrm{Slice}(B_X, x_n^*, 4^{-n})$ with $x_n^* \in S_{X^*}$ ($n \in \mathbb{N}$), satisfying that for every $n \in \mathbb{N}$, $e_n \in S_n$ and that every element of $S_{n+1} \cap S_X$ is $(4^{-n}, 1)$-orthogonal to $\mathrm{lin}\{e_1, \ldots, e_n\}$. The sequence (e_n) is then equivalent to the unit vector basis in ℓ_1.

Let us now provide a related result: the dual of a Banach space with the Daugavet property contains *isometric* copies of ℓ_1. We first need a version of Lemma 3.1.14 valid in the dual of a Banach space with the Daugavet property, which is of interest by itself.

LEMMA 3.2.4. *Let X be a Banach space with the Daugavet property, V a separable subspace of X^*, and S^* a weak-star slice of B_{X^*}. Then, there is an element $x^* \in S^* \cap S_{X^*}$ which is quasi-codirected to all the elements of V, that is, the equality*

$$\|x^* + v^*\| = 1 + \|v^*\|$$

holds for all $v^* \in V$.

PROOF. Take a dense sequence (v_n^*) in S_V and use repeatedly assertion (x)* of Theorem 3.1.11 to get a sequence (S_n^*) of weak-star slices of B_{X^*}, which we can consider weak-star closed and hence weak-star compact, satisfying that

$$\|x^* + v_k^*\| \geqslant 2 - \frac{1}{n}$$

for all $x^* \in S_n^*$ and $k = 1, \dots, n$. Clearly, any $x^* \in \bigcap_{n \in \mathbb{N}} S_n^*$ (which is non-empty by weak-star compactness) works. $\square$

Now, an isometric copy of ℓ_1 can be produced in the dual of a Banach space with the Daugavet property by an obvious inductive procedure (similar to the one that we indicate in the paragraph after Theorem 3.2.3). We state the result for future reference.

Corollary 3.2.5. *Let X be a Banach space with the Daugavet property. Then X^* contains an isometric copy of ℓ_1. In particular, X^* is neither strictly convex nor smooth.*

The next result reveals the most surprising feature of the Daugavet property. Although the definition requires the validity of the Daugavet equation only for operators of rank one, this equation is then automatically satisfied by all compact operators (as in the original Daugavet theorem) and many other operators as well. We will return to such kind of results many times at proper instances, but now we are ready for the very first sample result of this type. Recall that the strong Radon-Nikodým property of operators and corresponding geometric properties of sets were addressed in Section 2.7 (see Theorem 2.7.12 and Definition 2.7.15).

Theorem 3.2.6. *Let $X \in \mathrm{DPr}$. Then, every strong Radon-Nikodým operator $T \in L(X)$ (in particular, every compact or weakly compact operator) satisfies the Daugavet equation.*

Proof. As we have already remarked, it is sufficient to consider the case of $\|T\| = 1$. According to the definition of strong Radon-Nikodým operators, $K := \overline{T(B_X)}$ possesses the Radon-Nikodým property. Consequently, K is equal to the closed convex hull of the set $\mathrm{dent}(K)$ of its denting points. Hence, for every $\varepsilon > 0$ there is $y_0 \in \mathrm{dent}\, K$ such that

$$\|y_0\| > \sup\{\|y\| \colon y \in K\} - \varepsilon = 1 - \varepsilon.$$

By the definition of denting point, there is a slice S of K such that $y_0 \in S$ and $\mathrm{diam}(S) < \varepsilon$.

By Proposition 2.6.5, the set $\widetilde{S} := T^{-1}(S) \cap B_X$ is a slice of B_X. Then, for every $x \in \widetilde{S}$, we have $Tx \in S$ and, consequently, $\|Tx - y_0\| \leqslant \varepsilon$.

Applying (iv) of Theorem 3.1.11 to $x = \frac{y_0}{\|y_0\|} \in S_X$ and to the slice $\widetilde{S}$, we obtain some $x_0 \in \widetilde{S}$ such that

$$\left\| x_0 + \frac{y_0}{\|y_0\|} \right\| \geqslant 2 - \varepsilon.$$

Then $\|x_0 + y_0\| \geqslant 2 - \delta - \varepsilon > 2 - 2\varepsilon$, and we have that

$$\|\mathrm{Id} + T\| \geqslant \|x_0 + Tx_0\| \geqslant \|x_0 + y_0\| - \|Tx_0 - y_0\| > 2 - 3\varepsilon.$$

It follows that $\|\mathrm{Id} + T\| = 2$, since $\varepsilon > 0$ was arbitrary. $\square$

3.3. Basic examples. The Daugavet property and duality

As we have already mentioned in the Preface and proved in Chapter 1, two basic examples of spaces with the Daugavet property are $C[0,1]$ and $L_1[0,1]$. Below we shall prove these results in a somewhat more general form using the methods from the present chapter.

We start with spaces of continuous functions. Given a locally compact Hausdorff topological space L, we write $C_0(L)$ for the Banach space of those continuous functions from L to $\mathbb{K}$ vanishing at infinity (i.e., they are uniform limits of continuous functions with compact support). Observe that when L is actually compact, $C_0(L) \cong C(L)$. Recall that a Hausdorff topological space is *perfect* if it does not contain isolated points.

Theorem 3.3.1. *Let L be a locally compact Hausdorff topological space.*

(a) *If L is a perfect, then $C_0(L) \in \mathrm{DPr}$.*

(b) *If L has an isolated point, then $C_0(L)$ does not have the Daugavet property.*

Proof. (a) Let $x \in S_{C_0(L)}$ and $x^* \in S_{C_0(L)^*}$. Then there is a regular Borel (signed or complex) measure μ on L such that $x^*(z) = \int_L z\,d\mu$ for all $z \in C_0(L)$. Let $t_0 \in L$ be a point at which $|x(t_0)| = \|x\| = 1$. For $\varepsilon > 0$, find a non-empty open set $B \subset \{t \in L\colon |x(t_0) - x(t)| < \varepsilon/2\}$ with $|\mu|(B) < \varepsilon/4$. (Indeed, by the absence of isolated points and the Hausdorff condition, there is a disjoint sequence of non-void open subsets B_n of $\{t \in L\colon |x(t_0) - x(t)| < \varepsilon/2\}$. By the countable additivity of the measure $|\mu|$, $\lim_{n\to\infty} |\mu|(B_n) = 0$, so B_n with n large enough can be taken as B.)

Take $z \in C_0(L)$ such that $\operatorname{Re} x^*(z) = \operatorname{Re} \int_L z\,d\mu > 1 - \varepsilon/4$, select $t_1 \in B$ and define $y \in B_{C_0(L)}$ in such a way that $y(t_1) = x(t_1)$ and $y|_{L\setminus B} = z|_{L\setminus B}$ (Tietze extension theorem). Then

$$|x^*(y) - x^*(z)| \leqslant \int_B |y - z|\,d\mu < \varepsilon/2,$$

consequently $\operatorname{Re} x^*(y) > 1 - \varepsilon$. Also,

$$\|x + y\| \geqslant |x(t_1) + y(t_1)| = 2\,|x(t_1)| \geqslant 2\,|x(t_0)| - 2\,|x(t_1) - x(t_0)| > 2 - \varepsilon.$$

By (ii) of Theorem 3.1.11, this completes the proof.

(b) Let $t \in L$ be an isolated point. Then $\mathbb{1}_{\{t\}} \in C_0(L)$. Consider the operator T that maps each $z \in C_0(L)$ to $T(z) = z(t)\mathbb{1}_{\{t\}}$. This T is a rank-one projector with $\|T\| = 1$, but

$$\|\mathrm{Id} - T\| = 1 \neq 1 + \|T\|. \qquad \square$$

It is now time to deal with spaces of integrable functions. Recall that an *atom* of a measure space (Ω, Σ, μ) is a subset $A \in \Sigma$ with $\mu(A) > 0$ such that whenever $A = B \cup (A \setminus B)$ with $B \in \Sigma$, then $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. If the measure space is σ-finite, then it is immediate that atoms have finite measure. The measure space (Ω, Σ, μ) is *non-atomic* or *atomless* if there are no atoms.

Theorem 3.3.2. *Let (Ω, Σ, μ) be a measure space.*

(a) *If (Ω, Σ, μ) is non-atomic and finite or σ-finite, then $L_1(\mu) \in \mathrm{DPr}$.*

(b) *If (Ω, Σ, μ) has an atom of finite measure, then $L_1(\mu)$ does not have the Daugavet property.*

PROOF. (a) Let $x \in S_{L_1(\mu)}$ and $x^* \in S_{L_1(\mu)^*}$. Recall that $L_1(\mu)^* = L_\infty(\mu)$, so x^* can be considered as a measurable function $\varphi \in L_\infty(\mu)$ with $\|\varphi\|_{L_\infty} = 1$. For $\varepsilon > 0$, find $B \in \Sigma$ with $\mu(B) > 0$ such $|\varphi(\omega)| > 1 - \varepsilon/2$ for almost all $\omega \in B$ (here we use the definition of the norm in $L_\infty(\mu)$) and find a smaller $\tilde{B} \subset B$ with $\mu(\tilde{B}) > 0$ such that $\displaystyle\int_B |x|\, d\mu < \varepsilon/2$ (here we use the non-atomicity of μ). Pick $y \in S_{L_1(\mu)}$ so that the support of y lies in $\tilde{B}$ and $\operatorname{Re} x^*(y) > 1 - \varepsilon$. Since, clearly,

$$\begin{aligned}\|x + y\| = \int_\Omega |x + y|\, d\mu &= \int_{\Omega\setminus B} |x|\, d\mu + \int_B |x + y|\, d\mu \\ &> \int_\Omega |x|\, d\mu - \frac{\varepsilon}{2} + \int_B |y|\, d\mu - \frac{\varepsilon}{2} \geqslant 2 - \varepsilon,\end{aligned}$$

the condition (ii) of Theorem 3.1.11 is fulfilled.

(b) Let $A \in \Sigma$ be an atom of finite measure. Then, the restriction of any $z \in L_1(\mu)$ on A is a constant function (a.e.). Consequently, the operator T that maps each $z \in L_1(\mu)$ to $T(z) = z\mathbb{1}_A$ is well-defined and a rank-one projector. For this T, we clearly have $\|T\| = 1$, but

$$\|\mathrm{Id} - T\| = 1 \neq 1 + \|T\|. \qquad \square$$

Now, we are ready to approach the duality questions for the Daugavet property.

THEOREM 3.3.3.

(a) *Let X be a Banach space such that $X^* \in$* DPr. *Then, $X \in$* DPr.

(b) $C[0,1]^* \notin$ DPr. *Consequently, there exists a Banach space $X \in$* DPr *such that $X^* \notin$* DPr *(namely, $X = C[0,1]$).*

PROOF. (a) Let $T \in L(X)$ be a rank-one operator, then $T^* \in L(X^*)$ is also of rank one, so our assumption gives $\|\mathrm{Id}_{X^*} + T^*\| = 1 + \|T^*\|$. Consequently, $\|\mathrm{Id}_X + T\| = \|\mathrm{Id}_{X^*} + T^*\| = 1 + \|T^*\| = 1 + \|T\|$, which proves the Daugavet property of X. (Cf. Lemma 1.3.5.)

(b) By the Riesz representation theorem, $C[0,1]^* = M[0,1]$ (the space of regular Borel (signed or complex) measures). Consider δ_0, the norm-one purely atomic positive measure that is concentrated at the point 0. For every $\mu \in M[0,1]$ denote $T\mu = \mu(\{0\})\delta_0$. This $T \in L(M[0,1])$ is a rank-one projector of norm $\|T\| = 1$, and $\mathrm{Id} - T$ is a norm-one projector onto the subspace of those measures that do not have an atom at 0, so

$$\|\mathrm{Id} - T\| = 1 \neq 1 + \|T\|.$$

This demonstrates that $M[0,1] \cong C[0,1]^* \notin$ DPr. $\square$

Remark that there are also spaces $X \in$ DPr for which $X^* \in$ DPr as well, so the first statement of Theorem 3.3.3 does not deal with a void collection of spaces. A good example is $X = L_1[0,1]$. Indeed, $L_1[0,1]^* = L_\infty[0,1]$, but $L_\infty[0,1]$ is isometric to some $C(K)$ on a perfect compact K, so it possesses the Daugavet property. It was Pełczyński who first remarked this fact and noticed that Theorem 3.3.2 may be deduced from Theorem 3.3.1 using the duality argument; cf. Theorem 1.1.2 and Corollary 1.3.4.

In connection with the above paragraph, the existence of a bidual space with the Daugavet property remains open (see question (3.1) in Section 3.7).

3.4. More examples. L_1-Dispersed and L_∞-dispersed spaces

In this section we introduce two large classes of Banach spaces with the Daugavet property, viz. the L_1-dispersed and L_∞-dispersed spaces, which extend the classes of non-atomic L_1- and $C(K)$-spaces.

Definition 3.4.1. Let X be a Banach space. A finite collection of subspaces $(Z_k)_{k=1}^n \subset X$ is said to be *ε-equivalent to their ℓ_1-sum* if for every collection $(z_k)_{k=1}^n$, $z_k \in Z_k$,

$$(1-\varepsilon)\sum_{k=1}^n \|z_k\| \leqslant \left\|\sum_{k=1}^n z_k\right\| \leqslant \sum_{k=1}^n \|z_k\|. \tag{3.4.1}$$

A Banach space X is said to be *L_1-dispersed* if for every $x, g \in S_X$ and $\varepsilon > 0$ there is a finite collection of subspaces $(Z_k)_{k=1}^n \subset X$ which is ε-equivalent to their ℓ_1-sum such that $x, g \in \operatorname{lin}\big(\bigcup_{k=1}^n Z_k\big)$ and in the corresponding representation $x = \sum_{k=1}^n x_k$ with $x_k \in Z_k$, we have that $\|x_k\| < \varepsilon$ for $k = 1, \dots, n$.

Theorem 3.4.2. *Every L_1-dispersed Banach space X possesses the Daugavet property.*

Proof. Let us begin by fixing $x \in S_X$, $x^* \in S_{X^*}$ and $\delta > 0$. According to condition (ii) of Theorem 3.1.11, our goal is to find some $y \in B_X$ such that $\operatorname{Re} x^*(y) > 1 - \delta$ and $\|x + y\| > 2 - \delta$.

Choose some $\varepsilon \in (0, 1)$ in such a way that $2(1-\varepsilon)^2 > 2 - \delta$ and hence also $(1-\varepsilon)^2 > 1 - \delta$.

Let $g \in S_X$ be an element at which $\operatorname{Re} x^*(g) > 1 - \varepsilon$. Let the subspaces $(Z_k)_{k=1}^n \subset X$ be taken from the above Definition 3.4.1 for these x, g and ε, and let $x = \sum_{k=1}^n x_k$, $x_k \in Z_k$ with $\|x_k\| < \varepsilon$, and $g = \sum_{k=1}^n g_k$, $g_k \in Z_k$, be the corresponding representations.

Then $1 = \|g\| \geqslant (1-\varepsilon)\sum_{k=1}^n \|g_k\|$ and

$$\sum_{k=1}^n \|g_k\| \operatorname{Re} x^*\Big(\frac{g_k}{\|g_k\|}\Big) = \operatorname{Re} x^*(g) > 1 - \varepsilon.$$

Consequently, there is $j \in \overline{1, n}$ such that $\operatorname{Re} x^*\Big(\frac{g_j}{\|g_j\|}\Big) > (1-\varepsilon)^2 > 1 - \delta$. Denote $y = \frac{g_j}{\|g_j\|}$. Then $y \in S_X$, $\operatorname{Re} x^*(y) > 1 - \delta$ and, denoting $A := \overline{1, n} \setminus \{j\}$, we obtain the following chain of inequalities:

$$\begin{aligned}
\frac{1}{1-\varepsilon}\|x + y\| &\geqslant \sum_{k=1}^n \|x_k + y_k\| = \|x_j + y\| + \sum_{k \in A} \|x_k\| \\
&\geqslant \|y\| - \|x_j\| + \sum_{k \in A} \|x_k\| \\
&\geqslant \|y\| + \|x\| - 2\|x_j\| > 2 - 2\varepsilon,
\end{aligned}$$

so $\|x + y\| > 2 - \delta$. □

The next definition and the subsequent theorem provide a wide range of Daugavet spaces that are similar to L_1.

Definition 3.4.3. A Banach space X is said to be *ℓ_1-dispersed* if for every $x \in S_X$ and $\varepsilon > 0$ there is a representation of X in the form of a finite ℓ_1-sum

$X = \left(\bigoplus_{k=1}^{n} X_k\right)_{\ell_1}$ of its subspaces such that in the corresponding representation $x = \sum_{k=1}^{n} x_k$, where $x_k \in X_k$, we have that $\|x_k\| < \varepsilon$ for $k = 1, \dots, n$.

We are now able to present the main examples of L_1-dispersed spaces. The spaces of type $\overline{L}_1$ were introduced in Definition 2.7.5.

Theorem 3.4.4.

(a) *Every ℓ_1-dispersed Banach space is L_1-dispersed, and consequently possesses the Daugavet property.*

(b) *Let X be a Banach space, (Ω, Σ, μ) be a non-atomic measure space, then the spaces $L_1(\mu)$, $\overline{L}_1(\Omega, \Sigma, \mu)$, $L_1(\mu, X)$ and $\overline{L}_1(\Omega, \Sigma, \mu, X)$ are ℓ_1-dispersed and, consequently, have the Daugavet property.*

Proof. The first statement is evident. For the second one, it is enough to remark that as (Ω, Σ, μ) is non-atomic, for every $f \in L_1(\Omega, \Sigma, \mu)$ and every $\varepsilon > 0$ there is a finite partition $\Omega = \bigsqcup_{k=1}^{n} A_k$ into measurable sets of non-zero measure in such a way that $\int_{A_k} |f(t)|\, d\mu(t) < \varepsilon$ for all k. With this partition in hand, we can write a representation $L_1(\mu) = \left(\bigoplus_{k=1}^{n} X_k\right)_{\ell_1}$, where X_k consists of all A_k-supported elements of $L_1(\mu)$. The other spaces Y from the list can be dealt in a similar way, taking for $y \in Y$ a positive $f \in L_1(\Omega, \Sigma, \mu)$ that dominates y: $\|y(t)\| \leqslant f(t)$ for almost all $t \in \Omega$. $\square$

Let us now deal with the L_∞ counterpart of the above results.

Definition 3.4.5. Let X be a Banach space. A finite collection of subspaces $(Z_k)_{k=1}^{n} \subset X$ is said to be ε-equivalent to their ℓ_∞-sum if for every collection $(z_k)_{k=1}^{n}$, $z_k \in Z_k$,

$$\left\|\sum_{k=1}^{n} z_k\right\| \leqslant (1+\varepsilon) \max_{k \in \overline{1,n}} \|z_k\|. \tag{3.4.2}$$

X is said to be *L_∞-dispersed* if for every $x, g \in S_X$, every $m \in \mathbb{N}$ and $\varepsilon > 0$, there is a finite collection of subspaces $(Z_k)_{k=1}^{n} \subset X$ which is ε-equivalent to their ℓ_∞-sum such that $x, g \in \operatorname{lin}\left(\bigcup_{k=1}^{n} Z_k\right)$ and in the corresponding representation $x = \sum_{k=1}^{n} x_k$, $x_k \in Z_k$, the set $D := \{k \in \overline{1,n} \colon \|x_k\| > 1 - \varepsilon\}$ has at least m elements.

Lemma 3.4.6. *Let $(Z_k)_{k=1}^{n} \subset X$ be a collection of subspaces which is ε-equivalent to their ℓ_∞-sum. Then for every collection $(z_k)_{k=1}^{n}$, $z_k \in Z_k$ for $k \in \overline{1,n}$,*

$$\left\|\sum_{k=1}^{n} z_k\right\| \geqslant (1-\varepsilon) \max_{k \in \overline{1,n}} \|z_k\|, \tag{3.4.3}$$

in particular, $(Z_k)_{k=1}^{n}$ is a linearly independent collection of subspaces.

Proof. Pick $j \in \overline{1,n}$ such that $\|z_j\| = \max_{k \in \overline{1,n}} \|z_k\|$. Denote $\tilde{z}_k = z_k$ for $k \neq j$ and $\tilde{z}_j = -z_j$. Then $\|\tilde{z}_k\| = \|z_k\|$ for all $k \in \overline{1,n}$, so

$$\left\|\sum_{k=1}^{n} \tilde{z}_k\right\| \leqslant (1+\varepsilon) \max_{k \in \overline{1,n}} \|z_k\| = (1+\varepsilon)\|z_j\|.$$

Now,

$$\left\|\sum_{k=1}^{n} z_k\right\| = \left\|\sum_{k=1}^{n} \tilde{z}_k + 2z_j\right\| \geqslant 2\|z_j\| - \left\|\sum_{k=1}^{n} \tilde{z}_k\right\|$$
$$\geqslant 2\|z_j\| - (1+\varepsilon)\|z_j\| = (1-\varepsilon)\|z_j\| = (1-\varepsilon)\max_{k\in\overline{1,n}} \|z_k\|. \qquad \square$$

THEOREM 3.4.7. *Every L_∞-dispersed Banach space X possesses the Daugavet property.*

PROOF. As before, for given $x \in S_X$, $x^* \in S_{X^*}$ and $\delta > 0$, our goal is to find $y \in X$ such that $\|y\| < 1 + \delta$, $\operatorname{Re} x^*(y) > 1 - \delta$ and $\|x + y\| > 2 - \delta$ (here we use Remark 3.1.12).

Choose $\varepsilon \in (0, 1)$ and $m \in \mathbb{N}$ in such a way that

$$2(1-\varepsilon)^2 > 2 - \delta, \quad \frac{1+\varepsilon}{1-\varepsilon} < 1 + \delta, \quad \text{and} \quad \varepsilon + \frac{2}{m}\frac{1+\varepsilon}{1-\varepsilon} < \delta.$$

Let $g \in S_X$ be an element at which $\operatorname{Re} x^*(g) > 1-\varepsilon$ and let subspaces $Z_1, \ldots, Z_n \subset X$ be taken from the above Definition 3.4.5 for these x, g, m and ε. Let $x = \sum_{k=1}^{n} x_k$, and $g = \sum_{k=1}^{n} g_k$, $x_k, g_k \in Z_k$ be the corresponding representations, and so that the corresponding $D = \{k \in \overline{1,n} \colon \|x_k\| > 1 - \varepsilon\}$ has at least m elements.

Taking into account that, thanks to Lemma 3.4.6,

$$\|x_k\| < \frac{1}{1-\varepsilon}, \quad \|g_k\| < \frac{1}{1-\varepsilon} \tag{3.4.4}$$

and consequently $\|g_k - x_k\| < \frac{2}{1-\varepsilon}$ for all $k \in \overline{1,n}$, we have the following estimate:

$$\sum_{k\in D} |\operatorname{Re} x^*(g_k - x_k)| = \max_{\pm} \operatorname{Re} x^* \left(\sum_{k\in D} \pm(g_k - x_k)\right)$$
$$\leqslant \max_{\pm} \left\|\sum_{k\in D} \pm(g_k - x_k)\right\| < 2\frac{1+\varepsilon}{1-\varepsilon}.$$

Hence, there is $j \in D$ such that

$$|\operatorname{Re} x^*(g_j - x_j)| < \frac{2}{m}\frac{1+\varepsilon}{1-\varepsilon}.$$

Denote $y = g + (x_j - g_j)$. In other words, if $y = \sum_{k=1}^{n} y_k$, $y_k \in Z_k$, is the corresponding representation, then $y_j = x_j$ and $y_k = g_k$ for $k \neq j$.

Let us check that this y is what we need. The ℓ_∞-equivalence condition (3.4.2) and inequalities (3.4.4) imply that

$$\|y\| \leqslant (1+\varepsilon) \max_{k\in\overline{1,n}} \|y_k\| \leqslant \frac{1+\varepsilon}{1-\varepsilon} < 1 + \delta.$$

Next,

$$\operatorname{Re} x^*(y) = \operatorname{Re} x^*(g) + \operatorname{Re} x^*(x_j - g_j) > 1 - \varepsilon - \frac{2}{m}\frac{1+\varepsilon}{1-\varepsilon} > 1 - \delta.$$

Finally, taking into account that $j \in D$, we deduce that

$$\frac{1}{1-\varepsilon}\|x + y\| \geqslant \|x_j + y_j\| = 2\|x_j\| > 2(1-\varepsilon),$$

so $\|x + y\| > 2 - \delta$. $\square$

A straightforward application of the above theorem is the Daugavet property of non-atomic L_∞ spaces (in both the scalar- and vector-valued case).

DEFINITION 3.4.8. A Banach space X is said to be *ℓ_∞-dispersed* if for every $x \in S_X$, every $m \in \mathbb{N}$ and $\varepsilon > 0$ there is a representation of X in the form of a finite ℓ_∞-sum $X = \left(\bigoplus_{k=1}^n X_k\right)_{\ell_\infty}$ of some subspaces such that in the corresponding representation $x = \sum_{k=1}^n x_k$, $x_k \in X_k$, the set $D := \{k \in \overline{1,n}\colon \|x_k\| > 1-\varepsilon\}$ has at least m elements.

REMARK 3.4.9. *Evidently, every ℓ_∞-dispersed space is L_∞-dispersed, and consequently possesses the Daugavet property. Also, in the above definition one may always take $n = m$. Indeed, take some distinct $k_j \in D$, $j \in \overline{1,m}$, and denote $\tilde{X}_1 = \left(\bigoplus_{k\in\overline{1,n}\setminus\{k_2,k_3,\dots,k_m\}} X_k\right)_{\ell_\infty}$, put $\tilde{X}_j = X_{k_j}$ for $j \in \overline{2,m}$ and write a new representation $X = \left(\bigoplus_{j=1}^m \tilde{X}_j\right)_{\ell_\infty}$. The possibility of n being greater than m is left in the definition because this way we have a bit more flexibility when demonstrating this property for concrete spaces.*

THEOREM 3.4.10. *Let X be a Banach space, and let (Ω, Σ, μ) be a non-atomic measure space. Then the space $L_\infty(\mu, X)$ is ℓ_∞-dispersed and consequently has the Daugavet property. In particular, $L_\infty(\mu) \in \mathrm{DPr}$.*

PROOF. Fix $x \in S_{L_\infty(\mu,X)}$, $m \in \mathbb{N}$ and $\varepsilon > 0$. Then, there is a set $A \in \Sigma$ of positive measure such that $\|x(t)\| > 1-\varepsilon$ for all $t \in A$. By the non-atomicity, there is a finite partition $A = \bigsqcup_{k=1}^m A_k$ into measurable sets of non-zero measure. Denote $A_{m+1} := \Omega \setminus A$. Take $n = m+1$ and Z_k consisting of all A_k-supported elements of $L_\infty(\mu, X)$, $k = 1, \dots, n$. Then $L_\infty(\mu, X) = (\bigoplus_{k=1}^n Z_k)_{\ell_\infty}$, and all the conditions of Definition 3.4.8 are satisfied. □

A little bit more work is needed in order to deduce the Daugavet property of spaces of continuous functions from Theorem 3.4.7. This is so because not even $C[0,1]$ is ℓ_∞-dispersed; hence we need to work with the technically less convenient L_∞-dispersity.

Recall that a topological space Ω is said to be *completely regular* if for every closed subset $C \subset \Omega$ and every point $t \in \Omega \setminus C$, there is a continuous function $\varphi\colon \Omega \to [0,1]$ such that $\varphi(t) = 1$ and $\varphi(C) = \{0\}$. Every compact or locally compact Hausdorff topological space is completely regular.

The next result extends the original Daugavet theorem about the Daugavet property of $C[0,1]$ and of $C(K)$ on perfect compact spaces K (i.e., compact Hausdorff topological spaces without isolated points) to a wide range of spaces of vector-valued functions. The first step in this direction was taken in [**173**, Theorem 4.4]. We need some notation. Given a Banach space X and a completely regular topological space Ω, $C_b(\Omega, X)$ denotes the Banach space of all bounded continuous functions acting from Ω to X equipped with the standard sup-norm. If L is a Hausdorff locally compact topological space, $C_0(L, X)$ denotes the closure in $C_b(\Omega, X)$ of the space of those functions with compact support (recall that a locally compact Hausdorff space is completely regular by virtue of Urysohn's lemma). If K is a compact Hausdorff topological space, then we write $C(K, X) = C_0(K, X) = C_b(K, X)$.

THEOREM 3.4.11. *Let X be a Banach space, Ω be a completely regular Hausdorff topological space without isolated points. Then, $C_b(\Omega, X)$ is L_∞-dispersed and* *consequently has the Daugavet property. Besides, the same happens with $C_0(L,X)$ if L is a locally compact Hausdorff topological space without isolated points. In particular, if K is a perfect Hausdorff compact topological space, then $C(K,X) \in \mathrm{DPr}$.*

PROOF. We first deal with the case of $C_b(\Omega,X)$. Fix $x,g \in S_{C_b(\Omega,X)}$, $m \in \mathbb{N}$ and $\varepsilon > 0$. Choose $\tilde{\varepsilon} \in (0,\varepsilon)$ in such a way that $\frac{1}{1-\tilde{\varepsilon}} < 1+\varepsilon$. Denote $Y := \mathrm{lin}\{x,g\}$. By the standard argument that uses finite δ-nets in S_Y (for example, one may apply Theorem 1 of [**156**, Section 17.2.4] with $G := \{y^* \circ \delta_t \colon y^* \in S_{Y^*},\ t \in \Omega\}$), there is a finite subset $A \subset \Omega$ such that for every $f \in Y$

$$\max_{t\in A} \|f(t)\| \geqslant (1-\tilde{\varepsilon})\|f\|. \tag{3.4.5}$$

Select a finite set $\{t_k \colon k = 1,\dots,m\}$ of distinct elements in $\Omega \setminus A$ in such a way that $\|x(t_k)\| > 1-\tilde{\varepsilon}$, $k \in \overline{1,m}$ (here we use the absence of isolated points), disjoint open neighbourhoods $U_k \subset \Omega \setminus A$ of t_k, and finite subsets $A_k \subset U_k$ with $t_k \in A_k$, in such a way that for every $f \in Y$

$$\max_{t\in A_k} \|f(t)\| \geqslant (1-\tilde{\varepsilon}) \sup_{t\in U_k} \|f(t)\|. \tag{3.4.6}$$

Using complete regularity, select continuous functions $\varphi_k \colon \Omega \to [0,1]$ such that $\varphi_k(A_k) = \{1\}$ and $\varphi_k(\Omega \setminus U_k) = \{0\}$, $k \in \overline{1,m}$. Denote $\varphi_{m+1} = 1 - \sum_{k=1}^m \varphi_k$, $U_{m+1} = \Omega \setminus \bigcup_{k=1}^n A_k$ and define $n = m+1$. With this definition, $\sum_{j=1}^n \varphi_j = \mathbb{1}$, $\varphi_j \geqslant 0$ for all $j \in \overline{1,n}$. The requested Z_k, $k \in \overline{1,n}$, are defined as follows:

$$Z_k = \{\varphi_k f \colon f \in Y\}.$$

It remains to verify the conditions of Definition 3.4.5.

Let us start with the ε-equivalence of $(Z_k)_{k=1}^n$ to their ℓ_∞-sum. For every collection $(z_k)_{k=1}^n$, $z_k \in Z_k$, we may write them in the form of $z_k = \varphi_k f_k$ with $f_k \in Y$. By our construction, for $k \in \overline{1,m}$ we have

$$\begin{aligned}\|z_k\| &= \sup_{t\in U_k} \varphi_k(t)\|f_k(t)\| \geqslant \sup_{t\in A_k} \varphi_k(t)\|f_k(t)\| \\ &= \sup_{t\in A_k} \|f_k(t)\| \geqslant (1-\tilde{\varepsilon}) \sup_{t\in U_k} \|f_k(t)\|.\end{aligned}$$

Analogously,

$$\|z_n\| \geqslant \sup_{t\in A} \varphi_n(t)\|f_n(t)\| \geqslant (1-\tilde{\varepsilon})\|f_n\| \geqslant (1-\tilde{\varepsilon}) \sup_{t\in U_n} \|f_n(t)\|.$$

Consequently,

$$\begin{aligned}\left\|\sum_{k=1}^n z_k\right\| &= \sup_{t\in\Omega} \left\|\sum_{k=1}^n \varphi_k(t) f_k(t)\right\| \leqslant \sup_{t\in\Omega} \sum_{k=1}^n \varphi_k(t) \sup_{t\in U_k} \|f_k(t)\| \\ &\leqslant \max_{k\in\overline{1,n}} \sup_{t\in U_k} \|f_k(t)\| \leqslant (1+\varepsilon) \max_{k\in\overline{1,n}} \|z_k\|.\end{aligned}$$

With this, the ε-equivalence of $(Z_k)_{k=1}^n$ to their ℓ_∞-sum is demonstrated.

For each $f \in Y$ we have $f = \sum_{j=1}^n \varphi_j y \in \mathrm{lin}\bigl(\bigcup_{k=1}^n Z_k\bigr)$ so, in particular, $x,g \in \mathrm{lin}\bigl(\bigcup_{k=1}^n Z_k\bigr)$. Finally, in the representation $x = \sum_{k=1}^n x_k$, $x_k \in Z_k$, we have $x_k = \varphi_k x$, so $\|x_k\| \geqslant \varphi_k(t_k)\|x(t_k)\| > 1-\varepsilon$ for $k \in \overline{1,m}$, so the set $D := \{k \in \overline{1,n} \colon \|x_k\| > 1-\varepsilon\}$ contains $\overline{1,m}$.

Observe that the previous proof can be adapted to the case of $C_0(L,X)$ as Urysohn's lemma allows one to find the functions φ_k, $k \in \overline{1,m}$, with compact support, and this assures that when starting with $x, g \in C_0(L, X)$, all the functions involved in the proof are also in $C_0(L, X)$. $\square$

Although the above demonstration is ideologically valuable, ironically, a direct proof of the Daugavet property for $C_b(\Omega, X)$ and $C_0(L, X)$ can be given in a much shorter way. Let us present it as well.

An alternative proof. Fix $y \in S_Y$, $y^* \in S_{C_b(K,X)^*}$ and $\varepsilon > 0$. Select $z \in S_{C_b(K,X)}$ with $\operatorname{Re} y^*(z) > 1-\varepsilon$. Let $t_0 \in K$ be a point at which $\|y(t_0)\| > 1-\varepsilon$, and let $U \subset K$ be an open set such that $\|z(t)-z(t_0)\| < \varepsilon$ and $\|y(t)-y(t_0)\| < \varepsilon$ for all $t \in U$. Using the absence of isolated points and the complete regularity, we may pick a sequence of nonnegative functions $(f_n) \subset S_{C(K)}$ whose supports are disjoint and lie in U and points $t_n \in U$ with $f_n(t_n) = 1$. It is plain that for any selection of $w_n \in X$, the sequence $(w_n f_n)$ in $C_b(K, X)$ is equivalent to the canonical basis of c_0 and, in particular, it tends weakly to 0.

Then, the sequence $v_n = (y(t_0) - z(t_0))f_n$ tend weakly to 0 as $n \to \infty$, so $y^*(v_n) \to 0$. Consequently, $\operatorname{Re} y^*(z + v_n) > 1 - \varepsilon$ when n is big enough.

Let us remark that $\|z + v_n\| < 1 + \varepsilon$. Indeed, for $t \in K \setminus \operatorname{supp} v_n$ we have $|(z + v_n)(t)| = |z(t)| \leqslant 1$, and for $t \in \operatorname{supp} v_n$ we know that $\|z(t) - z(t_0)\| < \varepsilon$, so

$$\begin{aligned}\|(z + v_n)(t)\| &< \|z(t_0) + v_n(t)\| + \varepsilon\\ &= \|y(t_0)f_n(t) + z(t_0)(1 - f_n(t))\| + \varepsilon \leqslant 1 + \varepsilon.\end{aligned}$$

Also,

$$\begin{aligned}\|(z + v_n) + y\| &\geqslant \|((z + v_n) + y)(t_n)\|\\ &\geqslant \|z(t_0) + y(t_0) + v_n(t_n)\| - 2\varepsilon\\ &= \|2y(t_0)\| - 2\varepsilon = 2 - 4\varepsilon.\end{aligned}$$

Hence, denoting $h = z + v_n$ with n big enough, for given $y \in S_{C_b(K,X)}$, $y^* \in S_{C_b(K,X)^*}$, and $\varepsilon > 0$, we obtain the existence of $h \in C_b(K, X)$ that satisfies the conditions $\operatorname{Re} y^*(h) > 1-\varepsilon$, $\|h\| < 1+\varepsilon$ and $\|h+y\| > 2-4\varepsilon$. According to Remark 3.1.12, this ensures the Daugavet property of Y.

Again, the case of $C_0(L, X)$ follows in the same way as we may choose the functions f_n, $n \in \mathbb{N}$, with compact support (by Urysohn's lemma). $\square$

Remark that it is easy to verify that the classes of ℓ_1- and L_1-dispersed spaces are stable with respect to ℓ_1-sums, and ℓ_∞- and L_∞-dispersed spaces are stable with respect to ℓ_∞-sums. Later, we will demonstrate that the Daugavet property itself is stable with respect to both ℓ_1- and ℓ_∞-sums (see Corollary 7.4.6 and Corollary 7.2.5).

We conclude the section with a duality result which gives a hint at "why" the dual of $L_1[0, 1]$ has the Daugavet property, although in general the Daugavet property does not pass to the dual space.

Theorem 3.4.12. *Let X be an ℓ_1-dispersed Banach space. Then, X^* is ℓ_∞-dispersed and hence has the Daugavet property.*

Proof. Fix $x^* \in S_{X^*}$, $m \in \mathbb{N}$ and $\varepsilon > 0$. Select $\delta \in (0, \varepsilon)$ so small that $(\varepsilon - \delta)/\delta \geqslant m$ and choose $x \in S_X$ such that $x^*(x) \in (1 - \delta, 1]$. According to the definition of ℓ_1-dispersity, there are $n \in \mathbb{N}$ and a representation of X in the form of $X = \left(\bigoplus_{k=1}^n X_k\right)_{\ell_1}$ such that, in the corresponding representation, $x = \sum_{k=1}^n x_k$ where $x_k \in X_k$ with $\|x_k\| < \delta$ for $k = 1, \ldots, n$.

Then, by a standard duality argument, X^* identifies with $\left(\bigoplus_{k=1}^n X_k^*\right)_{\ell_\infty}$. Our goal is to check that, in the corresponding representation, $x^* = \sum_{k=1}^n x_k^*$, $x_k^* \in X_k^*$, and the set $D := \{k \in \overline{1,n} \colon \|x_k^*\| > 1 - \varepsilon\}$ has at least m elements.

Indeed, denote by $|D|$ the number of elements in D. Since $\|x_k\| < \delta$ and $\sum_{k \in \overline{1,n}} \|x_k\| = \|x\| = 1$, we have

$$\begin{aligned} 1 - \delta \leqslant x^*(x) &= \sum_{k=1}^n x_k^*(x_k) \\ &\leqslant \sum_{k \in D} |x_k^*(x_k)| + \sum_{k \in \overline{1,n} \setminus D} |x_k^*(x_k)| \\ &< \delta |D| + (1 - \varepsilon) \sum_{k \in \overline{1,n} \setminus D} \|x_k\| \\ &\leqslant \delta |D| + (1 - \varepsilon). \end{aligned}$$

This means that $|D| > (\varepsilon - \delta)/\delta \geqslant m$, and the job is done. □

3.5. Further examples of Banach spaces with the Daugavet property

Our aim in this section is to provide many examples of Banach spaces with the Daugavet property which complement those of Sections 3.3 and 3.4. Actually, we will characterise the Daugavet property in some natural families. We divide this section into three subsections devoted, respectively, to C^*-algebras and von Neumann preduals, to uniform algebras and preduals of $L_1(\mu)$ spaces and, finally, to representable Banach spaces.

3.5.1. C^*-algebras and von Neumann preduals. We start with a useful sufficient condition for the Daugavet property which will be the key to characterise the property for C^*-algebras and von Neumann preduals.

THEOREM 3.5.1. *Let X be a Banach space such that there are two one-norming subspaces Y and Z of X^* such that $X^* = Y \oplus_1 Z$. Then, X has the Daugavet property.*

PROOF. We fix $x_0 \in S_X$, $f_0 \in S_{X^*}$ and $\varepsilon > 0$. We write $f_0 = y_0^* + z_0^*$ such that $y_0^* \in Y$, $z_0^* \in Z$, $\|f_0\| = \|y_0^*\| + \|z_0^*\|$. Take $z^* \in B_Z \cap \operatorname{Slice}(B_{X^*}, x_0, \varepsilon/2)$ (which is not empty since B_Z is weak-star dense in B_{X^*}). Clearly, $\|z^*\| > 1 - \varepsilon/2$. As B_Y is also weak-star dense in B_{X^*} and $z^* \in \operatorname{Slice}(B_{X^*}, x_0, \varepsilon/2)$, we may find a net $(y_\lambda^*)_{\lambda \in \Lambda}$ in $B_Y \cap \operatorname{Slice}(B_{X^*}, x_0, \varepsilon)$ for which z^* is its weak-star limit. Now, since $(y_\lambda^* + y_0^*)_{\lambda \in \Lambda}$ weak-star converges to $z^* + y_0^*$ and the norm is weak-star-lower semi-continuous, we get that

$$\liminf \|y_\lambda^* + y_0^*\| \geqslant \|z^* + y_0^*\| = \|z^*\| + \|y_0^*\| > 1 + \|y_0^*\| - \varepsilon/2.$$

Therefore, we may find μ such that

$$\|y_\mu^* + y_0^*\| \geqslant 1 + \|y_0^*\| - \varepsilon/2.$$

Finally,

$$\begin{aligned} \|f_0 + y_\mu^*\| = \|(y_0^* + y_\mu^*) + z_0^*\| &= \|y_0^* + y_\mu^*\| + \|z_0^*\| \\ &> 1 + \|y_0^*\| - \varepsilon/2 + \|z_0^*\| = 2 - \varepsilon. \end{aligned}$$

As we already have that $\operatorname{Re} y_\mu^*(x_0) > 1-\varepsilon$ since $y_\mu^* \in \operatorname{Slice}(B_{X^*}, x_0, \varepsilon)$, X has the Daugavet property by item (ii)* in Theorem 3.1.11. $\square$

Just using the Goldstine and Krein-Milman Theorems, we obtain the following useful particular case. We refer to Subsection 2.9.2 for the basics on L-embedded spaces.

COROLLARY 3.5.2. *Let X be a non-null L-embedded Banach space without extreme points. Then, X^* (and hence X) has the Daugavet property.*

PROOF. By definition, $X^{**} = X \oplus_1 X_s$ for some subspace X_s. On the one hand, since B_X has no extreme points and $\operatorname{ext}(B_{X^{**}}) = \operatorname{ext}(B_X) \cup \operatorname{ext}(B_{X_s})$ (see Remark 2.9.10), we have $\operatorname{ext}(B_{X^{**}}) = \operatorname{ext}(B_{X_s})$, and the Krein-Milman Theorem gives us that B_{X_s} is weak-star dense in $B_{X^{**}}$. On the other hand, Goldstine's Theorem gives us that B_X is weak-star dense in $B_{X^{**}}$, and then the result follows from Theorem 3.5.1. $\square$

Let us present some immediate consequences of the above result.

COROLLARY 3.5.3. *If X is a non-null L-embedded space with $\operatorname{ext}(B_X) = \emptyset$ and $Y \subsetneq X$ is also an L-embedded space, then $(X/Y)^*$ (and hence X/Y) has the Daugavet property.*

PROOF. On the one hand, X/Y is a non-null L-embedded space by [**133**, Corollary IV.1.3]. On the other hand, [**133**, Propositions IV.1.12 and IV.1.14] give us that $\operatorname{ext}(B_{X/Y}) = \emptyset$. Therefore, Corollary 3.5.2 applies. $\square$

As a particular case of the above corollary we have the following result.

COROLLARY 3.5.4. *If Y is an L-embedded space which is a subspace of $L_1 := L_1[0,1]$, then $(L_1/Y)^*$ has the Daugavet property. In particular, $(L_1/Y)^*$ has the Daugavet property for every reflexive subspace Y of L_1 and so do H^∞ and its predual L_1/H_0^1.*

PROOF. The space L_1 is an L-embedded space with $\operatorname{ext}(B_{L_1}) = \emptyset$, so the result follows from Corollary 3.5.3. For the particular cases, reflexive spaces are trivially L-embedded and the space $H_0^1 \subset L_1$ is an L-embedded space (see Example 2.9.8). $\square$

Let us now apply the results of this section to C^*-algebras and von Neumann preduals. Recall that a *C^*-algebra* is a complex Banach algebra A with an involution, $*$, satisfying that $\|x^*x\| = \|x\|^2$ for every $x \in A$ (hence, in particular, $\|x^*\| = \|x\|$ for every $x \in A$). A *projection* in A is an element $p \in A$ such that $p^* = p$ and $p^2 = p$. A projection is said to be *atomic* or *minimal* if $p \neq 0$ and $pAp = \mathbb{C}p$. When no minimal projections exist, we say that the C^*-algebra A is *diffuse* or *non-atomic*. A *von Neumann algebra* is a C^*-algebra which is (isometrically isomorphic to) a dual Banach space. The *unique* predual of a von Neumann algebra A is denoted by A_*.

Let us start by characterising the Daugavet property for preduals of von Neumann algebras, which is a consequence of Corollary 3.5.2 and some known results on the geometry of von Neumann algebras and their preduals.

THEOREM 3.5.5. *Let A be a von Neumann algebra and let A_* be its unique predual. Then, the following assertions are equivalent:*

(i) $A \in \mathrm{DPr}$.

(ii) $A_* \in \mathrm{DPr}$.
(iii) *Every nonvoid relatively weakly open subset of B_{A_*} has diameter two.*
(iv) *The norm of A is not Fréchet differentiable at any point.*
(v) *The norm of A is not smooth at any point.*
(vi) *B_{A_*} has no strongly exposed points.*
(vii) *B_{A_*} has no extreme points.*
(viii) *A is diffuse.*

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) is a consequence of Lemma 3.1.15. (iii) $\Rightarrow$ (iv) is clear by Shmulyan's test (Lemma 2.10.5). (iv) $\Leftrightarrow$ (v) is a consequence of [**289**, Theorem]. (iv) $\Leftrightarrow$ (vi) is again Shmulyan's test. (vi) $\Leftrightarrow$ (vii) is a consequence of [**290**, Theorem on p. 342]. (iv) $\Leftrightarrow$ (viii) is a consequence of [**289**, Theorem].

Finally, (vii) $\Rightarrow$ (i) follows from Corollary 3.5.2 as von Neumann preduals are L-embedded, see Example 2.9.8. $\square$

The $L_\infty(\mu)$ spaces are examples of commutative von Neumann algebras, so the previous theorem applies to their preduals $L_1(\mu)$. We can view then Theorem 3.5.5 as a non-commutative generalisation to Lozanovskiĭ's Theorem 1.2.1 and its generalisations to $L_1(\mu)$ when μ is atomless (see Theorems 3.3.2 and 3.4.4).

The characterisation of the Daugavet property for C^*-algebras is a little harder to get.

Theorem 3.5.6. *Let A be a C^*-algebra. Then, the following assertions are equivalent:*
(i) $A \in \mathrm{DPr}$.
(ii) *The norm of A is not Fréchet differentiable at any point.*
(iii) *A is diffuse.*

Proof. (i) $\Rightarrow$ (ii) follows from Corollary 3.1.13.

By [**290**, Theorem 4.1], A has minimal projections if and only if the norm of A is Fréchet differentiable at some element of A. This shows (ii) $\Leftrightarrow$ (iii).

Let us finally prove that (iii) $\Rightarrow$ (i). By the general theory of C^*-algebras, A^{**} is a von Neumann algebra with (unique) predual A^*. Write Z for the norm closed linear span of the extreme points of B_{A^*}. It is known that there is a closed subspace W of A^* such that B_W has no extreme points and $A^* = Z \oplus_1 W$ (it is a consequence of the "splitting into atomic and non-atomic parts" of the state space of a JBW-algebra, see [**20**, Corollary 5.8] where it is claimed that the result is well known for C^*-algebras; an explicit formulation can be found in [**113**, Theorem 1 on p. 84] in the setting of JB*-triples). Then, $A^{**} = \mathcal{Z} \oplus_\infty \mathcal{W}$, where $\mathcal{Z} = W^\perp \cong Z^*$ and $\mathcal{W} = Z^\perp \cong W^*$ are von Neumann algebras which are actually ideals of A^{**} (that is, $\mathcal{Z}A^{**} \subset \mathcal{Z}$ and $\mathcal{W}A^{**} \subset \mathcal{W}$ and the same when multiplying from the right). Write $\pi_{\mathcal{Z}}\colon A^{**} \to \mathcal{Z}$ and $\pi_{\mathcal{W}}\colon A^{**} \to \mathcal{W}$ for the natural projections, which are algebra homomorphism. Observe that Z is one-norming for A as it contains $\operatorname{ext}(B_{A^*})$. We claim that W is also one-norming for X and this will give (i) by using Theorem 3.5.1. Indeed, write $Y = A \cap \mathcal{Z}$ which is a closed ideal of A, so it has no minimal projections (since minimal projections of an ideal lift to the whole of A which is diffuse). Next, it is routine to show that Y^{**} is contained in $\mathcal{Z}$, hence Y^* is the predual of an atomic von Neumann algebra, so Y^* has the RNP [**70**, Proposition 3.7], and Y is Asplund. But then, if $Y \neq \{0\}$, the norm of Y is Fréchet differentiable on a dense (hence nonempty) subset and this would imply that Y has minimal projections by [**290**, Theorem 4.1], a contradiction. Therefore, $Y = \{0\}$ and so $\pi_{\mathcal{W}} \circ J_A$ is injective. Being an algebra homomorphism, it is actually an isometric embedding, so for every $x \in A$,

$$\|x\| = \|\pi_{\mathcal{W}}(J_A(x))\| = \sup\bigl\{\bigl|\langle \pi_{\mathcal{W}}(J_X(x)), x^*\rangle\bigr|\colon x^* \in B_{X^*}\bigr\}$$

and, as $\mathcal{W} = Z^\perp$, we have

$$= \sup\bigl\{\bigl|\langle \pi_{\mathcal{W}}(J_X(x)), x^*\rangle\bigr|\colon x^* \in B_W\bigr\} = \sup\bigl\{|x^*(x)|\colon x^* \in B_W\bigr\}.$$

This proves that W is one-norming for A, as desired. □

Commutative C^*-algebras are just $C_0(L)$ spaces for locally compact Hausdorff spaces L, hence the above theorem reproves the known fact that $C_0(L)$ has the Daugavet property if and only if L has no isolated points; cf. Remark 1.4.5.(2) or Theorem 3.3.1. Also, it is immediate from the above theorem that neither $L(H)$ nor $K(H)$ have the Daugavet property when H is a Hilbert space, since both C^*-algebras are atomic, hence not diffuse. As all minimal projections in $L(H)$ actually belong to $K(H)$, it is reasonable to think that the Calkin algebra $Q(H) = L(H)/K(H)$ has the Daugavet property: this is the case, as $Q(H)$ even lacks abelian elements (i.e., positive elements a such that $aQ(H)a$ is commutative), see [**245**, pp. 191–192].

Corollary 3.5.7. *For every infinite dimensional Hilbert space H, the Calkin algebra over H, $Q(H) := L(H)/K(H)$, has the Daugavet property.*

3.5.2. Norming ℓ_1-structures: L_1-predual spaces and uniform algebras. Our aim is to characterise the Daugavet property in a wide class of Banach spaces which includes isometric preduals of L_1-spaces and uniform algebras.

Definition 3.5.8. A Banach space X has a *norming ℓ_1-structure* if there exists a subset A of S_{X^*} which is one-norming for X such that $\mathbb{K}a^*$ is an L-summand of X^* for every $a^* \in A$.

Some remarks on this definition need to be mentioned and will be useful in the sequel. Item (e) below justifies the name "norming ℓ_1-structure."

Remarks 3.5.9. *Let X be a Banach space with norming ℓ_1-structure witnessed by $A \subset S_{X^*}$.*

(a) *Given $a^* \in A$, since $\mathbb{K}a^*$ is an L-summand of X^*, it follows that there is an L-projection $P_{a^*}\colon X^* \to X^*$ such that $P_{a^*}(X^*) = \mathbb{K}a^*$. Hence, $X^* = \mathbb{K}a^* \oplus_1 \ker P_{a^*}$.*
(b) *It is immediate from (a) that $a^* \in \operatorname{ext}(B_{X^*})$ for every $a^* \in A$ (see Remark 2.9.10). On the other hand, the fact that A is one-norming means that $\operatorname{ext}(B_{X^*}) \subset \overline{\mathbb{T}A}^{w^*}$ with $\mathbb{T} = \{\lambda \in \mathbb{K}\colon |\lambda| = 1\}$.*
(c) *Given $a_1^*, \ldots, a_n^* \in A$ pairwise linearly independent and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, one has*
$$\Bigl\|\sum_{k=1}^n \lambda_k\, a_k^*\Bigr\| = \sum_{k=1}^n |\lambda_k|.$$
In particular, $\{a_1^*, \ldots, a_n^*\}$ is linearly independent.
(d) *We may always suppose, modifying the set A if needed, that the elements in A are pairwise linearly independent (which is equivalent to being linearly independent by (c) above).*

(e) *If A is taken pairwise linearly independent, then writing $Y = \overline{\operatorname{lin}(A)}$, we have that $Y \cong \ell_1(A)$ and $X^* = Y \oplus_1 Z$ for some subspace Z of X^*.*

Proof. (a) is just the definition of an L-summand and (b) follows from Remark 2.9.10. (c) follows from Lemma 2.9.4.

(d) An easy way is the following: fix $x_0^{**} \in \operatorname{ext}(B_{X^*})$ and observe that $|x_0^{**}(a^*)| = 1$ for every $a^* \in A$. For each $a^* \in A$, pick $\theta_{a^*} \in \mathbb{T}$ such that $x_0^{**}(\theta_{a^*} a^*) = 1$. Now, the set

$$A' := \{\theta_{a^*} a^* \colon a^* \in A\}$$

is pairwise linearly independent and $\mathbb{T}A' = \mathbb{T}A$, hence A' is one-norming and $\mathbb{K}\theta_{a^*} a^* = \mathbb{K}a^*$ is an L-summand for every $\theta_{a^*} a^* \in A'$.

(e) That $Y \cong \ell_1(A)$ follows from item (c). By Proposition 2.9.3, the set of L-projections on X^* forms a complete Boolean algebra; so, in particular,

$$P := \sup\{P_{a^*} \colon a^* \in A\}$$

is an L-projection. As it is shown in Lemma 2.9.4, $P_{a^*} P_{b^*} = 0$ if $a^*, b^* \in A$, $a^* \neq b^*$, so it is routine to show that $P = \sum_{a^* \in A} P_{a^*}$ pointwise, hence $P(X^*) \subset Y$. On the other hand, $P(a^*) = a^*$ for every $a^* \in A$, hence $Y \subset P(X^*)$. It is enough to consider $Z = \ker P$ to get that $X^* = Y \oplus_1 Z$. $\square$

The following result gives some equivalent reformulations of the previous concept.

Proposition 3.5.10. *Let X be a Banach space. Then, the following are equivalent:*

(i) *X has a norming ℓ_1-structure.*

(ii) *$X^* \cong Y \oplus_1 Z$ such that $Y \cong \ell_1(\Gamma)$ for some set Γ and is one-norming for X.*

(iii) *X is* nicely embedded *into a $C_b(S)$-space, that is, there is a Hausdorff topological space S and an isometric embedding $J\colon X \to C_b(S)$ such that for all $s \in S$ the following properties are satisfied:*

  (N1) *for $p_s := J^*(\delta_s) \in X^*$ we have $\|p_s\| = 1$,*

  (N2) *$\mathbb{K}p_s$ is an L-summand in X^*.*

Proof. (i) $\Rightarrow$ (ii) is just item (e) of Remark 3.5.9.

(ii) $\Rightarrow$ (i) is immediate, as the composition of two L-projections is again an L-projection.

(i) $\Rightarrow$ (iii). Suppose that X has norming ℓ_1-structure witnessed by a set $A \subset S_{X^*}$. Let S be the set A endowed with the restriction of the weak-star topology and define $J\colon X \to C_b(S)$ by $[J(x)](a^*) = a^*(x)$ for every $a^* \in S$ and every $x \in X$. As A is one-norming, it follows that J is an isometric embedding. Moreover, $p_{a^*} = J^*(\delta_{a^*}) = a^* \in X^*$ has norm one and $\mathbb{K}p_{a^*} = \mathbb{K}a^*$ is an L-summand by definition of a norming ℓ_1-structure. Hence, (N1) and (N2) hold.

(iii) $\Rightarrow$ (i). Consider $A = \{p_s \colon s \in S\}$. Observe that $A \subset S_{X^*}$ by (N1) and that $\mathbb{K}a^*$ is an L-summand in X^* by (N2). Besides, A is one-norming since J is an isometry. $\square$

Let us present the main examples of Banach spaces with a norming ℓ_1-structure. The first family of examples is the one of isometric preduals of $L_1(\mu)$ spaces, also known as *isometric L_1-preduals*.

EXAMPLE 3.5.11. Isometric preduals of $L_1(\mu)$ spaces have norming ℓ_1-structure.

PROOF. Let X be a Banach space such that $X^* \cong L_1(\mu)$ and let $A = \operatorname{ext}(B_{X^*})$, which is one-norming by the Krein-Milman Theorem. Recall that the extreme points of the unit ball of an $L_1(\mu)$ space are of the form $\theta \frac{\mathbb{1}_A}{\mu(A)}$ where $\theta \in \mathbb{T}$ and A is an atom of μ with $0 < \mu(A) < \infty$. It readily follows that when $f_0 \in \operatorname{ext}(B_{L_1(\mu)})$, then $\mathbb{K}f_0$ is an L-summand of $L_1(\mu)$. Actually, $L_1(\mu) = \mathbb{K}f_0 \oplus_1 Z$ where Z is just the subspace of those functions of $L_1(\mu)$ whose supports do not intersect A, and the projection is given by $P(f) = \frac{1}{\mu(A)} \int f \mathbb{1}_A$ for every $f \in L_1(\mu)$. $\square$

The second family of spaces having a norming ℓ_1-structure is the one of uniform algebras. Recall that a *unital uniform algebra* is a closed subalgebra X of $C(K)$ for a compact Hausdorff topological space K that separates the points of K (that is, if $t \neq s$ in K, there exists $f \in X$ such that $f(t) \neq f(s)$) and contains the constant functions. The *Choquet boundary* of X is

$$\partial X := \{s \in K\colon \delta_s|_X \in \operatorname{ext}(B_{X^*})\}$$

endowed with the topology induced by K.

REMARK 3.5.12. *By the Hahn-Banach and Krein-Milman theorems, and the well-known description of the extreme points of $B_{C(K)^*}$, it is immediate that*

$$\operatorname{ext}\bigl(\{x^* \in B_{X^*}\colon x^*(\mathbb{1}) = 1\}\bigr) = \{\delta_s|_X\colon s \in \partial X\}$$

and that

$$\operatorname{ext}(B_{X^*}) = \mathbb{T}\{\delta_s|_X\colon s \in \partial X\}.$$

We refer the reader to [**249**, Chapter 6] for background. Consider the operator $J\colon X \to C_b(\partial X)$ by $J(f) = f|_{\partial X}$ for every $f \in X$ and let us see that X is nicely embedded into $C_b(\partial X)$ by means of the map J. It is well known that for every $f \in X$ there is $s \in \partial X$ such that $|f(s)| = \|f\|$ [**249**, Proposition 6.3] (that is, that the Choquet boundary is a boundary in the sense introduced on page 14). In particular, J is an isometric embedding. Besides, for every $s \in \partial X$, $p_s := J^*(\delta_s) = \delta_s|_X$ has norm one, so (N1) holds. Moreover, it is also known that $\mathbb{K}\delta_s$ is an L-summand in X^* [**136**], [**133**, Th. V.4.2]. Therefore, the following result follows.

EXAMPLE 3.5.13. Every unital uniform algebra X has a norming ℓ_1-structure. Moreover, X is nicely embedded into $C_b(\partial X)$ by means of the isometric embedding $J\colon X \to C_b(\partial X)$ defined by $J(f) = f|_{\partial X}$ for every $f \in X$ and $p_s := J^*(\delta_s) = \delta_s|_X$ for every $s \in \partial X$.

Our next result describes Daugavet points of a space with a norming ℓ_1-structure. We need some notation. Given a Banach space X and $x \in S_X$, we write

$$\operatorname{ext}_x^+(B_{X^*}) := \{x^* \in \operatorname{ext}(B_{X^*})\colon \operatorname{Re} x^*(x) = |x^*(x)|\}.$$

Observe that $\mathbb{T}\operatorname{ext}_x^+(B_{X^*}) = \operatorname{ext}(B_{X^*})$. We write

$$D(X,x) := \{x^* \in S_{X^*}\colon x^*(x) = 1\} = \{x^* \in S_{X^*}\colon \operatorname{Re} x^*(x) = 1\}$$

for the set of *states* of the point $x \in X$.

THEOREM 3.5.14. *Let X be a Banach space with norming ℓ_1-structure witnessed by $A \subset S_{X^*}$. Then, the following assertions about a point $x \in S_X$ are equivalent:*

(1) *x is a Daugavet point.*
(2) *For every $\varepsilon > 0$, the set*
$$\{a^* \in A\colon |a^*(x)| > 1 - \varepsilon\}$$
contains infinitely many (pairwise) linearly independent elements.
(3) *$D(X,x) \cap \big(\operatorname{ext}_x^+(B_{X^*}), w^*\big)' \neq \emptyset$, that is, $D(X,x)$ contains a weak-star cluster point of $\operatorname{ext}_x^+(B_{X^*})$.*
(4) *For every $y \in B_X$, there exists a sequence $(x_n^{**}) \subset B_{X^{**}}$ satisfying that $\limsup_n \|x - x_n^{**}\| = 2$ and that*
$$\left\| \sum_{k=1}^{m} \lambda_k (x_k^{**} - y) \right\| \leqslant 2 \max\{|\lambda_1|, \ldots, |\lambda_m|\}$$
for every $m \in \mathbb{N}$ and all $\lambda_1, \ldots, \lambda_m \in \mathbb{K}$ (that is, the linear operator T from c_0 to X^{**} defined by $T(e_n) = x_n^{**} - y$ for all $n \in \mathbb{N}$ is continuous with $\|T\| \leqslant 2$).
(5) *For every $y \in B_X$ there exists a sequence $(x_n^{**}) \subset B_{X^{**}}$ satisfying that $\limsup \|x - x_n^{**}\| = 2$ and that $(x_n^{**}) \to y$ weakly.*
(6) *For every $y \in B_X$ there exists a sequence $(x_n^{**}) \subset B_{X^{**}}$ satisfying that $\limsup \|x - x_n^{**}\| = 2$ and that $(x_n^{**}) \to y$ in the weak-star topology.*

Proof. (1) $\Rightarrow$ (2). Suppose that (2) does not hold. This means that there is $0 < \varepsilon_0 < 1$ such that the set
$$B = \{a^* \in A\colon |a^*(x)| > 1 - \varepsilon_0\}$$
is finite up to rotations. By taking a smaller ε_0 if needed, we actually have that there are $a_1^*, \ldots, a_k^* \in A$ and $\theta_1, \ldots, \theta_k \in \mathbb{T}$ such that $\theta_i a_i^*(x) = 1$ for $i = 1, \ldots, k$ and
$$|a^*(x)| \leqslant 1 - \varepsilon_0 \quad \text{for } a^* \in A \setminus \mathbb{T}\{a_1^*, \ldots, a_k^*\}.$$
Define $x^* = \frac{1}{k} \sum_{i=1}^{k} \theta_i a_i^* \in X^*$ and observe that $\|x^*\| = 1$ since $x^*(x) = 1$. We claim that $\|x - y\| \leqslant 2 - \varepsilon_0$ for every $y \in \operatorname{Slice}\big(B_X, x^*, \frac{\varepsilon_0}{2k}\big)$, which shows that (1) fails by the definition of Daugavet point. Indeed, a convexity argument shows that $\operatorname{Re} \theta_i a_i^*(y) > 1 - \frac{\varepsilon_0}{2}$ for every $i = 1, \ldots, k$, so, in particular,
$$|a_i^*(x - y)| = |\theta_i a_i^*(x - y)| = |1 - \theta_i a_i^*(y)| \leqslant \sqrt{\varepsilon_0}.$$
For $a^* \in A \setminus \mathbb{T}\{a_1^*, \ldots, a_k^*\}$, we have that
$$|a^*(x - y)| \leqslant |a^*(x)| + |a^*(y)| \leqslant 1 - \varepsilon_0 + 1 = 2 - \varepsilon_0.$$
As A is one-norming, we deduce that
$$\|x - y\| \leqslant \max\{\sqrt{\varepsilon_0}, 2 - \varepsilon_0\} = 2 - \varepsilon_0,$$
as desired.

(2) $\Rightarrow$ (3). Pick a sequence (a_n^*) of (pairwise) linearly independent elements of A and a sequence (θ_n) in $\mathbb{T}$ such that
$$\theta_n a_n^*(x) = |a_n^*(x)| > 1 - \tfrac{1}{n}.$$
Then, $\theta_n a_n^* \in \operatorname{ext}_x^+(B_{X^*})$ for every $n \in \mathbb{N}$. The set $\{\theta_n a_n^*\colon n \in \mathbb{N}\}$ being infinite,
$$\emptyset \neq \{\theta_n a_n^*\colon n \in \mathbb{N}\}' \subset \operatorname{ext}_x^+(B_{X^*})'.$$
Clearly, any element in $\{\theta_n a_n^*\colon n \in \mathbb{N}\}'$ belongs to $D(X,x)$.

(3) $\Rightarrow$ (4). Since $D(X,x)\cap \operatorname{ext}_x^+(B_{X^*})' \neq \emptyset$, we may find a sequence (a_n^*) of pairwise linearly independent elements of A and a sequence (θ_n) in $\mathbb{T}$ such that

$$\theta_n a_n^*(x) = |a_n^*(x)| > 1 - \frac{1}{n}.$$

For every $n\in\mathbb{N}$, consider the L-projection P_n such that $X^* = \mathbb{K}a_n^* \oplus_1 \ker P_n$ of Remark 3.5.9(a). Define the linear functional $x_n^{**}\colon X^* = \mathbb{K}a_n^* \oplus_1 \ker P_n \to \mathbb{K}$ by

$$x_n^{**}(\lambda a_n^* + z^*) = -\lambda\overline{\theta_n} + z^*(y).$$

Clearly,

$$|x_n^{**}(\lambda a_n^* + z^*)| \leqslant |\lambda| + |z^*(y)| \leqslant |\lambda| + \|z^*\| = \|\lambda a_n^* + z^*\|,$$

hence x_n^{**} is continuous and, actually, $x_n^{**}\in B_{X^{**}}$. Let us show that the sequence (x_n^{**}) fulfills our requirements. On the one hand,

$$\|x - x_n^{**}\| \geqslant \big|\theta_n a_n^*(x) - x_n^{**}(\theta_n a_n^*)\big| = \big||p_{s_n}(x)| + 1\big| > 2 - \frac{1}{n}.$$

It follows that $\limsup \|x - x_n^{**}\| = 2$, as desired. On the other hand, pick $m\in\mathbb{N}$ and $\lambda_1,\ldots,\lambda_m\in\mathbb{K}$. We consider $P = P_1 + \cdots + P_m$ and use Lemma 2.9.4 to get that P is an L-projection, that $X^* = P(X^*)\oplus_1 \ker P$, that $\ker P = \bigcap_{k=1}^m \ker P_k$, and that $B_{P(Z)} = \operatorname{aconv}\big(\{p_{s_1},\ldots,p_{s_n}\}\big)$. With this in mind, and taking into account that $x_k^{**}(z^*) - z^*(y) = 0$ whenever $z^*\in\ker P\subset \ker P_k$ for $k=1,\ldots,m$, we have that

$$\left\|\sum_{k=1}^m \lambda_k(x_k^{**} - y)\right\| = \sup_{j=1,\ldots,m}\left|\sum_{k=1}^m \lambda_k\big(x_k^{**}(p_{s_j}) - p_{s_j}(y)\big)\right|.$$

But now, as $x_k^{**}(p_{s_j}) - p_{s_j}(y) = 0$ whenever $k,j\in\{1,\ldots,m\}$ with $k\neq j$, it follows that

$$\begin{aligned}\left\|\sum_{k=1}^m \lambda_k(x_k^{**} - y)\right\| &= \max_{j=1,\ldots,m}\big|\lambda_j\big(x_j^{**}(p_{s_j}) - p_{s_j}(y)\big)\big|\\ &\leqslant \max_{j=1,\ldots,m}\big|\lambda_j\big(|x_j^{**}(p_{s_j})| + |p_{s_j}(y)|\big)\big| \leqslant 2\max_{1\leqslant j\leqslant n}|\lambda_j|.\end{aligned}$$

(4) $\Rightarrow$ (5) is immediate since the basis (e_n) of c_0 converges weakly to 0, hence $(T(e_n)+y) = (x_n^{**})$ converges weakly to y.

(5) $\Rightarrow$ (6) is obvious.

Finally, (6) $\Rightarrow$ (1) follows from Lemma 3.1.19. $\square$

An interesting consequence of the above theorem is the following characterisation of the Daugavet property for nicely embedded spaces. We need the following definition.

DEFINITION 3.5.15. Given a Banach space X, we define an equivalence relation in $\operatorname{ext}(B_{X^*})$ by $f\sim g$ if and only if f and g are linearly dependent. We write E_X to denote the quotient space $\operatorname{ext}(B_{X^*})/\!\sim$ endowed with the quotient topology of the weak-star topology. Write $\pi\colon \operatorname{ext}(B_{X^*})\to E_X$ to denote the quotient projection, which is onto, continuous and open when $\operatorname{ext}(B_{X^*})$ is endowed with the weak-star topology.

COROLLARY 3.5.16. *Let X be a Banach space with a norming ℓ_1-structure. Then, the following assertions are equivalent:*

(i) *X has the Daugavet property.*

(ii) *The norm of X is not Fréchet differentiable at any point.*

(iii) *E_X does not contain isolated points.*

We will use the following sufficient condition to get weak-star strongly exposed points.

PROPOSITION 3.5.17. *Let X be a Banach space and let $A \subset \operatorname{ext}(B_{X^*})$ be a one-norming set for X. If $x^* \in A$ satisfies that $\pi(x^*)$ is isolated in $\pi(A) \subset E_X$, then x^* is a weak-star strongly exposed point.*

PROOF. As π is continuous and onto and $\mathbb{T}A$ is weak-star dense in $\operatorname{ext}(B_{X^*})$ by the converse of the Krein-Milman theorem, $\pi(A) = \pi(\mathbb{T}A)$ is dense in E_X. Hence, if $\pi(x^*)$ is isolated in $\pi(A)$, it is a fortiori isolated in E_X. Therefore, we may find a weak-star neighbourhood U of x^* in B_{X^*} such that whenever $z^* \in \operatorname{ext}(B_{X^*})$ belongs to U, then $x^* \sim z^*$. By Choquet's Lemma (see Theorem 2.6.16), we may certainly suppose that U is a weak-star open slice of B_{X^*}; i.e., there are $x \in S_X$ and $0 < \alpha_0 < 1$ such that

$$z^* \in \operatorname{ext}(B_{X^*}),\ z^* \in \operatorname{Slice}(B_{X^*}, x, \alpha_0) \quad \Longrightarrow \quad z^* \sim x^*. \tag{3.5.1}$$

We claim that, for $0 < \alpha \leqslant \alpha_0$ and $y \in S_X$ satisfying $\|y - x\| < \alpha$, there exists $\omega_y \in \mathbb{T}$ such that $D(X, y)$ reduces to the singleton $\{\omega_y\, x^*\}$ and

$$\|\omega_y\, x^* - \omega_x\, x^*\| < \sqrt{2\alpha}.$$

Let us observe that this claim finishes the proof since it implies that every selector of the duality mapping is norm to norm continuous at x, which gives that the norm of X is Fréchet-smooth at x (see [**96**, Theorem II.2.1]) and then $\omega_x\, x^*$ (and hence x^*) is weak-star strongly exposed (by Shmulyan's test, Lemma 2.10.5).

Let us prove the claim. If $\|y - x\| < \alpha$, then every $y^* \in D(X, y)$ satisfies

$$\operatorname{Re} y^*(x) = \operatorname{Re} y^*(y) - \big(\operatorname{Re} y^*(y) - \operatorname{Re} y^*(x)\big) \geqslant 1 - \|x - y\| > 1 - \alpha, \tag{3.5.2}$$

and so $D(X, y)$ is contained in $\operatorname{Slice}(B_{X^*}, x, \alpha) \subset \operatorname{Slice}(B_{X^*}, x, \alpha_0)$. Then, every extreme point of the weak-star closed face $D(X, y)$ (remaining extreme in B_{X^*}) is a multiple of x^* by Eq. (3.5.1). Since only one multiple of x^* can be in the face $D(X, y)$ and, being weak-star compact, $D(X, y)$ is the weak-star closed convex hull of its extreme points, we get $D(X, y) = \{\omega_y\, x^*\}$ for a suitable modulus-one scalar ω_y. Finally, on the one hand, since $|x^*(x)| = 1$, we have that

$$\|\omega_x\, x^* - \omega_y\, x^*\| = |\omega_x - \omega_y| = |\omega_x\, x^*(x) - \omega_y\, x^*(x)| = |1 - \omega_y\, f(x)|.$$

On the other hand, Eq. (3.5.2) says that $\operatorname{Re} \omega_y\, x^*(x) > 1 - \alpha$ and so, a straightforward computation gives that

$$|1 - \omega_y\, x^*(x)| < \sqrt{2\alpha}.$$

Finally, if $\pi^{-1}(\pi(z^*)) \cap A$ is finite for every z^*, then $\pi(x^*)$ is isolated in $\pi(A)$ if and only if x^* is isolated in A. $\square$

We are now ready to provide the pending proof.

PROOF OF COROLLARY 3.5.16. By Remark 3.5.9(d), we may consider a one-norming subset $A \subset S_{X^*}$ such that its elements are pairwise linearly independent and such that $\mathbb{K}a^*$ is an L-summand in X^* for every $a^* \in A$.

(i) $\Rightarrow$ (ii) is immediate since a point $x \in S_X$ at which the norm is Fréchet differentiable produces weak-star strongly exposed points in B_{X^*} and hence, weak-star slices of B_{X^*} with arbitrarily small diameter.

(ii) $\Rightarrow$ (iii). This is Proposition 3.5.17.

(iii) $\Rightarrow$ (i). For every $x \in S_X$ and $\varepsilon > 0$, consider the weak-star open set $U = \{x^* \in X^*\colon |x^*(x)| > 1-\varepsilon\}$ and let $V = U \cap \operatorname{ext}(B_{X^*})$. Then, $\pi(V)$ is open in E_X, hence infinite as E_X has no isolated points. Since A is weak-star dense in $\operatorname{ext}(B_{X^*})$ and π is open, $\pi(A)$ is dense in E_X. Hence,

$$\pi(A) \cap \pi(V) = \{[a^*] \in E_X\colon a^* \in A,\ |x^*(x)| > 1-\varepsilon\},$$

is also infinite. It follows that the set

$$\{a^* \in A\colon |a^*(x)| > 1-\varepsilon\}$$

contains infinitely many pairwise linearly independent elements. Now, Theorem 3.5.14 shows that x is a Daugavet point. ☐

The following result gives a useful sufficient condition to get the Daugavet property of a Banach space with a norming ℓ_1-structure. It is written in terms of nice embeddings with the notation from Proposition 3.5.10.

Corollary 3.5.18. *Let S be Hausdorff topological space without isolated points and let X be a Banach space nicely embedded into $C_b(S)$ such that the set $\{p_s\colon s \in S\}$ is pairwise linearly independent. Then, $X \in \mathrm{DPr}$.*

Proof. Let $J\colon X \to C_b(S)$ be the corresponding isometric embedding and write $A := \{p_s\colon s \in S\} \subset X^*$. Consider the map $\Psi\colon S \to A$ given by $\Psi(s) = p_s = J^*(\delta_s)$ for every $s \in S$. Observe that Ψ is continuous when A is endowed with the weak-star topology (as it is the composition of J^* with the weak-star continuous map $s \mapsto \delta_s$). Therefore, for every $x \in X$ and every $\varepsilon > 0$, the set

$$V = \{s \in S\colon |p_s(x)| > 1-\varepsilon\} = \Psi^{-1}\big(\{p_s \in A\colon |p_s(x)| > 1-\varepsilon\}\big)$$

is open in S, hence infinite. As the set A is pairwise linearly independent, it follows that the set $\{p_s \in A\colon |p_s(x)| > 1-\varepsilon\}$ contains infinitely many pairwise linearly independent elements. Now, Theorem 3.5.14 gives the result. ☐

Some remarks are pertinent.

Remarks 3.5.19.

(a) *Corollary 3.5.18 does not hold if we do not require the set $\{p_s\colon s \in S\}$ to be pairwise linearly independent. Indeed, consider any compact Hausdorff topological space K (with or without isolated points) and consider the mapping J: $C(K) \to C(K \times [0,1])$ given by $[J(f)](k,t) = f(k)$ for every $(k,t) \in K \times [0,1]$ and every $f \in C(K)$. Then, J embeds $C(K)$ nicely into $C(K \times [0,1])$ and $K \times [0,1]$ has no isolated points.*

(b) *A Banach space X that is nicely embedded in $C_b(S)$ for some Hausdorff topological space S with isolated points and such that the set $\{p_s\colon s \in S\}$ is pairwise linearly independent may still have the Daugavet property.* Indeed, let $S = \{-1\} \cup (0,1] \subset \mathbb{R}$ and consider the embedding $J\colon C[0,1] \to C_b(S)$ given by $[J(f)](-1) = f(0)$ and $[J(f)](s) = f(s)$ for $s \in (0,1]$. It is immediate that J is an isometric embedding. Observe that $J^*(\delta_{-1}) = \delta'_0 \in C[0,1]^*$ and $J^*(\delta_s) = \delta'_s \in C[0,1]^*$ for $s \in (0,1]$. It follows that $C[0,1]$ is nicely embedded in $C_b(S)$ and that $\{p_s\colon s \in S\}$ is pairwise linearly independent.

Let us particularise the results of the subsection to the cases of isometric preduals of $L_1(\mu)$ spaces and of unital uniform algebras. The following is a special case of Corollary 3.5.16.

Corollary 3.5.20. *Let X be an isometric predual of an $L_1(\mu)$ space or a unital uniform algebra. Then, the following assertions are equivalent:*

(i) *X has the Daugavet property.*
(ii) *The norm of X is not Fréchet differentiable at any point.*
(iii) *E_X does not contain isolated points.*

In the case of a unital uniform algebra X, we may give a characterisation of the Daugavet property in terms of the Choquet boundary ∂X which complements Corollary 3.5.16.

Corollary 3.5.21. *Let X be a unital uniform algebra with Choquet boundary ∂X. Then, the following are equivalent:*

(i) *X has the Daugavet property.*
(ii) *The norm of X is not Fréchet differentiable at any point.*
(iii) *E_X does not contain isolated points.*
(iv) *∂X does not contain isolated points.*

Proof. The equivalence between (i), (ii), and (iii) is given in Corollary 3.5.16.

(iii) $\Rightarrow$ (iv). Suppose that $s_0 \in \partial X$ is isolated in ∂X. Then, there is an open set U of K such that $U \cap \partial X = \{s_0\}$. It is well known that the elements in ∂X are strong boundary points [**86**, Theorem 4.3.4], so there is $f \in X$ with $\|f\| = f(s_0) = 1$ such that $|f(s)| \leqslant 1/2$ for $s \in K \setminus U$ (an alternative direct proof of this fact is given in [**76**, Lemma 2.1]). This implies that the set

$$\begin{aligned} W &= \{x^* \in \operatorname{ext}(B_{X^*})\colon |x^*(f)| > 3/4\} \\ &= \{\theta\delta_s|_X\colon s \in \partial X,\ |f(s)| > 3/4,\ \theta \in \mathbb{T}\} \end{aligned}$$

reduces to $\mathbb{T}\{\delta_{s_0}|_X\}$. It clearly follows that $\pi(\delta_{s_0}|_X)$ is isolated in E_X.

Finally, to get (iv) $\Rightarrow$ (i), first observe that (using the notation given in Example 3.5.13) for $s, t \in \partial X \subset K$ with $s \neq t$, as X separates the points of K, $p_s = J^*(\delta_s) = \delta_s|_X$ and $p_t = J^*(\delta_t) = \delta_t|_X$ are linearly independent. Therefore, Corollary 3.5.18 gives the implication. $\square$

A paradigmatic example of a unital uniform algebra is the *disk algebra* $\mathbb{A}$, the space of those functions in $C\big(\overline{\mathbb{D}}\big)$ which are holomorphic in $\mathbb{D}$, endowed with the supremum norm. The Choquet boundary of $\mathbb{A}$ is $\mathbb{T}$ (see for instance Example I.1.4(b) in [**133**]), so Corollary 3.5.21 shows that $\mathbb{A}$ has the Daugavet property.

3.5.3. Representable Banach spaces. Our aim in this section is to present a class of Banach spaces having the Daugavet property which is stable by passing to some spaces of operators and to injective tensor products, so getting a number of new examples.

Let us present the needed notation to give the proper definition of the new class. Let K be a compact Hausdorff topological space and let $\{X_k\colon k \in K\}$ be a family of Banach spaces. We write

$$\left(\bigoplus_{k\in K} X_k\right)_{\ell_\infty}$$

for the Banach space of all bounded families $(x_k)_{k\in K}$ with $x_k \in X_k$ for every $k \in K$, endowed with the supremum norm. This space is a $C(K)$-*module* in the sense that given $f \in C(K)$ and $(x_k)_{k\in K}$, the family $(f(k)x_k)_{k\in K}$ still belongs to the space.

DEFINITION 3.5.22. Let K be a compact Hausdorff topological space and let X be a Banach space. We say that X is *K-representable* if there is a family of Banach spaces $\{X_k\colon k \in K\}$ such that X is isometrically isomorphic to a closed $C(K)$-submodule of $\left(\bigoplus_{k\in K} X_k\right)_{\ell_\infty}$ satisfying that for every $x \in S_X$ and every $\varepsilon > 0$, the set

$$\{k \in K\colon \|x(k)\| > 1 - \varepsilon\}$$

is infinite. When the compact set K is not relevant, we just say that X is *representable.*

It is immediate to check that the only compact Hausdorff topological spaces K for which there are K-representable spaces are the perfect ones. Indeed, if X is K-representable and if there is an isolated point $t_0 \in K$ then, for fixed $x \in S_X$, the element $\mathbb{1}_{\{t_0\}}x$ belongs to X and, clearly, the set $\{t \in K\colon \|x(t)\| > 0\}$ is finite.

The main result here is that representable Banach spaces have the Daugavet property.

THEOREM 3.5.23. *Let X be a representable Banach space. Then, X has the Daugavet property.*

We need a folklore result on compact Hausdorff topological spaces. (Part (b) will be used in Chapter 11.)

LEMMA 3.5.24.

(a) *Let K be a compact Hausdorff topological space and let S be an infinite subset of K. Then, there are a sequence $(k_n)_{n\in\mathbb{N}}$ in S and a sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint nonempty open subsets of K such that $k_n \in U_n$ for every $n \in \mathbb{N}$.*

(b) *Let (M, d) be a metric space let S be an infinite subset of M. Then, there are a sequence $(k_n)_{n\in\mathbb{N}}$ in S and a sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint nonempty balls in M with centre k_n for every $n \in \mathbb{N}$.*

PROOF. (a) Let $w \in K$ be a cluster point of S. Choose $k_1 \in S\setminus\{w\}$ arbitrarily. By the Hausdorff condition, we may choose disjoint open sets $U_1, W_1 \subset K$ in such a way that $k_1 \in U_1$ and $w \in W_1$. So k_1 and U_1 are already constructed. By the construction, $S \cap W_1$ is infinite. Choose $k_2 \in (S \cap W_1) \setminus \{w\}$ arbitrarily. By the Hausdorff condition, we may choose disjoint open sets $U_2, W_2 \subset W_1$ such that $k_2 \in U_2$ and $w \in W_2$. Then $U_2 \cap U_1 = \emptyset$. Proceeding this way, we obtain the desired sequences $(k_n)_{n\in\mathbb{N}}$ and $(U_n)_{n\in\mathbb{N}}$.

(b) It is enough to apply part (a) to the Stone-Čech compactification $K = \beta M$ of M. (Admittedly, a direct proof would be possible, too.) $\square$

PROOF OF THEOREM 3.5.23. Let K and $\{X_k\colon k \in K\}$ be as in Definition 3.5.22. Fix $x \in S_X$, $z \in B_X$, and $\varepsilon > 0$. As the set $S = \{k \in K\colon \|x(k)\| > 1 - \varepsilon/2\}$ is infinite, the above lemma provides a sequence $(k_n)_{n\in\mathbb{N}}$ in S and a sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint nonempty open subsets of K such that $k_n \in U_n$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, Urysohn's lemma allows us to find $f_n \in C(K)$ such that $0 \leqslant f_n \leqslant 1$, $f_n(k_n) = 1$ and $\operatorname{supp}(f_n) \subset U_n$. The elements of the sequence $(f_n)_{n\in\mathbb{N}}$ have pairwise disjoint supports by construction, so the operator from c_0 to X given by $e_n \mapsto f_n(x - z)$ is well-defined and continuous, hence the sequence $(f_n(x - z))_{n\in\mathbb{N}}$ weakly converges to 0. Therefore, writing

$z_n := f_n x + (1-f_n)z$ for every $n\in\mathbb{N}$, we have that
$$\|z_n\| = \sup_{k\in K}\|z_n(k)\| = \sup_{k\in K}\|f_n(k)x + (1-f_n(k))z\| \leqslant 1,$$
that the sequence $(z_n)_{n\in\mathbb{N}}$ weakly converges to z (as the $f_n's$ have disjoint support), and that
$$\|x+z_n\| \geqslant \|x(k_n)+z_n(k_n)\| = \|2x(k_n)\| \geqslant 2-\varepsilon.$$
Now, Lemma 3.1.19 shows that $X\in \mathrm{DPr}$. □

The main advantage of the previous result is that we can identify spaces of operators with the Daugavet property with its help.

Proposition 3.5.25. *Let X be a Banach space and let Y be a representable space. Then every closed subspace M of $L(X,Y)$ such that $L(Y)\circ M \subset M$ is representable.*

Proof. Let K and $\{Y_k\colon k\in K\}$ be as in Definition 3.5.22 for Y and define $M_k := L(X,Y_k)$. We claim that M is K-representable in $\left(\bigoplus_{k\in K} M_k\right)_{\ell_\infty}$. Indeed, given $T\in M$, we denote by $\widehat{T}$ the element of $\left(\bigoplus_{k\in K} M_k\right)_{\ell_\infty}$ defined by $[\widehat{T}(k)](x) = [Tx](k)$ for every $k\in K$ and every $x\in X$. It is immediate that the mapping $T\mapsto\widehat{T}$ is an isometric embedding from M onto the subspace $\widehat{M} := \{\widehat{T}\colon T\in M\}$ of $\left(\bigoplus_{k\in K} M_k\right)_{\ell_\infty}$. As $L(Y)\circ M\subset M$ and multiplication by elements of $C(K)$ belongs to $L(Y)$, it is straightforward to show that $\widehat{M}$ is a $C(K)$-submodule of $\left(\bigoplus_{k\in K} M_k\right)_{\ell_\infty}$. Finally, fix $T\in S_M$ and $\varepsilon>0$. There is $x\in S_X$ such that $\|Tx\|>1-\varepsilon$ and, since Y is representable, the set
$$\{k\in K\colon \|[Tx](k)\|>1-\varepsilon\}$$
is infinite. Hence, the set
$$\{k\in K\colon \|\widehat{T}(k)\|>1-\varepsilon\}$$
is infinite. □

In particular, this applies to injective tensor products; $X\otimes Y$ should be identified with the weak-star continuous finite-rank operators from X^* to Y.

Corollary 3.5.26. *Let X be a Banach space and let Y be a representable space. Then, $X\mathbin{\widehat{\otimes}_\varepsilon} Y$ is representable.*

It is immediate that the space $C(K)$ is K-representable if K is perfect. Actually, the same is true for $C(K,Y)$ for every Banach space Y. Our next aim is to extend this latter result to continuous functions from K to Y with other topologies. Fix a perfect compact Hausdorff topological space K and a Banach space Y, consider a one-norming subspace Z of Y^* and let τ be a vector topology on Y satisfying that $\sigma(Y,Z)\leqslant\tau\leqslant n$ (where n stands for the norm topology). We write $C(K,(Y,\tau))$ for the space of those functions from K to Y which are continuous when Y is endowed with the topology τ. For every element $f\in C(K,(Y,\tau))$, $f(K)$ is compact in (Y,τ), hence compact in $(Y,\sigma(Y,Z))$ and therefore norm bounded in Y by the uniform boundedness principle (as Z is one-norming). Therefore, we can view $C(K,(Y,\tau))$ as a subspace of $\left(\bigoplus_{k\in K} Y\right)_{\ell_\infty}$. As $\tau\leqslant n$, $C(K,(Y,\tau))$ is actually a closed subspace. Applying again that Z is one-norming, it follows that for every $f\in S_{C(K,(Y,\tau))}$, the function $k\mapsto\|f(k)\|$ is lower semicontinuous from K into $\mathbb{R}$. Therefore, for every

$\varepsilon > 0$, the set $\{k\colon \|f(k)\| > 1-\varepsilon\}$ is open, hence infinite as K is perfect. We have shown the following.

EXAMPLE 3.5.27. Let K be a perfect Hausdorff topological space, let Y be a Banach space, let Z be a subspace of Y^* which is one-norming for Y, and let τ be a vector topology on Y such that $\sigma(Y,Z) \leqslant \tau \leqslant n$. Then, the Banach space $C(K,(Y,\tau))$ is representable and hence has the Daugavet property.

3.6. Notes and remarks

Section 3.1. The "modern" approach to the Daugavet equation started with the paper [**178**] (see also [**177**]) that rephrased the problem in terms of slices of the unit ball and thus made it susceptible for geometric methods of Banach space theory. Shortly afterwards, Roman Shvidkoy (Shvydkoy in the transliteration of the Ukrainian spelling of the name), solved in [**285**] a number of problems left open in [**178**]. In the process, he rediscovered Bourgain's lemma 2.6.19 from [**62**] that was unbeknownst to him and which proved to be invaluable in many follow-up papers on the subject.

The notion of Daugavet point of Definition 3.1.7 is considered in [**4**] as a pointwise version of the characterisation of the Daugavet property in terms of the behaviour of slices. Let us mention that in a similar way two new notions appeared in [**224**, Definition 2.5] in order to crystallise a pointwise version of the characterisation of the Daugavet property in terms of the behaviour of weakly open sets and convex combinations of slices (ccs for short).

Let X be a Banach space and let $x \in S_X$. We say that

(1) x is a *Daugavet point* if $\sup_{y\in S}\|x-y\| = 2$ for every slice S of B_X [**4**],
(2) x is a *super Daugavet point* if $\sup_{y\in V}\|x-y\| = 2$ for every non-empty relatively weakly open subset V of B_X [**224**],
(3) x is a *ccs Daugavet point* if $\sup_{y\in C}\|x-y\| = 2$ for every ccs C of B_X [**224**].

Observe that Lemma 3.1.15 implies that a Banach space X has the Daugavet property if, and only if, every point of S_X is a Daugavet point (respectively a super Daugavet point, a ccs Daugavet point). However, it turns out that these notions, which are equivalent when all the points of S_X are considered, are different from each other in the pointwise versions. We refer the interested reader to [**224**, Subsection 4.7] for counterexamples and further background. In Section 12.2 we shall report on the various diameter 2 properties that have their origin in Lemma 3.1.15.

Section 3.2 builds on the results of [**178**] where the existence of ℓ_1-subspaces was proved without explicit recourse to ℓ_1-type sequences. The paper also contains the first example of a space with the Daugavet property without $L_1[0,1]$-subspaces. Theorem 3.2.1 vastly generalises Wojtaszczyk's [**302**] observation that the unit ball of a space with the Daugavet property doesn't contain strongly exposed points, and it has been the starting point for the study of diameter 2 properties that will be surveyed in Section 12.2.

Section 3.3 reproves the Daugavet property for the classical $C(K)$- and $L_1(\mu)$-spaces; see Chapter 1 for references to the original works.

Section 3.4. The concept of ℓ_1-dispersed space and Theorem 3.4.4 in the part that concerns $\overline{L}_1$ are, to the best of our knowledge, new. The concepts of L_1-dispersed and L_∞-dispersed spaces and their applications to the Daugavet property are new as well. The notion of L_1- and ℓ_1-dispersedness appears custom-made to deal with Bochner L_1-spaces and their generalisations. The first proof of the Daugavet property of $L_1(\mu, X)$ was given by I. Nazarenko in his unpublished Master diploma thesis (V. N. Karazin National University Kharkiv, 1999) and was published in [**178**].

Section 3.5. The Daugavet property for C^*- and von Neumann algebras was investigated in [**240**] and [**44**]; the first reference contains a proof that Daugavet property and diffuseness are equivalent for C^*-algebras which is completely algebraic. It is the latter paper's approach making use of the L-structure of the duals that is reproduced here. Actually, [**44**] even deals with a vast generalisation of C^*-algebras, the JB*-triples, and we refer to that paper for details including the somewhat technical definition of a JB*-triple; besides, this paper deals with both complex and real C^*-algebras and JB*-triples. In many senses, JB*-triples are a nice setting to study geometric properties of C^*-algebras: the absence of nice algebraic structures forces one to look for other weaker algebraic notions (such as the one of "tripotent") which have an important geometric meaning. This is specially the case when dealing with the real case, for which even the theory of C^*-algebras is not so nice as in the complex case. We refer the reader to [**34, 70, 105, 113, 248**] as a sample of papers on geometric properties of JB*-triples which may give a taste of them. The paper [**226**] contains further results on the geometry of preduals of von Neumman algebras with the Daugavet property (i.e., preduals of diffuse von Neumann algebras, Theorem 3.5.5). Indeed, the characterisation given in the aforementioned theorem, showing that the predual M_* of a von Neumann algebra M has the Daugavet property if and only if B_{M_*} fails to contain extreme points, is carried further. In fact, if M is diffuse, there is a dense subset of elements of S_{M_*} which are the starting point of a girth curve (that is, a simple curve lying in S_{M_*} connecting an element and its opposite and with length two); this set is the whole S_{M_*} if M does not contain any type III_1 factor as a direct summand, and it is strictly contained in S_{M_*} at least when M is the unique hyperfinite type III_1 factor.

The idea of norming ℓ_1-structure is implicit in [**299**], while the name only appears in this book. The proof of the fundamental Theorem 3.5.14 is taken from [**225**], where it is proved for L_1-preduals first and extended to nicely embedded spaces. The global characterisation of the Daugavet property for spaces with a norming ℓ_1-structure given in Corollary 3.5.16 (the absence of Fréchet differentiable points is equivalent to the Daugavet property) appeared in [**45**] for L_1-preduals, based on the results of [**299**], but seems to be new for uniform algebras. The same kind of ideas have also been used in [**151**] to show that some algebras of holomorphic functions defined on a bounded open balanced convex subset of a Banach space have the Daugavet property.

Finally, the notion of a representable Banach space and its relation to the Daugavet property are discussed in [**46**]; see also [**47**].

3.7. Open questions

In this subsection we collect different open questions and possible future research lines derived from the results of this chapter.

The following problem formulated more than two decades ago remains open:

(3.1) Does there exist a bidual space with the Daugavet property, i.e., does there exist a Banach space X such that $X^{**} \in \mathrm{DPr}$?

The following question has to do with the existence of a minimal subspace that Daugavet spaces must contain.

(3.2) Let us call a Banach space X a *universal Daugavet subspace* if it embeds isomorphically in every space with the Daugavet property. We know that ℓ_1 and, consequently, the subspaces of ℓ_1 are universal Daugavet subspaces. Is it true that every universal Daugavet subspace is isomorphic to a subspace of ℓ_1?

Even though it seems that a Banach space with the Daugavet property has to be very far from being strictly convex or smooth, we do not know if this is really the case.

(3.3) Is there a strictly convex Banach space with the Daugavet property?

(3.4) Is there a smooth Banach space with the Daugavet property?

Let us give some comments on the previous two questions. First, in absence of completeness, the answer to question (3.3) is positive: in [**173**, §5] a non-complete strictly convex normed space with the Daugavet property is constructed. A similar construction is done in [**161**, §3] to get a (non-complete) strictly convex normed space which is lush and so it has the alternative Daugavet property (see Section 12.3 for the definition and background of these properties), and the same proof actually shows that the constructed space has the Daugavet property. In fact, the construction in [**161**, §3] gives a strictly convex normed space with the Daugavet property containing an arbitrary fixed Banach space; the completion of this second example is a predual of an L_1-space, hence very far away from being strictly convex. On the other hand, it is shown in Corollary 3.2.5 that the dual of a Banach space with the Daugavet property can neither be smooth nor strictly convex.

Finally, we do not even know if it is possible to construct a non-complete smooth normed space with the Daugavet property.

(3.5) Is there a smooth (non-complete) normed space with the Daugavet property?

CHAPTER 4

Further results on the Daugavet property

The present chapter contains more specialised or advanced topics: separable determination of the Daugavet property, tensor products, L-orthogonal elements, L-embedded spaces, rearrangement invariant spaces, and the polynomial Daugavet property are studied.

4.1. Separable determination

Roughly speaking, in this section we are going to demonstrate that a non-separable Banach space possesses the Daugavet property if and only if it has "a lot of" separable subspaces with the same property.

We start with a small perturbation of Proposition 3.1.8.

PROPOSITION 4.1.1. *$X \in \mathrm{DPr}$ if and only if the set of Daugavet points of S_X is dense in S_X.*

PROOF. In one direction there is nothing to prove: if $X \in \mathrm{DPr}$, then the set of Daugavet points of S_X is equal to S_X. So, we have to concentrate on the converse implication. Assume that the set of Daugavet points of S_X is dense. Let $x \in S_X$ be an arbitrary element. We are going to demonstrate, using (iii) of Lemma 3.1.9, that x is a Daugavet point. Fix $\varepsilon > 0$ and select a Daugavet point $v \in S_X$ with $\|x - v\| < \varepsilon/2$. For every slice S of B_X there is $y \in S \cap S_X$ with $\|v - y\| \geqslant 2 - \varepsilon/2$; then $\|x - y\| \geqslant \|v - y\| - \|x - v\| \geqslant 2 - \varepsilon$. □

Combining this with (viii) of Lemma 3.1.9, we obtain one more reformulation.

PROPOSITION 4.1.2. *$X \in \mathrm{DPr}$ if and only if there is a dense subset A of S_X such that for every $v \in A$ and every $\delta \in (0, 1)$*

$$\overline{\mathrm{conv}}(A \setminus (v + (2 - \delta)B_X)) \supset A. \tag{4.1.1}$$

PROOF. Let us start with the "if" part. By the density of A in S_X and convexity and closedness of the left hand side of (4.1.1), the inclusion (4.1.1) is equivalent to

$$\overline{\mathrm{conv}}(A \setminus (v + (2 - \delta)B_X)) \supset B_X. \tag{4.1.2}$$

Since $A \subset S_X$, (4.1.1) implies that

$$\overline{\mathrm{conv}}(S_X \setminus (v + (2 - \delta)B_X)) \supset B_X,$$

which, thanks to (viii) of Lemma 3.1.9, means that v is a Daugavet point. Therefore, our condition implies that the set of Daugavet points of S_X contains A and so, it is dense. By the previous Proposition, $X \in \mathrm{DPr}$.

Conversely, let $X \in \mathrm{DPr}$. In this case $A = S_X$ satisfies (4.1.1) for all $v \in A$ and every $\delta \in (0, 1)$ (item (ix) of Theorem 3.1.11). □

Definition 4.1.3. A family $\mathcal{G}$ of subsets of a given set X is said to be *upward filtering* if for every pair $E_1, E_2 \in \mathcal{G}$ there is $E_3 \in \mathcal{G}$ such that $E_3 \supset E_1 \cup E_2$.

Lemma 4.1.4. *Let $\mathcal{G}$ be an upward filtering family of linear subspaces of a Banach space X, and let $Y = \overline{\bigcup_{V \in \mathcal{G}} V}$. Let $\mathcal{G}$ possess the following property: for every $\delta > 0$ and $E \in \mathcal{G}$ there is $Z \in \mathcal{G}$ such that for every $z \in S_E$*

$$\overline{\operatorname{conv}}(S_Z \setminus (z + (2-\delta)B_X)) \supset S_E. \tag{4.1.3}$$

Then $Y \in \mathrm{DPr}$.

Proof. The upward filtering of $\mathcal{G}$ implies that Y is a subspace. In order to demonstrate the Daugavet property of Y, let us apply Proposition 4.1.2 with $A := \bigcup_{V \in \mathcal{G}} S_V$. The density of A in S_Y is evident. Next, for arbitrary $v, z \in A$ choose $E \in \mathcal{G}$ that contains both elements v and z. By assumption on $\mathcal{G}$, for this E and every $\delta > 0$ there is $Z \in \mathcal{G}$ for which (4.1.3) holds true. Then,

$$\overline{\operatorname{conv}}(A \setminus (z + (2-\delta)B_X)) \supset \overline{\operatorname{conv}}(S_Z \setminus (z + (2-\delta)B_X)) \supset S_E \supset \{v\}.$$

By the arbitrariness of $v \in A$ we obtain the desired inclusion

$$\overline{\operatorname{conv}}(A \setminus (z + (2-\delta)B_Y)) \supset A. \qquad \square$$

Corollary 4.1.5. *Let $X_1 \subset X_2 \subset X_3 \subset \dots$ be an increasing chain of linear subspaces of a Banach space X such that for every $n \in \mathbb{N}$ and $z \in S_{X_n}$*

$$\overline{\operatorname{conv}}\left(S_{X_{n+1}} \setminus \left(z + \left(2 - \frac{1}{n}\right) B_X\right)\right) \supset S_{X_n}. \tag{4.1.4}$$

Then $\overline{\bigcup_{n \in \mathbb{N}} X_n} \in \mathrm{DPr}$.

Proof. It is sufficient to apply Lemma 4.1.4 with $\mathcal{G} = (X_n)_{n \in \mathbb{N}}$. In order to verify (4.1.3), for given $\delta > 0$ and $E = X_m \in \mathcal{G}$, as the needed $Z \in \mathcal{G}$ one should take X_n with $n > m$ so large that $1/n < \delta$. $\square$

Corollary 4.1.6. *Let $X_1 \subset X_2 \subset X_3 \subset \dots$ be an increasing chain of subspaces of a Banach space X such that all X_n have the Daugavet property and $\bigcup_{n \in \mathbb{N}} X_n$ is dense in X. Then $X \in \mathrm{DPr}$.*

Proof. This follows from the previous Corollary. The validity of (4.1.4) follows from item (ix) of Theorem 3.1.11. $\square$

Theorem 4.1.7. *For a Banach space X the following assertions are equivalent:*

(i) *$X \in \mathrm{DPr}$.*

(ii) *Every separable subspace of X is contained in another separable subspace with the Daugavet property.*

Proof. (ii) $\Rightarrow$ (i). The condition (ii) means that the family $\mathcal{G}$ of separable subspaces of X that possess the Daugavet property is upward filtering and its union is the whole space X. It remains to apply Lemma 4.1.4: the corresponding condition 4.1.3 holds true even with $Z = E$ (item (ix) of Theorem 3.1.11 for $E \in \mathrm{DPr}$).

(i) $\Rightarrow$ (ii). Let $E \subset X$ be a separable subspace. Denoting $X_1 := E$, our goal is to construct inductively a sequence $X_1 \subset X_2 \subset X_3 \subset \dots$ of linear subspaces that fulfills the conditions of Corollary 4.1.5. If we succeed, $Y := \overline{\bigcup_{n \in \mathbb{N}} X_n}$ will be the needed separable subspace with the Daugavet property that contains E. In order to perform the construction, we have to demonstrate that, for the given separable X_n, there is a separable X_{n+1} fulfilling the condition (4.1.4) for every

$z \in S_{X_n}$. Indeed, take a sequence $(z_k)_{k\in\mathbb{N}} \subset S_{X_n}$ that is dense in S_{X_n}. Applying the Daugavet property of X in the form from (ix) of Theorem 3.1.11, for each pair $(k,j) \in \mathbb{N}\times\mathbb{N}$ select a sequence

$$(v_{k,j,m})_{m\in\mathbb{N}} \subset S_X \setminus \left(z_k + \left(2 - \tfrac{1}{2n}\right) B_X\right),$$

in such a way that $z_j \in \overline{\operatorname{conv}}\big(\{v_{k,j,m}\colon m \in \mathbb{N}\}\big)$. Put $X_{n+1} := \operatorname{lin}\{v_{k,j,m}\colon k,j,m \in \mathbb{N}\}$. The separability of X_{n+1} is evident; let us check (4.1.4). For a given $z \in S_{X_n}$ choose $k_0 \in \mathbb{N}$ such that $\|z - z_{k_0}\| < \frac{1}{2n}$. Then for every $j,m \in \mathbb{N}$

$$v_{k_0,j,m} \in S_X \setminus \left(z + \left(2 - \tfrac{1}{n}\right) B_X\right),$$

and $z_j \in \overline{\operatorname{conv}}\big(\{v_{k_0,j,m}\colon m \in \mathbb{N}\}\big)$, $j = 1,2,\dots$. Consequently,

$$\overline{\operatorname{conv}}\left(S_{X_{n+1}} \setminus \left(z + \left(2 - \tfrac{1}{n}\right) B_X\right)\right) \supset \overline{\{z_j\colon j \in \mathbb{N}\}} \supset S_{X_n}. \qquad \square$$

4.2. The Daugavet property in tensor product spaces

In this section we address the question of when the Daugavet property is preserved by taking injective or projective tensor products. Here is an example where this is so; indeed, the identification $C(K) \mathbin{\widehat{\otimes}_\varepsilon} Y = C(K,Y)$ together with the results of Section 3.4 show that sometimes $X \mathbin{\widehat{\otimes}_\varepsilon} Y$ has the Daugavet property whenever X or Y has the Daugavet property. Similarly, the natural identification $L_1(\mu) \mathbin{\widehat{\otimes}_\pi} Y = L_1(\mu, Y)$ shows the same about the projective tensor product, again thanks to the results of Section 3.4.

However, the tensor product does not inherit the Daugavet property from one of its factors, in general.

THEOREM 4.2.1. *Let $1 < p < 2$, $1/p + 1/p^* = 1$, and $n \geqslant 3$. Then, neither $L_1 \mathbin{\widehat{\otimes}_\varepsilon} \ell_p^{(n)}$ nor $L_\infty \mathbin{\widehat{\otimes}_\pi} \ell_{p^*}^{(n)}$ enjoys the Daugavet property.*

We need the following result. For the concept of finite representability, we refer to Definition 2.2.8.

LEMMA 4.2.2. *Let X and Y be Banach spaces and assume that Y^* is uniformly convex. Assume also that there exists a closed subspace H of $L(Y^*, X)$ such that $X \otimes Y \subset H$ and H has the Daugavet property. Then, Y^* is finitely representable in X.*

PROOF. Recall that the modulus of uniform convexity of Y^* is given by

$$\delta_{Y^*}(\varepsilon) = \inf\left\{1 - \left\|\frac{f+g}{2}\right\| \colon f,g \in B_{Y^*},\ \|f-g\| \geqslant \varepsilon\right\}.$$

Note that if $f,g \in B_{Y^*}$ satisfy $\operatorname{Re} f(y) > 1 - \delta_{Y^*}(\varepsilon)$ and $\operatorname{Re} g(y) > 1 - \delta_{Y^*}(\varepsilon)$, for some $y \in S_Y$, then $\|f - g\| < \varepsilon$.

Let $\varepsilon > 0$ and choose $\nu > 0$ so small that $(1+\nu)(1-3\nu)^{-1} < 1 + \varepsilon$. Pick $0 < \eta < \nu/2$ such that $\delta_{Y^*}(\eta) < \nu/2$.

Let $F \subset Y^*$ be a finite-dimensional subspace. Pick a ν-net $(f_i)_{i=1}^n$ for S_F. Choose $y_i \in S_Y$ such that $f_i(y_i) = 1$ for $i \in \overline{1,n}$.

Let $x \in S_X$. Since H has the Daugavet property, by Lemma 3.1.14 there exists $T \in S_H$ such that

$$\|y_i \otimes x + T\| > 2 - \delta_{Y^*}(\eta)$$

holds for every $i \in \{1,\dots,n\}$.

We want to show that F is $(1+\varepsilon)$-isometric to a subspace of X. On the one hand, we have $\|T(f)\| \leqslant \|f\|$ since T has norm one. Conversely, for every $i \in \{1, \ldots, n\}$ we choose $\varphi_i \in S_{Y^*}$ such that

$$\|\varphi_i(y_i)x + T(\varphi_i)\| > 2 - \delta_{Y^*}(\eta).$$

By the triangle inequality $|\varphi_i(y_i)| > 1 - \delta_{Y^*}(\eta)$ and $\|T(\varphi_i)\| > 1 - \delta_{Y^*}(\eta)$. We may assume that $\varphi_i(y_i) > 1 - \delta_{Y^*}(\eta)$. Since $f_i(y_i) = 1$ we get from the uniform convexity of Y^* that $\|f_i - \varphi_i\| < \eta < \nu/2$. We also get

$$\|T(f_i)\| \geqslant \|T(\varphi_i)\| - \|T\|\|f_i - \varphi_i\| > 1 - \delta_{Y^*}(\eta) - \frac{\nu}{2} > 1 - \nu.$$

We then get that T restricted to F is a $(1+\varepsilon)$-isometry using [**108**, Lemma 12.1.11]. $\square$

Proof or Theorem 4.2.1. Notice that $(\ell_p^{(n)})^* = \ell_{p^*}^{(n)}$ and $2 < p^* < \infty$. Observe that [**101**, Corollary] implies that $(\ell_p^{(n)})^*$ does not embed isometrically in L_1; hence $(\ell_p^{(n)})^*$ is not finitely representable in L_1. Indeed, if $(\ell_p^{(n)})^*$ were finitely representable in L_1, then it would be an isometric subspace of the ultrapower $(L_1)^{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$ by virtue of [**108**, Proposition 12.1.12]. But $(L_1)^{\mathcal{U}}$ is isometric to $L_1(\nu)$ for some measure ν by [**134**, Theorem 3.3(ii)]. Since $(\ell_p^{(n)})^*$ is separable, we can assume that $L_1(\nu)$ is separable by [**303**, Proposition III.A.2] and, consequently, $L_1(\nu)$ is isometric to a subspace of $L_1[0,1]$ by [**150**, pp. 14–15], obtaining that $(\ell_p^{(n)})^*$ is isometric to a subspace of $L_1[0,1]$, a contradiction.

Consequently, $(\ell_p^{(n)})^*$ is not finitely representable in L_1. This means that $L_1 \widehat{\otimes}_\varepsilon \ell_p^{(n)} = K(\ell_{p^*}^{(n)}, L_1)$ does not have the Daugavet property by Lemma 4.2.2. Consequently, its dual space, which is $L_\infty \widehat{\otimes}_\pi \ell_{p^*}^{(n)}$ by virtue of Theorem 2.11.6, also fails the Daugavet property by Theorem 3.3.3. $\square$

In view of Theorem 4.2.1, the only possibility to obtain a stability result of the Daugavet property by taking tensor product spaces is to require that both spaces have the Daugavet property. More precisely, given two Banach spaces X and Y, we wonder the following:

(1) If X and Y have the Daugavet property, does $X \widehat{\otimes}_\pi Y$ have the Daugavet property?
(2) If X and Y have the Daugavet property, does $X \widehat{\otimes}_\varepsilon Y$ have the Daugavet property?

As far as we know, the previous questions have remained open (see Question (4.1) and (4.3) in Section 4.7). We will, however, obtain partial positive answers in the case of concrete Banach spaces, which cover the classical cases of $C(K)$ and L_1 spaces. In order to do so, we need to introduce a strengthening of the Daugavet property.

Definition 4.2.3. Let X be a Banach space. We say that X has the *weak operator Daugavet property* (*WODP* for short) if, given $x_1, \ldots, x_n \in S_X$, $\varepsilon > 0$, a slice S of B_X and $x' \in B_X$, we can find $x \in S$ and $T\colon X \to X$ with $\|T\| \leqslant 1 + \varepsilon$, $\|T(x_i) - x_i\| < \varepsilon$ for every $i \in \{1, \ldots, n\}$ and $\|T(x) - x'\| < \varepsilon$.

Remark 4.2.4. *If X is a Banach space with the WODP, then X has the Daugavet property.* Indeed, given $x_1 = \xi \in S_X$, a slice S of B_X, and $\varepsilon > 0$, taking $x' = -\xi$ we can find, by the definition of WODP, an element $x \in S$ and an operator

$T\colon X \to X$ with $\|T\| \leqslant 1+\varepsilon$ and such that $\max\{\|T(\xi)-\xi\|, \|T(x)+\xi\|\} < \varepsilon$. It is not difficult to prove that $\|\xi + x\| \geqslant \frac{2-2\varepsilon}{1+\varepsilon}$.

Our first interest in the WODP is that it is stable by taking projective tensor products; so, in particular, WODP is a sufficient condition on X and Y in order to guarantee that $X \widehat{\otimes}_\pi Y$ enjoys the Daugavet property.

THEOREM 4.2.5. *Let X and Y be two Banach spaces with the WODP. Then, $X \widehat{\otimes}_\pi Y$ has the WODP.*

We need the following technical lemma.

LEMMA 4.2.6. *Let X be a Banach space with the WODP. Then, for all $x_1, \ldots, x_n \in S_X$, for all $y_1', \ldots, y_k' \in B_X$, all slices $S_1, \ldots, S_k$ of B_X, and every $\varepsilon > 0$, we can find $y_j \in S_j$, $1 \leqslant j \leqslant k$, and an operator $T\colon X \to X$ with $\|T\| \leqslant 1+\varepsilon$ satisfying that*

$$\|T(x_i) - x_i\| < \varepsilon \ \text{ for } 1 \leqslant i \leqslant n \quad \text{and} \quad \|T(y_j) - y_j'\| < \varepsilon \ \text{ for } 1 \leqslant j \leqslant k.$$

PROOF. Let us prove the result by induction on k. The case $k=1$ follows from the definition of the WODP. Now, assume by induction hypothesis that the result holds for k, and let us prove the case $k+1$. To this end, pick $x_1, \ldots, x_n \in S_X$, $\varepsilon > 0$, slices $S_1, \ldots, S_{k+1}$ of B_X and $y_1', \ldots, y_{k+1}' \in B_X$, and let us find an operator ϕ witnessing the thesis of the lemma.

To this end, by the induction hypothesis, we can find $y_i \in S_i$ for $1 \leqslant i \leqslant k$ and an operator $T\colon X \to X$ with $\|T\| \leqslant 1+\varepsilon$ and such that

(1) $\|T(x_i) - x_i\| < \varepsilon$ for every $1 \leqslant i \leqslant n$ and $\|T(y_{k+1}') - y_{k+1}'\| < \varepsilon$,
(2) $\|T(y_i) - y_i'\| < \varepsilon$ holds for every $1 \leqslant i \leqslant k$.

Now, by the definition of the WODP we can find $y_{k+1} \in S_{k+1}$ and an operator $G\colon X \to X$ with $\|G\| \leqslant 1+\varepsilon$ and such that

(3) $\|G(x_i) - x_i\| < \varepsilon$ for $1 \leqslant i \leqslant n$ and $\|G(y_j) - y_j\| < \varepsilon$ for $1 \leqslant j \leqslant k$,
(4) $\|G(y_{k+1}) - y_{k+1}'\| < \varepsilon$.

Define $\phi := T \circ G\colon X \to X$ and let us prove that ϕ meets our purposes. First, $\|\phi\| \leqslant (1+\varepsilon)^2$. Next, given $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} \|\phi(x_i) - x_i\| &= \|T(G(x_i)) - T(x_i) + T(x_i) - x_i\| \\ &\leqslant \|T(G(x_i) - x_i)\| + \|T(x_i) - x_i\| \\ &\leqslant \|T\|\|G(x_i) - x_i\| + \varepsilon < (1+\varepsilon)\varepsilon + \varepsilon = (2+\varepsilon)\varepsilon \end{aligned}$$

by combining (1) and (3). Moreover, given $i \in \{1, \ldots, k\}$, we obtain

$$\begin{aligned} \|\phi(y_i) - y_i'\| &= \|T(G(y_i)) - T(y_i) + T(y_i) - y_i'\| \\ &\leqslant \|T(G(y_i) - y_i)\| + \|T(y_i) - y_i'\| \\ &\leqslant \|T\|\|G(y_i) - y_i\| + \varepsilon < (1+\varepsilon)\varepsilon + \varepsilon = (2+\varepsilon)\varepsilon \end{aligned}$$

by combining (2) and (3). Finally,

$$\begin{aligned} \|\phi(y_{k+1}) - y_{k+1}'\| &= \|T(G(y_{k+1})) - T(y_{k+1}') + T(y_{k+1}') - y_{k+1}'\| \\ &\leqslant \|T(G(y_{k+1}) - y_{k+1}')\| + \|T(y_{k+1}') - y_{k+1}'\| \\ &< \|T\|\|G(y_{k+1}) - y_{k+1}'\| + \varepsilon < (1+\varepsilon)\varepsilon + \varepsilon = (2+\varepsilon)\varepsilon \end{aligned}$$

by combining (1) and (4). This proves, up to making a choice of a smaller ε, that ϕ is our desired operator. ∎

We are now ready to give the pending proof.

Proof of Theorem 4.2.5. Let $Z := X \widehat{\otimes}_\pi Y$. Fix $z_1, \ldots, z_n \in B_Z$, $\varepsilon > 0$, $z' \in B_Z$, and a slice $S = \mathrm{Slice}(B_Z, B, \alpha)$ for a certain norm-one bilinear form B: $X \times Y \to \mathbb{K}$, i.e., $B \in Z^*$.

By a density argument, we can assume with no loss of generality that

$$z_i = \sum_{j=1}^{n_i} \lambda_{ij} a_{ij} \otimes b_{ij} \in \mathrm{conv}(S_X \otimes S_Y), \qquad i \in \{1, \ldots, n\}$$

and, in a similar way, that $z' = \sum_{k=1}^{t} \mu_k x'_k \otimes y'_k \in \mathrm{conv}(S_X \otimes S_Y)$.

Take $u_0 \otimes v_0 \in S$ with $u_0 \in B_X$ and $v_0 \in B_Y$, which means $\operatorname{Re} B(u_0, v_0) > 1-\alpha$ or, equivalently, that $u_0 \in S' := \{z \in B_X\colon \operatorname{Re} B(z, v_0) > 1 - \alpha\}$, which is a slice of B_X. By Lemma 4.2.6, for every $1 \leqslant k \leqslant t$ we can find an element $x_k \in S'$ (which implies that $x_k \otimes v_0 \in S$) and an operator T: $X \to X$ with $\|T\| \leqslant 1 + \varepsilon$, satisfying that

$$\|T(a_{ij}) - a_{ij}\| < \varepsilon \ \text{ for every } i, j \quad \text{and} \quad \|T(x_k) - x'_k\| < \varepsilon \ \text{ for every } k.$$

Notice that $v_0 \in S_k := \{z \in B_Y\colon \operatorname{Re} B(x_k, z) > 1 - \alpha\}$ for every $k \in \{1, \ldots, t\}$. Again, by the previous lemma, for every $k \in \{1, \ldots, t\}$ we can find $y_k \in S_k$ (which means that $x_k \otimes y_k \in S$) and an operator U: $Y \to Y$ with $\|U\| \leqslant 1 + \varepsilon$ satisfying that

$$\|U(b_{ij}) - b_{ij}\| < \varepsilon \ \text{ for every } i, j \quad \text{and} \quad \|U(y_k) - y'_k\| < \varepsilon \text{ for } 1 \leqslant k \leqslant t.$$

Now, define $z := \sum_{k=1}^{t} \mu_k x_k \otimes y_k$. Notice that $z \in S$ since

$$\operatorname{Re} B(z) = \sum_{k=1}^{t} \mu_k \operatorname{Re} B(x_k, y_k) > (1 - \alpha) \sum_{k=1}^{t} \mu_k = 1 - \alpha.$$

Finally define $\phi := T \otimes U$: $Z \to Z$. By item 3 in Proposition 2.11.2, $\|\phi\| = \|T\|\|U\| \leqslant (1 + \varepsilon)^2$. On the other hand, given $1 \leqslant i \leqslant n$, we get

$$\begin{aligned}
\|\phi(z_i) - z_i\| &= \left\| \sum_{j=1}^{n_i} \lambda_{ij} (T(a_{ij}) \otimes T(b_{ij}) - a_{ij} \otimes b_{ij}) \right\| \\
&\leqslant \sum_{j=1}^{n_i} \lambda_{ij} \|T(a_{ij}) \otimes T(b_{ij}) - T(a_{ij}) \otimes b_{ij} + T(a_{ij}) \otimes b_{ij} - a_{ij} \otimes b_{ij}\| \\
&\leqslant \sum_{j=1}^{n_i} \lambda_{ij} (\|T(a_{ij})\|\|T(b_{ij}) - b_{ij}\| + \|T(a_{ij}) - a_{ij}\|\|b_{ij}\|) \\
&< \sum_{j=1}^{n_i} \lambda_{ij} ((1 + \varepsilon)\varepsilon + \varepsilon) = (2 + \varepsilon)\varepsilon \sum_{j=1}^{n_i} \lambda_{ij} = (2 + \varepsilon)\varepsilon.
\end{aligned}$$

Similar estimates prove that $\|\phi(z) - z'\| < (2 + \varepsilon)\varepsilon$. □

We do not know whether the Daugavet property implies the WODP (see Question (4.2) in Section 4.7). We can, however, point out classes of Banach spaces with the Daugavet property which even enjoy the WODP. The first class is provided by L_1-preduals.

PROPOSITION 4.2.7. *If X is an L_1-predual with the Daugavet property, then X has the WODP.*

PROOF. Fix $x_1, \ldots, x_n \in S_X$, $x' \in B_X$, $\varepsilon > 0$, $\alpha > 0$, and a slice $S = \operatorname{Slice}(B_X, x^*, \alpha)$. Hence, by Lemma 3.1.14 there exists an element $y \in S$ such that, denoting $E := \operatorname{lin}\{x_1, \ldots, x_n\}$, we have that

$$\|e + \lambda y\| > (1 - \varepsilon)(\|e\| + |\lambda|)$$

holds for every $e \in E$ and every $\lambda \in \mathbb{K}$. Define $T\colon E \oplus \mathbb{K}y \to X$ by

$$T(e + \lambda y) := e + \lambda x'.$$

Notice that

$$\|T(e + \lambda y)\| = \|e + \lambda x'\| \leqslant \|e\| + |\lambda| \leqslant \frac{1}{1-\varepsilon}\|e + \lambda y\|,$$

so $\|T\| \leqslant \frac{1}{1-\varepsilon}$. Since X is an L_1-predual, T can be extended to the whole of X (still denoted by T) with norm $\|T\| \leqslant \frac{1+\varepsilon}{1-\varepsilon}$ (the real case follows from [**205**, Theorem 6.1] and the complex case from [**140**], see [**203**, p. 3]). $\square$

Another family of examples is given by vector-valued L_1-spaces.

PROPOSITION 4.2.8. *Let μ be an atomless σ-finite positive measure and let Y be a Banach space. Then, $L_1(\mu, Y)$ has the WODP.*

For the proof we need the following technical lemma.

LEMMA 4.2.9. *Let μ be an atomless σ-finite positive measure and let Y be a Banach space. Let S be a slice of $B_{L_1(\mu,Y)}$. Then, for every $\varepsilon > 0$ there exists $g \in S$ with $\mu(\operatorname{supp}(g)) < \varepsilon$.*

PROOF. Take $f_0 \in S \cap S_{L_1(\mu,Y)}$. Since S is relatively open and μ is σ-finite we can assume that $A = \operatorname{supp}(f_0)$ has finite measure. Let $n \in \mathbb{N}$ so that $\frac{\mu(A)}{n} < \varepsilon$. Since μ is atomless, we can find subsets $A_1, \ldots, A_n \subset A$ so that $\mu(A_i) = \frac{\mu(A)}{n}$ for every $i \in \{1, \ldots, n\}$ and so that $\mu(A_i \cap A_j) = 0$ whenever $i \neq j$. Consequently, $\bigcup_{i=1}^n A_i$ is a subset of A of total measure $\mu(A)$. Define $f_i := \frac{f_0 \mathbb{1}_{A_i}}{\|f_0 \mathbb{1}_{A_i}\|}$ and let $\lambda_i := \|f_0 \mathbb{1}_{A_i}\|$. Observe that $\lambda_i \geqslant 0$ for every i and that

$$\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \int_{A_i} \|f_0(t)\| \, d\mu = \int_{\bigcup_{i=1}^n A_i} \|f_0(t)\| = \int_\Omega \|f_0(t)\| \, d\mu = \|f_0\| = 1,$$

where the third equality holds since $\bigcup_{i=1}^n A_i$ is a set of total measure in $A = \operatorname{supp}(f_0)$. It is immediate that f_0 equals $\sum_{i=1}^n \lambda_i f_i$ almost everywhere, so they are equal in $L_1(\mu, Y)$. It is plain that $\mu(\operatorname{supp}(f_i)) = \frac{\mu(A)}{n} < \varepsilon$. Finally, since $B_{L_1(\mu,Y)} \setminus S$ is convex and $\sum_{i=1}^n \lambda_i f_i \in S$, there is $i \in \{1, \ldots, n\}$ so that $f_i \in S$, and the lemma is proved. $\square$

PROOF OF PROPOSITION 4.2.8. Fix $x_1, \ldots, x_n \in S_{L_1(\mu,Y)}$, $x' \in B_{L_1(\mu,Y)}$, $\varepsilon > 0$ and a slice S of $B_{L_1(\mu,Y)}$. From the finiteness of $\{x_1, \ldots, x_n\}$ and the fact that μ is atomless, we may find $\delta > 0$ satisfying that

$$A \in \Sigma,\ \mu(A) < \delta \implies \int_A \|x_i\| \, d\mu < \frac{\varepsilon}{2}.$$

By Lemma 4.2.9 there is $g \in S \cap S_{L_1(\mu,Y)}$ satisfying
$$\mu(\operatorname{supp}(g)) < \delta.$$
Write $B := \operatorname{supp}(g)$. As $L_\infty(\mu, Y^*)$ is norming for $L_1(\mu, Y)$ (because $L_1(\mu)^* = L_\infty(\mu)$ and simple functions are dense in $L_1(\mu, Y)$), we can find $h \in S_{L_\infty(\mu,Y^*)}$ such that
$$\operatorname{supp}(h) \subset B \quad \text{and} \quad \operatorname{Re}\langle h, g\rangle = \operatorname{Re} \int_B \langle h(t), g(t)\rangle \, d\mu(t) > 1 - \varepsilon.$$
By using again the denseness of simple functions, and taking into account that $\int_B |x_i| \, d\mu < \frac{\varepsilon}{2}$, we can find pairwise disjoint sets $C_1, \ldots, C_t \in \Sigma$ with positive and finite measure, all of them included in $\Omega \setminus B$, and $a_i^j \in Y$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, t\}$, such that $x_i' := \sum_{k=1}^t a_i^k \mathbb{1}_{C_k}$ satisfies
$$\|x_i - x_i'\| < \frac{\varepsilon}{2}.$$
Define now $T\colon L_1(\mu, Y) \to L_1(\mu, Y)$ by the equation
$$T(f) := \sum_{k=1}^t \left(\frac{1}{\mu(C_k)} \int_{C_k} f \, d\mu \right) \mathbb{1}_{C_k} + \left(\int_B \langle h(t), f(t)\rangle \, d\mu(t) \right) x'.$$
It is not difficult to see that $\|T\| \leqslant 1$ and that $T(x_i') = x_i'$, so
$$\|T(x_i) - x_i\| \leqslant \|T(x_i - x_i')\| + \|x_i' - x_i\| < \varepsilon.$$
Also, since $C_i \cap B = \emptyset$ and $\operatorname{supp}(g) = B$, we get
$$\|T(g) - x'\| \leqslant \left| 1 - \int_B \langle h(t), g(t)\rangle \, d\mu(t) \right| \|x'\| < \sqrt{2\varepsilon}.$$
This concludes the proof. □

We should mention that the $\otimes_\pi$-stability of the Daugavet property is clear when one of the spaces involved is a Bochner L_1-space, which has the Daugavet property by Theorem 3.4.4; indeed $L_1(\mu, Y) \mathbin{\widehat{\otimes}_\pi} Z \cong L_1(\mu, Y \mathbin{\widehat{\otimes}_\pi} Z)$.

We finish the section with the following result for the injective tensor product.

Theorem 4.2.10. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be atomless measure spaces. Then, the space $L_1(\mu_1) \mathbin{\widehat{\otimes}_\varepsilon} L_1(\mu_2)$ has the Daugavet property.*

Proof. For brevity, let $X = L_1(\mu_1) \mathbin{\widehat{\otimes}_\varepsilon} L_1(\mu_2)$. Let $\alpha \in X$ and $\varphi \in X^*$ with $\|\alpha\|_X = 1 = \|\varphi\|_{X^*}$, and let $\varphi \otimes \alpha$ denote, as usual, the rank-one operator given by $[\varphi \otimes \alpha](x) = \varphi(x)\alpha$ for $x \in X$. We will show that
$$\|\mathrm{Id} + \varphi \otimes \alpha\| = 1 + \|\varphi \otimes \alpha\| = 2.$$
To this end fix $\varepsilon > 0$. Since simple functions are dense in any L_1 space, we can, up to perturbation, assume without loss of generality that there are two collections of pairwise disjoint sets of finite positive measure $(A_i)_{i=1}^n \subset \Sigma_1$ and $(B_j)_{j=1}^n \subset \Sigma_2$, and scalars $(a_{ij})_{i,j=1}^n$, $(b_{ij})_{i,j=1}^n$ such that
$$\alpha = \sum_{i,j=1}^n a_{ij} \mathbb{1}_{A_i} \otimes \mathbb{1}_{B_j},$$
and the element
$$\beta := \sum_{i,j=1}^n b_{ij} \mathbb{1}_{A_i} \otimes \mathbb{1}_{B_j},$$

satisfies

$$\|\beta\| = 1, \quad 1 - \frac{\varepsilon}{2} < \operatorname{Re}\varphi(\beta) \leqslant 1.$$

Also note that it follows immediately from the definition of the injective norm that the set

$$\begin{aligned} N &= \big\{h_1 \otimes h_2\colon\ h_i \in \operatorname{ext}(B_{L_\infty(\mu_i)}) \text{ for } i = 1, 2\big\} \\ &= \big\{h_1 \otimes h_2\colon\ h_i \in L_\infty(\mu_i),\ |h_i| = 1 \text{ a.e.}\big\} \end{aligned}$$

is norming for X (cf. [**274**, p. 46]).

We will need the following:

Claim: For every $\delta > 0$ there exist $(A'_i)_{i=1}^n \subset \Sigma_1$, $(B'_j)_{j=1}^n \subset \Sigma_2$ such that

(1) $A'_i \subset A_i$, $\mu_1(A'_i) < \delta$, for $1 \leqslant i \leqslant n$.
(2) $B'_j \subset B_j$, $\mu_2(B'_j) < \delta$, for $1 \leqslant j \leqslant n$.
(3) If we denote

$$\beta' = \sum_{i,j=1}^n b_{ij} \frac{\mu_1(A_i)\mu_2(B_j)}{\mu_1(A'_i)\mu_2(B'_j)} \mathbb{1}_{A'_i} \otimes \mathbb{1}_{B'_j},$$

then we have

$$\|\beta'\|_X = 1 \qquad \text{and} \qquad \operatorname{Re}\varphi(\beta') > 1 - \varepsilon.$$

Indeed, fix $1 \leqslant i \leqslant n$. Let $r_i = \sum_{j=1}^n b_{ij}\varphi(\mathbb{1}_{A_i} \otimes \mathbb{1}_{B_j})$. Note that

$$\operatorname{Re}\sum_{i=1}^n r_i = \operatorname{Re}\varphi(\beta) > 1 - \frac{\varepsilon}{2}.$$

For $f \in L_1(\Omega_1, \Sigma_1, \mu_1)$, let

$$\varphi_i(f) = \sum_{j=1}^n b_{ij}\mu_1(A_i)\varphi(f\mathbb{1}_{A_i} \otimes \mathbb{1}_{B_j}).$$

Clearly, φ_i is linear and

$$|\varphi_i(f)| \leqslant \|\varphi\|_{X^*} \sum_{j=1}^n |b_{ij}|\mu_1(A_i)\mu_2(B_j)\|f\|_{L_1} \leqslant C\|f\|_{L_1},$$

for some finite C. Since $\varphi_i(f) = 0$ whenever $f\mathbb{1}_{A_i} = 0$, we have that $\varphi_i(f) = \varphi_i(f\mathbb{1}_{A_i})$. Thus, as $f\mathbb{1}_{A_i} \in L_1(\mu_1|_{A_i})$ and $\mu_1(A_i) < \infty$, there is $g_i \in L_1(\mu_1|_{A_i})^* = L_\infty(\mu_1|_{A_i})$ such that

$$\varphi_i(f) = \int_{A_i} g_i f\, d\mu_1.$$

Since

$$\frac{1}{\mu_1(A_i)} \int_{A_i} g_i\, d\mu_1 = \varphi_i\Big(\frac{\mathbb{1}_{A_i}}{\mu_1(A_i)}\Big) = r_i,$$

it follows that $\operatorname{Re} g_i > \operatorname{Re} r_i - \varepsilon/2n$ on a subset of A_i with positive measure. Let A'_i be such a set satisfying the additional requirement that $\mu_1(A'_i) < \delta$ (we are using the absence of atoms of μ_1). We have that

$$\operatorname{Re}\varphi_i\Big(\frac{\mathbb{1}_{A'_i}}{\mu_1(A'_i)}\Big) = \frac{1}{\mu_1(A'_i)} \int_{A'_i} \operatorname{Re} g_i\, d\mu_1 > \operatorname{Re} r_i - \frac{\varepsilon}{2n}.$$

Let now

$$\beta_0' = \sum_{i,j=1}^n b_{ij} \frac{\mu_1(A_i)}{\mu_1(A_i')} \mathbb{1}_{A_i'} \otimes \mathbb{1}_{B_j}.$$

It follows that

$$\begin{aligned}\operatorname{Re}\varphi(\beta_0') &= \operatorname{Re} \sum_{i,j=1}^n b_{ij} \frac{\mu_1(A_i)}{\mu_1(A_i')} \varphi(\mathbb{1}_{A_i'} \otimes \mathbb{1}_{B_j}) \\ &= \sum_{i=1}^n \operatorname{Re}\varphi_i\Big(\frac{\mathbb{1}_{A_i'}}{\mu_1(A_i')}\Big) \\ &> \sum_{i=1}^n \Big(\operatorname{Re} r_i - \frac{\varepsilon}{2n}\Big) > 1 - \varepsilon.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\|\beta_0'\|_X &= \sup_{h_1 \otimes h_2 \in N} \big|\langle h_1 \otimes h_2, \beta_0'\rangle\big| \\ &= \sup_{h_1 \otimes h_2 \in N} \left|\sum_{i,j=1}^n b_{ij} \frac{\mu_1(A_i)}{\mu_1(A_i')} \int_{A_i'} h_1\, d\mu_1 \int_{B_j} h_2\, d\mu_2\right| \\ &= \sum_{i,j=1}^n |b_{ij}| \mu_1(A_i)\mu_2(B_j) \\ &= \sup_{h_1 \otimes h_2 \in N} \left|\sum_{i,j=1}^n b_{ij} \int_{A_i} h_1\, d\mu_1 \int_{B_j} h_2\, d\mu_2\right| = \|\beta\|_X.\end{aligned}$$

Finally, if we perform the same argument starting with β_0' and interchanging the role of i and j, then the claim follows.

Now, let $h_1 \otimes h_2 \in N$ be such that

$$\operatorname{Re}\langle h_1 \otimes h_2, \alpha\rangle > 1 - \varepsilon. \tag{4.2.1}$$

Take

$$0 < \delta < \frac{\varepsilon}{4\left(\max\limits_{1\leqslant i\leqslant n} \mu_1(A_i) + \max\limits_{1\leqslant j\leqslant n} \mu_2(B_j)\right) \sum\limits_{i,j=1}^n |a_{ij}|}, \tag{4.2.2}$$

and let $(A_i')_{i=1}^n \subset \Sigma_1$, $(B_j')_{j=1}^n \subset \Sigma_2$, and β' as given in the claim. Let also $h_1' \otimes h_2' \in N$ be such that

$$\operatorname{Re}\langle h_1' \otimes h_2', \beta'\rangle > 1 - \varepsilon. \tag{4.2.3}$$

Now, let us define

$$\tilde h_1(x) = \begin{cases} h_1'(x) & \text{for } x \in \bigcup_{i=1}^n A_i' \\ h_1(x) & \text{elsewhere,} \end{cases}$$

$$\tilde h_2(y) = \begin{cases} h_2'(y) & \text{for } y \in \bigcup_{j=1}^n B_j' \\ h_2(y) & \text{elsewhere.} \end{cases}$$

First, note that by our choice of δ, we have

$$\begin{aligned}\Big|\sum_{i,j=1}^{n} a_{ij}\ &\Big(\langle (h_1'-h_1)\otimes h_2, \mathbb{1}_{A_i'}\otimes \mathbb{1}_{B_j}\rangle + \langle h_1\otimes (h_2'-h_2), \mathbb{1}_{A_i}\otimes \mathbb{1}_{B_j'}\rangle\Big)\Big|\\
&\leqslant \sum_{i,j=1}^{n} |a_{ij}|\Big(\int_{A_i'}(|h_1'|+|h_1|)\,d\mu_1\int_{B_j}|h_2|\,d\mu_2\\
&\qquad+\int_{A_i}|h_1|\,d\mu_1\int_{B_j'}(|h_2'|+|h_2|)\,d\mu_2\Big)\\
&\leqslant \sum_{i,j=1}^{n}|a_{ij}|\cdot 2(\mu_1(A_i')\mu_2(B_j)+\mu_1(A_i)\mu_2(B_j')) \overset{(4.2.2)}{\leqslant} \varepsilon.\end{aligned}$$

Analogously,

$$\begin{aligned}\Big|\sum_{i,j=1}^{n} a_{ij}\ &\Big(\langle (h_1'-h_1)\otimes (h_2-h_2'), \mathbb{1}_{A_i'}\otimes \mathbb{1}_{B_j'}\rangle + \langle h_1\otimes (h_2'-h_2), \mathbb{1}_{A_i}\otimes \mathbb{1}_{B_j'}\rangle\Big)\Big|\\
&\leqslant \sum_{i,j=1}^{n} |a_{ij}|\Big(\int_{A_i'}(|h_1'|+|h_1|)\,d\mu_1\int_{B_j'}(|h_2|+|h_2'|)\,d\mu_2\\
&\leqslant \sum_{i,j=1}^{n}|a_{ij}|\cdot 4\mu_1(A_i')\mu_2(B_j')\leqslant \varepsilon.\end{aligned}$$

From the above two estimates, calling $z_1 := \langle h_1'\otimes h_2', \beta'\rangle$, $z_2 := \varphi(\beta')$, $z_3 := \langle h_1\otimes h_2, \alpha\rangle = \sum_{i,j=1}^{n} a_{ij}\langle h_1\otimes h_2, \mathbb{1}_{A_i}\otimes\mathbb{1}_{B_j}\rangle$, and taking into account that $\operatorname{Re} z_1 > 1-\varepsilon$ by (4.2.3), that $\operatorname{Re} z_2 > 1-\varepsilon$ by (3) in the claim, and that $\operatorname{Re} z_2 > 1-\varepsilon$ by (4.2.1), it follows that

$$\begin{aligned}\|\mathrm{Id}+\varphi\otimes\alpha\| &\geqslant \|\beta'+\varphi(\beta')\alpha\|_X \geqslant \operatorname{Re}\langle \tilde h_1\otimes\tilde h_2, \beta'+\varphi(\beta')\alpha\rangle\\
&= \operatorname{Re}\sum_{i,j=1}^{n}\Big(b_{ij}\tfrac{\mu_1(A_i)\mu_2(B_j)}{\mu_1(A_i')\mu_2(B_j')}\langle \tilde h_1\otimes\tilde h_2, \mathbb{1}_{A_i'}\otimes\mathbb{1}_{B_j'}\rangle\\
&\qquad+\varphi(\beta')a_{ij}\langle \tilde h_1\otimes\tilde h_2, \mathbb{1}_{A_i}\otimes\mathbb{1}_{B_j}\rangle\Big)\\
&= \operatorname{Re}\langle h_1'\otimes h_2', \beta'\rangle + \operatorname{Re}\Bigg[\varphi(\beta')\Big(\sum_{i,j=1}^{n} a_{ij}\big(\langle h_1\otimes h_2, \mathbb{1}_{A_i}\otimes\mathbb{1}_{B_j}\rangle\\
&\qquad+\langle (h_1'-h_1)\otimes h_2, \mathbb{1}_{A_i'}\otimes\mathbb{1}_{B_j}\rangle + \langle h_1\otimes(h_2'-h_2), \mathbb{1}_{A_i}\otimes\mathbb{1}_{B_j'}\rangle\\
&\qquad\qquad+\langle (h_1'-h_1)\otimes(h_2'-h_2), \mathbb{1}_{A_i'}\otimes\mathbb{1}_{B_j'}\rangle\big)\Big)\Bigg]\\
&\geqslant \operatorname{Re} z_1 + \operatorname{Re}(z_2z_3) - 2\varepsilon\\
&> 1-\varepsilon+(1-\varepsilon-\sqrt{2\varepsilon})-2\varepsilon = 2-4\varepsilon-\sqrt{2\varepsilon}.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, we get that $\|\mathrm{Id}+\varphi\otimes\alpha\|\geqslant 2$ as claimed. $\square$

4.3. L-orthogonal elements in Daugavet spaces

In this section we will show that there exists a strong relation between the Daugavet property and the abundance of L-orthogonal elements (see Section 2.8 for the formal definition) in Banach spaces with small density character.

Let us start by commenting that it clearly follows from Lemma 3.1.19 that the weak-star density of L-orthogonal elements in the bidual ball implies the Daugavet property. Actually, the following result shows that a little less is needed.

Lemma 4.3.1. *Let X be a Banach space satisfying that for every non-empty weak-star slice S' of $B_{X^{**}}$, there exists $u \in S_{X^{**}} \cap S'$ such that the equality*

$$\|x+u\| = 1 + \|x\|$$

holds for every $x \in X$. Then, X has the Daugavet property.

Proof. To show that X has the Daugavet property we will apply Lemma 3.1.9. In order to do so, pick $x \in S_X$, $\varepsilon > 0$ and consider a slice S of B_X, and let us find $y \in S$ with $\|x+y\| > 2-\varepsilon$. Let S^{**} be the slice of $B_{X^{**}}$ such that $S^{**} \cap B_X = S$ and consider $u \in S^{**} \cap S_{X^{**}}$ as in the assumption. Pick a net $\{x_s\}$ in B_X which is weak-star convergent to u in $B_{X^{**}}$. On the one hand, because of the weak-star convergence condition, we can find s_0 such that $s \geqslant s_0$ implies $x_s \in S^{**}$, and hence $x_s \in S^{**} \cap B_X = S$. On the other hand, by the weak-star lower semicontinuity of the norm of X^{**}, we get

$$2 = \|x+u\| \leqslant \liminf_s \|x_s + x\|,$$

so we can find $s \geqslant s_0$ such that $\|x_s + x\| > 2 - \varepsilon$, and taking $y = x_s$ finishes the proof. $\square$

Remark 4.3.2. *Observe that the hypotheses of the previous result are clearly satisfied if the set of L-orthogonal elements is weak-star dense. But, in this case,* $X \in \mathrm{DPr}$ *also follows from Lemma 3.1.19 as previously commented.*

Our goal now is to provide L-orthogonal elements for Banach spaces with the Daugavet property. The first result deals with the separable case and it is slightly stronger than Corollary 3.1.18.

Lemma 4.3.3. *Let X be a separable Banach space with the Daugavet property. Let $u \in B_{X^{**}}$ and $\{g_n \colon n \in \mathbb{N}\} \subset X^*$. Then, there exists an element $v \in S_{X^{**}}$ satisfying that:*

1. *the equality $\|x+v\| = 1 + \|x\|$ holds for every $x \in X$ (in other words, v is an L-orthogonal element).*
2. *$v(g_n) = u(g_n)$ holds for every $n \in \mathbb{N}$.*

Proof. Let $\{x_n \colon n \in \mathbb{N}\} \subset S_X$ be a dense subset of S_X and, for every $n \in \mathbb{N}$, define

$$V_n := \left\{ x \in B_X \colon |g_i(x) - u(g_i)| < \frac{1}{n} \quad (1 \leqslant i \leqslant n) \right\}.$$

Note that V_n is a weakly open subset of B_X which is non-empty because of the w^*-denseness of B_X in $B_{X^{**}}$. Since X has the Daugavet property, by Lemma 3.1.14 there exists $y_n \in V_n$ so that

$$\|x_i + y_n\| > 2 - \frac{1}{n}$$

holds for every $i \in \{1, \dots, n\}$. From the above condition and the density of $\{x_n \colon n \in \mathbb{N}\}$ in S_X, it is not difficult to prove that $(y_n)_{n\in\mathbb{N}}$ is an ℓ_1-type sequence. By Lemma 2.8.9 and Theorem 2.8.10 there exists $v \in \{y_n \colon n \in \mathbb{N}\}'$, i.e., a w^*-cluster point of $\{y_n \colon n \in \mathbb{N}\}$, which is an L-orthogonal element. It remains to prove that v satisfies condition (2). To this end, pick $n \in \mathbb{N}$. Given $k \in \mathbb{N}$ we can find, since v is a w^*-cluster point of $\{y_\nu\}$, a natural number $p \geqslant k$ so that $|g_n(y_p) - v(g_n)| < \frac{1}{k}$. Note also that, since $y_p \in V_p$, then $|g_n(y_p) - u(g_n)| < \frac{1}{p} \leqslant \frac{1}{k}$. Consequently, $|v(g_n) - u(g_n)| \leqslant \frac{2}{k}$. Since $k \in \mathbb{N}$ was arbitrary we conclude that $u(g_n) = v(g_n)$, as desired. $\square$

The main result deals with non-separable spaces with a small density character.

THEOREM 4.3.4. *Let X be a Banach space with the Daugavet property and* $\operatorname{dens}(X) \leqslant \omega_1$. *Let $u \in B_{X^{**}}$ and $\{g_n \colon n \in \mathbb{N}\} \subset X^*$. Then, there exists an element $v \in S_{X^{**}}$ satisfying that:*

(1) *the equality $\|x + v\| = 1 + \|x\|$ holds for every $x \in X$ (in other words, v is an L-orthogonal element).*
(2) *$v(g_n) = u(g_n)$ holds for every $n \in \mathbb{N}$.*

PROOF. In order to construct v, pick $\{x_\beta \colon \beta < \omega_1\} \subset S_X$ to be a dense subset of S_X. Let us construct by transfinite induction on $\omega_0 \leqslant \beta < \omega_1$ a family $\{(Z_\beta, \varphi_\beta, \{f_{\beta,\gamma} \colon \gamma < \beta\}, v_\beta) \colon \omega_0 \leqslant \beta < \omega_1\}$ satisfying the following assertions:

(a) Z_β is a separable almost isometric ideal in X containing $\bigcup_{\omega_0 \leqslant \gamma < \beta} Z_\gamma \cup \{x_\beta\}$ and $\{x_n \colon n \in \mathbb{N}\} \cup \{x_{\omega_0}\} \subset Z_{\omega_0}$. (Almost isometric ideals were introduced in Definition 2.9.11.)
(b) $\varphi_\beta \colon Z_\beta^* \to X^*$ is an almost isometric Hahn-Banach extension operator such that $\{f_{\gamma,\delta} \colon \delta < \gamma < \beta,\ \omega_0 \leqslant \gamma\} \cup \{g_n \colon n \in \mathbb{N}\} \subset \varphi_\beta(Z_\beta^*)$.
(c) $v_\beta \in S_{X^{**}}$ satisfies that
$$\|z + v_\beta\| = 1 + \|z\|$$
for every $z \in Z_\beta$, and $\{f_{\beta,\gamma} \colon \gamma < \beta\} \subset S_{X^*}$ is norming for $Z_\beta \oplus \mathbb{R} v_\beta$.
(d) For every $\delta < \gamma < \beta < \omega_1$ and $\omega_0 \leqslant \gamma$ it follows that
$$v_\beta(f_{\gamma,\delta}) = v_\gamma(f_{\gamma,\delta}),$$
and that the equality
$$v_\beta(g_n) = u(g_n)$$
holds for every $n \in \mathbb{N}$.

The construction of the family will be completed by transfinite induction on β. To this end, in the case $\beta = \omega_0$ define, using Theorem 2.9.13, a separable almost isometric ideal Z_{ω_0} of X containing $\{x_n \colon n \in \mathbb{N}\} \cup \{x_{\omega_0}\}$ and an almost isometric Hahn-Banach extension operator $\varphi_{\omega_0} \colon Z_{\omega_0}^* \to X^*$ such that $\{g_n \colon n \in \mathbb{N}\} \subset \varphi_{\omega_0}(Z_{\omega_0})$. Now, find an element v_{ω_0} satisfying the conclusion of Lemma 4.3.3. Finally, defining $\{f_{\omega_0,n} \colon n \in \mathbb{N}\} \subset S_{X^*}$ to be a norming subset for Z_{ω_0}, we get the initial step proved.

Next assume that $(Z_\gamma, \varphi_\gamma, \{f_{\gamma,\delta} \colon \delta < \gamma\}, v_\gamma)$ has already been constructed for every $\omega_0 \leqslant \gamma < \beta$, and let us construct $(Z_\beta, \varphi_\beta, \{f_{\beta,\gamma} \colon \gamma < \beta\}, v_\beta)$. Pick w to be a w^*-cluster point of the net $\{v_\gamma \colon \omega_0 \leqslant \gamma < \beta\}$ (where the order in $[0, \beta)$ is the classical order). Notice that, by induction hypothesis, for every $\omega_0 \leqslant \gamma_0 < \gamma < \beta$ and $\delta_0 < \gamma_0$ we have that

$$v_\gamma(f_{\gamma_0,\delta_0}) = v_{\gamma_0}(f_{\gamma_0,\delta_0}).$$

Then, since w is a w^*-cluster point of $\{v_\gamma\colon \omega_0 \leqslant \gamma < \beta\}$, we get that

$$w(f_{\gamma_0,\delta_0}) = v_{\gamma_0}(f_{\gamma_0,\delta_0}). \tag{4.3.1}$$

Because of the same reason, given $n \in \mathbb{N}$, we obtain that

$$w(g_n) = v_{\omega_0}(g_n) = u(g_n). \tag{4.3.2}$$

Now, notice that the set

$$\{f_{\gamma,\delta}\colon \delta < \gamma < \beta, \omega_0 \leqslant \gamma\} \cup \{g_n\colon n \in \mathbb{N}\}$$

is countable because β is a countable ordinal. Also, $\bigcup_{\omega_0 \leqslant \gamma < \beta} Z_\gamma$ is separable. Then, by Theorem 2.9.13, there exist an almost isometric ideal Z_β in X containing $\bigcup_{\omega_0 \leqslant \gamma < \beta} Z_\gamma \cup \{x_\beta\}$ and an almost isometric Hahn-Banach extension operator φ_β: $Z_\beta^* \to X^*$ such that

$$\varphi_\beta(Z_\beta^*) \supset \{f_{\gamma,\delta}\colon \delta < \gamma < \beta, \omega_0 \leqslant \gamma\} \cup \{g_n\colon n \in \mathbb{N}\}.$$

Let us construct v_β. To this end, since Z_β is separable, Lemma 4.3.3 applies for $w \in B_{X^{**}}$. Consequently, we can find $v_\beta \in S_{X^{**}}$ such that

(a) $\|z + v_\beta\| = 1 + \|x\|$ for every $x \in Z_\beta$, and
(b) $v_\beta(f_{\gamma,\delta}) = w(f_{\gamma,\delta})$ for $\delta < \gamma < \beta$, and $v_\beta(g_n) = w(g_n)$ for every $n \in \mathbb{N}$.

Take $\{f_{\beta,\gamma}\colon \gamma < \beta\} \subset S_{X^*}$ to be a norming set for $Z_\beta \oplus \mathbb{R}v_\beta$. It follows as before that $\{(Z_\gamma, \varphi_\gamma, \{f_{\gamma,\delta}\colon \delta < \gamma\}, v_\gamma)\colon \omega_0 \leqslant \gamma \leqslant \beta\}$ satisfies our purposes. This settles the transfinite induction.

Now, consider a w^*-cluster point v of $\{v_\beta\colon \omega_0 \leqslant \beta < \omega_1\}$. Let us prove that v satisfies the thesis of the theorem.

(1) Given $x \in S_Z$ we show that

$$\|x + v\| = 2.$$

To this end, pick $\varepsilon > 0$. Since $\{x_\beta\colon \beta < \omega_1\}$ is dense in S_Z find $\omega_0 \leqslant \beta < \omega_1$ such that $\|x - x_\beta\| < \frac{\varepsilon}{3}$. Since $\|x_\beta + v_\beta\| = 2$, find $\gamma < \beta$ such that

$$[x_\beta + v_\beta](f_{\beta,\gamma}) > 2 - \frac{\varepsilon}{3}.$$

Now, given any $\beta' > \beta$ we have that

$$[x_\beta + v_{\beta'}](f_{\beta,\gamma}) = [x_\beta + v_\beta](f_{\beta,\gamma}) > 2 - \frac{\varepsilon}{3}.$$

Since v is a w^*-cluster point of $\{v_\beta\colon \omega_0 \leqslant \beta < \omega_1\}$ we obtain that

$$2 - \frac{\varepsilon}{3} \leqslant [x_\beta + v](f_{\beta,\gamma}) \leqslant \|x_\beta + v\| \leqslant \|x + v\| + \frac{\varepsilon}{3},$$

so $\|x + v\| > 2 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we also conclude that $\|x + v\| = 2$. Finally, since $x \in S_Z$ was arbitrary, one in fact has

$$\|x + v\| = 1 + \|x\|$$

for every $x \in Z$ (cf. Remark 2.6.2).

(2) Let us prove that $v(g_n) = u(g_n)$ for every $n \in \mathbb{N}$. To this end, pick $\varepsilon > 0$, $n \in \mathbb{N}$, and find $\gamma > \omega_0$ so that $|[v - v_\gamma](g_n)| < \varepsilon$. Since $v_\delta(g_n) = u(g_n)$ holds for every $\delta \geqslant \omega_0$, it follows that

$$|[v - u](g_n)| = |[v - v_\gamma](g_n)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done. □

As a direct consequence of the previous theorem we get the following corollary.

Corollary 4.3.5. *Let X be a Banach space with the Daugavet property and* $\operatorname{dens}(X) \leqslant \omega_1$. *Then the set*

$$\{u \in X^{**}\colon \|x + u\| = \|x\| + \|u\| \ \forall x \in X\}$$

is weak-star dense in X^{**}.

A natural question here is whether Theorem 4.3.4 holds without any restriction on the density character of the space. We will see that the answer is negative and, in fact, we will prove that the restriction $\operatorname{dens}(X) \leqslant \omega_1$ is sharp under some set-theoretic assumptions. To begin with, let us obtain a necessary condition for an injective tensor product to have L-orthogonal elements.

Proposition 4.3.6. *Let X be a uniformly smooth Banach space and let Y be a Banach space. Assume that either X^* or Y^* has the approximation property and that there exists an element $T \in (X \widehat{\otimes}_\varepsilon Y)^{**} = (X^* \widehat{\otimes}_\pi Y^*)^* = L(X^*, Y^{**})$ with $\|T\| = 1$ and satisfying that the equality*

$$\|T + S\| = 1 + \|S\|$$

holds for every $S \in X \widehat{\otimes}_\varepsilon Y = X^{**} \widehat{\otimes}_\varepsilon Y = K(X^*, Y)$. Then, T is an isometric embedding.

Compare this proposition (and its proof) with Lemma 4.2.2.

Proof. Recall that a uniformly smooth space is reflexive and that its dual space is uniformly convex. Fix $x^* \in S_{X^*}$ and let us prove that $\|T(x^*)\| = 1$. To this end, take $x \in S_X$ with $x^*(x) = 1$ and $y \in S_Y$ arbitrary, and define $S := x \otimes y \in X \widehat{\otimes}_\varepsilon Y$. By the assumption, $\|T + S\| = 2$. Consequently, for every $n \in \mathbb{N}$, there exists $x_n^* \in S_{X^*}$ such that

$$2 - \frac{1}{n} < \|T(x_n^*) + S(x_n^*)\| \leqslant \|T(x_n^*)\| + \|S(x_n^*)\| = \|T(x_n^*)\| + |x_n^*(x)|.$$

The previous estimate implies that $\|T(x_n^*)\| \to 1$ and $|x_n^*(x)| \to 1$. This implies that there is a sequence (θ_n) in $\mathbb{T}$ such that $\operatorname{Re} \theta_n x_n^*(x) \to 1$. Therefore

$$\|\theta_n x_n^* + x^*\| \geqslant \operatorname{Re} \theta_n x_n^*(x) + x^*(x) \to 2.$$

Then, the uniform convexity of X^* implies that $\|\theta_n x_n^* - x^*\| \to 0$. As $T(\theta_n x_n^*) \to T(x^*)$ by continuity, we have that $\|T(x_n^*)\| \to \|T(x^*)\|$, hence $\|T(x^*)\| = 1$. □

We are now able to exhibit the announced example.

Example 4.3.7. Let Γ be a set with cardinality $2^{\mathfrak{c}}$. Consider the space $X = \ell_2(\Gamma) \widehat{\otimes}_\varepsilon C[0,1] \cong C([0,1], \ell_2(\Gamma))$. Notice that, by Theorem 3.4.11, X has the Daugavet property. However, X^{**} does not contain any element which is L-orthogonal on X.

Indeed, assume for contradiction that there exists an L-orthogonal $T \in X^{**} = L(\ell_2(\Gamma)), C[0,1]^{**})$. By Proposition 4.3.6, T is an isometric embedding. Moreover, notice that T is an adjoint operator (say $T = S^*$) because $\ell_2(\Gamma)$ is reflexive. Now, $S\colon C[0,1]^* \to \ell_2(\Gamma)$ is surjective because S^* is an isometry. However, this is a contradiction because

$$\operatorname{card}\bigl(C[0,1]^*\bigr) = \mathfrak{c} < 2^{\mathfrak{c}} = \operatorname{card}(\Gamma) = \operatorname{card}\bigl(\ell_2(\Gamma)\bigr).$$

If we asume the CH, then the example above shows that Theorem 4.3.4 cannot be extended to larger cardinals.

We end this section with some applications to L-embedded Banach spaces with the Daugavet property. We already know from Theorem 3.4.4 that for an atomless measure, $L_1(\mu, Y) = L_1(\mu) \mathbin{\widehat{\otimes}_\pi} Y$ has the Daugavet property regardless of Y. Our aim is to show a similar result with $L_1(\mu)$ replaced with certain L-embedded Banach spaces. First, we state a preliminary result which is interesting by itself.

Theorem 4.3.8. *Let X be an L-embedded Banach space with* $\operatorname{dens}(X) \leqslant \omega_1$. *Assume that $X^{**} = X \oplus_1 Z$. Then, the following are equivalent:*
(1) *X^* has the Daugavet property.*
(2) *X has the Daugavet property.*
(3) *B_Z is w^* dense in $B_{X^{**}}$.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). Let W be a non-empty w^* open subset of $B_{X^{**}}$ and let us prove that $B_Z \cap W \neq \emptyset$. By Theorem 4.3.4, we can find $u \in W \cap S_{X^{**}}$ such that

$$\|x + u\| = 1 + \|x\|$$

for every $x \in X$. Write $u \in X^{**}$ as $u = x + z$ for suitable $x \in X$ and $z \in Z$. Now,

$$1 \geqslant \|z\| = \|-x + (x + z)\| = 1 + \|x\|.$$

This implies that $x = 0$ and, consequently, $u \in B_Z$. So $W \cap B_Z \neq \emptyset$, as desired.

(3) $\Rightarrow$ 1 follows from Theorem 3.5.1 and Goldstine's Theorem. ☐

This result generalises, for spaces with small density character, Theorem 3.5.5 where it is shown that a von Neumann algebra X has the Daugavet property if, and only if, its predual X_* (which is an L-embedded Banach space) does. We do not know Theorem 4.3.8 holds without the assumption than $\operatorname{dens}(X) \leqslant \omega_1$ (see Question (4.5) in Section 4.7).

Moreover, Theorem 4.3.8 allows us to provide another result about the Daugavet property for projective tensor products.

Theorem 4.3.9. *Let X be an L-embedded Banach space with the Daugavet property and* $\operatorname{dens}(X) \leqslant \omega_1$, *and let Y be a non-zero Banach space. If either X^{**} or Y has the metric approximation property, then $X \mathbin{\widehat{\otimes}_\pi} Y$ has the Daugavet property.*

Before beginning the proof, we refer the reader to the paragraph after Proposition 2.11.3 for background about the natural identification $(X \mathbin{\widehat{\otimes}_\pi} Y)^* = L(X, Y^*)$.

Proof. Write $X^{**} = X \oplus_1 Z$. In order to prove that $X \mathbin{\widehat{\otimes}_\pi} Y$ has the Daugavet property it is enough to show, by Lemma 4.3.1, that every w^*-slice of $B_{(X \widehat{\otimes}_\pi Y)^{**}}$ contains an element which is L-orthogonal to $X \mathbin{\widehat{\otimes}_\pi} Y$. To this end, pick $G \in S_{L(X,Y^*)} = S_{(X \widehat{\otimes}_\pi Y)^*}$ and $\alpha > 0$, and it suffices to find elements $u \in S_{X^{**}}$ and

$y \in S_Y$ such that $\operatorname{Re} u(y \circ G) > 1 - \alpha$ (in other words, the element $u \otimes y \in S(B_{(X \widehat{\otimes}_\pi Y)^{**}}, G, \alpha)$ and

$$\|z + u \otimes y\|_{(X \widehat{\otimes}_\pi Y)^{**}} = 1 + \|z\| \tag{4.3.3}$$

for every $z \in X \widehat{\otimes}_\pi Y$). To do so, by the assumption that either X^{**} or Y has the MAP, it follows that $X^{**} \widehat{\otimes}_\pi Y$ is an isometric subspace of $(X \widehat{\otimes}_\pi Y)^{**}$ by [**197**, Proposition 2.3], so it suffices to prove that

$$\|z + u \otimes y\|_{X^{**} \widehat{\otimes}_\pi Y} = 1 + \|z\|$$

for every $z \in X \widehat{\otimes}_\pi Y$ (in other words, it suffices to prove (4.3.3) in the space $X^{**} \widehat{\otimes}_\pi Y$ instead of in $(X \widehat{\otimes}_\pi Y)^{**}$).

So let us find u and y. Find $x \in S_X$ and $y \in S_Y$ such that $\operatorname{Re} G(x)(y) > 1 - \alpha$. This means that

$$x \in S(B_X, y \circ G, \alpha).$$

Since $S(B_{X^{**}}, y \circ G, \alpha)$ is a non-empty w^* open subset of $B_{X^{**}}$ and X is an L-embedded Banach space with the Daugavet property, Theorem 4.3.8 gives us some $u \in S_Z$ such that $\operatorname{Re} u(y \circ G) > 1 - \alpha$. Let us prove that

$$\|z + u \otimes y\|_{X^{**} \widehat{\otimes}_\pi Y} = 1 + \|z\|$$

for every $z \in X \widehat{\otimes}_\pi Y$. To this end, pick $z \in X \widehat{\otimes}_\pi Y$ and $\varepsilon > 0$. By a density argument, let us assume with no loss of generality that z is a finite sum of basic tensors, that is, $z = \sum_{i=1}^n x_i \otimes y_i$. By the Hahn-Banach theorem, take $T \in S_{L(X,Y^*)} = S_{(X \widehat{\otimes}_\pi Y)^*}$ such that $T(z) = \sum_{i=1}^n \operatorname{Re} T(x_i)(y_i) = \|z\|$. Since $\|u\| = 1$ choose $x^* \in S_{X^*}$ such that $\operatorname{Re} u(x^*) > 1 - \varepsilon$. Pick $y^* \in S_{Y^*}$ such that $y^*(y) = 1$ and define $\hat{T}\colon X^{**} = X \oplus_1 Z \to Y^*$ by the equation

$$\hat{T}(\tilde{x} + \tilde{z}) = T(\tilde{x}) + \tilde{z}(x^*)y^*.$$

Observe that $\hat{T} \in L(X^{**}, Y^*) = (X^{**} \widehat{\otimes}_\pi Y)^*$, $\|\hat{T}\| = 1$ and $\hat{T}(x) = T(x)$ holds for every $x \in X$ (in particular, $\hat{T}(z) = \sum_{i=1}^n \operatorname{Re} T(x_i)(y_i) = \|z\|$). Hence

$$\begin{aligned} \|z + u \otimes y\|_{X^{**} \widehat{\otimes}_\pi Y} &\geqslant \operatorname{Re} \hat{T}(z + u \otimes y) = \operatorname{Re}\big(T(z) + u(x^*)y^*(y)\big) \\ &= \|z\| + u(x^*) > 1 + \|z\| - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude the proof of the theorem. $\square$

We do not know whether the above theorem holds without the assumption that $\operatorname{dens}(X) \leqslant \omega_1$ (see Question (4.4) in Section 4.7).

We finish the section with some consequences about u-structure in Banach spaces with the Daugavet property, see Subsection 2.9.4 for the needed definitions. We have the following consequence of Theorem 4.3.4.

Proposition 4.3.10. *Let X be a Banach space with the Daugavet property and* $\operatorname{dens}(X) \leqslant \omega_1$. *Assume that X is a u-summand in its bidual, say $X^{**} = X \oplus Z$. Then, B_Z is w^*-dense in $B_{X^{**}}$.*

Proof. By Theorem 4.3.4, it is enough to prove that every element $u \in S_{X^{**}}$ such that the norm equality

$$\|x + u\| = 1 + \|x\|$$

holds for every $x \in X$ belong to Z. To this end, pick such an element $u \in S_{X^{**}}$. By the decomposition $X^{**} = X \oplus Z$ we get that there exist (unique) $x \in X$ and $z \in Z$ such that $u = x + z$. Let us prove that $x = 0$. Notice that

$$1 + 2\|x\| = \|u - 2x\| = \|u - 2P(u)\| \leqslant \|\mathrm{Id} - 2P\| \leqslant 1.$$

By the above inequality, we obtain $x = 0$ or, equivalently, that $u = z \in Z$. □

4.4. The Daugavet property in separable r.i. function spaces

The following section is devoted to a study of the Daugavet property in certain function spaces.

We first recall some basic definitions and facts about real or complex rearrangement invariant spaces and the properties we are going to investigate. For background on rearrangement invariant spaces (and on Köthe spaces in general) we refer the reader to the classical book by J. Lindenstrauss and L. Tzafriri [**207**] for the real case, and to [**242**] for the complex case. In the sequel we follow the notation of [**207**]. Let (Ω, Σ, μ) be a complete σ-finite measure space. A real or complex Banach space X consisting of equivalence classes, modulo equality almost everywhere, of locally integrable scalar valued functions on Ω is a *Köthe function space* if the following conditions hold.

(1) X is *solid*, i.e., if $|f| \leqslant |g|$ a.e. on Ω with f measurable and $g \in X$, then $f \in X$ and $\|f\| \leqslant \|g\|$.
(2) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function $\mathbb{1}_A$ of A belongs to X.

Let us comment that the definition of a Köthe space is usually given in the real case (this is the case of [**207**]), but it extends to the complex case in an obvious way. Most of the basic properties we are going to use are known in the real case but their proofs extend without many problems to the complex case.

If X is a Köthe function space, then every measurable function g on Ω so that $gf \in L_1(\mu)$ for every $f \in X$ defines an element x_g^* in X^* by $x_g^*(f) = \int_\Omega fg\, d\mu$. Any functional on X of the form x_g^* is called an *integral functional* and the linear space of all integral functionals is denoted by X'; X' is called the *associate space* of X. In the norm induced on X' by X^*, this space is also a Köthe function space on (Ω, Σ, μ). The space X is *order continuous* if whenever (f_n) is a decreasing sequence of positive functions which converges to 0 a.e., then (f_n) converges to 0 in norm. (We note that for general Banach lattices, the above defines σ-order continuity, which is weaker than order continuity in this more general context.) If X is order continuous, then every continuous linear functional on X is an integral functional, i.e., $X^* = X'$.

From now on, we will consider (Ω, Σ, μ) to be the interval $[0, 1]$ equipped with the Lebesgue measure. A Köthe function space on $[0, 1]$ is a *rearrangement invariant space* (*r.i. space*) or *symmetric space* if the following conditions hold.

(1) If $\tau\colon [0, 1] \to [0, 1]$ is an automorphism, i.e., a measure-preserving bijection, and f is a measurable function on $[0, 1]$, then $f \in X$ if and only if $f \circ \tau \in X$, and in this case $\|f\| = \|f \circ \tau\|$.
(2) X' is a norming subspace of X^* and thus X is isometric to a subspace of X''. As a subspace of X'', either $X = X''$, or X is the closed linear span of the simple integrable functions of X''.

(3) As sets,

$$L_\infty[0,1] \subset X \subset L_1[0,1]$$

and the inclusion maps are of norm one, i.e., if $f \in L_\infty[0,1]$ then $\|f\|_X \leqslant \|f\|_\infty$, and if $f \in X$ then $\|f\|_1 \leqslant \|f\|_X$.

An r.i. space X is order continuous if and only if it is separable (cf. [**207**, p. 118]). In this case, all bounded linear functionals on X are integrals (i.e., $X^* = X'$).

In this section we show that, in the real case, the only separable r.i. function space on $[0,1]$ with the Daugavet property is $L_1[0,1]$ endowed with its canonical norm; the nonseparable situation will be discussed in the Notes and Remarks section (Section 4.6).

Below X is a separable (hence order continuous) real r.i. function space on $[0,1]$. We remark that order continuity implies that both the subspace of simple functions and the subspace of continuous functions are dense in X. Denote by ϕ the *fundamental function* of X, that is, $\phi(t) = \|\mathbb{1}_{[0,t]}\|_X$. Let us list here some known properties of ϕ:

(a) ϕ is non-decreasing,
(b) $t \leqslant \phi(t) \leqslant 1$,
(c) $\phi(t+\tau) \leqslant \phi(t) + \phi(\tau)$,
(d) $\lim_{t\to 0} \phi(t) = 0$ (see [**50**, Chapter 2, Theorem 5.5], for instance).

We need several preliminary results. The first one is certainly known, but we haven't been able to locate a reference. It characterises $L_1[0,1]$ among separable r.i. function spaces on $[0,1]$.

LEMMA 4.4.1. *Let X be a separable r.i. function space on $[0,1]$ and let ϕ be its fundamental function. If $\liminf_{\tau\to 0} \phi(\tau)/\tau = 1$, then $X = L_1[0,1]$ endowed with its canonical norm.*

PROOF. It is sufficient to prove that $\phi(t) = t$ for all $t \in [0,1)$. Indeed, in this case for every simple function $f = \sum_{k=1}^n a_k \mathbb{1}_{A_k}$ we have that

$$\|f\|_{L_1} \leqslant \|f\|_X \leqslant \sum_{k=1}^n |a_k|\phi(\mu(A_k)) = \|f\|_{L_1}.$$

So fix $t \in [0,1)$ and select a sequence $(\tau_n) \subset \mathbb{R}^+$ converging to 0 such that $\phi(\tau_n)/\tau_n \to 1$. Denote $m(n)$ the smallest positive integer such that $m(n)\tau_n \geqslant t$ and observe that $t \leqslant m(n)\tau_n < t + \tau_n$. Then

$$t \leqslant \phi(t) \leqslant \phi(m(n)\tau_n) \leqslant m(n)\phi(\tau_n) = \tau_n m(n)\phi(\tau_n)/\tau_n \to t$$

as $n \to \infty$. □

The following lemma prepares an elementary proof of Corollary 4.4.3, which also follows from the contractivity of conditional expectations in r.i. spaces [**207**, Theorem 2.a.4].

LEMMA 4.4.2. *Let $\Delta = [0,a] \subset [0,1]$ be a subinterval. Define for every $\tau \in \Delta$ the Δ-circling shift operator T_τ by $[T_\tau f](t) = f(t)$ for $t > a$, $[T_\tau f](t) = f(t+\tau)$ for $0 \leqslant t \leqslant a-\tau$, and $[T_\tau f](t) = f(t-a+\tau)$ for $a-\tau < t \leqslant a$. Then, for every $f \in X$*

the map $\tau \mapsto T_\tau f$ is continuous in the norm topology of X and hence is Riemann integrable. Moreover,

$$\frac{1}{a}\int_0^a T_\tau f\,d\tau = \left(\frac{1}{a}\int_0^a f(t)\,dt\right)\mathbb{1}_\Delta + f\mathbb{1}_{[0,1]\setminus\Delta}.$$

PROOF. The fact is evident when f is continuous and fulfills $f(0) = f(a)$. Since, as remarked above, such functions form a dense subset of X, we are done. □

COROLLARY 4.4.3. *Let $[0,1]$ be split into a disjoint union of measurable subsets Δ_1 and Δ_2. Then for every $g \in X$*

$$\left\| |g|\mathbb{1}_{\Delta_1} + \left(\frac{1}{\mu(\Delta_2)}\int_{\Delta_2} g(t)\,dt\right)\mathbb{1}_{\Delta_2}\right\|_X \leqslant \|g\|_X.$$

PROOF. We can assume without loss of generality $\Delta_2 = [0,a]$ and apply the previous lemma. □

COROLLARY 4.4.4. *Let $g \in X$. Then for every $t \geqslant \mu(\operatorname{supp} g)$*

$$\frac{1}{t}\phi(t)\|g\|_1 \leqslant \|g\|_X$$

PROOF. We may assume without loss of generality that $\Delta_2 := [0,t] \supset \operatorname{supp} g$ and apply the previous corollary. □

LEMMA 4.4.5. *Let $g \in L_\infty[0,1]$. Then for every $\alpha > 0$*

$$\|g\|_X \leqslant \alpha + \|g\|_\infty \phi(\alpha^{-1}\|g\|_1).$$

In particular, if $f_n \in L_\infty[0,1]$, $\sup_n \|f_n\|_\infty < \infty$ and $\lim_{n\to\infty}\|f_n\|_1 = 0$, then $\lim_{n\to\infty}\|f_n\|_X = 0$.

PROOF. Remark that

$$|g| \leqslant \alpha + \|g\|_\infty \mathbb{1}_{\{\tau\in[0,1]\colon\ |g(\tau)|>\alpha\}},$$

and that $\mu(\{\tau \in [0,1]\colon |g(\tau)| > \alpha\}) \leqslant \alpha^{-1}\|g\|_1$. □

THEOREM 4.4.6. *Let X be a separable real r.i. space on $[0,1]$ with the following property: for every $\varepsilon > 0$ there is $f = f_\varepsilon \in X$ such that*

(a) $\|f\|_X = 1$
(b) $\int_0^1 f(t)\,dt < -1 + \varepsilon$
(c) $\|f + \mathbb{1}\|_X \geqslant 2 - \varepsilon$.

Then $X = L_1[0,1]$ (endowed with its canonical norm).

Before giving the proof, we first record the main result of this section as an immediate consequence.

COROLLARY 4.4.7. *The only separable real r.i. function space on $[0,1]$ with the Daugavet property is $L_1[0,1]$ in its canonical norm.*

Indeed, the characterisation of the Daugavet property in terms of slices (Theorem 3.1.5) allows us to deduce this corollary from Theorem 4.4.6 by putting $x = \mathbb{1}$, $x^* = -\mathbb{1}$ and taking as f the corresponding y.

Proof of Theorem 4.4.6. Fix $\varepsilon > 0$ and $f = f_\varepsilon \in X$ with the properties (a), (b) and (c). Consider the following partition:

$$[0,1] = A \cup B = A_1 \cup A_2 \cup B_1 \cup B_2,$$

where

$$\begin{aligned} A &= \{t \in [0,1] \colon f(t) \leqslant 0\}, & B &= \{t \in [0,1] \colon f(t) > 0\}, \\ A_1 &= \{t \in A \colon |f(t)| \leqslant 2\}, & A_2 &= \{t \in A \colon |f(t)| > 2\}, \\ B_1 &= \{t \in B \colon |f(t)| \leqslant 2\}, & B_2 &= \{t \in B \colon |f(t)| > 2\} \end{aligned}$$

(all these sets depend on ε). Remark first that (a) and (b) imply

$$\int_B f\,d\mu < \varepsilon, \tag{4.4.1}$$

otherwise $g = f\mathbb{1}_A - f\mathbb{1}_B$ would be a norm-one function with $|\int_0^1 g(t)\,dt| > 1$. In particular,

$$\int_{B_1} f\,d\mu < \varepsilon,$$

and Lemma 4.4.5 says that

$$u(\varepsilon) := \|f\mathbb{1}_{B_1}\|_X \to 0 \tag{4.4.2}$$

as $\varepsilon \to 0$. Since also $\int_{B_2} f\,d\mu < \varepsilon$ and $f \geqslant 2$ on B_2 we have

$$\mu(B_2) < \frac{\varepsilon}{2}. \tag{4.4.3}$$

From $\|f\|_1 \leqslant \|f\|_X = 1$ we deduce

$$\mu(A_2 \cup B_2) < \frac{1}{2}. \tag{4.4.4}$$

Now, using the facts $|\mathbb{1}_{A_1} + f\mathbb{1}_{A_1}| \leqslant \mathbb{1}_{A_1}$ and $|\mathbb{1}_{A_2} + f\mathbb{1}_{A_2}| \leqslant |f|\mathbb{1}_{A_2}$, it is easy to check that

$$|\mathbb{1} + f| \leqslant \mathbb{1}_{A_1} + |f|\mathbb{1}_{A_2} + \mathbb{1}_{B_1} + |f|\mathbb{1}_{B_2} + |f|\mathbb{1}_{B_1} + \mathbb{1}_{B_2}$$

and, therefore, one can write

$$\begin{aligned} 2 - \varepsilon \leqslant \|\mathbb{1} + f\| &\leqslant \|\mathbb{1}_{A_1} + |f|\mathbb{1}_{A_2} + \mathbb{1}_{B_1} + |f|\mathbb{1}_{B_2}\| + \||f|\mathbb{1}_{B_1}\| + \|\mathbb{1}_{B_2}\| \\ &\leqslant \||f|\mathbb{1}_{A_2 \cup B_2} + \mathbb{1}_{A_1 \cup B_1}\| + u(\varepsilon) + \phi(\varepsilon) \end{aligned}$$

by (4.4.2) and (4.4.3). An application of Corollary 4.4.3 with $\Delta_1 = A_2 \cup B_2$, $\Delta_2 = A_1 \cup B_1$, and

$$g = |f| + \left(1 - \frac{1}{\mu(\Delta_2)} \int_{\Delta_2} |f|\,d\mu\right) \mathbb{1}_{\Delta_2},$$

implies that

$$\begin{aligned} 2 - \varepsilon &\leqslant \|g\|_X + u(\varepsilon) + \phi(\varepsilon) \\ &\leqslant 1 + \phi(\mu(\Delta_2)) \left(1 - \frac{1}{\mu(\Delta_2)} \int_{\Delta_2} |f|\,d\mu\right) + u(\varepsilon) + \phi(\varepsilon). \end{aligned} \tag{4.4.5}$$

Since we have by (4.4.4) $\mu(\Delta_2) \geqslant \frac{1}{2}$ for all values of ε, the last inequality implies $\lim_{\varepsilon \to 0} \int_{\Delta_2} |f|\,d\mu = 0$. Together with (4.4.1), this means that

$$\lim_{\varepsilon \to 0} \int_{A_2} |f|\,d\mu = 1. \tag{4.4.6}$$

Condition (4.4.5) also implies that

$$\lim_{\varepsilon\to 0}\phi(\mu(\Delta_2)) = 1. \tag{4.4.7}$$

Since $\mu(A_2) \leqslant \mu(\Delta_1) \leqslant \mu(\Delta_2)$, we can apply Corollary 4.4.4 for $g = |f|\mathbb{1}_{A_2}$ and $t = \mu(\Delta_2)$. Then,

$$1 \geqslant \frac{1}{\mu(\Delta_2)}\phi(\mu(\Delta_2))\int_{A_2}|f|\,d\mu. \tag{4.4.8}$$

By (4.4.6), (4.4.7), and (4.4.8), this implies $\mu(\Delta_2)\to 1$ and, consequently, $\mu(A_2)\to 0$ as $\varepsilon\to 0$. Now, we can apply again the same Corollary 4.4.4 but for $t = \mu(A_2)$ and $g = |f|\mathbb{1}_{A_2}$. This gives us that $\liminf_{\varepsilon\to 0}\phi(t)/t = 1$, and since $t\to 0$ as $\varepsilon\to 0$, an application of Lemma 4.4.1 completes the proof. $\square$

REMARK 4.4.8. *Theorem 4.4.6 also implies that $L_1[0,1]$ is the only separable real r.i. space on $[0,1]$ with "bad projections" (defined in* [**144**] *to mean that $\|\mathrm{Id} - P\| \geqslant 2$ for every rank-one projection) and the only separable real r.i. space on $[0,1]$ with the property that $\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$ for every rank-one operator T (the last property, the plus-minus property, appears in Chapter 12, cf. Definition 12.1.7).* This is so since the latter property is stronger than the former one, and since spaces with "bad projections" fit the conditions of Theorem 4.4.6 by using a characterisation of this property in terms of slices from [**144**]: X is a space with "bad projections" if and only if for every $x^*\in S_{X^*}$, every $\varepsilon > 0$ and every $x\in S_X$ with $\operatorname{Re} x^*(x) > 1-\varepsilon$, there is $y\in S_X$ such that $\|x-y\| > 2-\varepsilon$ and $\operatorname{Re} x^*(y) > 1-\varepsilon$. For more background on this phenomenon we refer the reader to Subsection 12.2.1.

4.5. The polynomial Daugavet property

The norm $\|T\|$ of a bounded linear operator T can be interpreted from a broader perspective in (at least) two different ways:

(1) $\|T\|$ is the sup norm of the bounded function $x\mapsto T(x)$ on the unit ball,
(2) $\|T\|$ is the optimal Lipschitz constant of the Lipschitz map T.

While we shall adopt the second point of view later, in Chapter 11, we are now going to explore the approach in (1) for certain nonlinear maps, viz., for the polynomials.

Let us start by defining this class of maps. First we define homogeneous polynomials. Let $m\in\mathbb{N}$. A mapping $P\colon X\to Y$ between Banach spaces is called a (continuous) *m-homogeneous polynomial*, $P\in \mathrm{Pol}(^m X, Y)$, if there exists an m-linear continuous mapping $L\colon X^m = X\times\cdots\times X\to Y$ such that $P(x) = L(x,\dots,x)$ for $x\in X$. Since L is continuous, which is equivalent to an estimate $\|L(x_1,\dots,x_m)\|\leqslant C\|x_1\|\cdots\|x_m\|$ for some constant C, we know that

$$\sup_{\|x\|\leqslant 1}\|P(x)\| < \infty.$$

Note that 1-homogeneous polynomials are just linear operators, and constant maps can be considered as 0-homogeneous polynomials.

A (continuous) *polynomial* $P\colon X\to Y$ is, by definition, a finite sum of homogeneous polynomials

$$P = \sum_{m=0}^{M} P_m, \qquad P_m\in \mathrm{Pol}(^m X, Y).$$

The set $\mathrm{Pol}(X,Y)$ of all polynomials has the structure of a vector space, and the sup norm

$$\|P\| := \sup_{\|x\|\leqslant 1} \|P(x)\|$$

is well-defined on $\mathrm{Pol}(X,Y)$ making it a normed space. An introduction to the theory of polynomials on Banach spaces can for instance be found in [**100**].

We say that $P \in \mathrm{Pol}(X,Y)$ is *weakly compact* if $P(B_X)$ is a relatively weakly compact subset of Y.

Since polynomials are bounded when restricted to the unit ball, they are amenable to (1) above; thus it makes sense to study polynomials $P \in \mathrm{Pol}(X,X)$ that satisfy the *polynomial Daugavet equation*

$$\|\mathrm{Id} + P\| = 1 + \|P\| \tag{4.5.1}$$

and Banach spaces where this is so for all weakly compact polynomials. We are going to develop the corresponding theory in parallel with the linear case. So the following definition and lemma shouldn't come as a surprise.

For $p_0 \in \mathrm{Pol}(X,\mathbb{K})$ and $x_0 \in X$ denote by $p_0 \otimes x_0$ the rank-one polynomial $x \mapsto p_0(x)x_0$; note that $\|p_0 \otimes x_0\| = \|p_0\|\|x_0\|$.

Definition 4.5.1. A Banach space X has the *polynomial Daugavet property* if all rank-one polynomials $P = p_0 \otimes x_0$ ($p_0 \in \mathrm{Pol}(X,\mathbb{K})$, $x_0 \in X$) satisfy the polynomial Daugavet equation (4.5.1).

As before (cf. Remark 3.1.2) it is enough to check this in the case $\|P\| = 1$.

Since linear operators are polynomials, the polynomial Daugavet property implies the Daugavet property.

The next lemma is the analogue of Lemma 3.1.3 and Theorem 3.1.5.

Lemma 4.5.2. *Let $p_0 \in \mathrm{Pol}(X,\mathbb{K})$, $x_0 \in X$, $\|p_0\| = \|x_0\| = 1$. Then the following are euivalent:*
(i) $\|\mathrm{Id} + p_0\| = 1 + \|p_0\|$.
(ii) *For all $\varepsilon > 0$ and $\delta > 0$ there exist $y \in B_X$ and $\omega \in \mathbb{T}$ such that*

$$\mathrm{Re}\, \omega p_0(y) > 1 - \delta \quad \textit{and} \quad \|x_0 + \omega y\| > 2 - \varepsilon.$$

Proof. (i) $\Rightarrow$ (ii). Let $\varepsilon' = \min\{\delta, \varepsilon\}$. By (i), there exists $y \in B_X$ such that

$$\|y + p_0(y)x_0\| > 2 - \frac{\varepsilon'}{2}.$$

Then $|p_0(y)| > 1 - \varepsilon'/2$ and for a suitable $\omega \in \mathbb{T}$, viz., $\omega = |p_0(y)|/p_0(y)$, we have $\mathrm{Re}\, \omega p_0(y) > 1 - \varepsilon'/2 > 1 - \delta$. On the other hand,

$$\begin{aligned}\|x_0 + \omega y\| &\geqslant \|\omega p_0(y)x_0 + \omega y\| - \|x_0 - \omega p_0(y)x_0\| \\ &> \left(2 - \frac{\varepsilon'}{2}\right) - \frac{\varepsilon'}{2} \geqslant 2 - \varepsilon\end{aligned}$$

since $|1 - \omega p_0(y)| = 1 - \mathrm{Re}\, \omega p_0(y) < \varepsilon'/2$.

(ii) $\Rightarrow$ (i). Let $\varepsilon = \delta > 0$ and pick ω and y according to (ii). Then

$$\begin{aligned}\|\mathrm{Id} + p_0\| &\geqslant \|y + p_0(y)x_0\| \\ &\geqslant \|x_0 + \omega y\| - \|x_0 - \omega p_0(y)x_0\| \\ &\geqslant (2 - \varepsilon) - |1 - \omega p_0(y)| \geqslant 2 - \varepsilon - \sqrt{2\varepsilon}.\end{aligned}$$

(For a complex number w, if $1-\varepsilon \leqslant \operatorname{Re} w \leqslant |w| \leqslant 1$, then $|1-\operatorname{Re} w| \leqslant \sqrt{2\varepsilon}$.) This proves (i), since $\varepsilon > 0$ was arbitrary. □

The next proposition is the polynomial analogue of Theorem 3.2.6.

PROPOSITION 4.5.3. *If X has the polynomial Daugavet property and $P \in \operatorname{Pol}(X, X)$ is weakly compact, then*

$$\|\mathrm{Id} + P\| = 1 + \|P\|.$$

PROOF. The proof is almost the same as the one for Theorem 3.2.6. We may assume that $\|P\| = 1$.

Let $\varepsilon > 0$. Since $P(B_X)$ is relatively weakly compact, so is its absolutely convex hull (Krein's theorem; [**109**, Theorem 3.133]). The weakly compact set $K = \overline{\operatorname{aconv}}(P(B_X))$ is the closed convex hull of its denting points (cf. Theorem 2.7.12); therefore, there exists a denting point y_0 of K with $\|y_0\| > 1-\varepsilon$ (recall that $\|P\| = 1$). Consequently, there is a slice S of K of diameter less than ε containing y_0. Let us write

$$S = \{y \in K\colon\ \operatorname{Re} y_0^*(y) > 1-\delta\}$$

for some $y_0^* \in X^*$ such that $\sup \operatorname{Re} y_0^*(K) = \sup |y_0^*(K)| = 1$ and some $\delta > 0$. By the diameter condition,

$$z \in K,\ \operatorname{Re} y_0^*(z) > 1-\delta \quad \Rightarrow \quad \|z - y_0\| < \varepsilon. \tag{4.5.2}$$

Now, let $p_0 = y_0^* \circ P$ and observe that

$$\|p_0\| = \sup_{x \in B_X} |y_0^*(P(x))| = \sup_{z \in K} |y_0^*(z)| = 1.$$

Also, let $x_0 = y_0/\|y_0\|$. By the polynomial Daugavet property (cf. Lemma 4.5.2) there exist $y \in B_X$ and $\omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega p_0(y) > 1-\delta \quad \text{and} \quad \|x_0 + \omega y\| > 2-\varepsilon.$$

We observe that

$$\operatorname{Re} y_0^*(\omega P(y)) = \operatorname{Re} \omega p_0(y) > 1-\delta,$$

so $\|\omega P(y) - y_0\| < \varepsilon$ by (4.5.2). On the other hand,

$$\begin{aligned}\|y_0 + \omega y\| &\geqslant \|x_0 + \omega y\| - \|x_0 - y_0\| \\ &\geqslant (2-\varepsilon) - (1 - \|y_0\|) > 2 - 2\varepsilon;\end{aligned}$$

finally

$$\|\mathrm{Id} + P\| \geqslant \|\omega(y + Py)\| \geqslant \|y_0 + \omega y\| - \|\omega P(y) - y_0\| > 2 - 3\varepsilon.$$

This completes the proof. □

Next, we shall discuss a number of classes of Banach spaces with the polynomial Daugavet property. Several of the forthcoming arguments rely on a weak continuity property of polynomials on certain Banach spaces that we isolate first.

PROPOSITION 4.5.4. *If $X = c_0$ or $X = L_1(\mu)$, then every polynomial $p \in \operatorname{Pol}(X, \mathbb{K})$ is weakly sequentially continuous, that is,*

$$x_n \to x \text{ weakly} \quad \Rightarrow \quad p(x_n) \to p(x).$$

More generally, this is so whenever X has the Dunford-Pettis property.

This result is due to Ryan [**273**]; for the proof see also [**100**, Prop. 2.34 (or Prop. 1.59 for c_0)]. For the Dunford-Pettis property we refer to e.g. [**109**, Section 13.7].

We shall first address the class of Banach spaces with a norming ℓ_1-structure, cf. Definition 3.5.8 in Subsection 3.5.2. We need the following variant of part (6) of Theorem 3.5.14.

LEMMA 4.5.5. *Let X be a Banach space with the following property:*

- *For all $x \in S_X$, $y \in B_X$, and $\omega \in \mathbb{T}$ there exists a sequence (x_n^{**}) in $B_{X^{**}}$ such that*
$$\limsup_{n\to\infty} \|x + \omega x_n^{**}\| = 2$$
and such that there is a bounded linear operator from c_0 to X^{**} mapping e_n to $x_n^{**} - y$.

Then X has the polynomial Daugavet property.

PROOF. We shall verify the condition of Lemma 4.5.2. Let $x \in S_X$, $\delta, \varepsilon > 0$, and let $p \in \mathrm{Pol}(X, \mathbb{K})$ be a norm-one polynomial; we have to produce some $z \in B_X$ and $\omega \in \mathbb{T}$ such that
$$\operatorname{Re} \omega p(z) > 1 - \delta \quad \text{and} \quad \|x + \omega z\| > 2 - \varepsilon.$$
To this end we use some sophisticated tools from the theory of polynomials on Banach spaces including Proposition 4.5.4.

Since $\|p\| = 1$, we can find $y \in B_X$ and $\omega \in \mathbb{T}$ with
$$\operatorname{Re} \omega p(y) > 1 - \delta.$$
Apply the assumptions of the lemma to these x, y, ω to obtain (x_n^{**}) in $B_{X^{**}}$ with
$$\limsup_{n\to\infty} \|x + \omega x_n^{**}\| = 2$$
and such that the mapping $e_1 \mapsto y$, $e_{n+1} \mapsto x_n^{**} - y$ extends to a bounded linear operator $T\colon c_0 \to X^{**}$. Now, we employ the Aron-Berner extension $\widehat{p}$ of p, $\widehat{p} \in \mathrm{Pol}(X^{**}, \mathbb{K})$; see [**100**, Prop. 1.51] for this topic. Let $q = \widehat{p} \circ T \in \mathrm{Pol}(c_0, \mathbb{K})$. Since $e_n + e_1 \to e_1$ weakly in c_0, Proposition 4.5.4 implies that
$$\widehat{p}(x_{n-1}^{**}) = q(e_n + e_1) \to q(e_1) = p(y);$$
hence
$$\operatorname{Re} \omega \widehat{p}(x_n^{**}) \to \operatorname{Re} \omega p(y) \ (> 1 - \delta).$$
Therefore, for some $N \in \mathbb{N}$,
$$\operatorname{Re} \omega \widehat{p}(x_N^{**}) > 1 - \delta \quad \text{and} \quad \|x + \omega x_N^{**}\| > 2 - \varepsilon.$$

The final step is to use a result due to Davie and Gamelin [**89**] that produces a net (z_α) in B_X such that $r(z_\alpha) \to \widehat{r}(x_N^{**})$ for all polynomials r, in particular, $z_\alpha \to x_N^{**}$ weakly-star. Since the norm is weak-star lower-semicontinuous, we conclude for some sufficiently large α that
$$\operatorname{Re} \omega p(z_\alpha) > 1 - \delta \quad \text{and} \quad \|x + \omega z_\alpha\| > 2 - \varepsilon,$$
completing the proof. $\square$

Thus we obtain:

THEOREM 4.5.6. *Let X be a Banach space with a norming ℓ_1-structure having the Daugavet property. Then X also has the polynomial Daugavet property.*

PROOF. In order to verify the conditions of Lemma 4.5.5 we just apply (6) of Theorem 3.5.14 with $-\omega x$ in place of x; since X has the Daugavet property, this is a Daugavet point. $\square$

COROLLARY 4.5.7. *The following classes of Banach spaces have the polynomial Daugavet property.*

(a) *Spaces $C_0(L)$ if the locally compact space L doesn't have isolated points;*
(b) *L_1-preduals for which E_X (see Definition 3.5.15) doesn't have isolated points; that is, those for which the norm is not Fréchet differentiable at any point.*
(c) *unital uniform algebras whose Choquet boundaries don't have isolated points.*

PROOF. Combine Theorem 4.5.6 with the results in Subsection 3.5.2. $\square$

We now resume the discussion of representable Banach spaces; see Definition 3.5.22 in Subsection 3.5.3. By a variant of the proof of Theorem 3.5.23 we are going to establish the polynomial Daugavet property for these spaces as well.

THEOREM 4.5.8. *Every representable Banach space has the polynomial Daugavet property.*

PROOF. Let $X \subset \prod_{k\in K} X_k$ be K-representable. In order to apply Lemma 4.5.2, suppose x_0, p_0, δ, and ε are given as described in that lemma. Let us fix an auxiliary vector $z \in B_X$ and some $\omega \in \mathbb{T}$ such that $\operatorname{Re} \omega p_0(z) > 1 - \delta$.

By definition, the set $S = \{k \in K\colon \|x_0(k)\| > 1 - \varepsilon/2\}$ is infinite. Pick a sequence (k_n) in S together with pairwise disjoint open sets $U_n \subset K$ containing k_n (cf. Lemma 3.5.24). Further, define Urysohn functions $f_n\colon K \to [0, 1]$ with $f_n(k_n) = 1$ and $\operatorname{supp}(f_n) \subset U_n$. Put

$$z_n = f_n \cdot (\omega^{-1} x_0 - z),$$

which is an element of X by its module property. Since the supports of the f_n are pairwise disjoint, it is clear that (z_n) is equivalent to the unit vector basis of c_0, and the closed linear span of $\{z, z_1, z_2, \dots\}$ is isomorphic to c_0. In particular, $z_n \to 0$ weakly.

For $k \in K$ we can estimate

$$\begin{aligned}\|(z + z_n)(k)\| &= \|(1 - f_n(k))z(k) + f_n(k)\omega^{-1} x_0(k)\| \\ &\leqslant (1 - f_n(k))\|z(k)\| + f_n(k)\|x_0(k)\| \leqslant 1\end{aligned}$$

and so $\|z + z_n\| \leqslant 1$. Therefore

$$\begin{aligned}\|x_0 + \omega(z + z_n)\| &\geqslant \|[x_0 + \omega(z + z_n)](k_n)\| \\ &= 2\|x_0(k_n)\| > 2 - \varepsilon\end{aligned}$$

since $k_n \in S$. Finally, we again make use of Proposition 4.5.4 (since $\overline{\operatorname{lin}}\{z, z_1, z_2, \dots\}$ is isomorphic to c_0) and deduce from $z_n \to 0$ weakly that $\operatorname{Re} \omega p_0(z + z_n) \to \operatorname{Re} \omega p_0(z) > 1 - \delta$.

Consequently, for large enough n, the vector $y = z + z_n$ and the scalar ω determined above meet the requirements of Lemma 4.5.2. $\square$

We mention in particular that, as a special case of the examples of representable spaces from Subsection 3.5.3, the space of vector-valued functions $C(K, E)$ has the polynomial Daugavet property if K is perfect; cf. Example 3.5.27.

We will now deal with L_1-spaces over nonatomic measure spaces and their vector-valued cousins, the Bochner spaces $L_1(\mu, E)$. We begin with two technical lemmas; the norms appearing there are the L_1-norms.

LEMMA 4.5.9. *Let (Ω, Σ, μ) be a nonatomic measure space, and let $p \in \mathrm{Pol}(L_1(\mu), \mathbb{K})$. If $u \in L_1(\mu)$ is a step function satisfying $|p(u)| > \alpha$ for some $\alpha > 0$, then there is another step function $v \in L_1(\mu)$ with $\|v\| = \|u\|$, $\mu(\mathrm{supp}(v)) = \frac{1}{2}\mu(\mathrm{supp}(u))$ that satisfies $|p(v)| > \alpha$ as well.*

PROOF. Let us first pretend that u is an indicator function, $u = \mathbb{1}_A$. Since μ is nonatomic, A supports a *Rademacher sequence* (r_n), that is, a sequence of independent random variables on the probability space $(A, \Sigma|_A, \frac{\mu}{\mu(A)})$ with

$$\mu(\{r_n = 1\}) = \mu(\{r_n = -1\}) = \mu(A)/2.$$

By Khinchin's inequality, the r_n span a copy of ℓ_2, and it follows that $r_n \to 0$ weakly in $L_1(\mu)$. (Here is an indication of how to construct such functions. Since μ is nonatomic, we can find two disjoint subsets $A_+, A_- \subset A$ with $\mu(A_\pm) = \frac{1}{2}\mu(A)$; define $r_1 = \mathbb{1}_{A_+} - \mathbb{1}_{A_-}$. Then split A_+ and A_- in the same way, producing A_{++}, A_{+-} and A_{-+}, A_{--}; define $r_2 = \mathbb{1}_{A_{++} \cup A_{-+}} - \mathbb{1}_{A_{+-} \cup A_{--}}$. Etc.)

Let $v_n = (1 + r_n)\mathbb{1}_A$. Then $v_n \to u$ weakly and by Proposition 4.5.4, $p(v_n) \to p(u)$. Since obviously $\|v_n\| = \|u\|$ and $\mu(\mathrm{supp}(v_n)) = \frac{1}{2}\mu(\mathrm{supp}(u))$ for all n, choosing n large enough will yield some $v = v_n$ as required in the lemma.

In the general case, u can be represented as a finite series

$$u = \sum_{k=1}^{N} a_k \mathbb{1}_{A_k} \tag{4.5.3}$$

with pairwise disjoint $A_1, \dots, A_N$. Choose Rademacher sequences $(r_n^{(k)})$ supported on A_k, for $k = 1, \dots, N$. Then again

$$v = v_n := \sum_{k=1}^{N} a_k (1 + r_n^{(k)}) \mathbb{1}_{A_k}$$

will be the required step function if n is large enough; note that $v_n \to u$ weakly. □

The point of the following lemma is to guarantee that one can find a function as below of norm one.

LEMMA 4.5.10. *Let (Ω, Σ, μ) be a nonatomic measure space, $p \in \mathrm{Pol}(L_1(\mu), \mathbb{K})$ be a polynomial and $\delta > 0$. Then there exists a step function $z \in L_1(\mu)$ with $\|z\| = 1$ and $|p(z)| > \|p\| - \delta$.*

PROOF. Without loss of generality we assume that $\|p\| = 1$. Since the step functions are dense, we can find a step function $u \in B_{L_1(\mu)}$ such that $|p(u)| > 1 - \delta$. If necessary, we will now modify u to obtain a step function v not having full support, still satisfying this inequality, as follows.

Take any sequence (B_k) of subsets of the support of u with $\mu(B_k) > 0$ and $\mu(B_k) \to 0$; this is possible because μ is nonatomic. Let $v_k = \mathbb{1}_{\Omega \setminus B_k} u$. These are step functions converging to u, which implies $p(v_k) \to p(u)$. Choosing k big enough, we obtain a step function $v = v_k$ with $|p(v)| > 1 - \delta$ and a set B of positive measure disjoint from the support of v.

We next consider a Rademacher sequence (r_n) supported on B, we define $\beta = \frac{1-\|v\|}{\mu(B)}$ and let $z_n = v + \beta r_n$. By choice of β we have $\|z_n\| = 1$. Now, $z_n \to v$ weakly, and by Proposition 4.5.4, $p(z_n) \to p(v)$. Again choosing n big enough, $z = z_n$ will be the required step function. □

We remark that the previous lemma remains valid in all *complex* Banach spaces, but may fail in real spaces (e.g., for $p(x) = 1 - \|x\|^2$ on a real Hilbert space). To see the former, let $p \in \mathrm{Pol}(X, \mathbb{C})$ be a polynomial on a complex Banach space, and let $\delta > 0$. Choose $\zeta \in B_X \setminus \{0\}$ such that $|p(\zeta)| > \|p\| - \delta$ and let $\xi = \zeta/\|\zeta\|$. Let $q\colon \mathbb{C} \to \mathbb{C}$ be the polynomial defined by $q(\lambda) = p(\lambda\xi)$; note that

$$\|p\| - \delta < |p(\zeta)| = |p(\|\zeta\|\xi)| = |q(\|\zeta\|)| \leqslant \|q\| \leqslant \|p\|.$$

By the maximum modulus principle, we have $\|q\| = |q(\lambda_0)|$ for some $|\lambda_0| = 1$; so $\xi_0 = \lambda_0\xi$ is a norm-one element with $|p(\xi_0)| > \|p\| - \delta$.

The following theorem establishes the polynomial Daugavet property for L_1-spaces over nonatomic measure spaces.

THEOREM 4.5.11. *If μ is a nonatomic measure, then $L_1(\mu)$ has the polynomial Daugavet property.*

PROOF. Let x_0, p_0, δ, and ε be given as in Lemma 4.5.2. With the help of Lemma 4.5.10 we can find a step function z of norm 1 such that

$$|p_0(z)| > 1 - \delta.$$

Let $\eta > 0$ be such that

$$A \in \Sigma,\ \mu(A) < \eta \quad \Longrightarrow \quad \int_A |x_0|\, d\mu < \frac{\varepsilon}{2},$$

pick $m \in \mathbb{N}$ such that $2^{-m}\mu(\mathrm{supp}(z)) < \eta$ and apply Lemma 4.5.9 m times to obtain a step function y with

$$\|y\| = \|z\| = 1, \quad \mu(\mathrm{supp}(y)) < \eta, \quad |p_0(y)| > 1 - \delta.$$

Then there is some $\omega \in \mathbb{T}$ such that

$$\mathrm{Re}\, \omega p_0(y) = |p_0(y)| > 1 - \delta.$$

It remains to estimate $\|x_0 + \omega y\|$; for this let us write $S = \mathrm{supp}(y)$. It follows

$$\begin{aligned} \|x_0 + \omega y\| &= \int_{\Omega\setminus S} |x_0|\, d\mu + \int_S |x_0 + \omega y|\, d\mu \\ &\geqslant \int_{\Omega\setminus S} |x_0|\, d\mu + \int_S |y|\, d\mu - \int_S |x_0|\, d\mu \\ &= 2 - 2\int_S |x_0|\, d\mu > 2 - \varepsilon \end{aligned}$$

since $\|x_0\| = \|y\| = 1$ and $\mu(S) < \eta$. □

We finally extend this theorem to the vector-valued setting.

THEOREM 4.5.12. *If μ is a nonatomic measure and E is a Banach space, then $L_1(\mu, E)$ has the polynomial Daugavet property.*

PROOF. The proof is virtually the same as before, with one caveat. Although $L_1(\mu, E)$ need not have the Dunford-Pettis property, the application of Proposition 4.5.4 in the variant of Lemma 4.5.9 and Lemma 4.5.10 for $L_1(\mu, E)$ is still possible. To see this, the crucial point is the representation (4.5.3) in which now $a_1, \dots, a_N \in E$. So the whole argument is set in the space $L_1(\mu, F)$ with $F = \operatorname{lin}\{a_1, \dots, a_N\}$, a finite-dimensional subspace of E, say $\dim(F) = d$. Now, $L_1(\mu, F)$ is isomorphic to $L_1(\mu, \ell_1^{(d)})$, which is isometric to another $L_1(\nu)$-space; hence it has the Dunford-Pettis property.

The argument for the previous theorem now carries over verbatim. □

4.6. Notes and remarks

Section 4.1. Let us comment that we cannot relax the hypothesis of norm denseness of Daugavet points in Proposition 4.1.1 to weak denseness, as there is a Banach space with a 1-unconditional basis (hence failing the Daugavet property, see Section 5.3) and containing a weakly dense subset of Daugavet points in its unit ball [**6**].

Theorem 4.1.7, saying that the Daugavet property is separably determined in the wide sense, was first established in [**179**]. (We are saying "in the wide sense" since the classical version that a property P is separably determined means that a Banach space has P if and only if every separable subspace has P, which fails in the case of the Daugavet property.)

Observe that an alternative proof of Theorem 4.1.7 can be provided by making use of the notion of almost isometric ideal. Indeed, thanks to Theorem 2.9.13, given any Banach space X with the Daugavet property and any subspace $Y \subset X$, we can find an almost isometric ideal Z in X with the properties $Y \subset Z$ and $\operatorname{dens}(Z) = \operatorname{dens}(Y)$. Since ai-ideals inherit the Daugavet property [**8**, Proposition 3.8], Z has the Daugavet property. This is an alternative proof to Theorem 4.1.7 which shows that this result still holds if we replace a separable Y with any Y.

Among the heredity properties of the Daugavet property we would like to mention that M-ideals and L-summands inherit the Daugavet property; conversely, the Daugavet property passes to ℓ_1-, ℓ_∞-, and c_0-sums, and if an M-ideal $Y \subset X$ and the quotient space X/Y have the Daugavet property, then so does X [**178**]. See Section 7.2, Section 7.3 and Section 7.5 for these matters and generalisations.

Section 4.2. In [**158**, Section 4] an example is given of a complex finite-dimensional Banach space F so that $L_1^{\mathbb{C}}[0,1] \mathbin{\widehat{\otimes}_\varepsilon} F$ and $L_\infty^{\mathbb{C}}[0,1] \mathbin{\widehat{\otimes}_\pi} F^*$ fail the Daugavet property. The example given in Theorem 4.2.1 is taken from [**197**]. The reason why we included the latter example is that it fails weaker requirements than the Daugavet property as octahedrality (see Definition 12.2.10 for the definition of octahedral norm).

The rest of the section is based on [**225**]. Observe that the definition of the WODP is motivated by the stronger property defined in [**272**], the Operator Daugavet Property (ODP). Using the ODP, it was proved in [**272**] that the projective tensor product of two L_1-predual spaces with the Daugavet property has the Daugavet property. The big goal of the WODP is that it is stable under taking projective tensor products, which makes it a promising tool for giving a possible positive answer to the (open) question whether the Daugavet property is inherited by projective tensor products.

Theorem 4.2.10 is taken from [**272**].

Section 4.3. The connection between the Daugavet property and the abundance of L-orthogonal elements was probably initiated in [**265**], where it was proved that in separable Banach spaces with the Daugavet property the set of L-orthogonal elements is weakly dense. The main motivation in this paper was to obtain Theorem 4.3.9 in the separable case.

The extension of the above results to spaces with density character less than or equal to ω_1 was obtained in [**211**], together with the Example 4.3.7. The main aim of this paper was, however, to obtain a counterexample for [**121**, Lemma 9.1], that is, to find a Banach space X without non-zero L-orthogonal elements but with the following property: for every finite subset $F \subset S_X$ and every $\varepsilon > 0$ there exists $x \in S_X$ so that $\|y+x\| > 2-\varepsilon$ holds for every $y \in F$. In particular, Example 4.3.7 is one such example since it does not have any non-trival L-orthogonal element but enjoys the Daugavet property.

Let us notice that the way in which the separable case is obtained in both results (by making use of the ball topology) differs from our approach (using techniques of Maurey types).

Moreover, see Theorem 5.5.1 for results about L-orthogonal elements and Daugavet centres. See also Theorems 6.7.1 and 6.7.2 for the connection between L-orthogonal elements and narrow operators.

Let us also mention that, in analogy with L-embedded spaces whose density character is $\leqslant \omega_1$, it is true for the class of Lipschitz-free spaces that the Daugavet property of $\mathcal{F}(M)$ passes to $\mathcal{F}(M) \widehat{\otimes}_\pi X$ for every non-zero Banach space X. We refer the reader to Chapter 11 for notation and to Section 11.3 for the proof.

Section 4.4. Corollary 4.4.7, on separable rearrangement invariant spaces, is the main theorem of this section; it was proved in [**163**], improving results in the previous paper [**18**]. The nonseparable case was tackled in [**19**]. Among the nonseparable rearrangement invariant spaces with the DP is the space L_∞, so the statement of Corollary 4.4.7 does not extend verbatim to the general case; rather, the conjecture is that a rearrangement invariant space with the Daugavet property must be isometric to an L_1- or an L_∞-space.

The authors of [**19**] make important steps toward the solution of that problem. They demonstrate the correctness of this conjecture for every non-separable r.i. space E on a finite measure space, under the assumption of the weak Fatou property of E, i.e., under the assumption that for every increasing sequence $(f_n) \subset E$ of non-negative functions and for every $f \in E$ the condition $f_n \to f$ a.e. implies that $\|f_n\| \to \|f\|$. (Note that in the framework of [**207**] the axioms of an r.i. space imply the weak Fatou property.)

They also prove that a uniformly monotone rearrangement invariant space over an infinite atomless measure space with the Daugavet property is isometric to L_1. As an application, they obtain that an Orlicz space over an atomless measure space has the Daugavet property if and only if it is isometrically isomorphic to L_1.

Another contribution in this direction is the paper [**183**]. There, it is shown that among Musielak-Orlicz function spaces on a σ-finite non-atomic complete measure space equipped with either the Luxemburg norm or the Orlicz norm, the only examples with the Daugavet property are of the form L_1, L_∞, $L_1 \oplus_1 L_\infty$ or $L_1 \oplus_\infty L_\infty$. In particular, this leads to complete characterisations of the Daugavet property in the weighted interpolation spaces, the variable exponent Lebesgue spaces (Nakano spaces) and the Orlicz spaces.

Other papers dealing with the Daugavet property of certain function spaces are [**184**] and [**216**].

Section 4.5. The Daugavet equation for polynomials was first studied in [**81**]; indeed the point of view of item (1) from the beginning of Section 4.5 was taken up there, and many results were obtained in this context. Other papers dealing with the Daugavet equation for bounded functions on the unit ball include [**82**], [**275**], [**69**]; see [**87**] for a comprehensive survey on the nonlinear Daugavet theory.

In [**81**], the polynomial Daugavet property for spaces like $C(K)$ or $C_0(L)$, including their vector-valued counterparts, was obtained whereas [**80**] did this for uniform algebras. In this section, we have deduced these results from the general approach to the polynomial Daugavet property in Banach spaces with a norming ℓ_1-structure ([**225**]; our Theorem 4.5.6), respectively, in representable Banach spaces ([**61**]; our Theorem 4.5.8). The case of scalar or Bochner L_1-spaces (Theorems 4.5.11 and 4.5.12) comes from [**221**]; we have taken this opportunity to correct an oversight there. Related papers are [**201**] and [**278**].

In Subsection 3.5.1, we have discussed the Daugavet property for the class of C^*-algebras. After a first attempt by Santos in [**277**], the general result that C^*-algebras with the Daugavet property actually have the polynomial Daugavet property was achieved in [**72**]; indeed, that paper even covers the case of JB*-triples. Since the techniques in this proof are rather different from those in Subsection 3.5.1, we refer the interested reader directly to [**72**] for details; suffice it to say that the arguments there provide a new proof of the (classical) Daugavet property for diffuse C^*-algebras and non-atomic JB*-triples.

By now, no example of a space with the Daugavet property failing the polynomial Daugavet property is known. This leads us to ask in Question (4.7) whether the two properties are actually equivalent.

Let us finally point out that an operator version of the polynomial Daugavet property, the so-called polynomial weak operator Daugavet property, is considered in [**225**, Definition 5.7] in order to obtain a sufficient condition for projective symmetric tensor product spaces to enjoy the Daugavet property. (The m-fold projective symmetric tensor product is a predual of the space of m-homogeneous polynomials.) We refer the interested reader to [**225**].

4.7. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

(4.1) If X and Y have the Daugavet property, does $X \widehat{\otimes}_\pi Y$ have the Daugavet property?

(4.2) Does the Daugavet property imply the WODP? (see Definition 4.2.3).

Observe that a positive answer to this question would give a positive solution to the above one thanks to Theorem 4.2.5.

(4.3) If X and Y have the Daugavet property, does $X \widehat{\otimes}_\varepsilon Y$ have the Daugavet property?

(4.4) Given an L-embedded Banach space X with the Daugavet property, is it true that $X \widehat{\otimes}_\pi Y$ has the Daugavet property for every non-zero Banach space Y?

Observe that the answer is yes if $\operatorname{dens}(X) \leqslant \omega_1$ by Theorem 4.3.9.

(4.5) If X is an L-embedded Banach space, say $X^{**} = X \oplus_1 Z$, is it true that X has the Daugavet property if, and only if, B_Z is w^*-dense in $B_{X^{**}}$?

Observe that the answer is yes if $\operatorname{dens}(X) \leqslant \omega_1$ by Theorem 4.3.8. Observe also that an affirmative answer to this question would imply a positive solution to the above one by repeating the proof of Theorem 4.3.9.

(4.6) Does Theorem 4.4.6 hold true in the complex case?

In order to get this extension, one would have to replace (b) of Theorem 4.4.6 with $\int_0^1 \operatorname{Re} f(t)\,dt < -1 + \varepsilon$. Unfortunately we haven't succeeded in proving this.

(4.7) Does every Banach space with the Daugavet property have the polynomial Daugavet property?

CHAPTER 5

Daugavet centres and unconditional decompositions

This section is devoted to an extension of the idea of the Daugavet property, namely that of Daugavet centres. We also find a connection to certain unconditional expansions, respectively, the impossibility of these. This leads to the corollary that a Banach space with the Daugavet property does not embed isomorphically into a space with an unconditional basis, extending the classical result by Pełczyński for $L_1[0,1]$ (and hence for $C[0,1]$). Finally, we take a look at PP-narrow operators.

5.1. Daugavet centres and Daugavet pairs of spaces

We shall now look at the Daugavet property from a broader perspective, considering it as a property of Id_X rather than of X. Therefore we present the following definition.

Definition 5.1.1. Let X, Y be Banach spaces. An operator $G \in L(X,Y)$ is said to be a *Daugavet centre* if the norm equality

$$\|G+T\| = \|G\| + \|T\| \tag{5.1.1}$$

holds for every rank-one $T \in L(X,Y)$.

The condition (5.1.1) is a direct extension of the Daugavet equation: the identity operator is just substituted by a general $G \in L(X,Y)$. With a little abuse of notation, when G is fixed, we use the name *Daugavet equation* for (5.1.1) as well. By the same reason as with the original Daugavet equation it is sufficient to deal with the case of $\|T\| = \|G\| = 1$ (for quasi-codirected G and T the operators aG and bT remain quasi-codirected for all $a, b \in [0,+\infty)$, see Remark 2.6.2).

Let us start with an extension of Theorem 3.1.5 and of a part of Theorem 3.1.11 to the case of Daugavet centres.

Theorem 5.1.2. *Let X, Y be Banach spaces, $G \in L(X,Y)$, $\|G\| = 1$. Then the following assertions are equivalent:*

(i) *G is a Daugavet centre.*
(ii) *For every $y \in S_Y$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there is $x \in S_X$ such that $\operatorname{Re} x^*(x) > 1-\varepsilon$ and $\|Gx + y\| > 2-\varepsilon$.*
(iii) *For every $y \in S_Y$, $x^* \in S_{X^*}$, and $\varepsilon > 0$, there is $x \in \mathrm{Slice}(B_X, x^*, \varepsilon)$ such that*

$$\|Gx + y\| > 2-\varepsilon. \tag{5.1.2}$$

(iv) *For every $y \in S_Y$, $\delta \in (0,1)$, and every slice S_0 of B_X, there is a smaller slice $S_1 \subset S_0$ such that all $x \in S_1$ satisfy the condition $\|Gx + y\| > 2-\delta$.*
(v) *For every $\varepsilon > 0$ and every $y \in S_Y$*

$$\overline{\mathrm{conv}}(\{x \in B_X \colon \|y + Gx\| > 2-\varepsilon\}) = B_X.$$

Proof. First, remark that (v) is just a Hahn-Banach style reformulation of (iii), and the implications (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) are evident (in the last one we just substitute the condition $x \in S_X$ by the weaker condition $x \in B_X$). In order to complete the proof, we will demonstrate the implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv), and (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) is included in Lemma 3.1.3 together with Remark 3.1.4.

The implication (iii) $\Rightarrow$ (iv) is almost a copy of Lemma 3.1.10. Indeed, let $S_0 = \mathrm{Slice}(B_X, x_0^*, \varepsilon_0)$, $x_0^* \in S_{X^*}$ and $\varepsilon_0 \in (0,1)$. Then, applying (iii) to the slice $\mathrm{Slice}(B_X, x_0^*, \varepsilon)$ with $\varepsilon \in (0, \min\{\delta/3, \varepsilon_0/2\})$, we obtain $x_0 \in S_X$ with $\operatorname{Re} x_0^*(x_0) > 1-\varepsilon$ and $\|Gx_0 + y\| \geqslant 2-\varepsilon$. Then, Gx_0 and y are ε-quasi-codirected and so Lemma 2.6.8 gives us a functional $y^* \in S_{Y^*}$ such that both $Gx_0, y \in \mathrm{Slice}(B_Y, y^*, \varepsilon)$. Denote $x^* := G^*y^*$ and

$$S_1 := \{z \in B_X \colon \operatorname{Re}(x_0^*(z) + x^*(z)) > 2 - 2\varepsilon\}.$$

S_1 is an intersection of a half-space with the unit ball and is not empty (as $x_0 \in S_1$), so S_1 is a slice of B_X. Let us check that $S_1 \subset S_0$. Indeed, for every $z \in S_1$

$$\operatorname{Re} x_0^*(z) > 2 - 2\varepsilon - \operatorname{Re} x^*(z) > 1 - 2\varepsilon > 1 - \varepsilon_0.$$

It remains to check that all $z \in S_1$ satisfy $\|Gx + z\| \geqslant 2 - \delta$. Indeed, for $z \in S_1$

$$\operatorname{Re} y^*(Gz) = \operatorname{Re} x^*(z) = \operatorname{Re}(x_0^*(z) + x^*(z)) - \operatorname{Re}(x_0^*(z)) > 2 - 2\varepsilon - 1 = 1 - 2\varepsilon$$

and, consequently, $\|Gz + y\| \geqslant \operatorname{Re} y^*(Gz + y) > 2 - 3\varepsilon \geqslant 2 - \delta$.

For the last implication (iii) $\Rightarrow$ (i) consider an operator $T = x^* \otimes y$, where $y \in Y$, $x^* \in X^*$, and $\|T\| = 1$. The representation $T = x^* \otimes y$ can be taken in such a way that $y \in S_Y$ and $x^* \in S_{X^*}$. It remains to apply Lemma 3.1.3. $\square$

Analogously to Corollary 3.1.6 the above characterisation in terms of slices permits, when it is convenient, to reduce the study to the real case.

Corollary 5.1.3. *A complex-linear operator G between complex Banach spaces X, Y is a Daugavet centre if and only if the same operator G is a Daugavet centre when considered as a real-linear operator between the underlying real spaces $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$.*

The next two lemmas are complete analogues of Lemmas 3.1.14 and 3.1.15, so we permit ourselves to omit the details, which an interested reader can easily check. The detailed demonstrations can be found in [**58**, Lemma 2.11]) and [**60**, Lemma 2.1]).

Lemma 5.1.4. *Let $G\colon X \to Y$ be a Daugavet centre with $\|G\| = 1$. Then, for every finite-dimensional subspace $Y_0 \subset Y$ and for every $\varepsilon > 0$, every slice S of B_X contains a smaller slice $\tilde{S} \subset S$ of B_X such that the inequality*

$$\|y + tGx\| \geqslant (1 - \varepsilon_0)(\|y\| + |t|) \tag{5.1.3}$$

holds for all $y \in Y_0$, $x \in \tilde{S}$ and $t \in \mathbb{K}$.

Lemma 5.1.5. *For an operator $G \in S_{L(X,Y)}$ the following assertions are equivalent:*

(i) *G is a Daugavet centre.*

(ii) *For every $\varepsilon > 0$, $y \in S_Y$, and every subset U of B_X that contains a convex combination of slices of the ball, there exists $x \in U$ such that $\|Gx+y\| > 2-\varepsilon$.*

(iii) *For every $\varepsilon > 0$, $y \in S_Y$ and every nonvoid relatively weakly open subset U of B_X there exists some $x \in U$ such that $\|Gx + y\| > 2 - \varepsilon$.*

In order to give one more geometric description of Daugavet centres, let us introduce more notation. For a bounded subset $A \subset Y$, its (outer) *radius at the point $y \in Y$* is $r_y(A) = \sup\{\|y - a\|\colon a \in A\}$ (see Figure 5.1).

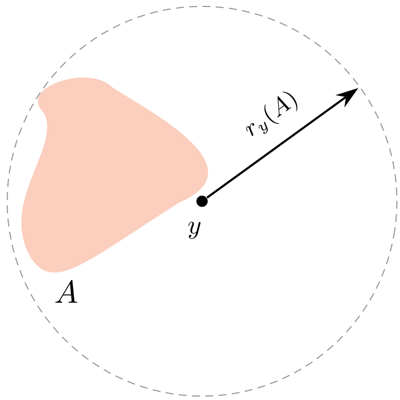

FIGURE 5.1. Outer radius

Clearly, $r_y(A) = r_y(\overline{\mathrm{conv}}(A))$ for every $y \in X$ and every bounded set A. Remark that, by the triangle inequality,

$$r_y(A) \leqslant \|y\| + \sup\{\|a\|\colon a \in A\} = \|y\| + r_0(A) \tag{5.1.4}$$

and the reciprocal inequality holds if A is a ball centred at 0 or even if $\overline{\mathrm{conv}}(A)$ is a ball centred at 0. This observation leads to the following definition.

DEFINITION 5.1.6. A bounded subset $A \subset Y$ is said to be a *quasiball* if for every $y \in Y$

$$r_y(A) = \|y\| + r_0(A). \tag{5.1.5}$$

By the comment before the definition, every subset whose closed convex hull is a ball centred at 0 is a quasiball, but the converse result is false, as Figure 5.2 shows. (We caution the reader that the notion of a quasiball with an entirely different meaning has appeared elsewhere, cf. [**133**, page 79].)

DEFINITION 5.1.7. A bounded subset $A \subset Y$ is said to be *antidentable* if for each $y \in Y$ and every $r \in [0, r_y(A))$

$$\overline{\mathrm{conv}}\,(A \setminus B_Y(y, r)) \supset A. \tag{5.1.6}$$

Our goal is to characterise Daugavet centres by means of quasiballs and antidentable sets. The first result in this line is the following necessary condition.

THEOREM 5.1.8. *If $G \in S_{L(X,Y)}$ is a Daugavet centre, then $A := G(B_X)$ is a quasiball.*

PROOF. Lemma 5.1.4 applied to the one-dimensional subspace $Y_0 = \mathrm{lin}\{y\}$ and $t = 1$ implies, in particular, that for every $y \in Y$ and $\varepsilon > 0$ there is $x \in B_X$ with $\|y - Gx\| > \|y\| + 1 - \varepsilon$. Consequently, $r_y(A) > \|y\| + 1 - \varepsilon \geqslant \|y\| + r_0(A) - \varepsilon$. $\square$

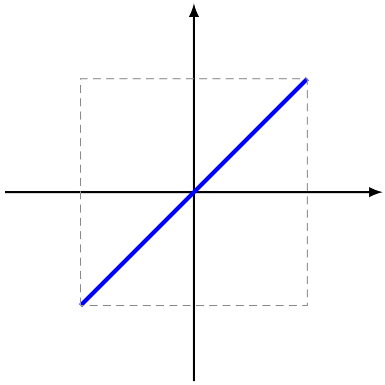

FIGURE 5.2. Closed convex quasiball in $(\mathbb{R}^2, \|\cdot\|_\infty)$ which is not a ball

The second result is another necessary condition for Daugavet centres.

THEOREM 5.1.9. *If $G \in S_{L(X,Y)}$ is a Daugavet centre, then for every $y \in Y$ and every $r \in [0, r_y(G(B_X)))$,*

$$V := \overline{\text{conv}}\left(B_X \setminus G^{-1}(B_Y(y,r))\right) \supset B_X. \tag{5.1.7}$$

PROOF. Let us assume to the contrary the existence of $y \in Y$ and $r \in [0, r_y(G(B_X)))$ for which V from (5.1.7) does not contain the whole unit ball B_X. By the Hahn-Banach theorem in the form of Lemma 2.6.7, there is a slice S_0 of B_X disjoint from V. For that slice we have $S_0 \subset G^{-1}(B_Y(y,r))$. Since, according to (5.1.4), $r < r_y(G(B_X)) \leqslant \|y\| + 1$, we may choose some $\delta_0 > 0$ so small that $\|y\| + 1 - \delta_0 > r$. With the help of Lemma 5.1.4 applied to the one-dimensional subspace $Y_0 = \text{lin}\{y\}$ and $t = 1$, we obtain some $x \in S_0 \subset G^{-1}(B_Y(y,r))$ with $\|Gx - y\| > \|y\| + 1 - \delta_0 > r$. But then, $Gx \notin B_Y(y,r)$, that is, $x \notin G^{-1}(B_Y(y,r))$. This contradiction concludes the proof. $\square$

In particular, we get this necessary condition in terms of antidentability.

COROLLARY 5.1.10. *For a Daugavet centre $G \in S_{L(X,Y)}$ the corresponding set $A := G(B_X)$ is antidentable.*

PROOF. By the previous theorem, for each $y \in Y$ and every $r \in [0, r_y(A))$, the inclusion (5.1.7) holds true. Consequently,

$$A \subset G(V) \subset \overline{\text{conv}}\left(G\left(B_X \setminus G^{-1}(B_Y(y,r))\right)\right) = \overline{\text{conv}}\left(A \setminus B_Y(y,r)\right). \qquad \square$$

Next, we are going to demonstrate that the properties from Theorems 5.1.8 and 5.1.9 together give a characterisation of Daugavet centres. We also extend to Daugavet centres the effect remarked previously in Theorem 3.2.6.

THEOREM 5.1.11. *An operator $G \in S_{L(X,Y)}$ is a Daugavet centre if and only if it simultaneously satisfies the following two conditions:*

(A) *$A := G(B_X)$ is a quasiball;*
(B) *the inclusion (5.1.7) holds true for all $y \in Y$ and all $r \in [0, r_y(G(B_X)))$.*

Moreover, if G is a Daugavet centre, then the identity (5.1.1) remains valid for all strong Radon-Nikodým operators $T \in L(X,Y)$, in particular, for all compact and all weakly compact T.

PROOF. A part of the job is already done in Theorems 5.1.8 and 5.1.9. It remains to show that (A) and (B) together imply (5.1.1) for all strong Radon-Nikodým operators $T \in L(X,Y)$. It will be a little bit more convenient for us to demonstrate the Daugavet equation (5.1.1) in the form of

$$\|G - T\| = 1 + \|T\|,$$

that is, with $-T$ instead of T. Fix $\varepsilon > 0$. By the definition of a strong Radon-Nikodým operator, $K = \overline{T(B_X)}$ is a set with the Radon-Nikodým property. So, K is equal to the closed convex hull of its denting points. Hence, there is a denting point u of K with $\|u\| > \|T\| - \varepsilon$. Select a slice $\tilde{S}$ of K that contains u and has $\operatorname{diam} \tilde{S} < \varepsilon$. Denote

$$r = r_u(G(B_X)) - \varepsilon.$$

Remark that $T^{-1}(\tilde{S}) \cap B_X$ is a slice of B_X (Proposition 2.6.5). By this reason, (B) implies that

$$\big(T^{-1}(\tilde{S}) \cap B_X\big) \cap \big(B_X \setminus G^{-1}(B_Y(u,r))\big) \neq \emptyset.$$

This means the existence of some $x_0 \in B_X$ such that $Tx_0 \in \tilde{S}$ (and, consequently, $\|Tx_0 - u\| < \varepsilon$), but $Tx_0 \notin B_Y(u,r)$, that is, $\|Gx_0 - u\| > r$. Then,

$$\|G - T\| \geqslant \|Gx_0 - Tx_0\| \geqslant \|Gx_0 - u\| - \varepsilon > r - \varepsilon = r_u(G(B_X)) - 2\varepsilon.$$

Applying (A), we may proceed as follows:

$$\|G - T\| \geqslant \|u\| + r_0(G(B_X)) - 2\varepsilon = \|u\| + \|G\| - 2\varepsilon \geqslant \|T\| + \|G\| - 3\varepsilon. \quad \square$$

The next result is an extension of Theorem 4.1.7.

THEOREM 5.1.12 ([**147**, Theorem 1]). *For $G \in S_{L(X,Y)}$, the following assertions are equivalent:*

(i) *G is a Daugavet centre;*
(ii) *for all separable subspaces $X_1 \subset X$ and $Y_1 \subset Y$ there exist separable subspaces $X_2 \subset X$ and $Y_2 \subset Y$ such that $X_1 \subset X_2$, $Y_1 \subset Y_2$, $G(X_2) \subset Y_2$ and the restriction $G|_{X_2}\colon X_2 \to Y_2$ of G is a norm-one Daugavet centre.*

PROOF. (i) $\Rightarrow$ (ii). In order to construct the subspaces X_2 and Y_2 we will inductively select two countable sets: $\{x_{n,m}\colon n,m \in \mathbb{N}\} \subset B_X$ such that the set of all linear combinations of its elements with rational coefficients is dense in X_2 (i.e., X_2 will be the closed linear span of the $x_{n,m}$), and $\{y_{n,m}\colon n,m \in \mathbb{N}\} \subset S_Y$ such that $\overline{\{y_{n,m}\colon n,m \in \mathbb{N}\}} = S_{Y_2}$. As the starting point, we take a countable dense subset of B_{X_1}, say $\{x_{1,m}\colon m \in \mathbb{N}\}$, and a countable dense sequence in the unit sphere of $\overline{\operatorname{lin}}\,(Y_1 \cup \{Gx_{1,m}\colon m \in \mathbb{N}\})$, say $(y_{1,m})_{m\in\mathbb{N}}$.

Assume that, for some natural k, we have constructed all the sequences $(x_{n,m})_{m\in\mathbb{N}}$ and $(y_{n,m})_{m\in\mathbb{N}}$ for every $n = 1,\dots,k$. Now, we construct the sequences $(x_{k+1,m})_{m\in\mathbb{N}}$ and $(y_{k+1,m})_{m\in\mathbb{N}}$. Take a sequence of positive numbers (ε_i) which converges to zero. Consider the Cartesian product

$$A_k := \left\{(x,y)\colon x \in \bigcup_{n=1}^{k} \{x_{n,m}\colon m \in \mathbb{N}\},\ y \in \bigcup_{n=1}^{k} \{y_{n,m}\colon m \in \mathbb{N}\}\right\}.$$

Using (v) of Theorem 5.1.2, for every $i \in \mathbb{N}$ and every $(x, y) \in A_k$ we find a finite convex combination $\sum \lambda_{ij} \hat{x}_{ij}$, where $\hat{x}_{ij} \in B_X$, such that $\|x - \sum \lambda_{ij} \hat{x}_{ij}\| < \varepsilon_i$ and for every $\hat{x}_{ij}$ the inequality $\|y + G\hat{x}_{ij}\| > 2 - \varepsilon_i$ holds true.

Let the symbol D_k denote the set of all $\hat{x}_{ij}$ that we have chosen in this way for all $i \in \mathbb{N}$ and all $(x, y) \in A_k$. Note that D_k is a countable subset of B_X. Let us extend D_k to a countable dense subset of the unit ball of the space $\overline{\mathrm{lin}}\, D_k$ and take the set obtained this way as $(x_{k+1,m})_{m\in\mathbb{N}}$. As $(y_{k+1,m})_{m\in\mathbb{N}}$ we take a countable dense subset of the unit sphere of the space

$$\overline{\mathrm{lin}} \left\{ Y_1 \cup \bigcup_{n=1}^{k+1} \{G x_{n,m} \colon m \in \mathbb{N}\} \right\}.$$

Now, let us show that these sets $\{x_{n,m} \colon n, m \in \mathbb{N}\}$ and $\{y_{n,m} \colon n, m \in \mathbb{N}\}$ satisfy our requirements. Consider $X_2 := \overline{\mathrm{lin}}\, \{x_{n,m} \colon n, m \in \mathbb{N}\}$ and $Y_2 := \overline{\mathrm{lin}}\{y_{n,m} \colon n, m \in \mathbb{N}\}$ which are clearly separable subspaces of X and Y respectively. It is easy to see that $X_1 \subset X_2$, $Y_1 \subset Y_2$ and $G(X_2) \subset Y_2$.

Let us prove that $G|_{X_2} \colon X_2 \to Y_2$ is a Daugavet centre. By (v) of Theorem 5.1.2 it is sufficient to show that for every $\varepsilon > 0$ and every $y \in S_{Y_2}$

$$B_{X_2} \subset \overline{\mathrm{conv}}\big(\{z \in B_{X_2} \colon \|y + Gz\| > 2 - \varepsilon\}\big).$$

Take $x \in B_{X_2}$ and $\delta > 0$. There is $\hat{x} \in \{x_{n,m} \colon n, m \in \mathbb{N}\}$ with $\|x - \hat{x}\| < \delta/2$, and there is $\hat{y} \in \{y_{n,m} \colon n, m \in \mathbb{N}\}$ with $\|y - \hat{y}\| < \varepsilon/3$. Let $k \in \mathbb{N}$ be such that $\hat{x} \in \bigcup_{n=1}^{k} \{x_{n,m} \colon m \in \mathbb{N}\}$ and $\hat{y} \in \bigcup_{n=1}^{k} \{y_{n,m} \colon m \in \mathbb{N}\}$. Then there exists a convex combination $\sum \lambda_j \hat{x}_j$ with $\hat{x}_j \in \bigcup_{n=1}^{k+1} \{x_{n,m} \colon m \in \mathbb{N}\}$ such that $\|\hat{x} - \sum \lambda_j \hat{x}_j\| < \delta/2$ and for every $\hat{x}_j$ the inequality $\|\hat{y} + G\hat{x}_j\| > 2 - \varepsilon/3$ holds true. Then

$$\|y + G\hat{x}_j\| > \|\hat{y} + G\hat{x}_j\| - \varepsilon/3 > 2 - 2\varepsilon/3 > 2 - \varepsilon$$

and $\|x - \sum \lambda_j \hat{x}_j\| < \delta$. Hence $x \in \overline{\mathrm{conv}}\big(\{z \in B_{X_2} \colon \|y - Gz\| > 1 + \|y\| - \varepsilon\}\big)$.

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$, $T \in L(X, Y)$ be a rank-one operator of norm 1, $x \in B_X$ be such that $\|Tx\| > 1 - \varepsilon$ and $z \in B_X$ be such that $\|Gz\| > 1 - \varepsilon$. For $X_1 := \mathrm{lin}\{x, z\}$ and $Y_1 := T(X)$ pick separable subspaces X_2 and Y_2 as in (ii). Consider the restriction $T|_{X_2} \colon X_2 \to Y_2$. Since $x, z \in X_2$, we have

$$\|T|_{X_2}\| \geqslant \|T|_{X_2} x\| = \|Tx\| \geqslant 1 - \varepsilon$$

and by the analogous argument $\|G|_{X_2}\| \geqslant 1 - \varepsilon$. By (ii) $G|_{X_2} \colon X_2 \to Y_2$ is a Daugavet centre, hence

$$\|G + T\| \geqslant \|G|_{X_2} + T|_{X_2}\| = \|G|_{X_2}\| + \|T|_{X_2}\| > 2 - 2\varepsilon.$$

Since the above inequality holds true for an arbitrary $\varepsilon > 0$, G is a Daugavet centre. □

REMARK 5.1.13. *From the definition it follows immediately that if $G \in S_{L(X,Y)}$ is a Daugavet centre, $E \subset Y$ is a subspace such that $G(X) \subset E$, then G considered as an operator acting from X to E remains a Daugavet centre.*

This remark implies that for a Daugavet centre $G \in S_{L(X,Y)}$ the statement (ii) of the previous theorem remains valid if one demands additionally that

$$Y_2 = \overline{\mathrm{lin}}(Y_1 \cup G(X_2)) \tag{5.1.8}$$

This helps to understand why the difficult part of Theorem 4.1.7 (the implication (i) $\Rightarrow$ (ii) of that theorem) follows from Theorem 5.1.12: for $X \in \mathrm{DPr}$ and for a separable subspace $X_1 \subset X$ apply Theorem 5.1.12 with $Y = X$, $G = \mathrm{Id}$ and

$Y_1 = X_1$. One obtains an $X_2 \supset X_1$ and $Y_2 = \overline{\mathrm{lin}}(Y_1 \cup \mathrm{Id}(X_2)) = X_2$, and that is all. Why the implication (ii) $\Rightarrow$ (i) of Theorem 4.1.7 follows from Theorem 5.1.12 is left as an exercise for the interested reader.

The most important examples of Daugavet centres for us are of course the identity operators in spaces with the Daugavet property. Another important class can be formalised with the help of the following definition.

DEFINITION 5.1.14. Let a Banach space X be a subspace of a Banach space Y. The pair (X, Y) is said to be a *Daugavet pair* if the natural embedding operator J: $X \to Y$ is a Daugavet centre.

A number of examples of Daugavet pairs can be found in [**170, 174, 178**]. Popov in [**255**] demonstrated that every isometric embedding of $L_1[0,1]$ into itself is a Daugavet centre. Several examples of Daugavet centres that are not embedding operators are described in [**58, 92**], viz., certain composition operators and weighted composition operators on $C(K)$ and the disk algebra.

5.2. The renorming theorem

In this section we are dealing the following situation. G: $X \to Y$ is a Daugavet centre, Y is a subspace of some Banach space E, and J: $Y \to E$ is the corresponding natural embedding. We want to know whether the operator $J \circ G$: $X \to E$ is a Daugavet centre. In general, the answer is negative. This happens, for example, if E is of the form $E = Y \oplus_\infty Z$ for some non-zero subspace Z. On the other hand, if $E = Y \oplus_1 Z$, then the answer is positive. Moreover, taking into account that in the case of complemented $Y \subset E$, E can be equivalently renormed to be equal to $Y \oplus_1 Z$, in this case we see that there exists an equivalent norm $|||\cdot|||$ on E that extends the original norm of Y and such that in this new norm the operator $J \circ G$: $X \to (E, |||\cdot|||)$ is a Daugavet centre. The aim of this section is to demonstrate that this renorming result remains true without the additional assumption of complementability of Y in E. The result is not easy, and requires some preparatory work.

DEFINITION 5.2.1. Let E be a linear space equipped with a seminorm $\|\cdot\|$ (seminormed space), $A \subset B_E$, $\mathfrak{U}$ be a free ultrafilter on a set Γ and f: $\Gamma \to A$ be a function. The triple $(\Gamma, \mathfrak{U}, f)$ is said to be an *A-valued E-atom* if for every $w \in E$

$$\lim_{\mathfrak{U}} \|f + w\| = 1 + \|w\|. \tag{5.2.1}$$

It is clear that for $A \subset B \subset B_E$ each A-valued E-atom is at the same time a B-valued E-atom. We will call B_E-valued E-atoms just *E-atoms*.

The following characterisation of Daugavet centres in terms of E-atoms is a consequence of Theorem 5.1.2 and Lemma 5.1.4.

THEOREM 5.2.2. *Let X, Y be Banach spaces. An operator $G \in S_{L(X,Y)}$ is a Daugavet centre if and only if for every slice S of B_X there exists a $G(S)$-valued Y-atom.*

PROOF. Let us start with the "if" part. Assume that for every slice S of B_X there is a $G(S)$-valued Y-atom. Our goal is to demonstrate that G is a Daugavet centre. For this, let us verify the condition (iii) of Theorem 5.1.2. Fix $y \in S_Y$, $x^* \in S_{X^*}$ and $\varepsilon > 0$. Denote $S = \mathrm{Slice}(B_X, x_0^*, \varepsilon)$. Our assumption produces a $G(S)$-valued Y-atom $(\Gamma, \mathfrak{U}, f)$. Substituting $w = y$ in (5.2.1), we obtain that in particular $\|f(t)+y\| > 2-\varepsilon$ for some $t \in \Gamma$. Since $f(t) \in G(S)$, there is $x \in S$ such that $f(t) = Gx$. This x fulfills (iii) of Theorem 5.1.2: $\operatorname{Re} x^*(x) > 1-\varepsilon$ and $\|Gx+y\| > 2-\varepsilon$. So, the "if" part is done.

Let us demonstrate the "only if" part. Assume that G is a Daugavet centre and S is a slice of B_X. Put $\Gamma = \mathrm{FIN}(Y)$, the family of finite subsets of Y, and denote by $\mathfrak{F}$ the natural filter on Γ induced by the ordering by inclusion, that is, the filter whose base is formed by the family $\{\hat{A} \subset \mathrm{FIN}(Y)\colon A \in \mathrm{FIN}(Y)\}$, where $\hat{A} := \{B \in \mathrm{FIN}(Y)\colon A \subset B\}$. By Lemma 5.1.4, for every $A \in \mathrm{FIN}(Y)$ there is $x(A) \in S$ such that for all $y \in A$

$$\|y + G(x(A))\| > \left(1 - \frac{1}{|A|}\right)(\|y\|+1).$$

Let us define the mapping $f\colon \Gamma \to G(S)$ by the formula $f(A) := G(x(A))$. Then, for every ultrafilter $\mathfrak{U}$ that dominates $\mathfrak{F}$, the triple $(\Gamma, \mathfrak{U}, f)$ will be the desired $G(S)$-valued Y-atom. □

The next few lemmata include some preparatory work for the renorming theorem promised in the title of this section.

Lemma 5.2.3. *Let (E,p) be a seminormed space, Y be a subspace of E and $(\Gamma,\mathfrak{U},f)$ be a Y-atom. Define for each $x \in E$ and $r > 0$ the quantity*

$$p_r(x) = \mathfrak{U}\text{-}\lim_t p(x + rf(t)) - r.$$

Then, p_r possesses the following properties:

(a) $0 \leqslant p_r(x) \leqslant p(x)$ *for all* $x \in E$,
(b) $p_r(y) = p(y)$ *for all* $y \in Y$,
(c) *the function* $x \mapsto p_r(x)$ *is convex for every fixed* $r > 0$,
(d) *the function* $r \mapsto p_r(x)$ *is convex for every fixed* $x \in E$,
(e) $p_r(tx) = tp_{r/t}(x)$ *for all* $t > 0$.

Proof. The property (b) follows from (5.2.1) with $w = \frac{x}{r}$. Indeed,

$$p_r(y) = r\left(\mathfrak{U}\text{-}\lim_t p\left(\frac{y}{r} + f(t)\right) - 1\right) = rp\left(\frac{y}{r}\right) = p(y).$$

Among the remaining properties the only non-evident one is the inequality $p_r \geqslant 0$. Let us demonstrate it. Substituting $w = 0$ in (5.2.1) we obtain in particular that $\mathfrak{U}\text{-}\lim_t p(f(t)) = 1$. Fix $\varepsilon > 0$ and select $t_\varepsilon \in \Gamma$ in such a way that $p(f(t_\varepsilon)) > 1-\varepsilon$ and

$$p(x + rf(t_\varepsilon)) \leqslant \mathfrak{U}\text{-}\lim_t p(x + rf(t)) + \varepsilon.$$

Then

$$\begin{aligned}
\mathfrak{U}\text{-}\lim_t p(x + rf(t)) &\geqslant \mathfrak{U}\text{-}\lim_t p(-rf(t_\varepsilon) + rf(t)) - p(x + rf(t_\varepsilon)) \\
&= rp(f(t_\varepsilon)) + r - p(x + rf(t_\varepsilon)) \\
&\geqslant 2r - r\varepsilon - \mathfrak{U}\text{-}\lim_t p(x + rf(t)) - \varepsilon.
\end{aligned}$$

Consequently,

$$\mathfrak{U}\text{-}\lim_t p(x + rf(t)) \geqslant \frac{1}{2}(2r - \varepsilon - r\varepsilon)$$

and, by the arbitrariness of ε, $p_r(x) \geqslant 0$. □

LEMMA 5.2.4. *Under the conditions of Lemma 5.2.3, the function $r \mapsto p_r(x)$ is monotonically non-increasing for each x. The quantity*

$$\bar{p}(x) := \lim_{r\to\infty} p_r(x) = \inf_r p_r(x)$$

possesses the following properties:

(a) $0 \leqslant \bar{p}(x) \leqslant p(x)$ *for all* $x \in E$,

(b) $\bar{p}(y) = p(y)$ *for all* $y \in Y$,

(c) *the function* $x \mapsto \bar{p}(x)$ *is convex,*

and, moreover,

$$\bar{p}(tx) = t\bar{p}(x) \quad \textit{for all} \quad t > 0,\ x \in E. \tag{5.2.2}$$

PROOF. Due to (a) and (d) of Lemma 5.2.3, the function $r \mapsto p_r(x)$ is non-negative, convex and bounded, hence it is monotonically non-increasing. This explains the correctness of the definition of $\bar{p}$. The items (a)–(c) of the current lemma follow from the corresponding items of Lemma 5.2.3, and the condition (5.2.2) follows from (e) of Lemma 5.2.3. $\square$

DEFINITION 5.2.5. Let $(E, \|\cdot\|)$ be a Banach space, Y be a subspace of E. An equivalent norm $|||\cdot|||$ on E is said to have the Y*-atomic property* if it satisfies the following two conditions:

(1) $|||y||| = \|y\|$ for each $y \in Y$;

(2) every Y-atom is at the same time an $(E, |||\cdot|||)$-atom.

The construction of the renorming and its basic properties are gathered in the next lemma.

LEMMA 5.2.6. *Let Y be a subspace of a Banach space E. Then, there exists an equivalent norm on E that possesses the Y-atomic property.*

PROOF. Denote $\mathcal{P}$ the family of all those seminorms q on E such that $q(x) \leqslant \|x\|$ for all $x \in E$ and $q(y) = \|y\|$ for all $y \in Y$. By Zorn's lemma, $\mathcal{P}$ possesses a minimal element p. Let us demonstrate first that every Y-atom $(\Gamma, \mathfrak{U}, f)$ is at the same time an (E, p)-atom, that is, that for every $x \in E$

$$\lim_{\mathfrak{U}} p(f + x) = 1 + p(x). \tag{5.2.3}$$

Indeed, consider the quantity $\bar{p}$ from Lemma 5.2.4 that is built on the seminorm p and the Y-atom $(\Gamma, \mathfrak{U}, f)$. Recall that $0 \leqslant \bar{p} \leqslant p$, $\bar{p}(y) = p(y) = \|y\|$ for all $y \in Y$, but $\bar{p}$ is not necessarily a seminorm. Nevertheless, the expression

$$q(x) = \frac{\bar{p}(x) + \bar{p}(-x)}{2}$$

already defines a seminorm. Moreover, $q \in \mathcal{P}$ and $q \leqslant p$. The minimality of p implies that

$$q(x) = p(x) \qquad \forall x \in E. \tag{5.2.4}$$

Further, taking into account that $p(x) \geqslant \bar{p}(x)$ and $p(x) = p(-x) \geqslant \bar{p}(-x)$, the equality (5.2.4) means that $p(x) = \bar{p}(x)$. Finally, from (a) of Lemma 5.2.3 and from the definition of $\bar{p}$ follows that $p(x) = p_r(x)$ for all $r > 0$; in particular, $p(x) = p_1(x)$, and this is exactly the desired (5.2.3).

Now, let us define the requested norm on E by the formula

$$|||x||| := p(x) + \|[x]\|_{E/Y},$$

and let us show that it fulfills the conditions of our lemma. First of all,

$$|||x||| \leqslant 2\|x\|.$$

Let us demonstrate that

$$|||x||| \geqslant \frac{1}{3}\|x\|.$$

By the homogeneity, it is sufficient to verify the above condition for $\|x\| = 1$. If $\|[x]\|_{E/Y} \geqslant \frac{1}{3}$, everything is clear. In the opposite case, we may select $y \in Y$ in such a way that $\|x - y\| < \frac{1}{3}$. Then $p(y) = \|y\| > \frac{2}{3}$ and

$$|||x||| \geqslant p(x) \geqslant p(y) - p(x - y) > \frac{2}{3} - \|x - y\| > \frac{1}{3}.$$

We have already demonstrated that $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms. Also, for all $y \in Y$

$$|||y||| = p(y) = \|y\|.$$

So, it remains to demonstrate that in this new norm each Y-atom $(\Gamma, \mathfrak{U}, f)$ is also an E-atom. Indeed, for every $x \in E$

$$\begin{aligned}\lim_{\mathfrak{U}} |||f + x||| &= \lim_{\mathfrak{U}} \big(p(f + x) + \|[f + x]\|_{E/Y}\big)\\ &= \lim_{\mathfrak{U}} p(f + x) + \|[x]\|_{E/Y}\\ &= 1 + p(x) + \|[x]\|_{E/Y} = 1 + |||x|||,\end{aligned}$$

which completes the proof. □

Now, we are ready for the main renorming theorem.

Theorem 5.2.7. *Let G: $X \to Y$ be a Daugavet centre, Y be a subspace of a Banach space E, and J: $Y \to E$ be the corresponding natural embedding. Then there exists an equivalent norm $|||\cdot|||$ on E that extends the original norm of Y and such that in this new norm the operator $J \circ G$: $X \to (E, |||\cdot|||)$ is a Daugavet centre. In the role of this norm, one can take any equivalent norm on E that possesses the Y-atomic property (Lemma 5.2.6).*

Proof. Take any equivalent norm $|||\cdot|||$ on E that possesses the Y-atomic property as the requested norm (which exists thanks to Lemma 5.2.6). We already know that it is an equivalent norm on E that extends the original norm of Y. So, it remains to prove that $J \circ G$: $X \to E$ is a Daugavet centre. We are going to do this with the help of Theorem 5.2.2. Indeed, let S be an arbitrary slice of B_X. According to Theorem 5.2.2, it is sufficient to find a $G(S)$-valued $(E, |||\cdot|||)$-atom.

Since G: $X \to Y$ is a Daugavet centre, the same Theorem 5.2.2 gives the existence of a $G(S)$-valued Y-atom $(\Gamma, \mathfrak{U}, f)$. Then, according to Lemma 5.2.6, $(\Gamma, \mathfrak{U}, f)$ is at the same time a $G(S)$-valued $(E, |||\cdot|||)$-atom, and the job is done. □

5.3. Pointwise unconditional convergence of operator series and the Daugavet equation

An operator series $\sum_{n=1}^{\infty} T_n$, $T_n \in L(X, Y)$, is said to be *pointwise unconditionally convergent* if for every $x \in X$ the series $\sum_{n=1}^{\infty} T_n x$ converges unconditionally.

The uniform boundedness principle and Proposition 2.4.1 imply the validity of condition

$$\sup\left\{\left\|\sum_{n\in A} T_n\right\| : A \in \mathrm{FIN}(\mathbb{N})\right\} < \infty \tag{5.3.1}$$

for every pointwise unconditionally convergent operator series $\sum_{n=1}^{\infty} T_n$.

Now, we are ready for the main result of the section.

THEOREM 5.3.1. *Let X, Y be Banach spaces, $G \in L(X,Y)$, and $\mathcal{M} \subset L(X,Y)$ be a linear space of operators such that the identity*

$$\|G - T\| = 1 + \|T\| \tag{5.3.2}$$

(that we still call the Daugavet equation) holds true for every $T \in \mathcal{M}$. Assume that an operator $V \in L(X,Y)$ can be represented as the sum of a pointwise unconditionally convergent series of elements of $\mathcal{M}$. Then $\|G - V\| \geqslant 1$. In particular, G cannot be expanded into a pointwise unconditionally convergent series of operators $T_n \in \mathcal{M}$.

PROOF. Let $V = \sum_{n=1}^{\infty} T_n$ be an expansion of V as a pointwise unconditionally convergent series of operators $T_n \in \mathcal{M}$. Denote

$$\alpha = \sup\left\{\left\|\sum_{n\in A} T_n\right\| : A \in \mathrm{FIN}(\mathbb{N})\right\}.$$

According to (5.3.1), $\alpha < \infty$. Also, for every $B \subset \mathbb{N}$ we have

$$\left\|\sum_{n\in B} T_n\right\| \leqslant \sup\left\{\left\|\sum_{n\in A} T_n\right\| : A \in \mathrm{FIN}(\mathbb{N}),\ A \subset B\right\} \leqslant \alpha. \tag{5.3.3}$$

For a given $\varepsilon > 0$ choose $A_\varepsilon \in \mathrm{FIN}(\mathbb{N})$ in such a way that $\|\sum_{n\in A_\varepsilon} T_n\| \geqslant \alpha - \varepsilon$. Then

$$\|G - V\| \geqslant \left\|G - \sum_{n\in A_\varepsilon} T_n\right\| - \left\|\sum_{n\notin A_\varepsilon} T_n\right\| \geqslant 1 + \left\|\sum_{n\in A_\varepsilon} T_n\right\| - \alpha \geqslant 1 - \varepsilon.$$

by (5.3.2) and (5.3.3). By the arbitrariness of ε this gives what we need. □

Remark that in [**58**, Theorem 2.9] one can find that the above theorem extends to uncountable pointwise unconditionally convergent summable families with the same proof, and moreover, if the condition (5.3.2) is replaced by the following one: there is $C > 0$ such that $\|G - T\| \geqslant C + \|T\|$ for all $T \in \mathcal{M}$, then the conclusion modifies to $\|G - V\| \geqslant C$.

COROLLARY 5.3.2. *Let X, Y be Banach spaces, $G \in S_{L(X,Y)}$ be a Daugavet centre. Then neither X nor Y has an unconditional basis. In particular, a Banach space with an unconditional basis cannot possess the Daugavet property.*

PROOF. Case 1: Assume X has an unconditional basis. Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis of X, $(e_n^*)_{n=1}^{\infty}$ be the corresponding coordinate functionals. Then $\mathrm{Id}_X = \sum_{n=1}^{\infty} e_n^* \otimes e_n$, where the series converges pointwise unconditionally, so the formula

$$G = \sum_{n=1}^{\infty} G \circ (e_n^* \otimes e_n) \tag{5.3.4}$$

gives a representation of G as a pointwise unconditionally convergent series of compact operators. Since compact operators form a linear space and satisfy the equation (5.3.2) (Theorem 5.1.11), this leads to a contradiction with Theorem 5.3.1.

Case 2: Assume Y has an unconditional basis $(g_n)_{n=1}^{\infty}$, and let $(g_n^*)_{n=1}^{\infty} \subset Y^*$ be the corresponding coordinate functionals. Then the formula

$$G = \sum_{n=1}^{\infty} (g_n^* \otimes g_n) \circ G$$

leads to a contradiction in the same way as (5.3.4) in the Case 1. □

Corollary 5.3.2 demonstrates that the Daugavet property is the common reason for the absence of an unconditional basis in the classical spaces $C[0,1]$ and $L_1[0,1]$. The next result shows that every basis in a space with the Daugavet property stays in some sense as far as possible from an unconditional one. Let us recall some definitions.

Let $(e_n)_{n=1}^{\infty}$ be a basis of a Banach space X, $(e_n^*)_{n=1}^{\infty}$ be the corresponding coordinate functionals. As we mentioned in Section 2.3, the partial sum operators $S_n = \sum_{k=1}^{n} e_k^* \otimes e_k$ are uniformly bounded, and the quantity $C = \sup_{n\in\mathbb{N}} \|S_n\|$ is called the basis constant of the basis (e_n).

For each $A \in \mathrm{FIN}(\mathbb{N})$, denote $\nu(A)$ the variation of $\mathbb{1}_A$ (in other words, $\nu(A)$ is the number of jumps from 1 to 0 and back). For example, $\nu(\{1,2,3\}) = 1$, $\nu(\{2,3,4\}) = 2$, $\nu(\{1,3,4,5\}) = 3$, etc. Let us introduce

$$\alpha_n = \alpha_n\big((e_k)_{k=1}^{\infty}\big) = \sup\left\{\left\|\sum_{k\in A} e_k^* \otimes e_k\right\| : A \in \mathrm{FIN}(\mathbb{N}),\ \nu(A) \leqslant n\right\}.$$

These quantities have an easy estimate from above: $\alpha_n \leqslant Cn$, where C is the basis constant of (e_n). In order to be unconditional, the basis needs to satisfy $\sup_n \alpha_n < \infty$. The next theorem demonstrates that the Daugavet property of the space implies a linear growth (that is, the maximal possible one) of α_n as $n \to \infty$ for every basis of the space.

Theorem 5.3.3. *Let $(e_k)_{k=1}^{\infty}$ be a basis of a Banach space $X \in \mathrm{DPr}$. Then, the above quantities $\alpha_n = \alpha_n\big((e_k)_{k=1}^{\infty}\big)$ satisfy the inequality $\alpha_{n+1} \geqslant \alpha_n + 1$ for all $n \in \mathbb{N}$.*

Proof. For a given $\varepsilon > 0$ choose $A \in \mathrm{FIN}(\mathbb{N})$ with $\nu(A) \leqslant n$ in such a way that $\|\sum_{k\in A} e_k^* \otimes e_k\| \geqslant \alpha_n - \varepsilon$. Then the Daugavet inequality implies that

$$\left\|\sum_{k\in\mathbb{N}\setminus A} e_k^* \otimes e_k\right\| = \left\|\mathrm{Id} - \sum_{k\in A} e_k^* \otimes e_k\right\| \geqslant 1 + \alpha_n - \varepsilon.$$

Since the sequence of operators $\sum_{k\in(\mathbb{N}\setminus A)\cap\{1,2,\dots,m\}} e_k^* \otimes e_k$ converges pointwise to $\sum_{k\in\mathbb{N}\setminus A} e_k^* \otimes e_k$ as $m \to \infty$ and, since $\nu\big((\mathbb{N}\setminus A)\cap\{1,2,\dots,m\}\big) \leqslant n+1$, we obtain that

$$\alpha_{n+1} \geqslant \sup_m \left\|\sum_{k\in(\mathbb{N}\setminus A)\cap\{1,2,\dots,m\}} e_k^* \otimes e_k\right\| \geqslant \left\|\sum_{k\in\mathbb{N}\setminus A} e_k^* \otimes e_k\right\| \geqslant 1 + \alpha_n - \varepsilon.$$

By the arbitrariness of ε, this completes the proof. □

Corollary 5.3.2 may be developed further in several directions, and for that the renorming theorem from the previous section is an invaluable tool. We give a sample result in Theorem 5.3.6 below.

Let Y be a Banach space. Consider the dual unit ball B_{Y^*} and the Banach space

$$\ell_\infty(B_{Y^*}) = \left\{ f\colon B_{Y^*} \to \mathbb{R}\colon \|f\|_\infty = \sup_{s \in B_{Y^*}} |f(s)| < \infty \right\}.$$

Every $y \in Y$ gives rise, via $f_y(y^*) = y^*(y)$, to a bounded function on B_{Y^*} with $\|f_y\|_\infty = \|y\|$; so, in a standard sense, $Y \subset \ell_\infty(B_{Y^*})$. (Indeed, $Y \subset C(B_{Y^*})$ if the dual unit ball is equipped with the weak* topology.)

Let us fix an equivalent norm $|||\cdot|||$ on $\ell_\infty(B_{Y^*})$ possessing the Y-atomic property (use Lemma 5.2.6), and denote $l_Y := (\ell_\infty(B_{Y^*}), |||\cdot|||)$. Recall that the pointwise application of the Hahn-Banach extension theorem generalises this theorem to operators acting to (renormings of) $\ell_\infty(\Gamma)$-spaces; this property is called *injectivity* of the space. For future reference we formulate this as a remark.

REMARK 5.3.4. *Let Y, E be Banach spaces and Z be a subspace of E. Then, for every $V \in L(Z, Y)$ there is an extension $\widetilde{V} \in L(E, l_Y)$, that is, $\widetilde{V}z = Vz$ for every $z \in Z$.*

LEMMA 5.3.5. *Let Z, E be Banach spaces, and $V\colon Z \to E$ be an isomorphic embedding. Then, there is an equivalent norm $\|\cdot\|_1$ on E such that $V\colon Z \to (E, \|\cdot\|_1)$ is an isometric embedding.*

PROOF. It is sufficient to define $\|y\|_1 = \|V^{-1}(y)\|$ for $y \in V(Z)$ and extend $\|\cdot\|_1$ to the whole of E using Theorem 2.10.1. □

Here is the main consequence of our study of Daugavet centres.

THEOREM 5.3.6. *If $G \in S_{L(X,Y)}$ is a Daugavet centre, then neither X nor Y can be isomorphically embedded into a space E in which the identity operator Id_E has a representation as a pointwise unconditionally convergent series of weakly compact operators.*

PROOF. Thanks to Lemma 5.3.5, an isomorphic embedding in our assumptions may be substituted by an isometric one. This way we reduce our problem to the case of $Z \subset E$, with V being the natural embedding operator $V\colon Z \to E$ of the subspace Z into the bigger space E.

Let $\mathrm{Id}_E = \sum_{n\in\mathbb{N}} T_n$, where the series is pointwise unconditionally convergent, and all the $T_n\colon E \to E$ are weakly compact. At first assume $Y \subset E$, and denote $J \in L(Y, E)$ the natural embedding operator. Equip E with the equivalent norm making $J \circ G$ a Daugavet centre, given by Theorem 5.2.7. Then $J \circ G = \sum_{n\in\mathbb{N}} T_n \circ J \circ G$, the series is pointwise unconditionally convergent, and all the operators $T_n \circ J \circ G$ are weakly compact. Since weakly compact operators form a linear space whose elements enjoy the Daugavet equation, this contradicts Theorem 5.3.1.

Now, assume $X \subset E$. Consider the isometric embedding $J\colon Y \to l_Y$ described above. According to Remark 5.3.4, there is an operator $U\colon E \to l_Y$ such that $U|_X = J \circ G$. Then

$$J \circ G = (U \circ \mathrm{Id}_E)|_X = \sum_{n\in\mathbb{N}} U \circ T_n|_X.$$

This representation leads to a contradiction in the same way as in the previous case. □

From the previous theorem, we can largely extend Pełczyński's result on the non-embedability of $L_1[0,1]$ into a Banach space with unconditional basis. We need some definition.

DEFINITION 5.3.7. The space E is an *unconditional direct sum* $\bigoplus_{n=1}^{\infty} E_n$ of spaces E_n provided that $E_n \subset E$, $n \in \mathbb{N}$, each $z \in E$ has a unique expansion $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in E_n$, and the series $\sum_{n=1}^{\infty} x_n$ converges unconditionally.

The next result is the extension of Pełczyński's result.

COROLLARY 5.3.8. *If $G \in S_{L(X,Y)}$ is a Daugavet centre, then neither X nor Y can be embedded into a space E having an unconditional basis or having a representation as an unconditional direct sum of reflexive subspaces. In particular, a Banach space with the Daugavet property cannot be isomorphically embedded in a space with an unconditional basis.*

PROOF. Assume on the contrary that either X or Y embeds isomorphically into an unconditional direct sum of reflexive spaces $E = \bigoplus_{n=1}^{\infty} X_n$. Denote by P_n the natural projections of the space E on X_n. By the reflexivity of X_n, all the P_n are weakly compact. According to Definition 5.3.7, $\sum_{n=1}^{\infty} P_n$ is pointwise unconditionally convergent to Id_Y. It remains to apply Theorem 5.3.6. □

REMARK 5.3.9. *In the sequel, for some important spaces and operator classes $\mathcal{M}$ we will demonstrate that, under the conditions of Theorem 5.3.1, the corresponding V satisfies the Daugavet equation $\|G - V\| = 1 + \|V\|$ and not only $\|G - V\| \geqslant 1$. We don't know whether such a result is true in general (see Question (5.1) in Section 5.6). The above claim can be confirmed if, for every $\varepsilon > 0$, there is an expansion $V = \sum_{n=1}^{\infty} T_n$, $T_n \in \mathcal{M}$, with the following property:*

$$\|V\| > \left\| \sum_{n=1}^{\infty} \pm T_n \right\| - \varepsilon \text{ for every choice of } \pm . \tag{5.3.5}$$

(This property holds true, in particular, if the original expansion is 1-unconditional, that is, if $\|V\| = \|\sum_{n=1}^{\infty} \pm T_n\|$ for every choice of $\pm$.) Indeed, let α and A_ε be as in the proof of Theorem 5.3.1, and let $\lambda_n = -1$ for $n \in A_\varepsilon$ and $\lambda_n = 1$ for $n \in \mathbb{N} \setminus A_\varepsilon$. Then we have under the condition (5.3.5)

$$\begin{aligned} \|G - V\| &= \left\| G - 2 \sum_{n \in A_\varepsilon} T_n - \sum_{n \in \mathbb{N}} \lambda_n T_n \right\| \\ &\geqslant 1 + 2 \left\| \sum_{n \in A_\varepsilon} T_n \right\| - \left\| \sum_{n \in \mathbb{N}} \lambda_n T_n \right\| \\ &\geqslant 1 + 2(\alpha - \varepsilon) - \|V\| - \varepsilon \geqslant 1 + \|V\| - 3\varepsilon. \end{aligned}$$

One more way to apply the ideas exposed in this section is shown below.

THEOREM 5.3.10. *Let X, Y be Banach spaces, $X \in$ DPr, and suppose $V \in L(X,Y)$ has an expansion $V = \sum_{n=1}^{\infty} T_n$ as a sum of a pointwise unconditionally convergent series of weakly compact operators $T_n \in L(X,Y)$. Then V is not an isomorphic embedding (i.e., V is not bounded below).*

PROOF. Assume on the contrary that V is an isomorphic embedding. Without loss of generality we may suppose that V is an isometric embedding (Lemma 5.3.5). Then X is isometric to $V(X)$ and we may reduce our problem to the case of $X \subset Y$, with V being the natural embedding operator $V\colon X \to Y$ of the subspace X into the bigger space Y. Since $X \in \mathrm{DPr}$, $\mathrm{Id}_X\colon X \to X$ is a Daugavet centre, and according to Theorem 5.2.7, we may introduce another equivalent norm on Y in such a way that $V\colon X \to Y$ will be a Daugavet centre as well. Then, due to Theorem 5.1.11, in the new norm every weakly compact operator $T\colon X \to Y$ satisfies the Daugavet equation $\|V-T\| = 1+\|T\|$. Consequently, by Theorem 5.3.1, V is not representable as a pointwise unconditionally convergent series of weakly compact operators, as opposed to our assumption. □

5.4. PP-narrow operators, hereditarily PP-narrow operators, and pointwise unconditional convergence

In the previous section we looked at pointwise unconditionally convergent series of operators; now we are going to iterate this idea and will first introduce a suitable notation for this.

Definition 5.4.1. Let E, Y be Banach spaces, $\mathcal{M} \subset L(E, Y)$ be a linear subspace. By $\mathrm{unc}(\mathcal{M})$ we denote the bigger linear subspace consisting of all operators $T\colon E \to Y$ that can be represented as a sum of a pointwise unconditionally convergent series of some operators $T_n \in \mathcal{M}$.

In this notation, Theorem 5.3.10 says, in particular, that for $E \in \mathrm{DPr}$ and Y arbitrary, the corresponding subspace $\mathrm{unc}(K(E, Y))$ does not contain any bounded below operator.

Iterating the operation "unc", we may introduce the classes

$$\mathrm{unc}_2(K(E,Y)) = \mathrm{unc}(\mathrm{unc}(K(E,Y))),$$
$$\mathrm{unc}_3(K(E,Y)) = \mathrm{unc}(\mathrm{unc}_2(K(E,Y))),$$

and so on. We are interested in analysing whether it is true that, given two Banach spaces E, Y with $E \in \mathrm{DPr}$, none of the classes $\mathrm{unc}_n(K(E,Y))$, $n \in \mathbb{N}$, contains a bounded below operator (it is indeed an open question, see Question (5.2) in Section 5.6).

A natural approach to the above problem consists in finding, for given E and Y, a "convenient" class $\mathcal{M} \subset L(E, Y)$ that is stable under the operation "unc", contains all compact operators, and does not contain any bounded below operator.

Although we are not able to solve the above problem for a general $E \in \mathrm{DPr}$, for the spaces $C[0,1]$ and $L_1[0,1]$ a positive answer can be found with the help of the idea that is described above. In Chapter 8 we will demonstrate that for $E = C[0,1]$ and $E = C(K, X)$, such a "convenient" class is formed by the C-narrow operators (Theorem 8.3.5).

In this section we are going to construct a "convenient" class for $E = L_1[0,1]$. This class consists of the hereditarily PP-narrow operators introduced in [**159**], and a big part of the section literally repeats the exposition from [**159**]. A similar problem for pointwise absolute convergent expansions was approached in [**153**].

In 1990 Plichko and Popov [**252**] introduced the important notion of a narrow operator. We prefer to call such operators *PP-narrow*, that is, "narrow in the sense of Plichko and Popov". Nowadays the theory of PP-narrow operators is deeply developed in many directions. A good account of it can be found in the monograph [**254**]. In the sequel, we will refer to PP-narrow operators many times, because the analogy with this class gave rise to the extremely important concept of a narrow operator on a space with the Daugavet property, to be discussed in Chapter 6. Let us introduce the necessary notation and definitions.

In this section (Ω, Σ, μ) is a finite or σ-finite nonatomic measure space. We denote by Σ^+ the collection of those $A \in \Sigma$ for which $0 < \mu(A) < \infty$. For $1 \leqslant p < \infty$, L_p means $L_p(\Omega, \Sigma, \mu)$, and $L_p^0(\Omega, \Sigma, \mu) \subset L_p$ is the subspace of all $f \in L_p$ with $\int_\Omega f\, d\mu = 0$.

Definition 5.4.2. Let $A \in \Sigma^+$. A measurable function $f\colon \Omega \to \mathbb{R}$ is said to be a *sign on* A if it is of the form $f = \mathbb{1}_{B_1} - \mathbb{1}_{B_2}$, where B_1 and B_2 form a partition of A into two measurable subsets with $\mu(B_1) = \mu(B_2)$.

An operator $T \in L(L_p, X)$ is said to be *PP-narrow* if for every $A \in \Sigma^+$ and $\varepsilon > 0$ there is a sign f on A such that $\|Tf\| \leqslant \varepsilon$.

For the space L_1 the complement of this class of operators was studied under the name "sign-preserving operators" in papers by Ghoussoub and Rosenthal [**118**], [**262**] and [**263**].

From the definition it follows that a PP-narrow operator cannot be bounded below. Also, every compact operator $T \in L(L_p, X)$, $1 \leqslant p < \infty$, is PP-narrow [**252**]. Indeed, for $A \in \Sigma^+$ consider a Rademacher sequence (r_n) on A, that is, a sequence of independent random variables on the probability space $\left(A, \Sigma|_A, \frac{\mu}{\mu(A)}\right)$ with $\mu(\{r_n = 1\}) = \mu(\{r_n = -1\}) = \mu(A)/2$. (Cf. page 119.) All r_n are signs on A, $r_n \to 0$ weakly in L_p, so $\|Tr_n\| \to 0$ by compactness of T.

It is known [**176, 286**] that the class of PP-narrow operators in $L(L_1)$, is stable under the operation unc. Unfortunately, this result does not extend to those PP-narrow operators that act from L_1 into another space. Moreover, Mykhaylyuk and Popov [**236**] constructed a space Y and a couple of PP-narrow operators $T_1, T_2 \in L(L_1[0,1], Y)$ whose sum is not PP-narrow; so in general it does not make sense to speak about infinite sums of PP-narrow operators. In order to solve this problem, we introduce a subclass of PP-narrow of operators, called hereditarily PP-narrow (HPP-narrow for short).

In this section we consider only *real* spaces. Since the question about operator classes that we study is of isomorphic character, and complex L_1, considered as a real space, is isomorphic to the real L_1, we don't lose any generality excluding the complex case from our consideration.

Although below we are speaking about a general fixed non-atomic measure space (Ω, Σ, μ) (finite or σ-finite), taking into account that L_1 on a non-atomic countably generated measure space is isometric to $L_1[0,1]$, for our main task the reader doesn't lose anything considering only $\Omega = [0,1]$ equipped with the σ-algebra Σ of Lebesgue measurable subsets and the Lebesgue measure.

5.4.1. Haar-like systems and hereditarily PP-narrow operators. Denote $\mathcal{A}_0 = \{\emptyset\}$, $\mathcal{A}_n = \{-1, 1\}^n$, $\mathcal{A}_\infty = \bigcup_{n=0}^\infty \mathcal{A}_n$. The elements of $\mathcal{A}_n$ are n-tuples of the form $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_k = \pm 1$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{A}_n$ and $\alpha_{n+1} \in \{-1, 1\}$ denote by α, α_{n+1} the $(n+1)$-tuple $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \in \mathcal{A}_{n+1}$; let us agree also that $\emptyset, \alpha_1 = (\alpha_1)$ (observe that this operation is nothing but the concatenation of finite sequences). The elements of $\mathcal{A}_\infty$ can be written as a sequence in the following *natural ordering*:

$$\emptyset,\ -1,\ 1,\ (-1,-1),\ (-1,1),\ (1,-1),\ (1,1),\ (-1,-1,-1),\ \ldots.$$

Definition 5.4.3. Let $A \in \Sigma^+$. A family $\{A_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ of measurable subsets of A is said to be *a tree of subsets on* A if $A_\emptyset = A$, and for every $\alpha \in \mathcal{A}_\infty$ the corresponding couple of sets $A_{\alpha,1}$, $A_{\alpha,-1}$ form a partition of A_α in two subsets of equal measure. The family of functions $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$, defined by the formula

$$h_\alpha = \mathbb{1}_{A_{\alpha,1}} - \mathbb{1}_{A_{\alpha,-1}},$$

is said to be *a Haar-like system on* A (relative to the tree $\{A_\alpha\colon \alpha \in \mathcal{A}_\infty\}$).

Remark that after deleting the constant function, the classical Haar system on $[0,1]$ gives an example of a Haar-like system. Moreover, each Haar-like system is equivalent to the one just mentioned.

Remark 5.4.4. (a) *Let $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ be a Haar-like system on A, relative to the tree $\{A_\alpha\colon \alpha \in \mathcal{A}_\infty\}$, and let $1 \leqslant p < \infty$. Then $\|h_\alpha\| = \big(2^{-n}\mu(A)\big)^{1/p}$ for $\alpha \in \mathcal{A}_n$. Denote by Σ_1 the σ-algebra on A generated by the sets A_α. Then the system $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ in its natural order forms a monotone basis for $L_p^0(A, \Sigma_1, \mu)$.*

(b) *Consequently, if $\varepsilon > 0$ and $\{\varepsilon_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ is a family of positive numbers with $\sum_\alpha \varepsilon_\alpha/\|h_\alpha\| < \varepsilon/2$, and if $\{x_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ are elements of a Banach space X with $\|x_\alpha\| \leqslant \varepsilon_\alpha$, then the mapping $h_\alpha \mapsto x_\alpha$ uniquely extends to a linear bounded operator $U\colon L_p^0(A, \Sigma_1, \mu) \to X$ of norm $\|U\| < \varepsilon$.*

Lemma 5.4.5. *Let $1 \leqslant p < \infty$ and let $T\colon L_p \to X$ be a PP-narrow operator. Then,*

(a) *For every $A \in \Sigma^+$ and every family $\{\varepsilon_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ of positive numbers, there is a Haar-like system $\{h_\alpha\colon \alpha \in \mathcal{A}_\infty\}$ on A such that $\|Th_\alpha\| \leqslant \varepsilon_\alpha$ for $\alpha \in \mathcal{A}_\infty$.*

(b) *For every $\varepsilon > 0$ and every $A \in \Sigma^+$ there is a σ-algebra $\Sigma_\varepsilon \subset \Sigma$ on A such that $(A, \Sigma_\varepsilon, \mu)$ is a nonatomic measure space and $\Big\|T|_{L_p^0(A,\Sigma_\varepsilon,\mu)}\Big\| < \varepsilon$.*

Proof. To construct a tree of subsets and the corresponding Haar-like system for (a) we repeatedly apply the definition of a PP-narrow operator. Namely, let $h_\emptyset$ be a sign supported on A with $\|Th_\emptyset\| \leqslant \varepsilon_\emptyset$. Put, using the standard notation $\{h = x\} = \{\omega \in \Omega\colon h(\omega) = x\}$,

$$A_{-1} = \{h_\emptyset = -1\}, \quad A_1 = \{h_\emptyset = 1\}.$$

Let h_{-1} and h_1 be signs supported on A_{-1} and A_1 respectively with $\|Th_{\pm 1}\| \leqslant \varepsilon_{\pm 1}$; put

$$A_{-1,-1} = \{h_{-1} = -1\}, \quad A_{-1,1} = \{h_{-1} = 1\},$$
$$A_{1,-1} = \{h_1 = -1\}, \quad A_{1,1} = \{h_1 = 1\}$$

and continue in the above fashion. This yields part (a).

Part (b) follows from (a) and Remark 5.4.4. □

For $1 \leqslant p < \infty$ the class of PP-narrow operators on L_p is not stable under taking sums (see [**252**, p. 59] for the case of $1 < p < \infty$ and [**236**] for $p = 1$); this is why we have to consider a smaller class of operators that we introduce next.

Definition 5.4.6. An operator $T \in L(L_p, X)$ is said to be *hereditarily PP-narrow* (*HPP-narrow* for short) if for every $A \in \Sigma^+$ and every nonatomic sub-σ-algebra $\Sigma_1 \subset \Sigma$ on A the restriction of T to $L_p(A, \Sigma_1, \mu)$ is PP-narrow.

Since every compact operator on L_p is PP-narrow and compactness is inherited by restrictions, compact operators on L_p are HPP-narrow. On the other hand, the following example shows that a PP-narrow operator need not be HPP-narrow.

EXAMPLE 5.4.7. The averaging operator

$$T\colon L_p([0,1]^2) \to L_p[0,1], \quad (Tf)(s) = \int_0^1 f(s,t)\,dt$$

is PP-narrow operator but not HPP-narrow.

Indeed, for $A = [0,1]^2$ and for the sub-σ-algebra Σ_1 of those measurable subsets $B \subset [0,1]^2$ that do not depend on the second coordinate (i.e., if $(s,\tau) \in B$ for some $\tau \in [0,1]$, then $(s,t) \in B$ for all $t \in [0,1]$), the corresponding $T|_{L_p(A,\Sigma_1,\mu)}$ is an isometry, so it is not PP-narrow. This shows that T is not HPP-narrow. On the other hand, T is PP-narrow because for each $A \subset [0,1]^2$ of positive measure we may divide each vertical section of A into two subsets of equal linear Lebesgue measure, thus generating a partition $A = B_1 \sqcup B_2$ such that $T(\mathbb{1}_{B_1} - \mathbb{1}_{B_2}) = 0$ (see [**254**, Lemma 4.11] for the details).

We now show that the set of HPP-narrow operators forms a subspace of $L(L_p, X)$.

PROPOSITION 5.4.8. *Let $1 \leqslant p < \infty$ and let $U, V\colon L_p \to X$. Then*

(a) *if U is PP-narrow and V is HPP-narrow, then $U + V$ is PP-narrow;*

(b) *if U and V are both HPP-narrow, then $U + V$ is HPP-narrow as well.*

PROOF. (a) Let $A \in \Sigma^+$ and $\varepsilon > 0$. By (b) of Lemma 5.4.5, there is a sub-σ-algebra $\Sigma_\varepsilon \subset \Sigma$ on A such that $(A, \Sigma_\varepsilon, \mu)$ is a nonatomic measure space and the restriction of U to $L_p^0(A, \Sigma_\varepsilon, \mu)$ has norm $\leqslant \varepsilon$. Since V is HPP-narrow, there is a Σ_ε-measurable sign f supported on A for which $\|Vf\| \leqslant \varepsilon$. Then $\|(U+V)f\| \leqslant \varepsilon\mu(A)^{1/p} + \varepsilon \leqslant 2\varepsilon$.

(b) follows from (a). $\square$

5.4.2. Unconditionally convergent series of HPP-narrow operators. We begin with a factorisation lemma for unconditional sums of HPP-narrow operators.

LEMMA 5.4.9. *Let $1 \leqslant p < \infty$, X be a Banach space, $T_n\colon L_p \to X$ be HPP-narrow operators with $\sum_{n=1}^\infty T_n$ converging pointwise unconditionally to an operator T and let $M = \sup_\pm \|\sum_{n=1}^\infty \pm T_n\|$. Given $0 < \varepsilon < 1/2$, there exist a Banach space Y and a factorisation*

$$\begin{tikzcd} L_p \arrow[rr, "T"] \arrow[dr, "\tilde{T}"'] & & X \\ & Y \arrow[ur, "W"'] & \end{tikzcd}$$

with $\|\tilde{T}\| \leqslant M$, $\|W\| \leqslant 1$, and there are a nonatomic sub-σ-algebra $\Sigma_1 \subset \Sigma$, a Haar-like system (h_α) forming a basis for $L_p^0(\Omega, \Sigma_1, \mu)$, and operators $U, V\colon L_p^0(\Omega, \Sigma_1, \mu) \to Y$ with $U + V = \tilde{T}$ on $L_p^0(\Omega, \Sigma_1, \mu)$ such that U maps (h_α) to a 1-unconditional basic sequence and $\|V\| \leqslant \varepsilon$.

PROOF. Define Y as the space of all sequences $y = (y_n) \in X^{\mathbb{N}}$ such that $\sum_{n=1}^{\infty} y_n$ converges unconditionally in X. Equip Y with the natural norm

$$\|y\| = \sup_{\pm} \Big\| \sum_{n=1}^{\infty} \pm y_n \Big\|.$$

Put $\tilde{T} f = (T_1 f, T_2 f, \dots)$ and $Wy = \sum_{n=1}^{\infty} y_n$. Then Y, $\tilde{T}$, and W satisfy the desired factorisation scheme.

Our main task is now to define for this $\tilde{T}$ a Haar-like system (h_α) and operators U, V as claimed in the lemma. To do this, one uses a standard blocking technique and the stability of HPP-narrow operators under summation (Proposition 5.4.8). Namely, for every $1 \leqslant n < m \leqslant \infty$ define a projection operator $P_{n,m} \colon Y \to Y$ as follows:

$$P_{n,m}(y_1, y_2, \dots) = (0, 0, \dots, 0, y_n, y_{n+1}, \dots, y_{m-1}, 0, 0, \dots).$$

Let (ε_α) be positive numbers. Select an arbitrary sign $h_\emptyset$ supported on Ω and find $n_\emptyset \in \mathbb{N}$ for which

$$\|P_{n_\emptyset, \infty} \tilde{T} h_\emptyset\| \leqslant \varepsilon_\emptyset.$$

Put

$$U h_\emptyset = P_{1, n_\emptyset} \tilde{T} h_\emptyset, \quad V h_\emptyset = P_{n_\emptyset, \infty} \tilde{T} h_\emptyset.$$

The sign $h_\emptyset$ generates a partition of Ω, i.e.,

$$A_{-1} = \{h_\emptyset = -1\}, \quad A_1 = \{h_\emptyset = 1\}.$$

Since the operator $P_{1,n_\emptyset} \tilde{T}$ is PP-narrow by Proposition 5.4.8, there is a sign h_{-1} supported on A_{-1} for which

$$\|P_{1,n_\emptyset} \tilde{T} h_{-1}\| \leqslant \frac{1}{2} \varepsilon_{-1}.$$

Find $n_{-1} > n_\emptyset$ such that

$$\|P_{n_{-1},\infty} \tilde{T} h_{-1}\| \leqslant \frac{1}{2} \varepsilon_{-1}.$$

Put

$$U h_{-1} = P_{n_\emptyset, n_{-1}} \tilde{T} h_{-1}, \quad V h_{-1} = (P_{1,n_\emptyset} + P_{n_{-1},\infty}) \tilde{T} h_{-1}.$$

Continuing in this fashion, we obtain a Haar-like system (h_α) and operators U, V: $\mathrm{lin}\{h_\alpha \colon \alpha \in \mathcal{A}_\infty\} \to Y$ such that $U + V = \tilde{T}$ on $\mathrm{lin}\{h_\alpha \colon \alpha \in \mathcal{A}_\infty\}$, U maps (h_α) to disjoint elements of the sequence space Y and hence to a 1-unconditional basic sequence, and V maps (h_α) to elements whose norms are controlled by the numbers ε_α; therefore $\|V\| \leqslant \varepsilon$ by Remark 5.4.4(b) if $\varepsilon_\alpha \to 0$ sufficiently fast. $\square$

Remark that the above factorisation lemma was one of the key ingredients of the following result [**159**, Theorem 3.3]: there is a Banach space X for which

$$\mathrm{Id} \in \mathrm{unc}_2(K(X,X)) \setminus \mathrm{unc}(K(X,X)).$$

From now on we concentrate on the space L_1 and go in the direction of the announced main result that the sum of a pointwise unconditionally convergent series of HPP-narrow operators on L_1 is again an HPP-narrow operator. The first lemma on this way implies that the operator U from Lemma 5.4.9 factors through c_0.

LEMMA 5.4.10. *Let (h_α) be a Haar-like system in L_1, $U\colon L_1 \to X$ be an operator which maps (h_α) into an unconditional basic sequence. Then, there is a constant C such that for every element of the form $f = \sum_\alpha a_\alpha h_\alpha$ one has*

$$\|Uf\| \leqslant C \sup_\alpha |a_\alpha|. \tag{5.4.1}$$

PROOF. Without loss of generality we can assume that $\|U\| = 1$, $\|h_\emptyset\| = 1$ and that the unconditional constant of (Uh_α) also equals 1 (one can achieve all these goals by an equivalent renorming of X and by multiplication of μ by a constant).

Let us first remark that for every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathcal{A}_n$

$$\|\alpha_1 h_\emptyset + 2\alpha_2 h_{\alpha_1} + 4\alpha_3 h_{\alpha_1,\alpha_2} + \dots + 2^{n-1}\alpha_n h_{\alpha_1,\dots,\alpha_{n-1}}\| \leqslant 2;$$

indeed, it is easy to check by induction over n that this sum equals

$$2^n \chi_{A_{\alpha_1,\dots,\alpha_n}} - \chi_{A_\emptyset}.$$

Hence

$$\|\alpha_1 Uh_\emptyset + 2\alpha_2 Uh_{\alpha_1} + \dots + 2^{n-1}\alpha_n Uh_{\alpha_1,\dots,\alpha_{n-1}}\| \leqslant 2,$$

and, since (Uh_α) is a 1-unconditional basic sequence,

$$\|Uh_\emptyset + 2Uh_{\alpha_1} + \dots + 2^{n-1}Uh_{\alpha_1,\dots,\alpha_{n-1}}\| \leqslant 2.$$

Passing from $n-1$ to n in the last inequality and averaging over $\alpha \in \mathcal{A}_n$ we obtain that

$$2 \geqslant \left\| \frac{1}{2^n} \sum_{\alpha \in \mathcal{A}_n} (Uh_\emptyset + 2Uh_{\alpha_1} + \dots + 2^{n-1}Uh_{\alpha_1,\dots,\alpha_n}) \right\| = \left\| \sum_{k=0}^{n} \sum_{\alpha \in \mathcal{A}_k} Uh_\alpha \right\|.$$

Again by 1-unconditionality of (Uh_α), the last inequality implies that for all $a_\alpha \in [-1, 1]$

$$\left\| \sum_{k=0}^{n} \sum_{\alpha \in \mathcal{A}_k} a_\alpha Uh_\alpha \right\| \leqslant 2,$$

which gives (5.4.1)) with $C = 2$. □

An inspection of the proof of Lemma 5.4.10 shows that

$$\|Uf\| \leqslant 2\|U\|\beta^2 \sup_\alpha |a_\alpha|,$$

where β denotes the unconditional constant of the basic sequence (Uh_α).

In the proof of the next lemma, a reader experienced in probability theory will recognise the "stopping time" technique from martingale theory.

LEMMA 5.4.11. *For every Haar-like system (h_α) in L_1 supported on A and every $\delta > 0$, there is a sign*

$$f = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathcal{A}_k} a_\alpha h_\alpha \tag{5.4.2}$$

supported on A with $\sup_\alpha |a_\alpha| \leqslant \delta$.

PROOF. Fix $m \in \mathbb{N}$ such that $1/m \leqslant \delta$ and define

$$f_k = \sum_{\alpha \in \mathcal{A}_k} a_\alpha h_\alpha$$

as follows: $f_0 = \frac{1}{m} h_\emptyset$, and for every $\alpha \in \mathcal{A}_n$ put $a_\alpha = 1/m$ if $|\sum_{k=0}^{n-1} f_k| < 1$ on $\operatorname{supp} h_\alpha$ and $a_\alpha = 0$ if $|\sum_{k=0}^{n-1} f_k| = 1$ on $\operatorname{supp} h_\alpha$. Under this construction all the partial sums of the series $\sum_{k=0}^{\infty} f_k$ are bounded by 1 in modulus. Since $(f_k)_{k=0}^{\infty}$ is an orthogonal system, the series $\sum_{k=0}^{\infty} f_k$ converges in L_2 (and hence in L_1) to a function f supported on A that can be represented as in (5.4.2) with $\sup_\alpha |a_\alpha| \leqslant \delta$. We shall prove that f is a sign.

Obviously $\int_A f\,d\mu = 0$. Consider $B = \{t \in A\colon |f(t)| \neq 1\}$. By our construction we have for each $n \in \mathbb{N}$

$$B \subset \{t \in A\colon f_n(t) \neq 0\} = \Big\{t \in A\colon |f_n(t)| = \frac{1}{m}\Big\},$$

so $\mu(B) \leqslant m\|f_n\|$, and since $\|f_n\| \to 0$, we conclude that $\mu(B) = 0$. Therefore f is a sign. □

We are now ready for the main result of this section.

THEOREM 5.4.12. *Let $T_n\colon L_1 \to X$ be HPP-narrow operators, and suppose that $\sum_{n=1}^{\infty} T_n$ converges pointwise unconditionally to some operator T. Then T is HPP-narrow.*

PROOF. Let $A \in \Sigma^+$, and let $\tilde{\Sigma}$ be a nonatomic sub-σ-algebra of $\Sigma|_A$. We have to show that for every $\varepsilon > 0$ there is a sign $f \in L_1(A, \tilde{\Sigma}, \mu)$ supported on A with $\|Tf\| \leqslant \varepsilon$.

Applying Lemma 5.4.9 to the restrictions of T_n and T to $L_1(A, \tilde{\Sigma}, \mu)$ we get a Haar-like system (h_α) forming a basis for some $L_1^0(A, \Sigma_1, \mu)$ and we obtain operators $U, V\colon L_1^0(A, \Sigma_1, \mu) \to Y$, $W\colon Y \to X$ such that $\|W\| \leqslant 1$, $T = W(U + V)$ on $L_1^0(A, \Sigma_1, \mu)$, $\|V\| \leqslant \varepsilon/2$ and U maps (h_α) to a 1-unconditional basic sequence. Let C be the constant from (5.4.1). Taking a sign

$$f = \sum_{k=0}^{\infty} \sum_{\alpha \in \mathcal{A}_k} a_\alpha h_\alpha$$

supported on A with $\sup_\alpha |a_\alpha| \leqslant \varepsilon/(2C)$ (Lemma 5.4.11) we obtain from (5.4.1) that $\|Uf\| \leqslant \varepsilon/2$. Therefore $\|Tf\| \leqslant \|Uf\| + \|Vf\| \leqslant \varepsilon$. □

COROLLARY 5.4.13. *For any Banach space X, no embedding operator is contained in* $\mathrm{unc}_n(K(X,X))$ *for any $n \in \mathbb{N}$.*

PROOF. This follows from the fact that compact operators are HPP-narrow. □

The next corollary is due to Rosenthal [**260**].

COROLLARY 5.4.14. *Every operator T from L_1 into a Banach space X with an unconditional basis is HPP-narrow; in particular it is PP-narrow.*

PROOF. If e_n, e_n^*, $n = 1, 2, \ldots$, are the elements of an unconditional basis of X and the corresponding coordinate functionals, then $T = \sum_{n=1}^{\infty} (e_n^* \otimes e_n) T$ is a pointwise unconditionally convergent series of rank one operators. □

Rosenthal deduces from this corollary that L_1 does not even sign-embed into a space with an unconditional basis.

5.5. Notes and remarks

Section 5.1. The content of this section comes mainly from [**58**]. The separable determination theorem 5.1.12 is due to Ivashyna [**147**, Theorem 1].

In [**268**, Section 3], a strong relation between L-orthogonal elements and Daugavet centres for small density characters was exhibited, as a consequence of the techniques used in Lemma 4.3.3 and Theorem 4.3.4. The result is the following:

THEOREM 5.5.1 ([**268**, Theorems 3.5 and 3.6]). *Let X and Y be two Banach spaces. Let $G\colon X \to Y$ be a Daugavet centre and assume that $G(X)$ is separable. If* $\operatorname{dens}(Y) \leqslant \omega_1$ *then, for every $u \in B_{X^{**}}$ and every sequence $(g_n)_{n\in\mathbb{N}} \subset S_{X^*}$, there exists $v \in B_{X^{**}}$ so that*

(1) *$u(g_n) = v(g_n)$ holds for every $n \in \mathbb{N}$;*
(2) *$\|G^{**}(v) + y\| = 1 + \|y\|$ holds for every $y \in Y$; in other words, $G^{**}(v)$ is a non-zero L-orthogonal element in Y^{**}.*

In particular, the set of those $v \in B_{X^{**}}$ satisfying (2) is weak-star dense in $B_{X^{**}}$.

The existence of Daugavet spaces without L-orthogonal elements (Example 4.3.7) reveals that we cannot remove the assumption on the density character of Y.

Section 5.2. The main renorming theorem is taken from [**58**, Theorem 1.3]. The idea of the construction appeared first in [**169**].

The direct predecessor of [**58**, Theorem 1.3] was the following result that was later extended to non-separable spaces by Shvydkoy [**285**]; see Definition 5.1.14 for the notion of a Daugavet pair.

THEOREM 5.5.2 ([**178**, Theorem 2.5]). *Let $X \subset Y$ be separable Banach spaces,* $X \in \mathrm{DPr}$. *Then there is an equivalent norm p on Y which extends the original norm of X in such a way that $(X, (Y, p))$ is a Daugavet pair.*

Comparing with [**58**, Theorem 1.3], the original demonstration of Theorem 5.5.2 was based on a completely different technique originating in [**174**] and developed further in [**170**]. Namely, for a Daugavet pair (X, Y) denote by $J\colon X \to Y$ the natural embedding. Introduce the compact topological space K formed by the w^*-closure in Y^* of the set $\operatorname{ext}(B_{Y^*})$ and equipped with the weak-star topology. Then Y embeds into $C(K)$ by means of the operator $J_0\colon Y \to C(K)$ that acts by the formula $(J_0 y)(y^*) = y^*(y)$.

Denote by $\ell_\infty(K)$ the space of all bounded scalar functions on K, equipped with the sup-norm, and let $m(K)$ be the closed subspace of $\ell_\infty(K)$, consisting of those $f \in \ell_\infty(K)$ for which the support $\{t\colon f(t) \neq 0\}$ is a meagre subset of K. For the quotient space

$$m_0(K) := \ell_\infty(K)/m(K),$$

the fact that compact spaces are Baire spaces implies for the quotient map $Q\colon \ell_\infty(K) \to m_0(K)$ that its restriction $Q_0 = Q|_{C(K)}\colon C(K) \to m_0(K)$ is an isometric embedding.

Consequently, X embeds into $m_0(K)$ with the corresponding embedding operator being $\tilde{J} = Q_0 J_0 J$. Then, in the above notation, one can demonstrate that $(X, m_0(K))$ is a Daugavet pair.

Further, one can construct an isomorphic embedding $U\colon Y \to m_0(K)$ in such a way that $U|_X = \tilde{J}$. Finally, the requested new norm p on Y may be defined as $p(y) := \|U(y)\|$.

Section 5.3. The main Theorem 5.3.1 of the section is essentially contained in [**173**], although formally it was demonstrated there only for the identity operator and without the estimation of $\|\mathrm{Id} - V\|$; its extension to Daugavet pairs [**178**] and Daugavet centres [**58**] does not require much effort. Theorem 5.3.3 is taken from [**178**]. Remark that it was new even for the classical spaces $C[0,1]$ and $L_1[0,1]$.

Section 5.4. We have already mentioned that this section follows [**159**] and that what we call PP-narrow operators was first introduced by Plichko and Popov in [**252**] under the name narrow operators. Rosenthal's Corollary 5.4.14 is from his unpublished, though occasionally cited paper [**260**]; actually, not only is this paper unpublished, as a matter of fact it has never been written up, as Rosenthal pointed out to the authors of [**158**]. A recent important paper along these lines is [**228**].

Previously, operators on L_1 that are not PP-narrow were studied by Ghoussoub and Rosenthal [**118**], [**262**] and [**263**], and Popov and Randrianantoanina have published a monograph on the topic of PP-narrow operators [**254**].

5.6. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

(5.1) Is it true that if $G\colon X \to Y$ is a Daugavet centre, $\|G\| = 1$, $\mathcal{M} \subset L(X,Y)$ is a linear space of operators such that the Daugavet equation $\|G - T\| = 1 + \|T\|$ holds true for every $T \in \mathcal{M}$, then the Daugavet equation remains valid for every operator that can be represented as the sum of a pointwise unconditionally convergent series of elements of the space $\mathcal{M}$?

Thanks to Remark 5.3.9, a positive solution may be possible if, in the above context, one can try for every $V \in L(X,Y)$ that can be is represented as a sum of pointwise unconditionally convergent series of operators from $\mathcal{M}$ and every $\varepsilon > 0$ to search for another representation that satisfies (5.3.5). We do not know whether this is possible, for example, for $\mathcal{M}$ being the set of all finite-rank operators.

(5.2) Let E, Y be Banach spaces, $E \in \mathrm{DPr}$. Is it true that none of the classes $\mathrm{unc}_n(K(E,Y))$, $n \in \mathbb{N}$, may contain a bounded below operator?

(5.3) Can one describe $\mathrm{unc}(K(L_1,X))$ for general X? What about $X = L_1$?

(5.4) Describe the smallest class of operators $\mathcal{M} \subset L(L_1,X)$ that contains the compact operators and is stable under pointwise unconditional sums. In particular, is $\mathrm{unc}(K(L_1,L_1)) = \mathrm{unc}_2(K(L_1,L_1))$?

Note that X does not embed into a space with an unconditional basis if $\mathcal{M} \neq L(L_1,X)$.

(5.5) Can one develop a similar theory for operators on the James space or other spaces that do not embed into spaces with unconditional bases?

(5.6) Is there a space $X \in \mathrm{DPr}$ such that $\mathrm{Id}_X \in \mathrm{unc}_n(K(X,X))$ for some $n \in \mathbb{N}$?

Question (5.6) makes sense for "infinite" n: one can define $\mathrm{unc}_n(K(X,X))$ for arbitrary ordinals using transfinite induction. In this generalised form, the question reformulates as follows:

(5.7) Is it true that for every Banach space $X \in \mathrm{DPr}$ there is a linear space of operators $\mathcal{M} \subset L(X,X)$ that contains the compact operators, is stable under pointwise unconditional sums and such that $\mathrm{Id}_X \notin \mathcal{M}$?

CHAPTER 6

Narrow operators and rich subspaces

This section is devoted to the study of a class of operators that are very well suited to investigate the Daugavet property and the Daugavet equation; these are the narrow operators on a space with the Daugavet property. (We have met a variant of this idea, viz. the PP-narrow operators, in Definition 5.4.2.) They form a large class of operators satisfying the Daugavet equation and arguably the largest sensible one. Dual to this notion is the one of a rich subspace; it turns out that rich subspaces inherit the Daugavet property.

The definition of narrow operators is somewhat technical, and we take a little detour and introduce the semigroup $\mathcal{OP}(X)$ first. Section 6.1 will study this semigroup in algebraic terms.

6.1. The semigroup $\mathcal{OP}(X)$

In this section we introduce a formalism that helps to deal with those properties of an operator which depend only on the norms of the images of elements. We define corresponding equivalence classes and their formal sums and differences, which is reminiscent of certain procedures in the theory of operator ideals.

Definition 6.1.1. We say that two operators T_1 and T_2 are *equivalent* (in symbols $T_1 \sim T_2$) if $\|T_1x\| = \|T_2x\|$ for every $x \in X$. A class $\mathcal{M}$ of operators is said to be *admissible* if for every $T \in \mathcal{M}$ all the members of the equivalence class of T also belong to $\mathcal{M}$.

In other words, the operators T_1 and T_2 are equivalent if there is an isometry $U: T_1(X) \to T_2(X)$ such that $T_2 = UT_1$. For example, the classes of finite-rank operators, compact operators, weakly compact operators, operators bounded from below are admissible; surjections, isomorphisms, projections are examples of non-admissible operator classes.

Definition 6.1.2. We say that $T_1 \leqslant T_2$ if $\|T_1x\| \leqslant \|T_2x\|$ for every $x \in X$. A class $\mathcal{M}$ of operators forms an *order ideal* if for every $T \in \mathcal{M}$ every operator $T_1 \leqslant T$ also belongs to $\mathcal{M}$.

In other words, $T_1 \leqslant T_2$ if there is a bounded operator $U: T_2(X) \to T_1(X)$ of norm $\leqslant 1$ such that $T_1 = UT_2$. Order ideals are clearly admissible. The classes of finite-rank operators, compact operators, weakly compact operators are order ideals.

Definition 6.1.3. A sequence (T_n) of operators is said to be $\sim$*convergent* to an operator T if $\|T_nx\| \to \|Tx\|$ uniformly on B_X. In terms of $\sim$convergence we define the notions of a $\sim$closed set of operators, $\sim$closure, etc. in a natural way.

Of course, the $\sim$limit of a sequence is not unique, but it is unique up to equivalence of operators.

For example, the class $\mathrm{FinRan}(X)$ of finite-rank operators on an infinite-dimensional space X is not $\sim$closed: its $\sim$closure contains all compact operators. Indeed, let $T\colon X \to Y$ be compact. Then, for the canonical isometry U from Y into $C(B_{Y^*})$, $T_1 := UT$ is compact, too, and by definition $T_1 \sim T$. Since $C(B_{Y^*})$ has the approximation property, T_1 can be approximated by finite-rank operators in the above sense (up to equivalence of operators).

In fact, the $\sim$closure of $\mathrm{FinRan}(X)$ coincides with the class $K(X)$ of all compact operators since $K(X)$ is $\sim$closed. To see this suppose that (T_n) is a $\sim$convergent sequence of compact operators on X with limit T. Let (x_n) be a bounded sequence in X; using a diagonal procedure one can find a subsequence (x'_n) such that $(T_k x'_n)_n$ is convergent for each k. But $\|T_k x\| \to \|Tx\|$ uniformly on bounded sets as $k \to \infty$; hence (Tx'_n) is a Cauchy sequence and thus convergent.

Definition 6.1.4. Let $\mathcal{N}$ be a collection of subsets in X. We define a class of operators $\mathcal{N}^\sim$ as follows: $T \in \mathcal{N}^\sim$ if for every $A \in \mathcal{N}$, T is unbounded from below on A; i.e.,

$$\forall \varepsilon > 0 \ \exists x \in A\colon \ \ \|Tx\| \leqslant \varepsilon.$$

Remark that if $\emptyset \in \mathcal{N}$ then $\mathcal{N}^\sim = \emptyset$. In the case of $\emptyset \notin \mathcal{N}$ we have $0 \in \mathcal{N}^\sim$, so $\mathcal{N}^\sim \neq \emptyset$.

Evidently, $\mathcal{N}^\sim$ is a $\sim$closed order ideal, and it is *homogeneous* in the sense that $\lambda T \in \mathcal{N}^\sim$ whenever $\lambda \in \mathbb{K}$ and $T \in \mathcal{N}^\sim$. For example, if $\mathcal{N} = \{S_X\}$, then $\mathcal{N}^\sim = \mathcal{UB}_X$, the class of operators that are *unbounded from below* which is defined by

$$T \in \mathcal{UB}_X \iff \inf\{\|Tx\|\colon \|x\| = 1\} = 0.$$

A significant example for us is the class of all C-narrow operators on the space $C(K)$ that was introduced in [**175**]. We are going to speak a lot about this class in Chapter 8. Here is the definition.

Definition 6.1.5. An operators $T\colon C(K) \to Y$ is said to be *C-narrow* if for every proper closed subset F of K, T is unbounded from below on the unit sphere of the subspace $J_F := \{f \in C(K)\colon f|_F = 0\}$.

Taking $\mathcal{N}$ to be the collection of these unit spheres S_{J_F}, we see that the class of C-narrow operators is just $\mathcal{N}^\sim$.

Another important example is the class of all *PP-narrow operators* on the space $L_1 = L_1(\Omega, \Sigma, \mu)$, see Definition 5.4.2. For every $A \in \Sigma^+$ denote by Sign_A the set of all signs on A, and by SIGN the collection of these sets Sign_A, $A \in \Sigma^+$. Then the class of PP-narrow operators on L_1 is again $\mathrm{SIGN}^\sim$.

We now define $\mathcal{OP}(X)$ as the class of all operators on X with the convention that equivalent operators will be identified. Hence $\mathcal{OP}(X)$ is actually a collection of equivalence classes, and in fact it is a set. Namely, for an operator T on X its equivalent class can be identified with the seminorm $x \mapsto \|Tx\|$, and the collection of seminorms on X is clearly a set. Thus, admissible families of operators can be identified with subsets of $\mathcal{OP}(X)$, and it makes sense to write $T \in \mathcal{OP}(X)$ or $\mathcal{M} \subset \mathcal{OP}(X)$.

We now introduce addition and subtraction on $\mathcal{OP}(X)$. If $T_1\colon X \to Y_1$ and $T_2\colon X \to Y_2$ are two operators, define

$$T_1 \mathbin{\tilde{+}} T_2\colon X \to Y_1 \oplus_1 Y_2, \quad x \mapsto (T_1 x, T_2 x);$$

i.e.,

$$\|(T_1 \tilde{+} T_2)x\| = \|T_1x\| + \|T_2x\|.$$

Definition 6.1.6. If $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{OP}(X)$ are non-empty, then their $\sim$*sum* is defined by $\mathcal{M}_1 \tilde{+} \mathcal{M}_2 = \{T_1 \tilde{+} T_2 \colon T_1 \in \mathcal{M}_1,\ T_2 \in \mathcal{M}_2\}$. Their $\sim$*difference* is defined by $\mathcal{M}_2 \tilde{-} \mathcal{M}_1 = \{T \in \mathcal{OP}(X) \colon T \tilde{+} T_1 \in \mathcal{M}_2 \text{ whenever } T_1 \in \mathcal{M}_1\}$.

The operation $\tilde{+}$ is a commutative and associative operation on $\mathcal{OP}(X)$, and we have $0 \in \mathcal{M}_2 \tilde{-} \mathcal{M}_1$ if and only if $\mathcal{M}_1 \subset \mathcal{M}_2$.

Let us give some examples.

Example 6.1.7. Let K be a compact Hausdorff space and let $\mathcal{MUB}(C(K))$ denote the class of operators equivalent to some multiplication operator $M_h \colon f \mapsto hf$ on $C(K)$ which is unbounded from below; i.e., where h has a zero. Then $\mathcal{UB}(C(K)) \tilde{-} \mathcal{MUB}(C(K))$ consists exactly of the C-narrow operators described above.

Proof. Let $T \colon C(K) \to Y$ be C-narrow. If h has a zero, we have to show that, given $\varepsilon > 0$, there is some $f \in S_{C(K)}$ such that both $\|Tf\| \leqslant \varepsilon$ and $\|hf\|_\infty \leqslant \varepsilon$. Now, if $F = \{|h| \geqslant \varepsilon\}$, which is a proper subset of K, and $f \in S(J_F)$ such that $\|Tf\| \leqslant \varepsilon$, then $\|hf\|_\infty \leqslant \varepsilon$ as well.

Conversely, if a closed proper subset $F \subset K$ is given, pick some $h \in S_{C(K)}$ such that $h = 1$ on F, $h = 0$ off a neighbourhood V of F. Use that $T + M_h$ is unbounded below to find $f \in S_{C(K)}$ with $\|Tf\| + \|hf\|_\infty \leqslant \varepsilon$. Since $\|f\|_\infty \leqslant 1$, $\|Tf\| \leqslant \varepsilon$ and $\|hf\|_\infty \leqslant \varepsilon$, then in particular $|f| \leqslant \varepsilon$ on F. Hence it is possible to replace f by a function $g \in S(J_F)$ such that $\|Tg\| \leqslant 2\varepsilon$, which proves that T is C-narrow. □

More examples can be found in [**179**, Example 2.7]: the class $\mathcal{UB}(X) \tilde{-} \operatorname{FinRan}(X)$ consists of all operators that are not left semi-Fredholm operators; and the class $\mathcal{UB}(X) \tilde{-} (\mathcal{UB}(X) \tilde{-} \operatorname{FinRan}(X))$ consists of all strictly singular operators.

Let us list some elementary properties of the operation $\tilde{-}$ that follow directly from the definition.

Proposition 6.1.8. *Suppose that $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{OP}(X)$ contain the zero operator.*

(a) *$\mathcal{M}_2 \tilde{-} \mathcal{M}_1$ is an order ideal, respectively, $\sim$closed, whenever $\mathcal{M}_2$ is.*
(b) *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are order ideals, then $\mathcal{M}_2 \tilde{-} \mathcal{M}_1$ is homogeneous whenever $\mathcal{M}_2$ is.*

Of particular relevance are subsets of $\mathcal{OP}(X)$ that are semigroups with respect to the operation $\tilde{+}$.

Proposition 6.1.9. *Suppose that $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{OP}(X)$ contain the zero operator.*

(a) *$\mathcal{M}_1$ is a subsemigroup of $\mathcal{OP}(X)$ if and only if $\mathcal{M}_1 \tilde{-} \mathcal{M}_1 \supset \mathcal{M}_1$, in which case $\mathcal{M}_1 \tilde{-} \mathcal{M}_1 = \mathcal{M}_1$.*
(b) *Let $\mathcal{M}_1$ be a subsemigroup of $\mathcal{OP}(X)$, and let $\mathcal{M}_1 \subset \mathcal{M}_2$. Then $\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{M}_1)$ is again a subsemigroup.*
(c) *$\mathcal{M}_2 \tilde{-} \mathcal{M}_2$ is always a subsemigroup of $\mathcal{OP}(X)$.*

Proof. (a) is clear from the definition.

For (b) we note first that

$$\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{M}_1)) = \mathcal{M}_2 \tilde{-} \mathcal{M}_1. \tag{6.1.1}$$

Indeed, by definition of $\tilde{-}$ we have

$$\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{M}_1) \supset \mathcal{M}_1, \tag{6.1.2}$$

whence

$$\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{M}_1)) \subset \mathcal{M}_2 \tilde{-} \mathcal{M}_1.$$

On the other hand, an application of (6.1.2) with $\mathcal{M}_1$ replaced with $\mathcal{M}_2 \tilde{-} \mathcal{M}_1$ gives "$\supset$" in (6.1.1). Now, by elementary arithmetic involving $\tilde{+}$ and $\tilde{-}$ we have, writing $\mathcal{D} = \mathcal{M}_2 \tilde{-} \mathcal{M}_1$ for short,

$$\begin{aligned}
(\mathcal{M}_2 \tilde{-} \mathcal{D}) \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{D}) &= \mathcal{M}_2 \tilde{-} (\mathcal{D} \tilde{+} (\mathcal{M}_2 \tilde{-} \mathcal{D})) \\
&= \mathcal{M}_2 \tilde{-} ((\mathcal{M}_2 \tilde{-} \mathcal{D}) \tilde{+} \mathcal{D}) \\
&= (\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{D})) \tilde{-} \mathcal{D} = \mathcal{D} \tilde{-} \mathcal{D} \qquad \text{(by (6.1.1))} \\
&= (\mathcal{M}_2 \tilde{-} \mathcal{M}_1) \tilde{-} \mathcal{D} = \mathcal{M}_2 \tilde{-} (\mathcal{M}_1 \tilde{+} \mathcal{D}).
\end{aligned}$$

Because $\mathcal{M}_1$ is a semigroup, one can easily deduce that $\mathcal{M}_1 \tilde{+} \mathcal{D} \subset \mathcal{D}$; indeed,

$$\begin{aligned}
\mathcal{M}_1 \tilde{+} \mathcal{D} &= (\mathcal{M}_2 \tilde{-} \mathcal{M}_1) \tilde{+} \mathcal{M}_1 \\
&= (\mathcal{M}_2 \tilde{-} (\mathcal{M}_1 \tilde{+} \mathcal{M}_1)) \tilde{+} \mathcal{M}_1 \\
&= ((\mathcal{M}_2 \tilde{-} \mathcal{M}_1) \tilde{-} \mathcal{M}_1) \tilde{+} \mathcal{M}_1 \subset \mathcal{M}_2 \tilde{-} \mathcal{M}_1.
\end{aligned}$$

Therefore

$$(\mathcal{M}_2 \tilde{-} \mathcal{D}) \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{D}) \supset \mathcal{M}_2 \tilde{-} \mathcal{D},$$

completing the proof that $\mathcal{M}_2 \tilde{-} (\mathcal{M}_2 \tilde{-} \mathcal{M}_1)$ is a semigroup.

Finally, (c) is the special case $\mathcal{M}_1 = \{0\}$ of (b). □

Some operator classes that are important for us are not stable with respect to addition (neither the usual one, nor the $\tilde{+}$ addition). This motivates us to introduce the following concept.

DEFINITION 6.1.10. Let $\mathcal{M} \subset \mathcal{OP}(X)$, and let $\mathcal{M}_1 \subset \mathcal{M}$ be a subsemigroup of $\mathcal{OP}(X)$. $\mathcal{M}_1$ is called a *maximal subsemigroup* of $\mathcal{M}$ if every subsemigroup $\mathcal{M}_2 \subset \mathcal{M}$ which includes $\mathcal{M}_1$ coincides with $\mathcal{M}_1$. We call the intersection of all maximal subsemigroups of $\mathcal{M}$ the *central part* of $\mathcal{M}$ and denote it by $\operatorname{cp}(\mathcal{M})$.

Here is a characterisation of the central part of $\mathcal{M}$.

THEOREM 6.1.11. *Let $\mathcal{M} \subset \mathcal{OP}(X)$ have the following properties: $0 \in \mathcal{M}$ and every element of $\mathcal{M}$ is contained in a subsemigroup of $\mathcal{M}$ (this happens for example if $\mathcal{M}$ is homogeneous). Then* $\operatorname{cp}(\mathcal{M}) = \mathcal{M} \tilde{-} \mathcal{M}$.

PROOF. Let $\mathcal{M}_1$ be a maximal subsemigroup of $\mathcal{M}$. Put $\mathcal{M}_2 = \mathcal{M} \tilde{-} \mathcal{M}$. We have proved above in Proposition 6.1.9(c) that $\mathcal{M}_2$ is a subsemigroup, so $\mathcal{M}_2 \tilde{+} \mathcal{M}_1$ is a subsemigroup, too. By definition of $\mathcal{M}_2$ we have $\mathcal{M}_2 \tilde{+} \mathcal{M}_1 \subset \mathcal{M}$. So the maximality of $\mathcal{M}_1$ implies that $\mathcal{M}_1 \supset \mathcal{M}_2$. This proves the inclusion $\operatorname{cp}(\mathcal{M}) \supset \mathcal{M} \tilde{-} \mathcal{M}$.

Let us now prove the inverse inclusion. Let $T \in \operatorname{cp}(\mathcal{M}) \setminus (\mathcal{M} \tilde{-} \mathcal{M})$. Then there is some $T_1 \in \mathcal{M}$ such that $T_1 \tilde{+} T$ does not belong to $\mathcal{M}$. Consider the maximal subsemigroup $\mathcal{M}_3$ of $\mathcal{M}$ which contains T_1. Then $\mathcal{M}_3$ cannot contain T, so $\operatorname{cp}(\mathcal{M})$ cannot contain T either. □

For every operator T and $\varepsilon > 0$ we define the *tube*

$$U_{T,\varepsilon} = \{x \in X \colon \|Tx\| < \varepsilon\}.$$

Let $\mathcal{M} \subset \mathcal{OP}(X)$. Put

$$\mathcal{M}_\sim = \{U_{T,\varepsilon} \cap S_X \colon T \in \mathcal{M},\ \varepsilon > 0\}$$

Then $(\mathcal{M}_\sim)^\sim = \mathcal{UB}(X) \mathrel{\tilde{-}} \mathcal{M}$.

Proposition 6.1.12. *Let $\mathcal{M} \subset \mathcal{OP}(X)$ and let $\mathcal{N}$ be a collection of subsets in X. Then $\mathcal{N}^\sim \mathrel{\tilde{-}} \mathcal{M} = {\mathcal{N}_1}^\sim$, where $\mathcal{N}_1$ consists of all intersections of the form $U_{T,\varepsilon} \cap A$, $T \in \mathcal{M}$, $A \in \mathcal{N}$, $\varepsilon > 0$. In particular, if $\mathcal{N}^\sim \mathrel{\tilde{-}} \mathcal{M}$ is non-empty, then all the intersections $U_{T,\varepsilon} \cap A$ are non-empty and $\mathcal{N}^\sim \supset \mathcal{M}$.*

Proof. Let $T_1 \in \mathcal{N}^\sim \mathrel{\tilde{-}} \mathcal{M}$. Then for every $T \in \mathcal{M}$ we have $T_1 \mathrel{\tilde{+}} T \in \mathcal{N}^\sim$. This means that for every $A \in \mathcal{N}$ and $\varepsilon > 0$ there is an element $x \in A$ such that $\|(T_1 \mathrel{\tilde{+}} T)x\| < \varepsilon$. This in turn implies that $x \in A \cap U_{T,\varepsilon}$ and $\|T_1 x\| < \varepsilon$. So $T_1 \in {\mathcal{N}_1}^\sim$.

Now, let $T_1 \in {\mathcal{N}_1}^\sim$. Then for every $T \in \mathcal{M}$, every $A \in \mathcal{N}$ and $\varepsilon > 0$ there is an element $x \in A \cap U_{T,\varepsilon/2}$ such that $\|T_1 x\| < \varepsilon/2$. But by the definition of tubes, $\|Tx\| < \varepsilon/2$. So $\|(T_1 \mathrel{\tilde{+}} T)x\| < \varepsilon$ and $T_1 \in \mathcal{N}^\sim \mathrel{\tilde{-}} \mathcal{M}$. □

6.2. Strong Daugavet and narrow operators

In this section we define the class of narrow operators on a Banach space with the Daugavet property, and, more generally of G-narrow operators, which are narrow with respect to a Daugavet centre G. But first we need to introduce a closely related class of operators.

Definition 6.2.1. Let $G \in S_{L(X,Y)}$. An operator $T \in \mathcal{OP}(X)$ is said to be a *G-strong Daugavet operator* if for every $x \in S_X$, $y \in S_Y$ and $\varepsilon > 0$ there is an element $z \in (x + U_{T,\varepsilon}) \cap B_X$ such that $\|Gz + y\| > 2 - \varepsilon$. We denote the class of all G-strong Daugavet operators by $\mathcal{SD}_G(X)$.

In particular, T is said to be a *strong Daugavet operator* if $T \in \mathcal{SD}_{\mathrm{Id}_X}(X)$, that is, for every two elements $x, y \in S_X$ and for every $\varepsilon > 0$ there is $z \in (x + U_{T,\varepsilon}) \cap B_X$ such that $\|z + y\| > 2 - \varepsilon$. We denote for short $\mathcal{SD}(X) := \mathcal{SD}_{\mathrm{Id}_X}(X)$.

Analogously to Remark 3.1.12, the condition on $\|z\|$ in the above definition can be modified a little without affecting the result, which makes it more flexible.

Remark 6.2.2. *Let $G \in S_{L(X,Y)}$, $T \in \mathcal{OP}(X)$. Then the following assertions are equivalent:*

(i) *$T \in \mathcal{SD}_G(X)$.*
(ii) *For every $x \in S_X$, $y \in S_Y$ and $\varepsilon > 0$ there is $z \in (x + U_{T,\varepsilon}) \cap (1 + \varepsilon)B_X$ such that $\|Gz + y\| > 2 - \varepsilon$.*
(iii) *For every $x \in S_X$, $y \in S_Y$ and $\varepsilon > 0$ there is $z \in (x + U_{T,\varepsilon}) \cap S_X$ such that $\|Gz + y\| > 2 - \varepsilon$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are evident, so it remains to demonstrate that (ii) $\Rightarrow$ (iii). Fix $\varepsilon > 0$ and take $\delta \in (0, \varepsilon/2)$ so small that $\delta\|T\| + \delta < \varepsilon$ and apply (ii) with δ instead of ε. We obtain $z_0 \in X$ such that $\|Tz_0 - Tx\| < \delta$, $\|z_0\| \leqslant 1 + \delta$ and $\|Gz_0 + y\| > 2 - \delta$. The last inequality implies that

$$\|z_0\| \geqslant \|Gz_0\| \geqslant \|Gz_0 + y\| - 1 > 1 - \delta.$$

Denoting $z = \frac{z_0}{\|z_0\|}$ we have $z \in S_X$ and $\|z - z_0\| \leqslant \delta$. Consequently, $\|Tz - Tx\| \leqslant |Tz - Tz_0\| + \|Tz_0 - Tx\| < \delta\|T\| + \delta < \varepsilon$, so $z \in (x + U_{T,\varepsilon}) \cap S_X$. Also, $\|Gz + y\| \geqslant \|Gz_0 + y\| - \delta > 2 - 2\delta > 2 - \varepsilon$, as desired. □

It follows from Lemma 5.1.5 for a Daugavet centre $G \in S_{L(X,Y)}$ that every finite-rank operator on X is G-strongly Daugavet (indeed, in this case $U_{T,\varepsilon}$ is weakly open), and conversely, if every rank-one operator on X is G-strongly Daugavet, then G is a Daugavet centre.

There is an obvious connection between strong Daugavet operators and the Daugavet equation.

LEMMA 6.2.3. *Let $G \in S_{L(X,Y)}$. If $T\colon X \to Y$ is a G-strong Daugavet operator, then $\|G + T\| = 1 + \|T\|$.*

PROOF. We assume without loss of generality that $\|T\| = 1$. Given $\varepsilon > 0$ pick $x \in S_X$ such that $\|Tx\| \geqslant 1 - \varepsilon$. If $y = Tx/\|Tx\|$ and z is chosen according to Definition 6.2.1, then

$$2 - \varepsilon < \|Gz + y\| \leqslant \|Gz + Tx\| + \varepsilon \leqslant \|Gz + Tz\| + 2\varepsilon,$$

hence

$$\|G + T\| \geqslant \|Gz + Tz\| > 2 - 3\varepsilon,$$

which proves the lemma. □

We now relate the G-strong Daugavet property to a collection of subsets of X.

DEFINITION 6.2.4. Let $G \in S_{L(X,Y)}$. For every $x \in S_X$, every $y \in S_Y$, and every $\varepsilon > 0$, let us define the set

$$D_G(x, y, \varepsilon) := \{z \in X\colon \|Gz + Gx + y\| > 2 - \varepsilon \ \&\ \|z + x\| < 1 + \varepsilon\}.$$

By $\mathcal{D}_G(X)$ we denote the collection of all sets $D_G(x, y, \varepsilon)$, where $x \in S_X$, $y \in S_Y$ and $\varepsilon > 0$. In the case of $G = \mathrm{Id}_X$ we simplify $D(x, y, \varepsilon) := D_{\mathrm{Id}_X}(x, y, \varepsilon)$ and $\mathcal{D}(X) := \mathcal{D}_{\mathrm{Id}_X}(X)$.

PROPOSITION 6.2.5. *$\mathcal{SD}_G(X) = \mathcal{D}_G(X)^{\sim}$.*

PROOF. $T \in \mathcal{D}_G(X)^{\sim}$ if and only if for every $x \in S_X$, $y \in S_Y$ and $\varepsilon > 0$ there is an element $z \in D_G(x, y, \varepsilon)$ such that $\|Tz\| < \varepsilon$. This in turn is equivalent to the following condition: for every $x \in S_X$, $y \in S_Y$ and $\varepsilon > 0$ there is an element v such that $\|v\| < 1 + \varepsilon$, $\|Gv + y\| > 2 - \varepsilon$ and v belongs to the tube $x + U_{T,\varepsilon}$ (just put $v = z + x$). Then T is a G-strong Daugavet operator by Remark 6.2.2. □

For general G it may happen that some of $D_G(x, y, \varepsilon)$ is empty and $\mathcal{SD}_G(X) = \emptyset$. In the case of $G = \mathrm{Id}_X$ this, fortunately, is impossible.

REMARK 6.2.6. *If G maps the open unit ball of X surjectively onto the open unit ball of Y (i.e., G is a quotient map), then all $D_G(x, y, \varepsilon)$ are nonempty.* Indeed, in this case there is $v \in X$ with $\|v\| < 1 + \varepsilon$ such that $Gv = y$. Then $z := v - x \in D_G(x, y, \varepsilon)$. Consequently, in this case $\mathcal{SD}_G(X) \neq \emptyset$: at least, the zero operator belongs to $\mathcal{SD}_G(X)$. In particular, for every Banach space X all $D(x, y, \varepsilon) \neq \emptyset$ and $0 \in \mathcal{SD}(X)$.

Actually, a somewhat smaller class of operators turns out to be crucial.

DEFINITION 6.2.7. Let $G \in S_{L(X,Y)}$. Define the class of *G-narrow operators* by $\mathcal{NAR}_G(X) = \mathcal{SD}_G(X) \tilde{-} X^*$. An Id_X-narrow operator is called *narrow*, and the corresponding class of narrow operators on X is $\mathcal{NAR}(X) := \mathcal{NAR}(\mathrm{Id}_X, X) = \mathcal{SD}(X) \tilde{-} X^*$.

In other words, an operator T is said to be G-narrow if, for every $x^* \in X^*$, $T \tilde{+} x^*$ is a G-strong Daugavet operator; and T is said to be narrow if, for every $x^* \in X^*$, $T \tilde{+} x^*$ is a strong Daugavet operator.

Decoding the definition of G-strong Daugavet operator we see that T is G-narrow if for every $\varepsilon > 0$, every $x \in S_X$, every $y \in S_Y$ and every $x^* \in X^*$ there exists $z \in B_X$ satisfying the inequalities $\|T(x - z)\| + |x^*(x - z)| < \varepsilon$ and $\|y + Gz\| > 2 - \varepsilon$.

It is easy to see that if G is a Daugavet centre then all sets $D_G(x, y, \varepsilon)$ are nonempty, so the zero operator is G-narrow.

REMARK 6.2.8. Since $\|G\| = 1$, for every small $\varepsilon > 0$ there are $x \in S_X$ and $y \in S_Y$ such that $\|y - Gx\| < \varepsilon$. Then, for all elements $z \in D_G(x, -y, \varepsilon)$,

$$\|z\| \geqslant \|Gz\| \geqslant \|Gz + Gx - y\| - \|Gx - y\| > 2 - 2\varepsilon.$$

Taking into account that every G-strong Daugavet operator is unbounded from below on this $D(x, -y, \varepsilon)$, we deduce that *every G-strong Daugavet operator is unbounded from below*. Also, $\mathcal{NAR}_G(X) \subset \mathcal{SD}_G(X)$, so *every G-narrow operator is unbounded below*. Finally, Proposition 6.2.5 and Proposition 6.1.8 imply that $\mathcal{NAR}_G(X)$ *is a $\sim$closed homogeneous order ideal* and hence, $\mathrm{cp}(\mathcal{NAR}_G(X))$ *is a $\sim$closed homogeneous order ideal, which is a $\tilde{+}$-semigroup*.

REMARK 6.2.9. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, Z be an arbitrary Banach space. Denote $\mathrm{cp}(\mathcal{NAR}_G(X, Z))$ the subset of $L(X, Z)$ consisting of all those $T \in L(X, Z)$ that belong to $\mathrm{cp}(\mathcal{NAR}_G(X))$. Then $\mathrm{cp}(\mathcal{NAR}_G(X, Z))$ forms a linear subspace of $L(X, Z)$.*

PROOF. The homogeneity of $\mathrm{cp}(\mathcal{NAR}_G(X, Z))$ is evident, so we only have to show that for all $T_1, T_2 \in \mathrm{cp}(\mathcal{NAR}_G(X, Z))$ their sum belongs to $\mathrm{cp}(\mathcal{NAR}_G(X, Z))$. Indeed, since $\mathrm{cp}(\mathcal{NAR}_G(X))$ is a $\tilde{+}$-semigroup, $T_1 \tilde{+} T_2 \in \mathrm{cp}(\mathcal{NAR}_G(X))$. At the same time $T_1 + T_2 \leqslant T_1 \tilde{+} T_2$, and it remains to use the fact that $\mathrm{cp}(\mathcal{NAR}_G(X))$ is an order ideal. $\square$

The above remark combined with Theorem 5.3.1 clarify the importance of $\mathrm{cp}(\mathcal{NAR}_G(X, Z))$ for unconditional representations: each time when we discover a class of operators that lies in $\mathrm{cp}(\mathcal{NAR}_G(X, Y))$, we deduce as a corollary that the Daugavet centre G is not representable as a pointwise unconditional sum of operators from that class. This, for us, is a good motivation for the deep study of narrow operators in general and in concrete spaces, which we will perform in the sequel.

Since in general $\mathcal{NAR}_G(X)$ itself is not a $\tilde{+}$-semigroup, the following simple observation is sometimes of use.

REMARK 6.2.10. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, and suppose $T_1, T_2 \in L(X, Y)$ are G-narrow. Then $T_1 + T_2 \neq G$.*

PROOF. Suppose that, on the contrary, $T_1 + T_2 = G$. According to Lemma 6.2.3, $\|G - T_i\| = 1 + \|T_i\|$. Then with

$$\|T_1\| = \|G - T_2\| = 1 + \|T_2\| = 1 + \|G - T_1\| = 2 + \|T_1\|$$

we arrive at a contradiction. $\square$

In Chapter 8 we will study in detail strong Daugavet and narrow operators in concrete spaces with the Daugavet property, and will demonstrate that in some basic spaces like $C(K)$ these classes are $\tilde{+}$-semigroups, although in general this is not true. We will show (cf. Proposition 8.4.1) that if K is a perfect compact and T is an operator on $X = C(K)$, then the following assertions are equivalent:

(1) $T \in \mathcal{SD}(X)$.
(2) $T \in \mathcal{NAR}(X)$.
(3) T is C-narrow.

Also, we will show below that for general $X \in \mathrm{DPr}$ it may happen that $\mathcal{SD}(X) \neq \mathcal{NAR}(X)$. Although in general $\mathcal{NAR}(X)$ is not a subsemigroup of $\mathcal{OP}(X)$, we will show that its central part $\mathrm{cp}(\mathcal{NAR}(X))$ is always large. It contains, in particular, all strong Radon-Nikodým operators and all operators which do not fix copies of ℓ_1. Hence all the operators which are majorised by linear combinations of strong Radon-Nikodým operators and operators not fixing copies of ℓ_1, as well as $\sim$limits of sequences of such operators belong to $\mathrm{cp}(\mathcal{NAR}(X))$.

LEMMA 6.2.11. *Let $G \in S_{L(X,Y)}$ and $T \in \mathcal{NAR}_G(X)$. Then for every $x \in S_X$, $y \in S_Y$, $\varepsilon > 0$ and every slice $S = \mathrm{Slice}(B_X, x^*, \alpha)$, with $x^* \in S_{X^*}$, containing x there is an element $v \in S$ with $\|v\| = 1$, such that $\|Gv + y\| > 2 - \varepsilon$ and $\|T(x - v)\| < \varepsilon$.*

PROOF. Since $x \in S$ then there is $\varepsilon_1 > 0$ such that $\operatorname{Re} x^*(x) > 1 - \alpha + \varepsilon_1$. By Proposition 6.1.12, for every $0 < \delta < \varepsilon$ there is an element $u \in U_{x^*,\delta} \cap D_G(x, y, \delta)$ such that $\|Tu\| < \delta$. This means that $|x^*(u)| < \delta$, $\|Tu\| < \delta$, $\|x + u\| < 1 + \delta$ and $\|Gx + Gu + y\| > 2 - \delta$. From the last inequality, $\|x + u\| > 1 - \delta$. Put $v := (x + u)/\|x + u\|$; then

$$\begin{aligned}
\operatorname{Re} x^*(v) &> \frac{1}{1+\delta}(1 - \alpha + \varepsilon_1 - \delta), \\
\|T(x - v)\| &= \left\| T\left(x - \frac{x+u}{\|x+u\|} \right) \right\| \\
&\leqslant \frac{\big|\|x+u\| - 1\big| \|Tx\| + \|Tu\|}{\|x+u\|} < \frac{\delta(\|T\| + 1)}{1 - \delta},
\end{aligned}$$

and

$$\|Gv + y\| \geqslant \|Gx + Gu + y\| - \left\| G(x+u) - \frac{G(x+u)}{\|x+u\|} \right\| > 2 - 2\delta.$$

If δ is small enough, then v satisfies our requirements. $\square$

Using the previous lemma and Theorem 5.1.2 one can easily prove the following fact.

COROLLARY 6.2.12. *If for an operator G there exists at least one G-narrow operator, then G is a Daugavet centre.*

A result for strong Daugavet operators analogous to the above does not hold true.

EXAMPLE 6.2.13. *The formula $x^*_\sigma(x) := \sum_{n=1}^{\infty} \sigma_n x(n)$ defines a strong Daugavet functional on ℓ_1 whenever σ is a sequence of signs, i.e., if $|\sigma_n| = 1$ for all n.*

Indeed, let $x \in S_{\ell_1}$, $y \in S_{\ell_1}$ and $\varepsilon > 0$. Pick N such that $\sum_{n=1}^{N} |x(n)| > 1-\varepsilon$ and define $u \in S_{\ell_1}$ by $u(n) = 0$ for $n \leqslant N$ and $u(n) = \sigma_{n-N} y(n-N)/\sigma_n$ for $n > N$. Then $\|u\| = 1$, $x_\sigma^*(u) = x_\sigma^*(y)$ and $\|x+u\| > 2-\varepsilon$; hence $z := u - y \in D(x,y,\varepsilon)$ and $x^*(z) = 0$.

In the next theorem we collect various geometrical descriptions of G-narrow operators. The reader may notice that some of the reformulations are almost identical and are listed here just for the sake of convenience, but others are essential.

THEOREM 6.2.14. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre and $T \in \mathcal{OP}(X)$. Then the following assertions are equivalent:*

(i) $T \in \mathcal{NAR}_G(X)$.

(ii) *For every $x \in S_X$, $y \in S_Y$, $\varepsilon > 0$ and every slice $S = \mathrm{Slice}(B_X, x^*, \alpha)$, $x^* \in S_{X^*}$, with $x \in S$, there is $v \in S$ such that $\|Gv + y\| > 2-\varepsilon$ and $\|T(x-v)\| < \varepsilon$.*

(iii) *For every $x \in S_X$, $y \in S_Y$, $\varepsilon > 0$ and every slice $S = \mathrm{Slice}(B_X, x^*, \alpha)$, $x^* \in S_{X^*}$, with $x \in S$, there is $v \in S$ such that $\|Gv + y\| > 2-\varepsilon$, $\|T(x-v)\| < \varepsilon$, and, additionally to (ii), $v \in S_X$.*

In other words, for every $y \in S_Y$, $\varepsilon > 0$, every slice S_0 of S_X, and every $x \in S_0$ there is $v \in S_0$ such that $\|Gv + y\| > 2-\varepsilon$ and $\|T(x-v)\| < \varepsilon$.

(iv) *For every $\varepsilon > 0$, every $W \subset B_X$ that is a convex combination of slices of the unit sphere, every $y_1 \in S_Y$ and every $w \in W$ there is $u \in W$ such that $\|Gu + y_1\| > 2-\varepsilon$ and $\|T(w-u)\| < \varepsilon$.*

(v) *For every $\varepsilon > 0$, every relatively weakly open subset $U \subset B_X$, every $y_1 \in S_Y$ and every $w \in U$ there is $u \in U$ such that $\|Gu+y_1\| > 2-\varepsilon$ and $\|T(w-u)\| < \varepsilon$.*

(v)′ *For every $\varepsilon > 0$, every relatively weakly open subset $U \subset B_X$, every $y_1 \in S_Y$ and every $w \in U$ there is $u \in U$ such that $\|Gu+y_1\| > 2-\varepsilon$ and $\|T(w-u)\| < \varepsilon$ and, additionally to (v), $u \in S_X$.*

(vi) *For every $x \in B_X$, $y \in S_Y$, $\varepsilon > 0$ and every slice $S = \mathrm{Slice}(B_X, x^*, \alpha)$, $x^* \in S_{X^*}$, containing x there is $v \in S$ such that $\|Gv + y\| > 2-\varepsilon$ and $\|T(x-v)\| < \varepsilon$.*

PROOF. The implications (iii) ⇒ (ii), (v) ⇒ (vi), (v)′ ⇒ (vi) ⇒ (ii) are evident, (i) ⇒ (iii) was proved in Lemma 6.2.11. So, it is sufficient to prove that (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i) and (v) ⇒ (v)′. Without loss of generality, in all those implications we may and do assume $\|T\| = 1$.

(ii) ⇒ (iii). Fix $y \in S_Y$, $x \in S_X$, $\varepsilon > 0$ and a slice $S = \mathrm{Slice}(B_X, x^*, \alpha)$ generated by $x^* \in S_{X^*}$ with $x \in S$. Our goal is to find $v \in S$ such that $\|Gv + y\| > 2-\varepsilon$ and $\|T(x-v)\| < \varepsilon$, and $v \in S_X$.

Select $\delta \in (0, \min\{\varepsilon/2, \alpha, 1\})$. By the Diminishing of slices lemma 2.6.13, there is a slice $\tilde{S} = \mathrm{Slice}(B_X, \tilde{x}^*, \delta)$, $\tilde{x}^* \in S_{X^*}$, such that $x \in \tilde{S} \subset S$. Let us apply (ii) to these $x, y, \tilde{S}$ and δ. We obtain $\tilde{v} \in S$ such that $\|G\tilde{v}+y\| > 2-\delta$ and $\|T(x-\tilde{v})\| < \delta$. Then $1 \geqslant \|\tilde{v}\| \geqslant \mathrm{Re}\, \tilde{x}^*(\tilde{v}) > 1-\delta$, so for $v := \tilde{v}/\|\tilde{v}\| \in S_X$ we have $\|v - \tilde{v}\| < \delta$. For this v we also have that

$$\mathrm{Re}\, \tilde{x}^*(v) = \frac{1}{\|\tilde{v}\|} \mathrm{Re}\, \tilde{x}^*(\tilde{v}) \geqslant \mathrm{Re}\, \tilde{x}^*(\tilde{v}) > 1-\delta.$$

So $v \in \tilde{S} \subset S$. Further, $\|Gv + y\| \geqslant \|G\tilde{v} + y\| - \delta > 2 - 2\delta > 2-\varepsilon$ and $\|T(x-v)\| < \|T(x-\tilde{v})\| + \delta < \varepsilon$.

(iii) $\Rightarrow$ (iv). Since $W \subset B_X$ from item (iv) is a convex combination of slices, there are slices of the unit sphere $S_1, \dots, S_n \subset S_X$, and there are $\lambda_k \geqslant 0$, $k = 1, \dots, n$, $\sum_{k=1}^n \lambda_k = 1$, such that $\lambda_1 S_1 + \dots + \lambda_n S_n = W$. Take $x_k \in S_k$ such that $\lambda_1 x_1 + \dots + \lambda_n x_n = w$. Also fix $\varepsilon_j > 0$, $j \in \overline{1,n}$, with $\sum_{k=1}^n \varepsilon_k < \varepsilon/2$. Applying consecutively the condition (iii) with ε_j to the slice S_j, element $x_j \in S_j$ and the element

$$y_j = \left(y_1 + \sum_{k=1}^{j-1} \lambda_k G v_k\right) \Big/ \left\| y_1 + \sum_{k=1}^{j-1} \lambda_k G v_k \right\|,$$

we select $v_j \in S_j$ such that $\|T(x_j - v_j)\| < \varepsilon$, $j \in \overline{1,n}$, and for every $j = 1, \dots, n$

$$\|y_j + G v_j\| > 2 - \varepsilon_j.$$

Then, due to the Lemma 2.6.4,

$$\begin{aligned}\left\| y_1 + \sum_{k=1}^{j} \lambda_k G v_k \right\| &= \left\| \left\| y_1 + \sum_{k=1}^{j-1} \lambda_k G v_k \right\| y_j + \lambda_j G v_j \right\| \\ &> \left\| y_1 + \sum_{k=1}^{j-1} \lambda_k G v_k \right\| + |\lambda_j| - 2\varepsilon_j.\end{aligned}$$

Under this inductive construction,

$$\left\| y_1 + \sum_{k=1}^{n} \lambda_k v_k \right\| > 1 + \sum_{k=1}^{n} \lambda_k - 2 \sum_{k=1}^{n} \varepsilon_k > 2 - \varepsilon,$$

so $u := \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$ will be the desired element.

(iv) $\Rightarrow$ (v). This follows from Lemma 2.6.21.

(v) $\Rightarrow$ (i). Let us demonstrate that for every $x^* \in X^*$ the corresponding $T \tilde{+} x^*$ is G-strongly Daugavet. For arbitrary fixed $x \in S_X$, $y \in S_Y$ and $\varepsilon > 0$ apply (v) with parameter $\varepsilon/2$ to the weak neighbourhood of x

$$U := \{z \in B_X \colon |x^*(z - x)| < \varepsilon/2\}.$$

This gives us $u \in U$ such that $\|Gu + y\| > 2 - \frac{\varepsilon}{2}$ and $\|T(u - x)\| < \frac{\varepsilon}{2}$. Then $\|(T \tilde{+} x^*)(u - x)\| < \varepsilon$. Due to Definition 6.2.1, $T \tilde{+} x_0^* \in \mathcal{SD}_G(X)$.

(v) $\Rightarrow$ (v)$'$. Find in U a subset $(w + \widetilde{U}) \cap B_X$, where $\widetilde{U}$ is a convex balanced weak neighbourhood of 0. Fix $r \in (0,1)$ such that $r\widetilde{U} + rB_X \subset \widetilde{U}$ and apply (v) to $U_1 = (w + r\widetilde{U}) \cap B_X$ and $\tilde{\varepsilon} := \min\{\varepsilon/2, r\}$. We obtain $u_1 \in U_1$ such that $\|Gu_1 + y_1\| > 2 - \tilde{\varepsilon}$ and $\|T(w - u_1)\| < \tilde{\varepsilon}$. Then $\|u_1\| \geqslant \|Gu_1\| \geqslant \|Gu_1 + y_1\| - 1 > 1 - \tilde{\varepsilon}$, so for $u := u_1/\|u_1\| \in S_X$ we have $\|u_1 - u\| < \tilde{\varepsilon}$, which ensures all the required properties. □

REMARK 6.2.15. Analogously to Corollaries 3.1.6 and 5.1.3, the reformulation (ii) of the last theorem implies that *a complex-linear $T \colon X \to E$ between complex spaces is G-narrow if and only if the same operator between $X_{\mathbb{R}}$ and $E_{\mathbb{R}}$ is G-narrow.*

REMARK 6.2.16. The Diminishing of Slices Lemma 2.6.13 gives additional flexibility to the usage of items (ii) and (iii) of Theorem 6.2.14: each of these items just has to be verified for small values of α, for example, for $\alpha < \varepsilon$. This agrees with the picture of "thin" slices that we usually have in mind. Also, thanks to the quantifier "for every $\varepsilon > 0$", these statements with $\alpha < \varepsilon$ may be deduced from analogous statements with $\alpha = \varepsilon$, so sometimes we use the following equivalent reformulations (ii)$'$ and (iii)$'$: *For every $x \in S_X$, $y \in S_Y$, $\varepsilon > 0$ and $x^* \in S_{X^*}$ such* *that* $\operatorname{Re} x^*(x) > 1-\varepsilon$*, there is* $v \in S_X$ *(*$v \in B_X$ *in the version of* (iii)$'$*) such that* $\operatorname{Re} x^*(v) > 1-\varepsilon$*,* $\|Gv+y\| > 2-\varepsilon$ *and* $\|T(x-v)\| < \varepsilon$*.*

Recall that $T \in L(X,E)$ is a strong Radon-Nikodým operator if $\overline{T(B_X)} \in \mathrm{RNP}$, see Definition 2.7.15. Remark that in this case, for every $x^* \in X^*$ the image of B_X under $T_1 := x^* \tilde{+} T \colon X \to \mathbb{K} \oplus E$ lies in $B_{\mathbb{K}} \oplus T(B_X) \in \mathrm{RNP}$ (see Corollary 2.7.14), so T_1 is a strong Radon-Nikodým operator as well.

Let $G \in S_{L(X,Y)}$ be a Daugavet centre and $T \in \mathcal{OP}(X)$ be a strong Radon-Nikodým operator. Our goal is to show that such an operator is G-narrow, and even more, that it belongs to the central part of the class of G-narrow operators.

For $\varepsilon > 0$, consider the subset $A(T,\varepsilon) \subset B_X$ defined by $y \in A(T,\varepsilon)$ if there exists a convex combination U of slices of B_X such that $y \in U$ and $U \subset y + U_{T,\varepsilon}$.

LEMMA 6.2.17. *Let* $T \in L(X,E)$ *be a strong Radon-Nikodým operator. Then the set* $A(T,\varepsilon)$ *introduced above is a convex dense subset of* B_X*.*

PROOF. The convexity is evident. To prove the density we need to show, by the Hahn-Banach theorem in the form of Lemma 2.6.7, that for every $x^* \in S_{X^*}$ and every $0 < \delta < \varepsilon$ there is an element $y \in A(T,\varepsilon)$ such that $\operatorname{Re} x^*(y) > 1-\delta$, in other words, $y \in S := \operatorname{Slice}(B_X, x^*, \delta)$. Let us fix $x \in B_X$ with $\operatorname{Re} x^*(x) > 1-\delta/2$ and consider the operator $T_1 = x^* \tilde{+} T$. Consider further $\overline{T_1(B_X)}$ and a $\delta/2$-neighbourhood W of $T_1 x$ in $\overline{T_1(B_X)}$. By the Radon-Nikodým property of the set $\overline{T_1(B_X)}$ there is a convex combination W_1 of slices of $\overline{T_1(B_X)}$ in W. The preimages in B_X of these slices of $\overline{T_1(B_X)}$ are slices in B_X. The corresponding convex combination U of these slices in B_X lies in the preimage of W in B_X, so this convex combination is contained in $(x + U_{T_1,\delta/2}) \cap B_X$. Fix an element $y \in U$. By our construction $y \in U \subset (x + U_{T_1,\delta/2}) \cap B_X \subset S$. On the other hand,

$$U \subset x + U_{T_1,\delta/2} \subset y + U_{T_1,\delta} \subset y + U_{T,\delta} \subset y + U_{T,\varepsilon},$$

so $y \in A(T,\varepsilon)$. □

The following result is a generalisation of Theorem 3.2.6 and the "moreover" part of Theorem 5.1.11. It can be understood as a transfer theorem: in Definition 6.2.7 one can pass from one-dimensional operators to a much wider class of operators. Let us denote the class of strong Radon-Nikodým operators on X by $\mathcal{SRN}(X)$.

THEOREM 6.2.18. *Let* $G \in S_{L(X,Y)}$ *be a Daugavet centre,* T *be* G*-narrow and* T_1 *be a strong Radon-Nikodým operator on* X*. Then* $T \tilde{+} T_1$ *is* G*-narrow; that is, we have* $\mathcal{NAR}_G(X) \tilde{+} \mathcal{SRN}(X) = \mathcal{NAR}_G(X)$*. In particular every strong Radon-Nikodým operator* T_1 *on* X *is* G*-narrow.*

PROOF. Let us fix $x \in S_X$, $y \in S_Y$, $\varepsilon > 0$, and $x_1 \in A(T_1,\varepsilon)$ satisfying that $\|x - x_1\| < \varepsilon$. According to the definition of $A(T_1,\varepsilon)$ there exists a convex combination U of slices of the unit ball such that $x_1 \in U$ and $U \subset x_1 + U_{T_1,\varepsilon}$. By (iv) of Theorem 6.2.14 there is an element $z \in U$ such that $\|Gz + y\| > 2-\varepsilon$ and $\|T(x_1 - z)\| < \varepsilon$. But the inclusion $z \in x_1 + U_{T_1,\varepsilon}$ means that $\|T_1(x_1 - z)\| < \varepsilon$. So

$$\|(T \tilde{+} T_1)(x-z)\| < \varepsilon\|T \tilde{+} T_1\| + \|(T \tilde{+} T_1)(x_1 - z)\| < \varepsilon\|T \tilde{+} T_1\| + 2\varepsilon.$$

Because ε is arbitrarily small, the last inequality shows that $T \tilde{+} T_1$ satisfies the definition of a G-strong Daugavet operator.

Now, let $x^* \in X^*$ and consider $T_2 = T_1 \tilde{+} x^*$. This is a strong Radon-Nikodým operator, too. So $(T \tilde{+} T_1) \tilde{+} x^* = T \tilde{+} T_2$ is a G-strong Daugavet operator by what we have just proved; by definition, this says that $T \tilde{+} T_1$ is G-narrow. □

COROLLARY 6.2.19. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre. Then*

(a) $\mathcal{NAR}_G(X) \tilde{+} X^* = \mathcal{NAR}_G(X)$.

(b) $\operatorname{cp}(\mathcal{NAR}_G(X)) = \mathcal{SD}_G(X) \tilde{-} \mathcal{NAR}_G(X)$.

(c) $\mathcal{SRN}(X) \subset \operatorname{cp}(\mathcal{NAR}_G(X))$.

PROOF. (a) follows from the previous theorem, because every finite-rank operator is a strong Radon-Nikodým operator.

For (b) use Theorem 6.1.11 and note that

$$\begin{aligned}\mathcal{SD}_G(X) \tilde{-} \mathcal{NAR}_G(X) &= \mathcal{SD}_G(X) \tilde{-} (\mathcal{NAR}_G(X) \tilde{+} X^*)\\ &= (\mathcal{SD}_G(X) \tilde{-} X^*) \tilde{-} \mathcal{NAR}_G(X)\\ &= \mathcal{NAR}_G(X) \tilde{-} \mathcal{NAR}_G(X).\end{aligned}$$

(c) is a restatement of Theorem 6.2.18. □

The next theorem develops the ideas of separable determination further that we addressed in Theorems 4.1.7 and 5.1.12. These effects will play an important role in Chapter 10, where the possibility of reduction to the separable case makes the whole theory more applicable. First, a technical lemma.

LEMMA 6.2.20. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, $T \in \mathcal{OP}(X)$ be G-narrow. Then, for any $\varepsilon \in (0,1)$, $x \in S_X$, and $y \in S_Y$, there exists a finite-dimensional subspace $\tilde{X} = \tilde{X}(x,y,\varepsilon) \subset X$ with $x \in \tilde{X}$ such that for every slice $S = \operatorname{Slice}(B_X, x^*, \varepsilon/2)$, $x^* \in S_{X^*}$, containing x there is some $x_1 \in S_{\tilde{X}} \cap S$ with $\|Tx_1 - Tx\| < \varepsilon$ such that $\|Gx_1 + y\| > 2 - \varepsilon$.*

PROOF. Assume to the contrary the existence of $\varepsilon \in (0,1)$, $x \in S_X$, and $y \in S_Y$ such that for every finite-dimensional subspace $\tilde{X} = \tilde{X}(x,y,\varepsilon) \subset X$ with $x \in \tilde{X}$, there is a functional $x^*_{\tilde{X}} \in S_{X^*}$ such that the corresponding slice $S = \operatorname{Slice}(B_X, x^*_{\tilde{X}}, \varepsilon/2)$ contains x but for every $x_1 \in S_{\tilde{X}} \cap S$ either $\|Tx_1 - Tx\| \geqslant \varepsilon$ or $\|Gx_1 + y\| \leqslant 2 - \varepsilon$.

Consider the directed set Γ whose elements are finite-dimensional subspaces of X containing x, ordered by increasing inclusion. By w^*-compactness of B_{X^*} there is a weak* cluster point x^* of the net $(x^*_{\tilde{X}}, \tilde{X} \in \Gamma)$, which satisfies $x^*(x) \geqslant 1 - \frac{\varepsilon}{2}$. In particular $x^* \neq 0$ and we can define $x_0^* = x^*/\|x^*\|$. We have $\operatorname{Re} x^*(x) \geqslant 1 - \varepsilon/2$ since $x \in \operatorname{Slice}(B_X, x^*_Y, \varepsilon/2)$ and therefore $\|x^*\| \geqslant 1 - \varepsilon/2$. Now, if $x_1 \in \operatorname{Slice}(B_X, x_0^*, \varepsilon/2)$, then $\operatorname{Re} x^*(x_1) \geqslant \|x^*\|(1 - \varepsilon/2) > 1 - \varepsilon$ and therefore $\operatorname{Re} x^*_{\tilde{X}_1}(x_1) > 1 - \varepsilon$ for some $\tilde{X}_1 \in \Gamma$ that contains x_1. So by assumption either $\|Tx_1 - Tx\| \geqslant \varepsilon$ or $\|Gx_1 + y\| \leqslant 2 - \varepsilon$, which contradicts (ii) of Theorem 6.2.14 when applied to the slice $\operatorname{Slice}(B_X, x_0^*, \varepsilon/2)$. □

THEOREM 6.2.21. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, $T \in L(X,E)$. The operator T is G-narrow if and only if for every pair of separable subspaces $X_1 \subset X$ and $Y_1 \subset Y$ there are bigger separable subspaces X_2 and Y_2, $X_1 \subset X_2 \subset X$, $Y_1 \subset Y_2 \subset Y$ such that the restriction $G|_{X_2}\colon X_2 \to Y_2$ is a Daugavet centre and the restriction of T to X_2 is a $G|_{X_2}$-narrow operator.*

PROOF. First, let us demonstrate the easier "only if" part. Take $\varepsilon > 0$, $x \in S_X$, $y \in S_Y$ and an $x^* \in S_{X^*}$. Using our assumption, pick separable subspaces X_2 and Y_2 for $X_1 := \mathrm{lin}\{x\}$ and $Y_1 := \mathrm{lin}\{y\}$. Since $G|_{X_2}\colon X_2 \to Y_2$ is a Daugavet centre and the restriction of T to X_2 is a $G|_{X_2}$-narrow operator, there is $z \in B_{X_2} \subset B_X$ such that $\|T(x-z)\| + |x^*(x-z)| < \varepsilon$ and $\|y+Gz\| > 2-\varepsilon$, which is exactly what we need.

Now, let us concentrate on the "if" part. Suppose that $T \in \mathcal{NAR}_G(X)$ with $\|T\| = 1$. Let (v_n) be a dense sequence in X_1 and (w_n) be a dense sequence in Y_1 We select sequences of finite-dimensional subspaces $V_1 \subset V_2 \subset \dots$ in X and $W_1 \subset W_2 \subset \dots$ in Y by the following inductive procedure. Put $V_1 = \mathrm{lin}\{v_1\}$, $W_1 = \mathrm{lin}\{w_1\} \cup G(V_1)$. Suppose V_n, W_n have already been constructed. Fix a 2^{-n}-net (x_k^n, y_k^n), $k \in \overline{1, N_n}$, in $S_{V_n} \times S_{W_n}$ provided with the sum norm, select by Lemma 6.2.20 $\tilde{X}_k^n = \tilde{X}(x_k^n, y_k^n, 2^{-n}) \subset X$ with $x_k^n \in \tilde{X}_k^n$ such that for every slice $S = \mathrm{Slice}(B_X, x^*, 2^{-n-1})$, $x^* \in S_{X^*}$, containing x_k^n there is some $v \in S_{\tilde{X}_k^n} \cap S$ with $\|Tv - Tx_k^n\| < 2^{-n}$ such that $\|Gv + y_k^n\| > 2 - 2^{-n}$.

Define

$$V_{n+1} = \mathrm{lin}(\{v_{n+1}\} \cup \tilde{X}_1^n \cup \dots \cup \tilde{X}_{N_n}^n) \text{ and } W_{n+1} = \mathrm{lin}(W_n \cup \{w_{n+1}\} \cup G(V_{n+1})).$$

Finally, define X_2 to be the closure of the union of all the V_n and Y_2 to be the closure of the union of all the W_n, and the job is done.

Indeed, by item (iii) of Theorem 6.2.14 it is sufficient to demonstrate, for every $x \in S_{X_2}$, $y \in S_{Y_2}$, $\varepsilon > 0$ and every slice $S = \mathrm{Slice}(S_{X_2}, x^*, \alpha)$, $x^* \in S_{X^*}$, with $x \in S$, the existence of $v \in S$ such that $\|Gv + y\| > 2 - \varepsilon$ and $\|T(x-v)\| < \varepsilon$. In order to do this, fix $\delta \in (0, \min\{\varepsilon/2, \alpha\})$ find $n_0 \in \mathbb{N}$ such that $2^{-n_0} < \delta$. By the construction, the set $\{(x_k^n, y_k^n)\colon n > n_0,\ k \in \overline{1, N_n}\}$ is δ-dense in $S_{X_2} \times S_{Y_2}$, so, taking into account that S is relatively open in S_{X_2}, there is $n > n_0$ and $k \in \overline{1, N_n}$ such that $x_k^n \in S$, $\|x - x_k^n\| < \delta$ and $\|y - y_k^n\| < \delta$. By the Diminishing of Slices Lemma 2.6.13, there is a slice $\tilde{S} = \mathrm{Slice}(S_X, x^*, 2^{-n-1})$, $x^* \in S_{X^*}$, containing x_k^n, such that $\tilde{S} \subset S$. Then, by the construction, there is some $v \in S_{\tilde{X}_k^n} \cap \tilde{S} \subset S$ with $\|Tv - Tx_k^n\| < 2^{-n} < \delta$ such that $\|Gv + y_k^n\| > 2 - 2^{-n} > 2 - \delta$. Then

$$\|Gv + y\| \geqslant \|Gv + y_k^n\| - \|y - y_k^n\| > 2 - 2\delta > 2 - \varepsilon,$$

and $\|T(x-v)\| \leqslant \|Tv - Tx_k^n\| + \|T\|\|x - x_k^n\| < 2\delta < \varepsilon$. □

6.3. Operators that do not fix copies of ℓ_1

Recall that $T \in \mathcal{OP}(X)$ does not fix a copy of ℓ_1 if there is no subspace $E \subset X$ isomorphic to ℓ_1 on which the restriction $T\colon E \to T(E)$ is an isomorphism. By Rosenthal's ℓ_1-theorem, this is equivalent to saying that for every bounded sequence $(x_n) \subset X$, the sequence of images (Tx_n) admits a weak Cauchy subsequence. We shall investigate the class of operators not fixing a copy of ℓ_1 in the present context. In this section we are going to prove that for every Daugavet centre G any operator not fixing a copy of ℓ_1 is G-narrow (Theorem 6.3.5 below).

Next we introduce a topology related to an order ideal of operators.

DEFINITION 6.3.1. Let $\mathcal{M} \subset \mathcal{OP}(X)$ be an order ideal of operators, closed under the operation $\tilde{+}$. Then the system of tubes $U_{T,\varepsilon}$, $T \in \mathcal{M}$, $\varepsilon > 0$, defines a base of neighbourhoods of 0 for some locally convex topology on X. We denote this topology by $\sigma(X, \mathcal{M})$.

If $\mathcal{M} = \mathrm{FinRan}(X)$, the class of all finite-rank operators, then $\sigma(X, \mathcal{M})$ coincides with the weak topology; if $\mathcal{M} = \mathcal{OP}(X)$, then $\sigma(X, \mathcal{M})$ coincides with the norm topology. For classes which are in between, one gets topologies which are between the weak and the norm topology. If $\mathcal{N}$ is a collection of subsets in X such that $\mathcal{N}^{\sim}$ is closed under the operation $\tilde{+}$, then $\sigma(X, \mathcal{N}^{\sim})$ is the strongest locally convex topology on X which is dominated by the norm topology and in which the zero vector belongs to the closure of every element of $\mathcal{N}$.

Definition 6.3.2. Let X, Y be Banach spaces, $G \in L(X, Y)$. A locally convex topology τ on X is said to be a *G-Daugavet topology* if for every two elements $x \in S_X$, $y \in S_Y$, for every $\varepsilon > 0$ and every τ-neighbourhood U of y there is an element $z \in U \cap S_X$ such that $\|Gz + y\| > 2 - \varepsilon$.

Of course, $\sigma(X, \mathcal{M})$ is a Daugavet topology if and only if every operator $T \in \mathcal{M}$ is a G-strong Daugavet operator.

Lemma 6.3.3. *Let X, Y be Banach spaces, $G \in L(X, Y)$ be a Daugavet centre, $T \in \mathcal{OP}(X)$ a G-narrow operator, $A = \{a_1, \dots, a_n\} \subset S_Y$, $\varepsilon > 0$ and $x \in B_X$. Then for every $\sigma(X, \mathrm{cp}(\mathcal{NAR}_G(X)))$-neighbourhood W of x there is an element $w \in W \cap S_X$ such that $\|T(w - x)\| < \varepsilon$ and $\|Gw + a\| > 2 - \varepsilon$ for every $a \in A$.*

In particular, the lemma is applicable to any ordinary weak neighbourhood W of x.

Proof. We shall argue by induction on n. First of all consider $n = 1$. Every $\sigma(X, \mathrm{cp}(\mathcal{NAR}_G(X)))$-neighbourhood of x can be represented as $W = x + U_{R,\delta}$, where $R \in \mathrm{cp}(\mathcal{NAR}_G(X))$. Since $T_1 = R \tilde{+} T$ is G-narrow (by definition of the central part), there is an element $w \in S_X$ such that $\|Gw + a_1\| > 2 - \varepsilon$ and $\|T_1(w - x)\| < \min(\delta, \varepsilon)$ (Theorem 6.2.14(v)$'$ with $U = B_X$). The last inequality means, in particular, that $\|T(w - y)\| < \varepsilon$ and $w \in W$.

Now, suppose our assertion is true for n, let us prove it for $n + 1$. Let $A = \{a_1, \dots, a_n, a_{n+1}\} \subset S_Y$, and let us assume that, by the inductive step, an element $w_1 \in W \cap S_X$ such that $\|T(w_1 - x)\| < \varepsilon/2$ and $\|Gw_1 + a_k\| > 2 - \varepsilon$, $k = 1, \dots, n$, has already been selected. Then there is a weak neighbourhood U of Gw_1 such that the inequalities $\|Gu + a_k\| > 2 - \varepsilon$, $k = 1, \dots, n$, hold for every $u \in U$ (Proposition 2.6.10). $\tilde{U} := G^{-1}U$ is a weak neighbourhood of w_1. The intersection $\tilde{U} \cap W$ is a $\sigma(X, \mathrm{cp}(\mathcal{NAR}_G(X)))$-neighbourhood of w_1, so according to our inductive assumption for $n = 1$, there is an element $w \in S_X \cap \tilde{U} \cap W$ such that $\|Gw + a_{n+1}\| > 2 - \varepsilon$ and $\|T(w - w_1)\| < \varepsilon/2$. This element w satisfies all the requirements. $\square$

Using an ε-net of the unit ball of the finite-dimensional subspace Z below, one can easily deduce the following result (see Lemma 2.8.8 for an analogous reasoning).

Proposition 6.3.4. *Let X, Y be Banach spaces, $G \in L(X, Y)$ be a Daugavet centre, $R \in \mathcal{NAR}_G(X)$ and $Z \subset Y$ be a finite-dimensional subspace. Then for every $\varepsilon > 0$, every $x \in B_X$ and every $\sigma(X, \mathrm{cp}(\mathcal{NAR}_G(X)))$-neighbourhood W of x there is an element $w \in W \cap S_X$ such that $\|R(w - x)\| < \varepsilon$ and $\|z + Gw\| > (1 - \varepsilon)(\|z\| + \|w\|)$ for every $z \in Z$.*

Theorem 6.3.5. *Let X, Y be Banach spaces, $G \in L(X, Y)$ be a Daugavet centre, and let $T \in \mathcal{OP}(X)$ be an operator on X which does not fix a copy of ℓ_1. Then $T \in \mathrm{cp}(\mathcal{NAR}_G(X))$ so, in particular, T is a G-narrow operator.*

Proof. By Theorem 6.2.21 we may assume that X and Y are separable. Lemma 1(xii) of [**90**] combined with Rosenthal's Alternative implies that every operator which does not fix a copy of ℓ_1 can be factored through a space without ℓ_1-subspaces. So every operator which does not fix a copy of ℓ_1 can be majorised by an operator which maps into a space without ℓ_1-subspaces. Since the class of narrow operators is an order ideal, it is enough to prove our theorem for $T\colon X \to E$, where E has no ℓ_1-subspaces, and we may for our convenience assume $E = \overline{T(X)}$, so E is also separable.

Let us fix a G-narrow operator R on X, $\varepsilon > 0$, $x \in S_X$ and $y \in S_Y$. Let us introduce a directed set $(\Gamma, \prec)$ as follows: the elements of Γ are finite sequences in S_X of the form $\gamma = (x_1, \dots, x_n)$, $n \in \mathbb{N}$. The (strict) ordering is defined by

$$(x_1, \dots, x_n) \prec (y_1, \dots, y_m) \iff n < m \;\&\; \{x_1, \dots, x_n\} \subset \{y_1, \dots, y_{m-1}\}$$

and of course $\gamma_1 \preceq \gamma_2$ non-strictly if $\gamma_1 \prec \gamma_2$ or $\gamma_1 = \gamma_2$. Now, define a bounded function $F\colon \Gamma \to E \times \mathbb{R} \times \mathbb{R}$ by

$$F(\gamma) = (Tx_n, \alpha(\gamma), \|R(x - x_n)\|),$$

where

$$\begin{aligned}\alpha(\gamma) = \sup\bigl\{a > 0\colon \|z + Gx_n\| > a(\|z\| + 1)\\ \text{for all } z \in \operatorname{lin}\{y, Gx_1, Gx_2, \dots, Gx_{n-1}\}\bigr\}.\end{aligned}$$

Due to Proposition 6.3.4, for every weak neighbourhood U of Tx in E, every $\varepsilon > 0$ and every finite collection $\{v_1, \dots, v_n\} \subset X$ there is some $v_{n+1} \in (T^{-1}U) \cap S_X$ for which $\alpha\bigl((v_1, \dots, v_{n+1})\bigr) > 1 - \varepsilon$ and $\|R(x - v_{n+1})\| < \varepsilon$. This means that $(Tx, 1, 0)$ is a weak limit point of the function F. So, by Theorem 2.5.6 there is a strictly $\prec$-increasing sequence $(\gamma_j) = \bigl((x_1, \dots, x_{n(j)})\bigr)$ for which $(Tx_{n(j)})$ tends weakly to Tx, $(\|R(y - x_{n(j)})\|)$ tends to 0 and $(\alpha(\gamma_j))$ tends to 1. Passing to a subsequence we can select points $x_{n(j)}$ in such a way that the sequence $\{y, Gx_{n(1)}, Gx_{n(2)}, \dots\}$ is ε-equivalent to the canonical basis of ℓ_1.

According to Mazur's Theorem 2.1.2, there is a sequence (z_n) with $z_n \in \operatorname{conv}(\{x_{n(j)}\colon j > n\})$ for all $n \in \mathbb{N}$ such that $\|Tx - Tz_n\| \to 0$. Evidently $\|Gz_n + y\| > 2 - \varepsilon$ and $\|(R \tilde{+} T)(x - z_n)\| \to 0$, which means that $R \tilde{+} T \in \mathcal{SD}(X)$ and thus proves the theorem by Corollary 6.2.19(b). ☐

Corollary 6.3.6. *If X is a Banach space with the Daugavet property and $T\colon X \to X$ does not fix a copy of ℓ_1, then T satisfies the Daugavet equation.*

Remark that on the way we have demonstrated the existence of $(x_n) \subset S_X$ for which (Gx_n) is equivalent to the canonical basis of ℓ_1. Then the sequence $(x_n) \subset S_X$ is also equivalent to the canonical basis of ℓ_1 (Lemma 2.3.7), which gives the following result.

Theorem 6.3.7. *Every Daugavet centre fixes a copy of ℓ_1. In particular, if $G \in L(X, Y)$ is a Daugavet centre, then both X and Y have subspaces isomorphic to ℓ_1.*

6.4. A refinement to the renorming theorem with applications to unconditional sums

The class $\mathcal{NAR}_G(X)$ depends on the Daugavet centre G, moreover, this may happen in very classical spaces X, like $C[0, 1]$. Let us cite two enlightening examples by Bosenko [**60**]. For the understanding of these examples it would be useful to read first the reformulations listed in Proposition 8.4.1.

EXAMPLE 6.4.1 ([**60**, Example 1]). Let $\mathcal{K}$ be the Cantor set in $[0,1]$ and G: $C[0,1] \to C(\mathcal{K})$, $Gf = f\big|_{\mathcal{K}}$. Then G is $\mathrm{Id}_{C[0,1]}$-narrow. But G is a Daugavet centre and hence is not G-narrow.

EXAMPLE 6.4.2 ([**60**, Example 2]). Consider compact sets $K_1 \subset [0,1]$ and $K_2 \subset [0,1]$ with $K_1 \cap K_2 = \emptyset$. Let K_1 contain some open set $U \subset [0,1]$ and let K_2 have no isolated points. Consider the restriction operators T: $C[0,1] \to C(K_1)$ and G: $C[0,1] \to C(K_2)$. Then T is a G-narrow operator, but T is not $\mathrm{Id}_{C[0,1]}$-narrow.

So, there exist G-narrow operators on $C(K)$ which are not narrow, and there are narrow operators on $C(K)$ which are not G-narrow for some Daugavet centre G: $C(K) \to Y$.

This leads to the question whether the renorming of the extended codomain space from Theorem 5.2.7 preserves narrowness of those operators that were narrow before the extension and renorming. Our goal is to demonstrate that nothing bad happens and the narrowness remains intact. For this we need a lemma.

LEMMA 6.4.3. *Let X, Y be Banach spaces, $G \in S_{L(X,Y)}$ be a Daugavet centre. $T \in \mathcal{OP}(X)$ is G-narrow if and only if for every $\varepsilon > 0$, every relatively weakly open subset $U \subset B_X$ and every $x \in U$ there exists a $G((x + U_{T,\varepsilon}) \cap U)$-valued Y-atom.*

PROOF. Let us start with the "if" part. Let $T \in \mathcal{OP}(X)$ satisfy the condition. Out goal is to show that T is G-narrow. For this, we will verify the condition (v) of Theorem 6.2.11. Fix $y \in S_Y$, $\varepsilon > 0$, and a relatively weakly open subset $U \subset B_X$.

Our assumption produces a $G((x + U_{T,\varepsilon}) \cap U)$-valued Y-atom $(\Gamma, \mathfrak{U}, f)$. Then $\lim_{\mathfrak{U}} \|f + y\| = 1 + \|w\|$, in particular $\|f(t) + y\| > 2 - \varepsilon$ for some $t \in \Gamma$. Since $f(t) \in G((x + U_{T,\varepsilon}) \cap U)$, there is $u \in (x + U_{T,\varepsilon}) \cap U$ such that $f(t) = Gu$. This u fulfills $u \in U$, $\|Gu + y\| > 2 - \varepsilon$ and $\|T(x - u)\| < \varepsilon$.

Let us demonstrate the "only if" part.

Assume that T is G-narrow, $\varepsilon > 0$, $U \subset B_X$ is relatively weakly open, and $x \in U$. Put $\Gamma = \mathrm{FIN}(Y)$, denote by $\mathfrak{F}$ the natural filter on Γ induced by the ordering by inclusion. By Lemma 6.3.3, for every $A \in \mathrm{FIN}(Y)$ there is $x(A) \in (x + U_{T,\varepsilon}) \cap U$ such that for all $y \in A$

$$\|y + G(x(A))\| > \left(1 - \frac{1}{|A|}\right)(\|y\| + 1).$$

Let us define the mapping f: $\Gamma \to G((x + U_{T,\varepsilon}) \cap U)$ by the formula $f(A) := G(x(A))$. Then for every ultrafilter $\mathfrak{U}$ that dominates $\mathfrak{F}$, the triple $(\Gamma, \mathfrak{U}, f)$ will be the desired $G((x + U_{T,\varepsilon}) \cap U)$-valued Y-atom. $\square$

THEOREM 6.4.4. *Let X, Y be Banach spaces, $G \in S_{L(X,Y)}$ be a Daugavet centre, $Y \subset E$, and J: $Y \to E$ be the corresponding natural embedding, and let $|||\cdot|||$ be an equivalent norm on E with the Y-atomic property (the existence of such a renorming was demonstrated in Lemma 5.2.6). Then the Daugavet centre $\tilde{G} := J \circ G$: $X \to (E, |||\cdot|||)$ has the additional property that $\mathcal{NAR}_G(X) = \mathcal{NAR}_{\tilde{G}}(X)$ and, consequently,* $\mathrm{cp}(\mathcal{NAR}_G(X)) = \mathrm{cp}(\mathcal{NAR}_{\tilde{G}}(X))$.

PROOF. The implication $T \in \mathcal{NAR}_{\tilde{G}}(X) \Rightarrow T \in \mathcal{NAR}_G(X)$ is evident: from the viewpoint of (ii) of Theorem 6.2.14 the first condition demands something for all $y \in S_E$, and the second one demands the same but for the smaller collection of $y \in S_Y$.

The implication $T \in \mathcal{NAR}_G(X) \Rightarrow T \in \mathcal{NAR}_{\tilde{G}}(X)$ follows from Lemma 6.4.3 that describes G-narrowness in terms of Y-atoms and $\tilde{G}$-narrowness in terms of $(E, |||\cdot|||)$-atoms, and from the Y-atomic property that says that every Y-atom is at the same time an $(E, |||\cdot|||)$-atom. $\square$

Our next goal is to develop, in Theorem 6.4.6, an extension of Theorem 5.3.6 on unconditional representations to the case of G-narrow operators. Below we use the same notation l_Y for the injective space from Remark 5.3.4 that contains Y and is equipped with a norm having the Y-atomic property. We already used this space in the proof of Theorem 5.3.6.

Theorem 6.4.5. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, E be an arbitrary Banach space, and take G-narrow operators $T_1, T_2 \in L(X, E)$. Then the operator $T = T_1 + T_2$ is unbounded from below.*

Proof. Let $J\colon Y \to l_Y$ be the natural embedding operator. Suppose T is bounded below. Then Lemma 5.3.5, with the help of an equivalent norm on E, reduces our problem to the case of $X \subset E$, with T being the natural embedding operator $T\colon X \to E$.

By Remark 5.3.4, there is an extension of G to $\widetilde{G} \in L(E, l_Y)$. Then

$$\widetilde{G} \circ T_1 + \widetilde{G} \circ T_2 = \widetilde{G} \circ T = J \circ G.$$

Since narrow operators form an order ideal, $\widetilde{G} \circ T_1$ and $\widetilde{G} \circ T_2$ are G-narrow. By Theorem 6.4.4, the operator $J \circ G$ is a Daugavet centre, and the operators $\widetilde{G} \circ T_1$ and $\widetilde{G} \circ T_2$ are $J \circ G$-narrow, which contradicts Remark 6.2.10. $\square$

Theorem 6.4.6. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, and let $\mathcal{M}$ be a set of G-narrow operators acting from X to some other fixed Banach space E and forming a linear subspace of $L(X, E)$. Let $T_n \in \mathcal{M}$ be such that the series $\sum_{n\in\mathbb{N}} T_n$ converges pointwise unconditionally to some $T \in L(X, E)$. Then the operator T is unbounded from below.*

Proof. The proof resembles the one of Theorem 6.4.5. Let $J\colon Y \to l_Y$ be as before. Suppose T is bounded below. Then our problem reduces to the case of $X \subset E$, with T being the natural embedding operator $T\colon X \to E$. Let, as before, $\widetilde{G} \in L(E, l_Y)$ be an extension of G. Then

$$\sum_{n\in\mathbb{N}} \widetilde{G} \circ T_n = \widetilde{G} \circ T = J \circ G \tag{6.4.1}$$

and the series $\sum_{n\in\mathbb{N}} \widetilde{G} \circ T_n$ converges pointwise unconditionally. All operators of the form $\widetilde{G} \circ F$ with $F \in \mathcal{M}$ (in particular, all $\widetilde{G} \circ T_n$) are G-narrow and by Theorem 6.4.4 they are $J \circ G$-narrow as well. The set $\{\widetilde{G} \circ F\colon F \in \mathcal{M}\}$ is a linear space, and by Lemma 6.2.3 all its members V satisfy the Daugavet equation in the form $\|J \circ G - V\| = 1 + \|V\|$, so (6.4.1) contradicts Theorem 5.3.1. $\square$

Definition 6.4.7. Let $G \in S_{L(X,Y)}$ be a Daugavet centre, Z be a Banach space. An operator $T \in L(Z)$ is said to be a *left* $\mathrm{cp}\mathcal{NAR}_G$*-multiplier* if for every $V \in L(X, Z)$ the composition TV belongs to $\mathrm{cp}(\mathcal{NAR}_G(X))$. A Banach space Z is said to be a *strict* $\mathrm{cp}\mathcal{NAR}_G$*-range* if Id_Z is a left $\mathrm{cp}\mathcal{NAR}_G$-multiplier. In other words Z is a strict $\mathrm{cp}\mathcal{NAR}_G$-range if $L(X,Z) \subset \mathrm{cp}(\mathcal{NAR}_G(X))$. An operator $T \in L(Z)$ is said to be a *universal left* $\mathrm{cp}\mathcal{NAR}$*-multiplier* if for every pair of Banach spaces X, Y and every Daugavet centre $G \in S_{L(X,Y)}$ the operator T a left $\mathrm{cp}\mathcal{NAR}_G$-multiplier.

Remark that for a weakly compact operator T, strong Radon-Nikodým operator T, operator T not fixing copies of ℓ_1, the corresponding composition TV belongs to the same class; so such a T is a universal left $\mathrm{cp}\mathcal{NAR}$-multiplier.

Combining the above theorem with Remark 6.2.9, we obtain the following application, which extends Theorem 5.3.6 and Corollary 5.3.8 in the part that concerns the domain space X.

Corollary 6.4.8. *If $G \in S_{L(X,Y)}$ is a Daugavet centre, then X cannot be isomorphically embedded into a space E in which the identity operator Id_E has a representation $\mathrm{Id}_E = \sum_{n\in\mathbb{N}} V_n$ as a pointwise unconditionally convergent series of left $\mathrm{cp}\mathcal{NAR}_G$-multipliers V_n. In particular, X cannot be embedded into a space E having a representation as an unconditional direct sum $(\bigoplus_{n=1}^{\infty} E_n)$ of strict $\mathrm{cp}\mathcal{NAR}_G$-ranges.*

Proof. Let $T \in L(X,E)$. Then

$$T = \mathrm{Id}_E \circ T = \sum_{n\in\mathbb{N}} V_n T.$$

All $V_n T \in \mathrm{cp}(\mathcal{NAR}_G(X,E))$, and, due to Remark 6.2.9, $\mathrm{cp}(\mathcal{NAR}_G(X,E))$ is a linear subspace of $L(X,E)$. So, using Theorem 6.4.6 with $\mathcal{M} = \mathrm{cp}(\mathcal{NAR}_G(X,E))$ we obtain that T is unbounded from below, so it is not an isomorphic embedding. ☐

The next result speaks about the range space Y of a Daugavet centre $G \in S_{L(X,Y)}$ and extends Theorem 5.3.6 in the part that concerns the range space.

Theorem 6.4.9. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre. Then Y cannot be isomorphically embedded into a space E in which the identity operator Id_E has a representation $\mathrm{Id}_E = \sum_{n\in\mathbb{N}} V_n$ as a pointwise unconditionally convergent series of universal left $\mathrm{cp}\mathcal{NAR}$-multipliers V_n.*

Proof. Let $T \in L(Y,E)$. As before, Lemma 5.3.5, with the help of an equivalent norm on E, reduces our problem to the case of $Y \subset E$, with T being the natural embedding operator $T\colon Y \to E$. Equip E with another equivalent norm having the Y-atomic property. By Theorem 6.4.4, in this new norm on E the operator $T \circ G$ is a Daugavet centre. We have

$$T \circ G = \mathrm{Id}_E \circ T \circ G = \sum_{n\in\mathbb{N}} V_n \circ T \circ G,$$

and the series is pointwise unconditionally convergent. The compositions $V_n \circ (T \circ G)$ belong to the linear space $\mathrm{cp}(\mathcal{NAR}_{T\circ G}(X,E))$ of operators, satisfying the Daugavet equation for the Daugavet centre $T \circ G$. This contradicts Theorem 5.3.1. ☐

6.5. Rich subspaces

In this section we are going to demonstrate that every space with the Daugavet property has a large variety of subspaces with the same property. We extract a class of such subspaces, mentioned in the title, and give some general descriptions of this class. Rich subspaces of concrete spaces like L_1 or $C[0,1]$ will be addressed later.

THEOREM 6.5.1. *Let X, Y be Banach spaces, $G \in S_{L(X,Y)}$ be a Daugavet centre, $E \subset X$ be a subspace, $q\colon X \to X/E$ be the quotient map, $T \in \mathcal{OP}(X)$ be an operator such that $T \tilde{+} q \in \mathcal{NAR}_G(X)$. Denote $\tilde{T} = T|_E \in \mathcal{OP}(E)$, $\tilde{G} = G|_E\colon E \to Y$. Then $\|\tilde{G}\| = 1$ and $\tilde{T} \in \mathcal{NAR}_{\tilde{G}}(E)$.*

PROOF. We are going to use (v) of Theorem 6.2.14. Fix $\varepsilon > 0$, a relatively weakly open subset $U \subset B_E$, $y_1 \in S_Y$ and $w \in U$. Our goal is to find $u \in U$ such that $\|Gu + y_1\| > 2 - \varepsilon$ and $\|T(w - u)\| < \varepsilon$ (we write G and T without "tildes" because on E this does not matter). Remark that the existence for all $\varepsilon > 0$ of $u \in B_E$ with $\|Gu\| \geqslant \|Gu + y_1\| - \|y_1\| > 1 - \varepsilon$ implies that $\|\tilde{G}\| = 1$, so we don't need to prove this separately.

By the definition of a relatively weakly open subset, U contains a subset of the form $(w + U_0) \cap B_E$, where U_0 is a convex balanced weak neighbourhood of 0 in X. Fix $r \in (0,1)$ such that $rU_0 + rB_X \subset U_0$ and apply (v) to $T \tilde{+} q \in \mathcal{NAR}_G(X)$, $U_1 := (w + rU_0) \cap B_X$ and $\tilde{\varepsilon} := \min\{\varepsilon/4, r/2, \varepsilon/(1 + \|T\|)\}$. We obtain $u_1 \in U_1$ such that $\|Gu_1 + y_1\| > 2 - \tilde{\varepsilon}$ and $\|T(w - u_1)\| + \|q(w - u_1)\| < \tilde{\varepsilon}$.

In particular, $\|q(u_1)\| = \|q(w - u_1)\| < \tilde{\varepsilon}$. The last condition means that the distance from u_1 to E is smaller than $\tilde{\varepsilon}$, so there is an element $u \in B_E$ with $\|u - u_1\| < 2\tilde{\varepsilon}$. This u satisfies all the requirements from (v) of Theorem 6.2.14: $u = u_1 + (u - u_1) \in U_1 + 2\tilde{\varepsilon}B_X \subset w + rU_0 + B_X \subset w + U_0$. Consequently, $u \in (w + U_0) \cap B_E \subset U$ as we wanted. Also,

$$\|T(u - w)\| \leqslant \|T(w - u_1)\| + \|T(u - u_1)\| < (1 + \|T\|)\tilde{\varepsilon} \leqslant \varepsilon,$$

and

$$\|Gu + y_1\| \geqslant \|Gu_1 + y_1\| - \|u - u_1\| > 2 - 2\tilde{\varepsilon} > 2 - \varepsilon. \qquad \square$$

DEFINITION 6.5.2. Let X, Y be Banach spaces, $G \in S_{L(X,Y)}$ be a Daugavet centre, $E \subset X$ be a subspace. The subspace E is said to be *G-almost rich* if the quotient map $q\colon X \to X/E$ is a G-strong Daugavet operator. The subspace E is said to be *G-rich* if $q \in \mathcal{NAR}_G(X)$. In the particular case of $X = Y$ and $G = \mathrm{Id}$ the shorter names "almost rich" and "rich" are used: for $X \in \mathrm{DPr}$ a subspace $E \subset X$ is said to be *almost rich* if $q \in \mathcal{SD}(X)$, and is said to be *rich* if $q \in \mathcal{NAR}(X)$.

The necessity to distinguish rich and almost rich subspaces will become apparent later when we show that the following theorem does not extend to almost rich subspaces; see Theorem 6.6.5.

As a concrete example, we shall show in Example 8.4.4 that a uniform algebra represented on its Shilov boundary K, which is the closure of its Choquet boundary, is a rich subspace of $C(K)$ provided K has no isolated points. The following theorem then provides another proof for Corollary 3.5.21.

THEOREM 6.5.3. *Let X, Y be Banach spaces, $G \in S_{L(X,Y)}$ be a Daugavet centre, $E \subset X$ be a G-rich subspace. Then $G|_E \in S_{L(E,Y)}$ is a Daugavet centre as well. In particular, a rich subspace E of a Banach space $X \in$* DPr *has the Daugavet property itself. Moreover, (E, X) is a Daugavet pair.*

PROOF. Just apply Theorem 6.5.1 with $T = 0$. $\square$

REMARK 6.5.4. *If the quotient map $q\colon X \to X/E$ belongs to $\mathrm{cp}(\mathcal{NAR}_G(X))$, then the restriction to E of every G-narrow operator on X is a $G|_E$-narrow operator itself. If a subspace $E \subset X$ is rich in $X \in \mathrm{DPr}$, then the restriction to E of every operator $T \in \mathrm{cp}(\mathcal{NAR}(X))$ is $\mathrm{Id}_X|_E$-narrow and, in particular, narrow.*

PROOF. The same Theorem 6.5.1 combined with the definition of the central part gives the result. □

DEFINITION 6.5.5. We say that a subspace $Y \subset X$ with the Daugavet property is *wealthy* if Y itself and every subspace Z of X containing Y have the Daugavet property (i.e., $(Y \subset Z \subset X) \Rightarrow (Z \in \mathrm{DPr})$).

It is plain that if Y is an (almost) rich subspace of a space X with the Daugavet property, then every bigger subspace is (almost) rich, too: if $Y \subset Z \subset X$, then $\|[x]_{X/Z}\| \leqslant \|[x]_{X/Y}\|$ for every $x \in X$, which gives the domination of the quotient map $\tilde{q}\colon X \to X/Z$ by $q\colon X \to X/Y$. Thus, if Y is rich, then it is wealthy. (See Theorem 6.5.12 for the final assessment of the taxonomy of affluence.)

This leads to some hereditary properties for the Daugavet property.

PROPOSITION 6.5.6. *Suppose Y is a subspace of a Banach space X with the Daugavet property.*

(a) *If the quotient space X/Y has the Radon-Nikodým property, then Y is rich.*
(b) *If the quotient space X/Y contains no copy of ℓ_1, then Y is rich.*
(c) *In particular, if $(X/Y)^*$ has the Radon-Nikodým property, then Y is rich.*
(d) *Most particularly, every finite-codimensional subspace Y of X is rich.*

In either case Y is wealthy, so it has the Daugavet property itself. Moreover, taking into account that operators that do not fix copies of ℓ_1 and strong Radon-Nikodým operators on X lie in $\mathrm{cp}(\mathcal{NAR}(X))$, in all the above cases restrictions of narrow operators on X to Y are narrow operators on Y.

PROOF. (a) follows from Theorem 6.2.18, (b) from Theorem 6.3.5, (c) follows from (b), and (d) follows from each of (a), (b) or (c). □

Our next goal is to demonstrate that rich subspaces and wealthy subspaces of $X \in \mathrm{DPr}$ are the same. After proving this, we will no longer use the temporarily useful name "wealthy subspace". For the demonstration we need a chain of lemmas.

LEMMA 6.5.7. *The following conditions for a subspace Y of a Banach space X with the Daugavet property are equivalent:*

(i) *Y is wealthy.*
(ii) *Every finite-codimensional subspace of Y is wealthy in X.*
(iii) *For every pair $x, y \in S_X$, the linear span of Y, x and y has the Daugavet property.*
(iv) *For every $x, y \in S_X$, for every $\varepsilon > 0$ and for every slice S of S_X which contains y there is an element $v \in \mathrm{lin}(\{x, y\} \cup Y) \cap S$ such that $\|x + v\| > 2 - \varepsilon$.*

PROOF. Due to Proposition 6.5.6 every finite-codimensional subspace of a space with the Daugavet property has the Daugavet property itself; this is the reason for the equivalence of (i) and (ii). The implication (i) $\Rightarrow$ (iii) follows immediately from the definition of a wealthy subspace; (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are consequences of Theorem 3.1.5. □

Below, $\mathrm{lin}^{\mathbb{R}}\{x, y\}$ denotes the set $\{ax + by\colon a, b \in \mathbb{R}\}$. Let us say that a pair of elements (x, y) in S_X is *ε-fine* if there is a slice S of S_X which contains y and the diameter of $S \cap \mathrm{lin}^{\mathbb{R}}\{x, y\}$ is less than ε.

LEMMA 6.5.8. *Let Y be a wealthy subspace of a Banach space X with the Daugavet property and let a pair (x, y) in S_X be ε-fine. Then Y intersects $D(y, x, 2\varepsilon)$. (For the definitions of the sets $D(y, x, \varepsilon)$ and the corresponding collection $\mathcal{D}(X)$, see Definition 6.2.4.)*

PROOF. First of all let us fix a slice $S = S(x^*, \varepsilon_1)$ from the definition of an ε-fine pair and fix $\delta > 0$ such that the set

$$W = \{w \in \mathrm{lin}^{\mathbb{R}}\{x, y\}\colon \|w\| < 1 + \delta,\ \mathrm{Re}\, x^*(w) > 1 - \varepsilon_1\}$$

still has diameter less than ε. Now, let us find a finite-codimensional subspace $E \subset Y$ such that

(1) $x^* = 0$ on E,
(2) if $e \in E$ and $w \in \mathrm{lin}\{x, y\}$, then $\|w\| < (1 + \delta)\|e + w\|$;

the last condition can be satisfied by Lemma 2.3.10. According to our assumptions, $\mathrm{lin}(\{x, y\} \cup E)$ has the Daugavet property, so its real-linear finite-codimensional subspace $\mathrm{lin}^{\mathbb{R}}\{x, y\} + E$ has the Daugavet property as well. So there is an element $v \in (\mathrm{lin}^{\mathbb{R}}\{x, y\} + E) \cap S$ such that $\|x + v\| > 2 - \varepsilon$. Let us represent v in the form $v = e + w$, where $e \in E$, $w \in \mathrm{lin}^{\mathbb{R}}\{x, y\}$. By choice of E this means that $\|w\| < 1 + \delta$ and $\mathrm{Re}\, x^*(w) = \mathrm{Re}\, x^*(v) > 1 - \varepsilon_1$. Thus, $w \in W$ and $\|y - w\| < \varepsilon$. Finally we have that $\|e + x + y\| = \|x + v + (y - w)\| > 2 - 2\varepsilon$ and $\|e + y\| = \|v + (y - w)\| \leqslant 1 + \varepsilon$; hence the element e belongs to $E \cap D(y, x, 2\varepsilon)$, which concludes the proof. $\square$

We now present two more lemmas.

LEMMA 6.5.9. *A subspace Y of a Banach space with the Daugavet property which is almost rich together with all of its 1-codimensional subspaces is rich.*

PROOF. Let $q\colon X \to X/Y$ be the quotient map and let $x^* \in S_{X^*}$; further let $Y_1 = Y \cap \ker x^*$ and let $q_1\colon X \to X/Y_1$ be the corresponding quotient map. Then $Y_1 = Y$ or Y_1 is 1-codimensional in Y. Now, in either case we have $\|q(x)\| + |x^*(x)| \leqslant 2\|q_1(x)\|$ for all $x \in X$. Since q_1 is a strong Daugavet operator by assumption, so is $q \tilde{+} x^*$, and q is narrow. $\square$

LEMMA 6.5.10. *A subspace Y of a Banach space X with the Daugavet property is almost rich if and only if Y intersects all the elements of $\mathcal{D}(X)$.*

PROOF. If Y intersects all the elements of $\mathcal{D}(X)$, then the quotient map q: $X \to X/Y$ is unbounded from below on every element of $\mathcal{D}(X)$. So the quotient map belongs to $\mathcal{D}(X)^{\sim}$, which coincides with the class of strong Daugavet operators by Proposition 6.2.5.

Now, consider the converse statement. If Y is almost rich, then for every $\varepsilon > 0$ the map q is unbounded from below on every set of the form $D(x, y, \varepsilon/2)$. This means that there is an element $z \in Y$ for which $\mathrm{dist}(z, D(x, y, \varepsilon/2)) < \varepsilon/2$. In this case z belongs to $D(x, y, \varepsilon)$, so the intersection of this set with Y is non-empty. $\square$

The following is the key result to establishing that wealthy subspaces are rich.

LEMMA 6.5.11. *Every wealthy subspace Y of a Banach space X having the Daugavet property is almost rich.*

PROOF. According to Lemma 6.5.10 we need to prove that for every positive $\varepsilon < 1/10$ and every pair $y, x \in S_X$ the subspace Y intersects $D(y, x, \varepsilon)$. To do this, according to Lemma 6.5.8, it is enough to show that for every $\varepsilon > 0$ and every pair $x, y \in S_X$ there is an ε-fine pair $x_1, y_1 \in S_X$ which approximates (x, y) well; i.e., $\|x - x_1\| + \|y - y_1\| < \varepsilon$. Let us fix a positive $\delta < \varepsilon^2/8$ and select an element $z \in S_X$ in such a way that for every $w \in \mathrm{lin}^{\mathbb{R}}\{x, y\}$ and for every $t > 0$

$$\|w + tz\| \geqslant (1 - \delta)(\|w\| + |t|)$$

(we use Lemma 3.1.14). Put $x_1 = x + \varepsilon z$, $y_1 = y$. To show that (x_1, y) is an ε-fine pair we have to demonstrate, that in $Z := \mathrm{lin}^{\mathbb{R}}\{x_1, y\}$ there is a slice S_ε of B_Z having $\operatorname{diam} S_\varepsilon < \varepsilon$, and such that $y \in S_\varepsilon$.

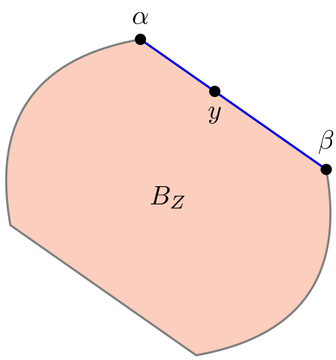

FIGURE 6.1. Some help on the proof of Lemma 6.5.11

In order to better understand the picture, recall that in the two-dimensional real space Z its sphere S_Z is a convex curve that may contain linear segments. Denote by $[\alpha, \beta]$ a maximal segment of S_Z that contains y (see Figure 6.1). It may happen that $\alpha = \beta = y$, then y is a denting point of S_Z since Z is finite dimensional, and the problem is solved. Without loss of generality we may assume that $\|\alpha - y\| \leqslant \|\beta - y\|$. If, moreover, $\|\alpha - y\| < \varepsilon$, one can cut a slice S_ε containing y from B_Z which is arbitrarily close to the segment $[\alpha, y]$, so it can be selected to have $\operatorname{diam} S_\varepsilon < \varepsilon$. Thus, there remains the last case for which we have not proved the existence of S_ε yet: $\varepsilon \leqslant \|\alpha - y\| \leqslant \|\beta - y\|$. Let us demonstrate that this case is impossible.

Denote $v := \alpha - y$. Then $y + v$ and $y - v$ lie on $[\alpha, \beta]$, so $\|y + v\| = \|y - v\| = 1$, and $\|v\| \geqslant \varepsilon$. Represent our $v \in Z = \mathrm{lin}^{\mathbb{R}}\{x + \varepsilon z, y\}$ as $v = ay + b(x + \varepsilon z)$. Then

$$\begin{aligned} 1 &= \max\{\|y + ay + b(x + \varepsilon z)\|, \|y - ay - b(x + \varepsilon z)\|\} \\ &\geqslant (1 - \delta)(\max\{\|y + ay + bx\|, \|y - ay - bx\|\} + |b|\varepsilon) \\ &\geqslant (1 - \delta)(1 + |b|\varepsilon). \end{aligned}$$

So $|b| \leqslant \delta/(\varepsilon(1 - \delta)) < \varepsilon/4$. But in this case $\|v - ay\| = \|b(x + \varepsilon z)\| < \varepsilon/3$, $|a| = \|ay\| \geqslant \|v\| - \|v - ay\| > 2\varepsilon/3$ and

$$\max\{\|y + v\|, \|y - v\|\} > \max\{\|y + ay\|, \|y - ay\|\} - \varepsilon/3 > 1 + \varepsilon/3,$$

which provides a contradiction. $\square$

THEOREM 6.5.12. *The following properties of a subspace Y of a space $X \in \mathrm{DPr}$ are equivalent:*
(i) *Y is wealthy.*
(ii) *Y is rich.*
(iii) *Every finite-codimensional subspace of Y is rich in X.*

PROOF. It is clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), see the remark following Definition 6.5.5. Now, suppose (i). Every 1-codimensional subspace of Y is wealthy by Lemma 6.5.7 and is hence almost rich by Lemma 6.5.11. An appeal to Lemma 6.5.9 completes the proof. $\square$

REMARK 6.5.13. *The above theorem is demonstrated simultaneously for real and complex spaces.* Indeed, for a complex space X and its complex subspace Y, there are two possible meanings for Y to be wealthy: real-wealthy if the intermediate subspace Z in Definition 6.5.5 is a real subspace, and complex-wealthy if the intermediate subspace Z is complex. Every real-wealthy subspace is complex-wealthy, so for a complex subspace $Y \subset X$ we have the implications

$$(Y \text{ is rich}) \Longrightarrow (Y \text{ is real-wealthy}) \Longrightarrow (Y \text{ is complex-wealthy}) \Longrightarrow (Y \text{ is rich}),$$

which means that we have demonstrated in passing the equivalence

$$(Y \text{ is complex-wealthy}) \Longleftrightarrow (Y \text{ is real-wealthy}).$$

Remark also the following Hahn-Banach style reformulation of richness analogous to the equivalence (i) $\Leftrightarrow$ (vii) of Theorem 3.1.11.

THEOREM 6.5.14. *The following assertions are equivalent for a subspace Z of a Banach space X.*
(i) *Z is rich in X.*
(ii) *For every $x, y \in S_X$ and every $\varepsilon > 0$,*
$$y \in \overline{\mathrm{conv}}\big(y + \big(D(x,y,\varepsilon) \cap Z\big)\big). \tag{6.5.1}$$
(iii) *For every $x, y \in S_X$ and every $\varepsilon > 0$,*
$$0 \in \overline{\mathrm{conv}}\big(D(x,y,\varepsilon) \cap Z\big). \tag{6.5.2}$$

PROOF. First, (6.5.2) is just a shift of (6.5.1), so the equivalence (ii) $\Leftrightarrow$ (iii) is plain.

(i) $\Rightarrow$ (iii). If Z is rich in X, then the previous Theorem 6.5.12 says that every finite-codimensional subspace of Z is rich in X as well. Then every finite-codimensional subspace Y of Z is almost rich in X, so, by Lemma 6.5.10, Y intersects $D(x,y,\varepsilon) \cap Z$. Since every weak neighbourhood of 0 in Z contains a subspace of finite codimension, this means that 0 is a weak cluster point of $D(x,y,\varepsilon) \cap Z$, which implies (6.5.2).

(ii) $\Rightarrow$ (i). By the definition of a rich subspace we need to demonstrate that the quotient map $q\colon X \to X/Z$ is narrow. For this we are going to show the validity of condition (iii)$'$ from Remark 6.2.16: for every $x, y \in S_X$, $\varepsilon > 0$ and $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(y) > 1 - \varepsilon$, there is $v \in B_X$ such that $\operatorname{Re} x^*(v) > 1 - \varepsilon$, $\|v + x\| > 2 - \varepsilon$ and $\operatorname{dist}(v - y, Z) = \|q(v - y)\| < \varepsilon$.

Indeed, select $\alpha \in \left(0, \min\left\{\frac{\operatorname{Re} x^*(y) - 1 + \varepsilon}{2}, \frac{\varepsilon}{2}\right\}\right)$. Then $\operatorname{Re} x^*(y) > 1 - \varepsilon + \alpha$, so the condition (ii) implies the existence of $w \in D(x,y,\alpha) \cap Z$ such that $\operatorname{Re} x^*(y + w) > 1 - \varepsilon + \alpha$. For this w we have $\|x + y + w\| > 2 - \alpha$ and $\|y + w\| < 1 + \alpha$. Define the desired v by the formula $v := \frac{y+w}{\|y+w\|} \in S_X$. Then $\|v - (y+w)\| < \alpha$, so $\operatorname{dist}(v-y, Z) \leqslant \|v-y-w\| < \varepsilon$, $\operatorname{Re} x^*(v) > \operatorname{Re} x^*(y+w) - \alpha > 1-\varepsilon$, and $\|v+x\| > \|x+y+w\| - \alpha > 2-2\alpha > 2-\varepsilon$. □

The analogous concept of G-wealthy subspaces was introduced in [**148**, Definition 4.4]:

Definition 6.5.15. Let X, Y be Banach spaces, $G \in S_{L(X,Y)}$ be a Daugavet centre. A subspace $E \subset X$ is said to be *G-wealthy* if for every subspace $Z \subset X$ containing Y the operator $G|_Z$ is a norm-one Daugavet centre.

In a similar manner one can show [**148**, Theorem 4.10]: For a subspace $E \subset X$ and a Daugavet centre $G \in S_{L(X,Y)}$ the following assertions are equivalent:

(i) E is G-wealthy.
(ii) E is G-rich.
(iii) Every finite-codimensional subspace of E is G-rich.

Although this result is analogous to Theorem 6.5.12, it does not imply the original theorem. The reason for this is a subtle difference between wealthy and Id_X-wealthy subspaces. For the first property of $E \subset X$ one demands for every $E \subset Z \subset X$ just the Daugavet property of Z, but for the Id_X-wealth one needs the stronger condition that (Z, X) is a Daugavet pair. In view of the already demonstrated Theorem 6.5.3, the proof of [**148**, Theorem 4.10] happens to be easier than that of Theorem 6.5.12. For this reason we have decided to include only the latter one here.

6.6. Narrow operators on L_1

In this section we shall study strong Daugavet and narrow operators on the *real* space L_1. The complex case will be covered by results from Section 8.6 where we focus on the general vector-valued case, namely by Corollary 8.6.11. We first introduce a technical definition.

Let (Ω, Σ, μ) be an atomless probability space. A function $f \in L_1 = L_1(\mu)$ is said to be a *balanced ε-peak* on $A \in \Sigma$ if $f \geqslant -1$, $\operatorname{supp} f \subset A$, $\int_\Omega f\,d\mu = 0$ and $\mu\{t\colon f(t) = -1\} > \mu(A) - \varepsilon$. The collection of all balanced ε-peaks on A will be denoted by $P(A, \varepsilon)$.

Theorem 6.6.1. $\mathcal{NAR}(L_1) = \{P(A,\varepsilon)\colon A \in \Sigma,\ \varepsilon > 0\}^{\sim}$.

Proof. Let $T \in \mathcal{NAR}(L_1)$, $\delta, \varepsilon > 0$, and $A \in \Sigma$. Consider a slice in L_1 of the form

$$S = \Big\{f \in B_{L_1}\colon \int_A f\,d\mu > 1-\delta\Big\}.$$

Applying Lemma 6.2.11 to this slice, the elements $x = -\mathbb{1}_A/\mu(A)$, $y = \mathbb{1}_A/\mu(A)$ and δ we get a function $v \in S$ such that

$$\|v - \mathbb{1}_A/\mu(A)\| > 2-\delta, \quad \|T(v - \mathbb{1}_A/\mu(A))\| < \delta. \tag{6.6.1}$$

Denote by B the set $\{t \in A\colon v(t) > 0\}$. The condition $v \in S$ implies that $\|v - \mathbb{1}_B v\| < \delta$, so

$$\|v\mathbb{1}_B - \mathbb{1}_A/\mu(A)\| > 2-2\delta.$$

Next, introduce $C = \{t \in A\colon v(t) > 1/\mu(A)\}$. By the last inequality

$$\|v\mathbb{1}_C - \mathbb{1}_A/\mu(A)\| > 2-2\delta, \quad \|v - \mathbb{1}_C v\| < 3\delta$$

and
$$\mu(C) < \delta\mu(A); \tag{6.6.2}$$
to see this observe that
$$\begin{aligned}
2 - 2\delta < \Big\| \mathbb{1}_B v - \frac{\mathbb{1}_A}{\mu(A)} \Big\| &\leqslant \int_C \Big(\mathbb{1}_B v - \frac{1}{\mu(A)} \Big)\, d\mu + \frac{1}{\mu(A)}(\mu(A) - \mu(C)) \\
&\leqslant 2 - 2\frac{\mu(C)}{\mu(A)}.
\end{aligned}$$
Put $f = (\mu(A)/\beta)\mathbb{1}_C v - \mathbb{1}_A$ with $\beta = \int_C v\, d\mu$ so that $\int_\Omega f\, d\mu = 0$. Since $\int_A v\, d\mu > 1 - \delta$ we have from $\|v - \mathbb{1}_C v\| < 3\delta$ that $\beta \geqslant 1 - 4\delta$. By (6.6.1) we conclude that
$$\|Tf\| = \mu(A) \Big\| T\Big(\frac{\mathbb{1}_C v}{\beta} - \frac{\mathbb{1}_A}{\mu(A)} \Big) \Big\| \leqslant \mu(A)\Big(\|T\| \, \Big\| \frac{\mathbb{1}_C v}{\beta} - v \Big\| + \delta \Big)$$
and
$$\Big\| \frac{\mathbb{1}_C v}{\beta} - v \Big\| \leqslant \Big\| \frac{\mathbb{1}_C v - v}{\beta} \Big\| + \Big\| \frac{v}{\beta} - v \Big\| \leqslant \frac{3\delta}{\beta} + \Big(\frac{1}{\beta} - 1 \Big) \leqslant \frac{7\delta}{1 - 4\delta},$$
and if δ is small enough, by (6.6.2) $f \in P(A, \varepsilon)$. This proves the inclusion $\mathcal{NAR}(L_1) \subset \{P(A, \varepsilon) \colon A \in \Sigma,\ \varepsilon > 0\}^{\sim}$.

To prove the opposite inclusion we use Theorem 6.2.14. Let us fix $T \in \{P(A, \varepsilon) \colon A \in \Sigma,\ \varepsilon > 0\}^{\sim}$. Let $x, y \in S_{L_1}$, $y^* \in S_{L_\infty}$ and $\varepsilon > 0$ be such that $\langle y^*, y \rangle > 1 - \varepsilon$. Without loss of generality we may assume that there is a partition $A_1, \ldots, A_n$ of Ω such that the restrictions of x, y and y^* on A_k are constants, say a_k, b_k and c_k respectively. By our assumption T is unbounded from below on each of the $P(A_k, \delta)$ for every $\delta > 0$, $k = 1, \ldots, n$. Let us fix functions $f_k \in P(A_k, \delta)$ such that $\|Tf_k\| < \delta$, $k = 1, \ldots, n$, and put
$$v = \sum_{k=1}^{n} b_k (\mathbb{1}_{A_k} + f_k).$$
By definition of balanced δ-peaks $\langle y^*, v \rangle > 1 - \varepsilon$, and moreover $\|v\| = 1$ (we will see this later), and $\|T(y - v)\|$ and $\mu(\operatorname{supp} v)$ become arbitrarily small when δ is small enough. Thus δ can be chosen so that v fulfills the conditions $\|T(y - v)\| < \varepsilon$ and $\|x + v\| > 2 - \varepsilon$.

In order to prove that $\|v\| = 1$ note that
$$\|v\| = \int_\Omega |v(t)|\, d\mu = \sum_{k=1}^{n} \int_{A_k} |b_k(\mathbb{1}_{A_k} + f_k)|\, d\mu = \sum_{k=1}^{n} |b_k| \int_{A_k} (\mathbb{1}_{A_k} + f_k)\, d\mu,$$
where the last equality follows since $\mathbb{1}_{A_k} + f_k$ is clearly positive. By the assumption on f_k, we get that $\|v\| = \sum_{k=1}^{n} |b_k| \mu(A_k) = \|y\| = 1$, as desired. □

The characterisation of narrow operators on real L_1 proved above looks similar to the definition of PP-narrow operators. It is easy to prove that every PP-narrow operator is narrow, as the following remark shows.

Remark 6.6.2. *Let (Ω, Σ, μ) be an atomless probability space and T be a PP-narrow operator on $L_1 = L_1(\mu)$. Then T is narrow.*

Proof. According to Theorem 6.6.1, we have to prove that T is unbounded below on the sets $P(A, \varepsilon)$, for every $\varepsilon > 0$ and every $A \in \Sigma$.

In order to do so, take such $A \in \Sigma$ and $\varepsilon > 0$. Let $\delta > 0$, and let us find $g \in P(A, \varepsilon)$ so that $\|g\| \geqslant \mu(A)$ and $\|T(g)\| < \delta$. This is enough up to a homogeneity argument since δ does not depend on A (and henceforth on $\mu(A)$).

Take $N \in \mathbb{N}$ so that $\frac{\mu(A)}{2^N} < \varepsilon$. Since T is a PP-narrow operator, take a sign $g_1 := \mathbb{1}_{P_1} - \mathbb{1}_{N_1}$ supported on A so that $\|T(g_1)\| < \frac{\delta}{N}$. By definition of a sign, $\mu(P_1) = \mu(N_1) = \frac{\mu(A)}{2}$. Repeat the process inductively to construct, for $i \in \{2, \ldots, N\}$, a sign $g_i = \mathbb{1}_{P_i} - \mathbb{1}_{N_i}$ supported on P_{i-1} so that $\|T(g_i)\| < \frac{\delta}{2^{i-1}N}$. Observe that $\mu(P_i) = \frac{\mu(A)}{2^i}$ by the inductive process. Define $g := \sum_{i=1}^{n} 2^{i-1} g_i$. Observe that

$$\|T(g)\| \leqslant \sum_{i=1}^{N} 2^{i-1} \|T(g_i)\| < \delta.$$

In order to finish we have to prove that $g \in P(A, \varepsilon)$. To do so, we claim that $g(t) = -1$ if $t \in A \setminus P_N$ and $g(t) = 2^N - 1$ on P_N. Observe that, if $t \in P_N$ then $g_i(t) = 1$ for every i and there is nothing to prove. On the other hand, if $t \in P_i \setminus P_{i+1}$ then $g_j(t) = 1$ for $j \leqslant i$, $g_{i+1} = -1$ and $g_j(t) = 0$ if $j > i+1$. Hence

$$g(t) = \sum_{j=0}^{i} 2^{j-1} - 2^i = -1.$$

Consequently, $\mu\{g = -1\} = \mu(A) - \mu(P_N) = \mu(A) - \frac{\mu(A)}{2^N} > \mu(A) - \varepsilon$. Moreover,

$$\begin{aligned} \int_0^1 g(t)\, dt &= 2^{N-1} \mu(P_N) - (\mu(A) - \mu(P_N)) \\ &= \frac{2^N - 1}{2^N} \mu(A) - \mu(A) \left(1 - \frac{1}{2^N}\right) = 0, \end{aligned}$$

so $g \in P(A, \varepsilon)$. Finally, observe that

$$\|g\| = \mu(A) \frac{2^N - 1}{2^{N-1}} = \mu(A) \left(2 - \frac{1}{2N - 1}\right) > \mu(A),$$

and the proof is finished. □

We do not know whether the classes of narrow operators and PP-narrow operators on L_1 coincide; see Question (6.5) in Section 6.8.

The aim of the remainder of this section is to construct an example of a strong Daugavet operator on L_1 which is not narrow. In fact, we shall define a subspace $Y \subset L_1[0, 1]$ so that the quotient map $q \colon L_1 \to L_1/Y$ is a strong Daugavet operator, but Y fails the Daugavet property. By Theorem 6.5.3, q cannot be narrow. Likewise, Y is almost rich, but not rich.

Let $I_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n})$ for $n \in \mathbb{N}_0$ and $k = 1, 2, \ldots, 2^n$. Fix $N \in \mathbb{N}$. We define

$$\begin{aligned}
g_{0,1} &= (2^N - 1)\mathbb{1}_{I_{N,1}} - \mathbb{1}_{I_{0,1}\setminus I_{N,1}}\\
g_{1,1} &= (2^{N^2-N} - 1)\mathbb{1}_{I_{N^2,1}} - \mathbb{1}_{I_{N,1}\setminus I_{N^2,1}}\\
g_{1,k} &= g_{1,1}(t - \tfrac{k-1}{2^n}), \qquad k = 2, \ldots, 2^N,\\
&\vdots\\
g_{n,1} &= (2^{N^{n+1}-N^n} - 1)\mathbb{1}_{I_{N^{n+1},1}} - \mathbb{1}_{I_{N^n,1}\setminus I_{N^{n+1},1}}\\
g_{n,k} &= g_{n,1}(t - \tfrac{k-1}{2^{N^n}}), \qquad k = 2, \ldots, 2^{N^n}.
\end{aligned}$$

Denote by P_n the "peak set" of the n'th generation, i.e.,

$$P_n = \Big\{ t \in [0,1] \colon \sum_{k=1}^{2^{N^n}} g_{n,k}(t) > 0 \Big\},$$

and $P = \bigcup_n P_n$. Clearly $|P_n| = 2^{N^n}/2^{N^{n+1}} = \big(1/2^{N-1}\big)^{N^n}$ and $|P| \leqslant 1/(2^N - 1)$. Notice also that $\int_0^1 g_{n,k}(t)\,dt = 0$ for all n and k.

First we formulate a lemma. All the norms appearing below are L_1-norms.

LEMMA 6.6.3. *Let*

$$g = \sum_{n=0}^{\infty} \sum_{k=1}^{2^{N^n}} a_{n,k} g_{n,k},$$

with finitely many non-zero $a_{n,k}$. *Then*

$$\|g\mathbb{1}_{[0,1]\setminus P}\| \leqslant 3\|g\mathbb{1}_P\|.$$

PROOF. Denote

$$\widetilde{\widetilde{g}} = \sum_{\operatorname{supp} g_{n,k} \subset P} a_{n,k} g_{n,k}, \quad \widetilde{g} = g - \widetilde{\widetilde{g}}.$$

Since $\widetilde{g}$ and g coincide off P, we clearly have

$$\|\widetilde{g}\mathbb{1}_{[0,1]\setminus P}\| = \|g\mathbb{1}_{[0,1]\setminus P}\|. \tag{6.6.3}$$

We also have that

$$\|\widetilde{g}\mathbb{1}_P\| \leqslant \|g\mathbb{1}_P\|. \tag{6.6.4}$$

Indeed, we can write P as a countable union of disjoint (half-open) intervals; denote by I any one of these. Then $\widetilde{g}$ is constant on I, and $\int_0^1 \widetilde{\widetilde{g}}(t)\,dt = 0$. Hence

$$\|\widetilde{g}\mathbb{1}_I\| = \Big|\int_0^1 \widetilde{g}(t)\mathbb{1}_I(t)\,dt\Big| = \Big|\int_0^1 (\widetilde{g}(t)\mathbb{1}_I(t) + \widetilde{\widetilde{g}}(t)\mathbb{1}_I(t))\,dt\Big| \leqslant \|g\mathbb{1}_I\|.$$

Summing up over all I gives the result.

Next, we claim that

$$\|\widetilde{g}\mathbb{1}_{[0,1]\setminus P}\| \leqslant 3\|\widetilde{g}\mathbb{1}_P\|. \tag{6.6.5}$$

To see this, we label the intervals I from the previous paragraph as follows. For every $l \in \mathbb{N}$ write $B_0 = P_0$ and $B_l = P_l \setminus \bigcup_{i=1}^{l-1} P_i$. Each B_l can be written as $\bigcup_{d\in D_l} I_{N^{l+1},d}$ where D_l is some subset of $\{1, \ldots, 2^{N^{l+1}}\}$ with cardinality $< 2^{N^l}$.

Let us write $\widetilde{g} = \sum_{n=0}^{\infty}\sum_{k=1}^{2^{N^n}} b_{n,k}g_{n,k}$ with finitely many non-zero $b_{n,k}$. We then have the estimates

$$\int_0^1 |\widetilde{g}(t)\mathbb{1}_{B_0}(t)|\,dt = |b_{0,1}|\frac{2^N-1}{2^N}$$

and

$$\begin{aligned}\int_0^1 |\widetilde{g}(t)\mathbb{1}_{B_l}(t)|\,dt &= \sum_{d\in D_l}\int_{I_{N^{l+1},d}}\Bigg|-b_{0,1}-\sum_{n=1}^{l-1}\sum_{k=1}^{2^{N^n}} b_{n,k}\mathbb{1}_{\operatorname{supp} g_{n,k}} \\ &\qquad + b_{l,(d-1)/(2^{N-1})^{N^l}+1}\big(2^{N^{l+1}-N^l}-1\big)\Bigg|\,dt \\ &\geqslant \sum_{k=1}^{2^{N^l}}\Big(\frac{1}{2^{N^l}}-\frac{1}{2^{N^{l+1}}}\Big)|b_{l,k}| \\ &\qquad -\frac{1}{(2^{N-1})^{N^l}}|b_{0,1}| - \frac{1}{(2^{N-1})^{N^l}}\sum_{n=1}^{l-1}\sum_{k=1}^{2^{N^l}}|b_{n,k}|.\end{aligned}$$

Summing up over all l gives us

$$\begin{aligned}\int_P |\widetilde{g}(t)|\,dt &\geqslant |b_{0,1}|\Bigg(\frac{2^N-1}{2^N}-\sum_{m=1}^{\infty}\frac{1}{(2^{N-1})^{N^m}}\Bigg) \\ &\qquad + \sum_{l=1}^{\infty}\Bigg(\frac{1}{2^{N^l}}-\frac{1}{2^{N^{l+1}}}-\sum_{m=l+1}^{\infty}\frac{1}{(2^{N-1})^{N^m}}\Bigg)\sum_{k=1}^{2^{N^l}}|b_{l,k}| \\ &\geqslant \frac{1}{2}|b_{0,1}| + \frac{1}{2}\sum_{l=1}^{\infty}\frac{1}{2^{N^l}}\sum_{k=1}^{2^{N^l}}|b_{l,k}|.\end{aligned}$$

On the other hand, by the triangle inequality

$$\int_0^1 |\widetilde{g}(t)|\,dt \leqslant 2\Bigg(|b_{0,1}| + \sum_{l=1}^{\infty}\frac{1}{2^{N^l}}\sum_{k=1}^{2^{N^l}}|b_{l,k}|\Bigg),$$

hence the claim follows.

The lemma now results from (6.6.3)–(6.6.5). □

THEOREM 6.6.4. *Let $Y_N \subset L_1[0,1]$ be the closed subspace generated by the system $\{g_{n,k}\colon n,k\in\mathbb{N}\}$ and the constants. Then the quotient map $q_N\colon L_1\to L_1/Y_N$ is a strong Daugavet operator for all N, but Y_N fails the Daugavet property if $N\geqslant 4$.*

PROOF. Let us fix $x,y\in S_{L_1}$ and $\varepsilon>0$. Without loss of generality we may assume that $x=\sum_{k=1}^{2^{N^n}} a_{n,k}\mathbb{1}_{I_{n,k}}$ for a big enough n to be chosen later.

Put $h=\sum_{k=1}^{2^{N^n}} a_{n,k}g_{n,k}$. Then

$$x+h = \sum_{k=1}^{2^{N^n}} 2^{N^{n+1}-N^n}\mathbb{1}_{N^{n+1},d_{n,k}}a_{n,k}$$

with $d_{n,k} = 1 + (k-1)(2^{N-1})^{N^n}$. So

$$\|x+h\| = \sum_{k=1}^{2^{N^n}} \frac{|a_{n,k}|}{2^{N^n}} = \|x\| = 1,$$

and $\operatorname{supp}(x+h) \subset P_n$. Since $|P_n| \to 0$ we can pick n big enough to satisfy $\|x+h+y\| > 2-\varepsilon$. This shows that q_N is a strong Daugavet operator.

To show that Y_N fails the Daugavet property if $N \geqslant 4$, take $g^* = \mathbb{1}_{[0,1]\setminus P} \in Y_N^*$ and $\varepsilon = 2|P|$. Since $\mathbb{1} \in S_{Y_N}$ and $g^*(\mathbb{1}) = 1 - \varepsilon/2 > 1-\varepsilon$, the set $S := \{y \in B_{Y_N}\colon g^*(y) > 1-\varepsilon\}$ is not empty, so it is a slice of B_{Y_N}. We show that there is no f in the slice S such that $\|f - \mathbb{1}\| > 2-\varepsilon$.

Suppose, on the contrary, that there is such an f. Without loss of generality we can assume that

$$f = a_0\mathbb{1} + g$$

where g is as in Lemma 6.6.3.

It follows from our conditions that

$$\|f\mathbb{1}_P\| = \int_P |f(t)|\,dt = \|f\| - g^*(|f|) \leqslant 1 - g^*(f) < \varepsilon. \tag{6.6.6}$$

Hence,

$$1 \geqslant \int_0^1 f(t)\,dt = \int_P f(t)\,dt + g^*(f) > 1 - 2\varepsilon,$$

and since $\int_0^1 f(t)\,dt = a_0$, we get

$$1 - 2\varepsilon < a_0 \leqslant 1. \tag{6.6.7}$$

By (6.6.6) and (6.6.7),

$$\|g\mathbb{1}_P\| \leqslant \varepsilon + |P| < 2\varepsilon, \tag{6.6.8}$$

thus (6.6.7) and (6.6.8) yield

$$\|g\mathbb{1}_{[0,1]\setminus P}\| \geqslant \|g\| - 2\varepsilon = \|f - a_0\mathbb{1}\| - 2\varepsilon \geqslant \|f - \mathbb{1}\| - 4\varepsilon > 2 - 5\varepsilon.$$

But now Lemma 6.6.3 and (6.6.8) imply

$$2 - 5\varepsilon < \|g\mathbb{1}_{[0,1]\setminus P}\| \leqslant 3\|g\mathbb{1}_P\| < 6\varepsilon,$$

which yields $\varepsilon > 2/11$, i.e., $|P| > 1/11$, which is false for $N \geqslant 4$. □

Theorems 6.6.4 and 6.5.3 immediately yield the following result.

THEOREM 6.6.5. *There is an almost rich subspace of $L_1[0,1]$ which fails the Daugavet property and hence fails to be rich. Thus, on $L_1[0,1]$ the class of strong Daugavet operators does not coincide with the class of narrow operators.*

6.7. Notes and remarks

Narrow operators were first introduced by Plichko and Popov [**252**] in the setting of Köthe function spaces, in particular for L_1; in this monograph we call them PP-narrow (cf. Definition 5.4.2) to distinguish them from other notions of narrowness. Then, Kadets and Popov [**175**] presented a version for $C(K)$-spaces; we call these operators C-narrow (cf. Definition 6.1.5). The abstract version of Definition 6.2.7 (for $G = \mathrm{Id}$) was suggested in [**179**]; and each of these versions comes with a corresponding notion of a rich subspace. In [**300**] some of the results of [**179**] were exhibited without reference to the algebraic superstructure of Section 6.1. Narrow operators with respect to Daugavet centres were first studied in [**60**].

Section 6.1. This part is taken almost literally from [**179**, Section 2].

Section 6.2. The interplay between L-orthogonal elements and narrow operators was observed in [**268**, Section 4]. Indeed, the following result is proved there.

THEOREM 6.7.1. *Let X be a separable Banach space with the Daugavet property, Y be a Banach space and $T\colon X \to Y$ be a narrow operator. Then, given any $y \in B_X$ and any subset $\{g_n\colon n \in \mathbb{N}\} \subset S_{X^*}$ we can find $u \in S_{X^{**}}$ such that*

(1) $\|x + u\| = 1 + \|x\|$ *holds for every $x \in X$;*
(2) $T^{**}(u) = T(y)$*;*
(3) $u(g_n) = g_n(y)$ *holds for every $n \in \mathbb{N}$.*

In particular, given any non-empty w^*-open subset W of $B_{X^{**}}$ there exists $u \in W$ satisfying (1) and (2).

It is not known whether or not the previous result extends to $\mathrm{dens}(X) = \omega_1$. However, there is a particular class of narrow operators where such an extension holds, as a consequence of a direct application of Theorem 4.3.4.

THEOREM 6.7.2. *Let X be a Banach space with the Daugavet property and Y be any Banach space. Let $T\colon X \to Y$ be a bounded operator such that $T^*(Y^*)$ is separable. Then, given $\{g_n\colon n \in \mathbb{N}\} \subset S_{X^*}$ and any $u \in B_{X^{**}}$ we can find $v \in S_{X^{**}}$ such that*

(1) $\|x + v\| = 1 + \|x\|$ *holds for every $x \in X$;*
(2) $T^{**}(v) = T^{**}(u)$*;*
(3) $v(g_n) = u(g_n)$ *holds for every $n \in \mathbb{N}$.*

In particular, given any non-empty w^*-open subset W of $B_{X^{**}}$ there exists $v \in W$ satisfying (1) and (2).

PROOF. Take $(y_n^*) \subset B_{Y^*}$ to be such that $\{T^*(y_n^*)\}$ is dense in $T^*(B_{Y^*})$. Now, we can apply Theorem 4.3.4 to find $v \in S_{X^{**}}$ satisfying that $u = v$ on the set $\{g_n\colon n \in \mathbb{N}\} \cup \{T^*(y_n^*)\colon n \in \mathbb{N}\}$ and such that the equality

$$\|x + v\| = 1 + \|x\|$$

holds for every $x \in X$. It only remains to prove that $T^{**}(v) = T^{**}(u)$. To this end notice that, since $\{T^*(y_n^*)\colon n \in \mathbb{N}\}$ is dense in $T^*(B_{Y^*})$, a density argument implies that $v = u$ on $T^*(B_{Y^*})$. Hence

$$T^{**}(v)(y^*) = v(T^*(y^*)) = u(T^*(y^*)) = T^{**}(u)(y^*)$$

holds for every $y^* \in B_{Y^*}$. From here it is immediate to get that $T^{**}(v) = T^{**}(u)$, and the proof is complete. □

REMARK 6.7.3. *Observe that, as a consequence of the previous theorem, we conclude that if $T^*(Y^*)$ is separable, then T is narrow. However, this is not surprising because it is not difficult to prove that in this situation such a T cannot fix a copy of ℓ_1, so in particular T is narrow by Theorem 6.3.5.*

Observe that the hypotheses of Theorems 6.7.1 and 6.7.2 imply the presence of L-orthogonal elements on the domain space. Since Example 4.3.7 provides an example of Banach spaces X with the Daugavet property and with no non-trivial L-orthogonal elements, none of the above mentioned theorems can be extended in complete generality for narrow operators $T\colon X \to Y$ so that $\mathrm{dens}(X) > \omega_1$.

Section 6.3. Shvydkoy [**285**] was the first to prove that an operator on a space with the Daugavet property not fixing a copy of ℓ_1 satisfies the Daugavet equation, solving a problem raised in [**178**]. In [**179**, Theorem 4.13], Theorem 6.3.5 appears formulated only for the case $G = \mathrm{Id}_X$. As far as the authors know, Theorem 6.3.5 is original in its current statement.

Section 6.4. The main results come from [**148**, Section 3]. The proofs of Theorems 6.4.5 and 6.4.6 develop some ideas from [**176**].

Section 6.5. In [**252**] a subspace Y of L_1 is called rich if the quotient map $q\colon L_1 \to L_1/Y$ is PP-narrow, and likewise a subspace Y of $C(K)$ is called rich in [**175**] if the quotient map $q\colon C(K) \to C(K)/Y$ is C-narrow.

Theorem 6.5.1 is new. Rich subspaces were introduced and studied in [**179**] and G-rich subspaces in [**148**].

A small remark on the papers [**60, 59, 58, 147, 148**]: T. Bosenko and T. Ivashyna are one and the same person.

Section 6.6. This part is taken almost literally from [**179**, Section 6].

6.8. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

Definition 6.8.1. Let $G \in S(L(X,Y))$ be a Daugavet centre, $T \in L(X,E)$. The operator T is said to be *hereditarily G-narrow* if for every pair of (separable if we want, this does not change the class) subspaces $X_1 \subset X$ and $Y_1 \subset Y$ such that the restriction $G|_{X_1}\colon X_1 \to Y_1$ is a Daugavet centre, the restriction of T to X_1 is a $G|_{X_1}$-narrow operator. In the case of $G = \mathrm{Id}_X$ we use the shorter name "*hereditarily narrow*" operators.

This definition is similar to hereditarily PP-narrow operators on L_1, but it looks like nobody studied it (even though we often deal with this class, because SCD-dominated operators (see Definition 10.4.14) are G-narrow for every Daugavet centre, so they are hereditarily G-narrow for every Daugavet centre).

(6.1) Is the class of hereditarily G-narrow operators on X a $\tilde{+}$-semigroup?

(6.2) What is the description of hereditarily narrow operators on $C[0,1]$?

This class may happen to be relatively small, because of incredibly large variety of Daugavet subspaces in $C[0,1]$ (by its universality).

(6.3) Let $E \in \mathrm{DPr}$ be a space on which the set $\mathcal{NAR}(E,X)$ of narrow operators from E to X is a linear space. Is it true that in this case $\mathcal{NAR}(E,X)$ is stable under the operation unc?

(6.4) Is it true that if $X \in \mathrm{DPr}$ and $Y \subset X$ is a subspace with a separable dual, then the quotient space X/Y also has the Daugavet property?

This question also appears in [**285**].

(6.5) Do the classes of narrow and PP-narrow operators on $L_1[0,1]$ coincide?

CHAPTER 7

Stability properties and ultrapowers

The main theme of this chapter is the investigation of stability properties of the Daugavet property with respect to direct sums, ideals, and ultraproducts. For the latter, we define the uniform Daugavet property and we give an example of a Banach space with the Daugavet property that fails the uniform Daugavet property.

7.1. Rigid versions of the Daugavet property, strong Daugavet and narrow operators

It will be technically convenient to have a version of the main notions in this monograph, viz. the Daugavet property and narrow operators, for the limiting case of $\varepsilon \to 0$. This will be studied in this first section.

DEFINITION 7.1.1. Let $G \in S_{L(X,Y)}$, $\Gamma \subset S_{X^*}$. G is said to be a Γ-*Daugavet centre* if for every $y \in S_Y$, $x^* \in \Gamma$ and $\varepsilon > 0$ there is $x \in S_X$ such that $\operatorname{Re} x^*(x) > 1 - \varepsilon$ and $\|Gx + y\| > 2 - \varepsilon$.

In particular, a Banach space X has the *Daugavet property with respect to a subset* $\Gamma \subset S_{X^*}$ ($X \in \mathrm{DPr}(\Gamma)$ for short) if for every $x \in S_X$, $x^* \in \Gamma$ and $\varepsilon > 0$ there exists some $y \in S_X$ such that $\operatorname{Re} x^*(y) > 1 - \varepsilon$ and $\|x + y\| > 2 - \varepsilon$.

If $E \subset X^*$ is a linear subspace, instead of saying S_E-Daugavet centre we say E-Daugavet centre, and instead of writing $X \in \mathrm{DPr}(S_E)$ we write for short $X \in \mathrm{DPr}(E)$ and read it "X has the *Daugavet property with respect to* E".

In this notation, $X \in \mathrm{DPr} \Leftrightarrow X \in \mathrm{DPr}(S_{X^*}) \Leftrightarrow X \in \mathrm{DPr}(X^*)$. Also, note that X has the Daugavet property if and only if X^* has the Daugavet property with respect to $X \subset X^{**}$.

A Γ-version of the definition of a narrow operator will also be useful for us.

DEFINITION 7.1.2. An operator T on a Banach space X is said to be *narrow with respect to a pair* (G, Γ), where $\Gamma \subset S_{X^*}$ and $G \in S_{L(X,Y)}$ ($T \in \mathcal{NAR}_G(X, \Gamma)$ for short) if for every two elements $x \in S_X$, $y \in S_Y$, for every $x^* \in \Gamma$ and for every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|Gz+y\| > 2-\varepsilon$ and $\|T(x-z)\|+|x^*(x-z)| < \varepsilon$.

In the case of $G = \mathrm{Id}_X$ we simplify the notation to "T is said to be narrow with respect to a subset $\Gamma \subset S_{X^*}$" and $T \in \mathcal{NAR}(X, \Gamma)$.

If $E \subset X^*$ is a linear subspace, instead of writing $T \in \mathcal{NAR}(X, S_E)$ we just write $T \in \mathcal{NAR}(X, E)$ and read it "T is narrow with respect to E".

Although we introduce the Γ-versions for arbitrary $\Gamma \subset S_{X^*}$, in applications we usually deal with one-norming Γ's, which enables reasoning like "for given $z \in S_X$ and $\varepsilon > 0$ consider a slice $S = \operatorname{Slice}(B_X, x^*, \varepsilon)$, $x^* \in \Gamma$, such that $z \in S$ and apply the definition to this slice".

It is technically convenient to work with the limiting case formally obtained by letting $\varepsilon \to 0$ in the above definitions. This limiting case gives rise to "rigid versions" of these notions.

DEFINITION 7.1.3.

(a) A Banach space X has the *rigid Daugavet property with respect to a subset* $\Gamma \subset S_{X^*}$ ($X \in \mathrm{DP}^{\mathrm{rid}}(\Gamma)$ for short) if for every $x \in S_X$ and $x^* \in \Gamma$ there exists some $y \in S_X$ such that $\operatorname{Re} x^*(y) = 1$ and $\|x + y\| = 2$.

(b) An operator T on a Banach space X is said to be a *rigid strong Daugavet operator* (in symbols $T \in \mathcal{SD}^{\mathrm{rid}}(X)$) if for every two elements $x, y \in S_X$ there is an element $z \in S_X$ such that $\|x + z\| = 2$ and $T(y - z) = 0$.

(c) An operator T is said to be *rigidly narrow with respect to a subset* $\Gamma \subset S_{X^*}$ (in symbols $T \in \mathcal{NAR}^{\mathrm{rid}}(X, \Gamma)$) if for every two elements $x, y \in S_X$ and for every $x^* \in \Gamma$ there is an element $z \in S_X$ such that $\|x + z\| = 2$ and $\|T(y - z)\| + |x^*(y - z)| = 0$.

(d) An operator $G \in S_{L(X,Y)}$ is said to be a *rigid* Γ*-Daugavet centre* if for every $y \in S_Y$, $x^* \in \Gamma$ and $\varepsilon > 0$ there is $x \in S_X$, such that $x^*(x) = 1$ and $\|Gx + y\| = 2$.

(e) An operator T on a Banach space X is said to be a *rigid* G*-strong Daugavet operator* (in symbols $T \in \mathcal{SD}_G^{\mathrm{rid}}(X)$) if for every two elements $x \in S_X$, $y \in S_Y$ there is an element $z \in S_X$ such that $\|Gz + y\| = 2$ and $T(x - z) = 0$.

(f) An operator T is said to be *rigidly narrow with respect to a pair* (G, Γ), where $\Gamma \subset S_{X^*}$ and $G \in S_{L(X,Y)}$ ($T \in \mathcal{NAR}_G^{\mathrm{rid}}(X, \Gamma)$ for short) if for every two elements $x \in S_X$, $y \in S_Y$, and for every $x^* \in \Gamma$ there is an element $z \in S_X$ such that $\|Gz + y\| = 2$ and $\|T(x - z)\| + |x^*(x - z)| = 0$.

REMARK 7.1.4. *Let us mention that every rigid* G*-strong Daugavet operator is necessarily non-injective.* To see this, one just has to proceed as in Remark 6.2.8: fix a pair $x \in S_X$ and $y \in S_Y$ such that $y = -Gx$. Then, according to the definition, there is $z \in S_X$ with $\|G(z - x)\| = \|Gz + y\| = 2$ and $T(x - z) = 0$. So, $x - z$ will be a nonzero element which T maps to 0.

Using this remark one can easily prove the following statement.

LEMMA 7.1.5. *Let* $G \in S_{L(X,Y)}$*. If* $T \in \mathcal{SD}_G^{\mathrm{rid}}(X)$*, then for every* $x \in B_X$ *and* $y \in S_Y$ *there is an element* $z \in S_X$ *such that* $\|y + Gz\| = 2$ *and* $T(x - z) = 0$.

PROOF. Using the non-injectivity of T one can find an element $x_1 \in S_X$ such that $T(x - x_1) = 0$. Then applying the definition of $\mathcal{SD}^{\mathrm{rid}}(X)$ to x_1 and y one obtains an element $z \in S_X$ such that $\|y + Gz\| = 2$ and $T(x_1 - z) = 0$. But for this element $T(x - z) = 0$, too. □

In particular, this works for $G = \mathrm{Id}_X$.

For many investigations in the context of the Daugavet property the study of the rigid notions above turns out to be sufficient, but is technically more feasible. The connection between the original versions and their rigid variants is made using ultrapowers. We refer to Section 2.2 for the basic definitions and notation.

Let $\mathfrak{U}$ be a nontrivial ultrafilter on $\mathbb{N}$, T be an operator acting from a Banach space X to a Banach space E, $\Gamma \subset S_{X^*}$. We recall from Definition 2.2.6 that $T^{\mathfrak{U}}$ is the natural operator between the ultrapowers $X^{\mathfrak{U}}$ and $E^{\mathfrak{U}}$ defined by $T^{\mathfrak{U}}[(x_n)] = [(Tx_n)]$, and by $\Gamma^{\mathfrak{U}}$ we denote the set of the linear functionals $F = (f_n)$, $f_n \in \Gamma$, of the form $F[(x_n)] = \lim_{\mathfrak{U}} f_n(x_n)$. Thanks to Proposition 2.2.7, for any $G \in S_{L(X,Y)}$ the corresponding $G^{\mathfrak{U}}$ is norm-attaining and by Proposition 2.2.10, for one-norming Γ the corresponding $\Gamma^{\mathfrak{U}}$ is a boundary, which helps in applications of the forthcoming Lemma 7.1.6.

Lemma 7.1.6.

(1) *If* $X \in \mathrm{DPr}(\Gamma)$, *then* $X^{\mathfrak{U}} \in \mathrm{DP}^{\mathrm{rid}}(\Gamma^{\mathfrak{U}})$. *If* $G \in S_{L(X,Y)}$ *is a* Γ*-Daugavet centre, then* $G^{\mathfrak{U}} \in S_{L(X^{\mathfrak{U}},Y^{\mathfrak{U}})}$ *is a rigid* $\Gamma^{\mathfrak{U}}$*-Daugavet centre.*

(2) *If* $X^{\mathfrak{U}} \in \mathrm{DPr}(\Gamma^{\mathfrak{U}})$, *then* $X \in \mathrm{DPr}(\Gamma)$. *If* $G^{\mathfrak{U}} \in S_{L(X^{\mathfrak{U}},Y^{\mathfrak{U}})}$ *is a rigid* $\Gamma^{\mathfrak{U}}$*-Daugavet centre, then* $G \in S_{L(X,Y)}$ *and* G *is a* Γ*-Daugavet centre.*

(3) *If* $T \in \mathcal{SD}(X)$, *then* $T^{\mathfrak{U}} \in \mathcal{SD}^{\mathrm{rid}}(X^{\mathfrak{U}})$. *If* $T \in \mathcal{SD}_G(X)$, *then* $T^{\mathfrak{U}} \in \mathcal{SD}^{\mathrm{rid}}_{G^{\mathfrak{U}}}(X^{\mathfrak{U}})$.

(4) *If* $T^{\mathfrak{U}} \in \mathcal{SD}(X^{\mathfrak{U}})$, *then* $T \in \mathcal{SD}(X)$. *If* $T^{\mathfrak{U}} \in \mathcal{SD}_{G^{\mathfrak{U}}}(X^{\mathfrak{U}})$, *then* $T \in \mathcal{SD}_G(X)$.

(5) *If* $T \in \mathcal{NAR}(X,\Gamma)$, *then* $T^{\mathfrak{U}} \in \mathcal{NAR}^{\mathrm{rid}}(X,\Gamma^{\mathfrak{U}})$.

(6) *If* $T^{\mathfrak{U}} \in \mathcal{NAR}(X,\Gamma^{\mathfrak{U}})$, *then* $T \in \mathcal{NAR}(X,\Gamma)$. *If* $T^{\mathfrak{U}} \in \mathcal{NAR}_{G^{\mathfrak{U}}}(X,\Gamma^{\mathfrak{U}})$, *then* $T \in \mathcal{NAR}_G(X,\Gamma)$.

Proof. All these statements don't differ too much in essence. Let us prove for example (7.1.6). Fix arbitrary elements $x = [(x_n)] \in S_{X^{\mathfrak{U}}}$, $y = [(y_n)] \in S_{Y^{\mathfrak{U}}}$, and $x^* = [(x_n^*)] \in \Gamma^{\mathfrak{U}}$. Without loss of generality (just replacing one representation of an element in $X^{\mathfrak{U}}$ by another) one may assume that $x_n \in S_X$, $y_n \in S_Y$ for all $n \in \mathbb{N}$. Applying the condition $T \in \mathcal{NAR}_G(X,\Gamma)$ for x_n, y_n, x_n^* and $\varepsilon = \frac{1}{n}$ we obtain elements $z_n \in S_X$ such that $\|y_n + Gz_n\| > 2 - \frac{1}{n}$ and

$$\|T(x_n - z_n)\| + |x_n^*(x_n - z_n)| < \frac{1}{n}.$$

This means that the conditions $\|G^{\mathfrak{U}}z + y\| = 2$ and $\|T^{\mathfrak{U}}(x - z)\| + |x^*(x - z)| = 0$ are fulfilled for $z = [(z_n)] \in S_{X^{\mathfrak{U}}}$. □

7.2. Strong Daugavet and narrow operators in ℓ_∞-sums

We first fix some notation. If T is an operator defined on $X = X_1 \oplus_\infty X_2$, we let T_1 stand for the restriction of T to X_1, i.e., $T_1x_1 = T(x_1, 0)$; and likewise $T_2x_2 = T(0, x_2)$ defines the restriction to X_2. Thus for $x = (x_1, x_2) \in X$, $Tx = T(x_1, x_2) = T_1x_1 + T_2x_2$.

The aim of this section is to prove that T is a strong Daugavet operator if and only if both restrictions T_1 and T_2 of T are strong Daugavet operators. The same is true for narrow operators.

Proposition 7.2.1. *If* $X = X_1 \oplus_\infty X_2$ *and* $T_i \in \mathcal{SD}(X_i)$ *(*$T_i \in \mathcal{SD}^{\mathrm{rid}}(X_i)$*) for* $i = 1, 2$, *then* $T \in \mathcal{SD}(X)$ *(*$T \in \mathcal{SD}^{\mathrm{rid}}(X)$, *respectively).*

Proof. By Lemma 7.1.6 it is sufficient to consider only the "rigid" version of the proposition. Indeed, we have $X^{\mathfrak{U}} = X_1^{\mathfrak{U}} \oplus_\infty X_2^{\mathfrak{U}}$ and $(T^{\mathfrak{U}})_i = (T_i)^{\mathfrak{U}}$. Therefore, if $T_i \in \mathcal{SD}(X_i)$, then $(T_i)^{\mathfrak{U}} \in \mathcal{SD}^{\mathrm{rid}}(X_i^{\mathfrak{U}})$ and, assuming the rigid version, we conclude that $T^{\mathfrak{U}} \in \mathcal{SD}^{\mathrm{rid}}(X^{\mathfrak{U}})$ which implies $T \in \mathcal{SD}(X)$.

Thus, we need to prove that for every $x = (x_1, x_2)$ with $\|x\| = \max\{\|x_1\|, \|x_2\|\} = 1$ and $y = (y_1, y_2)$ with $\|y\| = \max\{\|y_1\|, \|y_2\|\} = 1$, there is some $z = (z_1, z_2)$ with $\|z\| = \max\{\|z_1\|, \|z_2\|\} = 1$ such that $\|x + z\| = \max\{\|x_1 + z_1\|, \|x_2 + z_2\|\} = 2$ and $\|T(y - z)\| = \|T_1(y_1 - z_1) + T_2(y_2 - z_2)\| = 0$.

Without any loss of generality we may assume that $\|x_1\| = 1$. Using Lemma 7.1.5 for $T_1 \in \mathcal{SD}^{\mathrm{rid}}(X_1)$, we can find, given $x_1 \in S_X$ and $y_1 \in B_X$, some $z_1 \in S_X$ with $\|x_1 + z_1\| = 2$ and $\|T_1(y_1 - z_1)\| = 0$. Put $z_2 = y_2$, $z = (z_1, z_2)$; then $\|z\| = 1$, $\|x + z\| \geqslant \|x_1 + z_1\| = 2$, and

$$\|T(y - z)\| = \|T_1(y_1 - z_1) + T_2(y_2 - z_2)\| = \|T_1(y_1 - z_1)\| = 0,$$

which completes the proof. $\square$

Corollary 7.2.2. *If $X = X_1 \oplus_\infty X_2$ and $T_i \in \mathcal{NAR}(X_i)$ for $i = 1, 2$, then $T \in \mathcal{NAR}(X)$.*

Proof. We have to prove that for each $x^* = (x_1^*, x_2^*) \in X^* = X_1^* \oplus_1 X_2^*$, $T \tilde{+} x^*$ is a strong Daugavet operator. Let us consider the restriction of $T \tilde{+} x^*$ to X_1; then

$$\begin{aligned}\|(T \tilde{+} x^*)_1 x_1\| &= \|(T \tilde{+} x^*)(x_1, 0)\| = \|T(x_1, 0)\| + |x^*((x_1, 0))| \\ &= \|T_1 x_1\| + |x_1^*(x_1)|.\end{aligned}$$

Since T_1 is narrow, $T_1 \tilde{+} x_1^*$ is a strong Daugavet operator and hence, so is $(T \tilde{+} x^*)_1$. By symmetry, the same is true for the restriction to X_2, and Proposition 7.2.1 implies that $T \tilde{+} x^*$ is a strong Daugavet operator. Since x^* is arbitrary, T is narrow. $\square$

We now turn to the converse of Proposition 7.2.1. Recall that, by Definition 2.6.11, elements $x_1, \ldots, x_n$ of a normed space form a quasi-codirected n-tuple if

$$\|x_1 + \cdots + x_n\| = \|x_1\| + \cdots + \|x_n\|.$$

Theorem 7.2.3. *If $X = X_1 \oplus_\infty X_2$, then for every strong Daugavet operator T on X the restrictions T_1 and T_2 of T to X_1 and X_2 are strong Daugavet operators.*

Proof. As in Proposition 7.2.1 it is sufficient to prove that $T_1 \in \mathcal{SD}(X_1)$ whenever $T \in \mathcal{SD}^{\mathrm{rid}}(X)$.

So let $T \in \mathcal{SD}^{\mathrm{rid}}(X)$, $x_1, y_1 \in S_{X_1}$ and $\varepsilon > 0$. Apply the definition of a rigid strong Daugavet operator to $x = (x_1, 0)$, $y = (y_1, 0)$. We get some $z^1 = (z_1^1, z_2^1)$ for which $\|y_1 + z_1^1\| = 1$, $\|z_2^1\| \leqslant 1$, $\|x_1 + y_1 + z_1^1\| = 2$ and $Tz^1 = 0$. This means, in particular, that the vectors x_1 and $y_1 + z_1^1$ are quasi-codirected. Now, apply the definition of a rigid strong Daugavet operator to $x = (\frac{1}{2}(x_1 + y_1 + z_1^1), 0)$, $y = (y_1, z_2^1)$. We get some $z^2 = (z_1^2, z_2^2)$ for which $Tz^2 = 0$, $\|y_1 + z_1^2\| = 1$ (and consequently $\|z_2^1 + z_2^2\| \leqslant 1$) and $\|(x_1 + y_1 + z_1^1)/2 + (y_1 + z_1^2)\| = 2$. This again means, by Lemma 2.6.12, that the triple $(x_1,\ y_1 + z_1^1,\ y_1 + z_1^2)$ is quasi-codirected. Now, apply the same token to $x = ((x_1 + (y_1 + z_1^1) + (y_1 + z_1^2))/3, 0)$ and $y = (y_1, z_2^1 + z_2^2)$, etc.

Continuing this process, we obtain a sequence $z^n = (z_1^n, z_2^n)$ for which all the n-tuples $(x_1, y_1 + z_1^1, y_1 + z_1^2, \ldots, y_1 + z_1^{n-1})$, $n \in \mathbb{N}$, in S_{X_1} are quasi-codirected, $\|z_2^1 + \cdots + z_2^n\| \leqslant 1$ and $Tz^n = 0$. Consider $z = (z_1^1 + z_1^2 + \cdots + z_1^n)/n \in X_1$. By construction and Lemma 2.6.12, $\|x_1 + y_1 + z\| = 2$, $\|y_1 + z\| = 1$ and

$$\|T_1 z\| = \|T(z, 0)\| = \|T(0, \tfrac{1}{n}(z_2^1 + z_2^2 + \cdots + z_2^n))\| \leqslant \frac{\|T\|}{n}.$$

Because n can be taken arbitrarily large, this proves that $T_1 \in \mathcal{SD}(X_1)$. $\square$

Corollary 7.2.4. *If $X = X_1 \oplus_\infty X_2$, then for every narrow operator T on X, the restrictions T_1 and T_2 of T to X_1 and X_2 are narrow operators.*

PROOF. This follows directly from Theorem 7.2.3 and the definition of a narrow operator. ☐

Taking in account that $(X \in \mathrm{DPr}) \Leftrightarrow (0 \in \mathcal{NAR}(X))$, Corollaries 7.2.4 and 7.2.2 imply the following:

COROLLARY 7.2.5. *If $X = X_1 \oplus_\infty X_2$, then the following assertions are equivalent:*

(i) *X has the Daugavet property;*

(ii) *both X_1 and X_2 possess the Daugavet property.*

7.3. M-ideals and the Daugavet property

Our first result here is that the Daugavet property passes to M-ideals. Note that Corollary 7.2.5 shows that this is so for M-summands.

PROPOSITION 7.3.1. *The Daugavet property is inherited by M-ideals.*

PROOF. Suppose J is an M-ideal in a Banach space X with the Daugavet property. Let $y \in S_J$ and $\varepsilon > 0$, and let $x^* \in J^* \subset X^*$ with $\|x^*\| = 1$. Consider the slices $S_1 = \mathrm{Slice}(B_J, x^*, \varepsilon)$ and $S = \mathrm{Slice}(B_X, x^*, \varepsilon/3)$:

$$S_1 = \{\xi \in B_J\colon\ \mathrm{Re}\, x^*(\xi)\ > 1-\varepsilon\},\ S = \{\xi \in B_X\colon\ \mathrm{Re}\, x^*(\xi) > 1-\varepsilon/3\}$$

By the Daugavet property of X there is some $x \in S$ such that $\|x+y\| > 2-\varepsilon/3$; hence there is some $y^* \in S_{X^*}$ with $\mathrm{Re}\, y^*(x+y) > 2-\varepsilon/3$. Decompose $y^* = y_1^* + y_2^* \in J^* \oplus_1 J^\perp$ so that $1 = \|y^*\| = \|y_1^*\| + \|y_2^*\|$. Therefore we have

$$\mathrm{Re}\, y^*(x) + \mathrm{Re}\, y_1^*(y) > 2-\varepsilon/3$$

so that $\mathrm{Re}\, y^*(x) > 1-\varepsilon/3$ and $\mathrm{Re}\, y_1^*(y) > 1-\varepsilon/3$. Consequently, $\|y_1^*\| > 1-\varepsilon/3$ and thus $\|y_2^*\| < \varepsilon/3$.

By Lemma 2.9.5, we may find $\xi \in B_J$ satisfying $|y_1^*(\xi - x)| < \varepsilon/3$ and $|x^*(\xi - x)| < \varepsilon/3$, i.e., $\xi \in S_1$, and we have

$$\begin{aligned}
\|\xi+y\| = \mathrm{Re}\, y^*(\xi+y) \ &=\ \mathrm{Re}\, y_1^*(\xi) + \mathrm{Re}\, y_1^*(y)\\
&> \mathrm{Re}\, y_1^*(x) + \mathrm{Re}\, y_1^*(y) - \varepsilon/3\\
&> \mathrm{Re}\, y_1^*(x) + \mathrm{Re}\, y_2^*(x) + \mathrm{Re}\, y_1^*(y) - 2\varepsilon/3\\
&= \mathrm{Re}\, y^*(x) + \mathrm{Re}\, y_1^*(y) - 2\varepsilon/3 > 2-\varepsilon.
\end{aligned}$$

An application of Theorem 3.1.5 completes the proof of the proposition. ☐

Obviously, if X has the Daugavet property and $J \subset X$ is an M-ideal, then X/J need not have the Daugavet property; for example, if $X = C[0,1]$ and $J = \{f \in X\colon f(0) = 0\}$, then J is an M-ideal in $C[0,1]$ (see Example 2.9.2) and X/J is one-dimensional and thus, fails the Daugavet property.

We now prove a converse to Proposition 7.3.1, which can be regarded as a version of the three-space property for the Daugavet property under strong geometric assumptions.

PROPOSITION 7.3.2. *If J is an M-ideal in X such that J and X/J share the Daugavet property, then so does X.*

PROOF. Suppose that $y \in S_X$, $x^* \in S_{X^*}$ and $\varepsilon > 0$. We decompose

$$x^* = x_1^* + x_2^* \in J^* \oplus J^\perp, \quad \|x^*\| = \|x_1^*\| + \|x_2^*\|,$$

and from (2.9.3) on page 47 we deduce that

$$\|y\| = \max\left\{ \sup_{y^* \in B_{J^*}} |y^*(y)|, \sup_{y^* \in B_{J^\perp}} |y^*(y)| \right\} = 1.$$

We shall first assume that

$$\|[y]\|_{X/J} = \sup_{y^* \in B_{J^*}} |y^*(y)| = 1.$$

Since X/J has the Daugavet property and $(X/J)^* = J^\perp$, there is some $x_0 \in X$ satisfying

$$\|[x_0]\| = 1, \quad \operatorname{Re} x_2^*(x_0) \geqslant (1-\varepsilon)\|x_2^*\|, \quad \|[x_0 + y]\| > 2 - \varepsilon.$$

Next, pick $\xi \in B_J$ with

$$\operatorname{Re} x_1^*(\xi) \geqslant (1-\varepsilon)\|x_1^*\| \tag{7.3.1}$$

and use the 3-ball property of M-ideals (see Lemma 2.9.6) to find some $\eta \in J$ with

$$\|x_0 \pm \xi - \eta\| \leqslant 1 + \varepsilon. \tag{7.3.2}$$

Obviously, $x := x_0 + \xi - \eta$ has the properties

$$\begin{aligned} \|x\| &\leqslant 1 + \varepsilon, \\ \operatorname{Re} x_2^*(x) = \operatorname{Re} x_2^*(x_0) &\geqslant (1-\varepsilon)\|x_2^*\|, \\ \|x + y\| \geqslant \|[x+y]\| = \|[x_0+y]\| &> 2 - \varepsilon, \end{aligned}$$

and it is left to estimate $\operatorname{Re} x_1^*(x)$. Now, we get from (7.3.2)

$$|x_1^*(\xi) \pm x_1^*(x_0 - \eta)| \leqslant (1+\varepsilon)\|x_1^*\|$$

and hence from (7.3.1)

$$|x_1^*(x_0 - \eta)| \leqslant 2\varepsilon\|x_1^*\|$$

so that

$$\operatorname{Re} x_1^*(x) \geqslant (1-3\varepsilon)\|x_1^*\|$$

and finally

$$\operatorname{Re} x^*(x) \geqslant (1-3\varepsilon)\|x_1^*\| + (1-\varepsilon)\|x_2^*\| \geqslant 1 - 3\varepsilon.$$

After scaling x appropriately, we obtain (ii) of Theorem 3.1.5, and so X has the Daugavet property.

In the second part of the proof we suppose that

$$\theta := \sup_{y^* \in B_{J^\perp}} |y^*(y)| < \sup_{y^* \in B_{J^*}} |y^*(y)| = 1.$$

Claim: There is some $\xi \in S_J$ such that $\operatorname{Re} \xi^*(y) \geqslant 1 - 3\varepsilon$ whenever $\xi^* \in S_{J^*}$ and $\operatorname{Re} \xi^*(\xi) \geqslant 1 - \varepsilon$. Indeed, we have a decomposition $X^{**} = J^{\perp\perp} \oplus_\infty J^{*\perp}$ of the bidual space; denote the projection from X^{**} onto $J^{\perp\perp}$ by Q. Now,

$$1 = \|y\| = \max\{\|Qy\|, \|y - Qy\|\} = \max\{\|Qy\|, \theta\}$$

and thus $\|Qy\| = 1$. By the principle of local reflexivity, in the version of [48], there is a linear operator L: $\operatorname{lin}\{y, Qy\} \to X$ such that $\xi := L(Qy) \in S_J$, $Ly = y$ and $\|L\| \leqslant 1 + \varepsilon$; the point here is that L maps $Qy \in J^{\perp\perp}$ into J. Clearly $\xi = \frac{1}{2}y + \frac{1}{2}(2\xi - y)$ and

$$\|2\xi - y\| = \|L(2Qy - y)\| \leqslant (1+\varepsilon)\|2Qy - y\| = 1 + \varepsilon.$$

Hence, if $\xi^* \in S_{J^*}$, then $\operatorname{Re} \xi^*(y) \leqslant 1$ and $\operatorname{Re} \xi^*(2\xi - y) \leqslant 1 + \varepsilon$. Consequently, $\operatorname{Re} \xi^*(y) \geqslant 1 - 3\varepsilon$ whenever $\operatorname{Re} \xi^*(\xi) \geqslant 1 - \varepsilon$.

By the assumption on J and applying Theorem 3.1.11(v), there is $x_0 \in J$ such that

$$\|x_0\| = 1, \quad \operatorname{Re} x_1^*(x_0) \geqslant (1-\varepsilon)\|x_1^*\|, \quad \|x_0 + \xi\| \geqslant 2 - \varepsilon. \tag{7.3.3}$$

Next, pick $z \in B_X$ and $\xi_0^* \in S_{J^*}$ with the properties

$$\operatorname{Re} x_2^*(z) \geqslant (1-\varepsilon)\|x_2^*\|, \quad \operatorname{Re} \xi_0^*(x_0 + \xi) \geqslant 2 - \varepsilon$$

so that

$$\operatorname{Re} \xi_0^*(x_0) \geqslant 1 - \varepsilon, \quad \operatorname{Re} \xi_0^*(\xi) \geqslant 1 - \varepsilon. \tag{7.3.4}$$

By the construction of ξ, we therefore have $\operatorname{Re} \xi_0^*(y) \geqslant 1 - 3\varepsilon$. Using the 3-ball property of M-ideals again (see Lemma 2.9.6), we may find some $\eta \in J$ with

$$\|z \pm x_0 - \eta\| \leqslant 1 + \varepsilon,$$

and we let $x := z + x_0 - \eta$. As in the first part of the proof, we obtain from (7.3.3) and (7.3.4) that

$$|x_1^*(z - \eta)| \leqslant 2\varepsilon\|x_1^*\|, \quad |\xi_0^*(z - \eta)| \leqslant 2\varepsilon$$

and

$$\begin{aligned} \operatorname{Re} x_1^*(x) &\geqslant \operatorname{Re} x_1^*(x_0) - 2\varepsilon\|x_1^*\| \geqslant (1 - 3\varepsilon)\|x_1^*\|, \\ \operatorname{Re} x_2^*(x) &= \operatorname{Re} x_2^*(z) \geqslant (1-\varepsilon)\|x_2^*\|. \end{aligned}$$

Therefore,

$$\begin{gathered} \|x\| \leqslant 1 + \varepsilon, \\ \operatorname{Re} x^*(x) \geqslant 1 - 3\varepsilon, \end{gathered}$$

and

$$\|x + y\| \geqslant \operatorname{Re} \xi_0^*(x + y) = \operatorname{Re} \xi_0^*(z - \eta) + \operatorname{Re} \xi_0^*(x_0 + y) \geqslant 2 - 6\varepsilon.$$

Again, we see that (ii) of Theorem 3.1.11 in the form of Remark 3.1.12 is fulfilled, and hence X has the Daugavet property. ☐

Remark 7.3.3. Observe that, in general, *the Daugavet property is not a 3-space property*. Indeed, taking $X := C[0,1] \oplus_2 C[0,1]$, we have that $Y := C[0,1] \times \{0\}$ has the Daugavet property and $X/Y = C[0,1]$ has the Daugavet property, but X fails the Daugavet property by Corollary 7.5.7 below.

Let X_1 be an M-ideal of a Banach space X and T be a strong Daugavet operator on X. We haven't been able to decide whether the restriction of T to X_1 is a strong Daugavet operator again (see Question (7.2) in Section 7.10). This would give us the operator version of Proposition 7.3.1.

7.4. Strong Daugavet and narrow operators in ℓ_1-sums

We use the same notation concerning restrictions of operators as before, but for an ℓ_1-sum $X = X_1 \oplus_1 X_2$.

Proposition 7.4.1. *If $X = X_1 \oplus_1 X_2$ and $T_i \in \mathcal{SD}(X_i)$ (respectively, $T_i \in \mathcal{SD}^{\mathrm{rid}}(X_i)$) for $i = 1, 2$, then $T \in \mathcal{SD}(X)$ (respectively, $T \in \mathcal{SD}^{\mathrm{rid}}(X)$).*

PROOF. Again, by Lemma 7.1.6 it is sufficient to consider only the "rigid" version of the theorem. Thus, we need to prove that for every $x = (x_1, x_2)$ with $\|x\| = \|x_1\| + \|x_2\| = 1$ and $y = (y_1, y_2)$ with $\|y\| = \|y_1\| + \|y_2\| = 1$, there is some $z = (z_1, z_2)$ with $\|z\| = \|z_1\|+\|z_2\| = 1$ such that $\|x+z\| = \|x_1+z_1\|+\|x_2+z_2\| = 2$ and $\|T(y-z)\| = \|T_1(y_1 - z_1) + T_2(y_2 - z_2)\| = 0$.

For $i = 1, 2$, since $T_i \in \mathcal{SD}^{\mathrm{rid}}(X_i)$, we can produce, using Lemma 7.1.5, some $z_i \in \|y_i\| S_{X_i}$ with $\|x_i + z_i\| = \|x_i\| + \|z_i\|$ and $\|T_i(y_i - z_i)\| = 0$. Now, let us take $z = (z_1, z_2)$; then

$$\|z\| = \|z_1\| + \|z_2\| = \|y_1\| + \|y_2\| = 1,$$
$$\|x + z\| = \|x_1 + z_1\| + \|x_2 + z_2\| = \|x_1\| + \|z_1\| + \|x_2\| + \|z_2\| = 2,$$

and

$$T(y - z) = T_1(y_1 - z_1) + T_2(y_2 - z_2) = 0.$$

So, z satisfies all the conditions above, and the proposition is proved. $\square$

By the same argument as in Corollary 7.2.2 we obtain:

COROLLARY 7.4.2. *If $X = X_1 \oplus_1 X_2$ and $T_i \in \mathcal{NAR}(X_i)$ for $i = 1, 2$, then $T \in \mathcal{NAR}(X)$.*

Recall that a subset $\Gamma \subset S_{X^*}$ is a boundary for X if for every $x \in X$ there is some $x^* \in \Gamma$ such that $|x^*(x)| = \|x\|$. In the case of a balanced boundary Γ, one can find $x^* \in \Gamma$ with $x^*(x) = \|x\|$. The notion of a boundary is a "rigid" version of a one-norming set. It is easy to check (Proposition 2.2.10) that $\Gamma^{\mathfrak{U}}$ is a boundary for $X^{\mathfrak{U}}$ if and only if Γ is one-norming.

LEMMA 7.4.3. *Let $X = X_1 \oplus_1 X_2$, let $\Gamma_j \subset S_{X_j^*}$ be balanced boundaries for X_j for $j = 1, 2$, and let $\Gamma = \Gamma_1 \cup \Gamma_2$. If $T \in \mathcal{NAR}^{\mathrm{rid}}(X, \Gamma)$, then T_1 and T_2, the restrictions of T to X_1 and X_2, are rigid strong Daugavet operators.*

PROOF. Let us consider the case of T_1. We have to prove that for every $x_1, y_1 \in S_{X_1}$ there exists some $u_1 \in S_{X_1}$ such that $\|x_1 + u_1\| = 2$ and $T_1(u_1 - y_1) = 0$.

Let us take $x = (x_1, 0)$, $y = (y_1, 0) \in S_X$ and a functional $x_1^* \in \Gamma_1$ such that $x_1^*(y_1) = 1$. Let us further take $x^* = (x_1^*, 0) \in \Gamma$. Since T is narrow, we can apply Definition 7.1.3 with the elements x, y and x^* defined above; thus, there exists some $z = (z_1, z_2) \in S_X$ such that

$$\|x + z\| = \|x_1 + z_1\| + \|z_2\| = 2$$

and

$$\|T(z - y)\| + |x^*(z - y)| = \|T(z - y)\| + |x_1^*(z_1 - y_1)| = 0. \tag{7.4.1}$$

From the last condition we obtain $|x_1^*(z_1 - y_1)| = 0$. Keeping in mind that $x_1^*(y_1) = 1$, we get $x_1^*(z_1) = 1$. But $\|x_1^*\| = 1$, so $\|z_1\| = 1$. Then

$$\|z_2\| = 0, \tag{7.4.2}$$

because $\|z_1\| + \|z_2\| = 1$. So $\|x_1 + z_1\| = \|x + z\| = 2$ and by (7.4.1) and (7.4.2) $T_1(y_1) = T(y) = T(z) = T_1(z_1)$. Thus the definition of a rigid strong Daugavet operator is fulfilled for T_1. $\square$

We can now prove the converse of Corollary 7.4.2.

THEOREM 7.4.4. *Let $X = X_1 \oplus_1 X_2$ and $T \in \mathcal{NAR}(X)$. Then, T_1 and T_2, the restrictions of T to X_1 and X_2, are narrow operators.*

PROOF. If T is narrow, then so is $T \tilde{+} x^*$ for any $x^* \in X^*$, in particular for $x^* \in \Gamma = X_1^* \cup X_2^*$. By Lemma 7.1.6 we may pass to ultraproducts, apply the previous lemma, pass back to the original space and obtain that $T_1 \tilde{+} x_1^*$ is strongly Daugavet for every $x_1^* \in X_1^*$. Hence T_1 is narrow, and by symmetry, so is T_2. ☐

However, the analogue of Theorem 7.4.4 for strong Daugavet operators, i.e., the converse of Proposition 7.4.1, is false.

PROPOSITION 7.4.5. *Let $X = X_1 \oplus_1 X_2$ and $T \in \mathcal{SD}(X)$. Then T_1, the restriction of T to X_1, need not be a strong Daugavet operator.*

PROOF. The sum functional $Tx = \sum_{n=1}^{\infty} x(n)$ is a strong Daugavet operator on real $\ell_1 = \mathbb{R} \oplus_1 X_2$ (see Example 6.2.13), yet its restriction to $\mathbb{R}$ (i.e., the span of e_1) is not. ☐

We wish to indicate another counterexample that even works on a space with the Daugavet property, namely, on $L_1[0,1]$. For this, let us recall the main features of the example from Theorem 6.6.4. In this example subspaces $Y_1 \subset L_1[0,1]$ and $Y = Y_1 \oplus \operatorname{lin}\{\mathbb{1}\}$ and a measurable subset $P \subset [0,1]$ of measure $\mu(P) < 1/9$ with the following properties are constructed:

$$\|g\mathbb{1}_{[0,1]\setminus P}\| \leqslant 3\|g\mathbb{1}_P\| \qquad \forall g \in Y_1 \tag{7.4.3}$$

and the quotient map $q\colon L_1[0,1] \to L_1[0,1]/Y$ is a strong Daugavet operator.

Now, let $Q \subset [0,1]$, $\mu(Q) < 1/3$, $Q \cap P = \emptyset$. Then the restriction of q to $L_1(Q)$ is bounded from below. So in particular this restriction is not a strong Daugavet operator; observe that $L_1[0,1] = L_1(Q) \oplus_1 L_1([0,1] \setminus Q)$.

Indeed, let us assume to the contrary that the restriction of q to $L_1(Q)$ is unbounded from below. This means that for every $\varepsilon > 0$ there exist a function $f \in L_1(Q)$, a function $g_1 \in Y_1$ and a constant a such that

$$\|f - (g_1 + a)\| < \varepsilon.$$

Denote $[0,1] \setminus (P \cup Q)$ by S; then $\mu(S) > 1/2$. Then $\|(a + g_1)\mathbb{1}_{P\cup S}\| < \varepsilon$ and

$$\begin{aligned} a\mu(P) = \|a\mathbb{1}_P\| \geqslant \|g_1\mathbb{1}_P\| - \varepsilon &\geqslant \frac{1}{3}\|g_1\mathbb{1}_S\| - \varepsilon \qquad \text{(by 7.4.3)} \\ &\geqslant \frac{1}{3}\|a\mathbb{1}_S\| - 2\varepsilon = \frac{1}{3}a\mu(S) - 2\varepsilon, \end{aligned}$$

so $a < 40\varepsilon$. This means that $\|f - g_1\| < 41\varepsilon$. On the other hand

$$\begin{aligned} \|f - g_1\| &\geqslant \|(f - g_1)\mathbb{1}_P\| = \|g_1\mathbb{1}_P\| \\ &\geqslant \frac{1}{3}\|g_1\mathbb{1}_Q\| \geqslant \frac{1}{3}(\|f\| - \|(f - g_1)\mathbb{1}_Q\|) \geqslant \frac{1}{3}(1 - 41\varepsilon), \end{aligned}$$

which is a contradiction when ε is small enough.

Taking into account that $(X \in \mathrm{DPr}) \Leftrightarrow (0 \in \mathcal{NAR}(X))$, Theorem 7.4.4 and Corollary 7.4.2 imply the following stability theorem for the Daugavet property:

COROLLARY 7.4.6. *If $X = X_1 \oplus_1 X_2$, then the following assertions are equivalent:*

(i) *X has the Daugavet property;*

(ii) *both X_1 and X_2 have the Daugavet property.*

7.5. The Daugavet property in general absolute sums

Throughout this section F denotes a Banach space with a 1-unconditional normalised Schauder basis. We can think of the elements of F as sequences with the property that

$$\|(a_1, a_2, \dots)\|_F = \|(|a_1|, |a_2|, \dots)\|_F \qquad \forall (a_j) \in F.$$

Note that F is naturally endowed with the structure of a Banach lattice with respect to the pointwise operations.

Suppose that $X_1, X_2, \dots$ are Banach spaces. Their F-sum $X = (X_1, X_2, \dots)_F$ consists of all sequences (x_j) with $x_j \in X_j$ and $(\|x_j\|) \in F$, equipped with the norm $\|(x_j)\| = \|(\|x_j\|)\|_F$; see Section 2.9.1. We are going to characterise when such an F-sum has the Daugavet property.

Note that F^* can be represented by all sequences $(a_j^*) \in \mathbb{R}^{\mathbb{N}}$ such that

$$\sup_n \big\|(|a_1^*|, \dots, |a_n^*|, 0, 0, \dots)\big\|_{F^*} < \infty,$$

and X^* can be represented by all sequences (x_j^*), $x_j^* \in X_j^*$, such that

$$\|x^*\| = \sup_n \big\|(\|x_1^*\|, \dots, \|x_n^*\|, 0, 0, \dots)\big\|_{F^*} < \infty.$$

The key notion to deal with absolute sums and the Daugavet property is the following one.

Definition 7.5.1. A Banach lattice F is said to have the *positive Daugavet property* if $\|\mathrm{Id} + T\| = 1 + \|T\|$ for every $T\colon F \to F$ of the form $T = a^* \otimes a$ with $a \in F$, $a^* \in F^*$such that $a \geqslant 0$ and $a^* \geqslant 0$.

Remark that the positive Daugavet property may be characterised as is done for the Daugavet property in Theorem 3.1.5 with the same proof.

Lemma 7.5.2. *A Banach lattice has the positive Daugavet property if and only if for every positive $a \in S_F$, every positive $a^* \in S_{F^*}$ and every $\varepsilon > 0$ there is some positive $b \in S_F$ such that $a^*(b) \geqslant 1 - \varepsilon$ and $\|a + b\| \geqslant 2 - \varepsilon$.*

It is clear that c_0 and ℓ_1 have the positive Daugavet property, but there are other examples as well.

Example 7.5.3. The spaces c_0 and ℓ_1 have the positive Daugavet property. Indeed, to show the case of c_0, suppose $a \in c_0$, $a^* \in \ell_1$, both of norm 1 with nonnegative coordinates a_k and a_k^*. Choose N such that $\sup_{k \leqslant N} a_k = 1$ and $\sum_{k>N} a_k^* < \varepsilon$. Let b be the sequence $\mathbb{1}_{\{1,\dots,N\}}$ whose coordinates are $b_k = 1$ for $k \leqslant N$ and $b_k = 0$ otherwise. Then $a^*(b) = \sum_{k=1}^N a_k^* \geqslant 1 - \varepsilon$ and $\|a + b\| = 2$ so that the positive Daugavet property follows from Lemma 7.5.2. The case of ℓ_1 is handled similarly.

Theorem 7.5.4. *Let $X_1, X_2, \dots$ be Banach spaces, and F be a space with a 1-unconditional normalised basis. Then the F-sum $X := (X_1, X_2, \dots)_F$ has the Daugavet property if and only if the Banach lattice F has the positive Daugavet property and every X_n has Daugavet property.*

Proof. Suppose that X has the Daugavet property; we shall verify the condition of Lemma 7.5.2. Let $a = (a_j) \in S_F$ and $a^* = (a_j^*) \in S_{F^*}$ be positive elements and let $\varepsilon > 0$. Pick $x_j \in X_j$ and $x_j^* \in X_j^*$ such that $\|x_j\| = a_j$, $\|x_j^*\| = a_j^*$ and put $x = (x_j)$, $x^* = (x_j^*)$; then $\|x\| = \|x^*\| = 1$. Since X has the Daugavet property, we can find $y \in S_X$ such that $\operatorname{Re} x^*(y) \geqslant 1-\varepsilon$ and $\|x+y\| \geqslant 2-\varepsilon$. Write $y=(y_j)$ and $b=(\|y_j\|)$; then $\|b\|_F=1$ and

$$\begin{aligned}
&1-\varepsilon \leqslant \operatorname{Re} x^*(y)=\sum_{j=1}^{\infty} \operatorname{Re} x_j^*(y_j) \leqslant \sum_{j=1}^{\infty}\|x_j^*\|\|y_j\|=a^*(b), \\
&2-\varepsilon \leqslant \|x+y\|=\big\|(\|x_j+y_j\|)\big\|_F \leqslant \big\|(\|x_j\|+\|y_j\|)\big\|_F \leqslant \|a\|+\|b\|,
\end{aligned}$$

where we have used the fact that the norm of F is monotonic in each variable. Hence, F has the positive Daugavet property.

Now, let us prove that every X_n has the Daugavet property. In order to do so assume that, for some $n \in \mathbb{N}$, the space X_n fails the Daugavet property, and let us prove that X fails the Daugavet property too.

Since X_n fails the Daugavet property we can find $x_n \in S_{X_n}$, $\varepsilon_0>0$ and a slice $\operatorname{Slice}(B_{X_n}, x_n^*, \alpha_0)$ with satisfying that

$$y \in \operatorname{Slice}(B_{X_n}, x_n^*, \alpha_0) \implies \|x_n+y\| \leqslant \|x_n\|+\|y\|-\varepsilon_0. \tag{7.5.1}$$

Consider

$$x:=(0,0,\ldots,0,x_n,0,0,\ldots) \in S_X, \quad x^*:=(0,0,\ldots,0,x_n^*,0,\ldots) \in S_{X^*},$$

and the slice $\operatorname{Slice}(B_X, x^*, \alpha_0)$. Let us prove that $y \in \operatorname{Slice}(B_X, x^*, \alpha_0)$ implies $\|x+y\| \leqslant \|x\|+\|y\|-\varepsilon_0$, which implies that X fails the Daugavet property.

In order to do so, take $y=(y_k) \in \operatorname{Slice}(B_X, x^*, \alpha)$. By the definition of x^* this implies that $\operatorname{Re} x_n^*(y_n)>1-\alpha_0$, so $y_n \in \operatorname{Slice}(B_{X_n}, x_n^*, \alpha_0)$ (observe that $y_n \in B_{X_n}$ since

$$\|y_n\|=\|(0,0,\ldots,0,\|y_n\|,0,\ldots,)\|_F \leqslant \|(\|y_1\|,\ldots,\|y_n\|,\|y_{n+1}\|,\ldots)_F,$$

where the last inequality is justified by the monotonicity of the norm F).

The condition defining $\operatorname{Slice}(B_{X_n}, x_n^*, \alpha)$ implies $\|x_n+y_n\| \leqslant \|x_n\|+\|y_n\|-\varepsilon_0$. Consequently

$$\begin{aligned}
\|x+y\| &= \|(\|y_1\|,\ldots,\|x_n+y_n\|,\|y_{n+1}\|\ldots)\|_F \\
&\leqslant \|(\|y_1\|,\ldots,\|x_n\|+\|y_n\|-\varepsilon_0,\|y_{n+1}\|\ldots)\|_F \\
&= \|(\|y_1\|,\ldots,\|y_n\|,\ldots)+(0,0,\ldots,\|x_n\|-\varepsilon_0,0,\ldots)\|_F \\
&\leqslant \|y\|+\|(0,0,\ldots,\|x_n\|-\varepsilon_0,\ldots)\|_F \\
&= \|y\|+\left\|\left(0,0,\ldots\|x_n\|\left(1-\frac{\varepsilon_0}{\|x_n\|}\right),0,\ldots\right)\right\|_F \\
&\leqslant \|y\|+\|(0,0,\ldots\|x_n\|(1-\varepsilon_0),0,\ldots)\|_F \\
&= \|y\|+(1-\varepsilon_0)\|x\| \leqslant \|y\|+\|x\|-\varepsilon_0,
\end{aligned}$$

as desired.

Conversely, suppose that F has the positive Daugavet property. Let $x=(x_j) \in S_X$ and $x^*=(x_j^*) \in S_{X^*}$, define $a=(a_j)=(\|x_j\|) \in S_F$ and $a^*=(a_j^*)=(\|x_j^*\|) \in S_{F^*}$. Given $\varepsilon>0$, find using Lemma 7.5.2 some $b=(b_j) \in S_F$ such that $a^*(b) \geqslant 1-\varepsilon$ and $\|a+b\| \geqslant 2-\varepsilon$. Since X_j has the Daugavet property, one can find $y_j \in X_j$ such that

$$\|y_j\|=b_j, \quad \operatorname{Re} x_j^*(y_j) \geqslant (1-\varepsilon) a_j^* b_j, \quad \|x_j+y_j\| \geqslant (1-\varepsilon)(a_j+b_j);$$

just note that $\|\mathrm{Id} + (x_j^*/a_j^*) \otimes (x_j/b_j)\| = 1 + a_j/b_j$. Therefore, $y = (y_j) \in S_X$ satisfies

$$\operatorname{Re} x^*(y) = \sum_{j=1}^{\infty} \operatorname{Re} x_j^*(y_j) \geqslant (1-\varepsilon) \sum_{j=1}^{\infty} a_j^* b_j = (1-\varepsilon) a^*(b) \geqslant (1-\varepsilon)^2$$

and

$$\begin{aligned}\|x+y\| &= \big\|(\|x_j + y_j\|)\big\|_F \geqslant (1-\varepsilon)\big\|(\|x_j\| + \|y_j\|)\big\|_F \\ &= (1-\varepsilon)\|a+b\|_F \geqslant 2(1-\varepsilon)(1-2\varepsilon).\end{aligned}$$

Hence, X has the Daugavet property. □

Since c_0 and ℓ_1 have the positive Daugavet property, Theorem 7.5.4 is applicable in particular to $(\bigoplus_{i=1}^{\infty} X_i)_{\ell_1}$ and $(\bigoplus_{i=1}^{\infty} X_i)_{c_0}$ with $X_i \in \mathrm{DPr}$.

COROLLARY 7.5.5. *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of Banach spaces. Then:*

$$\Big(\bigoplus_{k=1}^{\infty} X_k\Big)_{\ell_1} \in \mathrm{DPr} \iff \Big(\bigoplus_{k=1}^{\infty} X_k\Big)_{c_0} \in \mathrm{DPr} \iff (X_k \in \mathrm{DPr} \textit{ for all } k \in \mathbb{N}).$$

Another way to show this is to apply the known fact about the finite sums together with Corollary 4.1.6 which says that for the Daugavet property of X it is sufficient to find an increasing chain of subspaces having the Daugavet property and whose union is dense in X.

If F is finite-dimensional, we can pass to the limit $\varepsilon = 0$ in Lemma 7.5.2 by compactness. Thus, we obtain the following variant of Theorem 7.5.4.

COROLLARY 7.5.6. *Let $\dim F = n$ and $X_1, \dots, X_n$ be Banach spaces. Then their F-sum $(X_1 \oplus \dots \oplus X_n)_F$ has the Daugavet property if and only if every X_i has the Daugavet property and for every positive $a \in S_F$ and every positive $a^* \in S_{F^*}$ there is some $b \in S_F$ such that $a^*(b) = 1$ and $\|a+b\| = 2$.*

This condition can be rephrased geometrically as follows. For any point $a \geqslant 0$ in S_F and any supporting hyperplane $H = \{a^* = 1\}$ of the positive part of the unit sphere, there is a line segment in the unit sphere that contains a and intersects $H \cap S_F$. From this the following corollary is evident.

COROLLARY 7.5.7. *If $X = (X_1 \oplus X_2)_F$ has the Daugavet property, then either $F = \ell_1^2$ or $F = \ell_\infty^2$, i.e., either $X = X_1 \oplus_1 X_2$ or $X = X_1 \oplus_\infty X_2$.*

It is easy to see that $F_1 \oplus_1 F_2$ and $F_1 \oplus_\infty F_2$ have the positive Daugavet property whenever F_1 and F_2 have; in fact, the proof of Theorem 7.5.4 shows that the F-sum $(F_1 \oplus F_2 \oplus \dots)_F$ of Banach lattices with the positive Daugavet property is a Banach lattice with the positive Daugavet property. Therefore, starting from the real line we can form ℓ_1-sums and ℓ_∞-sums consecutively to obtain finite-dimensional spaces with the positive Daugavet property, e.g., the 18-dimensional space

$$\big(\ell_\infty^{(3)} \oplus_1 \ell_\infty^{(4)}\big) \oplus_\infty \big(\ell_1^{(3)} \oplus_1 \ell_\infty^{(3)}\big) \oplus_\infty \ell_1^{(5)}.$$

However, there are other examples, even in the three-dimensional case; for example,

$$\|(a_1, a_2, a_3)\|_F = \max\Big\{|a_1| + \frac{|a_3|}{2}, |a_2| + |a_3|\Big\}$$

defines a norm on $\mathbb{R}^3$ with the positive Daugavet property.

In this example, the unit sphere intersected with the half-space $\{(s,t,u)\colon s,t \in \mathbb{R},\ u \geqslant 0\}$ looks like a hip roof, and the positive part of B_F, i.e., $B_F \cap \mathbb{R}_+^3$, is the convex hull of the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$ and $(1/2,0,1)$, see Figure 7.1. From this description it is easy to see (literally) that this norm has the positive Daugavet property.

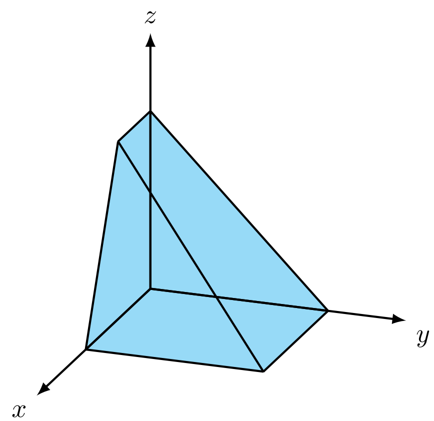

FIGURE 7.1. A three dimensional space with the positive Daugavet property

7.6. The uniform Daugavet property

For an element $x \in S_X$ and $\varepsilon > 0$, denote

$$l(x,\varepsilon) = l_X(x,\varepsilon) = \{y \in S_X\colon \|x+y\| \geqslant 2-\varepsilon\},$$
$$l^+(x,\varepsilon) = l^+_X(x,\varepsilon) = \{y \in X\colon \|y\| \leqslant 1+\varepsilon,\ \|x+y\| > 2-\varepsilon\}.$$

The next result that is important for this section is just (i) $\Leftrightarrow$ (vii) of Theorem 3.1.11 in the form mentioned in Remark 3.1.12. We'll isolate this as a lemma.

LEMMA 7.6.1. *The following assertions are equivalent:*
(1) $X \in \mathrm{DPr}$.
(2) *For every $x \in S_X$ and every $\varepsilon > 0$, $\overline{\mathrm{conv}}(l^+(x,\varepsilon))$ contains B_X.*
(3) *For every $x \in S_X$ and every $\varepsilon > 0$, $\overline{\mathrm{conv}}(l(x,\varepsilon))$ contains B_X.*

Lemma 7.6.1 suggests the following quantitative approach to the Daugavet property. For a subset $A \subset X$ denote by $\mathrm{conv}_n(A)$ the set of all convex combinations of all n-point collections of elements of A. Clearly

$$\mathrm{conv}_1(A) \subset \mathrm{conv}_2(A) \subset \dots, \tag{7.6.1}$$

and $\mathrm{conv}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{conv}_n(A)$. Denote

$$\mathrm{Daug}_n(X,\varepsilon) = \sup_{x,y\in S_X} \mathrm{dist}(y, \mathrm{conv}_n(l(x,\varepsilon))),$$
$$\mathrm{Daug}^+_n(X,\varepsilon) = \sup_{x,y\in S_X} \mathrm{dist}(y, \mathrm{conv}_n(l^+(x,\varepsilon))).$$

By (7.6.1) the sets $\mathrm{conv}_n(l(x,\varepsilon))$ increase with n, so for every $\varepsilon > 0$ the sequence $\big(\mathrm{Daug}_n(X,\varepsilon)\big)_{n\in\mathbb{N}}$ decreases (in the non-strict sense), and consequently it has a limit. The same happens with $\mathrm{conv}_n(l^+(x,\varepsilon))$ and $\left(\mathrm{Daug}_n^+(X,\varepsilon)\right)_{n\in\mathbb{N}}$. Also remark that $\mathrm{conv}_n(l^+(x,\varepsilon)) \supset \mathrm{conv}_n(l(x,\varepsilon))$, consequently

$$\mathrm{Daug}_n^+(X,\varepsilon) \leqslant \mathrm{Daug}_n(X,\varepsilon). \tag{7.6.2}$$

The subtle difference between the definitions with and without the "plus" is inessential (we will see this below), but sometimes one of them is technically more convenient to work with than the other.

Definition 7.6.2. A Banach space is said to have the *uniform Daugavet property* if

$$\lim_{n\to\infty} \mathrm{Daug}_n^+(X,\varepsilon) = 0 \tag{7.6.3}$$

for every $\varepsilon > 0$.

Proposition 7.6.3. *If X possesses the uniform Daugavet property, then $X \in$ DPr.*

Proof. Since $\mathrm{conv}_n(l^+(x,\varepsilon)) \subset \mathrm{conv}(l^+(x,\varepsilon))$ for all n, we have the inequality

$$\mathrm{Daug}_n(X,\varepsilon) \geqslant \sup_{x,y\in S_X} \mathrm{dist}(y, \mathrm{conv}(l^+(x,\varepsilon))).$$

So, the condition (7.6.3) implies that $\mathrm{dist}(y, \mathrm{conv}(l^+(x,\varepsilon))) = 0$ for all $x, y \in S_X$, which, in turn, implies (2) of Lemma 7.6.1. $\square$

Theorem 7.6.4. *Let $\mathfrak{U}$ be a free ultrafilter defined on $\mathbb{N}$, X_n $(n \in \mathbb{N})$ be a collection of Banach spaces and $(X_n)_{\mathfrak{U}}$ be the corresponding ultraproduct of the sequence (X_n). Then the following assertions are equivalent:*

1. *$(X_n)_{\mathfrak{U}}$ possesses the uniform Daugavet property.*
2. *$(X_n)_{\mathfrak{U}} \in$ DPr.*
3. *For every $\varepsilon > 0$ and every $\delta > 0$ there is $n \in \mathbb{N}$ such that the set of all k for which $\mathrm{Daug}_n(X_k,\varepsilon) < \delta$ belongs to the ultrafilter $\mathfrak{U}$.*
4. *For every $\varepsilon > 0$ and every $\delta > 0$ there is $n \in \mathbb{N}$ such that the set of all k for which $\mathrm{Daug}_n^+(X_k,\varepsilon) < \delta$ belongs to the ultrafilter $\mathfrak{U}$.*

Proof. The implication (1) $\Rightarrow$ (2) we have for free from Proposition 7.6.3, and (3) $\Rightarrow$ (4) thanks to (7.6.2).

(2) $\Rightarrow$ (3). Let us argue ad absurdum. Suppose there are $\varepsilon > 0$ and $\delta > 0$ such that $\{k \in \mathbb{N}\colon \mathrm{Daug}_n(X_k,\varepsilon) < \delta\} \notin \mathfrak{U}$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$

$$\{k \in \mathbb{N}\colon \mathrm{Daug}_n(X_k,\varepsilon) \geqslant \delta\} \in \mathfrak{U}.$$

Choose $A_n \in \mathfrak{U}$, $A_1 \supsetneqq A_2 \supsetneqq \ldots$, with $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ in such a way that $A_n \subset \{k \in \mathbb{N}\colon \mathrm{Daug}_n(X_k,\varepsilon) > \delta\}$. Denote $A_0 = \mathbb{N}$. Let us construct two elements $x = [(x_n)_{n\in\mathbb{N}}]$ and $y = [(y_n)_{n\in\mathbb{N}}]$ of $S_{(X_m)_{\mathfrak{U}}}$ in such a way that $x_k, y_k \in S_{X_k}$ and for every $k \in A_n \setminus A_{n-1}$

$$\mathrm{dist}(y_k, \mathrm{conv}_n(l_{X_k}(x_k,\varepsilon)) \geqslant \delta. \tag{7.6.4}$$

The conv_n-hull of a set is increasing when n is increasing, so (7.6.4) remains valid for every $n \in \mathbb{N}$ and every $k \in A_n = \bigcup_{m=n}^{\infty}(A_m \setminus A_{m+1})$. This means in turn that for every $n \in \mathbb{N}$, $\mathrm{dist}(y, \mathrm{conv}_n(l_{(X_m)_{\mathfrak{U}}}(x,\varepsilon))) \geqslant \delta$, so $\mathrm{dist}(y, \mathrm{conv}(l_{(X_m)_{\mathfrak{U}}}(x,\varepsilon))) \geqslant \delta$, which contradicts the Daugavet property of $(X_n)_{\mathfrak{U}}$.

(4) ⇒ (1). We know the existence of $n \in \mathbb{N}$ such that the set $A_n := \{k \in \mathbb{N}\colon \operatorname{Daug}_n^+(X_k, \varepsilon) < \delta\}$ belongs to the ultrafilter $\mathfrak{U}$. Consider arbitrary $[(x_n)], [(y_n)] \in (X_n)_{\mathfrak{U}}$, $x_j, y_j \in S_{X_j}$. For every $k \in A_n$ we have

$$\operatorname{dist}(y_k, \operatorname{conv}_n(l^+_{X_k}(x_k, \varepsilon)) < \delta,$$

so there is $z_k \in \operatorname{conv}_n(l^+_{X_k}(x_k, \varepsilon)$ with $\|z_k - y_k\| < \delta$. Write z_k in the form

$$z_k = \sum_{j=1}^{n} \lambda_{k,j} z_{k,j}, \ \|z_{k,j}\| \leqslant 1 + \varepsilon, \ \|z_{k,j} + x_k\| > 2 - \varepsilon. \tag{7.6.5}$$

For each $j \in \overline{1,n}$ denote by λ_j the $\mathfrak{U}|_{A_n}$-limit of the sequence $(\lambda_{k,j})_{k \in A_n}$ (the existence of $\mathfrak{U}$-limits follows from boundedness of the corresponding numerical sequences, Theorem 2.2.4). Consider $\tilde{z}_k := \sum_{j=1}^{n} \lambda_j z_{k,j}$. By construction,

$$\lim_{\mathfrak{U}|_{A_n}} \|z_k - \tilde{z}_k\| = 0,$$

so there is $B \in \mathfrak{U}$, $B \subset A_n$, such that $\|z_k - \tilde{z}_k\| < \delta$ for all $k \in B$. So, for all $k \in B$ we have $\|y_k - \tilde{z}_k\| < 2\delta$. Finally, for each $j \in \overline{1,n}$ the element $[(\tilde{z}_{k,j})] \in (X_n)_{\mathfrak{U}}$ is well-defined (it is sufficient to define the coordinates $z_{k,j}$ only for k from some element of the ultrafilter), $[(\tilde{z}_{k,j})] \in l^+_{(X_m)_{\mathfrak{U}}}([(x_k)], \varepsilon)$,

$$[(\tilde{z}_k)] = \sum_{j=1}^{n} \lambda_j [(z_{k,j})] \in \operatorname{conv}_n(l^+_{(X_m)_{\mathfrak{U}}}([(x_k)], \varepsilon)),$$

and $\|[(\tilde{z}_k)] - [(y_k)]\| < 2\delta$. In other words, for this n we have $\operatorname{Daug}_n^+((X_m)_{\mathfrak{U}}, \varepsilon) \leqslant 2\delta$. By monotonicity of $\operatorname{Daug}_n((X_m)_{\mathfrak{U}}, \varepsilon)$ in n, this means that $\operatorname{Daug}_n^+((X_m)_{\mathfrak{U}}, \varepsilon)$ tends to 0 when n tends to infinity, which proves the uniform Daugavet property for $(X_m)_{\mathfrak{U}}$. □

The next proposition is the main reason for considering, parallel to $l(x, \varepsilon)$ and $\operatorname{Daug}_n(X, \varepsilon)$, their more bulky versions $l^+(x, \varepsilon)$ and $\operatorname{Daug}_n^+(X, \varepsilon)$.

Proposition 7.6.5. *If $\lim_{n\to\infty} \operatorname{Daug}_n^+(X, \varepsilon) = 0$ for every $\varepsilon > 0$, then for every $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that $\operatorname{Daug}_n^+(X, \varepsilon) = 0$. More explicitly: If $\operatorname{Daug}_n^+(X, \varepsilon/2) < \varepsilon/2$, then $\operatorname{Daug}_n^+(X, \varepsilon) = 0$. Moreover, for every pair $x, y \in S_X$ not just $\operatorname{dist}(y, \operatorname{conv}_n(l^+(x, \varepsilon))) = 0$, but $y \in \operatorname{conv}_n(l^+(x, \varepsilon))$.*

Proof. Suppose $\operatorname{Daug}_n^+(X, \varepsilon/2) < \varepsilon/2$. Fix $x, y \in S_X$. There exist $y_1, \dots, y_n \in (1 + \varepsilon/2)B_X$, $\|x + y_n\| > 2 - \varepsilon/2$, and $a_1, \dots, a_n \geqslant 0, \sum_{k=1}^{n} a_k = 1$, for which $\|y - \sum_{k=1}^{n} a_k y_k\| < \varepsilon/2$. Define elements $z_j = y_j + y - \sum_{k=1}^{n} a_k y_k$. Then $z_j \in l^+(x, \varepsilon)$, $\sum_{j=1}^{n} a_j z_j = y$, so $y \in \operatorname{conv}_n(l^+(x, \varepsilon))$. □

So instead of $\operatorname{Daug}_n^+(X, \varepsilon)$ it is reasonable to consider the following notion, which seems to be a bit more convenient (at least it depends only on one parameter):

$$D_X(\varepsilon) = \inf\{n\colon \operatorname{conv}_n(l^+(x, \varepsilon)) \supset S_X \ \forall x \in S_X\}.$$

Corollary 7.6.6. *A Banach space X has the uniform Daugavet property if and only if $D_X(\varepsilon)$ is finite for every $\varepsilon > 0$.*

In this notation Theorem 7.6.4 can be reformulated in the following way.

Theorem 7.6.7. *Let $\mathfrak{U}$ be a free ultrafilter on $\mathbb{N}$, and (X_n) be a collection of Banach spaces. Then the following assertions are equivalent:*

(1) $(X_n)_{\mathfrak{U}}$ *possesses the uniform Daugavet property.*
(2) $(X_n)_{\mathfrak{U}} \in \mathrm{DPr}$.
(3) *For every* $\varepsilon > 0$ *there exists some* n *such that the set of all* k *for which* $D_{X_k}(\varepsilon) < n$ *belongs to the ultrafilter* $\mathfrak{U}$.

Applying all the above to the case when all X_n are one and the same space X, we obtain the central result of the section.

Theorem 7.6.8. *For a Banach space* X *the following assertions are equivalent:*
(1) $X^{\mathfrak{U}}$ *possesses the uniform Daugavet property.*
(2) $X^{\mathfrak{U}} \in \mathrm{DPr}$.
(3) *For every* $\varepsilon > 0$ *and every* $\delta > 0$ *there is* $n \in \mathbb{N}$ *such that* $\mathrm{Daug}_n^+(X, \varepsilon) < \delta$, *that is,* X *has the uniform Daugavet property.*
(4) *For every* $\varepsilon > 0$ *and every* $\delta > 0$ *there is* $n \in \mathbb{N}$ *such that* $\mathrm{Daug}_n(X, \varepsilon) < \delta$, *that is,* $\lim_{n\to\infty} \mathrm{Daug}_n(X, \varepsilon) = 0$ *for every* $\varepsilon > 0$.

Corollary 7.6.9. *X has the uniform Daugavet property if and only if* $\lim_{n\to\infty} \mathrm{Daug}_n(X, \varepsilon) = 0$ *for every* $\varepsilon > 0$.

Corollary 7.6.10. *X has the uniform Daugavet property if and only if $X^{\mathfrak{U}}$ has the Daugavet property.*

It follows from the canonical isometric isomorphism $(X \oplus_\infty Y)^{\mathfrak{U}} = X^{\mathfrak{U}} \oplus_\infty Y^{\mathfrak{U}}$ and Theorem 7.6.8 that the uniform Daugavet property is stable by taking ℓ_∞-direct sums and likewise by taking ℓ_1-direct sums.

Let us prove that the basic examples of spaces with the Daugavet property in fact are spaces with the uniform Daugavet property.

Lemma 7.6.11. *Let* $X = L_1[0,1]$. *If* $n > 2/\varepsilon$, *then* $\mathrm{Daug}_n^+(X, \varepsilon) = 0$; *if* $n \leqslant 2/\varepsilon$, *then* $\mathrm{Daug}_n^+(X, \varepsilon) \leqslant 1 - \varepsilon n/(2+\varepsilon)$. *Hence* $D_X(\varepsilon)$ *is of order* ε^{-1}.

Proof. Suppose $n > 2/\varepsilon$ and let us take arbitrary points x and y from S_X. There is a partitioning of $[0,1]$ into sets $E_1, \dots, E_n$ such that $\|x \cdot \mathbb{1}_{E_i}\| = 1/n < \varepsilon/2$. Define functions y_i by $y_i = \frac{1}{\|y\cdot\mathbb{1}_{E_i}\|} y \cdot \mathbb{1}_{E_i}$ if $\|y \cdot \mathbb{1}_{E_i}\| \neq 0$, and $y_i = 0$ if $\|y \cdot \mathbb{1}_{E_i}\| = 0$. Then $\sum_{i=1}^n y_i \lambda_i = y$, where $\lambda_i = \|y \cdot \mathbb{1}_{E_i}\|$. On the other hand, if $y_i \neq 0$, then

$$\|x + y_i\| \geqslant \|x \cdot \mathbb{1}_{[0,1]\setminus E_i}\| + \|y_i\| - \|x \cdot \mathbb{1}_{E_i}\| \geqslant 2 - 2\|x \cdot \mathbb{1}_{E_i}\| > 2 - \varepsilon.$$

So, $y_i \in l^+(x, \varepsilon)$.

If $n \leqslant 2/\varepsilon$, then proceeding as above, with $N = [2/\varepsilon]+1$ we get a decomposition $E_1, \dots, E_N$. Let us arrange the λ_i's in decreasing order and take the first n of them. Then

$$\left\|\sum_{i=1}^n y_i\lambda_i - y\right\| = \left\|\sum_{i=n+1}^N y_i\lambda_i\right\| \leqslant \sum_{i=n+1}^N \lambda_i = S.$$

We need to prove that $S \leqslant (N-n)/N$. Assume the opposite. Then

$$1 = \sum_{i=1}^N \lambda_i > \sum_{i=1}^n \lambda_i + \frac{N-n}{N};$$

hence $n/N > \sum_{i=1}^n \lambda_i \geqslant n\lambda_n$ and $1/N > \lambda_n$. Thus,

$$S = \sum_{i=n+1}^N \lambda_i \leqslant \lambda_n(N-n) < \frac{N-n}{N},$$

which is a contradiction. So,

$$S \leqslant \frac{N-n}{N} = 1 - \frac{n}{[\frac{2}{\varepsilon}]+1} \leqslant 1 - \frac{\varepsilon n}{2+\varepsilon}$$

and the proof of the lemma is finished. □

LEMMA 7.6.12. *If $X = C(K)$ for a compact Hausdorff space K without isolated points, then for every ε and n, $\operatorname{Daug}_n(X,\varepsilon) \leqslant 2/n$. Hence $D_X(\varepsilon)$ is of order ε^{-1}.*

PROOF. Let x and $y \in S_X$ be arbitrary. Without loss of generality, assume that x attains the value 1. Take an open neighbourhood U such that $x(u) > 1-\varepsilon$ for all $u \in U$. Now, pick n disjoint subneighbourhoods $V_1,\dots,V_n$ inside U. For each of them choose a positive function φ_i supported on V_i such that $\|\varphi_i\| \leqslant 2$, $\|y+\varphi_i\| \leqslant 1$ and $y+\varphi_i$ attains the value 1 in V_i. Obviously, $\|x+y+\varphi_i\| > 2-\varepsilon$, hence, $y+\varphi_i \in l^+(x,\varepsilon)$. On the other hand,

$$\left\|\frac{1}{n}\sum_{i=1}^{n}(y+\varphi_i) - y\right\| = \left\|\frac{1}{n}\sum_{i=1}^{n}\varphi_i\right\| \leqslant \frac{2}{n},$$

which proves the lemma. □

One can show that the same estimates for the Daug_n constants are valid for rich subspaces of $C(K)$-spaces (see Section 8.4 for the detailed description of rich subspaces in $C(K)$), for vector-valued $C(K)$- or L_1-spaces and for spaces of weakly continuous vector-valued functions with the sup-norm, which will be mentioned in Section 8.2.

REMARK 7.6.13. *For every Banach space X, the constants $D_X(\varepsilon)$ can be estimated from below by $(2+2\varepsilon)/(3\varepsilon)$, which is bigger than $2/(3\varepsilon)$. So the estimates from above which we have for L_1 and C are of optimal order.*

PROOF. Suppose $D_X(\varepsilon) = n < (2+2\varepsilon)/(3\varepsilon)$ for some $\varepsilon > 0$. This means in particular that for a fixed element $x \in S_X$ (taking $y = -x$) there are elements $y_1,\dots,y_n \in (1+\varepsilon)B_X$, $\|x+y_n\| > 2-\varepsilon$ and $a_1,\dots,a_n \geqslant 0$, $\sum_{k=1}^n a_k = 1$, for which $\sum_{k=1}^n a_k y_k = -x$. Without loss of generality we may assume that $a_1 \geqslant 1/n$ (otherwise just change the enumeration). Plugging in $\|x+y_1\| > 2-\varepsilon$ and $x = -\sum_{k=1}^n a_k y_k$, we obtain

$$\begin{aligned} 2-\varepsilon < \|x+y_1\| &= \left\|y_1(1-a_1) - \sum_{k=2}^{n} a_k y_k\right\| \\ &\leqslant (1+\varepsilon)(1-a_1) + (1+\varepsilon)(1-a_1) \\ &\leqslant 2(1+\varepsilon)(1-1/n) \leqslant 2-\varepsilon, \end{aligned}$$

which is a contradiction. □

Let us present other families of spaces for which the Daugavet property and the uniform Daugavet property are equivalent: isometric L_1-predual spaces, C^*-algebras, and uniform algebras. Contrary to the two examples above, the way of proving the result will be to show that, for these families, the Daugavet property passes to ultraproducts and the result will follow from Corollary 7.6.10.

PROPOSITION 7.6.14. *Let (X_n) be a sequence of Banach spaces with the Daugavet property. Each of the following conditions implies that $(X_n)_{\mathfrak{U}}$ has the Daugavet property:*

(a) *the spaces X_n are isometric L_1-predual spaces,*
(b) *the spaces X_n are C^*-algebras,*
(c) *the spaces X_n are uniform algebras.*

As a consequence, the Daugavet property and the uniform Daugavet property are equivalent for isometric L_1-predual spaces, C^*-algebras, and uniform algebras.

Proof. Having the Daugavet property, all the spaces X_n are extremely rough, see Corollary 3.1.13, so it follows from [**44**, Lemma 5.1] that so is $(X_n)_{\mathfrak{U}}$. In particular, there is no point of Fréchet differentiability of the norm in $(X_n)_{\mathfrak{U}}$. From now on, we separate the proofs of the three cases, which are indeed completely analogous.

(a) $(X_n)_{\mathfrak{U}}$ is an isometric predual of L_1 by [**135**, Proposition 2.1], and then it has the Daugavet property at the moment that its norm has no point of Fréchet differentiability, Corollary 3.5.20.

(b) $(X_n)_{\mathfrak{U}}$ is a C^*-algebra (this result is immediate by using the definition of a C^*-algebra, see [**134**, Proposition 3.1] for instance), and then it has the Daugavet property at the moment that its norm has no point of Fréchet differentiability, Theorem 3.5.6.

(c) $(X_n)_{\mathfrak{U}}$ is a uniform algebra. This result follows since a commutative complex Banach algebra A is a uniform algebra if and only if

(P) $\|a^2\| = \|a\|^2$ for every $a \in A$.

(This seems to be a very well-known fact to experts which is actually the definition in [**55**].) We present a short argument. First, that every uniform algebra satisfies (P) is clear. Conversely, consider the Gelfand transform $\Gamma\colon A \to C(\Delta(A))$ where $\Delta(A)$ is the spectrum or Gelfand space of A, which is compact since A is unital; then $\Gamma(A)$ is a closed subalgebra of $C(\Delta(A))$ which contains the constant functions and strongly separates the points of $\Delta(A)$; the property (P) implies that Γ is an isometry (see [**186**, Theorem 2.2.7] for all these facts).

Now, Corollary 3.5.21 shows that $(X_n)_{\mathfrak{U}}$ has the Daugavet property.

Finally, the last part of the proposition follows from Corollary 7.6.10. $\square$

7.7. The Bourgain-Rosenthal space

Recall that a Banach space has the Schur property if every weakly convergent sequence converges in norm; cf. Definition 2.5.8. It has been asked in [**158**] and [**300**], reiterating a question asked to one of us by A. Pełczyński, whether there exists a Banach space that has both the Schur and the Daugavet property. Given the isomorphic properties the Daugavet property entails, see Chapter 3, the two properties might appear to be mutually exclusive since a space with the Daugavet property should be thought of as rather large whereas a Schur space, which is hereditarily ℓ_1, could be imagined as rather thin. In the other direction, the examples of spaces with the Daugavet property listed so far support the conjecture that a space with the Daugavet property always contains a copy of ℓ_2 which for sure is impossible for a Schur space.

In this section we show that a certain subspace of L_1 that was constructed by Bourgain and Rosenthal [**64**] indeed has both the Schur and the Daugavet property. (Actually, minor modifications have to be implemented.) We start with an exposition of the Bourgain-Rosenthal construction in the form taken from Benyamini and Lindenstrauss's book [**51**], pointing out the modifications we need.

Afterwards, we show that direct sums of certain Bourgain-Rosenthal spaces serve as examples of spaces with the Daugavet property not having the uniform Daugavet property. With this we see that the Daugavet property in general does not pass to ultraproducts.

Below we consider the *real* space $L_1 = L_1(\Omega, \Sigma, \mu)$ over a countably generated nonatomic probability space. The symbol $\| \, . \, \|$ will refer to the L_1-norm. Besides the norm topology we will also consider the *topology of convergence in measure*, generated by the metric

$$d(f,g) = \inf\bigl\{\varepsilon > 0\colon \mu\{t\colon |f(t) - g(t)| \geqslant \varepsilon\} \leqslant \varepsilon\bigr\}.$$

Since all such (Ω, Σ, μ) are pairwise isomorphic, and the corresponding L_1-spaces are isometric to $L_1[0,1]$, we may choose to switch from one (Ω, Σ, μ) to another if it is convenient at that moment.

DEFINITION 7.7.1. A subset $D \subset L_1$ is said to be *uniformly integrable* if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for every $g \in D$ and $A \in \Sigma$ with $\mu(A) < \delta$

$$\int_A |g| \, d\mu < \varepsilon.$$

Uniform integrability of $D \subset L_1$ is equivalent to weak compactness of its weak closure (see [**98**] for an introduction to the subject).

We first observe a simple lemma on ℓ_1-orthogonality that will be used later. In particular, it applies to bounded subsets of finite-dimensional, indeed reflexive, subspaces of L_1.

LEMMA 7.7.2. *Let H be a uniformly integrable subset of L_1 and $\varepsilon > 0$. Then, there is $\delta > 0$ such that for every $g \in H$ and every $f \in L_1$ with $d(f,0) < \delta$ the following inequality holds:*

$$\|f + g\| \geqslant \|f\| + \|g\| - \varepsilon.$$

PROOF. Using the uniform integrability of H one can find $\delta > 0$ such that

$$2 \int_A |g| \, d\mu + 2\delta < \varepsilon$$

for every $g \in H$ and every measurable subset $A \subset \Omega$ with $\mu(A) < \delta$. Now, fix $f \in L_1$ with $d(f,0) < \delta$ and denote $A = \{t\colon |f(t)| \geqslant \delta\}$. Then for every $g \in H$ we have

$$\begin{aligned}
\|f + g\| &= \int_A |f + g| \, d\mu + \int_{\Omega \setminus A} |f + g| \, d\mu \\
&\geqslant \int_A |f| \, d\mu + \int_{\Omega \setminus A} |g| \, d\mu - \int_A |g| \, d\mu - \int_{\Omega \setminus A} |f| \, d\mu \\
&\geqslant \|f\| + \|g\| - 2 \int_A |g| \, d\mu - 2\delta \geqslant \|f\| + \|g\| - \varepsilon,
\end{aligned}$$

as requested. □

COROLLARY 7.7.3. *Let (f_n) be a sequence in L_1 that is convergent to 0 in measure and such that $\|f_n\| = 1$. Then (f_n) contains an ℓ_1-type subsequence. So, in particular, (f_n) contains a subsequence equivalent to the canonical basis of ℓ_1.*

Proof. In order to get an an ℓ_1-type subsequence just apply Lemma 2.8.3 to the set $A = \{f_n \colon n \in \mathbb{N}\}$. Then Lemma 2.8.9 gives a further subsequence equivalent to the canonical basis of ℓ_1. $\square$

Corollary 7.7.4. *Let $E \subset L_1$ be a subspace such that B_E is precompact for the metric d of convergence in measure in L_1. Then E has the Schur property.*

Proof. Assume that E does not have the Schur property. Then there is a weakly null sequence (f_n) in E such that $\|f_n\| = 1$ for all n. Using the d-precompactness of B_E we may find a subsequence $(g_n) \subset (f_n)$ that converges in measure (to 0 because the weak limit is 0). Then Corollary 7.7.3 gives us a further subsequence $(h_n) \subset (g_n)$ equivalent to the canonical basis of ℓ_1, but this is impossible because the canonical basis of ℓ_1 does not converge weakly. $\square$

We now quote a result from [**51**] that contains the key to the Bourgain-Rosenthal construction.

Lemma 7.7.5 ([**51**, Lemma 5.26]). *Let $0 < \varepsilon < 1$. Then there is a function $f \in L_1[0,1]$ with the following properties:*

(a) *$f \geqslant 0$, $\|f\| = 1$ and $\|f - \mathbb{1}\| \geqslant 2 - \varepsilon$.*

(b) *Let $(f_j)_{j=1}^{\infty} \subset L_1[0,1]$ be a sequence of independent random variables with the same distribution as f. Then for every $g \in \mathrm{lin}\{f_j \colon j \in \mathbb{N}\}$ with $\|g\| \leqslant 1$, there is a constant function c with $d(g,c) \leqslant \varepsilon$.*

(c) *$\|n^{-1}\sum_{j=1}^{n} f_j - \mathbb{1}\| \to 0$ as $n \to \infty$.*

Remark 7.7.6. *Property (a) of the previous lemma implies in particular that $d(f,0) \leqslant \sqrt{\varepsilon}$.* Indeed, for $A = \{t \colon f(t) \geqslant \sqrt{\varepsilon}\}$ we have (λ denotes the Lebesgue measure)

$$\int_A |f(t) - 1|\, dt \leqslant \int_A (f(t) + 1)\, dt - \sqrt{\varepsilon}\lambda(A)$$

since $(a+1) - |a-1| = 2\min\{a,1\} \geqslant \sqrt{\varepsilon}$ for $a \geqslant \sqrt{\varepsilon}$; so

$$2 - \varepsilon \leqslant \|f - \mathbb{1}\| \leqslant 2 - \sqrt{\varepsilon}\lambda(A)$$

and hence $\lambda(A) \leqslant \sqrt{\varepsilon}$.

In the lemma and in the construction below, (Ω, Σ, μ) will be the product of countably many copies of the probability space $[0,1]$. A subspace of $L_1 = L_1(\Omega, \Sigma, \mu)$ is said to *depend on finitely many coordinates* if all the elements of the subspace are functions depending only on a common finite set of coordinates.

The next lemma is a refinement of [**51**, Lemma 5.27]; the difference is that the latter lemma claims (a) only for $u = u_k$.

Lemma 7.7.7. *Let G be a finite-dimensional subspace of L_1 that depends on finitely many coordinates. Let $\{u_1, \dots, u_m\} \subset S_G$ and $\varepsilon > 0$. Then there is a finite-dimensional subspace $F \subset L_1$, also depending on finitely many coordinates and containing G, and there are an integer n and functions $(v_{k,j})_{k \leqslant m, j \leqslant n} \subset S_F$ such that:*

(a) *$\|u + v_{k,j}\| \geqslant 2 - \varepsilon$ for every $u \in S_G$ and all k and j.*

(b) *$\|u_k - n^{-1}\sum_{j=1}^{n} v_{k,j}\| \leqslant \varepsilon$ for all k.*

(c) *For every $\varphi \in B_F$ there is $\psi \in B_G$ with $d(\varphi, \psi) \leqslant \varepsilon$.*

PROOF. We shall first recall the construction of the proof in [**51**, Lemma 5.27] and then point out the necessary changes for our proof.

Let $\varepsilon_1 > 0$ be small enough, f be the function given by Lemma 7.7.5 for this ε_1, and n be such that $\|n^{-1}\sum_{j=1}^n f_j - \mathbb{1}\| \leqslant \varepsilon_1$ for f_j as in Lemma 7.7.5. (It will become apparent at the end of the proof how small ε_1 should be chosen.) Let G depend on the first N coordinates of Ω. For every $k \leqslant m$ choose $(f_{k,j})_{j\leqslant n}$ which depend on the $(N+k)$-th coordinate of Ω and are equidistributed with $(f_j)_{j\leqslant n}$. Put $v_{k,j} = f_{k,j}u_k$, and let F be the span of G and of $\{v_{k,j}\colon k \leqslant m, j \leqslant n\}$. Note that $\|f_{k,j}u_k\| = 1$ since the two functions are stochastically independent.

The properties (b) and (c) are the same as in Lemma 5.27 from [**51**] (cf. [**51**, p. 118] for the norm-one part of (c)), so we are not going to repeat their proofs here. We only have to deal with property (a). According to Remark 7.7.6, $d(f_{k,j}, 0) \leqslant \sqrt{\varepsilon_1}$. Denote

$$A_{k,j} = \{t\colon f_{k,j}(t) \geqslant \sqrt{\varepsilon_1}\}, \quad v^1_{k,j} = f_{k,j}u_k\mathbb{1}_{A_{k,j}}, \quad v^2_{k,j} = f_{k,j}u_k\mathbb{1}_{\Omega\setminus A_{k,j}}.$$

Then

$$v_{k,j} = v^1_{k,j} + v^2_{k,j},$$

where the first summand has a small support, viz.

$$\mu(\operatorname{supp} v^1_{k,j}) \leqslant \mu(A_{k,j}) \leqslant \sqrt{\varepsilon_1},$$

and the second summand has a small norm, namely

$$\|v^2_{k,j}\| \leqslant \int_{\Omega\setminus A_{k,j}} \sqrt{\varepsilon_1}|u_k|\,d\mu \leqslant \sqrt{\varepsilon_1}.$$

So for every $u \in S_G$

$$\|u + v_{k,j}\| \geqslant \|u + v^1_{k,j}\| - \sqrt{\varepsilon_1},$$

and $d(v^1_{k,j}, 0) \leqslant \sqrt{\varepsilon_1}$. To finish the proof it is enough to apply Lemma 7.7.2. $\square$

We now turn to the actual construction of the example. Fix a decreasing sequence $(\varepsilon_j) \subset (0,1)$ with $\sum_{j=N+1}^\infty \varepsilon_j < \varepsilon_N$ for all $N \in \mathbb{N}$ and select inductively finite-dimensional subspaces of L_1

$$\operatorname{lin}\{\mathbb{1}\} = E_1 \subset E_2 \subset E_3 \subset \dots,$$

each of them depending on finitely many coordinates, ε_N-nets $(u^N_k)_{k=1}^{m(N)}$ of S_{E_N}, and collections of elements $(v^N_{k,j})_{k\leqslant m(N),\, j\leqslant n(N)} \subset S_{E_{N+1}}$ in such a way that the conclusion of Lemma 7.7.7 holds with $\varepsilon = \varepsilon_N$, $G = E_N$, $F = E_{N+1}$,

$$(u_k)_{k=1}^m = (u^N_k)_{k=1}^{m(N)}, \quad \text{and} \quad (v_{k,j})_{k\leqslant m, j\leqslant n} = (v^N_{k,j})_{k\leqslant m(N), j\leqslant n(N)}.$$

Denote $E = \overline{\bigcup_{N=1}^\infty E_N}$.

THEOREM 7.7.8. *The space E constructed above has the following properties:*

(a) *E has the Daugavet property.*
(b) *For every $f \in B_E$ and for every $N \in \mathbb{N}$ there exists $g \in B_{E_N}$ such that $d(f,g) < \varepsilon_N$.*
(c) *E has the Schur property.*

PROOF. (a) According to Lemma 7.6.1 we need to show that for every $u \in S_E$ and $\varepsilon > 0$ the set $\operatorname{conv}(l^+(u,\varepsilon))$ is dense in B_E. By a small perturbation argument (as in Proposition 4.1.1), it is enough to check this condition only for u from the dense subset $S_{\bigcup_{N=1}^{\infty} E_N}$ of S_E.

Fix $N \in \mathbb{N}$, $u \in S_{E_N}$ and $\varepsilon > 0$. There is $M > N$ such that $\varepsilon_M < \varepsilon$. By construction (property (a) of Lemma 7.7.7) all the elements $v^L_{k,j}$ with $L > M$ belong to $l^+(u,\varepsilon)$. Taking into account property (b) of Lemma 7.7.7 and the fact that $\{u^L_k\colon k = 1,\dots,m(L)\}$ forms an ε_L-net of S_{E_L}, one can easily establish the density of $\operatorname{conv}(l^+(u,\varepsilon))$ in B_E.

(b) Fix $f \in B_{\bigcup_{J=1}^{\infty} E_J}$ and $N \in \mathbb{N}$. Then $f \in B_{E_{N+L}}$ for some L. Applying property (c) of Lemma 7.7.7 to f we find $f_1 \in S_{E_{N+L-1}}$ with $d(f,f_1) \leqslant \varepsilon_{N+L}$. Applying again property (c) of Lemma 7.7.7 to f_1 we find $f_2 \in S_{E_{N+L-2}}$ with $d(f_1,f_2) \leqslant \varepsilon_{N+L-1}$. Continuing in this fashion we obtain in the L-th step some $g = f_L \in S_{E_N}$ for which

$$d(f,g) \leqslant \varepsilon_{N+L} + \varepsilon_{N+L-1} + \cdots + \varepsilon_{N+1} < \varepsilon_N.$$

(c) It follows from (b) that the unit ball of E is a precompact in the metric d of convergence in measure. This, according to Corollary 7.7.4, implies the Schur property. $\square$

We observe that the space E cannot be a rich subspace of L_1. Indeed, the unit ball of E is precompact in the metric of convergence in measure, B_{L_1} is d-complete, hence the d-closure C_E of B_E is d-compact. But if E were rich, by the forthcoming Proposition 8.7.1, C_E would contain $\frac{1}{2}B_{L_1}$, and B_{L_1} would be d-compact as well, which is clearly false. This remark reveals that E is an essentially new specimen among the spaces with the Daugavet property.

Since a Banach space with the Schur property cannot contain infinite-dimensional reflexive subspaces, the following corollary holds.

COROLLARY 7.7.9. *There exists a Banach space with the Daugavet property that fails to contain infinite-dimensional reflexive subspaces; in particular it fails to contain a copy of* ℓ_2.

It follows as well that the space E from Theorem 7.7.8 does not contain a copy of L_1; the first space with the Daugavet property having this feature was constructed in [**178**] after an example given by Talagrand.

One can likewise express Corollary 7.7.9 in terms of narrow operators. Namely, the identity operator on E is an operator that does not fix a copy of ℓ_2, yet it is not narrow.

Schmidt [**283**] proved that every *Dunford-Pettis operator* T on L_1 (i.e., T maps weakly convergent sequences to norm convergent sequences) satisfies the Daugavet equation $\|\mathrm{Id} + T\| = 1 + \|T\|$. In fact, such an operator is easily seen to be narrow on L_1. However, on the Schur space E above, $-\mathrm{Id}$ is Dunford-Pettis, but it clearly fails the Daugavet equation. Therefore we have:

COROLLARY 7.7.10. *There is a Banach space with the Daugavet property and there is a Dunford-Pettis operator on that space which fails the Daugavet equation. Hence, Dunford-Pettis operators are in general not narrow.*

The Bourgain-Rosenthal spaces were constructed in order to provide an example of a Banach space which fails the Radon-Nikodým property (i.e., some uniformly bounded martingale diverges), yet every uniformly bounded dyadic martingale converges. Using more horticultural language, one can express this by saying that the unit ball contains some η-bush, but no η-trees (see [**51**, Chapter 5] for these concepts) or indeed no η-bushes with a fixed number of branches at each branching node.

This is reminiscent of the Daugavet property and its uniform variant. The Daugavet property means that every y of norm 1 is almost a convex combination of vectors from $l(x, \varepsilon)$ for any given x of norm 1; this enables one to find an η-bush for any $\eta < 2$. By contrast, the uniform Daugavet property is related to finding such bushes with a fixed number of branches at each level.

Precisely, we shall now prove the following theorem.

THEOREM 7.7.11.

(a) *For every $n \in \mathbb{N}$ there is a Banach space X_n with the Daugavet property such that*

$$\operatorname{Daug}_n\Big(X_n, \frac{1}{4}\Big) \geqslant \frac{1}{2}. \tag{7.7.1}$$

(b) *There is a Banach space X which has the Daugavet property, but does not have the uniform Daugavet property.*

(c) *An ultrapower of a space with the Daugavet property does not necessarily possess the Daugavet property.*

PROOF. (a) We take as X_n the space E from Theorem 7.7.8 with parameters $(\varepsilon_j)_{j\in\mathbb{N}}$, where ε_1 is selected in such a way that for every constant function $g \in [-2, 2]$ and every $f \in L_1$ with $d(f, 0) < n\varepsilon_1$, the inequality

$$\|g + f\| \geqslant \|g\| + \|f\| - \frac{1}{4} \tag{7.7.2}$$

holds (see Lemma 7.7.2). To prove (7.7.1), let us check that

$$\operatorname{dist}\big(\operatorname{conv}_n(l(x, \tfrac{1}{4})), y\big) \geqslant \frac{1}{2} \tag{7.7.3}$$

for the constant functions $x = -\mathbb{1}$ and $y = \mathbb{1}$.

Consider an arbitrary element $z \in \operatorname{conv}_n\big(l(x, \frac{1}{4})\big)$, $z = \sum_{k=1}^n \lambda_k z_k$, where $\lambda_k \geqslant 0$, $\sum_{k=1}^n \lambda_k = 1$, $\|z_k\| \leqslant 1$ and

$$\|z_k - \mathbb{1}\| \geqslant 7/4. \tag{7.7.4}$$

According to (b) of Theorem 7.7.8 (with $N = 1$ and $f = z_k$), for every $k \leqslant n$ there is a constant function $\alpha_k \in [-1, 1]$ such that $d(z_k - \alpha_k, 0) = d(z_k, \alpha_k) < \varepsilon_1$. Then, using (7.7.4) and (7.7.2) we conclude

$$\begin{aligned} 1 \geqslant \|z_k\| &= \|\alpha_k + (z_k - \alpha_k)\| \\ &\geqslant |\alpha_k| + \|z_k - \alpha_k\| - \frac{1}{4} \\ &\geqslant |\alpha_k| + \|z_k - \mathbb{1}\| - |1 - \alpha_k| - \frac{1}{4} \\ &\geqslant |\alpha_k| - |1 - \alpha_k| + \frac{3}{2}, \end{aligned}$$

therefore $|\alpha_k| - |1 - \alpha_k| \leqslant -\frac{1}{2}$. This implies that $\alpha_k \leqslant 1/4$, consequently

$$\sum_{k=1}^n \lambda_k \alpha_k \leqslant 1/4. \tag{7.7.5}$$

Since $d\big(\sum_{k=1}^n \lambda_k(z_k-\alpha_k),0\big) < n\varepsilon_1$, using (7.7.5) and (7.7.2) we deduce that

$$\begin{aligned}\|y-z\| &= \Big\|\mathbb{1}-\sum_{k=1}^n \lambda_k z_k\Big\| \\ &= \Big\|\Big(\mathbb{1}-\sum_{k=1}^n \lambda_k \alpha_k\Big)+\sum_{k=1}^n \lambda_k(\alpha_k-z_k)\Big\| \\ &\geqslant \Big\|\mathbb{1}-\sum_{k=1}^n \lambda_k \alpha_k\Big\| - \frac{1}{4} \geqslant \frac{1}{2},\end{aligned}$$

which proves (7.7.3).

(b) It is enough to take the ℓ_1-direct sum $X = X_1 \oplus_1 X_2 \oplus_1 \dots$; X has the Daugavet property by Corollary 7.5.5.

(c) This follows from (b) and Corollary 7.6.10. □

7.8. More on the Bourgain-Rosenthal example

The main aim of this section is to obtain a strengthening of Theorem 7.7.11. The reason for developing this extension in a separate section is that it needs a number of independent results and its own notation. However, the conclusion of the following theorem is far stronger than that of Theorem 7.7.11, which justifies this extra work. The construction will rely on the construction of the space of Theorem 7.7.11(b) with some variations.

The main result of this section is the following. Recall from Theorem 3.1.5 that every slice of the unit ball of a space with the Daugavet property has diameter 2; for more on this property see Section 12.2.

THEOREM 7.8.1. *For every $\varepsilon > 0$ there exists a Banach space X with the Daugavet property such that, for every free ultrafilter $\mathfrak{U}$ over $\mathbb{N}$, the space $X^{\mathfrak{U}}$ has a slice of diameter smaller than or equal to ε.*

As we have already pointed out, in order to prove the above theorem we will need a number of preliminary results in addition to extra notation. To begin with let us start with a characterisation of the fact that every slice of the unit ball of a Banach space has diameter $\geqslant \alpha > 0$ in the spirit of Theorem 7.6.1.

PROPOSITION 7.8.2. *Let X be a Banach space. The following are equivalent:*

(1) *Every slice of B_X has diameter at least α.*
(2) *$B_X = \overline{\operatorname{conv}}\big\{\frac{x+y}{2}\colon x,y\in B_X,\ \|x-y\|\geqslant \alpha-\varepsilon\big\}$ holds for every $\varepsilon>0$.*

PROOF. (1) ⇒ (2). Assume that (2) does not hold. Then there exist $\varepsilon > 0$ and $x_0 \in B_X$ such that $x_0 \notin \overline{\operatorname{conv}}\{\frac{x+y}{2}\colon x,y\in B_X,\ \|x-y\|\geqslant\alpha-\varepsilon\}$. Put $A := \{\frac{x+y}{2}\colon x,y\in B_X,\ \|x-y\|\geqslant\alpha-\varepsilon\}$. By the Hahn-Banach theorem we can find a slice S of B_X such that $x_0\in S$ and $S\cap A=\emptyset$. We claim that $\|u-v\|<\alpha-\varepsilon$ for $u,v\in S$. Indeed, if there existed $u,v\in S$ with $\|u-v\|\geqslant\alpha-\varepsilon$, then $\frac{u+v}{2}\in A$; on the other hand, $\frac{u+v}{2}\in S$ by the convexity of S and so $S\cap A\neq\emptyset$, which is impossible. This proves that $\|u-v\|\leqslant\alpha-\varepsilon$ holds for every $u,v\in S$, and thus the diameter of S is less than α, which proves the negation of (1).

(2) ⇒ (1). Take a slice $S := S(B_X, x^*, \beta)$, where $x^*\in S_{X^*}$ and $\beta>0$, and let $\varepsilon>0$, and let us prove that there are $u,v\in S$ such that $\|u-v\|\geqslant \alpha-\varepsilon$; the arbitrariness of ε will imply (1). In order to do so, consider the slice

$S(B_X, x^*, \frac{\beta}{2})$. Since $\overline{\text{conv}}\{\frac{x+y}{2}\colon x, y \in B_X,\ \|x-y\| \geqslant \alpha-\varepsilon\} = B_X$ we infer that $S(B_X, x^*, \frac{\beta}{2}) \cap \{\frac{x+y}{2}\colon x, y \in B_X,\ \|x-y\| \geqslant \alpha-\varepsilon\} \neq \emptyset$ (since the relative complement of a slice in B_X is clearly a closed convex set). Consequently, we can find $u, v \in B_X$ with $\|u-v\| \geqslant \alpha-\varepsilon$ and such that $\frac{u+v}{2} \in S\left(B_X, x^*, \frac{\beta}{2}\right)$. In order to finish the proof, let us prove that both $u, v \in S = S(B_X, x^*, \beta)$ which means, by definition, that $x^*(u) > 1-\beta$ and $x^*(v) > 1-\beta$. To this end observe that $\frac{u+v}{2} \in S(B_X, x^*, \frac{\beta}{2})$ means $x^*\left(\frac{u+v}{2}\right) > 1-\frac{\beta}{2}$. Now

$$1-\frac{\beta}{2} < \frac{x^*(u)+x^*(v)}{2} \leqslant \frac{x^*(u)+\|x^*\|}{2} = \frac{x^*(u)+1}{2}.$$

This implies $x^*(u) > 1-\beta$. In a similar way, one can prove that $x^*(v) > 1-\beta$. Therefore $u, v \in S$, as desired. $\square$

The above result motivates us to introduce the following notation, which will be useful throughout the section. Given a Banach space X and $\alpha > 0$, define

$$S^\alpha(X) := \left\{\frac{x+y}{2}\colon x, y \in B_X,\ \|x-y\| \geqslant \alpha\right\}.$$

Given $n \in \mathbb{N}$ we denote

$$S_n^\alpha(X) := \text{conv}_n(S^\alpha(X)) = \left\{\sum_{j=1}^n \lambda_j u_j\colon \lambda_j \in [0,1],\ \sum_{j=1}^n \lambda_j = 1,\ u_j \in S^\alpha(X)\right\}.$$

Finally, given $n \in \mathbb{N}$ and $\alpha > 0$, we define

$$C_n^\alpha(X) := \sup_{x \in S_X} \text{dist}(x, S_n^\alpha(X)) = \sup_{x \in S_X} \inf_{y \in S_n^\alpha(X)} \|x-y\|.$$

From the very definition of $C_n^\alpha(X)$ the following two properties follow:

(1) Given $0 < \alpha < \beta$, then $C_n^\alpha \geqslant C_n^\beta$.
(2) Given two natural numbers $n \geqslant m$, then $C_n^\alpha(X) \leqslant C_m^\alpha(X)$.

With the above notation in mind, we can establish a necessary condition for an ultrapower space to satisfy that every slice of its unit ball has diameter at least $\alpha > 0$. Compare this result with Theorem 7.6.4.

THEOREM 7.8.3. *Let (X_n) be a sequence of Banach spaces, $\mathfrak{U}$ be a free ultrafilter over $\mathbb{N}$ and $\alpha > 0$. Set $X := (X_n)_\mathfrak{U}$ and assume that every slice of B_X has diameter at least α. Then, for every $\delta > 0$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N}\colon C_n^{\alpha-\varepsilon}(X_k) < \delta\} \in \mathfrak{U}.$$

PROOF. Assume that there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ such that, for every $n \in \mathbb{N}$, we have

$$\{k \in \mathbb{N}\colon C_n^{\alpha-\varepsilon_0}(X_k) \geqslant \delta_0\} \in \mathfrak{U}.$$

(Recall that if a set does not belong to an ultrafilter, its complement does.) We can select, for every $n \geqslant 2$, a set $A_n \subset \{k \in \mathbb{N}\colon C_n^{\alpha-\varepsilon_0}(X_k) \geqslant \delta_0\}$ such that $A_n \in \mathfrak{U}$, $\bigcap_{n\geqslant 2} A_n = \emptyset$ and $A_{n+1} \subset A_n$ holds for $n \geqslant 2$. In addition, let $A_1 = \mathbb{N}$. Observe that $\{A_n \setminus A_{n+1}\colon n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$. Moreover, for every $n \geqslant 2$, for every $p \in A_n \setminus A_{n+1}$ we can find $x_p \in S_{X_p}$ satisfying that $\text{dist}(x_p, S_n^{\alpha-\varepsilon_0}(X_p)) \geqslant \frac{\delta_0}{2}$. For $p \in A_1 \setminus A_2$ select any $x_p \in S_{X_p}$.

Now, $x := [(x_p)] \in S_X$. We claim that $\text{dist}([(x_p)], \text{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))) \geqslant \frac{\delta_0}{2}$. Once this is proved, Proposition 7.8.2 implies that there exists a slice in $(X_n)_\mathfrak{U}$

of diameter smaller than α, which will finish the proof of the theorem. In order to prove the claim, take $z \in \operatorname{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))$; so there is $q \in \mathbb{N}$ such that $z \in \operatorname{conv}_q(S^{\alpha-\frac{\varepsilon_0}{2}}(X))$.

By definition we can find $\lambda_1, \ldots, \lambda_q \in [0,1]$ with $\sum_{i=1}^q \lambda_i = 1$ and $[(u_n^i)], [(v_n^i)] \in S_X$ with $\|[(u_n^i)] - [(v_n^i)]\| \geqslant \alpha - \frac{\varepsilon_0}{2}$ and $z = \sum_{i=1}^q \lambda_i \frac{[(u_n^i)]+[(v_n^i)]}{2}$. Let $\eta > 0$. Since $\|[(x_n)] - [(z_n)]\| = \lim_{\mathfrak{U}} \|x_n - z_n\|$, the set

$$B := \{n \in \mathbb{N}\colon |\|x_n - z_n\| - \|x - z\|| < \eta\} \in \mathfrak{U}.$$

On the other hand, given $1 \leqslant i \leqslant q$ it follows that $\lim_{\mathfrak{U}} \|u_n^i - v_n^i\| \geqslant \alpha - \frac{\varepsilon_0}{2} > \alpha - \varepsilon_0$. This implies that the set

$$C := \bigcap_{i=1}^{q} \left\{n \in \mathbb{N}\colon \|u_n^i - v_n^i\| > \alpha - \varepsilon_0\right\} \in \mathfrak{U}.$$

Select any $k \in A_q \cap B \cap C$. Then, since $k \in B$, we have

$$\|[(x_n)] - [(z_n)]\| \geqslant \|x_k - z_k\| - \eta.$$

On the other hand, $z_k = \sum_{i=1}^q \lambda_i \frac{u_k^i + v_k^i}{2}$ with $\|u_k^i - v_k^i\| \geqslant \alpha - \varepsilon_0$ since $k \in C$. Hence, $z_k \in \operatorname{conv}_q(S^{\alpha-\varepsilon_0}(X_k))$. Finally, since $k \in A_q$ we conclude by the choice of x_k that $\|x_k - z_k\| \geqslant \frac{\delta_0}{2}$, so

$$\|[(x_n)] - [(z_n)]\| \geqslant \frac{\delta_0}{2} - \eta.$$

The arbitrariness of $\eta > 0$ and $[(z_n)] \in \operatorname{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))$ implies that we can conclude $\operatorname{dist}([(x_n)], \operatorname{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))) \geqslant \frac{\delta_0}{2}$, as desired. □

In the following result we will make use of the main ideas behind Theorem 7.7.11(a).

THEOREM 7.8.4. *Let $n \in \mathbb{N}$ and $\eta > 0$. There exists a Banach space X with the Daugavet property such that*

$$C_n^{2\eta}(X) \geqslant \frac{\eta}{8}.$$

PROOF. Select $\delta > 0$ small enough so that

$$5\delta < \frac{\eta}{2}.$$

Let X be the space of Theorem 7.7.8 with $\varepsilon_1 > 0$ small enough to guarantee that given any constant function $g \in [-2, 2]$ (i.e., $g \in E_1$) and $f \in L_1$, the condition $d(f, 0) < 2n\varepsilon_1$ implies

$$\|f + g\| \geqslant \|f\| + \|g\| - \delta. \tag{7.8.1}$$

Our aim is to prove that

$$d\left(\mathbb{1}, S_n^{2\eta}(X)\right) \geqslant \frac{\eta}{8}. \tag{7.8.2}$$

In order to do so, take $z \in S_n^{2\eta}(X)$. Then $z = \sum_{k=1}^n \lambda_k z_k$ with $z_k \in S^{2\eta}(X)$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_{k=1}^n \lambda_k = 1$. Moreover, since $z_k \in S^{2\eta}(X)$ it follows that $z_k = \frac{u_k + v_k}{2}$ with $u_k, v_k \in B_X$ satisfying $\|u_k - v_k\| \geqslant 2\eta$, for every $1 \leqslant k \leqslant n$. Now, given k, the triangle inequality implies

$$2\eta \leqslant \|u_k - \mathbb{1} + \mathbb{1} - v_k\| \leqslant \|\mathbb{1} - u_k\| + \|\mathbb{1} - v_k\|.$$

The above inequality implies that either $\|\mathbb{1} - u_k\| \geqslant \eta$ or $\|\mathbb{1} - v_k\| \geqslant \eta$. Assume, up to a relabelling, that $\|\mathbb{1} - u_k\| \geqslant \eta$ holds for every $1 \leqslant k \leqslant n$.

Given $1 \leqslant k \leqslant n$, apply (b) of Theorem 7.7.8 (with $f = u_k$ and v_k respectively and $N = 1$) to find constant functions $\alpha_k, \beta_k \in [-1, 1]$ satisfying $d(u_k, \alpha_k) < \varepsilon_1$ and $d(v_k, \beta_k) < \varepsilon_1$.

Now, for $1 \leqslant k \leqslant n$, we have

$$1 \geqslant \|u_k\| = \|\alpha_k + (u_k - \alpha_k)\| \geqslant |\alpha_k| + \|u_k - \alpha_k\| - \delta$$

since α_k is a constant function and $d(u_k - \alpha_k, 0) = d(u_k, \alpha_k) < \varepsilon_1 < 2n\varepsilon_1$; so the inequality (7.8.1) holds. Now

$$\begin{aligned} 1 &\geqslant |\alpha_k| + \|u_k - \mathbb{1} + \mathbb{1} - \alpha_k\| - \delta \\ &\geqslant |\alpha_k| + \|\mathbb{1} - u_k\| - |1 - \alpha_k| - \delta \\ &= |\alpha_k| + \|\mathbb{1} - u_k\| - (1 - \alpha_k) - \delta, \end{aligned}$$

where the last equality follows from $\alpha_k \leqslant 1$. Taking into account that $\|\mathbb{1} - u_k\| \geqslant \eta$ the above inequality implies

$$1 \geqslant |\alpha_k| + \eta - (1 - \alpha_k) - \delta = |\alpha_k| + \alpha_k + \eta - 1 - \delta \geqslant 2\alpha_k - 1 + \eta - \delta.$$

Consequently,

$$\alpha_k \leqslant \frac{2 - \eta}{2} + \frac{\delta}{2}.$$

Since $\beta_k \in [-1, 1]$ holds for every k we get

$$\sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2} \leqslant \frac{\frac{2-\eta}{2} + \frac{\delta}{2} + 1}{2} = \frac{4 - \eta + \delta}{4}. \tag{7.8.3}$$

Now,

$$\begin{aligned} d\left(z - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2}, 0\right) &= d\left(\sum_{k=1}^{n} \frac{\lambda_k}{2}(u_k - \alpha_k + v_k - \beta_k), 0\right) \\ &\leqslant \sum_{k=1}^{n} (d(u_k - \alpha_k, 0) + d(v_k - \beta_k, 0)) < 2n\varepsilon_1. \end{aligned}$$

If we apply (7.8.1) to the constant function $\mathbb{1} - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2}$ and the function $z - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2}$, which is $2n\varepsilon_1$ close to 0 with respect to the distance d, we obtain

$$\begin{aligned} \|\mathbb{1} - z\| &= \left\| \left(\mathbb{1} - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2}\right) - \left(z - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2}\right) \right\| \\ &\geqslant \left\| \mathbb{1} - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2} \right\| + \left\| z - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2} \right\| - \delta \\ &\geqslant \left\| \mathbb{1} - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2} \right\| - \delta \geqslant 1 - \sum_{k=1}^{n} \lambda_k \frac{\alpha_k + \beta_k}{2} - \delta \\ &\geqslant 1 - \frac{4 - \eta + \delta}{4} - \delta \qquad \text{by (7.8.3)} \\ &= \frac{\eta - 5\delta}{4} > \frac{\eta}{8}. \end{aligned}$$

Now, the result follows by the arbitrariness of $z \in S_n^{2\eta}(X)$. $\square$

Let $\eta > 0$ and define, for every $n \in \mathbb{N}$, X_n to be the Banach space constructed in Theorem 7.8.4, and consider $X = (\bigoplus_{n=1}^{\infty} X_n)_1$. X has the Daugavet property as it is an ℓ_1-sum of Banach spaces with the Daugavet property (Corollary 7.5.5). Let $r > 0$ be small enough to guarantee that $2r < \eta$ and $\frac{r^2}{4} + r < \frac{\eta}{8}$. We claim that, given $n \in \mathbb{N}$, we have that

$$d\left((0,0,0,\ldots,\underbrace{\mathbb{1}}_{n},0,0,\ldots), S_n^{3\eta}(X)\right) \geqslant \frac{r^2}{4}.$$

In order to prove this write $x := (0,0,0,\ldots,\mathbb{1},0,0,\ldots)$ (with $\mathbb{1}$ in the n^{th} slot) and assume by contradiction that there is $z \in S_n^{3\eta}(X)$ such that $\|x - z\| < \left(\frac{r}{2}\right)^2$. Then

$$\|\mathbb{1} - z(n)\| = \|x(n) - z(n)\| \leqslant \sum_{k=1}^{\infty} \|x(k) - z(k)\| = \|x - z\| < \left(\frac{r}{2}\right)^2.$$

If we write $z = \sum_{i=1}^{n} \lambda_i z_i$ with $0 \leqslant \lambda_i \leqslant 1$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $z_i \in S^{3\eta}(X)$, we obtain from the above inequality that $\|\sum_{i=1}^{n} \lambda_i z_i(n)\| > 1 - \left(\frac{r}{2}\right)^2$. Set

$$G := \left\{ i \in \{1,\ldots,n\} \colon \|z_i(n)\| > 1 - \frac{r}{2} \right\}.$$

We claim that $\sum_{i \notin G} \lambda_i < \frac{r}{2}$. Indeed,

$$\begin{aligned} 1 - \left(\frac{r}{2}\right)^2 < \sum_{i=1}^{n} \lambda_i \|z_i(n)\| &= \sum_{i \in G} \lambda_i \|z_i(n)\| + \sum_{i \notin G} \lambda_i \|z_i(n)\| \\ &\leqslant \sum_{i \in G} \lambda_i + \sum_{i \notin G} \lambda_i \left(1 - \frac{r}{2}\right) = 1 - \frac{r}{2} \sum_{i \notin G} \lambda_i, \end{aligned}$$

from where $\sum_{i \notin G} \lambda_i < \frac{r}{2}$ follows.

On the other hand, since $z_i \in S^{3\eta}(X)$ for $1 \leqslant i \leqslant n$, there are $u_i, v_i \in B_X$ with $z_i = \frac{u_i + v_i}{2}$ and $\|u_i - v_i\| > 3\eta$. Given $i \in G$ we have $\|z_i(n)\| > 1 - \frac{r}{2}$, from where

$$1 - \frac{r}{2} < \frac{\|u_i(n) + v_i(n)\|}{2} \leqslant \frac{\|u_i(n)\| + \|v_i(n)\|}{2} \leqslant \frac{\|u_i(n)\| + 1}{2},$$

which implies $\|u_i(n)\| > 1 - r$ and likewise $\|v_i(n)\| > 1 - r$. Consequently, we have

$$1 - r < \|u_i(n)\| \leqslant \|u_i(n)\| + \sum_{k \neq n} \|u_i(k)\| = \|u_i\| \leqslant 1,$$

from where $\sum_{k \neq n} \|u_i(k)\| < r$. Similarly $\sum_{k \neq n} \|v_i(k)\| < r$. Since $\|u_i - v_i\| > 3\eta$ we obtain

$$3\eta < \|u_i(n) - v_i(n)\| + \sum_{k \neq n} (\|u_i(k)\| + \|v_i(k)\|) \leqslant \|u_i(n) - v_i(n)\| + 2r,$$

so $\|u_i(n) - v_i(n)\| > 3\eta - 2r > 2\eta$ since $2r < \eta$. Set $\lambda := 1 - \sum_{i \in G} \lambda_i$ and set $z' := \sum_{i \in G} \lambda_i z_i + \lambda z$ where $z = z_{i_0}$ for an arbitrarily chosen $i_0 \in G$. We clearly get that

$$z'(n) = \sum_{i \in G} \lambda_i \frac{u_i(n) + v_i(n)}{2} + \lambda \frac{u_{i_0}(n) + v_{i_0}(n)}{2}$$

where $\|u_i(n) - v_i(n)\| > 2\eta$ holds for every $i \in G$, including $\|u_{i_0}(n) - v_{i_0}(n)\| > 2\eta$. This shows $z'(n) \in S_n^{2\eta}(X_n)$. By (7.8.2) we obtain

$$\|\mathbb{1} - z'(n)\| \geqslant \frac{\eta}{8}.$$

As a result,

$$\begin{aligned}\frac{\eta}{8} &\leqslant \|x(n) - z'(n)\| \leqslant \|x - z'\| \leqslant \|x - z\| + \|z' - z\| \\ &\leqslant \frac{r^2}{4} + \sum_{i \notin G} \lambda_i \left\| z_i - \frac{u_{i_0} + v_{i_0}}{2} \right\| < \frac{r^2}{4} + r < \frac{\eta}{8},\end{aligned}$$

a contradiction, and the above claim is established.

This proves the estimate

$$C_n^{3\eta}(X) \geqslant \frac{r^2}{4} \qquad (n \in \mathbb{N}).$$

With the help of Theorem 7.8.3, we can now provide the pending proof of Theorem 7.8.1.

PROOF OF THEOREM 7.8.1. Given $\varepsilon > 0$, select $0 < \eta < \frac{\varepsilon}{3}$, and choose $r > 0$ small enough to guarantee $2r < \eta$ and $\frac{r^2}{4} + r < \frac{\eta}{8}$. We have proved that there exists a Banach space X such that $C_n^{3\eta}(X) \geqslant \frac{r^2}{4}$ for each $n \in \mathbb{N}$. According to Theorem 7.8.3 this implies that given any free ultrafilter $\mathfrak{U}$ over $\mathbb{N}$ there exists a slice of $B_{X^{\mathfrak{U}}}$ of diameter smaller than 3η. Since $3\eta < \varepsilon$ the conclusion follows. $\square$

7.9. Notes and remarks

Section 7.1. The text is based on [**54**, Section 2].

Section 7.2. Probably the starting point of the connection between the Daugavet property and absolute sums of Banach spaces goes back to [**11**], where it was proved that finite ℓ_∞-sums (respectively, ℓ_1-sums) of non-atomic $L_1(\mu)$-spaces (respectively, non-atomic $L_\infty(\mu)$-spaces) enjoy the Daugavet property. Later, the above results were extended in [**302**] to general finite ℓ_1- and ℓ_∞-sums of Banach spaces with the Daugavet property.

The text is based on [**54**, Section 3].

Corollary 7.2.5 appeared in [**178**, Lemma 2.15] in a bit more general form (for Daugavet pairs instead spaces with the Daugavet property). A version for Daugavet centres was given in [**58**, Lemma 3.3].

Section 7.3. The section follows rather verbatim [**178**, Propositions 2.10 and 2.11] and the text before it.

Section 7.4. The section is based on [**54**].

Section 7.5. The section is based on [**54**] as well.

A related circle of questions was addressed by Tetiana V. Bosenko (Ivashyna) in [**59**]. Bosenko introduced the following two concepts. A Banach space X is said to be a *Daugavet domain* if there exists a Daugavet centre $G\colon X \to Y$ for some Banach space Y and, analogously, X is said to be a *Daugavet range* if there exists a Daugavet centre $G\colon E \to X$ for some Banach space E. Bosenko gave a complete geometrical description (in terms of the shape and number of edges of the unit sphere) of those two-dimensional spaces F for which there exists a Daugavet domain of the form $X_1 \oplus_F X_2$, and she gave a description of those F for which there exists a Daugavet range of the form $X_1 \oplus_F X_2$. The classes of spaces F obtained happen to be different, which gives as a consequence examples of Daugavet domains that are not Daugavet ranges, and examples of Daugavet ranges that are not Daugavet domains.

Section 7.6. Again, the section is based on [**54**]. The authors thank Alicia Quero for pointing out a technical mistake in a previous version of Theorem 7.6.4.

The examples given in Proposition 7.6.14, while easy, seem to be new. Actually, there are two key facts. The first is the equivalence between the Daugavet property and the absence of Fréchet differentiability points of the norm in C^*-algebras, isometric L_1-preduals, and uniform algebras (for the latter family, this result seems to be new (see Section 3.6, the Notes and Remarks on Chapter 3). The second fact is the stability of these three classes of Banach spaces by ultrapowers (we thank Armando Villena for providing the short proof of this fact for uniform algebras, a result which seems to be very well known, but a reference is not easy to find). Let us mention that the same characterisation of the Daugavet property by the absence of Fréchet differentiability points works for Banach spaces with ℓ_1-norming structure, see Corollary 3.5.16. Hence, the only ingredient that one would need to mimic the proof of Proposition 7.6.14 for all Banach spaces with ℓ_1-norming structure is whether this class of Banach spaces is stable by taking ultraproducts (or just ultrapowers). We do not know if this is the case, see Question (7.8).

Finally, let us mention that it is claimed in [**44**, Theorem 5.6] that the Daugavet property and the uniform Daugavet property are equivalent for preduals of von Neumann algebras (and actually, for preduals of JBW*-triples). Unfortunately, the proof given there contains a gap and it is not valid. For preduals of von Neumann algebras, the result was proved in [**226**, Corollary 3.2] using techniques completely different from the ones used in the paper [**44**]. We do not know if the result is also true for preduals of JBW*-triples.

Section 7.7. The section is based on [**171**]. Theorem 7.7.8, apart from part (a), was obtained by Bourgain and Rosenthal [**64**]. Part (a) was first remarked in [**171**], but it clearly relies on the ideas of Bourgain and Rosenthal.

Section 7.8. This section is based on [**270**].

7.10. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

Yevhenii Kuznietsov extended the definition of uniform Daugavet property to Daugavet centres in his Master's thesis [**194**].

Definition 7.10.1. Let X, Y be Banach spaces. An operator $G \in S_{L(X,Y)}$ is said to be a *uniform Daugavet centre* if for every $\varepsilon > 0$

$$\operatorname{Daug}_n(G, \varepsilon) := \sup_{\substack{u \in S_X \\ v \in S_Y}} \operatorname{dist}\big(u, \operatorname{conv}_n(V(G, v, \varepsilon))\big) \xrightarrow[n\to\infty]{} 0,$$

where $V(G, v, \varepsilon) := \{x \in B_X \colon \|Gx + v\| > 2 - \varepsilon\}$.

Theorem 7.10.2. *Let X, Y be Banach spaces, $\mathfrak{U}$ be a free ultrafilter defined on $\mathbb{N}$, $X^{\mathfrak{U}}$, $Y^{\mathfrak{U}}$ be the corresponding ultrapowers and $G^{\mathfrak{U}} \colon X^{\mathfrak{U}} \to Y^{\mathfrak{U}}$ be the ultrapower of an operator $G \in S_{L(X,Y)}$. Then, the following assertions are equivalent:*

(1) *$G^{\mathfrak{U}}$ is a uniform Daugavet centre.*
(2) *$G^{\mathfrak{U}}$ is a Daugavet centre.*

(3) *G is a uniform Daugavet centre.*

This initial step is done, but the subject of uniform Daugavet centres is still awaiting its future explorers.

Concerning the results of Section 7.2, an analogous theory for G-narrow operators does not exist by now (and for narrow operators with respect to a pair (G, Γ) not either), which would be a natural continuation of the results developed there. The same happens with the results from Section 7.4.

It seems that narrow operators in absolute sums have not been studied yet. We now present a result in this line.

PROPOSITION 7.10.3. *Let (X_n) be a sequence of Banach space, F be a Banach space with a 1-unconditional normalised basis enjoying the positive Daugavet property. Let $X = (X_n)_F$ be its F-sum. Let $T\colon X \to Z$ be a bounded operator such that $T_{|X_j}\colon X_j \to Z$ is a narrow operator for every $j \in \mathbb{N}$. Then T is a narrow operator.*

PROOF. Let $x = (x_n), y = (y_n) \in S_X$, $x^* = (x_n^*) \in S_{X^*}$ and $\varepsilon > 0$. Let us find $z \in S_X$ satisfying that

(1) $x^*(z) > (1-\varepsilon)^2$,
(2) $\|x + z\| > 2(1-\varepsilon)(1-2\varepsilon)$ and,
(3) $\|T(y - z)\| < \varepsilon$.

In order to do so, define $a = (a_n) = (\|x_n\|) \in S_F$ and $a^* = (a_n^*) = (\|x_n^*\|) \in S_{F^*}$. The positive Daugavet property permits us to find $b = (b_n) \in S_F$ satisfying that $a^*(b) > 1 - \varepsilon$ and $\|a + b\| > 2 - \varepsilon$.

Select a sequence $(\varepsilon_n) \subset \mathbb{R}^+$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$.

Using that $T_{|X_n}$ is narrow for every $n \in \mathbb{N}$, we can select, for every $n \in \mathbb{N}$, some $z_n \in X_n$ with the following properties:

- $\|z_n\| = b_n$,
- $x_n^*(z_n) \geqslant (1-\varepsilon)a_n^* b_n$,
- $\|x_n + z_n\| \geqslant (1-\varepsilon)(a_n + b_n)$, and
- $\|T_{|X_n}(x_n - z_n)\| < \varepsilon_n$.

Following word-by-word the proof of Theorem 7.5.4 we can prove that conditions (1) and (2) are satisfied. Finally, the last condition allows us to prove that

$$\begin{aligned} \|T((y_n) - (z_n))\| &= \left\| T\left(\sum_{n=1}^{\infty} (0, 0, \dots, y_n - z_n, 0, \dots) \right) \right\| \\ &= \left\| \sum_{n=1}^{\infty} T((0, 0, \dots, y_n - z_n, 0, \dots) \right\| \\ &= \left\| \sum_{n=1}^{\infty} T_{|X_n}(y_n - z_n) \right\| \leqslant \sum_{n=1}^{\infty} \|T_{|X_n}(y_n - z_n)\| < \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon, \end{aligned}$$

which proves (3) and finishes the proof. □

The above proposition motivates the following question.

(7.1) Let (X_n) be a sequence of Banach spaces, F be a Banach space with a 1-unconditional normalised basis enjoying the positive Daugavet property. Let $X = (X_n)_F$ its F-sum. Let $T\colon X \to Z$ be a narrow operator. Is it true that $T_{|X_n}\colon X_n \to Z$ is narrow for every $n \in \mathbb{N}$?

Concerning Section 7.3, we have the following question.

(7.2) Let X_1 be an M-ideal of a Banach space X. If T is a strong Daugavet operator on X, is the restriction of T to X_1 a strong Daugavet operator again?

Concerning the uniform Daugavet property we have the following questions.

(7.3) What is the possible behaviour of $\mathrm{Daug}_n(X,\varepsilon)$?

(7.4) What operations with Banach spaces preserve the uniform Daugavet property? What happens with $\mathrm{Daug}_n(X,\varepsilon)$ under such operations?

(7.5) Does a rich subspace of a space with the uniform Daugavet property have the uniform Daugavet property? If not, then what is the right concept of rich subspace for the spaces with the uniform Daugavet property?

(7.6) What can one say about the duality for the uniform Daugavet property?

This does not look so easy because there is no good direct description of the dual of an ultrapower, so one needs some indirect methods here.

In order to pose another question about the uniform Daugavet property let us introduce a bit of notation.

Let X be a Banach space. We say that X has the *perfect Daugavet property* if, given $x \in S_X$ and given any slice S of B_X, there exists $y \in S$ such that $\|x+y\| = 2$. (This corresponds to $\varepsilon = 0$ in (3.1.5) in Theorem 3.1.5(ii).) It is immediate, by a standard Hahn-Banach argument, that X has the perfect Daugavet property if, and only if,

$$B_X := \overline{\mathrm{conv}}\{y \in B_X\colon \|y+x\| = 2\}.$$

Let us consider the following question:

(7.7) Let X be a Banach space. If $X^{\mathfrak{U}}$ has the Daugavet property, does $X^{\mathfrak{U}}$ actually enjoy the perfect Daugavet property?

This is a natural question because quite often when a geometric property is preserved by taking ultrapowers then the ultrapowers satisfy this property even with $\varepsilon = 0$.

We shall elaborate on this question, eventually reformulating it in (7.7*). In the following we shall characterise those ultraproduct spaces with the perfect Daugavet property.

THEOREM 7.10.4. *Let (X_k) be a sequence of Banach spaces and let $\mathfrak{U}$ be a free ultrafilter over $\mathbb{N}$. The following assertions are equivalent:*

(1) *$(X_k)_{\mathfrak{U}}$ has the perfect Daugavet property.*

(2) *For every $\delta > 0$ there exists $n \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N}\colon\ \mathrm{Daug}_n(X_k,\varepsilon) < \delta\} \in \mathfrak{U}$$

holds for every $\varepsilon > 0$.

PROOF. (1) $\Rightarrow$ (2). Assume that (2) does not hold and let us prove that (1) does not hold either.

By the assumption there exists $\delta_0 > 0$ such that, given any $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ satisfying that

$$\{k \in \mathbb{N}\colon\ \mathrm{Daug}_n(X_k,\varepsilon_n) < \delta_0\} \notin \mathfrak{U}.$$

Since $\mathfrak{U}$ is an ultrafilter, it follows that

$$\{k \in \mathbb{N}\colon\ \mathrm{Daug}_n(X_k,\varepsilon_n) \geqslant \delta_0\} \in \mathfrak{U}.$$

Since the ultrafilter $\mathfrak{U}$ is free, we can find, for every $n \in \mathbb{N}$, an element $A_n \in \mathfrak{U}$ such that $A_n \subset \{k \in \mathbb{N}\colon \operatorname{Daug}_n(X_k, \varepsilon_n) \geqslant \delta_0\}$, that $A_{n+1} \subset A_n$ holds for every $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

Now, given $k \in A_n \setminus A_{n+1}$ select $x_k, y_k \in S_{X_k}$ satisfying that

$$\operatorname{dist}\big(y_k, \operatorname{conv}_n(l_{X_k}(x_k, \varepsilon_n))\big) \geqslant \delta_0.$$

(Recall from Section 7.6 that $l_{X_k}(x_k, \varepsilon_n) = \{u \in S_{X_k}\colon \|x_k + u\| \geqslant 2 - \varepsilon_n\}$.) This defines two elements $[(x_k)], [(y_k)] \in S_{(X_k)_\mathfrak{U}}$ (except for those $k \notin A_1$, where it does not matter how they are defined).

Let us prove that

$$\operatorname{dist}\big([(y_k)], \operatorname{conv}(\{[(z_k)] \in S_{(X_k)_\mathfrak{U}}\colon \|[(x_k)] + [(z_k)]\| = 2\})\big) \geqslant \delta_0.$$

Put $V := \{[(z_k)] \in S_{(X_k)_\mathfrak{U}}\colon \|[(x_k)] + [(z_k)]\| = 2\}$ and select $[(z_k)] \in \operatorname{conv} V$, and let us prove that $\|(y_k) - (z_k)\| \geqslant \delta_0$. For some $n \in \mathbb{N}$ it follows that $[(z_k)] \in \operatorname{conv}_n V$. Consequently, we can write $[(z_k)] = \sum_{i=1}^n \lambda_i [(z_k^i)]$, where $\lambda_1, \ldots, \lambda_n \in [0,1]$, $\sum_{i=1}^n \lambda_i = 1$ and $\|[(x_k)] + [(z_k^i)]\| = 2$ holds for $1 \leqslant i \leqslant n$. In order to prove that $\big\|[(y_k)] - \sum_{i=1}^n \lambda_i [(z_k^i)]\big\| \geqslant \delta_0$, select $\eta > 0$ such that $\eta < \varepsilon_n$. Consider the following sets

$$B := \Big\{k \in \mathbb{N}\colon \Big|\|x_k + z_k^i\| - \|[(x_\kappa)] + [(z_\kappa^i)]\|\Big| < \eta\ \forall 1 \leqslant i \leqslant n\Big\} \in \mathfrak{U}$$

and

$$C := \left\{k \in \mathbb{N}\colon \left|\left\|y_k - \sum_{i=1}^n \lambda_i z_k^i\right\| - \left\|[(y_\kappa)] - \sum_{i=1}^n \lambda_i [(z_\kappa^i)]\right\|\right| < \eta\right\} \in \mathfrak{U}.$$

Select $k \in A_n \cap B \cap C$ (which is nonempty because it is an element of $\mathfrak{U}$). Since $k \in B$ we infer

$$\|x_k + z_k^i\| \geqslant \|[(x_\kappa)] + [(z_\kappa^i)]\| - \eta = 2 - \eta > 2 - \varepsilon_n$$

for every $1 \leqslant i \leqslant n$. Thus $\sum_{i=1}^n \lambda_i z_k^i \in \operatorname{conv}_n(l_{X_k}(x_k, \varepsilon_n))$ and, by the condition defining y_k, we get that $\big\|y_k - \sum_{i=1}^n \lambda_i z_k^i\big\| \geqslant \delta_0$ since $k \in A_n$. Finally, since $k \in C$ we get

$$\left\|[(y_\kappa)] - \sum_{i=1}^n \lambda_i [(z_\kappa^i)]\right\| > \left\|y_k - \sum_{i=1}^n \lambda_i z_k^i\right\| - \eta \geqslant \delta_0 - \eta.$$

Since $\eta > 0$ was arbitrary the proof is finished.

(2) $\Rightarrow$ (1). Let $[(x_k)], [(y_k)] \in S_{(X_k)_\mathfrak{U}}$ and let us prove that $[(y_k)] \in \overline{\operatorname{conv}}(\{[(z_k)]\colon \|[(x_k)] + [(z_k)]\| = 2\})$. In order to do so select $\delta > 0$. By the assumptions there exists $n \in \mathbb{N}$ satisfying that

$$\left\{k \in \mathbb{N}\colon \operatorname{Daug}_n\left(X_k, \frac{1}{p}\right) < \delta\right\} \in \mathfrak{U}$$

holds for every $p \in \mathbb{N}$. As before, since $\mathfrak{U}$ is a free ultrafilter, given $p \in \mathbb{N}$ find $A_p \in \mathfrak{U}$ satisfying $A_p \subset \left\{k \in \mathbb{N}\colon \operatorname{Daug}_n\left(X_k, \frac{1}{p}\right) < \delta\right\}$, that $A_{p+1} \subset A_p$ holds for every $p \in \mathbb{N}$ and $\bigcap_{p\in\mathbb{N}} A_p = \emptyset$.

Given $k \in A_p \setminus A_{p+1}$ select $\sum_{i=1}^n \lambda_k^i z_k^i \in \operatorname{conv}_n\left(l_{X_K}\left(x_k, \frac{1}{p}\right)\right)$ (i.e., $\|x_k + z_k^i\| \geqslant 2 - \frac{1}{p}$ holds for every $1 \leqslant i \leqslant n$) such that

$$\left\|y_k - \sum_{i=1}^n \lambda_k^i z_k^i\right\| < \delta.$$

For every $1 \leqslant i \leqslant n$ define $\lambda_i := \lim_{k,\mathfrak{U}} \lambda_k^i \in [0,1]$. It is immediate that $\sum_{i=1}^n \lambda_i = 1$.

We claim that $\|[(x_k)] + [(z_k^i)]\| = 2$ holds for $1 \leqslant i \leqslant n$ and that

$$\left\| [(y_k)] - \sum_{i=1}^n \lambda_i [(z_k^i)] \right\| \leqslant \delta.$$

Let us start by proving that, given $1 \leqslant i \leqslant n$, the equality

$$\|[(x_k)] + [(z_k^i)]\| = 2$$

holds. To this end, let $\eta > 0$ and select $p \in \mathbb{N}$ such that $\frac{1}{p} < \frac{\eta}{2}$. Define

$$B := \left\{ k \in \mathbb{N}\colon \left| \|x_k + z_k^i\| - \|[(x_\kappa)] + [(z_\kappa^i)]\| \right| < \frac{\eta}{2} \right\},$$

which is an element of $\mathfrak{U}$. Given $k \in A_p \cap B$ we get $\|x_k + z_k^i\| > 2 - \frac{1}{p}$ since $k \in A_p$. Since $k \in B$ we get $\lim_{\mathfrak{U}} \|x_\kappa + z_\kappa^i\| > \|x_k + z_k^i\| - \frac{\eta}{2}$. Thus

$$\|[(x_\kappa)] + [(z_\kappa^i)]\| \geqslant \|x_k + z_k^i\| - \frac{\eta}{2} > 2 - \frac{1}{p} - \frac{\eta}{2} > 2 - \eta.$$

Since $\eta > 0$ was arbitrary we conclude that $\|[(x_\kappa)] + [(z_\kappa^i)]\| = 2$.

In order to prove that $\|[(y_k)] - \sum_{i=1}^n \lambda_i [(z_k^i)]\| \leqslant \delta$, select $\eta > 0$ and define

$$C := \left\{ k \in \mathbb{N}\colon \left| \left\| [(y_\kappa)] - \sum_{i=1}^n \lambda_i [(z_\kappa^i)] \right\| - \left\| y_k - \sum_{i=1}^n \lambda_i z_k^i \right\| \right| < \frac{\eta}{2} \right\},$$

which is an element of $\mathfrak{U}$. The set

$$D := \left\{ k \in \mathbb{N}\colon |\lambda_i - \lambda_k^i| < \frac{\eta}{2n}\ \forall 1 \leqslant i \leqslant n \right\}$$

is also an element of $\mathfrak{U}$ by the definition of limit along $\mathfrak{U}$. Now, given $k \in A_p \cap C \cap D$ we get

$$\begin{aligned}
\|[(y_\kappa)] - \sum_{i=1}^n \lambda_i [(z_\kappa^i)]\| &\leqslant \left\| y_k - \sum_{i=1}^n \lambda_i z_k^i \right\| + \frac{\eta}{2} \\
&\leqslant \left\| y_k - \sum_{i=1}^n \lambda_k^i z_k^i \right\| + \sum_{i=1}^n |\lambda_k^i - \lambda_i| + \frac{\eta}{2} \\
&< \delta_0 + \sum_{i=1}^n \frac{\eta}{2n} + \frac{\eta}{2} = \delta + \eta.
\end{aligned}$$

The arbitrariness of $\eta > 0$ finishes the proof. □

Now, the following corollary is clear.

Corollary 7.10.5. *Let X be a Banach space. The following assertions are equivalent:*

(1) $X^{\mathfrak{U}}$ *has the perfect Daugavet property for every free ultrafilter* $\mathfrak{U}$ *over* $\mathbb{N}$.
(2) $\lim_n \operatorname{Daug}_n(X, \varepsilon) = 0$ *uniformly in* ε, *i.e., for every* $\delta > 0$ *there exists* $m \in \mathbb{N}$ *such that* $n \geqslant m$ *implies* $\operatorname{Daug}_n(X, \varepsilon) < \delta$ *for every* $\varepsilon > 0$.

In view of the definition of the uniform Daugavet property and the above Corollary, we can re-write Question (7.7) in the following terms.

(7.7*) Let X be a Banach space such that, for every $\varepsilon > 0$, $\lim_n \operatorname{Daug}_n(X, \varepsilon) = 0$. Does it follow that $\lim_n \operatorname{Daug}_n(X, \varepsilon) = 0$ uniformly in ε?

Even though we do not know whether the answer to the above question is affirmative, observe that this is the case for L_1-spaces (the estimates in Lemma 7.6.11), for $C(K)$-spaces (Lemma 7.6.12) and for spaces of Lipschitz functions (Remark 11.2.9).

Another interesting question is the following one.

(7.8) Are the Daugavet and uniform Daugavet properties equivalent for Banach spaces with ℓ_1-norming structure?

As commented in the Notes and Remark to Section 7.6, this would be the case if Banach spaces with norming ℓ_1-structure are stable by taking ultrapowers.

CHAPTER 8

Narrow operators in spaces of vector-valued functions

The basic examples of spaces with the Daugavet property are $C(K)$ on a perfect compact K together with its vector-valued extension $C(K, X)$ (Theorem 3.4.11) and $L_1(\mu)$ on a non-atomic measure space (Ω, Σ, μ) with its vector-valued analogue $L_1(\mu, X)$ (Theorem 3.4.4). In this chapter, we study in detail narrow operators on these spaces. Since the Daugavet property of a space X is equivalent to the Daugavet property of the underlying real space $X_{\mathbb{R}}$, we allow ourselves to consider only *real* spaces in this chapter. This does not reduce the amount of examples, because of the nature of the subject: the complex variants of $C(K)$ and $L_1(\mu)$ may be viewed as real $C(K, \ell_2^{(2)})$ and $L_1(\mu, \ell_2^{(2)})$, and every complex-linear operator on these spaces is at the same time real-linear.

The reader is already familiar with some information about narrow operators on $C(K)$ and $L_1(\mu)$. In Definition 6.1.5, we introduced C-narrow operators and announced without proof that on $C(K)$ the classes of strong Daugavet, narrow and C-narrow operators are the same. After completing some preliminary work in Section 8.1 and addressing general sup-normed spaces of functions in Section 8.2, we will demonstrate in Section 8.3 the announced result and give a complete description of those Banach spaces X for which an analogous result remains valid in $C(K, X)$. We will see that narrow operators on $C(K)$ form a linear space, that in some "bad" $C(K, X)$ this nice property is no longer true, but in "good" $C(K, X)$ not just the linearity of the class of narrow operators can be shown, but even the stability with respect to pointwise unconditionally convergent infinite sums. In Section 8.4 we turn to rich subspaces of $C(K)$ and deduce the following renorming result: if X is a separable Banach space containing a copy of $C[0, 1]$, then X possesses the Daugavet property in an equivalent norm.

The narrow operators on real $L_1(\mu)$ were described in Section 6.6 in terms of balanced ε-peaks. We announced that the complex case will follow from the results of Chapter 8, and we are going to complete this task in Section 8.6 using the above-mentioned identification of the complex $L_1(\mu)$ with the real $L_1(\mu, \ell_2^{(2)})$. Finally, in Section 8.7 we address rich subspaces of $L_1(\mu)$ and give some applications to Harmonic Analysis.

8.1. USD-nonfriendly spaces

In Section 12.4 we will study the class of anti-Daugavet spaces, which is in some sense opposite to the class of spaces with the Daugavet property. But, as mentioned at the end of that section, there is no mathematical definition of a "property that is opposite to the given one", so what "an opposite property" is depends on the standpoint and specific features of the problem under consideration.

In this section we introduce another class "opposite" to the class of spaces with the Daugavet property that will arise naturally in Section 8.3.

In Definition 6.2.4 we introduced the notation

$$D(x,y,\varepsilon) = \{z \in X\colon \|x+y+z\| > 2-\varepsilon,\ \|y+z\| < 1+\varepsilon\}$$

and

$$\mathcal{D}(X) = \{D(x,y,\varepsilon)\colon x \in S_X,\ y \in S_X,\ \varepsilon > 0\}$$

and remarked in Proposition 6.2.5 that $\mathcal{SD}(X) = \mathcal{D}(X)^{\sim}$, i.e., an operator $T \in \mathcal{OP}(X)$ is a strong Daugavet operator if and only if T is not bounded from below on any $D \in \mathcal{D}(X)$.

Below it will sometimes be more convenient to work with the bigger collection

$$\mathcal{D}_0(X) = \{D(x,y,\varepsilon)\colon x \in S_X,\ y \in B_X,\ \varepsilon > 0\}.$$

instead; therefore we formulate a lemma saying that this doesn't make any difference.

Lemma 8.1.1. *An operator $T \in \mathcal{OP}(X)$ is a strong Daugavet operator if and only if T is not bounded from below on any $D \in \mathcal{D}_0(X)$.*

Proof. We have to show that $T \in \mathcal{SD}(X)$ is not bounded from below on $D(x,y,\varepsilon)$ whenever $\|x\| = 1$, $\|y\| \leqslant 1$, $\varepsilon > 0$. By Remark 6.2.8, T is unbounded from below, hence, given $\varepsilon' > 0$, for some $\zeta \in S_X$ we have $\|T\zeta\| < \varepsilon'$. Now, pick $\lambda \geqslant 0$ such that $y + \lambda\zeta \in S_X$; then there is some $z' \in X$ such that

$$\|x + (y+\lambda\zeta) + z'\| > 2-\varepsilon, \quad \|(y+\lambda\zeta)+z'\| < 1+\varepsilon, \quad \|Tz'\| < \varepsilon';$$

i.e., $z := \lambda\zeta + z' \in D(x,y,\varepsilon)$ and $\|Tz\| < 3\varepsilon'$. □

Proposition 8.1.2. *The following conditions for a Banach space E are equivalent.*

(1) $\mathcal{SD}(E) = \{0\}$.
(2) *No nonzero linear functional on E is a strong Daugavet operator.*
(3) *For every $x^* \in S_{E^*}$ there exist some $\delta > 0$ and $D \in \mathcal{D}(E)$ such that $|x^*(z)| > \delta$ for all $z \in D$.*
(4) *The only closed absolutely convex subset $A \subset E$ with the property that $(\alpha A) \cap D \neq \emptyset$ for every $\alpha > 0$ and every $D \in \mathcal{D}(E)$ is the set $A = E$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are evident.

(3) ⇒ (4): Assume there is some closed absolutely convex subset $A \subset E$ with the property from (4) that does not coincide with the whole space E. By the Hahn-Banach theorem there are a functional $x^* \in S_{E^*}$ and a number $r > 0$ such that $|x^*(a)| \leqslant r$ for every $a \in A$. If $\delta > 0$ and $D \in \mathcal{D}(E)$ are arbitrary, pick $z \in (\frac{\delta}{r}A) \cap D$; this intersection is nonempty by assumption on A. It follows that $|x^*(z)| \leqslant \delta$, hence (3) fails.

(4) ⇒ (1): Suppose $T \in \mathcal{SD}(E)$ and put $A = \{e \in E\colon \|Te\| \leqslant 1\}$. By the definition of a strong Daugavet operator this A satisfies (4). So $A = E$ and hence $T = 0$. □

This proposition suggests the following definition.

Definition 8.1.3. A Banach space E is said to be an *SD-nonfriendly space* (i.e., *strong Daugavet-nonfriendly*) if $\mathcal{SD}(E) = \{0\}$. A space E is said to be a *USD-nonfriendly space* (i.e., *uniformly strong Daugavet-nonfriendly*) if there exists

$\alpha > 0$ such that every closed absolutely convex subset $A \subset E$ which intersects all the elements of $\mathcal{D}(E)$ contains αB_E. The largest admissible α is called the *USD-parameter* of E.

Proposition 8.1.2 shows that a USD-nonfriendly space is indeed SD-nonfriendly; but the converse is false as will be shown shortly. Also, SD-nonfriendliness is opposite to the Daugavet property in that the latter is equivalent to the condition that every functional is a strong Daugavet operator.

To further motivate the uniformity condition in the above definition, we supply a lemma.

LEMMA 8.1.4. *A Banach space E is USD-nonfriendly if and only if*

(3*) *There exists some $\delta > 0$ such that for every $x^* \in S_{E^*}$ there exists $D \in \mathcal{D}(E)$ such that $|x^*(z)| > \delta$ for all $z \in D$.*

PROOF. It is enough to prove the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) for the following assertions about a fixed number $\delta > 0$:

(a) There exists a closed absolutely convex set $A \subset E$ not containing δB_E that intersects all $D \in \mathcal{D}(E)$.

(b) There exists a functional $x^* \in S_{E^*}$ such that for all $D \in \mathcal{D}(E)$ there exists $z_D \in D$ satisfying $|x^*(z_D)| \leqslant \delta$.

(c) There exists a closed absolutely convex set $A \subset E$ that intersects all $D \in \mathcal{D}(E)$ and does not contain $\delta' B_E$ for any $\delta' > \delta$.

To see that (a) implies (b), pick $u \notin A$, $\|u\| \leqslant \delta$. By the Hahn-Banach theorem, we can separate u from A by means of a functional $x^* \in S_{E^*}$; then we shall have for some number $r > 0$ that $|x^*(z)| \leqslant r$ for all $z \in A$ and $x^*(u) > r$. On the other hand, $x^*(u) \leqslant \|x^*\|\,\|u\| \leqslant \delta$; hence (b) holds for x^*.

If we assume (b), we define A to be the closed absolutely convex hull of the elements z_D, $D \in \mathcal{D}(E)$, appearing in (b). Obviously A intersects each $D \in \mathcal{D}(E)$. If $\delta' B_E \subset A$ for some $\delta' > 0$, then since $|x^*| \leqslant \delta$ on A, we must have $|x^*| \leqslant \delta$ on $\delta' B_E$, i.e., $\delta' \leqslant \delta$. Therefore, A works in (c). $\square$

In Proposition 8.1.2 and Lemma 8.1.4 we may replace $\mathcal{D}(E)$ by $\mathcal{D}_0(E)$.

We now turn to some examples.

PROPOSITION 8.1.5.

(a) *The space c_0 is SD-nonfriendly, but not USD-nonfriendly.*

(b) *The space ℓ_1 is not SD-nonfriendly and hence not USD-nonfriendly either.*

PROOF. (a) Let $T \in \mathcal{SD}(c_0)$. Denote by e_k, $k \in \mathbb{N}$, the canonical basis vectors in c_0. Theorem 7.2.3 implies that for every $k \in \mathbb{N}$ the restriction of T to $\operatorname{lin} e_k$ is strongly Daugavet on that one-dimensional subspace. But then T is unbounded from below on $\operatorname{lin} e_k$, so $Te_k = 0$ for every k.

To show that c_0 is not USD-nonfriendly we shall exhibit a closed absolutely convex set A intersecting each $D \in \mathcal{D}(c_0)$, yet containing no ball. Let $A = 2B_{\ell_1} \subset c_0$, i.e.,

$$A = \left\{ (x(n)) \in c_0 \colon \sum_{n=1}^{\infty} |x(n)| \leqslant 2 \right\},$$

which is closed in c_0. Fix $x \in S_{c_0}$ and $y \in S_{c_0}$. If $|x(k)| = 1$, say $x(k) = 1$, pick a scalar β, $|\beta| \leqslant 2$, such that $y(k) + \beta = 1$. Then $\beta e_k \in D(x, y, \varepsilon) \cap A$ for every $\varepsilon > 0$. Obviously, A does not contain a multiple of B_{c_0}.

(b) According to Example 6.2.13, $x_\sigma^*(x) = \sum_{n=1}^\infty \sigma_n x(n)$ defines a strong Daugavet functional on ℓ_1 whenever σ is a sequence of signs, i.e., if $|\sigma_n| = 1$ for all n. □

Next, we wish to give some examples of USD-nonfriendly spaces. Recall (Definition 2.10.2) that a LUR-point of the unit sphere of a Banach space E is a point $x_0 \in S_E$ such that $x_n \to x_0$ whenever $\|x_n\| \leqslant 1$ and $\|x_n + x_0\| \to 2$.

Proposition 8.1.6. *If the unit sphere of E contains a LUR-point, then E is a USD-nonfriendly space with USD-parameter $\geqslant 1$.*

Proof. Let $x_0 \in S_E$ be a LUR-point and $A \subset E$ be a closed absolutely convex subset which intersects all the elements of $\mathcal{D}(E)$. In particular for every fixed $y \in S_E$ the set A intersects all the sets $D(x_0, y, \varepsilon) \subset E$, $\varepsilon > 0$. By definition of a LUR-point this means that all the points of the form $x_0 - y$, $y \in S_E$, belong to A, i.e., $B_E + x_0 \subset A$. But $-x_0$ is also a LUR-point, so $B_E - x_0 \subset A$ and, by convexity of A, $B_E \subset A$. □

Remark 8.1.7. *It is easy to construct an equivalent norm on $E := \ell_1 \oplus \mathbb{R}$ such that its restriction on ℓ_1 is equal to the original norm, but the unit sphere of E contains a LUR-point. In this norm, E is a USD-nonfriendly space which contains a subspace that is not USD-nonfriendly. So, the property of being a USD-nonfriendly space is not inherited by subspaces, even to one-codimensional ones.*

Corollary 8.1.8. *Every locally uniformly rotund space is USD-nonfriendly with USD-parameter 2. In particular, the spaces $L_p(\mu)$ are USD-nonfriendly for $1 < p < \infty$. Also, every separable space can be equivalently renormed to be USD-nonfriendly with USD-parameter 2.*

Proof. This follows from the previous proposition; that the USD-parameter is 2 is a consequence of $B_E + x_0 \subset A$ for all $x_0 \in S_E$; see the above proof. It is clear that $L_p(\mu)$ is LUR for $1 < p < \infty$ (it is actually uniformly convex). The last part is a consequence of the fact that separable Banach spaces admit LUR renormings; cf. Theorem 2.10.4 □

It is clear that no finite-dimensional space enjoys the Daugavet property, but more is true.

Proposition 8.1.9. *Every finite-dimensional Banach space E is a USD-nonfriendly space.*

Proof. Assume to the contrary that there is a finite-dimensional space E that is not USD-nonfriendly. By Lemma 8.1.4, we can find a sequence of functionals $(x_n^*) \subset S_{E^*}$ such that $\inf_{z \in D} |x_n^*(z)| \leqslant 1/n$ for each $D \in \mathcal{D}(E)$. By compactness of the ball we can pass to the limit and obtain a functional $x^* \in S_{E^*}$ with the property that $\inf_{z \in D} |x^*(z)| = 0$ for each $D \in \mathcal{D}(E)$.

Denote $K = \{e \in B_E\colon x^*(e) = 1\}$; this is a norm-compact convex set. Let $x_0 \in K$ be an arbitrary point. If we apply the above property to $D(x_0, -x_0, \varepsilon)$ for all $\varepsilon > 0$, we obtain, again by compactness, some z_0 such that $\|z_0 - x_0\| = 1$, $\|z_0\| = 2$ and $x^*(z_0) = 0$. We have $x^*(x_0 - z_0) = 1$, so $x_0 - z_0 \in K$. Therefore

$$2 \geqslant \operatorname{diam} K \geqslant \sup_{y \in K} \|x_0 - y\| \geqslant \|x_0 - (x_0 - z_0)\| = \|z_0\| = 2;$$

hence $\operatorname{diam} K = 2$ and x_0 is a diametral point of K, meaning

$$\sup_{y \in K} \|x_0 - y\| = \operatorname{diam} K.$$

But any compact convex set of positive diameter contains a nondiametral point [**156**, Section 15.3.1, Lemma 1]; thus we have reached a contradiction. □

We shall later estimate the worst possible USD-parameter of an n-dimensional normed space.

We haven't been able to decide whether every reflexive space is USD-nonfriendly (see Question (8.1) in Section 8.9). Proposition 8.1.12 below presents a necessary condition a hypothetical reflexive USD-friendly (= not USD-nonfriendly) space must fulfill.

First an easy geometrical lemma.

LEMMA 8.1.10. *Let $x, h \in E$, $\|x\| \leqslant 1+\varepsilon$, $\|h\| \leqslant 1+\varepsilon$, $\|x+h\| \geqslant 2-\varepsilon$. Let $f \in S_{E^*}$ be a supporting functional of $(x+h)/\|x+h\|$. Then $f(x)$ as well as $f(h)$ are estimated from below by $1-2\varepsilon$.*

PROOF. Denote $a = f(x)$, $b = f(h)$. Then $\max(a,b) \leqslant 1+\varepsilon$ but $a+b \geqslant 2-\varepsilon$. So, $\min(a,b) = a+b-\max(a,b) \geqslant 1-2\varepsilon$. □

Let E be a reflexive space, x_0^* be a strongly exposed point of S_{E^*} with strongly exposing evaluation functional x_0; i.e., the diameter of the slice $\{x^* \in S_{E^*} \colon x^*(x_0) > 1-\varepsilon\}$ tends to 0 when ε tends to 0 (Definition 2.7.11). Denote

$$\mathrm{Face}(S_E, x_0^*) = \{x \in S_E \colon x_0^*(x) = 1\}.$$

PROPOSITION 8.1.11. *Let E, x_0^*, x_0 be as above, let A be a closed convex set which intersects all the sets $D(x_0, 0, \varepsilon)$, $\varepsilon > 0$. Then A intersects* $\mathrm{Face}(S_E, x_0^*)$.

PROOF. For every $n \in \mathbb{N}$ select $h_n \in A \cap D(x_0, 0, \frac{1}{n})$. Then $\|h_n\| \leqslant 1+\frac{1}{n}$, $\|x_0+h_n\| \geqslant 2-\frac{1}{n}$. Denote by f_n a supporting functional of $(x_0+h_n)/\|x_0+h_n\|$. By the previous lemma, $f_n(x_0)$ tends to 1 when n tends to infinity. So by the definition of an exposing functional, f_n tends to x_0^*. By the same lemma, $f_n(h_n)$ tends to 1, so $x_0^*(h_n)$ also tends to 1. Hence, every weak limit point of the sequence (h_n) belongs to the intersection of A and $\mathrm{Face}(S_E, x_0^*)$, so this intersection is nonempty. □

PROPOSITION 8.1.12. *Let E be a reflexive space.*

(a) *If E is USD-nonfriendly with USD-parameter $< \alpha$, then there exists a functional $x^* \in S_{E^*}$ such that for every strongly exposed point x_0^* of B_{E^*} the numerical set $x^*(\mathrm{Face}(S_E, x_0^*))$ contains the interval $[-1+\alpha, 1-\alpha]$.*

(b) *If E is not USD-nonfriendly, then for every strongly exposed point x_0^* of B_{E^*} the set $\mathrm{Face}(S_E, x_0^*)$ has diameter 2. Moreover, for every $\delta > 0$ there exists a functional $x^* \in S_{E^*}$ such that for every strongly exposed point x_0^* of B_{E^*} the numerical set $x^*(\mathrm{Face}(S_E, x_0^*))$ contains the interval $[-1+\delta, 1-\delta]$.*

PROOF. (a) Let A be a closed absolutely convex set which intersects all the sets $D \in \mathcal{D}(E)$, but does not contain αB_E. By the Hahn-Banach theorem there exists a functional $x^* \in S_{E^*}$ such that $|x^*(a)| < \alpha$ for every $a \in A$. We fix $y \in S_E$ with $x^*(y) = -1$.

Let $x_0^* \in S_{E^*}$ be a strongly exposed point of B_{E^*}. As before, we denote a strongly exposing evaluation functional by x_0. Now, $A \cap D(x_0, y, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. By Proposition 8.1.11 and the evident equality $D(x_0, 0, \varepsilon) - y = D(x_0, y, \varepsilon)$, this implies that the set $A + y$ intersects $\mathrm{Face}(S_E, x_0^*)$. If z_1 is an element of this intersection, we see that $x^*(z_1) < \alpha - 1$.

Likewise, since $D(-x_0, 0, \varepsilon) = -D(x_0, 0, \varepsilon)$, we find some $z_2 \in (-A - y) \cap S_{x_0^*}$; hence $x^*(z_2) > -\alpha + 1$. Therefore, $[-1 + \alpha, 1 - \alpha] \subset x^*(S_{x_0^*})$.

(b) The argument is the same as in (a). $\square$

This proposition allows us to estimate the USD-parameter of finite-dimensional spaces.

PROPOSITION 8.1.13. *If E is n-dimensional, then its USD-parameter is $\geqslant 2/n$.*

PROOF. Assume that $\dim(E) = n$ and that its USD-parameter is $< 2/n$; then this parameter is strictly smaller than some $\alpha < 2/n$. Choose x^* as in Proposition 8.1.12 so that

$$[-1 + \alpha, 1 - \alpha] \subset x^*(S_{x_0^*}) \tag{8.1.1}$$

for every strongly exposed functional $x_0^* \in S_{E^*}$.

We now claim that in any ε-neighbourhood of x^* there is some $y^* \in B_{E^*}$ which can be represented as a convex combination of $\leqslant n$ strongly exposed points of B_{E^*}. First of all, the convex hull of the set $\operatorname{stexp}(B_{E^*})$ of strongly exposed points of B_{E^*} is norm-dense in B_{E^*}; in fact, this is true for any bounded closed convex set in a space with the Radon-Nikodým property, as we mentioned in Theorem 2.7.12, see [**51**, pp. 102–110] for a detailed exposition. Hence, for some $\|y_1^* - x^*\| < \varepsilon$, $\lambda_1', \ldots, \lambda_r' \geqslant 0$ with $\sum_{k=1}^r \lambda_k' = 1$ and $x_1^*, \ldots, x_r^* \in \operatorname{stexp} B_{E^*}$

$$y_1^* = \sum_{k=1}^r \lambda_k' x_k^*.$$

Let $C = \operatorname{conv}\{x_1^*, \ldots, x_r^*\}$ and let y^* be the point of intersection of the segment $[y_1^*, x^*]$ with the relative boundary of C, i.e., $y^* = \tau x^* + (1 - \tau) y_1^*$ with

$$\tau = \sup\{t \in [0, 1] \colon t x^* + (1 - t) y_1^* \in C\}.$$

Let F be the face of C generated by y^*; then F is a convex set of dimension $< n$. Therefore, an appeal to Carathéodory's theorem (see Theorem 2.6.15) shows that y^* can be represented as a convex combination of no more than n extreme points of F. But $\operatorname{ext}(F) \subset \operatorname{ext}(C) \subset \{x_1^*, \ldots, x_r^*\} \subset \operatorname{stexp} B_{E^*}$, and our claim is established.

We apply the claim with $\varepsilon < 2/n - \alpha$ to obtain some convex combination $y^* = \sum_{k=1}^n \lambda_k x_k^*$ of n strongly exposed points of B_{E^*} such that $\|y^* - x^*\| < \varepsilon$. One of the coefficients must be $\geqslant 1/n$, say $\lambda_n \geqslant 1/n$. Now, if $x \in S_{x_n^*}$,

$$\begin{aligned} x^*(x) \geqslant x^*(y) - \varepsilon &= \sum_{k=1}^{n-1} \lambda_k x_k^*(x) + \lambda_n - \varepsilon \\ &\geqslant -\sum_{k=1}^{n-1} \lambda_k + \lambda_n = -1 + 2\lambda_n - \varepsilon \geqslant -1 + 2/n - \varepsilon. \end{aligned}$$

By (8.1.1) we have $-1 + \alpha \geqslant -1 + 2/n - \varepsilon$ which contradicts our choice of ε. $\square$

For $\ell_\infty^{(n)}$ we can say more, namely, its USD-parameter is the worst possible.

PROPOSITION 8.1.14. *The USD-parameter of $\ell_\infty^{(n)}$ is $2/n$.*

PROOF. The argument of Proposition 8.1.5(a) implies in the setting of $\ell_\infty^{(n)}$ rather than c_0 that the USD-parameter of $\ell_\infty^{(n)}$ is $\leqslant 2/n$, and the converse estimate follows from Proposition 8.1.13. $\square$

8.2. Strong Daugavet and narrow operators in vector-valued sup-normed spaces

Let E be a Banach space and X be a closed subspace of the space $\ell_\infty(K,E)$ of all bounded E-valued functions defined on a set K, equipped with the sup-norm. It will be convenient to use the following notation: A disjoint pair (U,V) of subsets of K is said to be *interpolating* for X if for every $f,g \in X$ with $\|f\| < 1$ and $\|g\mathbb{1}_V\| < 1$, there exists $h \in B_X$ such that $h = f$ on U and $h = g$ on V.

Remark 8.2.1. *Let us describe a handy example of interpolating pairs. Let K be a compact Hausdorff space, $X = C(K,E)$, $U,V \subset K$ be closed and disjoint, then (U,V) is interpolating for X. Indeed, let $f,g \in X$ be as above. Then, by the continuity of g, there is an open set $V_0 \supset V$, $V_0 \cap U = \emptyset$, such that $\|g(t)\| < 1$ for all $t \in V_0$. There is a continuous function ϕ: $K \to [0,1]$ such that $\phi(t) = 0$ on $K \setminus V_0 \supset U$ and $\phi(t) = 1$ on V. Then the formula $h(t) = \phi(t)g(t) + (1-\phi(t))f(t)$ gives what we need.*

Also remark that for an interpolating pair (U,V) every pair of the form $(\widetilde{U},V)$ with $\widetilde{U} \subset U$ is interpolating. So, the above example generalises to $X = C(K,E)$, $U,V \subset K$ such that V is closed and $\overline{U} \cap V = \emptyset$. This kind of interpolating pair (U,V) appears in the proof of Theorem 8.2.7 when one takes an $f \in C(K,E)$ that vanishes on a closed set V and defines $U = \{t \in K\colon \|f(t)\| > \alpha\}$ for some $\alpha > 0$. In this case, the closure of U lies in $\{t \in K\colon \|f(t)\| \geqslant \alpha\}$, so $\overline{U} \cap V = \emptyset$.

For arbitrary $V \subset K$ denote by X_V the subspace of all functions from X vanishing on V. The next proposition extracts a property which the second component V of every interpolating pair (U,V) has.

Proposition 8.2.2. *Let $X \subset \ell_\infty(K,E)$ be as above and let (U,V) be an interpolating pair for X. Then for every $f \in X$*

$$\operatorname{dist}(f, X_V) \leqslant \sup_{t\in V} \|f(t)\|.$$

Proof. By the definition of an interpolating pair, for an arbitrary $\varepsilon > 0$ there exists an element $h \in X$, $\|h\| < \sup_{t\in V}\|f(t)\| + \varepsilon$, such that $h = 0$ on U and $h = f$ on V. Then the element $f - h$ belongs to X_V, so

$$\operatorname{dist}(f, X_V) \leqslant \|f - (f-h)\| = \|h\| < \sup_{t\in V}\|f(t)\| + \varepsilon,$$

which completes the proof. □

Lemma 8.2.3. *Let $X \subset \ell_\infty(K,E)$, $U,V \subset K$, $f \in S_{X_V}$, and $\varepsilon > 0$. Assume that $U \supset \{t \in K\colon \|f(t)\| > 1-\varepsilon\}$ and that (U,V) is an interpolating pair for X. If T is a strong Daugavet operator on X and $g \in B_X$, then there is a function $h \in X_V$, $\|h\| \leqslant 2+\varepsilon$, satisfying*

$$\|Th\| < \varepsilon,\ \|(g+h)\mathbb{1}_U\| < 1+\varepsilon \text{ and } \|(f+g+h)\mathbb{1}_U\| > 2-\varepsilon.$$

Before we enter the proof proper, we formulate a number of technical assertions that are easy to verify and will be needed later.

Sublemma 8.2.4. *If T is a strong Daugavet operator on a Banach space X, if $1-\eta < \|x\| < 1+\eta$ and $\|y\| < 1+\eta$, then there is some $z \in X$ such that*

$$\|x+y+z\| > 2-3\eta,\ \|y+z\| < 1+2\eta,\ \|Tz\| < \eta.$$

PROOF. Choose $x_0 \in S_X$ and $y_0 \in B_X$ such that $\|x_0-x\| < \eta$, $\|y_0-y\| < \eta$ and pick by Lemma 8.1.1 $z \in D(x_0, y_0, \eta)$ such that $\|Tz\| < \eta$; this z clearly works. $\square$

SUBLEMMA 8.2.5. *If $\|x\| < 1+\eta$, $\|y\| < 1+\eta$ and $\|(x+y)/2\| > 1-\eta$ in a normed space, then $\|\lambda x + (1-\lambda)y\| > 1-3\eta$ whenever $0 \leqslant \lambda \leqslant 1$.*

PROOF. If $\|\lambda x + (1-\lambda)y\| \leqslant 1-3\eta$ for some $0 \leqslant \lambda \leqslant 1/2$, then, since $\lambda_1 x + (1-\lambda_1)(\lambda x + (1-\lambda)y) = (x+y)/2$ for $\lambda_1 = (\frac{1}{2}-\lambda)/(1-\lambda) \in [0, 1/2]$,

$$\left\|\frac{x+y}{2}\right\| \leqslant \lambda_1(1+\eta) + (1-\lambda_1)(1-3\eta) = 1-(3-4\lambda_1)\eta \leqslant 1-\eta.$$

(The case $\lambda > 1/2$ is analogous.) $\square$

SUBLEMMA 8.2.6. *If $\|y\| < 1+\eta$ and $\|x+Ny\|/(N+1) > 1-3\eta$ in a normed space, then $\|(x+y)/2\| > 1-(2N+1)\eta$.*

PROOF. If $\|(x+y)/2\| \leqslant 1-(2N+1)\eta$, then

$$\begin{aligned}\left\|\frac{x+Ny}{1+N}\right\| &\leqslant \frac{2}{1+N}\left\|\frac{x+y}{2}\right\| + \Big(1-\frac{2}{1+N}\Big)\|y\| \\ &\leqslant \frac{2}{1+N}\big(1-(2N+1)\eta\big) + \Big(1-\frac{2}{1+N}\Big)(1+\eta) = 1-3\eta. \qquad \square\end{aligned}$$

PROOF OF LEMMA 8.2.3. We may assume that $\|T\| = 1$. Fix $N > 6/\varepsilon$ and $\delta > 0$ such that $2(2N+1)9^N\delta < \varepsilon$; let $\delta_n = 9^n\delta$ so that $(2N+1)\delta_N < \varepsilon/2$. Put $f_1 = f$, $g_1 = g$ and, using Lemma 8.1.1, pick $h_1 \in X$ such that

$$\|f_1+g_1+h_1\| > 2-\delta_1,\ \|g_1+h_1\| < 1+2\delta_0,\ \|Th_1\| < \delta_0.$$

We are going to construct functions $f_n, g_n, h_n \in X$ by induction so as to satisfy

(a) $f_{n+1} = \frac{1}{n+1}(f_1 + \sum_{k=1}^n (g_k+h_k)) = \frac{n}{n+1}f_n + \frac{1}{n+1}(g_n+h_n)$, with $1-3\delta_n < \|f_{n+1}\| < 1+\delta_n$,

(b) $g_{n+1} = g_1$ on U and $g_{n+1} = g_n+h_n$ $(= g_1+h_1+\cdots+h_n)$ on V, $\|g_{n+1}\| < 1+\delta_n$,

(c) $\|f_{n+1}+g_{n+1}+h_{n+1}\| > 2-\delta_{n+1}$, $1-2\delta_n < \|g_{n+1}+h_{n+1}\| < 1+6\delta_n < 1+\delta_{n+1}$, $\|Th_{n+1}\| < 3\delta_n$.

Suppose that these functions have already been constructed for the indices $1, \dots, n$. We then define f_{n+1} as in (a). Since, by induction hypothesis,

$$\|f_n\| < 1+\delta_{n-1} \quad\text{and}\quad \|g_n+h_n\| < 1+\delta_n,$$

we clearly have $\|f_{n+1}\| < 1+\delta_n$. From $\|f_n+g_n+h_n\| > 2-\delta_n$, we conclude using Sublemma 8.2.5 (with $\eta = \delta_n$), that $\|f_{n+1}\| > 1-3\delta_n$. Thus (a) is achieved. To achieve (b) it is enough to use that (U, V) is interpolating along with the induction hypothesis that $\|g_n+h_n\| < 1+\delta_n$. Finally, the existence of h_{n+1} that satisfies (c) follows from Sublemma 8.2.4 with $\eta = 3\delta_n$.

Next, we argue that

$$\left\|f_1 + \frac{1}{N}\sum_{k=1}^N (g_k+h_k)\right\| > 2-\varepsilon/2.$$

This follows from Sublemma 8.2.6, (c) and (a), and our choice of δ. But for $t \notin U$ we can estimate

$$\left\|f_1(t) + \frac{1}{N}\sum_{k=1}^N (g_k(t)+h_k(t))\right\| \leqslant 1-\varepsilon+1-\delta_N \leqslant 2-2\varepsilon,$$

therefore, letting $w = \frac{1}{N}\sum_{k=1}^{N} h_k$,

$$\|(f+g+w)\mathbb{1}_U\| = \left\|\left(f_1 + \frac{1}{N}\sum_{k=1}^{N}(g_k+h_k)\mathbb{1}_U\right)\right\| > 2-\varepsilon/2.$$

Furthermore, we have the estimates

$$\|(g+w)\mathbb{1}_U\| = \left\|\frac{1}{N}\sum_{k=1}^{N}(g_k+h_k)\mathbb{1}_U\right\| \leqslant 1+\delta_N < 1+\varepsilon/2,$$

$$\|Tw\| \leqslant \frac{1}{N}\sum_{k=1}^{N}\|Th_k\| < 3\delta_{N-1} = \frac{1}{3}\delta_N < \varepsilon/2,$$

$$\|h_k\| \leqslant \|g_k+h_k\| + \|g_k\| \leqslant 2+2\delta_k \leqslant 2+2\delta_N \leqslant 2+\varepsilon/2,$$

$$\|w\| \leqslant \frac{1}{N}\sum_{k=1}^{N}\|h_k\| \leqslant 2+\varepsilon/2$$

and, for $t \in V$,

$$\|w(t)\| = \frac{1}{N}\|g_{N+1}(t) - g_1(t)\| \leqslant \frac{2+\delta_N}{N} < \frac{3}{N} < \varepsilon/2.$$

By Proposition 8.2.2 and the above, we see that $\operatorname{dist}(w, X_V) < \varepsilon/2$. Hence it is left to replace w by an element $h \in X_V$, $\|h-w\| \leqslant \varepsilon/2$, to finish the proof. □

Remark that from the above lemma one can deduce Theorem 7.2.3: if $X = X_1 \oplus_\infty X_2$ and $T \in \mathcal{SD}(X)$, then $T|_{X_1} \in \mathcal{SD}(X_1)$. To see this, let $K = \operatorname{ext}(B_{X^*})$, $K_1 = \operatorname{ext}(B_{X_1^*})$, $K_2 = \operatorname{ext}(B_{X_2^*})$ so that $K = K_1 \cup K_2$ and $X \subset \ell_\infty(K)$ canonically. It is left to apply Lemma 8.2.3 with the interpolating pair (K_1, K_2).

In the sequel, for an element $y \in E$ and a function $f \in C(K)$, we use the notation $y \otimes f$ or $f \otimes y$ to denote the E-valued function on K acting by the rule $(y \otimes f)(t) := f(t)y$. For our convenience, when it does not lead to confusion, for $y \in E$ we write y instead of $y \otimes \mathbb{1}_K$. Recall also that when we speak about the space $C(K,E)$, K is always a compact Hausdorff space and E is a Banach space, so we will not repeat this in the statements of theorems involving $C(K,E)$.

THEOREM 8.2.7. *Let T be an operator on $X = C(K,E)$. Then, the following conditions are equivalent:*

(1) *$T \in \mathcal{SD}(X)$.*
(2) *For every closed subset $V \subsetneq K$, every $x \in S_E$, every $y \in B_E$ and every $\varepsilon > 0$ there exists an open subset $W \subset K \setminus V$, an element $e \in E$ with $\|e+y\| < 1+\varepsilon$, $\|e+y+x\| > 2-\varepsilon$, and a function $h \in X_V$, $\|h\| \leqslant 2+\varepsilon$, such that $\|Th\| < \varepsilon$ and $\|e - h(t)\| < \varepsilon$ for $t \in W$.*
(3) *For every closed subset $V \subsetneq K$, every $x \in S_E$, every $y \in B_E$ and every $\varepsilon > 0$ there exists a function $f \in X_V$ such that $\|Tf\| < \varepsilon$, $\|f+y\| < 1+\varepsilon$, $\|f+y+x\| > 2-\varepsilon$.*

If K has no isolated points, then these conditions are equivalent to

(4) *$T \in \mathcal{NAR}(X)$.*

PROOF. (1) ⇒ (2). Let us apply Lemma 8.2.3 to $\varepsilon/4 > 0$, $g = \mathbb{1}_K \otimes y$, $f = f_1 \otimes x \in S_X$, where f_1 is a positive scalar function vanishing on V, and $U = \{t \in K\colon \|f(t)\| > 1-\varepsilon/4\}$ (this kind of interpolating pair was described in

Remark 8.2.1). Then for $h \in X_V$, which we get from Lemma 8.2.3, let us find a point $t_0 \in U$ such that $\|(f+g+h)(t_0)\| = \|(f+h)(t_0)+y\| > 2-\varepsilon/4$. Because $\|h(t_0)+y\| < 1+\varepsilon/4$ we have $\|f(t_0)\| > 1-\varepsilon/2$, i.e., $\|f(t_0)-x\| < \varepsilon/2$. Now, select an open neighbourhood $W \subset U$ of t_0 such that $\|f(\tau)-x\| < \varepsilon/2$ for all $\tau \in W$ and put $e = h(t_0)$.

(2) $\Rightarrow$ (3). Let us fix a positive $\varepsilon < 1/10$, $\delta < \varepsilon/4$ and $N > 6+2/\varepsilon$. Now, apply inductively condition (2) to obtain elements x_k, y_k, e_k, $x_1 = x$, $y_k = y$, $k = 1,\dots,N$, open subsets $W_1 \supset W_2 \supset \dots$, closed subsets $V_{k+1} = K \setminus W_k$, $V_1 = V$ and functions $h_k \in X_{V_k}$ with the following properties:

(a) $x_{n+1} = \dfrac{x + \sum_{k=1}^n (y_k+e_k)}{\|x+\sum_{k=1}^n (y_k+e_k)\|} \in S_E$,

(b) $\|e_k + y_k\| < 1+\delta$, $\|e_k+y_k+x_k\| > 2-\delta$,

(c) $h_k \in X_{V_k}$, $\|h_k(t) - e_k\| < \varepsilon/4$ for all $t \in W_k$, $\|h_k\| \leqslant 2+\varepsilon$, and $\|Th_k\| < \varepsilon$.

By an argument similar to the one in Lemma 8.2.3, we have for a proper choice of δ

$$\Big\|x+y+\frac{1}{N}\sum_{k=1}^N e_k\Big\| = \Big\|x+\frac{1}{N}\sum_{k=1}^N (y_k+e_k)\Big\| > 2-\frac{\varepsilon}{2}.$$

Let us put $f = \frac{1}{N}\sum_{k=1}^N h_k$. Then, the last inequality and (c) of our construction yield that $f \in X_V$, $\|f+y+x\| > 2-\varepsilon$ and $\|Tf\| < \varepsilon$. The only thing left to do now is to estimate $\|f+y\|$ from above. If $t \in V$, then $\|f(t)+y\| = \|y\| \leqslant 1$. If $t \in W_n \setminus W_{n+1}$ for some n, then

$$\|f(t)+y\| = \Big\|\frac{1}{N}\sum_{k=1}^N h_k(t) + y\Big\| = \Big\|\frac{1}{N}\sum_{k=1}^N (h_k(t)+y)\Big\|.$$

In this last sum, all the summands except for the last one satisfy the inequality $\|h_k(t)+y\| \leqslant 1+\varepsilon/2$ and the last summand $h_n(t)+y$ is bounded by $3+\varepsilon$. So,

$$\|f(t)+y\| \leqslant \frac{1}{N}\sum_{k=1}^{N-1}\Big(1+\frac{\varepsilon}{2}\Big) + \frac{1}{N}(3+\varepsilon) \leqslant 1+\frac{\varepsilon}{2}+\frac{1}{N}(3+\varepsilon) \leqslant 1+\varepsilon.$$

The same estimate holds for $t \in W_N$.

(3) $\Rightarrow$ (1). Fix $f, g \in S_X$ and $0 < \varepsilon < 1/10$. Pick a point $t \in K$ with $\|f(t)\| > 1-\varepsilon/4$ and a neighbourhood U of t such that

$$\|f(t)-f(\tau)\| + \|g(t)-g(\tau)\| < \frac{\varepsilon}{4} \qquad \forall \tau \in U.$$

Denote $x = f(t)/\|f(t)\|$ and $y = g(t)$ and apply condition (3) to obtain a function $h \in X_V$ such that $\|Th\| < \varepsilon$, $\|h+y\| < 1+\varepsilon/4$ and $\|h+y+x\| > 2-\varepsilon/4$. For this h we have $\|h+g\| < 1+\varepsilon$ and $\|h+g+f\| > 2-\varepsilon$, so $T \in \mathcal{SD}(X)$.

Let us now pass to the case of a perfect compact K. The implication (4) $\Rightarrow$ (1) is evident.

The proof of the remaining implication (3) $\Rightarrow$ (4) is similar to that of (3) $\Rightarrow$ (1). Namely, let $f, g \in S_X$, $x^* \in X^*$ and let $\varepsilon > 0$ be small. We have to show that there is an element $h \in X$ such that

$$\|f+g+h\| > 2-\varepsilon, \quad \|g+h\| < 1+\varepsilon \tag{8.2.1}$$

and

$$\|Th\| + |x^*h| < \varepsilon. \tag{8.2.2}$$

To this end, using the absence of isolated points in K, let us pick a closed subset $V_0 \subsetneq K$, $V_0 \supset \{\tau \in K\colon \|f(\tau)\| \leqslant 1-\varepsilon/4\}$ such that

$$|x^*|_{X_{V_0}} < \frac{\varepsilon}{4}, \tag{8.2.3}$$

pick a point $t \in K \setminus V_0$ (then $\|f(t)\| > 1-\varepsilon/4$), and choose an open neighbourhood U of t, $U \subset K \setminus V_0$ such that for every $\tau \in U$

$$\|f(t) - f(\tau)\| + \|g(t) - g(\tau)\| < \frac{\varepsilon}{4}. \tag{8.2.4}$$

Denote $x = f(t)/\|f(t)\|$, $y = g(t)$ and apply condition (3) to $V := K \setminus U \supset V_0$. We obtain a function $h \in X_V \subset X_{V_0}$ such that $\|Th\| < \varepsilon/4$, $\|h+y\| < 1+\varepsilon/4$ and $\|h+y+x\| > 2-\varepsilon/4$. For this h, (8.2.1) follows from (8.2.4) and (8.2.2) follows from (8.2.3). $\square$

We are now in a position to provide an example, as promised in Chapter 6, of narrow operators with non-narrow sum, thus demonstrating in particular that sometimes $\mathcal{NAR}(X)$ is not a $\tilde{+}$-semigroup.

Let $T\colon E \to F$ be an operator between Banach spaces. By T^K let us denote the corresponding "multiplication" or "diagonal" operator $T^K\colon C(K,E) \to C(K,F)$ defined by

$$(T^K f)(t) = T(f(t)).$$

PROPOSITION 8.2.8. *$T^K \in \mathcal{SD}(C(K,E))$ if and only if $T \in \mathcal{SD}(E)$.*

PROOF. Criterion (3) of Theorem 8.2.7 immediately provides the proof. $\square$

Here is the announced example.

THEOREM 8.2.9. *There exists a Banach space X for which $\mathcal{NAR}(X)$ does not form a semigroup under the operation $\tilde{+}$; in fact, $C([0,1],\ell_1)$ is such a space, and, moreover, the set of narrow operators acting from $C([0,1],\ell_1)$ to $C([0,1],\ell_1)$ is not stable under the ordinary addition.*

PROOF. The key feature of ℓ_1 is that $\mathcal{SD}(\ell_1)$ is not a $\tilde{+}$-semigroup. According to Example 6.2.13, $x_1^*(x) = \sum_{n=1}^\infty x(n)$ and $x_2^*(x) = x(1) - \sum_{n=2}^\infty x(n)$ define strong Daugavet functionals on ℓ_1, but $x_1^* + x_2^*\colon x \mapsto 2x(1)$ is not in $\mathcal{SD}(\ell_1)$ and hence, $x_1^* \tilde{+} x_2^*$ is not, either.

Now, if $\mathcal{SD}(E)$ fails to be a $\tilde{+}$-semigroup, one can pick $T_1, T_2 \in \mathcal{SD}(E)$ with $T_1 \tilde{+} T_2 \notin \mathcal{SD}(E)$. Put $X = C([0,1],E)$; then by Proposition 8.2.8 and Theorem 8.2.7 $T_1^{[0,1]}, T_2^{[0,1]} \in \mathcal{NAR}(X))$, but $T_1^{[0,1]} \tilde{+} T_2^{[0,1]} \notin \mathcal{NAR}(X)$.

In order to make a similar example with the ordinary addition of operators on $C([0,1],\ell_1)$, fix some $e \in S_{\ell_1}$ and define $T_1, T_2 \in L(\ell_1)$ by the rule $T_j(x) = x_j^*(x)e$, $j = 1,2$. We have $T_j \sim x_j^*$, so they are strong Daugavet, but $T_1 + T_2 \sim x_1^* + x_2^*$ is not. Then $T_1^{[0,1]}, T_2^{[0,1]} \in L(C([0,1],\ell_1))$ are narrow but $T_1^{[0,1]} + T_2^{[0,1]} = (T_1+T_2)^{[0,1]} \in L(C([0,1],\ell_1))$ is not narrow. $\square$

8.3. C-narrow and narrow operators in $C(K,E)$

The following definition extends the notion of a C-narrow operator to the vector-valued setting. In order to distinguish between open and closed balls in a Banach space E, below we use the notation $B^0(x,\varepsilon) = \{z \in E\colon \|z-x\| < \varepsilon\}$.

DEFINITION 8.3.1. An operator $T \in L(C(K,E),W)$ is called *C-narrow* if there is a constant $\lambda > 0$ such that given $\varepsilon > 0$, $x \in S_E$ and $U \subset K$ open, there is a function $f \in C(K,E)$, $\|f\| \leqslant \lambda$, satisfying the following conditions:

(a) $\operatorname{supp}(f) \subset U$,
(b) $f^{-1}(B^0(x,\varepsilon)) \neq \emptyset$,
(c) $\|Tf\| < \varepsilon$.

As the following proposition shows, condition (b) of the previous definition can be substantially strengthened. In particular, the size of the constant λ is immaterial; but introducing this constant in the definition allows for more flexibility in applications. Also, Proposition 8.3.2 shows that for $E = \mathbb{R}$ the new notion of C-narrowness coincides with the one from Definition 6.1.5.

PROPOSITION 8.3.2. *If T is a C-narrow operator, then for every $\varepsilon > 0$, $x \in S_E$ and open set $U \subset K$ there is a function f of the form $g \otimes x$, where $g \in C(K)$,* $\operatorname{supp}(g) \subset U$, $\|g\| = 1$ *and g is nonnegative, such that $\|Tf\| < \varepsilon$.*

PROOF. Let us fix $\varepsilon > 0$, an open set U in K and $x \in S_E$. By Definition 8.3.1 we find a function $f_1 \in C(K,E)$ as described there corresponding to ε, U and x. Put $U_1 = U$ and $U_2 = f_1^{-1}(B^0(x,\frac{1}{2}))$. Then $\overline{U_2} \subset U_1$. As above, there is a function f_2 corresponding to ε, U_2 and x. We denote $U_3 = f_2^{-1}(B^0(x,\frac{1}{4}))$, $\overline{U_3} \subset U_2$ and continue the process. In the r^{th} step we get the set $U_r = f_{r-1}^{-1}(B^0(x,\frac{1}{2^{r-1}}))$ and apply Definition 8.3.1 to obtain a function f_r corresponding to U_r.

Choose $n \in \mathbb{N}$ so that $(\lambda+2)/n < \varepsilon$ and put $f = \frac{1}{n}(f_1 + \cdots + f_n)$. Now, using the Urysohn Lemma we find for each k a continuous real function g_k satisfying $0 \leqslant g_k(t) \leqslant 1$, $g_k|_{U_{k+1}} = 1$, $g_k|_{K\setminus U_k} = 0$, and consider $g = \frac{1}{n}(g_1 + \cdots + g_n)$. For this g we have $\frac{k-1}{n} \leqslant g(t) \leqslant \frac{k}{n}$ for all $t \in U_k \setminus U_{k+1}$, $k \in \overline{1,n}$; $g|_{U_{n+1}} = 1$ (so $\|g\| = 1$), and g vanishes outside U_1. We claim that $\|f - g \otimes x\| < \varepsilon$. Indeed, by our construction, if $t \in K \setminus U_1$, then $\|(f - g\otimes x)(t)\| = 0$; if $t \in U_{n+1}$, then

$$\begin{aligned}
\|(f - g\otimes x)(t)\| &= \Big\|\frac{1}{n}(f_1 + \cdots + f_n)(t) - g(t)\cdot x\Big\| \\
&= \Big\|\frac{1}{n}\big((f_1(t) - x) + \cdots + (f_n(t) - x)\big)\Big\| \\
&\leqslant \frac{1}{n}\Big(\frac{1}{2} + \cdots + \frac{1}{2^n}\Big) < \frac{1}{n} < \varepsilon.
\end{aligned}$$

Finally, if $t \in U_k \setminus U_{k+1}$, $k \in \overline{1,n}$, then

$$\begin{aligned}
\|(f - g\otimes x)(t)\| &= \Big\|\frac{1}{n}(f_1 + \cdots + g_k)(t) - g(t)\cdot x\Big\| \\
&\leqslant \Big\|\frac{1}{n}\big((f_1(t) - x) + \cdots + (f_{k-1}(t) - x) + f_k(t)\big)\Big\| + \frac{1}{n} \\
&\leqslant \frac{1}{n}\Big(\frac{1}{2} + \cdots + \frac{1}{2^{k-1}} + \lambda\Big) + \frac{1}{n} < \frac{\lambda+2}{n} < \varepsilon,
\end{aligned}$$

which demonstrates the announced inequality $\|f - g\otimes x\| < \varepsilon$. Moreover,

$$\|Tf\| \leqslant \frac{1}{n}\left(\|Tf_1\| + \|Tf_2\| + \cdots + \|Tf_n\|\right) < \varepsilon.$$

Thus, $\|T(g\otimes x)\| \leqslant \|Tf\| + \|T(f - g\otimes x)\| < \varepsilon + \varepsilon\|T\|$ and, since ε was chosen arbitrarily, we are done. $\square$

COROLLARY 8.3.3. *$T\colon C(K,E) \to W$ is C-narrow if and only if, for each $x \in E$, the restriction $T_x\colon C(K) \to W$, $T_x(g) = T(g \otimes x)$, is C-narrow.*

PROPOSITION 8.3.4.

(a) *Every C-narrow operator on $C(K,E)$ is a strong Daugavet operator. Hence, in the case of a perfect compact K every C-narrow operator on $C(K,E)$ is narrow.*

(b) *If E is a separable USD-nonfriendly space, then every strong Daugavet operator on $C(K,E)$ is C-narrow.*

(c) *If every strong Daugavet operator on $C(K,E)$ is C-narrow, then E is SD-nonfriendly.*

PROOF. (a) Let T be C-narrow. We will use criterion (3) of Theorem 8.2.7. Let $F \subset K$ be a closed subset, $x \in S_E$, $y \in B_E$ and $\varepsilon > 0$. According to Proposition 8.3.2 there exists a function f vanishing on F of the form $g \otimes (x - y)$, where $g \in C(K)$, $\|g\| = 1$ and g is nonnegative, such that $\|Tf\| < \varepsilon$. Evidently this f satisfies all the demands of criterion (3) in Theorem 8.2.7.

(b) Let T be a strong Daugavet operator, and suppose E is separable. Let $U \subset K$ be a nonvoid open subset. Given $x, y \in S_E$ and $\varepsilon' > 0$ we define

$$\begin{aligned} O(x,y,\varepsilon') = \{t \in U\colon\ \exists f \in C(K,E)\colon\ &\operatorname{supp} f \subset U, \\ &\|f + y\| < 1 + \varepsilon', \|f(t) + y + x\| > 2 - \varepsilon',\ \|Tf\| < \varepsilon'\}. \end{aligned}$$

This is an open subset of K, and by Theorem 8.2.7(3) it is dense in U. Now, pick a countable dense subset $\{(x_n, y_n)\colon n \in \mathbb{N}\}$ of $S_E \times S_E$ and a null sequence (ε_n). Then by Baire's theorem, $G := \bigcap_n O(x_n, y_n, \varepsilon_n)$ is nonempty.

Let $\varepsilon > 0$, and fix $t_0 \in G$. We denote by $A(U, \varepsilon)$ the closure of

$$\{f(t_0)\colon f \in C(K,E),\ \|f\| < 2 + \varepsilon,\ \|Tf\| < \varepsilon,\ \operatorname{supp} f \subset U\};$$

this is an absolutely convex set. We claim that $A(U,\varepsilon)$ intersects each set $D(x,y,\varepsilon') \in \mathcal{D}(E)$. Indeed, if $\|x_n - x\| < \varepsilon'/4$, $\|y_n - y\| < \varepsilon'/4$, $\varepsilon_n < \varepsilon'/2$ and $\varepsilon_n < \varepsilon$, then for a function f_n as appearing in the definition of $O(x_n, y_n, \varepsilon_n)$ we have $f_n(t_0) \in A(U,\varepsilon) \cap D(x_n, y_n, \varepsilon_n) \subset A(U,\varepsilon) \cap D(x,y,\varepsilon')$.

Since E is USD-nonfriendly, say with parameter α, the set $A(U,\varepsilon)$ contains αB_E. This implies that T satisfies the definition of a C-narrow operator with constant $\lambda = 3/\alpha$.

(c) Let $T \in \mathcal{SD}(E)$; then by Proposition 8.2.8 T^K is a strong Daugavet operator on $C(K,E)$. But

$$\big(T^K(g \otimes e)\big)(t) = T\big((g \otimes e)(t)\big) = g(t)Te,$$

hence T^K is not C-narrow by Proposition 8.3.2 unless $T = 0$. $\square$

The example $E = c_0$ shows that the converse of (b) is false. We have already pointed out in Proposition 8.1.5(a) that c_0 fails to be USD-nonfriendly; yet every strong Daugavet operator on $C(K, c_0)$ is C-narrow. To see this, we first remark that it is enough to check the condition spelled out in Proposition 8.3.2 for x in a dense subset of S_E. In our context, we may therefore assume that the sequence x vanishes eventually, say $x(n) = 0$ for $n > N$. If we write $c_0 = \ell_\infty^{(N)} \oplus_\infty Z$, with Z the space of null sequences supported on $\{N+1, N+2, \dots\}$, we also have $C(K, c_0) = C\left(K, \ell_\infty^{(N)}\right) \oplus_\infty C(K, Z)$. By Theorem 7.2.3 the restriction of any strong

Daugavet operator T on $C(K, c_0)$ to $C(K, \ell_\infty^{(N)})$ is again a strong Daugavet operator, and hence it is C-narrow, because $\ell_\infty^{(N)}$ is USD-nonfriendly (see Proposition 8.1.9). This implies that T is C-narrow.

We do not know whether (c) is actually an equivalence (see Question (8.2) in Section 8.9).

One of the fundamental properties of C-narrow operators is stated in our next theorem.

THEOREM 8.3.5. *Suppose that operators T, $T_n \in L(C(K,E),W)$ are such that the series $\sum_{n=1}^\infty w^*(T_n f)$ converges absolutely to $w^*(Tf)$, for every $w^* \in W^*$ and $f \in C(K,E)$. If all the T_n are C-narrow, then so is T. In particular, the sum of two C-narrow operators, or the sum of a pointwise unconditionally convergent series of C-narrow operators, is a C-narrow operator again.*

For the proof of Theorem 8.3.5 we need an auxiliary concept. A similar idea has appeared in [**175**].

DEFINITION 8.3.6. Let G be a closed G_δ-set in K and $T \in L(C(K),W)$. We say that G is a *vanishing set* of T if there is a sequence of open sets $(U_i)_{i\in\mathbb{N}}$ in K and a sequence of functions $(f_i)_{i\in\mathbb{N}}$ in $S_{C(K)}$ such that

(a) $G = \bigcap_{i=1}^\infty U_i$;
(b) $\operatorname{supp}(f_i) \subset U_i$;
(c) $\lim_{i\to\infty} f_i = \mathbb{1}_G$ pointwise;
(d) $\lim_{i\to\infty} \|Tf_i\| = 0$.

The collection of all vanishing sets of T is denoted by $\operatorname{van} T$.

Let $T \in L(C(K),W)$. By the Riesz Representation Theorem, T^*w^* can be viewed as a regular measure on the Borel subsets of K whenever $w^* \in W^*$. For convenience, we denote it by T^*w^* as well.

LEMMA 8.3.7. *Suppose G is a closed G_δ-set in K and $T \in L(C(K),W)$. Then $G \in \operatorname{van} T$ if and only if $T^*w^*(G) = 0$ for all $w^* \in W^*$. Moreover, in Definition 8.3.6 one can select the sequence $(U_k)_{k\in\mathbb{N}}$ with the additional property that $\overline{U}_{k+1} \subset U_k$, $k \in \mathbb{N}$.*

PROOF. Let $G \in \operatorname{van} T$, and pick functions $(f_k)_{k\in\mathbb{N}}$ as in Definition 8.3.6. Then by the Lebesgue Dominated Convergence Theorem, for any given $w^* \in W^*$ we have

$$T^*w^*(G) = \int_K \mathbb{1}_G \, dT^*w^* = \lim_{k\to\infty} \int_K f_k \, dT^*w^* = \lim_{k\to\infty} w^*(Tf_k) = 0.$$

Conversely, a closed G_δ-set in a Hausdorff compact K can be written as $G = \bigcap_{k=1}^\infty U_k$ where $(U_k)_{k\in\mathbb{N}}$ is a sequence of open sets in K such that $\overline{U}_{k+1} \subset U_k$ for all k. By the Urysohn Lemma, there exist functions $(g_k)_{k\in\mathbb{N}}$ having the following properties: $0 \leqslant g_k(t) \leqslant 1$ for all $t \in K$, $\operatorname{supp} g_k \subset U_k$, and $g_k(t) = 1$ if $t \in \overline{U}_{k+1}$. Clearly, $\lim_{k\to\infty} g_k = \mathbb{1}_G$ pointwise and

$$\lim_{k\to\infty} w^*(Tg_k) = \lim_{k\to\infty} T^*w^*(g_k) = T^*w^*(G) = 0$$

whenever $w^* \in W^*$. This means that the sequence $(Tg_k)_{k\in\mathbb{N}}$ is weakly null. Applying the Mazur Theorem 2.1.2 we finally obtain a sequence (f_n) of convex combinations of the functions $(g_k)_{k\in\mathbb{N}}$ that converges strongly to 0, $f_n \in \operatorname{conv}\big(\{g_k \colon k \geqslant n\}\big)$, $n = 1, 2, \ldots$, which satisfies all the conditions of Definition 8.3.6.

This completes the proof. $\square$

LEMMA 8.3.8. *An operator $T \in L(C(K),W)$ is C-narrow if and only if every nonvoid open set $U \subset K$ contains a nonvoid vanishing set of T. Moreover, if $(T_n)_{n\in\mathbb{N}} \subset L(C(K),W)$ is a sequence of C-narrow operators, every open set $U \neq \emptyset$ contains a set $G \neq \emptyset$ that is simultaneously a vanishing set for each T_n.*

PROOF. We first prove the more general "moreover" part. Put $U_{1,1} = U$. By the definition of a C-narrow operator and Proposition 8.3.2 there is a function $f_{1,1} \subset S_{C(K)}$ with $\operatorname{supp}(f_{1,1}) \subset U_{1,1}$, $U_{1,2} := f_{1,1}^{-1}(\frac{1}{2},1] \neq \emptyset$ and $\|T_1 f_{1,1}\| < \frac{1}{2}$. Obviously, $\overline{U}_{1,2} \subset f_{1,1}^{-1}[\frac{1}{2},1] \subset U_{1,1}$. Again applying the definition we find $f_{1,2} \in S_{C(K)}$ with $\operatorname{supp}(f_{1,2}) \subset U_{1,2}$, $U_{2,1} = f_{1,2}^{-1}(\frac{2}{3},1] \neq \emptyset$ and $\|T_1 f_{1,2}\| < \frac{1}{3}$. As above $\overline{U}_{2,1} \subset U_{1,2}$.

In view of the C-narrowness of T_2 there exists a function $f_{2,1} \in S_{C(K)}$ with $\operatorname{supp}(f_{2,1}) \subset U_{2,1}$, $U_{1,3} = f_{2,1}^{-1}(\frac{2}{3},1] \neq \emptyset$ and $\|T_2 f_{2,1}\| < \frac{1}{3}$. In the next step we construct $f_{1,3} \in S_{C(K)}$ such that $U_{2,2} = f_{1,3}^{-1}(\frac{3}{4},1] \neq \emptyset$ and $\|T_1 f_{1,3}\| < \frac{1}{4}$.

Proceeding in the same way, in the n^{th} step we find a set of functions $(f_{k,l})_{k+l=n} \subset S_{C(K)}$ and nonempty open sets $(U_{k,l})_{k+l=n}$ in K such that $\operatorname{supp}(f_{k,l}) \subset U_{k,l}$, $\|T_k f_{k,n-k}\| < \frac{1}{n}$ and $U_{k,l} = f_{k-1,l+1}^{-1}(\frac{n-1}{n},1]$, if $k \neq 1$. Then we put $U_{1,n} = f_{n-1,1}^{-1}(\frac{n-1}{n},1]$ to start the next step.

It remains to show that the set $G = \bigcap_{k,l\in\mathbb{N}} U_{k,l} = \bigcap_{k,l\in\mathbb{N}} \overline{U}_{k,l}$ is as desired. Indeed, G is clearly a nonempty closed G_δ-set and $G = \bigcap_{i=1}^\infty U_{n,i}$ for every $n \in \mathbb{N}$. It is easily seen that the sequences $(f_{n,i})_{i\in\mathbb{N}}$ and $(U_{n,i})_{i\in\mathbb{N}}$ meet the conditions of Definition 8.3.6 for the operator T_n. So, $G \in \operatorname{van} T_n$ for every $n \in \mathbb{N}$.

To prove the converse, let $U \neq \emptyset$ be any open set in K and let $\varepsilon > 0$. By assumption on $\operatorname{van} T$ we can find a closed G_δ-set $\emptyset \neq G \subset U$, $G \in \operatorname{van} T$. Consider the open sets $(U_i)_{i\in\mathbb{N}}$ and functions $(f_i)_{i\in\mathbb{N}}$ provided by Definition 8.3.6. For sufficiently large $i \in \mathbb{N}$ we have $U_i \subset U$ and $\|Tf_i\| < \varepsilon$ so that f_i may serve as a function as required in Definition 8.3.1.

This finishes the proof. $\square$

Now, we are in a position to prove Theorem 8.3.5.

PROOF OF THEOREM 8.3.5. By Corollary 8.3.3, we may assume that $E = \mathbb{R}$. By Lemma 8.3.8 it suffices to show that $\bigcap_{n=1}^\infty \operatorname{van} T_n \subset \operatorname{van} T$.

Suppose $G \in \bigcap_{n=1}^\infty \operatorname{van} T_n$. According to Lemma 8.3.7, we need to prove that $T^*w^*(G) = 0$ for all $w^* \in W^*$. By the condition of the theorem, the series $\sum_{n=1}^\infty T_n^* w^*$ is weakly* unconditionally Cauchy and hence is weakly unconditionally Cauchy (Proposition 2.4.4). Since $C(K)^*$ does not contain a copy of c_0, this series is actually norm convergent by the Bessaga-Pełczyński Theorem 2.4.3. This implies that for the bounded sequence of functions $(f_k)_{k\in\mathbb{N}}$ satisfying $f_k \to \mathbb{1}_G$ pointwise constructed in the proof of Lemma 8.3.7, we have

$$\begin{aligned} T^*w^*(G) &= \lim_{k\to\infty} T^*w^*(f_k) = \lim_{k\to\infty} \sum_{n=1}^\infty T_n^* w^*(f_k) \\ &= \sum_{n=1}^\infty T_n^* w^*(\mathbb{1}_G) = \sum_{n=1}^\infty T_n^* w^*(G) = 0. \end{aligned}$$ $\square$

We get the following consequence.

Corollary 8.3.9. *The class of C-narrow operators on $C(K, E)$ is a semigroup with respect to the operation $\tilde{+}$.*

Proof. Let $T_1 \in L(C(K.E), W_1)$, $T_2 \in L(C(K, E), W_1)$ be C-narrow. Define $W := W_1 \oplus_1 W_2$. Then, according to the definition, $T_1 \tilde{+} T_2 \in L(C(K, E), W)$, $(T_1 \tilde{+} T_2)f = (T_1 f, T_2 f)$. T_1 and T_2 are equivalent to the operators $\widetilde{T}_1$, $\widetilde{T}_2$ acting from $C(K, E)$ to W by the rule $\widetilde{T}_1 f := (T_1 f, 0)$, $\widetilde{T}_2 f := (0, T_2 f)$. These $\widetilde{T}_1$, $\widetilde{T}_2$ are C-narrow, so $T_1 \tilde{+} T_2 = \widetilde{T}_1 + \widetilde{T}_2$ is C-narrow as well. □

Corollary 8.3.10. *If E is separable and USD-nonfriendly, then $\mathcal{NAR}(C(K, E))$ is a $\tilde{+}$-semigroup. Moreover, in this case a pointwise unconditionally convergent sum of narrow operators on $C(K, E)$ acting to the same space W is a narrow operator itself. In particular, this happens for operators on the real or complex space $C(K)$.*

Indeed, this follows from Theorem 8.3.5 and Proposition 8.3.4; note that K is perfect if there exists a narrow operator defined on $C(K, E)$ in case E fails the Daugavet property. To see the latter assume that $K = \{k\} \cup K'$ for some isolated point k. If there exists a narrow operator on $C(K, E) \cong E \oplus_\infty C(K', E)$, then the restriction of this operator to the first summand E is narrow (Corollary 7.2.2), and so $E \in$ DPr. Finally, for the last statement, just recall that the complex space $C(K)$ is isometric to the real space $C(K, \ell_2^{(2)})$.

8.4. Strong Daugavet and narrow operators in $C(K)$. C-rich subspaces

Let us start with gathering, for the reader's convenience, the criteria of narrowness for operators on $C(K)$ that were demonstrated in the previous section for the case vector-valued function spaces (Propositions 8.3.4 and 8.3.2); C-narrow operators were introduced in Definition 6.1.5. Let us remark that the complex case in the proposition below is covered by the vector-valued real space $C(K, \ell_2^{(2)})$.

Below we denote $C_F(K) = \{f \in C(K)\colon f|_F = 0\}$ for $F \subset K$.

Proposition 8.4.1. *For an operator T on $C(K)$, the following assertions are equivalent:*

(i) *T is C-narrow.*

(ii) *T is unbounded from below on the unit sphere of each subspace $C_F(K)$, where $F \subsetneq K$ is closed in K. In other words, for every nonvoid open $U \subset K$ and every $\varepsilon > 0$ there is $f \in C(K)$ with $\|f\| = 1$ and $\operatorname{supp}(f) \subset U$ such that $\|Tf\| < \varepsilon$.*

(iii) *For every nonvoid open $U \subset K$ and every $\varepsilon > 0$ there is a non-negative function $f \in C(K)$ with $\|f\| = 1$ and $\operatorname{supp}(f) \subset U$ such that $\|Tf\| < \varepsilon$.*

If K has no isolated points, the above conditions are equivalent to T being narrow.

We now present the companion definition to C-narrow operators for subspaces.

Definition 8.4.2. Let K be a compact Hausdorff space. A closed subspace $E \subset C(K)$ is called *C-rich* if the quotient map $q\colon C(K) \to C(K)/E$ is C-narrow.

We reiterate the remark that only spaces with the Daugavet property can support narrow operators, respectively can have rich subspaces. However, the notions of C-narrowness, respectively C-richness, do not stipulate such a requirement. For example, c_0 is a C-rich subspace of c.

We have the following corollary to Proposition 8.4.1.

Corollary 8.4.3. *For a subspace $E \subset C(K)$, the following assertions are equivalent.*

(i) *E is C-rich.*
(ii) *E is almost rich.*
(iii) *For every proper closed subset $F \subsetneq K$ and every $\varepsilon > 0$, there is a function $f \in C_F(K)$ with $\|f\| = 1$ whose distance to E is less than ε.*

If K has no isolated points, the above conditions are equivalent to E being a rich subspace.

Example 8.4.4. With Corollary 8.4.3 in hand, we shall now look at uniform algebras. Let $A \subset C(K)$ be a uniform algebra whose *Shilov boundary*, i.e., the closure of its Choquet boundary, is K. Then A is a C-rich subspace of $C(K)$. Indeed, by Corollary 8.4.3 it is enough to show for a closed subset $F \subsetneq K$ and for $\varepsilon > 0$ that there is a continuous function f of norm 1 vanishing on F whose distance to A is $< \varepsilon$. But a fundamental theorem in the theory of uniform algebras ensures that there is some $g \in A$ with $\|g\| = 1$ and $|g| < \varepsilon$ on F ([**202**, p. 49 and p. 78]), hence an obvious modification of g yields a function f as requested.

If K has no isolated points, then A is even rich.

Remark 8.4.5. *For a perfect compact K, each $C_F(K)$ contains an isomorphic copy of $C[0,1]$. Consequently, in this case every operator on $C(K)$ not fixing copies of $C[0,1]$ is narrow. This implies the fact, obtained independently in [**175, 297**], that those $T\colon C(K) \to C(K)$ that do not fix copies of $C[0,1]$ satisfy the Daugavet equation.*

Remark also that among operators that fix copies of $C[0,1]$, one can find some narrow operators. Before giving the corresponding Example 8.4.6, recall that, according to Milutin's theorem, for every perfect metrisable compact K the corresponding $C(K)$ is isomorphic to $C[0,1]$; that, in this case, each $C_F[0,1]$ is isomorphic to $C[0,1]$ as well, and that, according to Pełczyński's theorem, each complemented subspace of $C[0,1]$ containing a copy of $C[0,1]$ is isomorphic to $C[0,1]$. For these results we refer to the memoir [**246**].

Example 8.4.6. Let $\mathcal{K} \subset [0,1]$ be the Cantor set. Define $P\colon C[0,1] \to C[0,1]$ as follows: for $t \in \mathcal{K}$ put $(Pf)(t) = f(t)$, and on each open interval (α, β) complementary to $\mathcal{K}$ interpolate $(Pf)(t)$ linearly:

$$(Pf)(t) = \frac{t-\beta}{\alpha-\beta} f(\alpha) + \frac{t-\alpha}{\beta-\alpha} f(\beta).$$

This P is a projection whose image is the subspace $E \subset C[0,1]$ consisting of functions that are linear on each open interval complementary to $\mathcal{K}$. The natural restriction operator $R\colon E \to C(\mathcal{K})$ is an isometry, so E is isometric to $C(\mathcal{K})$ and, by Milutin's theorem, is isomorphic to $C[0,1]$. The restriction of P to E is Id_E, which means that P fixes a copy of $C[0,1]$. At the same time, P vanishes on each function whose support lies in the complement of the Cantor set $\mathcal{K}$, which implies that P is narrow by Proposition 8.4.1.

The stability of narrow operators on $C(K)$ with respect to pointwise unconditionally convergent sums (Corollary 8.3.10) leads to the following much stronger version of Corollary 5.3.8 for $X = C[0,1]$ saying that a Banach space with the Daugavet property cannot be isomorphically embedded into a space E having an unconditional basis or having a representation as unconditional direct sum of reflexive subspaces.

THEOREM 8.4.7. *Let $C[0,1]$ be isomorphically embedded in a Banach space Y which is represented as an unconditional direct sum of its subspaces*

$$Y = \bigoplus_{n=1}^{\infty} X_n. \tag{8.4.1}$$

Then at least one of the X_n contains a copy of $C[0,1]$. In the particular case of $Y = C[0,1]$, at least one of the X_n in (8.4.1) is isomorphic to $C[0,1]$.

PROOF. Without loss of generality we may assume that $C[0,1] \subset Y$. Denote the natural embedding operator by $J\colon C[0,1] \to Y$ and denote the natural projections of Y on X_n by P_n. The operator J is bounded below, so it is not C-narrow. On the other hand, $\sum_{n=1}^{\infty} P_n J$ is pointwise unconditionally convergent to J, so Corollary 8.3.10 implies that some $P_m J$ is not C-narrow. This $P_m J$ must be bounded below on some $C_F[0,1]$, consequently the corresponding X_m contains an isomorphic copy of $C_F[0,1]$, which in its turn contains an isomorphic copy of $C[0,1]$. The improved statement for the particular case of $Y = C[0,1]$ follows from the fact that X_m is complemented in $Y = C[0,1]$. □

For $\Delta = [\alpha, \beta] \subset [0,1]$, denote by $C_0(\Delta)$ the subspace of those $f \in C[0,1]$ for which $\operatorname{supp} f \subset \Delta$. In other words, $C_0(\Delta) = C_{[0,1]\setminus\Delta}[0,1]$. Recall that $X \subset C[0,1]$ is rich if the quotient map $q\colon C[0,1] \to C[0,1]/X$ is narrow. Taking into account Proposition 8.4.1, we can say that $X \subset C[0,1]$ is rich if and only if for every segment $\Delta \subset [0,1]$ and every $\varepsilon > 0$ there exists a non-negative $g \in C_0(\Delta)$ with $\|g\| = 1$ for which $\|q(g)\| < \varepsilon$. With the help of the formula $\|q(g)\| = \operatorname{dist}(g, X)$, we obtain the following.

PROPOSITION 8.4.8. *A subspace $X \subset C[0,1]$ is rich if and only if for every segment $\Delta \subset [0,1]$ and every $\varepsilon > 0$ there is a non-negative function $g \in C_0(\Delta)$ with $\|g\| = 1$ and there is $f \in X$ such that $\|f - g\| < \varepsilon$.*

Remark that in the statement of Proposition 8.4.8 we may demand additionally that $f \in S_X$.

Our next goal is to give a complete isomorphic description of rich subspaces in $C[0,1]$. For simplicity, we are going to deal with the *real* space $C[0,1]$. The complex case can be managed similarly. We start with a technical lemma.

LEMMA 8.4.9. *Let $X \subset C[0,1]$ be a rich subspace, $\Delta = [\alpha, \beta] \subset [0,1]$, and $\varepsilon \in (0, 1/2)$. Then there are $f_{n,k} \in X$ and closed segments $\Delta_{n,k} \subset \Delta$, $n \in \mathbb{N}$, $k \in \overline{1, 2^{n-1}}$, such that*

(a) *$f_{n,k}(t) > 1 + \varepsilon 2^{-n-1}$ for all $t \in \Delta_{n,k}$;*

(b) *$\sum_{k=1}^{2^{n-1}} |f_{n,k}(t)| < 1 + \varepsilon - \varepsilon 2^{-n-1}$ for $t \in \Delta$ and $\sum_{k=1}^{2^{n-1}} |f_{n,k}(t)| < \varepsilon(1 - 2^{-n})$ for $t \in [0,1] \setminus \Delta$;*

(c) *for each n, the collection $(\Delta_{n,k})_{k \in \overline{1,2^{n-1}}}$ is disjoint;*

(d) *$f_{n,k} = f_{n+1,2k-1} + f_{n+1,2k}$;*

(e) *$\Delta_{n,k} \supset \Delta_{n+1,2k-1} \sqcup \Delta_{n+1,2k}$.*

PROOF. We will proceed by induction on n.

The start of the induction: $n = 1$. Using Proposition 8.4.8, select functions $g \in S_{C_0(\Delta)}$ and $f \in S_X$ such that $g \geqslant 0$ and $\|f - g\| < \varepsilon$. Put $f_{1,1} = (1 + \frac{\varepsilon}{2})f$, and in the neighbourhood of a point at which $f_{1,1}$ attains its maximum select a segment $\Delta_{1,1}$ on which $f_{1,1}(t) > 1 + \frac{\varepsilon}{4}$.

The inductive step. Assume that $f_{n,k} \in X$ and $\Delta_{n,k} \subset \Delta$, $n = 1, 2, \ldots, N$, $k \in \overline{1, 2^{n-1}}$ are already selected in such a way that (a), (b), (c) are satisfied for $n \leqslant N$, and (d), (e) are satisfied for $n \leqslant N-1$. Our goal is to construct $f_{N+1,k}$ and $\Delta_{N+1,k}$, $k \in \overline{1, 2^N}$. For this, let us consider $j \in \overline{1, 2^{N-1}}$ and denote $\Delta^*_{N,j}$ the left half $[a, \frac{a+b}{2}]$ of the segment $\Delta_{N,k} = [a, b]$. By richness of X there are $g \in S_{C_0(\Delta^*_{N,j})}$ and $f \in S_X$ such that $g \geqslant 0$ and $\|f - g\| < \varepsilon/2^{2N+3}$. For that f we have

$$-\frac{\varepsilon}{2^{2N+3}} \leqslant f(t) \leqslant 1; \max_{t \in \Delta^*_{N,j}} f(t) = 1 \text{ and } \sup_{t \in [0,1] \setminus \Delta^*_{N,j}} f(t) < \frac{\varepsilon}{2^{2N+3}}.$$

Denote by λ the maximal real coefficient for which $\lambda f \leqslant f_{N,j}$ on $\Delta^*_{N,j}$. Then $\lambda \in [1+\varepsilon 2^{-n-1}, 1+\varepsilon]$. Define $f_{N+1,2j-1} := \lambda f$ and $f_{N+1,2j} := f_{N,j} - \lambda f$. Performing this procedure for each $j \in \overline{1, 2^{N-1}}$, we obtain all the needed $f_{N+1,k}$, $k \in \overline{1, 2^N}$. With this definition, the condition (d) with $n = N$ is satisfied automatically. Next,

$$\max_{t \in \Delta^*_{N,j}} f_{N+1,2j-1}(t) \geqslant 1 + \frac{\varepsilon}{2^{N+1}}, \min_{t \in [0,1]} f_{N+1,2j-1}(t) \geqslant -\frac{\varepsilon}{2^{2N+2}} \tag{8.4.2}$$

and

$$\sup_{t \in [0,1] \setminus \Delta^*_{N,j}} f_{N+1,2j-1}(t) < \frac{\varepsilon}{2^{2N+2}}.$$

Denote $\Delta^{**}_{N,j} = \Delta_{N,j} \setminus \Delta^*_{N,j}$. The inequalities above imply, for all $t \in [0, 1]$,

$$|f_{N,j}(t)| \leqslant |f_{N+1,2j}(t)| + |f_{N+1,2j-1}(t)| \leqslant |f_{N,j}(t)| + \frac{\varepsilon}{2^{2N+1}}, \tag{8.4.3}$$

which, together with the inductive assumption (a) for the functions $f_{N,j}$, implies the inequality

$$\max_{t \in \Delta^{**}_{N,j}} f_{N+1,2j}(t) > 1 + \frac{\varepsilon}{2^{N+2}}. \tag{8.4.4}$$

The inductive assumption and (8.4.3) evidently imply (b) with $n = N + 1$. Finally, applying (8.4.2) and (8.4.4), we may select segments $\Delta_{N+1,2j-1} \subset \Delta^*_{N,j}$ and $\Delta_{N+1,2j} \subset \Delta^{**}_{N,j}$ near the points of maximum of $f_{N+1,2j-1}$ and $f_{N+1,2j}$ respectively in such a way that all the values of $f_{N+1,2j-1}$ on $\Delta_{N+1,2j-1}$ and all the values of $f_{N+1,2j}$ on $\Delta_{N+1,2j}$ are greater than $1 + \frac{\varepsilon}{2^{N+2}}$. Then (a), (c) and (e) will be fulfilled as well. $\square$

THEOREM 8.4.10. *Let $X \subset C[0, 1]$ be a rich subspace, $\Delta = [\alpha, \beta] \subset [0, 1]$ and $\theta \in (0, 1/2)$. Then there is a subspace $Y \subset X$ which is complemented in $C[0, 1]$, is isomorphic to $C[0, 1]$, and satisfies that*

$$\sup\{|y(t)|\colon t \in [0, 1] \setminus \Delta\} \leqslant \theta\|y\| \tag{8.4.5}$$

for all $y \in Y$.

PROOF. Applying Lemma 8.4.9 to X, Δ and $\varepsilon = \theta/4$, we obtain the corresponding $f_{n,k} \in X$ and $\Delta_{n,k} \subset \Delta$. We are going to demonstrate that $Y := \overline{\text{lin}} \left\{ f_{n,k}\colon n \in \mathbb{N}, \ k \in \overline{1, 2^{n-1}} \right\}$ is what we need.

Let us start with (8.4.5). By continuity, it is sufficient to check (8.4.5) for $y \in \operatorname{lin}\left\{f_{n,k}\colon n \in \mathbb{N},\ k \in \overline{1, 2^{n-1}}\right\}$. Thanks to (d), each $y \in \operatorname{lin}\left\{f_{n,k}\colon n \in \mathbb{N},\ k \in \overline{1, 2^{n-1}}\right\}$ can be written as $y = \sum_{k=1}^{2^{N-1}} \alpha_k f_{N,k}$ if N is big enough. Select $k_0 \in \overline{1, 2^{N-1}}$ in such a way that $|\alpha_{k_0}| = \max\{|\alpha_k|\colon k \in \overline{1, 2^{N-1}}\}$, and consider a $t_0 \in \Delta_{N,k_0}$. By the property (a) of the system $f_{n,k}$,

$$|f_{N,k_0}(t_0)| > 1 + \frac{\varepsilon}{2^{N+1}}.$$

Denote $A = \overline{1, 2^{N-1}} \setminus \{k_0\}$. Then from (b) we deduce that

$$\sum_{k \in A} |f_{N,k}(t_0)| = \sum_{k=1}^{2^{N-1}} |f_{N,k}(t_0)| - |f_{N,k_0}(t_0)| \leqslant \varepsilon \left(1 - \frac{1}{2^N}\right). \tag{8.4.6}$$

Then,

$$\begin{aligned}
\|y\| &\geqslant \left|\sum_{k=1}^{2^{N-1}} \alpha_k f_{N,k}(t_0)\right| \geqslant |\alpha_{k_0}||f_{N,k_0}(t_0)| - \sum_{k \in A} |\alpha_k||f_{N,k}(t_0)| \\
&\geqslant |\alpha_{k_0}| \left(1 + \frac{\varepsilon}{2^{N+1}}\right) - \max_{k \in A} |\alpha_k| \varepsilon \left(1 - \frac{1}{2^N}\right) \qquad (8.4.7)\\
&\geqslant (1 - \varepsilon) \max_{k \in \{1, \ldots, 2^{N-1}\}} |\alpha_k|.
\end{aligned}$$

On the other hand, according to (b) we have for $t \notin \Delta$

$$|y(t)| = \left|\sum_{k=1}^{2^{N-1}} \alpha_k f_{N,k}(t)\right| \leqslant \varepsilon \left(1 - \frac{1}{2^N}\right) \max_{k \in \{1, \ldots, 2^{N-1}\}} |\alpha_k|.$$

The last two inequalities together give the required condition (8.4.5).

Now, consider the following homeomorphic copy of the Cantor discontinuum:

$$\mathcal{K} := \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^{N-1}} \Delta_{n,k}.$$

Denote by $U\colon C[0,1] \to C(\mathcal{K})$ the natural restriction operator, $Uf := f|_{\mathcal{K}}$. We are going to show that $U|_Y$ performs an isomorphism of Y and $C(\mathcal{K})$. This will demonstrate that Y is isomorphic to $C(\mathcal{K})$ and, consequently, is isomorphic to $C[0,1]$. Clearly, $\|U\| \leqslant 1$, so we only need to show that U is bounded below on Y and $U(Y)$ is dense in $C(\mathcal{K})$.

As before, let $y = \sum_{k=1}^{2^{N-1}} \alpha_k f_{N,k}$ be an arbitrary element of the dense subset $\operatorname{lin}\left\{f_{n,k}\colon n \in \mathbb{N},\ k \in \overline{1, 2^{n-1}}\right\}$ of Y, $k_0 \in \overline{1, 2^{N-1}}$ be an index such that $|\alpha_{k_0}| = \max\{|\alpha_k|\colon k \in \overline{1, 2^{N-1}}\}$, and $t_0 \in \Delta_{N,k_0} \cap K$. According to (b),

$$\|y\| = \max_{t \in [0,1]} |y(t)| \leqslant \max_{k \in \overline{1, 2^{N-1}}} |\alpha_k| \max_{t \in [0,1]} \sum_{k=1}^{2^{N-1}} |f_{N,k}(t)| \leqslant (1 + \varepsilon) \max_{k \in \overline{1, 2^{N-1}}} |\alpha_k|$$

and

$$\|Uy\| \geqslant |y(t_0)| \overset{(8.4.7)}{\geqslant} (1 - \varepsilon) \max_{k \in \{1, \ldots, 2^{N-1}\}} |\alpha_k| \geqslant \frac{1 - \varepsilon}{1 + \varepsilon} \|y\| \geqslant (1 - \theta)\|y\|.$$

So, the boundedness from below of $U|_Y$ is demonstrated. It remains to show the density of $U(Y)$ in $C(\mathcal{K})$. Assume to the contrary the existence of $g \in S_{C(\mathcal{K})}$ with $\operatorname{dist}(g, U(Y)) > \frac{1}{2}$. Select $\delta > 0$ in such a way that $|g(t) - g(\tau)| < \varepsilon$ for very $t, \tau \in \mathcal{K}$ with $|t - \tau| \leqslant \delta$. Fix $N \in \mathbb{N}$ in such a way that $|\Delta_{N,k}| < \delta$ for all $k \in \overline{1, 2^{N-1}}$. Finally, for each $k \in \overline{1, 2^{N-1}}$ choose $t_k \in \Delta_{N,k}$. Then $|g(t) - g(t_k)| < \varepsilon$ for $t \in \Delta_{N,k} \cap K$. Define $f \in Y$ by means of the formula

$$f(t) = \sum_{k=1}^{2^{N-1}} g(t_k) \frac{f_{N,k}(t)}{f_{N,k}(t_k)}.$$

We are going to arrive at a contradiction demonstrating that $\|Uf - g\| < \theta < \frac{1}{2}$. Indeed, let $t \in \mathcal{K}$ be a point at which $|Uf - g|$ attains its maximum and j be that index for which $t \in \Delta_{N,j}$. In view of (a) and (b) of Lemma 8.4.9, all the values $f_{N,k}(t_k)$ and $f_{N,j}(t)$ lie between 1 and $1 + \varepsilon$. Applying (8.4.6) with $t_0 = t$, $k_0 = j$ and $A = \{1, \ldots, 2^{N-1}\} \setminus \{j\}$, we deduce that

$$\begin{aligned}
\|Uf - g\| = |(Uf - g)(t)| = |f(t) - g(t)| &\leqslant |f(t) - g(t_j)| + \varepsilon \\
&\leqslant \left| g(t_j) \frac{f_{N,j}(t)}{f_{N,j}(t_j)} - g(t_j) \right| + \left| \sum_{k \in A} g(t_k) \frac{f_{N,k}(t)}{f_{N,k}(t_k)} \right| + \varepsilon \\
&\leqslant \left| \frac{f_{N,j}(t)}{f_{N,j}(t_j)} - 1 \right| + \sum_{k \in A} \left| \frac{f_{N,k}(t)}{f_{N,k}(t_k)} \right| + \varepsilon \\
&\leqslant |f_{N,j}(t) - f_{N,j}(t_j)| + \sum_{k \in A} |f_{N,k}(t)| + \varepsilon \leqslant 3\varepsilon < \theta.
\end{aligned}$$

This completes the demonstration of the fact that Y and $C[0,1]$ are isomorphic. The remaining complementability of Y in $C[0,1]$ is plain. We have demonstrated that $U|_Y$ implements an isomorphism of Y and $C(\mathcal{K})$. The requested projection operator of $C[0,1]$ onto Y can be defined as $P := \left(U|_Y\right)^{-1} \circ U$. $\square$

The combination of the above theorem with Example 8.4.6 enables us to give the promised isomorphic description of rich subspaces in $C[0,1]$.

THEOREM 8.4.11. *For a Banach space E, the following assertions are equivalent:*

(1) *E is isomorphic to a rich subspace of $C[0,1]$.*
(2) *E is separable and contains an isomorphic copy of $C[0,1]$.*
(3) *E is isomorphic to a space of the form $C[0,1] \oplus Z$, where Z is separable.*

PROOF. According to [**247**], every separable Banach space X containing $C[0,1]$ contains another copy of $C[0,1]$ which is complemented. This gives us the equivalence of (2) and (3).

The fact that each rich subspace of $C[0,1]$ contains a copy of $C[0,1]$ is demonstrated in Theorem 8.4.10. So, it remains to show that (3) implies (1). Let $E = C[0,1] \oplus Z$ with separable Z. Consider the projector P from Example 8.4.6. Then $C[0,1] = \ker P \oplus P(C[0,1])$. Because of the narrowness of P, $\ker P$ is rich in $C[0,1]$. By Theorem 8.4.10, $\ker P$ contains a complemented isomorphic copy of $C[0,1]$ and is complemented in $C[0,1]$ itself. So, according to Pełczyński's theorem, $\ker P$ is isomorphic to $C[0,1]$. Next, $P(C[0,1])$ is isometric to the space $C(\mathcal{K})$

of continuous functions on the Cantor set. By the universality of $C(\mathcal{K})$ (Banach-Mazur theorem, see [**156**, 17.2.4, Exercise 7] for a sketch of the proof), $P(C[0,1])$ contains a subspace $\tilde{Z}$ which is isomorphic to Z. Consequently, on the one hand, $\tilde{E} := \ker P \oplus \tilde{Z} \subset C[0,1]$ is isomorphic to E and, on the other hand, $\tilde{E}$ is rich in $C[0,1]$, because $\tilde{E}$ contains the rich subspace $\ker P$. □

Corollary 8.4.12. *If a Banach space E is separable and contains an isomorphic copy of $C[0,1]$, then E is isomorphic to a Banach space with the Daugavet property.*

Proof. Rich subspaces inherit the Daugavet property (Theorem 6.5.3). □

The above result motivates the question whether $C[0,1]$ can be replaced with a general Banach space with the Daugavet property (see Question (8.4) in Section 8.9).

8.5. Some examples of small but rich spaces

In this section we provide examples of nonseparable Banach spaces possessing separable rich subspaces. We will consider spaces $C(K,E)$ embedded in a suitable space X; the type of spaces we have in mind will be defined next.

Definition 8.5.1. Let E be a Banach space and X be a sup-normed space of bounded E-valued functions on a compact space K. The space X is said to be a *$C(K,E)$-superspace* if it contains $C(K,E)$ and if for every $f \in X$, every $\varepsilon > 0$ and every nonvoid open subset $U \subset K$ there exist an element $e \in E$ with $\|e\| > (1-\varepsilon)\sup_U \|f(t)\|$ and a nonvoid open subset $V \subset U$ such that $\|e - f(\tau)\| < \varepsilon$ for every $\tau \in V$.

Basically, X is a $C(K,E)$-superspace if every element of X is large and almost constant on suitable open sets.

Here are some examples of this notion.

Proposition 8.5.2.

(a) *$D[0,1]$, the space of bounded real functions on $[0,1]$ that are right-continuous, have left limits everywhere, and are continuous at $t=1$, is a $C[0,1]$-superspace.*
(b) *Let K be a compact Hausdorff space and E be a Banach space. Then $C_w(K,E)$, the space of weakly continuous functions from K into E, is a $C(K,E)$-superspace.*

Proof. (a) $D[0,1]$ is the uniform closure of the span of the step functions $\mathbb{1}_{[a,b)}$, $0 \leqslant a < b < 1$, and $\mathbb{1}_{[a,1]}$, $0 \leqslant a < 1$; hence the result.

(b) Fix f, U and ε as in Definition 8.5.1; without loss of generality, we assume that $\sup_U \|f(t)\| = 1$. We are going to use results about the Radon-Nikodým property, described in Section 2.7. Consider the open set $U_0 = \{t \in U\colon \|f(t)\| > 1-\varepsilon\}$. Its image $f(U_0)$ is relatively weakly compact since f is weakly continuous; hence $\overline{\mathrm{conv}}(f(U_0))$ is weakly compact and, consequently, possesses the RNP. In particular, $\overline{\mathrm{conv}}(f(U_0))$ is equal to the closed convex hull of its denting points. Therefore, there exists a halfspace $H = \{x \in E\colon x^*(x) > \alpha\}$ such that $f(U_0) \cap H$ is nonvoid and has diameter $< \varepsilon$. Consequently, $V := f^{-1}(H) \cap U_0$ is an open subset of U for which $\|f(\tau_1) - f(\tau_2)\| < \varepsilon$ for all $\tau_1, \tau_2 \in V$. This shows that $C_w(K,E)$ is a $C(K,E)$-superspace. □

The following theorem explains the relevance of these ideas.

THEOREM 8.5.3. *If X is a $C(K,E)$-superspace and K is perfect, then $C(K,E)$ is rich in X; in particular, X has the Daugavet property.*

PROOF. We wish to verify condition (iii) of Theorem 6.5.14. Let $f,g \in S_X$ and $\varepsilon > 0$. We first find an open set V and an element $e \in E$, $\|e\| > 1-\varepsilon/4$, such that $\|e - f(\tau)\| < \varepsilon/4$ on V. Given $N \in \mathbb{N}$, find open nonvoid pairwise disjoint subsets $V_1,\dots,V_N$ of V. Applying the definition again, we obtain elements $e_j \in E$ and open subsets $W_j \subset V_j$ such that $\|e_j\| > (1-\varepsilon/4)\sup_{V_j}\|g(t)\|$ and $\|e_j - g(\tau)\| < \varepsilon/4$ on W_j. Let $x_j = e - e_j$, let $\varphi_j \in C(K)$ be a positive function supported on W_j of norm 1 and let $h_j = \varphi_j \otimes x_j$. Now, if $t_j \in W_j$ is selected to satisfy $\varphi_j(t_j) = 1$, then

$$\|f+g+h_j\| \geqslant \|(f+g+h_j)(t_j)\| > \|e+e_j+x_j\| - \varepsilon/2 > 2-\varepsilon.$$

Also,

$$\|g+h_j\| < 1+\varepsilon$$

since $\|g(t)\| = \|g(t)+h_j(t)\| \leqslant 1$ for $t \notin W_j$ and, for $t \in W_j$,

$$\|g(t)+h_j(t)\| \leqslant \|e_j + \varphi_j(t)x_j\| + \varepsilon/4 \leqslant (1-\varphi_j(t))\|e_j\| + \varphi_j(t)\|e\| + \varepsilon/4.$$

This shows that $h_j \in D(f,g,\varepsilon) \cap C(K,E)$. But the supports of the h_j are pairwise disjoint, hence $\|1/N\sum_{j=1}^N h_j\| \leqslant 2/N \to 0$. $\square$

COROLLARY 8.5.4.

(a) $C[0,1]$ *is a separable rich subspace of the nonseparable space* $D[0,1]$.

(b) *If K is perfect, then $C(K,E)$ is a rich subspace of $C_w(K,E)$. In particular, $C([0,1],\ell_p)$ is a separable rich subspace of the nonseparable space $C_w([0,1],\ell_p)$ if $1 < p < \infty$.*

Let us remark that there exist nonseparable spaces with the Daugavet property which have only nonseparable rich subspaces. Indeed, an ℓ_∞-sum of uncountably many spaces with the Daugavet property is an example of this phenomenon. To see this, we need (see Corollary 7.2.4) that whenever T is a narrow operator on $X_1 \oplus_\infty X_2$, then the restriction of T to X_1 is narrow too and, in particular, it is not bounded from below. Now, let X_γ, $\gamma \in \Gamma$, be Banach spaces with the Daugavet property and let X be their ℓ_∞-sum. If Z is a rich subspace of X then, by the result quoted above, there exist elements $x_\gamma \in S_{X_\gamma}$ and $z_\gamma \in Z$ with $\|x_\gamma - z_\gamma\| \leqslant 1/4$; hence $\|z_\gamma - z_\tau\| \geqslant 1/2$ for $\gamma \neq \tau$. If Γ is uncountable, this implies that Z is nonseparable.

REMARK 8.5.5. *The examples, like those from Corollary 8.5.4, give a hope for the validity of a version of Corollary 8.4.12 for nonseparable spaces, that is, that every space which contains a copy of $C[0,1]$ possesses the Daugavet property in an equivalent norm. By now, this direction remains completely unexplored (see Question (8.5) in Section 8.9).*

8.6. Narrow operators on vector-valued L_1-spaces

In this section we extend the theory of narrow operators on $L_1(\mu)$ from Section 6.6 to operators on spaces of vector-valued functions. Below μ and ν will be used for finite non-atomic σ-additive measures on a σ-algebra Σ of subsets of a fixed set Ω. By $L_1(\mu,X)$ we denote the space of X-valued Bochner integrable functions on Ω, where X is a *real* Banach spaces. The complex case is contained in the real one because every complex space can be considered as a real space, and a complex-linear narrow operator neither gains nor loses its narrowness when considered as a real-linear operator (see Remark 6.2.15). In particular, the complex $L_1(\mu)$ case is covered by $L_1(\mu, \ell_2^{(2)})$. By $L_1(A, \mu, X)$ we denote the subspace of $L_1(\mu, X)$, consisting of functions supported on A. Recall that we denote by Σ^+ the collection of sets $A \in \Sigma$ of finite non-zero measure.

At the beginning of Section 6.6 we introduced the following concept:

DEFINITION 8.6.1. A function $f \in L_1(\mu)$ is said to be a *balanced ε-peak* on $A \in \Sigma^+$ if there is a subset $A_1 \subset A$ with $\mu(A_1) < \varepsilon$ such that

(1) $f = -1$ for $t \in A \setminus A_1$, $\operatorname{supp} f \subset A$,
(2) $f \geqslant -1$,
(3) $\int_\Omega f \, d\mu = 0$.

These balanced ε-peaks were crucial for the characterisation of narrow operators on real non-atomic $L_1(\mu)$ that was proved in Theorem 6.6.1: an operator T: $L_1(\mu) \to E$ is narrow if and only if for every $\varepsilon > 0$ and every $A \in \Sigma$ there exists such a balanced ε-peak f on A that $\|T(f)\| < \varepsilon$.

One can find more about the characterisation of narrow operators on $L_1(\mu)$ as well as open problems in [**176**].

In this section we prove that for a wide class of spaces X, the narrow operators allow a description similar to Theorem 6.6.1. At the same time, there are spaces where the analogous description of narrow operators does not hold. More precisely, we introduce the following concept.

DEFINITION 8.6.2. Let $x \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma^+$. A function $f \in L_1(A, \mu, X)$ is said to be an (x, ε, A)-*peak* if there is a subset $A_1 \subset A$ with $\mu(A_1) < \varepsilon$ such that

(1) $f(t) = x$ for $t \in A \setminus A_1$;
(2) $\int_{A_1} \|f(t)\| \, d\mu(t) \leqslant (1 + \varepsilon)\mu(A)\|x\|$.

f is said to be an (x, x^*, ε, A)-*peak* if there is a subset $A_1 \subset A$ with $\mu(A_1) < \varepsilon$ such that the conditions (1) and (2) are fulfilled and additionally

(3) $\displaystyle \left| \int_A x^*(f(t)) \, d\mu(t) \right| < \varepsilon.$

An operator $T \in \mathcal{OP}(L_1(\mu, X))$ is said to be *L-narrow* (respectively, *almost L-narrow*) if for every $x \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma^+$ there is some (x, x^*, ε, A)-peak (respectively, (x, ε, A)-peak) function f with $\|T(f)\| < \varepsilon$.

The L-narrow operators form a class that is built on a generalisation of the property characterising narrow operators in the scalar case according to Theorem 6.6.1.

We will prove that under the condition $X^* \in \mathrm{RNP}$, every L-narrow operator on $L_1(\mu, X)$ is narrow (see Theorem 8.6.5), and in Theorem 8.6.10 we shall describe properties of X which are sufficient for the coincidence of the classes of L-narrow and narrow operators on $L_1(\mu, X)$.

Unfortunately, we cannot give the complete description of spaces where the coincidence takes place (see Question (8.6) in Section 8.9), but our results allow us to resolve the problem for all classical spaces.

The reader will find some similarities of the presented theory with the results about $C(K, X)$-spaces from Section 8.3. At the same time, the $L_1(\mu, X)$ case is, in our opinion, more difficult and has essential peculiarities compared to the $C(K,X)$ case. In particular, "good" spaces that we used in order to reduce narrow operators on $C(K,X)$ to C-narrow operators (USD non-friendly spaces from Section 8.1) do not help to deal with $L_1(\mu,X)$ and are substituted by "reasonable spaces" introduced in Subsection 8.6.2 below.

8.6.1. *L*-narrow operators. We start with a theorem that includes a criterion for an operator defined on $L_1(\mu,X)$ with $X^* \in \mathrm{RNP}$ to be narrow.

THEOREM 8.6.3. *Let X be a Banach space and $T \in \mathcal{OP}(L_1(\mu,X))$. If T is narrow, then it satisfies the following set of conditions called* (8.6.3-COND)*: For every $x,y \in X$, $x^* \in X^*$, $\varepsilon > 0$, and $A \in \Sigma^+$, there is a function $f \in L_1(A,\mu,X)$ with*

$$\|T(f - x\mathbb{1}_A)\| < \varepsilon, \ \|f\| = \mu(A)\|x\|, \tag{8.6.1}$$

$$\|f + y\mathbb{1}_A\| > (1-\varepsilon)\mu(A)(\|x\| + \|y\|), \tag{8.6.2}$$

and

$$\Big|\int_A x^*(f(t) - x)\, d\mu(t)\Big| < \varepsilon. \tag{8.6.3}$$

Conversely, if T satisfies the weakened (8.6.3-COND) *with the selection of f satisfying only* (8.6.1) *and* (8.6.2)*, then $T \in \mathcal{SD}(L_1(\mu,X))$; and, under the condition $X^* \in$ RNP, every operator T that satisfies the complete version of* (8.6.3-COND) *is narrow.*

PROOF. We start with the first part of the statement. The case $x = 0$ is trivial (just take $f = 0$), so we may exclude it from our considerations. Since $L_1(\Omega,\mu,X)$ can be represented as ℓ_1-sum of $L_1(A,\mu,X)$ and $L_1(\Omega \setminus A,\mu,X)$, the restriction of T to $L_1(A,\mu,X)$ is narrow (Theorem 7.4.4). To deduce the statement (8.6.3-COND), let us apply (v)$'$ of Theorem 6.2.14 to the restriction of T to $L_1(A,\mu,X)$, sufficiently small $\varepsilon_1 > 0$, the element $\hat{x} = \frac{x\mathbb{1}_A}{\|x\mathbb{1}_A\|} \in S_{L_1(A,\mu,X)}$, the weak neighbourhood W of $\hat{x}$ consisting of all functions $g \in L_1(A,\mu,X)$ with $|\int_A x^*(g(t) - \hat{x}(t))\, d\mu(t)| < \varepsilon_1$, and the element $\hat{y} = \frac{y\mathbb{1}_A}{\|x\mathbb{1}_A\|} \in S_{L_1(A,\mu,X)}$. Then we get an element $\hat{z} \in W \cap S_{L_1(A,\mu,X)}$ with the properties that $\|T(\hat{z}-\hat{x})\| < \varepsilon_1$ and $\|\hat{z}+\hat{y}\| > 2-\varepsilon_1$. Then $f = \|x\|\mu(A)\hat{z}$ will be what we need.

For the converse statement, we are going to use Definition 6.2.1 of strong Daugavet operators. Let $x,y \in S_{L_1(\mu,X)}$ and $\varepsilon > 0$. By a density argument we may assume without loss of generality that x and y are countably valued functions, that is, that there is a measurable partition $\Omega = \bigsqcup_{k=1}^\infty A_k$ such that $x = \sum_{k=1}^\infty x_k \mathbb{1}_{A_k}$, $y = \sum_{k=1}^\infty y_k \mathbb{1}_{A_k}$, where $x_k, y_k \in X$. For every $k \in \mathbb{N}$ apply the weakened condition (8.6.3-COND) to x_k, y_k and A_k with $\frac{\varepsilon}{2^k}$ instead of ε. We obtain $f_k \in L_1(A_k,\mu,X)$ with

$$\begin{aligned}
\|T(f_k - x_k\mathbb{1}_{A_k})\| &< \frac{\varepsilon}{2^k}, \\
\|f_k\| &= \mu(A_k)\|x_k\|, \\
\|f_k + y_k\mathbb{1}_{A_k}\| &> \left(1 - \frac{\varepsilon}{2^k}\right)\mu(A_k)(\|x_k\| + \|y_k\|).
\end{aligned}$$

Then for the element $v := \sum_{k=1}^{\infty} f_k \in L_1(\mu, X)$ we have

$$\|T(v-x)\| \leqslant \sum_{k=1}^{\infty} \|T(f_k - x_k \mathbb{1}_{A_k})\| < \varepsilon,$$

$$\|v\| = \sum_{k=1}^{\infty} \|f_k\| = \sum_{k=1}^{\infty} \mu(A_k)\|x_k\| = \|x\| = 1,$$

and

$$\|v+y\| = \sum_{k=1}^{\infty} \|f_k + y_k \mathbb{1}_{Ak}\| > (1-\frac{\varepsilon}{2}) \sum_{k=1}^{\infty} \mu(A_k)(\|x_k\| + \|y_k\|) = 2 - \varepsilon.$$

This demonstrates that $T \in \mathcal{SD}(L_1(\mu, X))$.

Now, we turn to the very last part of the statement which requires the assumption $X^* \in$ RNP. We are going to demonstrate that $T \in \mathcal{NAR}(L_1(\mu, X))$ using the Definition 6.2.7 of narrow operators. Let, as above, $x, y \in S_{L_1(\mu,X)}$, $\varepsilon > 0$, and additionally some $x^* \in L_1(\mu, X)^*$ be given. By Theorem 2.7.10, $L_1(\mu, X)^*$ identifies with the space $L_\infty(\mu, X^*)$ of all bounded strongly measurable X^*-valued functions on Ω. So, again as above, we may assume without loss of generality that there is a measurable partition $\Omega = \bigsqcup_{k=1}^{\infty} A_k$ such that $x = \sum_{k=1}^{\infty} x_k \mathbb{1}_{A_k}$, $y = \sum_{k=1}^{\infty} y_k \mathbb{1}_{A_k}$, where $x_k, y_k \in X$, and additionally $x^* = \sum_{k=1}^{\infty} x_k^* \mathbb{1}_{A_k}$, $x_k^* \in X^*$ (the last formal sum does not necessarily converge in norm, but converges pointwise on Ω and also in the w^*-sense in $L_1(\mu, X)^*$). Applying condition (8.6.3-COND) to x_k and y_k, x_k^* and A_k with $\frac{\varepsilon}{2^k}$ instead of ε we get all that we had above and, additionally, (thanks to (8.6.3))

$$\Big|\int_{A_k} x_k^*(f_k(t) - x_k)\, d\mu(t)\Big| < \frac{\varepsilon}{2^k}.$$

Then for the same selection of $v := \sum_{k=1}^{\infty} f_k \in L_1(\mu, X)$ we have all the properties required for the strong Daugavetness, and additionally the condition

$$|x^*(v-x)| = \Big|\int_{\Omega} x^*(v(t) - x(t))\, d\mu(t)\Big| \leqslant \sum_{k=1}^{\infty} \Big|\int_{A_k} x_k^*(f_k(t) - x_k)\, d\mu(t)\Big| < \varepsilon,$$

which makes the difference between strong Daugavet and narrow operators. □

REMARK 8.6.4. *Let $T \in \mathcal{OP}(L_1(\mu, X))$ be an L-narrow operator. Then for every $x \in X$, $x^* \in X^*$, $\varepsilon > 0$, and $A \in \Sigma^+$, there is some (x, x^*, ε, A)-peak function g with $\|T(g)\| < \varepsilon$ and with*

$$\int_{A_1} \|g(t)\|\, d\mu(t) = (1+\varepsilon)\mu(A)\|x\|$$

for a corresponding $A_1 \subset A$ from Definition 8.6.2. The same strengthening can be made for almost L-narrow operators.

PROOF. Let $\varepsilon < 1$, $\varepsilon_1 < \varepsilon/2$ and f be an $(x, x^*, \varepsilon_1, A)$-peak with corresponding $A_1 \in \Sigma^+$ such that $\|T(f)\| < \varepsilon_1$. For a positive $\delta < \min\{\mu(A)/2, \varepsilon\mu(A_1)/(16\mu(A))\}$, fix an (x, x^*, δ, A_1)-peak function h with $\|T(h)\| < \delta$. Consider $g_\lambda = f + \lambda h$ where $\lambda \geqslant 0$ is a parameter. Let us note that for $\lambda \in [0, \frac{\varepsilon}{2\delta}]$ the function g_λ is an (x, x^*, ε, A)-peak with the same A_1 as f and $\|T(g_\lambda)\| < \varepsilon$. In fact, for such a λ we have

$$\|T(g_\lambda)\| < \frac{\varepsilon}{2} + \lambda\delta \leqslant \varepsilon$$

and

$$\Big|\int_A x^*(g_\lambda(t))\,d\mu(t)\Big| < \frac{\varepsilon}{2} + \lambda\delta \leqslant \varepsilon.$$

Consider $F(\lambda) = \int_{A_1} \|g_\lambda(t)\|\,d\mu(t)$. If $\lambda = 0$ then $F(\lambda) < (1+\varepsilon)\mu(A)\|x\|$, and for $\lambda = \frac{\varepsilon}{2\delta}$ one has

$$F(\lambda) \geqslant \frac{\varepsilon}{2\delta}\|h\| - 2\mu(A)\|x\| \geqslant \frac{\varepsilon}{\delta}\|x\|\frac{1}{4}\mu(A_1) - 2\mu(A)\|x\| > 2\mu(A)\|x\|.$$

So there is $\lambda_0 \in [0, \frac{\varepsilon}{2\delta}]$ with $F(\lambda_0) = (1+\varepsilon)\mu(A)\|x\|$. Then $g = g_{\lambda_0}$ is the function we need. □

Theorem 8.6.5. *Let X be a Banach space. Then every almost L-narrow operator $T \in \mathcal{OP}(L_1(\mu, X))$ is a strong Daugavet operator. Under the additional assumption of $X^* \in$ RNP, every L-narrow operator $T \in \mathcal{OP}(L_1(\mu, X))$ is narrow.*

Proof. For our aim it is sufficient to demonstrate the conditions for the converse statement in Theorem 8.6.3. We will address the second part of our theorem that starts with the words "Under the additional assumption"; for the first part one just needs to omit all mentioning of x^*. Let $x, y \in X$, $x^* \in X^*$, $\varepsilon > 0$ and $A \in \Sigma^+$, let $g \in L_1(A, \mu, X)$ be an (x, x^*, δ, A)-peak with $\|T(g)\| < \delta$ for δ small enough, and let $A_1 \subset A$ be the corresponding subset from Definition 8.6.2. According to the previous remark we may assume that

$$\int_{A_1} \|g(t)\|\,d\mu(t) = (1+\delta)\mu(A)\|x\|.$$

Consider

$$f = -\frac{1}{1+\delta} g\mathbb{1}_{A_1}.$$

Then $\|f\| = \mu(A)\|x\|$ and

$$\begin{aligned} f - x\mathbb{1}_A &= -x\mathbb{1}_{A_1} - \frac{1}{1+\delta} g\mathbb{1}_A + \Big(\frac{1}{1+\delta} g - x\Big)\mathbb{1}_{A\setminus A_1} \\ &= -x\mathbb{1}_{A_1} - \frac{1}{1+\delta} g - \frac{\delta}{1+\delta} x\mathbb{1}_{A\setminus A_1}. \end{aligned}$$

Hence

$$\Big|\int_A x^*(f(t) - x)\,d\mu(t)\Big| \leqslant \delta\|x\| + \frac{\delta}{1+\delta} + \frac{\delta}{1+\delta}\mu(A)\|x\|.$$

By the same argument

$$\|T(f - x\mathbb{1}_A)\| \leqslant \delta\|x\|\|T\| + \frac{\delta}{1+\delta} + \frac{\delta}{1+\delta}\mu(A)\|x\|\|T\|.$$

So, when δ is small, the conditions (8.6.1) and (8.6.3) of Theorem 8.6.3 are satisfied. The remaining condition (8.6.2) follows from the fact that the support of f is of arbitrarily small measure δ, so $\|f + y\mathbb{1}_A\|$ almost equals the sum $\|f\| + \|y\mathbb{1}_A\|$. □

8.6.2. Reasonable spaces. The aim of the rest of this section is to prove the converse to Theorem 8.6.5 for a wide class of spaces X containing in particular all reflexive spaces.

LEMMA 8.6.6. *Let $u, v \in L_1(\Omega, \Sigma, \nu)$, $\Delta \in \Sigma^+$, $\delta > 0$, $u(t), v(t) \in (0, 2)$ for all $t \in \Omega$. Let us assume that*

$$\int_\Delta u\,d\nu \geqslant 2\nu(\Delta) - \delta, \tag{8.6.4}$$

$$\int_\Delta v\,d\nu \leqslant \delta \tag{8.6.5}$$

and that there are $\alpha > 0$ and $c < 2$ such that

$$\{t \in \Delta \colon v(t) < \alpha\} \subset \{t \in \Delta \colon u(t) < c\}. \tag{8.6.6}$$

Then

$$\nu(\Delta) \leqslant \frac{2\delta(1+\alpha)}{\alpha(2-c)}. \tag{8.6.7}$$

PROOF. Denote $\Delta_1 = \{t \in \Delta \colon v(t) < \alpha\}$, $\Delta_2 = \{t \in \Delta \colon v(t) \geqslant \alpha\}$. Then, according to (8.6.5),

$$\nu(\Delta_2) \leqslant \frac{\delta}{\alpha}.$$

Due to (8.6.4),

$$2\nu(\Delta) - \delta \leqslant \int_{\Delta_1} u\,d\nu + \int_{\Delta_2} u\,d\nu \leqslant c\nu(\Delta) + 2\frac{\delta}{\alpha}.$$

So $(2-c)\nu(\Delta) \leqslant 2\frac{\delta(1+\alpha)}{\alpha}$, which proves (8.6.7). □

We now introduce one more geometric condition that is in a sense opposite to the Daugavet property. We recall the following notions from Chapter 5. The (outer) *radius* of a subset $A \subset X$ at $y \in X$ is $r_y(A) = \sup\{\|a - y\| \colon a \in A\}$ (see Figure 5.1). The *Chebyshev radius* of A relative to another subset $B \subset X$ is

$$r_B(A) = \inf\{r_y(A) \colon y \in B\}$$

(see Figure 8.1).

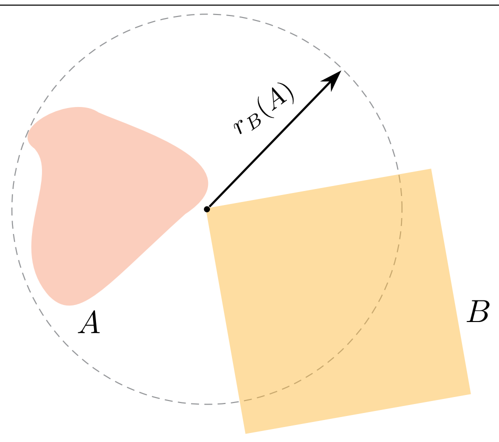

FIGURE 8.1. Chebyshev radius

DEFINITION 8.6.7. A point $x \in S_X$ is said to be *reasonable* if there is a slice $\mathrm{Slice}(S_X, x^*, \varepsilon)$ with $x^*(x) = 1$, and there is $y \in S_X$ such that $r_y(\mathrm{Slice}(S_X, x^*, \varepsilon)) < 2$. The set of all reasonable points $x \in S_X$ will be denoted by $\mathrm{Reas}(X)$. A Banach space X is said to be *reasonable* if the closed convex hull of $\mathrm{Reas}(X)$ contains the whole unit ball.

In other words, $x \in S_X$ is reasonable if $r_{S_X}(S) < 2$ for some slice $S = \mathrm{Slice}(S_X, x^*, \varepsilon)$ as above.

Evidently, every strongly exposed point of the unit ball is reasonable. Therefore, every Banach space with the Radon-Nikodým property is a reasonable space in every equivalent norm, because then every closed convex bounded subset is the closed convex hull of its strongly exposed points. Also, every locally uniformly convex space is reasonable. But no space with the Daugavet property is reasonable. Indeed, Lemma 3.1.9(i) implies that no point in the unit sphere of a Banach space X with the Daugavet property is reasonable: actually, a reformulation of that lemma is that $r_{S_X}(S) = 2$ for every slice S.

There are other nonreasonable spaces; for example, if X has the Daugavet property, then the only reasonable points of $Y = X \oplus_1 \mathbb{R}$, which fails the Daugavet property, are $(0, \pm 1)$. Indeed, $(0, \pm 1)$ are obviously strongly exposed points of B_Y. Now, let $(x, a) \in S_Y$ with $x \neq 0$, and let (x^*, b) be a functional in $S_{Y^*} = S_{X^* \oplus_\infty \mathbb{R}}$ attaining its norm at (x, a). Then $\|x^*\| = 1$. Consider the slice $S = \mathrm{Slice}(S_Y, (x^*, b), \varepsilon) \subset S_Y$ and the slice $\mathrm{Slice}(S_X, x^*, \varepsilon) \subset S_X$. By the Daugavet property, given a point $(y, \alpha) \in S_Y$, there is some $z \in \mathrm{Slice}(S_X, x^*, \varepsilon)$ such that $\|y - z\| \geqslant \|y\| + \|z\| - \varepsilon$. Then $(z, 0) \in S$, yet

$$\|(y, \alpha) - (z, 0)\| = \|y - z\| + |\alpha| \geqslant \|y\| + \|z\| + |\alpha| - \varepsilon = 2 - \varepsilon.$$

Hence, (x, a) is not reasonable.

There is a hierarchy of largeness conditions of slices of the unit ball. The strongest one is the Daugavet property, viz., $r_{S_X}(S) = 2$ for every slice. A strictly weaker property is $r_S(S) = 2$ for every slice. Still weaker is the condition that every slice has diameter 2. We refer the reader to Section 12.2 for background around these properties, where they receive the name of *diametral local diameter two property* (Definition 12.2.1) and the *slice diameter two property* (Definition 12.2.4), respectively. The following example shows that a relatively "bad" space can also be reasonable.

EXAMPLE 8.6.8. Although every slice of the unit sphere of c_0 is of diameter 2, every point of the unit sphere of c_0 is a reasonable point.

PROOF. We first present an elementary argument that every slice of S_{c_0} has diameter 2; see [**238**] for a more general statement. Let $x^* = (a_1, a_2, \dots) \in \ell_1$ with $\sum_n |a_n| = 1$ and consider the slice $\mathrm{Slice}(S_{c_0}, x^*, \varepsilon)$. Pick N so that $\sum_{n=1}^N |a_n| > 1 - \varepsilon/2$ and define $x, y \in S_{c_0}$ by $x_n = \operatorname{sign} a_n$ for $n < N$, $x_N = 1$, $x_n = 0$ for $n > N$ and $y_n = \operatorname{sign} a_n$ for $n < N$, $y_N = -1$, $y_n = 0$ for $n > N$. Then $x, y \in \mathrm{Slice}(S_{c_0}, x^*, \varepsilon)$ and $\|x - y\| = 2$.

Now, we show that every $x \in S_{c_0}$ is reasonable. Pick $k \in \mathbb{N}$ such that $|x_k| = 1$, say $x_k = 1$ without loss of generality. For the k^{th} unit vectors $e_k \in S_{c_0}$ and $e_k^* \in S_{\ell_1}$ we have $e_k^*(x) = 1$, and for $z = (z_1, z_2, \dots) \in S(e_k^*, \varepsilon)$ it follows $z_k > 1 - \varepsilon$ so that $\|z - e_k\| \leqslant 1$. □

The importance of reasonable points stems from the following lemma.

LEMMA 8.6.9. *Let $x \in \mathrm{Reas}(X)$. Then for every $U \in \mathcal{NAR}(L_1(\mu, X))$, every $\varepsilon > 0$, every $y^* \in S_{X^*}$ and every $A \in \Sigma$ there is some (x, y^*, ε, A)-peak function f with $\|U(f)\| < \varepsilon$.*

PROOF. Let $U \in \mathcal{OP}(L_1(\mu, X))$ be a narrow operator, $y^* \in S_{X^*}$. Consider the functional $F \in L_1(\mu, X)^*$ that acts by the rule $F(f) = \int_\Omega \langle y^*, f(t)\rangle \, d\mu(t)$ and introduce the auxiliary operator $T := U \tilde{+} F$. Being a $\sim$-sum of a narrow operator and a functional, T is narrow by Corollary 6.2.19, part (a).

We need to prove that for every $\varepsilon > 0$ and every $A \in \Sigma^+$ there is some $f \in L_1(A, \mu, X)$ with the following properties:

(1) $\mu\{t \in A\colon f(t) = x\} > \mu(A) - \varepsilon$;
(2) $\int_{\{t\in A\colon f(t)\neq x\}} \|f(t)\| \, d\mu(t) \leqslant \mu(A)$ and
(3) $\|T(f)\| < \varepsilon$.

According to the definition of $\mathrm{Reas}(X)$, there are $x^* \in S_{X^*}$, $y \in S_X$ and $\alpha \in (0, 1)$ such that $x^*(x) = 1$ and

$$r_y(S(x^*, \alpha)) =: c < 2. \tag{8.6.8}$$

Without loss of generality, one can assume $\mu(A) = 1$ (otherwise we multiply μ by an appropriate constant). Fix $\delta > 0$ and apply Theorem 8.6.3; hence there is a function $g \in L_1(A, \mu, X)$ with $\|g\|_1 = 1$ and

$$\int_A \langle x^*, g(t)\rangle \, d\mu(t) > 1 - \delta, \tag{8.6.9}$$

$$\|T(g - x\mathbb{1}_A)\| < \delta, \tag{8.6.10}$$

$$\|g - y\mathbb{1}_A\| > 2 - \delta. \tag{8.6.11}$$

Claim. Let $B = \{t \in A\colon \|g(t)\|_X < 1\}$, $D = \{t \in A\colon \|g(t)\|_X \geqslant 1\}$. Then

$$\int_B \|g(t)\|_X \, d\mu(t) < \frac{2\delta(1 + \alpha)}{\alpha(2 - c)}, \tag{8.6.12}$$

$$\mu(D) < \frac{2\delta(1 + \alpha)}{\alpha(2 - c)}. \tag{8.6.13}$$

Proof of the Claim. Since $g \in S_{L_1(A,\mu,X)}$, due to (8.6.9) we have

$$\|g\|_1 - \int_A \langle x^*, g(t)\rangle \, d\mu(t) < \delta,$$

i.e.,

$$\int_A \Big[1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle\Big] \|g(t)\| \, d\mu(t) < \delta. \tag{8.6.14}$$

Condition (8.6.11) can be rewritten as

$$\int_A (\|g(t)\| + 1 - \|y - g(t)\|) \, d\mu(t) < \delta. \tag{8.6.15}$$

Since the expressions under the integrals in (8.6.14) and (8.6.15) are non-negative, one can pass to a smaller set:

$$\int_B \Big[1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle\Big] \|g(t)\| \, d\mu(t) < \delta, \tag{8.6.16}$$

and $\int_B (\|g(t)\| + 1 - \|y - g(t)\|)\, d\mu(t) < \delta$. The last inequality means

$$\int_B \|y - g(t)\|\, d\mu(t) > \mu(B) + \int_B \|g(t)\|\, d\mu(t) - \delta. \tag{8.6.17}$$

By the triangle inequality,

$$\begin{aligned}\int_B \|y - g(t)\|\, d\mu(t) &\leqslant \int_B \big(\big\|\|g(t)\|y - g(t)\big\| + \big\|\|g(t)\|y - y\big\|\big)\, d\mu(t) \\ &\leqslant \int_B \|y - \frac{g(t)}{\|g(t)\|}\|\|g(t)\|\, d\mu(t) + \mu(B) - \int_B \|g(t)\|\, d\mu(t).\end{aligned}$$

Substituting this into (8.6.17), we obtain

$$\int_B \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\|\|g(t)\|\, d\mu(t) > 2\int_B \|g(t)\|\, d\mu(t) - \delta. \tag{8.6.18}$$

Using (8.6.16) and (8.6.18), we can apply Lemma 8.6.6 to

$$d\nu = \|g(t)\|\, d\mu,\ \Delta = B,\ u(t) = \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\|,\ v(t) = 1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle;$$

(condition (8.6.8) means exactly that (8.6.6) is fulfilled). This gives (8.6.12).

Let us now turn to the proof of (8.6.13). As before, passing in (8.6.14) and (8.6.15) to the smaller set D we obtain the inequalities

$$\int_D \Big[1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle\Big]\, d\mu(t) \leqslant \int_D \Big[1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle\Big]\|g(t)\|\, d\mu(t) < \delta, \tag{8.6.19}$$

and

$$\int_D \|y - g(t)\|\, d\mu(t) > \mu(D) + \int_D \|g(t)\|\, d\mu(t) - \delta. \tag{8.6.20}$$

By the triangle inequality,

$$\begin{aligned}\int_D \|y - g(t)\|\, d\mu(t) &\leqslant \int_D \Big(\Big\|y - \frac{g(t)}{\|g(t)\|}\Big\| + \Big\|g(t) - \frac{g(t)}{\|g(t)\|}\Big\|\Big)\, d\mu(t) \\ &\leqslant \int_D \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\|\, d\mu(t) + \int_D \|g(t)\|\, d\mu(t) - \mu(D)\end{aligned}$$

Substituting this into (8.6.20) we obtain

$$\int_D \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\|\, d\mu(t) > 2\mu(D) - \delta. \tag{8.6.21}$$

Using (8.6.19) and (8.6.21), we can apply Lemma 8.6.6 to

$$\nu = \mu,\ \Delta = D,\ u(t) = \Big\|y - \frac{g(t)}{\|g(t)\|}\Big\|,\ v(t) = 1 - \Big\langle x^*, \frac{g(t)}{\|g(t)\|}\Big\rangle.$$

This gives (8.6.13). The Claim is proved.

Now, we continue the proof of Lemma 8.6.9. Put $f = -g\mathbb{1}_D + x\mathbb{1}_B$. Let us prove the properties (1) to (3) formulated at the beginning of the proof for this f under the assumption that δ is small enough.

(1) $\mu\{t \in A\colon f(t) = x\} \geqslant \mu(B) = \mu(A) - \mu(D) > \mu(A) - \frac{2\delta(1-\alpha)}{\alpha(2-c)}$ (where we have used (8.6.13)).

(2) $\displaystyle\int_{\{t\in A\colon\, f(t)\neq x\}} \|f(t)\|\, d\mu(t) \leqslant \int_D \|g(t)\|\, d\mu \leqslant \|g\| = 1 = \mu(A).$

(3) $\|T(f)\| \leqslant \|T(g - x\mathbb{1}_A)\| + \|T\|\|g\mathbb{1}_B\| + \|T\|\mu(D)$. By (8.6.10), (8.6.12), and (8.6.13), this means

$$\|T(f)\| \leqslant \delta + \frac{4\delta(1-\alpha)}{\alpha(2-c)}\|T\|.$$

This completes the proof of the lemma. □

THEOREM 8.6.10. *Let X be a reasonable space. Then every narrow operator T acting from $L_1(\mu, X)$ to any other Banach space Y is L-narrow.*

PROOF. Let us fix $y^* \in S_{X^*}$ and $A \in \Sigma$, and denote by W the set of all $x \in X$ such that for every $\varepsilon > 0$ there is an (x, y^*, ε, A)-peak f with $\|Tf\| < \varepsilon$. We have to show that $W = X$. By homogeneity it is enough to check that $W \supset S_X$.

The previous lemma shows that $\mathrm{Reas}(X) \subset W$.

Now, let $x \in S_X$ be an arbitrary element. Fix a $\delta > 0$ and find a convex combination

$$e = \sum_{k=1}^{n} a_k y_k,$$

where $y_k \in \mathrm{Reas}(X)$, δ-approximating x, i.e., $\|x - e\| < \delta$. For every $k = 1, \ldots, n$ there is a $(y_k, y^*, \frac{\delta}{n}, A)$-peak g_k with $\|Tg_k\| < \delta$. Consider

$$g = \sum_{k=1}^{n} a_k g_k$$

and denote by B the set of all $t \in A$ with $g(t) = e$.

By our construction, $\mu(B) > \mu(A) - \delta$, $\|Tg\| < \delta$,

$$\int_{A \setminus B} \|g(t)\| \, d\mu(t) \leqslant (1+\delta)\mu(A) + \delta, \quad \text{and} \quad \left| \int_A x^*(g(t)) \, d\mu(t) \right| < \delta.$$

So, if δ is small enough, the function $f = g + (x - e)\mathbb{1}_B$ will be the (x, y^*, ε, A)-peak we need. □

Combining Theorems 8.6.10 and 8.6.5 we deduce the following:

COROLLARY 8.6.11. *Let X be a reasonable space with $X^* \in \mathrm{RNP}$. Then an operator T acting from $L_1(\mu, X)$ to any other Banach space Y is narrow if and only if it is L-narrow. Since $X = \mathbb{R}$ or $\mathbb{R}^2$ satisfies the above conditions, on both real and complex $L_1(\mu)$ with non-atomic μ, the sets of narrow and L-narrow operators are the same.*

Reflexive spaces are examples of Banach spaces X satisfying the assumptions of the above corollary.

8.7. Rich subspaces of L_1

Let $E \subset L_1(\mu, X)$ be a closed subspace where μ is a finite measure. We define C_E to be the closure of B_E in $L_1(\mu, X)$ with respect to the L_0*-topology*, the topology of convergence in measure. Note that for $f \in C_E$ there is a sequence (f_n) in B_E converging to f pointwise almost everywhere.

PROPOSITION 8.7.1. *Let X be a reasonable space, $E \subset L_1(\mu, X)$, and C_E be as above. If E is rich, then $\frac{1}{2} B_{L_1(\mu, X)} \subset C_E$.*

Proof. Since C_E is $L_1(\mu,X)$-closed, it is enough to show that $f_A := \mathbb{1}_A/\mu(A)x \in (2+r)C_E$ for every $A \in \Sigma^+$, every $x \in S_X$, and every $r > 0$. By the definition of a rich subspace, the quotient map q: $L_1(\mu,X) \to L_1(\mu,X)/E$ is narrow, and by Theorem 8.6.10 it is L-narrow and almost L-narrow as well. So, for every $n \in \mathbb{N}$ there is some $(x, 1/n, A)$-peak function $f_n \in L_1(A,\mu,X)$ with $\operatorname{dist}(f_n, E) < 1/n$. Pick $g_n \in E$ with $\|f_n - g_n\| < 1/n$. The definition of an $(x, 1/n, A)$-peak gives us the following property of f_n: there is a subset $A_n \subset A$ with $\mu(A_n) < 1/n$ such that $f_n(t) = x$ for $t \in A \setminus A_n$ and $\int_{A_n} \|f_n(t)\|\, d\mu(t) \leqslant (1+1/n)\mu(A)$. This implies that (f_n) converges in measure to $\mathbb{1}_A x$ and $\|f_n\| \leqslant (2+1/n)\mu(A)$. Then the $f_n/\mu(A)$ converge in measure to f_A and $\|f_n/\mu(A)\| \leqslant (2+1/n)$. Finally, $g_n/\mu(A) \in E$ are at distance $< \frac{1}{n\mu(A)}$ from the corresponding $f_n/\mu(A)$, so they also converge in measure to f_A as $n \to \infty$, and, for large values of n we have $\|g_n\| \leqslant 2 + 1/n + \frac{1}{n\mu(A)} < 2 + r$, so $g_n \in (2+r)B_E$ which demonstrates the desired inclusion $f_A \in (2+r)C_E$. □

Proposition 8.7.2. *If* $\frac{1}{2}B_{L_1(\mu,X)} \subset C_Y$ *for all* 1*-codimensional subspaces* Y *of* E*, then* E *is almost rich.*

Proof. In order to demonstrate the strong Daugavetness of the quotient map q: $L_1(\mu,X) \to L_1(\mu,X)/E$ it is sufficient to show that q is almost L-narrow (Theorem 8.6.5); that is, for every $x \in X$, $\varepsilon > 0$ and $A \in \Sigma^+$ we need to find an (x,ε,A)-peak function f with $\|q(f)\| < \varepsilon$.

Fix a supporting functional x^* for x, i.e., $\|x^*\| = 1$ and $x^*(x) = \|x\|$. Let $Y = \{f \in E\colon \int_A x^* f\, d\mu = 0\}$. By assumption, there is a sequence (f_n) in Y such that $\|f_n\| \leqslant 2\|x\|\mu(A)$ and $f_n \to x\mathbb{1}_A$ in measure. For a fixed small $\delta \in (0,\varepsilon)$ consider the subsets $B_n = \{t \in A\colon \|f_n(t) - x\| > \delta\}$ and $D_n = A \setminus B_n$. The convergence in measure gives us an $m = m(\delta)$ for which $\mu(B_m) < \delta$. Then

$$\begin{aligned}
\|f_m \mathbb{1}_{D_m}\| &= \int_{D_m} \|f_m(t)\|\, d\mu(t) \geqslant \int_{D_m} (\|x\| - \delta)\, d\mu \geqslant (\|x\| - \delta)(\mu(A) - \delta);\\
\|f_m \mathbb{1}_{B_m}\| &\geqslant \left| \int_{B_m} x^* f\, d\mu \right| = \left| \int_{D_m} x^* f\, d\mu \right| \geqslant \left| \int_{D_m} x^* x\, d\mu \right| - \delta\mu(D_m)\\
&= \|x\|\mu(D_m)(1-\delta) > \|x\|(\mu(A) - \delta)(1-\delta).
\end{aligned}$$

So,

$$\begin{aligned}
\|f_m \mathbb{1}_{\Omega\setminus A}\| &= \|f_m\| - \|f\mathbb{1}_{D_m}\| - \|f\mathbb{1}_{B_m}\|\\
&\leqslant 2\|x\|\mu(A) - (\|x\| - \delta)(\mu(A) - \delta) - \|x\|(\mu(A) - \delta)(1-\delta) \xrightarrow[\delta\to 0]{} 0.
\end{aligned}$$

Consider $f := x\mathbb{1}_{D_m} + f_m \mathbb{1}_{B_m}$. This f, when δ is small enough, is the desired (x,ε,A)-peak with $\|q(f)\| < \varepsilon$. Indeed, $\operatorname{supp} f \subset A$, the role of the corresponding A_1 is played B_m with $\mu(B_m) < \delta < \varepsilon$, because

$$\begin{aligned}
\int_{B_m} \|f(t)\|\, d\mu(t) &= \|f_m \mathbb{1}_{B_m}\| \leqslant \|f_m\| - \|f_m \mathbb{1}_{D_m}\|\\
&\leqslant 2\|x\|\mu(A) - (\|x\| - \delta)(\mu(A) - \delta) \xrightarrow[\delta\to 0]{} \mu(A)\|x\|
\end{aligned}$$

and $\|q(f)\| \leqslant \|f - f_m\| = \|(f - x)\mathbb{1}_{D_m} f_m\| + \|f_m \mathbb{1}_{\Omega\setminus A}\| \xrightarrow[\delta\to 0]{} 0.$ □

We sum this up in a theorem.

THEOREM 8.7.3. *Let X be a reasonable space, $E \subset L_1(\mu, X)$ a subspace. Then E is rich in $L_1(\mu, X)$ if and only if $\frac{1}{2}B_{L_1(\mu,X)} \subset C_Y$ for all 2-codimensional subspaces Y of E.*

PROOF. In one direction the result follows from Proposition 8.7.1 because a finite-codimensional subspace of a rich subspace is rich (Theorem 6.5.12). In the opposite direction we obtain the desired result from Proposition 8.7.2, taking into account Lemma 6.5.9: a subspace of a Banach space with the Daugavet property which is almost rich together with all of its 1-codimensional subspaces is rich. $\square$

The next proposition shows that the factor $\frac{1}{2}$ is optimal in the case of the real space $L_1 = L_1[0, 1]$. Below λ denotes the Lebesgue measure on $[0, 1]$.

PROPOSITION 8.7.4. *If, for some $r > \frac{1}{2}$, $rB_{L_1} \subset C_E$, then $E = L_1$.*

PROOF. Suppose $h \in L_\infty$, $\|h\|_\infty = 1$, and let $Y = \{f \in L_1\colon \int fh = 0\}$. Assume that $B_{L_1} \subset sC_Y$; we shall argue that $s \geqslant 2$. This will prove the proposition since every proper closed subspace E is contained in a closed hyperplane.

Assume, without loss of generality, that h takes the (essential) value 1. Let $\varepsilon > 0$, and put $A = \{|h - 1| < \varepsilon/2\}$; then A has positive measure. There is a sequence (f_n) converging to $\mathbb{1}_A$ in measure such that $\|f_n\| \leqslant s\,\lambda(A)$ and $\int f_n h = 0$ for all n. Since $f_n h \to \mathbb{1}_A h$ in measure as well, there is, if n is a sufficiently large index, a subset $A_n \subset A$ of measure $\geqslant (1 - \varepsilon)\lambda(A)$ such that $|f_n h - 1| < \varepsilon$ on A_n. For such an n,

$$\left|\int_{A_n} f_n h\right| = \left|\lambda(A_n) - \int_{A_n} (1 - f_n h)\right| \\ \geqslant \lambda(A_n) - \int_{A_n} |1 - f_n h| \geqslant (1 - \varepsilon)\lambda(A_n),$$

and, therefore,

$$\int_{A_n} |f_n h| \geqslant (1 - \varepsilon)\lambda(A_n)$$

and, if B_n denotes the complement of A_n,

$$\int_{B_n} |f_n h| \geqslant \left|\int_{B_n} f_n h\right| = \left|\int_{A_n} f_n h\right| \geqslant (1 - \varepsilon)\lambda(A_n)$$

so that

$$s\,\lambda(A) \geqslant \|f_n\| \geqslant \|f_n h\| \geqslant 2(1 - \varepsilon)^2\lambda(A).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $s \geqslant 2$. $\square$

Thus, the rich subspaces appear to be the next best thing in terms of size of a subspace E of L_1 after L_1 itself. At the other end of the spectrum are the *nicely placed* subspaces of L_1, defined by the condition that B_E is L_0-closed. It is shown in [**133**, Th. IV.3.5] that a subspace E of L_1 is nicely placed if and only if E is an L-summand in its bidual, i.e., $E^{**} = E \oplus_1 E_s$ for some closed subspace E_s of E^{**}.

We now look at the translation invariant case, and we consider $L_1 = L_1(\mathbb{T})$ (or $L_1(G)$ for a compact abelian group). As usual, for $\Lambda \subset \mathbb{Z}$ the space $L_{1,\Lambda}$ consists of those L_1-functions whose Fourier coefficients vanish off Λ (in other words, $\operatorname{supp} \widehat{f} \subset \Lambda$, with $\widehat{f}$ denoting the Fourier transform).

PROPOSITION 8.7.5. *Let $\Lambda \subset \mathbb{Z}$ and suppose that $L_{1,\Lambda}$ is rich in L_1. Then, for every measure μ on $\mathbb{T}$ and every $\varepsilon > 0$, there is a measure ν with $\|\nu\| \leqslant \|\mu\| + \varepsilon$ and $\widehat{\nu}(\gamma) = \widehat{\mu}(\gamma)$ for all $\gamma \notin \Lambda$ that is ε-almost singular in the sense that there is a set S with $\lambda(S) \leqslant \varepsilon$ and $|\nu|(\mathbb{T} \setminus S) \leqslant \varepsilon$.*

PROOF. Let $\mu = f\lambda + \mu_s$ be the Lebesgue decomposition of μ, and let $\delta > 0$. By Proposition 8.7.1, there is a function $g \in L_{1,\Lambda}$ such that $\|g\| \leqslant 2\|f\|$ and $A := \{|f - g| > \delta\}$ has measure $< \delta$. Let $B := \{|f - g| \leqslant \delta\}$. Then

$$\|g\mathbb{1}_A\| \leqslant 2\|f\| - \|g\mathbb{1}_B\| \leqslant 2\|f\| - \|f\mathbb{1}_B\| + \delta = \|f\| + \|f\mathbb{1}_A\| + \delta.$$

Therefore we have for $\nu := \mu - g\lambda$

$$\begin{aligned}\|\nu\| &= \|(f-g)\lambda + \mu_s\| \\ &\leqslant \|f\mathbb{1}_A\| + \|g\mathbb{1}_A\| + \|(f-g)\mathbb{1}_B\| + \|\mu_s\| \\ &\leqslant 2\|f\mathbb{1}_A\| + 2\delta + \|\mu\|\end{aligned}$$

and hence, $\|\nu\| \leqslant \|\mu\| + \varepsilon$ if δ is sufficiently small.

Clearly, $\widehat{\nu} = \widehat{\mu}$ on the complement of Λ, and if N is a null set supporting μ_s, then $S := A \cup N$ has the required properties if $\delta \leqslant \varepsilon$. □

We apply these ideas to *Sidon sets*, i.e., sets $\Lambda' \subset \mathbb{Z}$ such that all functions in $C_{\Lambda'} = \{f \in C(\mathbb{T})\colon \operatorname{supp} \widehat{f} \subset \Lambda'\}$ have absolutely sup-norm convergent Fourier series. (See [**182**] for results on this notion.) If Λ is the complement of a Sidon set, then $L_1/L_{1,\Lambda}$ is isomorphic to c_0 or finite-dimensional [**264**, p. 121]. Hence, $L_{1,\Lambda}$ is rich (the corresponding quotient operator does not fix copies of ℓ_1, so is narrow, see Theorem 6.3.5), and Proposition 8.7.5 applies. Thus, the following corollary holds.

COROLLARY 8.7.6. *If $\Lambda' \subset \mathbb{Z}$ is a Sidon set and μ is a measure on $\mathbb{T}$, then for every $\varepsilon > 0$ there is an ε-almost singular measure ν with $\|\nu\| \leqslant \|\mu\| + \varepsilon$ and $\widehat{\nu}(\gamma) = \widehat{\mu}(\gamma)$ for all $\gamma \in \Lambda'$.*

8.8. Notes and remarks

Section 8.1, **Section 8.2** and **Section 8.3.** These sections follow [**53**]. The part of Corollary 8.3.10 saying that a pointwise unconditionally convergent sum of narrow operators on the real space $C(K)$ acting to one the same space W is a narrow operator itself was first demonstrated in [**176**].

Section 8.4. Most of the results in this section can be found in [**175**].

Section 8.5. This part comes from [**158**, Section 3].

Section 8.6. The main results of this section come from the paper [**68**]. Theorem 8.6.3 is a corrected version of [**68**, Theorem 2.2] whose proof contains a gap. We don't know if the implication (2) ⇒ (1) in that theorem remains valid without additional assumptions like the RNP; see Question (8.7).

Section 8.7. This part originates from [**158**, Section 2]. We have extended some results from the original paper to the vector-valued case. Actually, the original scalar version of Theorem 8.7.3 is a little better: it suffices to deal with subspaces of codimension 1.

8.9. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

(8.1) Is every reflexive space USD-nonfriendly?

(8.2) Does the converse of Proposition 8.3.4(c) hold?

(8.3) Does Proposition 8.3.4(b) hold without the assumption of separability?

The difficulty when trying to remove the separability assumption is that the property of being USD-nonfriendly is not inherited by subspaces (see Remark 8.1.7).

(8.4) If a Banach space X possesses a subspace with the Daugavet property, is X isomorphic to a space with the Daugavet property (or, in other words, X has the Daugavet property in an equivalent norm)?

(8.5) Is Corollary 8.4.12 valid for non-separable Banach spaces E?

(8.6) What are the Banach spaces X for which every narrow operator T acting from $L_1(\mu, X)$ to any other Banach space Y is L-narrow? Does this class contain the Banach spaces X with the RNP?

(8.7) Can one remove the assumption $X^* \in \mathrm{RNP}$ from the last part of Theorem 8.6.3?

CHAPTER 9

The almost Daugavet property and duality

At the beginning of Chapter 7, in Definition 7.1.1, we introduced the concept of the Daugavet property with respect to a subspace of the dual space: a Banach space X has the Daugavet property with respect to a subspace $E \subset X^*$ ($X \in \mathrm{DPr}(E)$) if for every $x \in S_X$, $x^* \in S_E$ and $\varepsilon > 0$ there exists some $y \in S_X$ such that $\operatorname{Re} x^*(y) > 1-\varepsilon$ and $\|x+y\| > 2-\varepsilon$.

In this chapter we develop this idea further, concentrating on the most important case of the Daugavet property with respect to a one-norming subspace. The importance of this case comes from the following duality observation: if E is a one-norming subspace of X^*, then, under the standard notation $x(y) := y(x)$, $x \in X$, $y \in E \subset X^*$, X is isometric to a subspace of E^* which is identified with X and denoted by the same letter, X. Moreover, in this sense, X is a one-norming subspace of E^*. So, we may interchange X and E at our convenience.

We introduce the almost Daugavet property (that is, $\mathrm{DPr}(E)$ for some one-norming subspace $E \subset X^*$, see Definition 9.2.1 below), give a number of its, sometimes unexpected, characterisations, relating it to Whitley's thickness and presence of ℓ_1 subspaces; present an almost Daugavet property renorming result (for a separable space X such a remorming exists if and only if X contains a copy of ℓ_1), and after that we pass to duality considerations, which enable us to approach successfully the Daugavet property of quotient spaces.

The results in this chapter also allow us to tackle the following problem. According to Corollary 3.5.4, if $X = L_1[0,1]$ and Z is a reflexive subspace of X, then $X/Z \in \mathrm{DPr}$. In a private conversation with one of the authors, A. Pełczyński raised the question of whether this persists for subspaces with the RNP (see also [**285**, Problem 3]). Eventually, we shall give a negative answer in Section 9.7, see Corollary 9.7.14.

In [**285**, Theorem 6(b)] R. Shvydkoy proved the Daugavet property of X/Z for arbitrary $X \in \mathrm{DPr}$ and reflexive $Z \subset X$. This result may also be deduced from the duality argument that we develop in this chapter, see Corollary 9.6.9 below.

9.1. The Daugavet property with respect to a one-norming subspace

We start with a circle of reformulations, similar to those which we know for the ordinary Daugavet property. Let X be a Banach space. Analogously to Daugavet points, let us call a functional $z^* \in S_{X^*}$ a *Daugavet functional* if for every $x \in S_X$

$$\|\mathrm{Id} + z^* \otimes x\| = 2.$$

By a quasi-codirectness argument (by now presumably standard for the reader), this is the same as $\|\mathrm{Id} + a z^* \otimes x\| = 1 + \|a z^* \otimes x\|$ for every $x \in X$ and all scalars a.

LEMMA 9.1.1. *If E is a one-norming subspace of X^*, then the following assertions are equivalent.*

(i) $X \in \mathrm{DPr}(E)$.
(ii) *For every* $x \in S_X$, *every* $\varepsilon > 0$, *and every* $y^* \in S_E$, *there is some* $y \in \mathrm{Slice}(B_X, y^*, \varepsilon)$ *such that*

$$\|x + y\| > 2 - \varepsilon. \tag{9.1.1}$$

(iii) *For every* $x \in S_X$ *and* $y^* \in S_E$, *the identity* $\|\mathrm{Id}_X + y^* \otimes x\| = 2$ *holds true.*
(iv) *For every* $x \in S_X$, *every* $\varepsilon > 0$, *and every* $y^* \in S_E$, *there is a slice* $\mathrm{Slice}(B_X, y_1^*, \varepsilon_1) \subset \mathrm{Slice}(B_X, y^*, \varepsilon)$ *with* $y_1^* \in S_E$ *such that* (9.1.1) *holds for every* $y \in \mathrm{Slice}(B_X, y_1^*, \varepsilon_1)$.

Proof. (ii) is just a rephrasing of (i) in the language of slices up to an inessential technical difference between the conditions $y \in S_X$ and $y \in B_X$ (we already know how to deal with this difference); Lemma 3.1.3 gives us the equivalence (ii) $\Leftrightarrow$ (iii); (iv) evidently implies (ii), and the proof of the remaining implication (ii) $\Rightarrow$ (iv) repeats the one of Lemma 3.1.10 with just one minor modification: after the application of Lemma 2.6.8 one needs to use Remark 2.6.9 in order to get $x^* \in S_E$ instead of $x^* \in S_{X^*}$. $\square$

Applying the above lemma and a duality argument we obtain the following.

Lemma 9.1.2. *If E is a one-norming subspace of X^*, then the following assertions are equivalent.*
(i) $X \in \mathrm{DPr}(E)$.
(ii) $E \in \mathrm{DPr}(X)$.
(iii) *For every* $x^* \in S_E$, *every* $\varepsilon > 0$, *and every weak* slice* $\mathrm{Slice}(B_E, x, \varepsilon)$ *with* $x \in S_X$, *there is some* $y^* \in \mathrm{Slice}(B_E, x, \varepsilon)$ *such that* $\|x^* + y^*\| > 2 - \varepsilon$.
(iv) *For every* $x^* \in S_E$, *every* $\varepsilon > 0$, *and every weak* slice* $\mathrm{Slice}(B_E, x, \varepsilon)$, *there is another weak* slice* $\mathrm{Slice}(B_E, x_1, \varepsilon_1) \subset \mathrm{Slice}(B_E, x, \varepsilon)$ *such that* $\|x^* + y^*\| > 2 - \varepsilon$ *for every* $y^* \in \mathrm{Slice}(B_E, x_1, \varepsilon_1)$.

Proof. The mutual equivalence of the three last statements was demonstrated in the previous Lemma 9.1.1, so the only thing that we need to show is the equivalence (ii) $\Leftrightarrow$ (i). For this, we only need to remark that for $x \in S_X$, $y^* \in S_E$ the identity $\|\mathrm{Id}_X + y^* \otimes x\| = 2$ is equivalent to the identity $\|\mathrm{Id}_E + x \otimes y^*\| = 2$ by the same standard proof as in the standard fact that the norms of an operator and of its adjoint are the same:

$$\begin{aligned}\|\mathrm{Id}_X + y^* \otimes x\| &= \sup_{z \in S_X} \|z + y^*(z)x\| = \sup_{z \in S_X, f \in S_E} |f(z + y^*(z)x)| \\ &= \sup_{f \in S_E} \sup_{z \in S_X} |(f + f(x)y^*)z| = \sup_{f \in S_E} \|f + (x \otimes y^*)f\| \\ &= \|\mathrm{Id}_E + x \otimes y^*\|.\end{aligned}$$

(Here we used in the second step that E is one-norming.) $\square$

Observe that, for the case $E = X^*$, Lemmas 9.1.1 and 9.1.2 are contained in item (x)* of Theorem 3.1.11.

Two more reformulations come from the fact that S_E is weak* dense in B_{X^*} (statement 5 of Theorem 2.1.3).

Lemma 9.1.3. *Let E be a one-norming subspace of X^*; then the following assertions are equivalent.*
(i) $X \in \mathrm{DPr}(E)$.

(iii)′ *For every* $x^* \in S_E$*, for every* $\varepsilon > 0$*, and for every weak** *slice* $\mathrm{Slice}(B_{X^*}, x, \varepsilon)$*,* $x \in S_X$*, there is some* $y^* \in \mathrm{Slice}(B_{X^*}, x, \varepsilon)$ *such that* $\|x^* + y^*\| \geqslant 2 - \varepsilon$*.*

(iv)′ *For every* $x^* \in S_E$*, for every* $\varepsilon > 0$*, and for every weak** *slice* $\mathrm{Slice}(B_{X^*}, x, \varepsilon)$ *there is another weak** *slice* $\mathrm{Slice}(B_{X^*}, x_1, \varepsilon_1) \subset \mathrm{Slice}(B_{X^*}, x, \varepsilon)$ *such that* $\|x^* + y^*\| \geqslant 2 - \varepsilon$ *for every* $y^* \in \mathrm{Slice}(B_{X^*}, x_1, \varepsilon_1)$*.*

Proof. (iii)′ follows from (i) because the intersection of $\mathrm{Slice}(B_{X^*}, x, \varepsilon)$ with B_E is $\mathrm{Slice}(B_E, x, \varepsilon)$ and we have the requested $y^* \in \mathrm{Slice}(B_E, x, \varepsilon)$ thanks to (iii) of Lemma 9.1.2. (iv)′ follows from (iii)′ the same way as (ii) ⇒ (iv) in Lemma 9.1.1. Finally, (iv)′ implies (i) because $\mathrm{Slice}(B_{X^*}, x_1, \varepsilon_1) \cap S_E \neq \emptyset$ by the weak* density of S_E in B_{X^*}, which we mentioned just before the Lemma. □

9.2. A characterisation of almost Daugavet spaces by means of ℓ_1-sequences in the dual

Definition 9.2.1. A Banach space X is said to have the *almost Daugavet property* or to be an *almost Daugavet space* if it has $\mathrm{DPr}(E)$ for some one-norming subspace $E \subset X^*$.

Recall that according to Definition 2.8.5, for a subspace W of a Banach space Z and $\varepsilon > 0$, an element $w \in B_Z$ is $(\varepsilon, 1)$-orthogonal to W if $\|x + tw\| \geqslant (1-\varepsilon)(\|x\| + |t|)$ for every $x \in W$ and $t \in \mathbb{K}$.

Definition 9.2.2. Let E be a Banach space. A sequence $(e_n)_{n\in\mathbb{N}} \subset B_E \setminus \{0\}$ is said to be an *asymptotic* ℓ_1*-sequence* if there is a sequence (ε_n) of positive numbers with $\prod_{n\in\mathbb{N}}(1-\varepsilon_n) > 0$ such that e_{n+1} is $(\varepsilon_n, 1)$-orthogonal to $Y_n := \mathrm{lin}\{e_1, \dots, e_n\}$ for every $n \in \mathbb{N}$.

Evidently, every asymptotic ℓ_1-sequence is $1/\prod_{n\in\mathbb{N}}(1-\varepsilon_n)$-equivalent to the canonical basis of ℓ_1 and, moreover, every element of the unit sphere of $Z_m := \mathrm{lin}\{e_k \colon k \geqslant m+1\}$ is $\bigl(1 - \prod_{n\geqslant m}(1-\varepsilon_n), 1\bigr)$-orthogonal to Y_m for every $m \in \mathbb{N}$.

The following lemma is completely analogous to Lemma 3.1.14, just instead of (x) of Theorem 3.1.11 it uses (iv)′ of Lemma 9.1.3. So we state it without proof.

Lemma 9.2.3. *Let E be a one-norming subspace of X^*, $X \in \mathrm{DPr}(E)$, and let $E_0 \subset E$ be a finite-dimensional subspace. Then for every $\varepsilon_0 > 0$ and every weak** *slice* $\mathrm{Slice}(B_{X^*}, x_0, \varepsilon_0)$ *of* B_{X^*} *there is another weak** *slice* $\mathrm{Slice}(B_{X^*}, x_1, \varepsilon_1) \subset \mathrm{Slice}(B_{X^*}, x_0, \varepsilon_0)$ *of* B_{X^*} *such that every element* $e^* \in \mathrm{Slice}(B_{X^*}, x_1, \varepsilon_1)$ *is* $(\varepsilon_0, 1)$*-orthogonal to* E_0*. In particular, there is an element* $e_1^* \in \mathrm{Slice}(B_{X^*}, x_0, \varepsilon_0) \cap S_E$ *which is* $(\varepsilon_0, 1)$*-orthogonal to* E_0*.*

Changing the roles of X and E in the above lemma (or repeating once more the proof of Lemma 3.1.14, applying (iv) of Lemma 9.1.1 on the way), we obtain one more generalisation of Lemma 3.1.14.

Lemma 9.2.4. *If E is a one-norming subspace of X^* and $X \in \mathrm{DPr}(E)$, then, for every finite-dimensional subspace $X_0 \subset X$ and for every $\varepsilon > 0$, each slice* $\mathrm{Slice}(B_X, x_0^*, \varepsilon_0)$ *generated by* $x_0^* \in S_E$ *contains a smaller slice* $\mathrm{Slice}(B_X, x_1^*, \varepsilon_1) \subset \mathrm{Slice}(B_X, x_0^*, \varepsilon_0)$ *of* B_X *with* $x_1^* \in S_E$ *such that every* $x \in \tilde{\mathrm{Slice}}(B_X, x_1^*, \varepsilon_1) \cap S_X$ *is* $(\varepsilon, 1)$*-orthogonal to* X_0*.*

We need one more definition.

DEFINITION 9.2.5. A sequence $(e_n^*)_{n\in\mathbb{N}} \subset B_{X^*}$ is said to be *double-norming* if $\operatorname{lin}\bigl(\{e_k^*\colon k \geqslant n\}\bigr)$ is norming for every $n \in \mathbb{N}$.

Here is the main result of this section.

THEOREM 9.2.6. *A separable Banach space X is an almost Daugavet space if and only if X^* contains a double-norming asymptotic ℓ_1-sequence.*

PROOF. First we prove the "if" part. Let $(e_n^*)_{n\in\mathbb{N}} \subset B_{X^*}$ be a double-norming asymptotic ℓ_1-sequence, and let $\varepsilon_n > 0$ with $\prod_{n\in\mathbb{N}}(1-\varepsilon_n) > 0$ be such that e_{n+1}^* is $(\varepsilon_n, 1)$-orthogonal to $F_n := \operatorname{lin}\{e_1^*, \dots, e_n^*\}$ for every $n \in \mathbb{N}$. Let us prove that X has the Daugavet property with respect to $E = \overline{\operatorname{lin}}\{e_n^*\colon n \in \mathbb{N}\}$ where the closure is meant in the norm topology. To do this let us apply (iii)$'$ of Lemma 9.1.3.

Fix $x^* \in S_E$, $\varepsilon > 0$, and a weak* slice $\operatorname{Slice}(B_{X^*}, x, \varepsilon)$ of the dual ball B_{X^*}. Denote in addition to $F_m = \operatorname{lin}\{e_1^*, \dots, e_m^*\}$, $Z_m := \operatorname{lin}\{e_k^*\colon k \geqslant m+1\}$. Using the definition of E, select $m \in \mathbb{N}$ and $x_m^* \in F_m$ such that

$$\|x^* - x_m^*\| < \varepsilon/2 \ \text{ and } \ \prod_{n\geqslant m}(1-\varepsilon_n) > 1 - \varepsilon/2.$$

Since Z_m is norming, there is some $y^* \in \operatorname{Slice}(B_{X^*}, x, \varepsilon) \cap S_{Z_m}$. Taking into account that every element of S_{Z_m} is $(\varepsilon/2, 1)$-orthogonal to F_m, we obtain

$$\|x^* + y^*\| \geqslant \|x_m^* + y^*\| - \|x^* - x_m^*\| \geqslant 2 - \varepsilon.$$

For the "only if" part we proceed as follows. First we fix a sequence $(\varepsilon_n) \subset \mathbb{R}^+$ with $\prod_{n\in\mathbb{N}}(1-\varepsilon_n) > 0$ and a dense sequence (x_n) in S_X. We can choose these x_n in such a way that each of them appears in the sequence (x_n) infinitely many times. Assume now that $X \in \operatorname{DPr}(E)$, where $E \subset X^*$ is a one-norming subspace. Starting with $E_0 = \{0\}$, $\varepsilon_0 = 1$ and applying Lemma 9.2.3 step-by-step we can construct a sequence $(e_n^*)_{n\in\mathbb{N}} \subset S_E$ in such a way that each e_{n+1}^* belongs to $\operatorname{Slice}(B_{X^*}, x_n, \varepsilon_n)$ and is $(\varepsilon_n, 1)$-orthogonal to E_n, where $E_n = \operatorname{lin}\{e_1^*, \dots, e_n^*\}$ as before. This inductive construction ensures that the e_n^*, $n \in \mathbb{N}$, form an asymptotic ℓ_1-sequence. On the other hand, this sequence meets every slice $\operatorname{Slice}(B_{X^*}, x_n, \varepsilon_n)$ infinitely many times, and this implies by density of (x_n) that (e_n^*) is double-norming. $\square$

In Corollary 9.3.6 we shall observe a somewhat more pleasing version of the last result.

We conclude the section with two examples.

PROPOSITION 9.2.7. *The* real *space ℓ_1 is an almost Daugavet space.*

PROOF. To prove this statement we will construct a double-norming asymptotic ℓ_1-sequence $(f_n) \subset \ell_\infty = (\ell_1)^*$. At first consider a sequence $(g_n) \subset \ell_\infty$ of elements $g_n = (g_{n,j})_{j\in\mathbb{N}}$ with all $g_{n,j} = \pm 1$ satisfying the following independence condition: for arbitrary finite collections $\alpha_s = \pm 1$, $s = 1, \dots, n$, the set of those j that $g_{s,j} = \alpha_s$ for all $s = 1, \dots, n$ is infinite (for instance, put $g_{s,j} := r_s(t_j)$, where the r_s are the Rademacher functions and $(t_j)_{j\in\mathbb{N}}$ is a fixed sequence of irrationals that is dense in $[0, 1]$). These g_n, $n \in \mathbb{N}$, form an isometric ℓ_1-sequence and, moreover, if one changes a finite number of coordinates in each of the g_n to some other ± 1, the independence condition will survive, so the modified sequence will still be an isometric ℓ_1-sequence.

Now, let us define the vectors $f_n = (f_{n,j})_{j\in\mathbb{N}}$, $f_{n,j} = \pm 1$, in such a way that for $k = 1, 2, \dots$ and $n = 2^k+1, 2^k+2, \dots, 2^{k+1}$ the vectors $(f_{n,j})_{j=1}^k \in \ell_\infty^{(k)}$ run over all extreme points of the unit ball of $\ell_\infty^{(k)}$, i.e., over all possible k-tuples of ± 1; for the remaining values of indices we put $f_{n,j} = g_{n,j}$. As we have already remarked, the f_n form an isometric ℓ_1-sequence. Moreover, for every $k \in \mathbb{N}$ the restrictions of the f_n to the first k coordinates form a double-norming sequence over $\ell_1^{(k)}$, so $(f_n)_{n\in\mathbb{N}}$ is a double-norming sequence over ℓ_1. □

Some ideas of the previous proof will enter into the proof of Theorem 9.3.1. As a consequence of that theorem, the complex space ℓ_1 is almost Daugavet as well. It is worth noting that, because of the Radon-Nikodým property, ℓ_1 fails the Daugavet property and cannot even be renormed to have it.

Since ℓ_∞ is isomorphic to $L_\infty[0,1]$, which has the Daugavet property, ℓ_∞ can be equivalently renormed to possess the Daugavet property. Let us show that in the original norm it is not even an almost Daugavet space. This is a special case of the following proposition.

PROPOSITION 9.2.8. *No Banach space of the form $Z = X \oplus_\infty \mathbb{K}$ is an almost Daugavet space.*

PROOF. We shall show that $z_0^* = (x_0^*, b_0) \in S_{Z^*}$ is not a Daugavet functional if $b_0 \neq 0$. Hence all the Daugavet functionals lie in the weak* closed subspace $(\{0\} \oplus \mathbb{K})^\perp$ of $Z^* = X^* \oplus_1 \mathbb{K}$, which, by (iii) of Lemma 9.1.1, is impossible for an almost Daugavet space.

So let $x_0^* \in X^*$ and $b_0 \neq 0$ with $\|x_0^*\| + |b_0| = 1$, $z_0^* = (x_0^*, b_0)$ and let $z_0 = (0, -|b_0|/b_0)$. If $z = (x, a) \in B_Z$, i.e., $\|x\| \leqslant 1$ and $|a| \leqslant 1$, then

$$\begin{aligned}\|z + z_0^*(z) z_0\| &= \max\{\|x\|, |\, a - z_0^*(z)|b_0|/b_0\,|\} \\ &\leqslant \max\{1, |\, a - (x_0^*(x_0) + b_0 a)|b_0|/b_0\,|\} \\ &\leqslant \max\{1, \|x_0^*\| + (1 - |b_0|)\}.\end{aligned}$$

This shows that $\|\mathrm{Id} + z^* \otimes z\| \leqslant \max\{1, \|x_0^*\| + (1-|b_0|)\} < 2$ and so, z_0^* is not a Daugavet functional. □

If K is a compact Hausdorff space with an isolated point, then $C(K)$ is of the form $X \oplus_\infty \mathbb{K}$, hence it fails the almost Daugavet property. But if K is an uncountable metric space, then $C(K)$ is isomorphic to $C[0,1]$ by Milutin's theorem [**303**, Th. III.D.19], hence it can be renormed to have the Daugavet property.

9.3. The almost Daugavet property, ℓ_1-type sequences, and the Whitley parameter

For a Banach space X, R. Whitley [**301**] introduced the following parameter, called *thickness*, which is essentially the inner measure of non-compactness of the unit sphere S_X:

$$\mathrm{Thick}(X) = \inf\{\varepsilon > 0\colon \text{there exists a finite } \varepsilon\text{-net for } S_X \text{ in } S_X\},$$

or equivalently, $\mathrm{Thick}(X)$ is the infimum of those ε such that the unit sphere of X can be covered by a finite number of balls with radius ε and centres in S_X. He showed that $1 \leqslant \mathrm{Thick}(X) \leqslant 2$ in the infinite dimensional case and, in particular, that $\mathrm{Thick}(C(K)) = 1$ if K has isolated points and $\mathrm{Thick}(C(K)) = 2$ if not.

The main results of the section are the following.

THEOREM 9.3.1. *For a separable Banach space X the following assertions are equivalent:*

(a) $\mathrm{Thick}(X) = 2$;

(b) *there is a sequence $(e_n) \subset B_X$ such that for every $x \in X$*

$$\lim_{n\to\infty} \|x + e_n\| = \|x\| + 1;$$

(that is, (e_n) is an "ℓ_1-type sequence", see Definition 2.8.1).

(c) *X is an almost Daugavet space.*

Since the three properties considered in Theorem 9.3.1 hold for a complex Banach space X if and only if they hold for the underlying real space $X_{\mathbb{R}}$, we will tacitly assume in this section that we are dealing with *real* spaces.

We will accomplish the proof of Theorem 9.3.1 by means of the Propositions 9.3.2, 9.3.3 and 9.3.5 which give us implications (c) $\Rightarrow$ (a), (a) $\Rightarrow$ (b), and (b) $\Rightarrow$ (c) of Theorem 9.3.1, respectively.

PROPOSITION 9.3.2. *Every almost Daugavet space X has* $\mathrm{Thick}(X) = 2$.

PROOF. Let $E \subset X^*$ be a one-norming subspace with respect to which $X \in \mathrm{DPr}(E)$. According to the definition of $\mathrm{Thick}(X)$, we have to show that for every $\varepsilon_0 > 0$ there is no finite $(2-\varepsilon_0)$-net of S_X consisting of elements of S_X. In other words, we must demonstrate that for every collection $\{x_1, \ldots, x_n\} \subset S_X$ there is $y_0 \in S_X$ with $\|x_k - y_0\| > 2 - \varepsilon_0$ for all $k = 1, \ldots, n$. But this is an evident corollary of Lemma 9.2.4 applied to $X_0 := \mathrm{lin}\{x_1, \ldots, x_n\}$. □

Let us now turn to the implication (a) $\Rightarrow$ (b) of Theorem 9.3.1.

PROPOSITION 9.3.3. *If* $\mathrm{Thick}(X) = 2$ *and X is separable, then X contains an ℓ_1-type sequence.*

PROOF. Fix a dense countable set $A = \{a_n \colon n \in \mathbb{N}\} \subset S_X$ and a null-sequence (ε_n) of positive reals. Since for every $n \in \mathbb{N}$ the n-point set $\{-a_1, \ldots, -a_n\}$ is not a $(2-\varepsilon_n)$-net of S_X there is $e_n \in S_X$ with $\|e_n - (-a_k)\| > 2 - \varepsilon_n$ for all $k = 1, \ldots, n$. The constructed sequence (e_n) satisfies for every $k \in \mathbb{N}$ the condition

$$\lim_{n\to\infty} \|a_k + e_n\| = \|a_k\| + 1 = 2.$$

By the density of A in S_X and Remark 2.8.2, this yields that (e_n) is an ℓ_1-type sequence. □

By Lemma 2.8.9 we obtain:

COROLLARY 9.3.4. *Every almost Daugavet space contains an isomorphic copy of ℓ_1.*

It remains to prove the implication (b) $\Rightarrow$ (c) of Theorem 9.3.1.

PROPOSITION 9.3.5. *A separable Banach space X containing an ℓ_1-type sequence is an almost Daugavet space.*

PROOF. We will use Theorem 9.2.6. Fix an increasing sequence of finite-dimensional subspaces $E_1 \subset E_2 \subset E_3 \subset \ldots$ whose union is dense in X. Also, fix sequences $\varepsilon_n \searrow 0$ and $\delta_n > 0$ such that for all n

$$\prod_{k=n}^{\infty} (1 - \delta_k) \geqslant 1 - \varepsilon_n. \tag{9.3.1}$$

Passing to a subsequence if necessary, we can find an ℓ_1-type sequence (e_n) satisfying the following additional condition: For every $x \in \operatorname{lin}(E_n \cup \{e_1, \dots, e_n\})$ and every $\alpha \in \mathbb{R}$ we have

$$\|x + \alpha e_{n+1}\| \geqslant (1 - \delta_n)(\|x\| + |\alpha|). \tag{9.3.2}$$

Then, for every $x \in E_n$ and every $y = \sum_{k=n+1}^{M} a_k e_k$, we have by (9.3.1) and (9.3.2)

$$\|x + y\| \geqslant (1 - \varepsilon_n)\|x\| + \sum_{k=n+1}^{M} (1 - \varepsilon_{k-1})|a_k|. \tag{9.3.3}$$

Fix a dense sequence (x_n) in S_X such that $x_n \in E_n$ and every element of the range of the sequence is taken infinitely often, that is, the set $\{n\colon x_n = x_m\}$ is infinite for every $m \in \mathbb{N}$. Finally, fix an "independent" sequence $(g_n) \subset \ell_\infty$, $g_{n,j} = \pm 1$, as in the proof of Proposition 9.2.7.

Now, we are ready to construct a double-norming asymptotic ℓ_1-sequence $(f_n^*) \subset X^*$. First, we define $\tilde{f}_n^*$ on $F_n := \operatorname{lin}\{x_n, e_{n+1}, e_{n+2}, \dots\}$ by

$$\tilde{f}_n^*(x_n) = 1 - \varepsilon_n, \tag{9.3.4}$$

$$\tilde{f}_n^*(e_k) = (1 - \varepsilon_{k-1}) g_{n,k} \qquad (\text{if } k > n). \tag{9.3.5}$$

By (9.3.3), $\|\tilde{f}_n^*\| \leqslant 1$, and indeed $\|\tilde{f}_n^*\| = 1$ by (9.3.5). Define $f_n^* \in X^*$ to be a Hahn-Banach extension of $\tilde{f}_n^*$. Condition (9.3.4) and the choice of (x_n) ensure that (f_n^*) is double-norming. Let us show that it is an isometric ℓ_1-basis. Indeed, due to our definition of an "independent" sequence, for an arbitrary finite collection $A = \{a_1, \dots, a_n\}$ of non-zero coefficients the set J_A of those $j > n$ such that $g_{s,j} = \operatorname{sign} a_s$, $s = 1, \dots, n$, is infinite. So, by (9.3.5),

$$\left\| \sum_{s=1}^{n} a_s f_s^* \right\| \geqslant \sup_{j \in J_A} \left(\sum_{s=1}^{n} a_s f_s^* \right) e_j = \sup_{j \in J_A} (1 - \varepsilon_{j-1}) \sum_{s=1}^{n} |a_s| = \sum_{s=1}^{n} |a_s|. \qquad \square$$

Since we have constructed an isometric ℓ_1-basis (over the reals) in the last proof, we have obtained the following version of Theorem 9.2.6.

COROLLARY 9.3.6. *A* real *separable Banach space X is an almost Daugavet space if and only if X^* contains a double-norming isometric ℓ_1-sequence.*

We have shown in Corollary 4.4.7 that the only separable real r.i. function space on $[0, 1]$ with the Daugavet property is $L_1[0, 1]$ in its canonical norm. The situation for the almost Daugavet property is different: there are r.i. renormings X of $L_1[0, 1]$ with the almost Daugavet property that are different from $L_1[0, 1]$; in fact, the Banach-Mazur distance $\operatorname{dist}(X, L_1[0, 1])$ can be arbitrarily large.

THEOREM 9.3.7. *For every $\alpha \in (0, 1)$ denote by X_α the linear space $L_1[0, 1]$ equipped with the norm given by*

$$p_\alpha(f) = \frac{1}{\alpha} \sup\left\{ \int_A |f| \, d\mu\colon A \in \Sigma,\ \mu(A) \leqslant \alpha \right\} \qquad (f \in X).$$

Then, the following hold:

(1) *X_α is an almost Daugavet r.i. space;*
(2) *$\operatorname{dist}(X_\alpha, L_1[0, 1]) \to \infty$ as $\alpha \to 0$.*

Proof. By construction, X_α is rearrangement invariant. Denote $v_n = \frac{\alpha}{n}\mathbb{1}_{[0,1/n]}$. Evidently, $p_\alpha(v_n) = 1$ for all $n > \frac{1}{\alpha}$. If we show that $\lim_{n\to\infty} p_\alpha(f+v_n) = p_\alpha(f)+1$ for every $f \in X_\alpha$, then the almost Daugavet property of X_α will be proved, by virtue of Theorem 9.3.1. Indeed, fix $f \in X_\alpha$. By the definition of p_α, there is a sequence $(A_n) \subset \Sigma$ such that $\mu(A_n) \leqslant \alpha$ and $\frac{1}{\alpha}\int_{A_n} |f|\, d\mu \to p_\alpha(f)$. By the absolute continuity of the Lebesgue integral one can modify A_n in order to fulfill additionally the conditions $\mu(A_n) \leqslant \alpha - \frac{1}{n}$, $A_n \cap [0,1/n] = \emptyset$. Then

$$
\begin{aligned}
p_\alpha(f+v_n) &\geqslant \frac{1}{\alpha}\int_{A_n\cup[0,1/n]} |f+v_n|\, d\mu \\
&= \frac{1}{\alpha}\int_{A_n} |f|\, d\mu + \frac{1}{\alpha}\int_{[0,1/n]} |f+v_n|\, d\mu \\
&\geqslant \frac{1}{\alpha}\int_{A_n} |f|\, d\mu + 1 - \frac{1}{\alpha}\int_{[0,1/n]} |f|\, d\mu \\
&\to p_\alpha(f) + 1.
\end{aligned}
$$

So, (1) is proved.

To prove (2) it is enough to remark that X_α contains a subspace isometric to $\ell_\infty^{(m)}$, where m is the entire part of $1/\alpha$. This subspace is spanned by the functions $\mathbb{1}_{[0,\alpha]}, \mathbb{1}_{[\alpha,2\alpha]}, \dots, \mathbb{1}_{[(m-1)\alpha,m\alpha]}$. □

9.4. The almost Daugavet renorming theorem

In this short section we demonstrate the following theorem.

Theorem 9.4.1. *A separable Banach space X can be equivalently renormed to have thickness* $\mathrm{Thick}(X) = 2$ *(or, equivalently, the almost Daugavet property) if and only if X contains an isomorphic copy of ℓ_1.*

Since for separable spaces the condition $\mathrm{Thick}(X) = 2$ is equivalent to the presence of an ℓ_1-type sequence and an ℓ_1-type sequence contains a subsequence equivalent to the canonical basis of ℓ_1, to prove Theorem 9.4.1 it is sufficient to demonstrate the following:

Theorem 9.4.2. *Let X be a Banach space containing a copy of ℓ_1. Then X can be renormed to admit an ℓ_1-type sequence. Moreover if $(e_n) \subset X$ is an arbitrary sequence equivalent to the canonical basis of ℓ_1 in the original norm, then one can construct an equivalent norm on X in such a way that (e_n) is isometrically equivalent to the canonical basis of ℓ_1 and (e_n) forms an ℓ_1-type sequence in X in the new norm.*

Proof. Let Y be a subspace of X isomorphic to ℓ_1, and let (e_n) be its canonical basis. To begin with, we can apply the extension of norm Theorem 2.10.1 and renorm X in such a way that Y is isometric to ℓ_1 and (e_n) is an isometric ℓ_1-basis.

The rest of the proof is based on results of Section 5.2. By Lemma 5.2.6, the norm of Y can be extended to an equivalent norm $|||\cdot|||$ on X possessing the Y-atomic property. Let us demonstrate that $|||\cdot|||$ is what we need.

Indeed, the sequence $e := (e_n)$ can be considered as a function $e\colon \mathbb{N} \to Y$. Then, for every free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, the triple $(\mathbb{N}, \mathfrak{U}, e)$ fits the Definition 5.2.1 of a Y-atom. Definition 5.2.5 of the Y-atomic property says that $(\mathbb{N}, \mathfrak{U}, e)$ is at the same time an $(X, |||\cdot|||)$-atom. So, we have demonstrated that for every free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ and every $w \in X$

$$\lim_{\mathfrak{U}} \|e_n + w\| = 1 + \|w\|. \tag{9.4.1}$$

By the arbitrariness of $\mathfrak{U}$, (9.4.1) says that $\lim_{n\to\infty} \|e_n + w\| = 1 + \|w\|$. □

9.5. Narrow operators with respect to a one-norming subspace

At the beginning of Chapter 7, in Definition 7.1.2, we defined narrow operators with respect to a subspace $E \subset X^*$ of the dual space: $T \in \mathcal{NAR}(X, E)$ if for every two elements $x, y \in S_X$, every $x^* \in S_E$ and every $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|z + y\| > 2 - \varepsilon$ and $\|T(x - z)\| + |x^*(x - z)| < \varepsilon$.

In the spirit of the current chapter, we are going to elaborate further the theory of narrow operators with respect to a one-norming subspace. In the case of $E = X^*$, that is, in the case of ordinary narrow operators, we collected a number of reformulations in Theorem 6.2.14. Although the spirit of Theorem 6.2.14 survives for narrow operators with respect to a one-norming subspace, a significant technical difficulty appears on the way. Namely, in the complicated net of implications which we elaborated in that proof, we used the sharpening of Bourgain's Lemma 2.6.19 stated in Lemma 2.6.21. That sharpening used on the way Mazur's theorem that the weak closure of a convex set is the same as the closure in norm. If one tries to mimic all the theory in the general situation of narrow operators with respect to a one-norming subspace E, one gets stuck by the necessity of the usage of $\sigma(X, E)$-closures instead of weak ones, and the powerful tool of Mazur's theorem is lost. With this we lose the applicability of the beautiful Bourgain Lemma argument in the new setting. So, at some steps we have to search for new arguments.

On the other hand, some of the ε-δ technicalities can be avoided with the help of rigid versions, to which we may pass with the help of ultrapowers, as we sketched above in Lemma 7.1.6. All this explains the differences between the current exposition and the one from Section 6.2. Remark that the definition of strong Daugavet operators does not involve functionals, so the strong Daugavet part of Section 6.2 does not change. Every $T \in \mathcal{NAR}(X, E)$ is a strong Daugavet operator, so we may use the properties of strong Daugavet operators for such a T when we like. In particular, if $T \in \mathcal{NAR}(X, E)$ and acts from X to X, then it satisfies the Daugavet equation, see Lemma 6.2.3.

We start with a simple statement of school geometry nature (see Figure 9.1).

Lemma 9.5.1. *Let $\Delta = ABC$ be a right triangle in Euclidean space with lengths of legs $BC = a$ and $AC = b$ (the right angle is at C) such that $a, b \in [1, M]$. Denote by h the distance from C to AB. Then*

$$h \leqslant a \cdot \frac{M}{\sqrt{1 + M^2}}.$$

Proof. We have

$$h = \frac{ab}{\sqrt{a^2 + b^2}} = \frac{a}{\sqrt{(a/b)^2 + 1}}.$$

The last quantity is increasing in b, so

$$h \leqslant \frac{a}{\sqrt{(a/M)^2 + 1}} \leqslant \frac{a}{\sqrt{(1/M)^2 + 1}} = a\frac{M}{\sqrt{1 + M^2}},$$

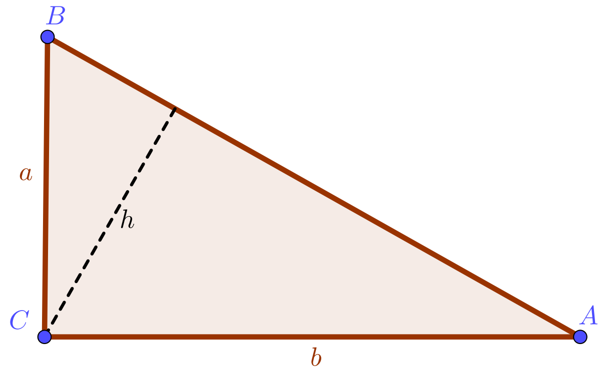

FIGURE 9.1. A school geometry statement

which completes the proof. □

LEMMA 9.5.2. *Let X be Banach space, $E \subset X^*$ be a one-norming subspace, and $T \in \mathcal{OP}(X)$. Then, the following assertions are equivalent:*

(i) *For every $x, y \in S_X$ and every $e^* \in E$, there is an element $v \in S_X$ such that*
$$\mathrm{Re}\langle e^*, v\rangle \geqslant \mathrm{Re}\langle e^*, x\rangle, \tag{9.5.1}$$
$\|v + y\| = 2$, and $Tv = Tx$.

(ii) *For every $x, y \in S_X$ and every $e^* \in E$, there is an element $z \in S_X$ such that*
$$\mathrm{Re}\langle e^*, z\rangle = \mathrm{Re}\langle e^*, x\rangle,$$
$\|z + y\| = 2$, and $Tz = Tx$.

Moreover, if the above properties hold true, then for every $x, y \in S_X$ and every $\sigma(X, E)$-neighbourhood U of x, there is $u \in U \cap S_X$ such that $\|u + y\| = 2$ and $Tu = Tx$.

PROOF. (i) ⇒ (ii). Applying (i) to $x, y \in S_X$ and $e^* \in E$ we obtain the corresponding $v \in S_X$ such that $Tv = Tx$, v is quasi-codirected with y and (9.5.1) holds. Now, let us apply (i) to x, $\frac{y+v}{2} \in S_X$ and $-e^*$. We get $\tilde{v} \in S_X$ such that $T\tilde{v} = Tx$, $\tilde{v}$ is quasi-codirected with $\frac{y+v}{2}$ and
$$\mathrm{Re}\langle e^*, \tilde{v}\rangle \leqslant \mathrm{Re}\langle e^*, x\rangle. \tag{9.5.2}$$
Consider $z(\lambda) = \lambda v + (1 - \lambda)\tilde{v}$, $\lambda \in [0, 1]$. First, $Tz(\lambda) = Tx$ for all $\lambda \in [0, 1]$. Then, according to Lemma 2.6.12(b), the vectors $y, v, \tilde{v}$ form a quasi-codirected triple so, $\|z(\lambda) + y\| = 2$ by Lemma 2.6.12(a). Finally, for the continuous function $g(\lambda) := \mathrm{Re}\langle e^*, z(\lambda)\rangle - \mathrm{Re}\langle e^*, x\rangle$ ($\lambda \in [0, 1]$), the conditions (9.5.1) and (9.5.2) mean that $g(0)g(1) \leqslant 0$. So, there is $\lambda_0 \in [0, 1]$ such that $g(\lambda_0) = 0$ by Bolzano's theorem. Then $z := z(\lambda_0)$ fulfills (ii).

The converse implication (ii) ⇒ (i) is evident, so let us pass to the more advanced "moreover" part of the lemma. Fix $x, y \in S_X$ and a $\sigma(X, E)$-neighbourhood U of x. There are $n \in \mathbb{N}$ and a surjective real-linear continuous operator F: $(X, \sigma(X, E)) \to \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with the standard Euclidean norm, such that
$$x + F^{-1}(B_{\mathbb{R}^n}) = \{v \in X\colon \|F(x) - F(v)\| \leqslant 1\} \subset U. \tag{9.5.3}$$

Denote for further convenience

$$M := \operatorname{diam} F(B_X), \quad \alpha := \frac{M}{\sqrt{1+M^2}}.$$

Our goal is to find $u \in \{v \in S_X : \|F(x)-F(v)\| \leqslant 1\} \subset U \cap S_X$ such that $\|u+y\| = 2$ and $Tu = Tx$.

Assume that such a u does not exist. We are going to arrive at a contradiction at some stage of the recurrent process that we are going to explain below.

Take an arbitrary $u_0 \in S_X$ such that $\|u_0+y\| = 2$ and $Tu_0 = Tx$ (the existence is guaranteed by (ii)). By our assumption, $\|F(x) - F(u_0)\| > 1$. Introduce a functional $e_0^* \in E$ on X by the formula

$$\operatorname{Re}\langle e_0^*, z\rangle := \langle F(z), F(x) - F(u_0)\rangle,$$

where the standard inner product in $\mathbb{R}^n$ appears on the right hand side. This e_0^* is $\sigma(X, E)$-continuous, so $e_0^* \in E$. Applying (ii) to x, $\frac{u_0+y}{2} \in S_X$ and e_0^* we get $\tilde{v} \in S_X$ such that $T\tilde{v} = Tx$, $\tilde{v}$ is quasi-codirected with $\frac{u_0+y}{2}$ and

$$\langle F(\tilde{v}), F(x) - F(u_0)\rangle = \operatorname{Re}\langle e_0^*, \tilde{v}\rangle = \operatorname{Re}\langle e_0^*, x\rangle = \langle F(x), F(x) - F(u_0)\rangle.$$

Consequently, $\langle F(\tilde{v}) - F(x), F(x) - F(u_0)\rangle = 0$. The last condition means that the vertices $F(x), F(u_0), F(\tilde{v})$ form a right triangle. Denote $u_0(\lambda) = \lambda u_0 + (1-\lambda)\tilde{v}$ and find $\lambda_1 \in [0, 1]$ at which $\|F(x) - u_0(\lambda)\|$ is minimal. Consider $u_1 = \lambda_1 u_0 + (1-\lambda_1)\tilde{v}$.

From Lemma 2.6.12 we see that $\tilde{v}, u_0$ and y form a quasi-codirected triple, so $\|u_1 + y\| = 2$ and $u_1 \in S_X$. The conditions $Tu_0 = Tx$ and $T\tilde{v} = Tx$ imply that $Tu_1 = Tx$. Finally, Lemma 9.5.1 applied to the triangle with the vertices $F(x), F(u_0), F(\tilde{v})$ gives

$$\|F(x) - F(u_1)\| \leqslant \alpha \|F(x) - F(u_0)\|.$$

Repeating the same construction, starting with u_1 instead of u_0, we obtain $u_2 \in S_X$ with $\|u_2 + y\| = 2$, $Tu_2 = Tx$ and

$$\|F(x) - F(u_2)\| \leqslant \alpha \|F(x) - F(u_1)\| \leqslant \alpha^2 \|F(x) - F(u_0)\|.$$

Recall that $\alpha < 1$, so it is impossible to keep $\alpha^j \|F(x) - F(u_0)\| > 1$ for all $j = 1, 2, 3, \ldots$. This gives the desired contradiction. ☐

THEOREM 9.5.3. *Let X be a Banach space, $E \subset X^*$ be a one-norming subspace, $\mathfrak{U}$ be a free ultrafilter on $\mathbb{N}$, and $T \in \mathcal{OP}(X)$. Then the following assertions are equivalent:*

(i) *$T \in \mathcal{NAR}(X, E)$.*

(ii) *For every $x, y \in S_X$, $\varepsilon > 0$ and every slice $S = \operatorname{Slice}(B_X, e^*, \alpha) \subset B_X$ generated by an element e^* of S_E and containing x there is an element $v \in S$ such that $\|y + v\| > 2 - \varepsilon$ and $\|Tv - Tx\| < \varepsilon$.*

(iii) *For every $[(x_n)], [(y_n)] \in S_{X^{\mathfrak{U}}}$, and every $[(e_n^*)] \in E^{\mathfrak{U}}$ there is an element $[(v_n)] \in S_{E^{\mathfrak{U}}}$ such that*

$$\operatorname{Re}\langle [(e_n^*)], [(v_n)]\rangle \geqslant \operatorname{Re}\langle [(e_n^*)], [(x_n)]\rangle, \tag{9.5.4}$$

$$\|[(v_n)] + [(y_n)]\| = 2, \tag{9.5.5}$$

and $T^{\mathfrak{U}}[(v_n)] = T^{\mathfrak{U}}[(x_n)]$.

(iv) *For every $[(x_n)], [(y_n)] \in S_{X^{\mathfrak{U}}}$, and every $\sigma(X^{\mathfrak{U}}, E^{\mathfrak{U}})$-neighbourhood $\widetilde{U}$ of $[(x_n)]$ there is $[(u_n)] \in \widetilde{U} \cap S_{X^{\mathfrak{U}}}$ such that $\|[(u_n)] + [(y_n)]\| = 2$ and $T^{\mathfrak{U}}[(u_n)] = T^{\mathfrak{U}}[(x_n)]$.*

(v) *For every* $x, y \in S_X$, $\varepsilon > 0$ *and every* $\sigma(X, E)$*-neighbourhood* U *of* x *there is* $u \in U \cap S_X$ *such that* $\|u + y\| > 2 - \varepsilon$ *and* $\|Tu - Tx\| < \varepsilon$.

(vi) *For every* $x, y \in S_X$, $\varepsilon > 0$ *and every* $\sigma(X, E)$*-neighbourhood* U *of* x *there is* $u \in U \cap (1 + \varepsilon)B_X$ *such that* $\|u + y\| > 2 - \varepsilon$ *and* $\|Tu - Tx\| < \varepsilon$.

PROOF. (i) $\Rightarrow$ (ii). Since $x \in S$ then there is $\varepsilon_1 \in (0, \varepsilon)$ such that

$$\operatorname{Re} e^*(x) > 1 - \alpha + \varepsilon_1. \tag{9.5.6}$$

By (i), there is $v \in S_X$ such that

$$\|v + y\| > 2 - \varepsilon_1 > 2 - \varepsilon \quad \text{and} \quad \|T(x - v)\| + |e^*(x - v)| < \varepsilon_1.$$

The last inequality implies that, first

$$\|T(x - v)\| < \varepsilon_1 < \varepsilon$$

and, second, $|e^*(x - v)| < \varepsilon_1$. This, together with (9.5.6) gives what remains to check:

$$\operatorname{Re} e^*(v) \geqslant \operatorname{Re} e^*(x) - |e^*(x - v)| > 1 - \alpha + \varepsilon_1 - \varepsilon_1 = 1 - \alpha,$$

that is, $v \in S$.

(ii) $\Rightarrow$ (iii) is a typical reformulation in the spirit of Lemma 7.1.6. Let us demonstrate it. Fix $[(x_n)], [(y_n)] \in S_{X^{\mathfrak{U}}}$, and $[(e_n^*)] \in E^{\mathfrak{U}}$. We are looking for $[(v_n)] \in S_{E^{\mathfrak{U}}}$ for which (9.5.4) and (9.5.5) hold true and $T^{\mathfrak{U}}[(v_n)] = T^{\mathfrak{U}}[(x_n)]$. For $[(e_n^*)] = 0$ (9.5.4) is trivial, so it is sufficient to consider $[(e_n^*)] \neq 0$, and thus we may assume $\|[(e_n^*)]\| = 1$. We fix representatives of the corresponding equivalence classes in such a way that $\|x_n\| = \|y_n\| = \|e_n^*\| = 1$ for each $n \in \mathbb{N}$. Applying for each $n \in \mathbb{N}$ our assumption (ii) for x_n, y_n, e_n^*, $\varepsilon_n = \frac{1}{n}$ and $\alpha_n = 1 - (\operatorname{Re} e_n^*(x_n) - \frac{1}{n})$, we obtain, for the corresponding slice

$$S_n = \operatorname{Slice}(B_X, e_n^*, \alpha_n) = \left\{x \in B_X\colon \operatorname{Re} e_n^*(x) > \operatorname{Re} e_n^*(x_n) - \frac{1}{n}\right\},$$

an element $v_n \in S_n$ such that $\|y_n + v_n\| > 2 - \varepsilon_n$ and $\|Tv_n - Tx_n\| < \varepsilon_n$. The corresponding $[(v_n)] \in B_{X^{\mathfrak{U}}}$ will satisfy the conditions (9.5.4), (9.5.5), and that $T^{\mathfrak{U}}[(v_n)] = T^{\mathfrak{U}}[(x_n)]$. It remains to remark that

$$\|[(v_n)]\| \geqslant \|[(v_n)] + [(y_n)]\| - \|[(y_n)]\| = 1,$$

so $[(v_n)] \in S_{X^{\mathfrak{U}}}$.

The implication (iii) $\Rightarrow$ (iv) follows from Lemma 9.5.2.

(iv) $\Rightarrow$ (v) is again a reformulation in the spirit of Lemma 7.1.6. Fix $x, y \in S_X$, $\varepsilon > 0$ and $\sigma(X, E)$-neighbourhood U of x. By the definition of the standard base of open neighbourhoods of 0 in $\sigma(X, E)$, there are $m \in \mathbb{N}$, $(e_k^*)_{k=1}^m \subset E$ and $\alpha > 0$ such that $x + U_{\alpha, (e_k^*)_{k=1}^m} \subset U$, where

$$U_{\alpha, (e_k^*)_{k=1}^m} = \Big\{z \in X\colon \max_{1 \leqslant k \leqslant m} |e_k^*(z)| < \alpha\Big\}.$$

Consider the equivalence classes $[(x)], [(y)] \in S_{X^{\mathfrak{U}}}$ and $[(e_k^*)] \in E^{\mathfrak{U}}$, $k \in \overline{1, m}$, of the corresponding constant sequences $(x, x, \dots)$, $(y, y, \dots)$ and $(e_k^*, e_k^*, \dots)$. Then

$$\widetilde{U} := \Big\{[(v_n)] \in X^{\mathfrak{U}}\colon \max_{1 \leqslant k \leqslant m} \big|[(e_k^*)]([(v_n - x)])\big| < \alpha\Big\}$$

is a $\sigma(X^{\mathfrak{U}}, E^{\mathfrak{U}})$-neighbourhood of $[(x)]$. According to the assumption (iv), there is $[(u_n)] \in \widetilde{U} \cap S_{X^{\mathfrak{U}}}$ such that $\|[(u_n)] + [(y)]\| = 2$ and $T^{\mathfrak{U}}[(u_n)] = T^{\mathfrak{U}}[(x_n)]$. Without loss of generality, we may assume that $\|u_n\| = 1$. The above conditions mean that the set of those $n \in \mathbb{N}$ for which

$$\|u_n + y\| > 2 - \varepsilon,\ \|Tu_n - Tx\| < \varepsilon \text{ and } |e_k^*(u_n - x)| < \alpha,\ k \in \overline{1, m},$$

belongs to the ultrafilter $\mathfrak{U}$, so the set contains at least one element n_0. Then $u := u_{n_0} \in S_X$ is the element we are searching for.

(v) $\Rightarrow$ (vi) is evident.

(vi) $\Rightarrow$ (i). For $x, y \in S_X$, $x^* \in S_E$ and $\varepsilon \in (0, 1)$ we are looking for $z \in S_X$ such that $\|z + y\| > 2 - \varepsilon$ and $\|T(x - z)\| + |x^*(x - z)| < \varepsilon$.

Without loss of generality, we may assume $\|T\| \leqslant 1$. Let us introduce the set $U := \{u \in X\colon |x^*(x - u)| < \varepsilon/2\}$. This U is a $\sigma(X, E)$-neighbourhood of x. Applying (v) with $\varepsilon/4$ we obtain $u \in U \cap (1 + \varepsilon)B_X$ such that $\|u + y\| > 2 - \varepsilon/4$ and $\|Tu - Tx\| < \varepsilon/4$. The element $z := \frac{u}{\|u\|}$ is what we need. □

Remark 9.5.4. *Using the description (9.5.3) of $\sigma(X, E)$-neighbourhoods and (vi) of the last theorem, one can easily see that for every $T \in \mathcal{NAR}(X, E)$ and every $\sigma(X, E)$-continuous operator $F \in \mathcal{OP}(X)$ of finite rank, the sum $T \tilde{+} F$ is narrow with respect to E. In particular, if $0 \in \mathcal{NAR}(X, E)$ (i.e., if $X \in \mathrm{DPr}(E)$), then every $\sigma(X, E)$-continuous operator of finite rank is narrow with respect to E. This is not so good as Theorem 6.2.18, but is sufficient for our needs.*

Remark 9.5.5. Analogously to Corollary 3.1.6 and Remark 6.2.15, the reformulation (ii) of the last theorem implies that *a complex-linear $T \in \mathcal{OP}(X)$ between complex spaces is narrow with respect to a one-norming subspace $E \subset X^*$ if and only if the same operator on $X_{\mathbb{R}}$ is narrow with respect to the one-norming subspace of $X_{\mathbb{R}}^*$ consisting of the real parts of elements of E.*

9.6. Rich subspaces with respect to a one-norming subspace

The material of this section is a generalisation of results from Section 6.5, so its structure to a large extend repeats that of Section 6.5 with some deviations.

Below we use the following abbreviations: if X is a Banach space, $E \subset X^*$ and $Y \subset X$ are subspaces, then the set of all restrictions to Y of the elements of E is denoted $E|_Y$. Remark that $E|_Y \subset Y^*$ is a linear subspace, and if E is a one-norming subspace, then $E|_Y$ is one-norming as well.

We start with the analogue of Theorem 6.5.1.

Theorem 9.6.1. *Let X be a Banach space, $E \subset X^*$ be a one-norming subspace, $X \in \mathrm{DPr}(E)$, $Y \subset X$ be a subspace, $q\colon X \to X/Y$ be the quotient map, $T \in \mathcal{OP}(X)$ be an operator such that $T \tilde{+} q \in \mathcal{NAR}(X, E)$. Denote $\tilde{T} = T|_Y \in \mathcal{OP}(Y)$. Then $\tilde{T} \in \mathcal{NAR}(Y, E|_Y)$.*

Proof. We are going to demonstrate that $\tilde{T}$ satisfies (vi) of Theorem 9.5.3. Fix $\varepsilon > 0$, $x, y \in S_Y$, and a $\sigma(Y, E|_Y)$-neighbourhood U of x. Our goal is to find $u \in U \cap (1 + \varepsilon)B_Y$ such that $\|u + y\| > 2 - \varepsilon$ and $\|Tu - Tx\| < \varepsilon$.

By the definition of $\sigma(Y, E|_Y)$, U contains a subset of the form $(x + U_0) \cap Y$, where U_0 is a convex balanced $\sigma(X, E)$-neighbourhood of 0 in X. Fix $r \in (0, 1)$ such that $rU_0 + rB_X \subset U_0$ and apply (v) of Theorem 9.5.3 to $T \tilde{+} q \in \mathcal{NAR}(X, E)$, $U_1 := x + rU_0$ and $\tilde{\varepsilon} := \min\{\varepsilon/4, r/2, \varepsilon/(1 + \|T\|)\}$. We obtain $u_1 \in U_1 \cap S_X$ such that $\|u_1 + y\| > 2 - \tilde{\varepsilon}$ and $\|T(x - u_1)\| + \|q(x - u_1)\| < \tilde{\varepsilon}$.

In particular, $\|q(u_1)\| = \|q(x - u_1)\| < \tilde{\varepsilon}$. The last condition means that the distance from u_1 to Y is smaller than $\tilde{\varepsilon}$, so there is an element $u \in Y$ with $\|u - u_1\| < \tilde{\varepsilon}$. This u satisfies all the demands from (vi) of Theorem 9.5.3:

$$u = u_1 + (u - u_1) \in U_1 + \tilde{\varepsilon} B_X \subset x + rU_0 + B_X \subset x + U_0.$$

Consequently, $u \in (x + U_0) \cap Y \subset U$ and $\|u\| \leqslant 1 + \tilde{\varepsilon} \leqslant 1 + \varepsilon$, so $u \in U \cap (1+\varepsilon)B_Y$ as we want. Also,

$$\|T(u - x)\| \leqslant \|T(x - u_1)\| + \|T(u - u_1)\| < (1 + \|T\|)\tilde{\varepsilon} \leqslant \varepsilon$$

and

$$\|u + y\| \geqslant \|u + y_1\| - \|u - u_1\| > 2 - 2\tilde{\varepsilon} > 2 - \varepsilon. \qquad \square$$

Recall (Definition 6.5.2) that for $X \in \mathrm{DPr}$, a subspace $Y \subset X$ is said to be almost rich if the corresponding quotient map q: $X \to X/Y$ is strongly Daugavet, and it is said to be rich if $q \in \mathcal{NAR}(X)$. Now, we are going to introduce the corresponding "E-versions". For almost richness nothing changes, apart from the relaxation of the condition $X \in \mathrm{DPr}$ to $X \in \mathrm{DPr}(E)$, so we don't change the name of the property.

Definition 9.6.2. Let X be a Banach space, $E \subset X^*$ be a one-norming subspace, $X \in \mathrm{DPr}(E)$. A subspace $Y \subset X$ is said to be *E-rich* if $q \in \mathcal{NAR}(X, E)$.

Theorem 9.6.3. *Under the above conditions an E-rich subspace Y of $X \in \mathrm{DPr}(E)$ has the Daugavet property with respect to $E|_Y$.*

Proof. The reformulation (ii) of E-narrowness in Theorem 9.5.3 implies that the zero operator is E-narrow if and only if the space possesses the Daugavet property with respect to E. So, in order to demonstrate our theorem it suffices to apply Theorem 9.6.1 with $T = 0$. $\square$

Now, we generalise Definition 6.5.5.

Definition 9.6.4. Let X be a Banach space, $E \subset X^*$ be a one-norming subspace, $X \in \mathrm{DPr}(E)$. A subspace $Y \subset X$ is said to be *E-wealthy* if every subspace Z of X containing Y has the Daugavet property with respect to $E|_Z$ (i.e., $(Y \subset Z \subset X) \Rightarrow (Z \in \mathrm{DPr}(E|_Z))$).

Remark 9.6.5. Recall that if $Y \subset Z \subset X$, then $\|[x]_{X/Z}\| \leqslant \|[x]_{X/Y}\|$ for every $x \in X$, which gives the domination of the quotient map $\tilde{q}$: $X \to X/Z$ by q: $X \to X/Y$. Thus, if Y is E-rich, then every bigger subspace Z is E-rich and, consequently, possesses the Daugavet property with respect to $E|_Z$. So, *every E-rich subspace is E-wealthy.*

Remark 9.6.6. A good example comes from Remark 9.5.4: if $X \in \mathrm{DPr}(E)$, then every $\sigma(X, E)$-continuous operator of finite rank is narrow with respect to E, in particular *every $\sigma(X, E)$-closed subspace of finite codimension is E-rich and E-wealthy.*

The above example of E-wealthy subspaces extends to subspaces of infinite codimension that have small annihilator.

Lemma 9.6.7. *Let $X \in \mathrm{DPr}(E)$, where $E \subset X^*$ is one-norming. Then for every $x \in S_X$, every $\varepsilon > 0$, and every separable subspace $V \subset E$, there is $x^* \in S_{X^*}$ such that* $\mathrm{Re}\, x^*(x) \geqslant 1 - \varepsilon$ *and* $\|x^* + f\| = 1 + \|f\|$ *for all* $f \in V$.

PROOF. Consider a dense sequence $(f_n)_{n=1}^{\infty} \subset V$ such that every element is repeated infinitely many times in the sequence. Applying (iv)$'$ of Lemma 9.1.3 to the slice $\mathrm{Slice}(B_{X^*}, x, \varepsilon)$ of B_{X^*} and to f_1 and then applying it step-by-step to f_n and to the slices obtained in the previous steps, we construct a sequence of closed slices $\overline{\mathrm{Slice}}(B_{X^*}, x, \varepsilon) \supset \overline{\mathrm{Slice}}(B_{X^*}, x_1, \varepsilon_1) \supset \overline{\mathrm{Slice}}(B_{X^*}, x_2, \varepsilon_2) \supset \cdots$ with $\varepsilon_n < 1/n$ such that $\|x^* + f_n\| \geqslant 2 - \varepsilon_{n-1}$ for all $x^* \in \mathrm{Slice}(B_{X^*}, x_n, \varepsilon_n)$. By w^*-compactness of all $\overline{\mathrm{Slice}(B_{X^*}, x_n, \varepsilon_n)}$, there is a point $x^* \in \bigcap_{n=1}^{\infty} \overline{\mathrm{Slice}(B_{X^*}, x_n, \varepsilon_n)} \subset \overline{\mathrm{Slice}(B_{X^*}, x, \varepsilon)}$. This is exactly the point we need. $\square$

THEOREM 9.6.8. *Let $X \in \mathrm{DPr}(E)$, where $E \subset X^*$ is one-norming, and let $Z \subset X$ be a subspace such that $Z^{\perp}$ lies in E and is separable. Then Z is E-wealthy.*

PROOF. If a subspace $Z \subset X$ satisfies the conditions of the theorem, then so do all the subspaces of X containing Z. So it is sufficient to show that the given conditions imply $Z \in \mathrm{DPr}(E|_Z)$. Recall that Z^* identifies with $X^*|_Z$ which in turn identifies with the quotient space $X^*/Z^{\perp}$, and $E|_Z$ identifies with $E/Z^{\perp}$. Let $z \in S_Z$, and let

$$S = \{[z^*] \in Z^* = X^*/Z^{\perp} \colon \|[z^*]\| \leqslant 1,\ \mathrm{Re}[z^*](z) \geqslant 1 - \varepsilon\}$$

be a closed w^*-slice of B_{Z^*}. Fix $[g] \in S_{E/Z^{\perp}}$, $g \in E$. According to (iii)$'$ of Lemma 9.1.3 it is sufficient to prove the existence of $[x^*] \in S$ such that $\|[x^* + g]\| = 2$. Applying Lemma 9.6.7 with $x = z$ and $V = \mathrm{lin}(\{g\} \cup Z^{\perp})$, we obtain $x^* \in S_{X^*}$ such that $x^*(z) \geqslant 1 - \varepsilon$ and

$$\|x^* + f\| = 1 + \|f\| \qquad \text{for all } f \in V.$$

Then $[x^*] \in S$ and

$$\|[x^* + g]\| = \inf_{f \in Z^{\perp}} \|x^* + g + f\| = \inf_{f \in Z^{\perp}} (1 + \|g + f\|) = 1 + \|[g]\| = 2. \qquad \square$$

As an application of Theorem 9.6.8 one gets the following result, due to Shvydkoy [**285**].

COROLLARY 9.6.9. *Let $W \in \mathrm{DPr}$, $\widetilde{W} \subset W$ be a reflexive subspace, then $W/\widetilde{W} \in \mathrm{DPr}$.*

PROOF. (Sketch) Indeed, in the case of a separable space W, we may apply Theorem 9.6.8 to $X = W^*$, $X^* = W^{**}$ and $Z = \widetilde{W}^{\perp}$ (the reflexivity of $\widetilde{W}$ gives that $Z^{\perp} = \widetilde{W}^{\perp\perp} = \widetilde{W} \subset W$, and the separability of $Z^{\perp}$ is evident). So, in this particular case, $Z = \widetilde{W}^{\perp} = (W/\widetilde{W})^*$ has the Daugavet property with respect to $W|_{\widetilde{W}^{\perp}} = W/\widetilde{W}$ and, consequently, $W/\widetilde{W} \in \mathrm{DPr}$.

The general case may be deduced with some effort with the help of a separable reduction argument. $\square$

Our goal is to demonstrate, analogously to Theorem 6.5.12 and along the same lines, that every $\sigma(X, E)$-closed E-wealthy subspace is E-rich.

In the proof we use finite-codimensional subspaces, and the convenience of the additional assumption of $\sigma(X, E)$-closedness comes from the fact that the $\sigma(X, E)$-closure Z of a finite-codimensional subspace $Y \in \mathrm{DPr}(E|_Y)$ may happen to be without the Daugavet property with respect to $E|_Z$, as the following example shows.

EXAMPLE 9.6.10. Let $X = Z$ be the space of all real functions on $[0, 1]$ of the form $f + a\mathbb{1}_{\{0\}}$, $f \in C[0, 1]$, $a \in \mathbb{R}$, equipped with sup-norm. Consider $Y = C[0, 1]$ and let $E := M[0, 1] \subset X^*$ be the space of all Borel (signed or complex) measures on $[0, 1]$ with the standard variation-norm, where the action of $\mu \in E$ on $x \in X$ is $\int_0^1 x \, d\mu$. Then $E = Y^*$, so the condition $Y \in \mathrm{DPr}(E)$ comes from the Daugavet property of $C[0, 1]$. The $\sigma(X, E)$-density of Y in X follows from the weak*-density of Y in its bidual. On the other hand, the slice $\mathrm{Slice}(B_X, \delta_0, \frac{1}{4})$ generated by the delta-measure δ_0 concentrated in 0 does not contain elements that are $\frac{1}{4}$-quasi-codirected to $x := -\mathbb{1}_{\{0\}}$.

In the Lemmas 9.6.11–9.6.15 below, X is a Banach space, $E \subset X^*$ is a one-norming subspace, and $X \in \mathrm{DPr}(E)$. We recall some notation from Definition 6.2.4: for $x \in S_X$, $y \in S_Y$, and $\varepsilon > 0$, we write

$$D(x, y, \varepsilon) := \{z \in X\colon \|z + x + y\| > 2 - \varepsilon \;\&\; \|z + x\| < 1 + \varepsilon\}$$

and $\mathcal{D}(X)$ is the collection of all sets $D(x, y, \varepsilon)$ with $x \in S_X$, $y \in S_Y$, and $\varepsilon > 0$.

We start with an extension of Lemma 6.5.7.

LEMMA 9.6.11. *The following conditions for a $\sigma(X, E)$-closed subspace $Y \subset X$ are equivalent:*

(i) *Y is E-wealthy.*
(ii) *Every $\sigma(X, E)$-closed finite-codimensional subspace $\widetilde{Y}$ of Y is E-wealthy in X.*
(iii) *For every pair $x, y \in S_X$, the space $W := \mathrm{lin}(\{x, y\} \cup Y)$ has the Daugavet property with respect to $E|_W$.*
(iv) *For every $x, y \in S_X$, for every $\varepsilon > 0$ and for every slice $S = \mathrm{Slice}(S_X, e^*, \alpha)$ generated by some $e^* \in E^*$ and such that $y \in S$ there is an element $v \in \mathrm{lin}(\{x, y\} \cup Y) \cap S$ such that $\|x + v\| > 2 - \varepsilon$.*

PROOF. Let us begin with the equivalence (i) $\Leftrightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) is evident: just take $\widetilde{Y} = Y$. So let us check the converse one. Consider an arbitrary subspace $\widetilde{Z}$ of X containing $\widetilde{Y}$ and denote $Z := \mathrm{lin}(\widetilde{Z} \cup Y)$. Since $Z \supset Y$, the condition (ii) implies that $Z \in \mathrm{DPr}(E|_Z)$. Next, $W := Z \cap \widetilde{Y}$ is a $\sigma(Z, E|_Z)$-closed subspace of finite codimension in Z. Applying Remark 9.6.6 we deduce that W is $E|_Z$-wealthy in Z, consequently the intermediate subspace $\widetilde{Z}$, $W \subset \widetilde{Z} \subset Z$, possesses the Daugavet property with respect to $E|_W$.

The implication (i) $\Rightarrow$ (iii) follows immediately from the Definition 9.6.4 of an E-wealthy subspace; (iii) $\Rightarrow$ (iv) follows from the Definition 7.1.1 of the Daugavet property with respect to a subspace, and (iv) $\Rightarrow$ (i) is a combination of Definitions 7.1.1 and 9.6.4. □

We recall more notation from Lemma 6.5.8: $\mathrm{lin}^{\mathbb{R}}\{x, y\} = \{ax + by\colon a, b \in \mathbb{R}\}$; a pair $x, y \in S_X$ is ε-fine if there is a slice S of S_X which contains y and the diameter of $S \cap \mathrm{lin}^{\mathbb{R}}\{x, y\}$ is less than ε. Now, remark that if $E \subset X^*$ is a one-norming subspace, then the dimension of $E|_{\mathrm{lin}\{x,y\}}$ is equal to 2, so every functional on $\mathrm{lin}\{x, y\}$ extends to an element of E. So, the slice S in the definition of an ε-fine pair may be assumed to be generated by an element of E. Also, the finite-codimensional ε-orthogonal subspace in Lemma 2.3.10 can be selected to be a finite intersection of kernels of elements from E. This permits us to extend Lemma 6.5.8 in the following way, keeping the same proof.

LEMMA 9.6.12. *Let Y be an E-wealthy subspace of X and let a pair $x, y \in S_X$ be ε-fine. Then Y intersects $D(y, x, 2\varepsilon)$.*

Lemmas 6.5.9 and 6.5.10 extend with the same proofs to the following versions.

LEMMA 9.6.13. *Let $Y \subset X$ be a $\sigma(X, E)$-closed subspace and let Y be almost rich together with all of its 1-codimensional $\sigma(X, E)$-closed subspaces. Then Y is E-rich.*

LEMMA 9.6.14. *A subspace $Y \subset X$ is almost rich if and only if Y intersects all the elements of $\mathcal{D}(X)$.*

The key Lemma 6.5.11 also has its natural extension. The only difference in the proof is that instead of the lemmas from Section 6.5 we should use their extensions given above, and instead of Lemma 3.1.14 one should refer to Lemma 9.2.4.

LEMMA 9.6.15. *Every E-wealthy subspace $Y \subset X$ is almost rich.*

Finally, we are ready to extend Theorem 6.5.12.

THEOREM 9.6.16. *Let X be a Banach space, $E \subset X^*$ be a one-norming subspace, $X \in \mathrm{DPr}(E)$. The following properties of a $\sigma(X, E)$-closed subspace $Y \subset X$ are equivalent:*
(i) *Y is E-wealthy.*
(ii) *Y is E-rich.*
(iii) *Every $\sigma(X, E)$-closed finite-codimensional subspace of Y is E-rich.*

PROOF. It is clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), see Remark 9.6.5. Now, supposing (i) let us demonstrate (iii). Let $Z \subset Y$ be a $\sigma(X, E)$-closed finite-codimensional subspace of Y. Every $\sigma(X, E)$-closed 1-codimensional subspace $\tilde{Z}$ of Z has finite codimension in Y, so it is E-wealthy by Lemma 9.6.11. This means that $\tilde{Z}$ is almost rich by Lemma 9.6.15. An appeal to Lemma 9.6.13 gives the desired E-richness of Z. $\square$

Like the rest of Section 6.5, Remark 6.5.13 has its E-counterpart. In order to formulate it, denote $\operatorname{Re} E = \{\operatorname{Re} x^* \colon x^* \in E\}$.

REMARK 9.6.17. *For a complex space X and its complex subspace Y, there are two possible meanings for Y to be E-wealthy: real $(\operatorname{Re} E)$-wealthy if the intermediate subspace from the corresponding definition is a real subspace, and complex-wealthy if the intermediate subspace Z is complex. Every real $(\operatorname{Re} E)$-wealthy subspace is complex E-wealthy, so for a complex subspace $Y \subset X$ we have the implications (Y is E-rich) $\Rightarrow$ (Y is real $(\operatorname{Re} E)$-wealthy) $\Rightarrow$ (Y is complex E-wealthy) $\Rightarrow$ (Y is E-rich), which means that on the way we have demonstrated the equivalence (Y is complex-wealthy) $\Leftrightarrow$ (Y is real $(\operatorname{Re} E)$-wealthy).*

9.7. The Daugavet property with respect to a one-norming subspace and duality. Poor subspaces

In this section we study the Daugavet property of quotient spaces using the duality argument that was mentioned at the beginning of the chapter: under the standard isometric inclusion $X \subset X^{**}$ the space X is a one-norming subspace of the dual to X^*. So, we may consider the Daugavet property of X^* with respect to X, X-rich subspaces of X^*, etc. Since all the questions of such kind reduce to the real case, we simplify notation considering in this section only *real* spaces.

9.7.1. Poverty as a dual property to richness.

DEFINITION 9.7.1. Let $X \in \mathrm{DPr}$. A subspace $Z \subset X$ is said to be *poor* if $X/\tilde{Z} \in \mathrm{DPr}$ for every subspace $\tilde{Z} \subset Z$.

Our study of poor subspaces uses duality, so let us start with a very simple observation that we state as a proposition for easy reference.

PROPOSITION 9.7.2. *A Banach space X has the Daugavet property if and only if $X^* \in \mathrm{DPr}(X)$. Hence, a subspace Z of a space X with the Daugavet property is poor if and only if for every subspace $\tilde{Z} \subset Z$ its dual $(X/\tilde{Z})^* = \tilde{Z}^\perp$ has the Daugavet property with respect to $X/\tilde{Z}$.*

Now, we are ready to give the basic characterisations of poverty.

THEOREM 9.7.3. *Let $X \in \mathrm{DPr}$. For a subspace $Z \subset X$ the following conditions are equivalent.*

(i) *Z is poor.*
(ii) *$X/\tilde{Z} \in \mathrm{DPr}$ for every subspace $\tilde{Z} \subset Z$ of codimension $\operatorname{codim}_Z(\tilde{Z}) \leqslant 2$.*
(iii) *$Z^\perp$ is a subspace of X^* that is rich with respect to X.*
(iv) *For every $x^*, e^* \in S_{X^*}$, $\varepsilon > 0$ and for every $x \in S_X$ such that $e^*(x) > 1 - \varepsilon$, there is an element $v^* \in B_{X^*}$ with the following properties:*
$$v^*(x) > 1 - \varepsilon, \ \ \|x^* + v^*\| > 2 - \varepsilon, \quad \text{and} \quad \|(e^* - v^*)|_Z\| < \varepsilon;$$
that is, the quotient map from X^* onto $X^*/Z^\perp$ is narrow with respect to X.

PROOF. (i) $\Rightarrow$ (ii) follows immediately from the definition of poor subspaces.

Let us prove (ii) $\Rightarrow$ (i). According to Proposition 9.7.2, we have to prove that for every subspace $Z_1 \subset Z$, its dual $Z_1^\perp$ has the Daugavet property with respect to X/Z_1. Fix $Z_1 \subset Z$. According to (iii)$'$ of Lemma 9.1.3 it is sufficient to find, for every $x^* \in S_{Z_1^\perp}$, $\varepsilon > 0$ and every $[x] \in S_{X/Z_1}$, an element $y^* \in S_{Z_1^\perp}$ such that $y^*([x]) \geqslant 1 - \varepsilon$ and $\|x^* + y^*\| \geqslant 2 - \varepsilon$. Since $[x] \in S_{X/Z_1}$, there exists $z^* \in S_{Z_1^\perp}$ such that $z^*([x]) = 1$. Denote $\tilde{Z} = Z \cap \ker x^* \cap \ker z^*$. Evidently, $\tilde{Z}$ is a subspace of Z of $\operatorname{codim}_Z \tilde{Z} \leqslant 2$ and $Z_1 \subset \tilde{Z}$. Also remark that
$$1 = \|[x]_{X/Z_1}\| \geqslant \|[x]\|_{X/\tilde{Z}} \geqslant z^*([x]) = 1,$$
which implies $[x]_{X/\tilde{Z}} \in S_{X/\tilde{Z}}$. By our assumption $\tilde{Z}^\perp$ has the Daugavet property with respect to $X/\tilde{Z}$, and hence for $x^* \in S_{\tilde{Z}^\perp}$ and $[x]_{X/\tilde{Z}} \in S_{X/\tilde{Z}}$ there is $y^* \in S_{\tilde{Z}^\perp}$ such that $y^*([x]) = y^*([x]_{X/\tilde{Z}}) \geqslant 1-\varepsilon$ and $\|x^*+y^*\| \geqslant 2-\varepsilon$. Then $y^* \in S_{\tilde{Z}^\perp} \subset S_{Z_1^\perp}$, and it meets all the requirements.

Now, we will prove that (ii) $\Leftrightarrow$ (iii). Theorem 9.6.16 implies that (iii) holds if and only if $Z^\perp$ is a subspace of X^* that is wealthy with respect to X; and this is equivalent to the claim that for every $x^*, y^* \in S_{X^*}$ the space $W = \operatorname{lin}(Z^\perp \cup \{x^*, y^*\})$ has the Daugavet property with respect to $X/W_\perp$ (Lemma 9.6.11, (iii)). But, for a space $\hat{Z} \supset Z^\perp$, the existence of $x^*, y^* \in S_{X^*}$ such that $W = \operatorname{lin}(Z^\perp \cup \{x^*, y^*\})$ is equivalent to the existence of a space $\tilde{Z} \subset Z$ such that $W = \tilde{Z}^\perp$ and $\operatorname{codim}_Z(\tilde{Z}) \leqslant 2$. Thus, we get that (iii) is equivalent to the claim that $\tilde{Z}^\perp \in \mathrm{DPr}(X/\tilde{Z})$ for every subspace $\tilde{Z} \subset Z$ of $\operatorname{codim}_Z \tilde{Z} \leqslant 2$, which is equivalent to (ii) according to Proposition 9.7.2.

The remaining equivalence (iii) $\Leftrightarrow$ (iv) is just a reformulation of the definition of a rich subspace. $\square$

9.7.2. Applications to the geometry of $C(K)$ and $L_1(\mu)$. For a compact Hausdorff space K denote by $M(K)$ the dual space of $C(K)$, i.e., $M(K)$ is the Banach space of all (not necessarily positive) finite regular Borel signed measures on K. (In the sequel, all measures on K will be tacitly assumed to be finite regular Borel measures.) We are going to prove a theorem which gives a characterisation of operators on $M(K)$ that are narrow with respect to $C(K)$. For this theorem we will need the following lemma in which ∂A denotes the (topological) boundary of a set $A \subset K$.

LEMMA 9.7.4. *Let K be compact, $f \in C(K)$, and μ be some positive measure on K. Then for every $\varepsilon > 0$ there exists a step function $\tilde{f} = \sum_{k=1}^{n} \beta_k \mathbb{1}_{A_k}$ on K such that $\mu(\partial A_k) = 0$ for $k = 1, \dots, n$, $A_1 \cup \dots \cup A_n = K$ and $\|f - \tilde{f}\|_\infty < \varepsilon$.*

PROOF. Since the image measure $\nu = \mu \circ f^{-1}$ on $\mathbb{R}$ has at most countably many atoms, it is possible to cover $f(K)$ by finitely many half-open intervals $I_k = (\beta_{k-1}, \beta_k]$ of length $< \varepsilon$ such that $\nu(\{\beta_0, \dots, \beta_n\}) = 0$. Let $A_k = f^{-1}(I_k)$; then $\tilde{f} = \sum_{k=1}^{n} \beta_k \mathbb{1}_{A_k}$ works. $\square$

THEOREM 9.7.5. *Let K be a perfect compact Hausdorff space. An operator T on $M(K)$ is narrow with respect to $C(K)$ if and only if for every open subset $U \subset K$, every two probability measures π_1, π_2 on U, and every $\varepsilon > 0$, there is a probability measure ν on U such that $\|T(\nu - \pi_1)\| < \varepsilon$ and $\|\pi_2 - \nu\| > 2 - \varepsilon$.*

PROOF. We will repeatedly use the reformulations of narrowness from Theorem 9.5.3, items (ii) and (v). We first prove the "only if" part. Since T is $C(K)$-narrow, for every $x, e \in S_{M(K)}$, $\varepsilon > 0$ and every weak* slice S of $B_{M(K)}$ containing e, there exists $v \in S$ (which, if we like, thanks to (v) of Theorem 9.5.3 can be selected to be of norm 1) such that $\|x + v\| > 2 - \varepsilon$ and $\|T(e - v)\| < \varepsilon$. Fix $\varepsilon_1 > 0$ and let $x = -\pi_2$ and $e = \pi_1$. Since U is open and $\pi_1(U) = 1$, we can find $f \in C(K)$ taking values in $[0, 1]$ with $\operatorname{supp} f \subset U$ and $\int f \, d\pi_1 > 1 - \varepsilon_1$. Applying narrowness of T with respect to $C(K)$ to the slice S generated by f with parameter ε_1, we get $\tilde{\nu} \in S_{M(K)}$ such that the following inequalities hold:

$$\int f \, d\tilde{\nu} > 1 - \varepsilon_1, \quad \|T(\tilde{\nu} - \pi_1)\| < \varepsilon_1, \quad \|\pi_2 - \tilde{\nu}\| > 2 - \varepsilon_1.$$

Let $\hat{\nu} = \tilde{\nu}^+|_U$. Using the properties of f we have $\|\tilde{\nu} - \hat{\nu}\| < 2\varepsilon_1$ and thus

$$1 - 3\varepsilon_1 < \|\hat{\nu}\| \leqslant 1 + 2\varepsilon_1, \quad \|T(\hat{\nu} - \pi_1)\| < \varepsilon_1(1 + 2\|T\|), \quad \|\pi_2 - \hat{\nu}\| > 2 - 3\varepsilon_1.$$

Hence for $\nu = \hat{\nu}/\|\hat{\nu}\|$ we have $\|\nu - \hat{\nu}\| = \big|1 - \|\hat{\nu}\|\big| < 3\varepsilon_1$ and consequently

$$\|\pi_2 - \nu\| \geqslant \|\pi_2 - \hat{\nu}\| - \|\nu - \hat{\nu}\| > 2 - 3\varepsilon_1 - 3\varepsilon_1 = 2 - 6\varepsilon_1,$$

and

$$\|T(\nu - \pi_1)\| \leqslant \|T(\hat{\nu} - \pi_1)\| + \|T(\hat{\nu} - \nu)\| < (1 + 5\|T\|)\varepsilon_1.$$

Then taking $\varepsilon_1 = \min\{\frac{\varepsilon}{6}, \frac{\varepsilon}{1+5\|T\|}\}$ completes the proof of the "only if" part.

Now, consider the "if" part. Given $\mu_1, \mu_2 \in S_{M(K)}$, $\varepsilon > 0$ and a weak* slice S of $B_{M(K)}$ containing μ_1, we have to find $\nu \in S$ such that $\|\mu_2 + \nu\| > 2 - \varepsilon$ and $\|T(\mu_1 - \nu)\| < \varepsilon$. Since one can wiggle the slice S a bit, there is no loss of generality in replacing S by a slice generated by a function of the form $f = \sum_{k=1}^{n} \beta_k \mathbb{1}_{A_k}$, where $A_1, \dots, A_n$ are measurable sets with $(|\mu_1| + |\mu_2|)(\bigcup_{k=1}^{n} \partial A_k) = 0$, as we may use Lemma 9.7.4. (Note that, in general, this new slice will not be relatively weak* open.) On the other hand, using the Hahn decomposition theorem, we have

$K = \bigcup_{i=1}^4 B_i$, where B_1 is a set on which μ_1 is positive and μ_2 is negative, B_2 is a set on which μ_2 is positive and μ_1 is negative, and B_3 (respectively, B_4) is a set where both μ_1 and μ_2 are positive (respectively, negative).

Fix $\varepsilon_1 > 0$ and let G_1 be an open set such that $G_1 \supset B_1$ and $|\mu_i|(G_1 \setminus B_1) < \varepsilon_1$ $(i = 1, 2)$. Define $C_k = G_1 \cap A_k$ and let $U_k = \operatorname{int} C_k$, $k = 1, \dots, n$. Clearly $C_k \setminus U_k \subset \partial A_k$, so the U_k are open sets with the following properties: $U_k \subset C_k$ and $(|\mu_1| + |\mu_2|)(C_k \setminus U_k) = 0$.

Consider those U_k for which $\mu_1(U_k \cap B_1) \neq 0$, $\mu_2(U_k \cap B_1) \neq 0$ and define two probability measures on U_k by

$$\mu_{i,k} = \frac{\mu_i|_{U_k \cap B_1}}{\mu_i(U_k \cap B_1)} \quad (i = 1, 2).$$

By assumption there exists a probability measure $\hat{\nu}_k$ on U_k such that

$$\|T(\hat{\nu}_k - \mu_{1,k})\| < \varepsilon_1 \ \text{ and } \ \|\mu_{2,k} - \hat{\nu}_k\| > 2 - \varepsilon_1.$$

Define $\nu_k = \mu_1(U_k \cap B_1) \cdot \hat{\nu}_k$. Then we have

$$\|\nu_k\| = \nu_k(U_k) = \mu_1(U_k \cap B_1), \quad \|\mu_1|_{U_k}\| - \varepsilon_1 \leqslant \|\nu_k\| \leqslant \|\mu_1|_{U_k}\| + \varepsilon_1 \tag{9.7.1}$$

and

$$\begin{aligned} \|\mu_2|_{U_k} + \nu_k\| &= \|\mu_2(U_k \cap B_1) \cdot \mu_{2,k} + \mu_2|_{U_k \setminus B_1} + \mu_1(U_k \cap B_1) \cdot \hat{\nu}_k\| \\ &\geqslant \| |\mu_2(U_k \cap B_1)| \cdot \mu_{2,k} - |\mu_1(U_k \cap B_1)| \cdot \hat{\nu}_k\| - \|\mu_2|_{U_k \setminus B_1}\| \\ &\geqslant |\mu_2|(U_k) + |\mu_1|(U_k) - 4\varepsilon_1 \end{aligned} \tag{9.7.2}$$

and

$$\begin{aligned} \|T(\nu_k - \mu_1|_{U_k})\| &\leqslant \|T(\mu_1(U_k \cap B_1) \cdot (\hat{\nu}_k - \mu_{1,k}))\| + \|T(\mu_1|_{U_k \setminus B_1})\| \\ &\leqslant \varepsilon_1(1 + \|T\|). \end{aligned} \tag{9.7.3}$$

For U_k with $\mu_1(U_k \cap B_1) = 0$ or $\mu_2(U_k \cap B_1) = 0$, the inequalities (9.7.1)–(9.7.3) hold with $\nu_k = \mu_1|_{U_k \cap B_1}$.

Now, define the measure μ_1^1 by

$$\mu_1^1|_{U_k} = \nu_k, \quad \mu_1^1|_{K \setminus \bigcup_{k=1}^n U_k} = \mu_1|_{K \setminus \bigcup_{k=1}^n U_k}.$$

From (9.7.1), (9.7.2), and (9.7.3), we obtain the following properties of μ_1^1:

$$\|\mu_1\| - n\varepsilon_1 \leqslant \|\mu_1^1\| \leqslant \|\mu_1\| + n\varepsilon_1, \quad \left| \int f \, d\mu_1^1 - \int f \, d\mu_1 \right| \leqslant n\varepsilon_1 \tag{9.7.4}$$

$$\begin{aligned} \|\mu_2|_{G_1} + \mu_1^1|_{G_1}\| &\geqslant \left\| \sum_{k=1}^n (\mu_2|_{U_k} + \nu_k) \right\| - (|\mu_2| + |\mu_1|) \left(\bigcup_{k=1}^n C_k \setminus U_k \right) \\ &\geqslant \sum_{k=1}^n (|\mu_2|(U_k) + |\mu_1|(U_k)) - 4n\varepsilon_1 \\ &\geqslant |\mu_1|(G_1) + |\mu_2|(G_1) - (4n + 2)\varepsilon_1, \end{aligned} \tag{9.7.5}$$

and

$$\|T(\mu_1^1 - \mu_1)\| = \left\| \sum_{k=1}^n T(\nu_k - \mu_1|_{U_k}) \right\| \leqslant (n + n\|T\|)\varepsilon_1. \tag{9.7.6}$$

Now, define $\tilde{B}_2 = B_2 \setminus G_1$. Notice that $\tilde{B}_2$ is a set of negativity for μ_1^1 and a set of positivity for μ_2. Following the same lines as above we define $G_2 \supset \tilde{B}_2$ and construct $\mu_1^2 \in M(K)$ such that

$$|\mu_2|(G_2 \setminus \tilde{B}_2) < \varepsilon_1, \quad |\mu_1^1|(G_2 \setminus \tilde{B}_2) < \varepsilon_1, \quad \|\mu_1^1\| - n\varepsilon_1 \leqslant \|\mu_1^2\| \leqslant \|\mu_1^1\| + n\varepsilon_1$$

and

$$\left|\int f\,d\mu_1^2 - \int f\,d\mu_1^1\right| \leqslant n\varepsilon_1,$$
$$\|T(\mu_1^2 - \mu_1^1)\| \leqslant (n + n\|T\|)\varepsilon_1,$$
$$\|\mu_2|_{G_2} + \mu_1^2|_{G_2}\| \geqslant |\mu_1|(G_2) + |\mu_2|(G_2) - (4n+2)\varepsilon_1.$$

From (9.7.4), (9.7.5), (9.7.6) and the above inequalities, we obtain the estimates

$$1 - 2n\varepsilon_1 \leqslant \|\mu_1^2\| \leqslant 1 - 2n\varepsilon_1, \quad \left|\int f\,d\mu_1^2 - \int f\,d\mu_1\right| \leqslant 2n\varepsilon_1$$

and $\|T(\mu_1^2 - \mu_1)\| \leqslant (2n + 2n\|T\|)\varepsilon_1$,

$$\|\mu_2|_{G_1\cup G_2} + \mu_1^2|_{G_1\cup G_2}\| \geqslant |\mu_1|(G_1 \cup G_2) + |\mu_2|(G_1 \cup G_2) - (8n+10)\varepsilon_1.$$

Finally, the definition of the sets B_3 and B_4 implies that

$$\|\mu_2 + \mu_1^2\| \geqslant \|\mu_1\| + \|\mu_2\| - (8n+10)\varepsilon_1 = 2 - (8n+10)\varepsilon_1.$$

Hence for ε_1 small enough, the normalised signed measure $\nu = \mu_1^2/\|\mu_1^2\|$ satisfies all the required conditions, which completes the proof of the theorem. $\square$

Applying this theorem to the operator $\mu \mapsto \mu|_Z$ yields by Theorem 9.7.3:

Corollary 9.7.6. *Let K be a perfect compact. A subspace $Z \subset C(K)$ is poor if and only if for every open subset $U \subset K$, for every two probability measures π_1, π_2 on U and for every $\varepsilon > 0$ there is a probability measure ν on U such that $\|\nu - \pi_1\|_{Z^*} < \varepsilon$ and $\|\pi_2 - \nu\| > 2 - \varepsilon$.*

For a closed subset K_1 of K denote by R_{K_1} the natural restriction operator $R_{K_1}\colon C(K) \to C(K_1)$. Note that for an operator $S\colon E \to F$ between Banach spaces the following assertions are equivalent, by the (proof of) the open mapping theorem: (i) S is onto; (ii) $S(B_E)$ is not nowhere dense; (iii) 0 is an interior point of $S(B_E)$.

Corollary 9.7.7. *Let K be a perfect compact Hausdorff space, $K_1 \subset K$ be a closed subset with non-empty interior, and let Z be a poor subspace of $C(K)$. Then $R_{K_1}(B_Z)$ is nowhere dense in $B_{C(K_1)}$.*

Proof. Apply Corollary 9.7.6 with $U = \operatorname{int} K_1$, $\pi_1 = \pi_2$ and a sufficiently small $\varepsilon > 0$ to see that $R_{K_1}(B_Z)$ cannot contain a ball $rB_{C(K_1)}$ of radius $r > 0$. $\square$

We now deal with poor subspaces of L_1. Let (Ω, Σ, μ) be a finite measure space. Denote by Σ^+ the collection of all $A \in \Sigma$ with $\mu(A) > 0$.

Theorem 9.7.8. *Let (Ω, Σ, μ) be a non-atomic finite measure space. An operator T on $L_\infty := L_\infty(\Omega, \Sigma, \mu)$ is narrow with respect to $L_1 := L_1(\Omega, \Sigma, \mu)$ if and only if for every $\Delta \in \Sigma^+$ and for every $\varepsilon > 0$ there is $g \in S_{L_\infty}$ such that $g = 0$ off Δ and $\|Tg\| < \varepsilon$. Moreover, in the statement above g can be selected non-negative.*

PROOF. First we prove the "if" part. By the narrowness of T with respect to L_1, for every $x, y \in S_{L_\infty}$, every $f \in S_{L_1}$ such that $\int f \cdot y \, d\mu > 1 - \delta$ (i.e., $y \in \operatorname{Slice}(B_{L_\infty}, f, \delta)$) and every $\varepsilon > 0$ we have to find $z \in \operatorname{Slice}(B_{L_\infty}, f, \delta)$ such that $\|x + z\| > 2 - \varepsilon$ and $\|T(y - z)\| < \varepsilon$. By density of step functions we may assume without loss of generality that there is a partition $A_1, \ldots, A_n$ of Ω such that the restrictions of x, y and f to A_k are constants, say a_k, b_k and c_k respectively. Fix some $\varepsilon_1 \in (0, \varepsilon)$ so small that $\int f \cdot y \, d\lambda > 1 - \delta + 2\varepsilon_1$. Since $\|x\| = 1$, there exists k such that $|a_k| > 1 - \varepsilon_1$. Let $B \in \Sigma^+$ be a subset of A_k with $\mu(B) \leqslant \varepsilon_1$ and $A_k \setminus B \in \Sigma^+$. By our assumption there exists $\hat{z} \in S_{L_\infty}$ such that $z \geqslant 0$, z is supported on B and $\|T(z)\| \leqslant \varepsilon_1/2$. Denote $\tilde{z} = y + (\operatorname{sign}(a_1) - b_1)\hat{z}$. It is easy to see that $\|\tilde{z}\| = 1$, $\|x + \tilde{z}\| > 2 - \varepsilon_1$, $\|T(y - \tilde{z})\| < \varepsilon_1$ and $\tilde{z} \in \operatorname{Slice}(B_{L_\infty}, f, \delta)$.

Now, we consider the "only if" part. Since T is narrow with respect to L_1, T is also a strong Daugavet operator. The commutative C^*-algebra L_∞ is a $C(K)$-space on its Gelfand compact K, so it remains to apply the characterisation of strong Daugavet operators on $C(K)$ given in Proposition 8.4.1 and particularise it to K being the Gelfand compact of L_∞. □

Again, specialising to the restriction operator $g \in L_\infty = (L_1)^* \mapsto g|_Z \in Z^*$ we obtain the following characterisation of poor subspaces.

COROLLARY 9.7.9. *Let (Ω, Σ, μ) be a non-atomic finite measure space. A subspace $Z \subset L_1(\Omega, \Sigma, \mu)$ is poor if and only if for every $\Delta \in \Sigma^+$ and for every $\varepsilon > 0$ there is $g \in S_{L_\infty}$ such that $g = 0$ off Δ and $\|g\|_{Z^*} < \varepsilon$. Moreover, in the statement above g can be selected non-negative.*

For a subset $A \in \Sigma^+$ denote by Q_A the natural restriction operator Q_A: $L_1(\Omega, \Sigma, \mu) \to L_1(A, \Sigma|_A, \mu|_A)$.

COROLLARY 9.7.10. *Let (Ω, Σ, μ) be a non-atomic finite measure space, $A \in \Sigma^+$ and let Z be a poor subspace of $L_1(\Omega, \Sigma, \mu)$. Then $Q_A(B_Z)$ is nowhere dense in $B_{L_1(A, \Sigma|_A, \mu)}$.*

PROOF. Apply Corollary 9.7.9 with $\Delta = A$ and a sufficiently small $\varepsilon > 0$ to see that $Q_A(B_Z)$ cannot contain a ball $rB_{L_1(A)}$ of radius $r > 0$. □

The Corollaries 9.7.7 and 9.7.10 look very similar. The next definition extracts the significant common feature.

DEFINITION 9.7.11. Let $X \in \mathrm{DPr}$. A subspace $E \subset X$ is said to be a *bank* if E contains an isomorphic copy of ℓ_1 and for every poor subspace Z of X, $q_E(B_Z)$ is nowhere dense in $B_{X/E}$ (here q_E denotes the natural quotient map q_E: $X \to X/E$). If $E \subset X$ is a bank, then $B_{X/E}$ will be called the *asset* of E.

In this terminology a poor subspace cannot cover a "significant part" of a bank's asset.

THEOREM 9.7.12. *Let $X \in \mathrm{DPr}$ and $E \subset X$ be a bank with separable asset. Then X contains a copy of ℓ_1 which is not poor in X.*

PROOF. Let $(e_n)_{n \in \mathbb{N}} \subset \frac{1}{2} B_E$ be equivalent to the canonical basis of ℓ_1 and let $(x_n)_{n \in \mathbb{N}} \subset B_E$ be a sequence such that $\{q_E(x_n) \colon n \in \mathbb{N}\}$ is dense in $B_{X/E}$. Then, if one selects a sufficiently small $\varepsilon > 0$, the sequence of $u_n = e_n + \varepsilon x_n \in B_E$ is still equivalent to the canonical basis of ℓ_1, and the image of this sequence under q_E equals $\{\varepsilon q_E(x_n) \colon n \in \mathbb{N}\}$, which is dense in $\varepsilon B_{X/E}$. This means that the closed linear span of $\{u_n \colon n \in \mathbb{N}\}$ is the copy of ℓ_1 we need. □

The next theorem is an immediate corollary of Theorem 9.7.12.

THEOREM 9.7.13. *In every $C(K)$-space with perfect metric compact K and in every separable $L_1(\Omega, \Sigma, \mu)$-space with non-atomic μ there is a subspace isomorphic to ℓ_1 that is not poor.*

PROOF. Corollary 9.7.7 implies that if K is a perfect compact and $K_1 \subset K$ is a proper closed subset with non-empty interior, then

$$C_0(K \setminus K_1) := \{f \in C(K)\colon f(t) = 0\, \forall t \in K_1\}$$

is a bank with $B_{C(K_1)}$ being its asset. Corollary 9.7.10 implies that if (Ω, Σ, μ) is a non-atomic finite measure space and $A \in \Sigma^+$, then $L_1(\Omega \setminus A)$ is a bank with $B_{L_1(A)}$ being its asset. Separability of these assets follows from the separability of the spaces $C(K)$ and $L_1(\Omega, \Sigma, \mu)$ considered. It is left to apply Theorem 9.7.12. □

Theorem 9.7.13 answers Pełczyński's question mentioned in the introduction of the chapter in the negative, since it provides a non-poor ℓ_1-subspace $Z \subset L_1[0, 1]$. By definition this means that for some subspace $\tilde{Z} \subset Z$, $L_1[0, 1]/\tilde{Z}$ fails the Daugavet property; but Z has the RNP and so does its subspace $\tilde{Z}$.

Let us sum up these considerations in a corollary.

COROLLARY 9.7.14. *There is a subspace $E \subset L_1[0, 1]$ that is isomorphic to ℓ_1 and hence has the RNP, but $L_1[0, 1]/E$ fails the Daugavet property.*

9.8. Notes and remarks

Section 9.1. The idea of a Daugavet space with respect to a one-norming subspace was first elaborated in [**168**].

Section 9.2 and **Section 9.3.** These sections are based on the follow-up paper [**169**]. We note that for spaces with the Daugavet property Proposition 9.3.2 was proved first in Barreno's PhD thesis, see [**257**, Prop. 4.1.6]. We also would like to mention that [**31**] contains a direct proof that a space with thickness $\mathrm{Thick}(X) = 2$ contains a copy of ℓ_1. Lücking studied the Daugavet property and the almost Daugavet property for translation invariant subspaces of $C(\mathbb{T})$ and $L_1(\mathbb{T})$ ([**213**], [**214**], [**215**]).

Section 9.4. The results of this section, also from [**169**], appeared before the results of Section 5.2 that we use in their proof. Even more, the results of Section 5.2 were inspired by the original proof of Theorem 9.4.2.

Let us also point out that, in connection with Theorem 9.4.2, there is another result of G. Godefroy: A Banach space X admits an equivalent renorming with a non-zero L-orthogonal element if, and only if, X contains an isomorphic copy of ℓ_1 [**119**, Theorem II.4].

Even though both renorming techniques are different, they are shown to have the same consequences. Indeed, given a Banach space X containing ℓ_1, then the renorming from [**119**, Theorem II.4] produces ℓ_1-type sequences. On the other hand, under the continuum hypothesis, every ℓ_1-type sequence has a non-zero L-orthogonal element in its w^*-closure. See the end of Section 5 in [**29**] for details.

Section 9.5, **Section 9.6** and **Section 9.7.** These sections again follow [**168**].

As far as the authors are aware, the concept of Daugavet centre with respect to a one-norming subspace has not been considered in the literature, which would be a natural continuation of the research of this chapter.

9.9. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

(9.1) Is it true that every separable space with the Daugavet property has an ℓ_1-subspace which is not poor?

(9.2) Can the separability condition in Theorem 9.7.12 be omitted?

(9.3) Is it true that every subspace without copies of ℓ_1 of a space with the Daugavet property is poor?

We don't even know the answer in the case of $C[0,1]$.

(9.4) Develop the concept of Daugavet centre with respect to a one-norming subspace.

(9.5) Is there a bidual space with the almost Daugavet property?

For the same problem for the Daugavet property, see Question (3.1).

(9.6) Is there a complex analogue of Corollary 9.3.6?

CHAPTER 10

Slicely countably determined sets and their applications

In this chapter we present an exposition of the relatively new geometric concepts of SCD sets, SCD spaces and SCD operators that have proved to be quite useful in studying Banach spaces with the Daugavet property and similar objects like the alternative Daugavet property, lushness, and non-linear generalisations of the Daugavet equation; SCD stands for “slicely countably determined”. In particular, the class of HSCD-dominated operators that appears on the way is the best known fix of the main drawback of the class of narrow operators – the instability with respect to the $\tilde{+}$ operation. Namely, HSCD-dominated operators form a $\tilde{+}$-stable two-sided operator ideal, which, in separable spaces with the Daugavet property, lies inside the class of narrow operators (Theorem 10.4.17) and contains the main previously extracted classes of narrow operators, like strong RN-operators or operator which do not fix copies of ℓ_1. On the other hand, the only known class of separable Banach spaces which doesn't possess the SCD property is the class of spaces that admit an equivalent renorming with the Daugavet property. Remark that spaces that are not SCD are the best known candidates for the isomorphic characterisation of the Daugavet property: it is a major open question whether every non-SCD space can be equivalently renormed to possess the Daugavet property in such an equivalent norm (see Questions (10.4) and (10.9) in Section 10.9).

10.1. SCD sets

DEFINITION 10.1.1. Let X be a Banach space and let A be a bounded subset of X. A countable family $\{V_n \colon n \in \mathbb{N}\}$ of non-empty subsets of A is called *determining* for A if $A \subset \overline{\operatorname{conv}}(B)$ for every $B \subset A$ intersecting all the sets V_n. Equivalently, $\{V_n \colon n \in \mathbb{N}\}$ is determining for A if for every sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \in V_n$ $(n \in \mathbb{N})$, one has $A \subset \overline{\operatorname{conv}}\big(\{v_n \colon n \in \mathbb{N}\}\big)$. We will also use the name *determining sequence.*

The above definition can be reformulated in the Hahn-Banach style.

PROPOSITION 10.1.2. *Let $U \subset X$ be a bounded set. A countable family $\{V_n \colon n \in \mathbb{N}\}$ of non-empty subsets of U is determining for U if and only if it has the following property $(*)$: every slice of U contains one of the V_n.*

PROOF. Let $(*)$ be fulfilled and let $B \subset U$ intersect all the V_n. Then B intersects all the slices of U, and then $\overline{\operatorname{conv}}(B) \supset U$ by Lemma 2.6.7. Now, the “only if” part. Assume that some slice $S = \operatorname{Slice}(U, x^*, \varepsilon)$ of U does not contain any of the V_n. Then, $U \setminus S$ intersects all the V_n. But

$$U \setminus S \subset \{x \in X \colon \operatorname{Re} x^*(x) \leqslant \sup_{a\in U} \operatorname{Re} x^*(a) - \varepsilon\},$$

hence $\overline{\operatorname{conv}}(U \setminus S) \subset \{x \in X\colon \operatorname{Re} x^*(x) \leqslant \sup_{a\in U} \operatorname{Re} x^*(a) - \varepsilon\}$, which means that $\overline{\operatorname{conv}}(U \setminus S) \not\supset U$, and consequently $\{V_n\colon n \in \mathbb{N}\}$ is not determining. □

We can now give the main definition of this section.

DEFINITION 10.1.3. A bounded subset A of a Banach space X is said to be *slicely countably determined* (*SCD set* for short) if there is a determining sequence of slices of A.

REMARK 10.1.4. *It is clear from the definition that every SCD set is separable.*

For a better understanding of the definition, we advise the reader to compare it with Lemma 2.6.7.

REMARK 10.1.5. *Let $A, U \subset X$ be bounded subsets, $A \subset U$ and S_n, $n \in \mathbb{N}$, be a determining sequence of slices of U. Then, if $S_n \cap A \neq \emptyset$ for all n, the $S_n \cap A$, $n \in \mathbb{N}$, form a determining sequence of slices of A.*

PROOF. An equivalent definition says that a slice of a set is a non-empty intersection of the set with an open half-space. This explains why $S_n \cap A$ are slices of A. Now, if B intersects all the $S_n \cap A$, we have that this B intersects all S_n, so $\overline{\operatorname{conv}}(B) \supset U \supset A$. □

We can restrict ourselves to study bounded, closed and convex sets because of the following results.

LEMMA 10.1.6. *A bounded subset U of a Banach space X is SCD if and only if the closure $\overline{U}$ of U is an SCD set.*

PROOF. Remark the following relationship between the slices of U and of $\overline{U}$: for every $x^* \in X^*$ and $\delta > 0$, denoting $a := \sup \operatorname{Re} x^*(U) = \sup \operatorname{Re} x^*(\overline{U})$ we have

$$\operatorname{Slice}(\overline{U}, x^*, \delta) \cap U = \{x \in U\colon \operatorname{Re} x^*(x) > a - \delta\} = \operatorname{Slice}(U, x^*, \delta),$$

and

$$\begin{aligned}\overline{\operatorname{Slice}(U, x^*, \delta)} &\subset \{x \in \overline{U}\colon \operatorname{Re} x^*(x) \geqslant a - \delta\} \\ &\subset \{x \in \overline{U}\colon \operatorname{Re} x^*(x) > a - 2\delta\} = \operatorname{Slice}\big(\overline{U}, x^*, 2\delta\big).\end{aligned}$$

Also, for every $x \in \overline{U}$ such that $\operatorname{Re} x^*(x) > a - \delta$ there is a sequence $(x_k) \subset U$ with $\operatorname{Re} x^*(x_k) > a - \delta$ tending to x. This demonstrates one more inclusion:

$$\overline{\operatorname{Slice}(U, x^*, \delta)} \supset \operatorname{Slice}(\overline{U}, x^*, \delta).$$

With all this in hand, let us show that $\overline{U}$ is SCD when U is. Consider a determining sequence of slices $S_n = \operatorname{Slice}(U, x_n^*, \varepsilon_n)$ $(n \in \mathbb{N})$ for U, and let us prove that the slices $S_n' = \operatorname{Slice}(\overline{U}, x_n^*, \varepsilon_n)$ $(n \in \mathbb{N})$ form a determining sequence for the closure of U. Consider an arbitrary slice $\operatorname{Slice}(\overline{U}, x^*, \varepsilon)$ of $\overline{U}$. Then, $\operatorname{Slice}(\overline{U}, x^*, \varepsilon/2) \cap U = \operatorname{Slice}(U, x^*, \varepsilon/2)$ is a slice of U, so there is $n \in \mathbb{N}$ such that $\operatorname{Slice}(U, x^*, \varepsilon/2) \supset S_n$ by Proposition 10.1.2. Therefore,

$$\operatorname{Slice}(\overline{U}, x^*, \varepsilon) \supset \overline{\operatorname{Slice}(U, x^*, \varepsilon/2)} \supset \overline{S_n} \supset S_n',$$

and again Proposition 10.1.2 gives us that $\{S_n'\colon n \in \mathbb{N}\}$ is determining for $\overline{U}$.

For the converse implication, we consider a determining sequence of slices for $\overline{U}$, and Remark 10.1.5 shows that the intersections with U form a determining sequence of slices for U. □

Lemma 10.1.7. *Let X be a Banach space. A bounded set $A \subset X$ is SCD if and only if its closed convex hull is SCD.*

Proof. If $(S_n)_{n\in\mathbb{N}}$ is a determining sequence of slices for $\overline{\operatorname{conv}}(A)$ then Proposition 2.6.6 implies that $S_n \cap A$ are non-empty, so by Remark 10.1.5 these intersections can be taken as a determining sequence of slices for A. So, if $\overline{\operatorname{conv}}(A)$ is SCD, then A is SCD.

For the converse implication, assume that A is SCD. In view of the previous lemma with $U = \operatorname{conv}(A)$, in remains to show that $\operatorname{conv}(A)$ is SCD. Let $\{\operatorname{Slice}(A, x_n^*, \varepsilon_n)\colon n \in \mathbb{N}\}$ be a family of slices determining for A. We consider the following countable family of slices of $\operatorname{conv}(A)$:

$$\mathcal{S} := \big\{\operatorname{Slice}\big(\operatorname{conv}(A), x_n^*, \varepsilon_n/k\big)\colon n, k \in \mathbb{N}\big\}.$$

Given any slice $\operatorname{Slice}(\operatorname{conv}(A), x^*, \varepsilon)$ of $\operatorname{conv}(A)$, where $\|x^*\| = 1$ without loss of generality, we will show that it contains an element of $\mathcal{S}$, thus proving that $\operatorname{conv}(A)$ is SCD by Proposition 10.1.2. Now, for the slice of A given by $\operatorname{Slice}(A, x^*, \varepsilon/2)$ we know that there is $n_0 \in \mathbb{N}$ such that $\operatorname{Slice}(A, x_{n_0}, \varepsilon_{n_0}) \subset \operatorname{Slice}(A, x^*, \varepsilon/2)$. Taking $k \in \mathbb{N}$ big enough, we will argue that

$$\operatorname{Slice}(\operatorname{conv}(A), x_{n_0}^*, \varepsilon_{n_0}/k) \subset \operatorname{conv}\big(\operatorname{Slice}(A, x_{n_0}^*, \varepsilon_{n_0})\big) + \frac{\varepsilon}{2} B_X.$$

To prove this inclusion we let $r := \sup_{a\in A} x_{n_0}^*(a)$ and $M := \sup_{a\in A} \|a\|$, hence also $\sup_{a\in\operatorname{conv}(A)} x_{n_0}^*(a) = r$. Consider a convex combination $a = \sum_{i=1}^n \lambda_i a_i$ of elements $a_i \in A$ such that $x_{n_0}^*(a) > r - \varepsilon_{n_0}/k$ where k is not yet specified. Let us denote $I = \{i\colon x_{n_0}^*(a_i) > r - \varepsilon_{n_0}\}$ and $J = \{i\colon x_{n_0}^*(a_i) \leqslant r - \varepsilon_{n_0}\}$. We then have

$$\begin{aligned} r - \frac{\varepsilon_{n_0}}{k} &< \sum_{i\in I} \lambda_i x_{n_0}^*(a_i) + \sum_{i\in J} \lambda_i x_{n_0}^*(a_i) \\ &\leqslant r \sum_{i\in I} \lambda_i + \sum_{i\in J} \lambda_i (r - \varepsilon_{n_0}) \leqslant r - \varepsilon_{n_0} \sum_{i\in J} \lambda_i, \end{aligned}$$

which implies

$$\sum_{i\in J} \lambda_i < \frac{1}{k} \qquad \text{and} \qquad \Lambda := \sum_{i\in I} \lambda_i > 1 - \frac{1}{k}.$$

Now, put $\mu_i := \lambda_i/\Lambda$ for $i \in I$ and consider the element

$$a' = \sum_{i\in I} \mu_i a_i \in \operatorname{conv}\big(\operatorname{Slice}(A, x_{n_0}^*, \varepsilon_{n_0})\big).$$

The estimate

$$\|a - a'\| = \Big\|(\Lambda - 1)\sum_{i\in I} \mu_i a_i + \sum_{i\in J} \lambda_i a_i\Big\| \leqslant |\Lambda - 1| M + \sum_{i\in J} \lambda_i M < \frac{2M}{k}$$

shows that the above inclusion holds true whenever $k \geqslant 4M/\varepsilon$.

It now follows for this choice of k that

$$\begin{aligned} \operatorname{Slice}(\operatorname{conv}(A), x_{n_0}^*, \varepsilon_{n_0}/k) &\subset \operatorname{conv}\big(\operatorname{Slice}(A, x_{n_0}^*, \varepsilon_{n_0})\big) + \frac{\varepsilon}{2} B_X \\ &\subset \operatorname{conv}\big(\operatorname{Slice}(A, x^*, \varepsilon/2)\big) + \frac{\varepsilon}{2} B_X \\ &\subset \operatorname{Slice}(\operatorname{conv}(A), x^*, \varepsilon/2) + \frac{\varepsilon}{2} B_X. \end{aligned}$$

Since trivially $\mathrm{Slice}(\mathrm{conv}(A), x^*_{n_0}, \varepsilon_{n_0}/k) \subset \mathrm{conv}(A)$, we finally get

$$\begin{aligned}\mathrm{Slice}(\mathrm{conv}(A), x^*_{n_0}, \varepsilon_{n_0}/k) &\subset \Big(\mathrm{Slice}(\mathrm{conv}(A), x^*, \varepsilon/2) + \frac{\varepsilon}{2} B_X\Big) \cap \mathrm{conv}(A)\\ &\subset \mathrm{Slice}(\mathrm{conv}(A), x^*, \varepsilon).\end{aligned}$$ □

We give two straightforward observations which will be useful later on.

REMARK 10.1.8. *Let X be a Banach space and let A be a convex bounded subset of X. Suppose that there is a sequence $(a_n)_{n\in\mathbb{N}}$ of points in A such that $A \subset \overline{\mathrm{conv}}\big(\{a_n : n \in \mathbb{N}\}\big)$ and that for every $n \in \mathbb{N}$, there is a countable family $\{V_{n,m} : m \in \mathbb{N}\}$ of subsets of A such that $a_n \in \overline{\mathrm{conv}}(B)$ whenever $B \subset A$ intersects $V_{n,m}$ for every $m \in \mathbb{N}$. Then, the family $\{V_{n,m} : n, m \in \mathbb{N}\}$ is determining for A.*

As an immediate consequence of the above result, we get the following.

REMARK 10.1.9. *Let X be a Banach space and let A be a separable convex bounded subset of X. Suppose that for every $a \in A$, there is a countable family $\{V^a_m : m \in \mathbb{N}\}$ of subsets of A such that $a \in \overline{\mathrm{conv}}(B)$ whenever $B \subset A$ intersects V^a_m for every $m \in \mathbb{N}$. Then, taking a countable dense subset $\{a_n : n \in \mathbb{N}\}$ in A, the family $\{V^{a_n}_m : n, m \in \mathbb{N}\}$ is determining for A.*

Our first goal is to present basic examples related to Definition 10.1.3: Radon-Nikodým and Asplund sets are SCD, whereas the unit ball of a Banach space with the Daugavet property is not.

We start with subsets having sufficiently many denting points (see Definition 2.7.11 for the corresponding notation).

PROPOSITION 10.1.10. *Let X be a Banach space and let A be a closed convex bounded subset of X. If A is separable and dentable, then A is SCD. In particular, every closed convex bounded separable Radon-Nikodým subset of X is SCD.*

PROOF. Since A is separable, so is the set of its denting points, and we may find a countable collection of denting points $\{a_n : n \in \mathbb{N}\}$ of A which is dense in $\mathrm{dent}(A)$. Now, for every $n, m \in \mathbb{N}$, we consider a slice $S_{n,m}$ of A containing a_n and having diameter less than $1/m$. Then, the family $\{S_{n,m} : n, m \in \mathbb{N}\}$ is determining for A. Indeed, if $B \subset A$ intersects all the $S_{n,m}$, then $a_n \in \overline{B}$ for every $n \in \mathbb{N}$, so

$$A \subset \overline{\mathrm{conv}}\big(\mathrm{dent}(A)\big) = \overline{\mathrm{conv}}\big(\{a_n : n \in \mathbb{N}\}\big) \subset \overline{\mathrm{conv}}(\overline{B}) = \overline{\mathrm{conv}}(B).$$ □

Recall that in LUR spaces B_X is dentable (Proposition 2.10.3). This explains the following example.

EXAMPLE 10.1.11. Let X be a separable Banach space with a LUR norm. Then B_X is SCD.

Since every separable Banach space admits a LUR renorming (Theorem 2.10.4), we get the following effect.

EXAMPLE 10.1.12. Every separable Banach space X admits an equivalent norm $|\cdot|$ such that $B_{(X,|\cdot|)}$ is an SCD set.

Our second family of elementary examples of SCD sets deals with the so-called Asplund property, a concept related to differentiability of convex continuous functions, which can equivalently be reformulated in terms of separability and duality

[65, §5]. A separable closed convex bounded subset A of a Banach space X has the *Asplund property* if and only if the semi-normed space (X^*, ρ_A) is separable, where

$$\rho_A(x^*) = \sup\{|x^*(a)|\colon a \in A\} \qquad (x^* \in X^*).$$

Of course, separable closed convex bounded subsets of Asplund spaces have the Asplund property.

Example 10.1.13. Let X be a Banach space and let A be a closed convex bounded subset of X. If A is separable and has the Asplund property, then A is SCD.

Proof. We take a ρ_A-dense countable family $\{x_n^*\colon n \in \mathbb{N}\}$ in (X^*, ρ_A), and consider the slices

$$S_{n,m} = \operatorname{Slice}(A, x_n^*, 1/m) \qquad (n, m \in \mathbb{N}).$$

We are done by just proving that if $\{v_{n,m}\colon n, m \in \mathbb{N}\}$ satisfies that $v_{n,m} \in S_{n,m}$ for every $n, m \in \mathbb{N}$, then

$$A \subset \overline{\operatorname{conv}}\left(\{v_{n,m}\colon n, m \in \mathbb{N}\}\right).$$

Indeed, suppose to the contrary that there are $a \in A$, $x^* \in X^*$, and $\delta > 0$ such that

$$\operatorname{Re} x^*(a) > \sup_{n,m} \operatorname{Re} x^*(v_{n,m}) + \delta.$$

Now, we may find $N \in \mathbb{N}$ such that $\rho_A(x_N^* - x^*) < \delta/2$ and so

$$\begin{aligned} \operatorname{Re} x_N^*(a) + \delta/2 > \operatorname{Re} x^*(a) &> \sup_{n,m} \operatorname{Re} x^*(v_{n,m}) + \delta \\ &\geqslant \sup_{m} \operatorname{Re} x^*(v_{N,m}) + \delta > \sup_{m} \operatorname{Re} x_N^*(v_{N,m}) + \delta/2 \\ &= \sup \operatorname{Re} x_N^*(A) + \delta/2, \end{aligned}$$

a contradiction. □

We now show that there are convex bounded subsets of separable Banach spaces which are not SCD.

Example 10.1.14. Let X be a separable Banach space with the Daugavet property. Then B_X is not an SCD set. In particular, $B_{C[0,1]}$ and $B_{L_1[0,1]}$ are not SCD sets.

Proof. Fix $x_0 \in S_X$ and an arbitrary sequence of slices $(S_n)_{n\in\mathbb{N}}$. We will get the result by showing that there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in S_n$ for every $n \in \mathbb{N}$ and such that $x_0 \notin \overline{\operatorname{lin}}\{x_n\colon n \in \mathbb{N}\}$. To do so, we use Lemma 3.1.14 which says, in particular, that for every finite-dimensional subspace $Y \subset X$, every $\varepsilon > 0$, and every slice S of B_X, there is $x \in S$ such that

$$\|y + tx\| \geqslant (1 - \varepsilon)(\|y\| + |t|) \qquad \forall y \in Y.$$

Using this result, one can inductively select elements $x_n \in S_n$, $n \in \mathbb{N}$, in such a way that

$$\|y + tx_n\| \geqslant \left(1 - \frac{1}{4^n}\right)(\|y\| + |t|) \qquad \big(y \in \operatorname{lin}\{x_k\colon k < n\}\big).$$

Then, $(x_n)_{n=0,1,\dots}$ forms a sequence equivalent to the unit vector basis of ℓ_1, so x_0 is not in the closure of $\operatorname{lin}\{x_n\colon n \in \mathbb{N}\}$, as desired. □

REMARK 10.1.15. *A subset of an SCD set is not necessarily SCD.* Indeed, let $X = C[0,1]$. As X is separable, it admits an equivalent LUR renorming $|\cdot|$ (see Theorem 2.10.4). By Example 10.1.12, $A = B_{(X,|\cdot|)}$ is SCD. Now, it is possible to find $\lambda > 0$ such that $C = \lambda B_{(X,\|\cdot\|_\infty)}$ is contained in A. Finally, C is not SCD by Example 10.1.14. □

Our next goal is to extend the above preliminary examples to more intriguing ones.

Let us recall that, in the case of convex sets, the Bourgain Lemma 2.6.19 lets us replace the sequence of slices with a sequence of relatively weakly open subsets in the definition of an SCD set. A non-convex set can be not SCD, but still possess a determining sequence of relatively weakly open subsets; this will be proved in Proposition 10.7.6 below.

PROPOSITION 10.1.16. *A set A in a Banach space is SCD if it possesses a determining sequence of convex combinations of slices. In particular, a* convex *set is SCD if it possesses a determining sequence of relatively weakly open subsets.*

PROOF. Let $(V_n)_{n\in\mathbb{N}}$ be a determining sequence of subsets of A formed by convex combination of slices of A. Now, for every $n \in \mathbb{N}$, there exists a collection of slices $\{S_{n,m}\colon m = 1, \ldots, k_n\}$ and positive numbers $\{\lambda_{n,m}\colon m = 1, \ldots, k_n\}$ with $\sum_{m=1}^{k_n} \lambda_{n,m} = 1$, such that $\sum_{m=1}^{k_n} \lambda_{n,m} S_{n,m} \subset V_n$. Then the collection of slices $\{S_{n,m}\colon n \in \mathbb{N},\ 1 \leqslant m \leqslant k_n\}$ is determining for A. Indeed, let B be a subset of A such that $B \cap S_{n,m} \neq \emptyset$ for all n, m, and consider $b_{n,m} \in B \cap S_{n,m}$ for every n, m. If we take $a_n = \sum_{m=1}^{k_n} \lambda_{n,m} b_{n,m}$, it is clear that $a_n \in \operatorname{conv}(B) \cap V_n$. So we know that $\operatorname{conv}(B) \cap V_n \neq \emptyset$ for all n, which by the assumption gives us that $\overline{\operatorname{conv}}(B) \supset A$.

Finally, if A is convex and has a determining sequence of relatively weakly open subsets $\{V_n\colon n \in \mathbb{N}\}$, Bourgain's Lemma 2.6.19 in the form of Remark 2.6.20 allows us to find convex combinations of slices inside the V_n's, and the proof above shows that A is SCD. □

DEFINITION 10.1.17. An element $y \in A$ is said to be a *point of weak-norm continuity for the identity map on A*, for short a *point of continuity of A* if for every $\varepsilon > 0$ there is a relatively weakly open subset S of A such that $y \in S$ and $\operatorname{diam} S < \varepsilon$. We say that A is *huskable* if A is equal to the closed convex hull of its points of continuity.

The first consequence of Proposition 10.1.16 is that Proposition 10.1.10 can be extended from dentable sets to huskable sets. With not much work, we are going to extend the result to the following more general setting. A closed convex bounded subset A of a Banach space X has *small combinations of slices* [**117**, **259**] if every slice of A contains convex combinations of slices of A with arbitrarily small diameter.

THEOREM 10.1.18. *Let X be a Banach space and let A be a separable closed convex bounded subset of X having small combinations of slices. Then A is an SCD set.*

PROOF. By [**117**, Corollary III.7], for every $x \in A$ and every $\varepsilon > 0$, there is a convex combination of slices of A contained in $B(x, \varepsilon)$. Now, we take a countable dense subset $\{x_n\colon n \in \mathbb{N}\}$ of A and for $(n,m) \in \mathbb{N} \times \mathbb{N}$, we take $V_{n,m}$ a convex combination of slices of A contained in $B(x_n, 1/m)$. Then, if $B \subset A$ intersects all the $V_{n,m}$, it intersects also all the balls $B(x_n, 1/m)$. Therefore, the set $\{x_n : n \in \mathbb{N}\}$ is contained in $\overline{B}$ and so, $A = \overline{\mathrm{conv}}(B)$. Finally, Proposition 10.1.16 gives us that A is SCD. $\square$

RNP sets have small combinations of slices, so the above result extends Proposition 10.1.10. Even more, strongly regular sets (in particular, huskable sets, CPCP sets) have small combinations of slices [**117**, Proposition III.5]. We recall that a closed convex bounded subset A of a Banach space is said to be *strongly regular* if every non-empty convex subset L of A contains a convex combination of slices of L of arbitrarily small diameter. A has the *convex point of continuity property* (*CPCP* in short) if every closed convex subset B of A contains a weak-to-norm point of continuity of the identity mapping. In this case, for every convex subset B of A and for every $\varepsilon > 0$, there is a relatively weakly open subset $C \subset B$ with $\mathrm{diam}(C) < \varepsilon$ [**63**].

Corollary 10.1.19. *Let X be a Banach space and let A be a closed convex bounded subset of X. If A is separable and strongly regular, then A is SCD. In particular, separable CPCP sets are SCD.*

Our next aim is to extend Example 10.1.13 to sets which do not contain ℓ_1 sequences. We need the following topological definition. By a *π-base* of a topological space (T, τ) we mean a family $\{O_i : i \in I\}$ of nonempty open sets such that every nonempty open subset O of T contains one of the elements of the family. The following result is another consequence of Bourgain's lemma.

Proposition 10.1.20. *Let X be a Banach space and let A be a convex bounded subset of X. If $(A, \sigma(X, X^*))$ has a countable π-base, then A is an SCD set.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be a countable π-base of $(A, \sigma(X, X^*))$. Since slices of A have non-empty weak interior, any of them contains some of the V_n. But then, Proposition 10.1.2 shows that the family $\{V_n : n \in \mathbb{N}\}$ is determining for A and Proposition 10.1.16 gives that A is SCD. $\square$

The main consequence of the above proposition is the following.

Theorem 10.1.21. *Let X be a Banach space and let A be a separable convex bounded subset of X which contains no ℓ_1-sequences. Then, $(A, \sigma(X, X^*))$ has a countable π-base. In particular, A is an SCD set.*

Proof. By [**292**, Theorem 3.11], $(A, \sigma(X, X^*))$ is a relatively compact subset of the space of first Baire class functions on $(B_{X^*}, \sigma(X^*, X))$, and we can apply [**291**, Lemma 4] by Todorčević, to deduce that $(A, \sigma(X, X^*))$ has a σ-disjoint π-base (i.e., a π-base $\{V_i : i \in I\}$ such that $I = \bigcup_{n \in \mathbb{N}} I_n$ and each subfamily $\{V_i : i \in I_n\}$ is a pairwise disjoint family). Now, it is clear that a σ-disjoint family of open subsets in a separable space has to be countable. Finally, A is SCD by Proposition 10.1.20. $\square$

This result obviously extends Example 10.1.13 since Asplund sets cannot contain ℓ_1-sequences.

10.2. ε-Accessible points. Hereditarily SCD sets

Definition 10.2.1. Let A be a convex bounded set in a Banach space X. A is called a *hereditarily SCD set* (*HSCD* set for short) if every subset B of A is an SCD set.

First note that since every SCD set is separable, every HSCD set is separable, too. Every convex subset of an HSCD set is HSCD.

REMARK 10.2.2. *Thanks to Lemmas 10.1.6 and 10.1.7, a bounded convex subset A of a Banach space X is HSCD if and only if $\overline{A}$ is and if and only if all (closed) convex subsets of A are SCD.* These facts will be used profusely when dealing with concrete examples, since most of the geometric definitions for sets we are using either are only defined for closed convex sets or admit useful characterisations which are only valid for closed convex sets.

Applying the results of Section 10.1, we get some classes of HSCD sets.

EXAMPLES 10.2.3.
(a) *If A is a separable closed convex bounded strongly regular set, then A is HSCD (Corollary 10.1.19 plus the evident fact that closures of convex subsets of a strongly regular set are strongly regular). In particular, sets with the RNP (more generally, with CPCP) are HSCD.*
(b) *Separable convex bounded sets which do not contain ℓ_1-sequences are HSCD (Theorem 10.1.21).*
(c) *Both families include convex weakly compact sets, so they are HSCD sets.*

The main result of this section will say that HSCD sets are stable with respect to direct sums. Although this result is a distant relative of the analogous fact about sets with the RNP (Corollary 2.7.14), its proof is relatively bulky. This proof needs the technical concept of ε-accessible point, which will be used a couple of times in other sections as well, so below we extract some properties of ε-accessible points for future applications.

DEFINITION 10.2.4. Let X be a Banach space, A be a convex set in X, $a \in A$ and ε be a positive real. A countable family $\{V_n\colon n \in \mathbb{N}\}$ of subsets of A is said to be *ε-determining for a in A* if for every $B \subset A$, if B intersects all the V_n, then $\operatorname{dist}(a, \operatorname{conv}(B)) < \varepsilon$. A point $a \in A$ is called an *ε-accessible point* of A if there is an ε-determining sequence $(V_n)_{n\in\mathbb{N}}$ for a consisting of relatively weakly open subsets of A.

REMARK 10.2.5. In the notation from the above definition we have:
(1) *A family $\{V_n\colon n \in \mathbb{N}\}$ of subsets of A is ε-determining for $a \in A$ if and only if for every convex $B \subset A$, if B intersects all the V_n, then $\operatorname{dist}(a, B) < \varepsilon$.*
(2) *If $C \subset A$ is non-empty, relatively weakly open and convex, $x \in C$ is an ε-accessible point of C, then x is an ε-accessible point of A.*

Indeed, let $(V_n)_{n\in\mathbb{N}}$ be an ε-determining sequence for x, relative to C, of relatively weakly open subsets of C. Then all V_n are relatively weakly open in A. Let us show that $\{V_n\colon n \in \mathbb{N}\}$ is ε-determining for x in A. Let $B \subset A$ be a set that intersects all the V_n. Then $B \cap C \subset C$ intersects all the V_n, so $\operatorname{dist}(x, \operatorname{conv}(B \cap C)) < \varepsilon$, which implies that $\operatorname{dist}(x, \operatorname{conv}(B)) < \varepsilon$.
(3) *The set A_ε of all ε-accessible points of A is convex.*

Let $x_1, x_2 \in A_\varepsilon$, $\lambda \in [0, 1]$, and let $\mathcal{V}_1 = (V_n^1)_{n\in\mathbb{N}}$, $\mathcal{V}_2 = (V_n^2)_{n\in\mathbb{N}}$ be corresponding ε-determining sequences of relatively weakly open subsets for x_1, x_2, respectively. Taking these two sequences together, we obtain a countable collection $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ of subsets with the property that for every $B \subset A$, if B intersects all the elements of $\mathcal{V}$, then $\operatorname{dist}(x_j, \operatorname{conv}(B)) < \varepsilon$, $j = 1, 2$. But then $\operatorname{dist}(\lambda x_1 + (1-\lambda)x_2, \operatorname{conv}(B)) < \varepsilon$, so $\lambda x_1 + (1-\lambda)x_2 \in A_\varepsilon$.

LEMMA 10.2.6. *Let X be a Banach space and let A be a separable convex bounded subset of X. Suppose that for every $\varepsilon > 0$ the set A_ε of all ε-accessible points of A is dense. Then A is an SCD set.*

PROOF. From our conditions it follows that for each $m \in \mathbb{N}$ there is a countable dense subset $\{a_{n,m}\colon n \in \mathbb{N}\}$ of A such that each $a_{n,m}$ is an $\frac{1}{m}$-accessible point of A. Select, for every $a_{n,m}$ the corresponding $\frac{1}{m}$-determining countable collection $\mathcal{V}_{n,m}$ of relatively weakly open subsets. Then $\mathcal{V} := \bigcup_{n,m\in\mathbb{N}} \mathcal{V}_{n,m}$ is a determining collection of relatively weakly open subsets which, according to Proposition 10.1.16, gives what we need. $\square$

LEMMA 10.2.7. *Let X be a Banach space and let A be a separable convex bounded subset of X. Suppose that for every non-empty relatively weakly open convex $C \subset A$ and every $\varepsilon > 0$, there is an ε-accessible point of C in C (or in A, which is the same according to the second statement of Remark 10.2.5). Then, A is an SCD set.*

PROOF. By Lemma 10.2.6, it is enough to show that for every $\varepsilon > 0$, the set A_ε of ε-accessible points of A is dense in A. Since A_ε is convex, it is enough to show that A_ε is weakly dense in A. Fix some convex relatively weakly open subset $V \subset A$. By the assumption, there is an ε-accessible point of V, so $A_\varepsilon \cap V \neq \emptyset$. $\square$

The main result of this section is:

THEOREM 10.2.8. *Let A_1 and A_2 be two convex bounded hereditarily SCD sets in Banach spaces X_1 and X_2, respectively. Then $A := A_1 \times A_2 \subset X_1 \oplus X_2$ is also a hereditarily SCD set (we suppose here that the direct sum $X_1 \oplus X_2$ is their Cartesian product endowed with some absolute norm).*

PROOF. Let B be a convex subset of A. What we need to prove is that B is an SCD set. Note that since A_1 and A_2 are hereditarily SCD sets, they are separable and thus A and B are also separable. So it is sufficient for us to prove that every $b \in B$ is 2ε-accessible for every $\varepsilon > 0$.

Fix some $b \in B$, $b = (b_1, b_2)$, and $\varepsilon > 0$. Denote

$$B_2 = \{x_2 \in A_2 : \exists x_1 \in A_1\colon (x_1, x_2) \in B,\ \|x_1 - b_1\| < \varepsilon\}.$$

Then B_2 is convex and $B_2 \subset A_2$. So using the fact that A_2 is a hereditarily SCD set we obtain a determining sequence $(S_n)_{n\in\mathbb{N}}$ of slices of B_2. Let S_n be generated by corresponding functionals $f_n^2 \in X_2^*$ and $S_n = \{y \in B_2\colon f_n^2(y) > \varepsilon_n\}$ $(n \in \mathbb{N})$. Then, for all natural n, denote by $\tilde{S}_n$ the slice of B generated by the functional $(0, f_n^2)$, i.e.,

$$\tilde{S}_n := \{z = (z_1, z_2) \in B\colon f_n^2(z_2) > \varepsilon_n\}.$$

Consider now

$$\tilde{S}_n^1 = \{x_1 \in A_1 : \exists x_2 \in A_2 \text{ with } (x_1, x_2) \in \tilde{S}_n\}, \quad n \in \mathbb{N}.$$

All these sets are convex and contained in A_1, and so using that A_1 is a hereditarily SCD set, we can find determining countable collection $\{S_{n,k}\colon k \in \mathbb{N}\}$ of slices of $\tilde{S}_n^1$. Let $S_{n,k} = \{x \in \tilde{S}_n^1\colon f_{n,k}^1(x) > \delta_{n,k}\}$ for some $f_{n,k}^1 \in X_1^*$ $(n, k \in \mathbb{N})$. Now, for all natural n, k consider

$$\tilde{S}_{n,k} = \left\{z = (z_1, z_2) \in B\colon f_{n,k}^1(z_1) > \delta_{n,k}\right\}$$

– the slice of B generated by $(f^1_{n,k}, 0)$. Then we can take

$$\{\tilde{S}_n \cap \tilde{S}_{n,k} \colon n, k \in \mathbb{N}\}$$

as the required sequence $(V_{\varepsilon,n})_{n=1}^{\infty}$ of relatively weakly open subsets of B. Remark that all these sets are nonempty: since $S_{n,k} \subset \tilde{S}^1_n$, by definition of $\tilde{S}^1_n$ for every $x_1 \in S_{n,k}$ there is $x_2 \in A_2$ such that $(x_1, x_2) \in \tilde{S}_n$. This (x_1, x_2) lies in $\tilde{S}_n \cap \tilde{S}_{n,k}$. Now, let's show that this is the 2ε-determining sequence we need.

Let $C \subset B$ be convex and suppose C intersects all the elements in $\{\tilde{S}_n \cap \tilde{S}_{n,k}\colon n, k \in \mathbb{N}\}$. Fix some $n \in \mathbb{N}$. We know that $(C \cap \tilde{S}_n) \cap \tilde{S}_{n,k} \neq \emptyset$, and so $(C \cap \tilde{S}_n)^1 \cap S_{n,k} \neq \emptyset$, where $(C \cap \tilde{S}_n)^1 = \{x_1 \in A_1 : \exists x_2 \in A_2 : (x_1, x_2) \in (C \cap \tilde{S}_n)\}$. Since the sequence $(S_{n,k})_{k=1}^{\infty}$ is determining for the set $\tilde{S}^1_n$ and $(C \cap \tilde{S}_n)^1 \subset \tilde{S}^1_n$, we obtain that $\overline{(C \cap \tilde{S}_n)^1} \supset \tilde{S}^1_n$. Let us now show that there exists $z_1 \in \tilde{S}^1_n$ such that $\|b_1 - z_1\| < \varepsilon$. We know that S_n is a slice of B_2, which implies that $S_n \cap B_2 \neq \emptyset$. Let $z_2 \in S_n \cap B_2$. Then from the definition of B_2 it follows that there exists z_1 such that $(z_1, z_2) \in B$ and $\|z_1 - b_1\| < \varepsilon$. We show that this z_1 meets our requirements. Indeed, we already know that $\|z_1 - b_1\| < \varepsilon$, and since $z_2 \in S_n$, we have that $(z_1, z_2) \in \tilde{S}_n$, whence $z_1 \in \tilde{S}^1_n$. Thus, using the fact that $\overline{(C \cap \tilde{S}_n)^1} \supset \tilde{S}^1_n$, we deduce that $z_1 \in \overline{(C \cap \tilde{S}_n)^1}$. This means that there exists $(c_{1,n}, c_{2,n}) \in C \cap \tilde{S}_n$ such that $\|c_{1,n} - z_1\| < \varepsilon - \|z_1 - b_1\|$, and so $\|c_{1,n} - b_1\| < \varepsilon$. It is easy to see that $c_{2,n} \in B_2$ and therefore $c_{2,n} \in S_n$. Let's now denote

$$C_2 = \{x_2 \in A_2\colon \exists x_1 \in A_1 \text{ with } (x_1, x_2) \in C,\ \|x_1 - b_1\| < \varepsilon\}.$$

Then what we have just proved is that $C_2 \cap S_n$ is not empty for all natural n. So since, evidently, $C_2 \subset B_2$ and the family $\{S_n\colon n \in \mathbb{N}\}$ is determining for B_2, we get that $\overline{C_2} \supset B_2$. In particular, there exists $x_2 \in C_2$ such that $\|x_2 - b_2\| < \varepsilon$. Then according to the definition of C_2, we obtain that there exists x_1 such that $c = (x_1, x_2) \in C$, $\|x_1 - b_1\| < \varepsilon$. Then for this c we have that $\|c - b\| \leqslant \|x_1 - b_1\| + \|x_2 - b_2\| < 2\varepsilon$, which implies that $\operatorname{dist}(C, b) < 2\varepsilon$ and we are done. □

10.3. SCD spaces

Definition 10.3.1. A separable Banach space X is said to be *slicely countably determined* (*SCD space* for short) if every convex bounded subset of X is an SCD set.

Remark that the definition implies that every convex bounded subset of X is HSCD. The main examples of SCD spaces come from Examples 10.2.3.

Examples 10.3.2.

(a) *If X is a separable strongly regular space, then X is SCD. In particular, RNP spaces (more generally, CPCP spaces) are SCD.*
(b) *Separable spaces which do not contain copies of ℓ_1 are SCD. In particular, if X^* is separable, then X is SCD.*
(c) *Both families include reflexive separable spaces, which are then SCD spaces.*

With respect to spaces which are not SCD we only know, thanks of Example 10.1.14, the Daugavet spaces.

Examples 10.3.3.

(a) *If X is a separable Banach space which admits an equivalent renorming with the Daugavet property, then X is not SCD.*

(b) *In particular, there is a Banach space with the Schur property which is not an SCD space.* Indeed, for this the space E from Theorem 7.7.8 serves, having the Schur and the Daugavet property.

Let us state the following immediate observations.

REMARKS 10.3.4.

(a) *Every subspace of an SCD space is SCD.*
(b) *For quotients the situation is different. For instance, because of the quotient universality of ℓ_1 (see Theorem 2.5.9), $C[0,1]$ is a non-SCD quotient of the SCD space ℓ_1.*

Our next aim is to show some stability results for the SCD spaces. The first one is the following.

THEOREM 10.3.5. *Let $X_1, \dots, X_n$ be SCD spaces. Then, $X_1 \oplus \dots \oplus X_n$ is SCD.*

PROOF. It is sufficient to consider $n = 2$. Since X_1, X_2 are SCD spaces, their balls B_{X_1}, B_{X_2} are HSCD sets. Theorem 10.2.8 says that the set $B_{X_1} \times B_{X_2} \subset X_1 \oplus X_2$ is also HSCD, but this set absorbs any bounded subset of $X_1 \oplus X_2$, so every bounded subset of $X_1 \oplus X_2$ is SCD. □

Our next goal is to deal with infinite sums. We use notation from Section 2.9.1.

THEOREM 10.3.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of SCD spaces and let E be a Banach space of sequences whose canonical basis is a 1-unconditional and shrinking basis (i.e., E does not contain copies of ℓ_1). Then, $X = \bigl(\bigoplus_{n\in\mathbb{N}} X_n\bigr)_E$ is also an SCD space.*

PROOF. For every $m \in \mathbb{N}$, we denote

$$Y_m = \bigl(X_1 \oplus X_2 \oplus \ldots \oplus X_m \oplus 0 \oplus 0 \oplus \cdots\bigr)_E \subset X$$

and let $P_m\colon X \to Y_m$ be the natural projection. Let A be a convex bounded subset of X. Now, for every $m \in \mathbb{N}$, $P_m(A)$ is a convex bounded subset of Y_m, which is an SCD space by Theorem 10.3.5. Hence, there is a determining sequence $(S_{m,k})_{k\in\mathbb{N}}$ of slices of $P_m(A)$. Consider $\widetilde{S}_{m,k} = P_m^{-1}\bigl(S_{m,k}\bigr) \cap A$. We will prove that $\{\widetilde{S}_{m,k}\colon k, m \in \mathbb{N}\}$ is a determining countable collection of slices of A.

Let B be a subset of A intersecting all the $\widetilde{S}_{m,k}$. We fix an arbitrary point $a \in A$ and we will prove that $a \in \overline{\mathrm{conv}}(B)$. Since B intersects all the $\widetilde{S}_{m,k}$, $P_m(B)$ intersects $S_{m,k}$ for every integer k. It follows that $\overline{\mathrm{conv}}\bigl(P_m(B)\bigr) \supset P_m(A)$. In particular, $\overline{\mathrm{conv}}\bigl(P_m(B)\bigr) \ni P_m(a)$. That means that there exists $b_m \in \mathrm{conv}(B)$ such that $\|P_m(b_m - a)\| < \frac{1}{m}$. Then, it is easy to see that b_m tends to a coordinatewise. But since the canonical basis of E is at the same time a shrinking basis, we get that b_m tends to a in the weak topology. So we can apply Mazur's theorem and get a sequence (b'_m) with $b'_m \in \mathrm{conv}\bigl(\{b_k\colon k \geqslant m\}\bigr) \subset \mathrm{conv}(B)$ which tends to a in the norm topology. But this exactly means that $a \in \overline{\mathrm{conv}}(B)$, which was to be proved. □

The next result deals with infinite sums when the natural basis of E is boundedly complete.

THEOREM 10.3.7. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of SCD spaces and let E be a space of sequences whose natural basis is a 1-unconditional and boundedly complete* *basis (i.e., E does not contain isomorphic copies of c_0). Then, $X = \big(\bigoplus_{n\in\mathbb{N}} X_n\big)_E$ is an SCD space.*

PROOF. Let a convex bounded subset A of X and $\varepsilon > 0$ be fixed. Consider the subset

$$A_E = \big\{(a_n)_{n\in\mathbb{N}} \in E\colon \exists x = (x_n)_{n\in\mathbb{N}} \in A \text{ with } \|x_n\| = |a_n| \text{ for all } n \in \mathbb{N}\big\}.$$

Since A_E is a bounded subset of a space with the RNP [**99**, p. 64], there are a functional $b = (b_n)_{n\in\mathbb{N}} \in E^*$ and a positive number α such that the slice

$$\operatorname{Slice}(A_E, b, \alpha) = \left\{(a_n)_{n\in\mathbb{N}} \in A_E\colon \sum_{n\in\mathbb{N}} b_n a_n > \alpha\right\}$$

is non-empty and has diameter smaller than $\varepsilon/4$ (Theorem 2.7.12). Taking into account that A_E is symmetric, we may assume that $b_n \geqslant 0$ (the slice of A_E defined by $|b| = (|b_n|)_{n\in\mathbb{N}}$ is isometric to $\operatorname{Slice}(A_E, b, \alpha)$). Fix $x \in A$ with $(\|x_n\|)_{n\in\mathbb{N}} \in \operatorname{Slice}(A_E, b, \alpha)$ and pick $x_n^* \in S_{X_n^*}$ such that $x_n^*(x_n) = \|x_n\|$. Write $f_n = b_n x_n^*$, $f = (f_n)_{n\in\mathbb{N}} \in X^*$. We claim that for the slice

$$S = \left\{(x_n)_{n\in\mathbb{N}} \in A\colon \sum_{n\in\mathbb{N}} f_n(x_n) > \alpha\right\}$$

there is $m \in \mathbb{N}$ with the following property

$$\big\|(0, \dots, 0, y_{m+1}, y_{m+2}, \dots)\big\| < \frac{\varepsilon}{2} \qquad \text{for all } (y_n)_{n\in\mathbb{N}} \in S. \tag{10.3.1}$$

To show this, it is sufficient to select m in such a way that

$$\|(0, \dots, 0, x_{m+1}, x_{m+2}, \dots)\| < \varepsilon/4$$

and to use that $\operatorname{diam}\operatorname{Slice}(A_E, b, \alpha) < \varepsilon/4$. In fact, with such a choice of m we get

$$\begin{aligned}\|(0, \dots, 0, y_{m+1}, y_{m+2}, \dots)\| &= \|(0, \dots, 0, \|y_{m+1}\|, \|y_{m+2}\|, \dots)\|\\ &\leqslant \big\|(0, \dots, 0, \|x_{m+1}\|, \|x_{m+2}\|, \dots)\big\|\\ &\quad + \big\|(0, \dots, 0, \big|\|x_{m+1}\| - \|y_{m+1}\|\big|, \big|\|x_{m+2}\| - \|y_{m+2}\|\big|, \dots)\big\|\\ &\leqslant \frac{\varepsilon}{4} + \big\|(\big|\|x_1\| - \|y_1\|\big|, \big|\|x_2\| - \|y_2\|\big|, \dots)\big\| \leqslant \frac{\varepsilon}{2}.\end{aligned}$$

Let us prove that x is an ε-accessible point of A. Consider

$$Y_m = \big(X_1 \oplus X_2 \oplus \ldots \oplus X_m \oplus 0 \oplus 0 \oplus \cdots\big)_E \subset X$$

and $P_m\colon X \to Y_m$ the natural projection. By Theorem 10.3.5, Y_m is an SCD space and, since $P_m(S)$ is a convex bounded set in Y_m, there exists a determining family $\{S_n\colon n \in \mathbb{N}\}$ of slices of $P_m(S)$. Notice that Y_m^* isometrically embeds into X^*. For every integer $n \in \mathbb{N}$, we consider $\widetilde{S}_n = P_m^{-1} S_n \cap S$, which is a slice of S and, obviously, relatively weakly open in A. Let B be a subset of A which intersects all the $\widetilde{S}_n$. We'll now prove that then $\operatorname{dist}\big(x, \operatorname{conv}(B)\big) < \varepsilon$.

Since B intersects all the $\widetilde{S}_n$, we can find a sequence $(y_n) \subset B$, such that $y_n \in \widetilde{S}_n$ for every $n \in \mathbb{N}$. This implies that $P_m(y_n) \in S_n$ for all $n \in \mathbb{N}$ and so $\overline{\operatorname{conv}}\big(\{P_m(y_n)\colon n \in \mathbb{N}\}\big) \supset P_m(S)$. In particular, $P_m(x) \in \overline{\operatorname{conv}}\big(\{P_m(y_n)\colon n \in \mathbb{N}\}\big)$. But (10.3.1) gives us that the m-th tails of x and of all the y_n are small, that is,

$$\|x - P_m(x)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|y_n - P_m(y_n)\| < \varepsilon/2 \qquad \big(\text{for all } n \in \mathbb{N}\big).$$

This gives us that $\operatorname{dist}\big(a, \operatorname{conv}(B)\big) < \varepsilon$, and the proof is complete. $\square$

An immediate consequence is the following.

EXAMPLE 10.3.8. *The spaces $c_0(\ell_1)$ and $\ell_1(c_0)$ are SCD.*

This result, together with those results of Section 10.1, gives us the following examples.

EXAMPLE 10.3.9. *The spaces $c_0 \widehat{\otimes}_\varepsilon c_0$, $c_0 \widehat{\otimes}_\pi c_0$, $c_0 \widehat{\otimes}_\varepsilon \ell_1$, $c_0 \widehat{\otimes}_\pi \ell_1$, $\ell_1 \widehat{\otimes}_\varepsilon \ell_1$, and $\ell_1 \widehat{\otimes}_\pi \ell_1$ are SCD.* Indeed, it is well known that $c_0 \widehat{\otimes}_\varepsilon c_0 \cong c_0$, $c_0 \widehat{\otimes}_\varepsilon \ell_1 \cong c_0(\ell_1)$, $c_0 \widehat{\otimes}_\pi \ell_1 \cong \ell_1(c_0)$, and $\ell_1 \widehat{\otimes}_\pi \ell_1 \cong \ell_1$ (see [**274**, Examples 2.19 and 3.3], for instance), so these cases are clear from the above example. For the remaining cases, just observe that $\big[c_0 \widehat{\otimes}_\pi c_0\big]^* \cong \ell_1 \widehat{\otimes}_\varepsilon \ell_1$ (since $[c_0 \widehat{\otimes}_\pi c_0]^* \cong L(c_0, \ell_1)$ [**274**, p. 24], $K(c_0, \ell_1) \cong \ell_1 \widehat{\otimes}_\varepsilon \ell_1$ [**274**, Corollary 4.13] and $K(c_0, \ell_1) = L(c_0, \ell_1)$ since ℓ_1 has the Schur property and c_0^* is separable), so $c_0 \widehat{\otimes}_\pi c_0$ is Asplund and $\ell_1 \widehat{\otimes}_\varepsilon \ell_1$ has the RNP.

Since for X and Y being c_0 or ℓ_1 one has $K(X, Y) \cong X^* \widehat{\otimes}_\varepsilon Y$ [**274**, Corollary 4.13], the following examples follow.

EXAMPLE 10.3.10. *The spaces $K(c_0)$ and $K(c_0, \ell_1)$ are SCD. The spaces $K(\ell_1)$ and $K(\ell_1, c_0)$ contain ℓ_∞ (this is because for a fixed vector $y_0 \in S_Y$ the operators of the form $x^* \otimes y_0$, $x^* \in X^*$, form a subspace of $K(X, Y)$ which is isometric to X^*) and so they are not separable, all the more not SCD.*

Another example in this line is the following.

EXAMPLE 10.3.11. *The spaces $\ell_2 \widehat{\otimes}_\pi \ell_2 \cong N(\ell_2)$, the space of nuclear operators, and $\ell_2 \widehat{\otimes}_\varepsilon \ell_2 \cong K(\ell_2)$ are SCD.* Indeed, the first space has the RNP and the second is an Asplund space.

In the following result we will enlarge the class of projective tensor products which are SCD.

THEOREM 10.3.12. *Let X and Y be separable Banach spaces. If $B_X = \overline{\operatorname{conv}}(\operatorname{dent}(B_X))$ and B_Y is an SCD set, then $B_{X \widehat{\otimes}_\pi Y}$ is an SCD set.*

We need the following useful (and technical) lemma from [**275**].

LEMMA 10.3.13 ([**275**, Lemma 3.4]). *Suppose that we have a norm one bilinear form $B \in \operatorname{Bil}(X \times Y) = (X \widehat{\otimes}_\pi Y)^*$ and $\varepsilon > 0$. Then,*

$$S(B_{X \widehat{\otimes}_\pi Y}, B, \varepsilon^2) \subset \overline{\operatorname{conv}}(\{x \otimes y \colon x \in B_X,\ y \in B_Y,\ B(x, y) > 1 - \varepsilon\}) + 4\varepsilon B_{X \widehat{\otimes}_\pi Y}.$$

PROOF OF THEOREM 10.3.12. As X is separable and $B_X = \overline{\operatorname{conv}}(\operatorname{dent}(B_X))$, we can find $\{x_n \colon n \in \mathbb{N}\} \subset \operatorname{dent}(B_X)$ such that $B_X = \overline{\operatorname{conv}}(\{x_n \colon n \in \mathbb{N}\})$.

Given $n \in \mathbb{N}$, since x_n is a denting point of B_X, we can find for each $k \in \mathbb{N}$ a slice $\operatorname{Slice}(B_X, x^*_{nk}, \alpha_k)$, where $\|x^*_{nk}\| = 1$ and

$$x_n \in \operatorname{Slice}(B_X, x^*_{nk}, \alpha_k) \subset B\Big(x_n, \frac{1}{k}\Big). \tag{10.3.2}$$

On the other hand, since B_Y is an SCD set, we can find a determining family of slices

$$\big\{\operatorname{Slice}(B_Y, y^*_m, \beta_m) \colon m \in \mathbb{N},\ \|y^*_m\| = 1\big\}, \tag{10.3.3}$$

for B_Y. Let us now define for each $n,m,k,p \in \mathbb{N}$ the following slices:

$$S^p_{n,k,m} = \operatorname{Slice}\Big(B_{X\widehat{\otimes}_\pi Y}, x^*_{nk} \otimes y^*_m, \frac{1}{p}\Big),$$

where, as usual,

$$(x^*_{nk} \otimes y^*_m)(x \otimes y) = x^*_{nk}(x) y^*_m(y)$$

for every $x \in X$ and $y \in Y$. Our goal is to prove that the countable collection of slices $\{S^p_{n,k,m}\colon n,m,k,p \in \mathbb{N}\}$ is determining for $B_{X\widehat{\otimes}_\pi Y}$. To this end, let $S = S(B_{X\widehat{\otimes}_\pi Y}, B, \alpha)$, where B is a norm one bounded bilinear form, and let us find a member of the family of slices $\{S^p_{n,k,m}\colon n,m,k,p \in \mathbb{N}\}$, which is contained in S.

Select $\gamma > 0$ and find $a \in B_X$, $b \in B_Y$ such that $B(a,b) > 1-\alpha+\gamma$. This in turn means that

$$a \in \{x \in B_X\colon B(x,b) > 1-\alpha+\gamma\},$$

where the set above is actually a slice of B_X, since it is not empty and the mapping $x \mapsto B(x,b)$ is clearly linear and continuous.

Since the above set is a slice and $B_X = \overline{\operatorname{conv}}(\{x_n\colon n \in \mathbb{N}\})$ we get that there exists $n \in \mathbb{N}$ such that $x_n \in \{x \in B_X\colon B(x,b) > 1-\alpha+\gamma\}$, i.e., $B(x_n,b) > 1-\alpha+\gamma$. Select $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\gamma}{32}$.

Then,

$$x_n \in \operatorname{Slice}(B_X, x^*_{nk}, \alpha_k) \subset B\Big(x_n, \frac{1}{k}\Big) \subset B\Big(x_n, \frac{\gamma}{32}\Big) \tag{10.3.4}$$

by (10.3.2). Using the fact that $B(x_n,b) > 1-\alpha+\gamma$, we see that similarly to the last case

$$b \in \{y \in B_Y\colon B(x_n,y) > 1-\alpha+\gamma\},$$

where the set is again a slice of B_Y. Since the sequence of slices in (10.3.3) determines B_Y, we can find $m \in \mathbb{N}$ such that

$$\operatorname{Slice}(B_Y, y^*_m, \beta_m) \subset \{y \in B_Y\colon B(x_n,y) > 1-\alpha+\gamma\}. \tag{10.3.5}$$

Consider now the following set:

$$S^{\otimes} := \{u \otimes v\colon u \in \operatorname{Slice}(B_X, x^*_{nk}, \alpha_k),\ v \in \operatorname{Slice}(B_Y, y^*_m, \beta_m)\}.$$

We claim that

$$S^{\otimes} \subset \Big\{z \in B_{X\widehat{\otimes}_\pi Y}\colon B(z) > 1-\alpha+\frac{31\gamma}{32}\Big\}. \tag{10.3.6}$$

Indeed, assume that $u \in \operatorname{Slice}(B_X, x^*_{nk}, \alpha_k)$ and $v \in \operatorname{Slice}(B_Y, y^*_m, \beta_m)$. By (10.3.5) we get $B(x_n,v) > 1-\alpha+\gamma$, and using (10.3.4), we obtain

$$\begin{aligned} B(u\otimes v) = B(u,v) &= B(x_n,v) - B(x_n-u,v) \\ &> 1-\alpha+\gamma-\|B\|\,\|v\|\,\|x_n-u\| > 1-\alpha+\gamma-\frac{1}{n} \\ &> 1-\alpha+\gamma-\frac{\gamma}{32} > 1-\alpha+\frac{31\gamma}{32}. \end{aligned}$$

To proceed further, pick for each $n,k,m \in \mathbb{N}$ another $p \in \mathbb{N}$ satisfying $1/p < \min\{\alpha_k, \beta_m\}$. Now, we claim that

$$\operatorname{Slice}\Big(B_X \otimes B_Y, x^*_{nk} \otimes y^*_m, \frac{1}{p}\Big) \subset S^{\otimes}. \tag{10.3.7}$$

Fix $x \otimes y \in \operatorname{Slice}(B_X \otimes B_Y, x^*_{nk} \otimes y^*_m, 1/p)$, i.e.,

$$(x^*_{nk} \otimes y^*_m)(x \otimes y) = (x^*_{nk}(x) y^*_m(y)) > 1-\frac{1}{p}.$$

Pick $\theta \in \mathbb{K}$ with $|\theta| = 1$ such that $x^*_{nk}(\theta x) = x^*_{nk}(\theta x)$; then $x \otimes y = \theta x \otimes \theta^{-1} y$ and

$$x^*_{nk}(\theta x) \cdot (y^*_m(\theta^{-1} y)) = (x^*_{nk}(\theta x) y^*_m(\theta^{-1} y)) > 1 - \frac{1}{p}$$

Since $\theta^{-1} y \in B_Y$, we have

$$x^*_{nk}(\theta x) \geqslant x^*_{nk}(\theta x) \cdot (y^*_m(\theta^{-1} y)) > 1 - \frac{1}{p} > 1 - \alpha_k,$$

which means that $\theta x \in \operatorname{Slice}(B_X, x^*_{nk}, \alpha_k)$. Analogously, $\theta^{-1} y \in S(B_Y, y^*_m, \beta_m)$. In conclusion,

$$x \otimes y = \theta x \otimes \theta^{-1} y \in S^{\otimes}.$$

Select $p \in \mathbb{N}$ be such that $4/p < \gamma/32$. In order to finish the proof, we will show that

$$\operatorname{Slice}\left(B_{X \widehat{\otimes}_\pi Y}, x^*_{nk} \otimes y^*_m, \frac{1}{p^2}\right) = S^{p^2}_{n,k,m} \subset S = \operatorname{Slice}(B_{X \widehat{\otimes}_\pi Y}, B, \alpha).$$

Indeed, (10.3.13) gives that

$$\operatorname{Slice}\left(B_{X \widehat{\otimes}_\pi Y}, x^*_{nk} \otimes y^*_m, \frac{1}{p^2}\right) \subset \overline{\operatorname{conv}}\Big(\operatorname{Slice}\left(B_X \otimes B_Y, x^*_{nk} \otimes y^*_m, \frac{1}{p}\right)\Big) + \frac{4}{p} B_{X \widehat{\otimes}_\pi Y}. \quad (10.3.8)$$

Pick an element $z \in \operatorname{Slice}(B_{X \widehat{\otimes}_\pi Y}, x^*_{nk} \otimes y^*_m, 1/p^2)$. By (10.3.8), we can write

$$z = g + \frac{4}{p} h \quad \text{with} \quad g \in \overline{\operatorname{conv}}\left(\operatorname{Slice}\left(B_X \otimes B_Y, x^*_{nk} \otimes y^*_m, \frac{1}{p}\right)\right), \quad h \in B_{X \widehat{\otimes}_\pi Y}.$$

This means that we can find $\widehat{g} \in \operatorname{conv}\big(\operatorname{Slice}\big(B_X \otimes B_Y, x^*_{nk} \otimes y^*_m, 1/p\big)\big)$ such that $\|g - \widehat{g}\| < \gamma/32$. By defining $\widehat{z} = \widehat{g} + (4/p)h$, it is obvious that

$$\|z - \widehat{z}\| = \|g - \widehat{g}\| < \frac{\gamma}{32}.$$

In addition, by (10.3.7), we have

$$\operatorname{conv}\big(\operatorname{Slice}(B_X \otimes B_Y, x^*_{nk} \otimes y^*_m, 1/p)\big) \subset \left\{z \in B_{X \widehat{\otimes}_\pi Y} \colon B(z) > 1 - \alpha + \frac{31\gamma}{32}\right\},$$

because slices are convex. With this, we obtain

$$B(\widehat{z}) = B(\widehat{g}) + \frac{4}{p} B(h) > 1 - \alpha + \frac{31\gamma}{32} - \frac{4}{p}.$$

Now, using the above estimations as well as (10.3.6) and (10.3.7), we get

$$\begin{aligned} B(z) &= B(z) - B(\widehat{z}) + B(\widehat{z}) = B(\widehat{z}) - B(\widehat{z} - z) \\ &> 1 - \alpha + \frac{31\gamma}{32} - \frac{4}{p} - \|B\|\, \|\widehat{z} - z\| > 1 - \alpha + \frac{31\gamma}{32} - \frac{4}{p} - \frac{\gamma}{32} \\ &= 1 - \alpha + \frac{30\gamma}{32} - \frac{4}{p} > 1 - \alpha + \frac{\gamma}{32} - \frac{4}{p} > 1 - \alpha. \end{aligned}$$

□

We are going to complete the collection of stability results for SCD space with the "three space property". For this we need the following technical lemma which shows that in Definition 10.3.1 it suffices to consider sets with nonempty interior.

Lemma 10.3.14. *Let X be a separable Banach space. If every* open *convex bounded subset of X is SCD, then X is SCD.*

PROOF. Our first observation is that our hypothesis forces that every bounded convex subset A of X with nonempty interior is SCD. Indeed, notice that since A is convex, the closure of the interior of A coincides with the closure of A, and we may apply Lemma 10.1.6 two times to get that A is SCD.

Now, let $A \subset X$ be bounded and convex. Since X is separable, we may find a countable subset $\{x_n \colon n \in \mathbb{N}\} \subset A$ which is dense in A. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive reals which tends to zero. For fixed $n, m \in \mathbb{N}$, we denote $A_{n,m} = \operatorname{conv}\big(B_{\varepsilon_m}(x_n) \cup A\big)$ which clearly contains A. Since the interior of $A_{n,m}$ is not empty, we may find a determining family $\{S^k_{n,m} \colon k \in \mathbb{N}\}$ of slices of $A_{n,m}$. Now, from the structure of $A_{n,m}$, it follows that either $S^k_{n,m} \cap B_{\varepsilon_m}(x_n) \neq \emptyset$, or $S^k_{n,m} \cap A \neq \emptyset$. Let $K_{n,m}$ be the set of all indices $k \in \mathbb{N}$ for which $S^k_{n,m}$ intersects A, and denote $\widetilde{S}^k_{n,m} = S^k_{n,m} \cap A$ for all $k \in K_{n,m}$, which are clearly slices of A. Also note that for every integer $k \notin K_{n,m}$, the slice $S^k_{n,m}$ intersects $B_{\varepsilon_m}(x_n)$. Finally, the family

$$\left\{\widetilde{S}^k_{n,m} \colon n, m \in \mathbb{N},\ k \in K_{n,m}\right\}$$

is determining for A. Indeed, let B be a subset of A intersecting all the $\widetilde{S}^k_{n,m}$ and fix some $\varepsilon > 0$. Since the set $\{x_n \colon n \in \mathbb{N}\}$ is dense in A, there is an integer $n_0 \in \mathbb{N}$ and $b \in B$ such that $\|b - x_{n_0}\| \leqslant \frac{\varepsilon}{2}$. Also, there is $m_0 \in \mathbb{N}$ such that $\varepsilon_{m_0} \leqslant \frac{\varepsilon}{2}$, as $\varepsilon_m \to 0$ when $m \to \infty$. We know that B intersects all $S^k_{n_0,m_0}$ with $k \in K_{n,m}$. On the other hand, we also know that the slice $S^k_{n_0,m_0}$ intersects the ball $B_{\varepsilon_{m_0}}(x_{n_0})$ for every $k \notin K_{n,m}$. Hence we can deduce that the set $B_{n_0,m_0} = B \cup B_{\varepsilon_{m_0}}(x_{n_0}) \subset A_{n,m}$ intersects all the $S^k_{n_0,m_0}$ which implies that

$$\overline{\operatorname{conv}}\big(B_{n_0,m_0}\big) \supset A_{n_0,m_0} \supset A.$$

Finally, notice that $B_{\varepsilon_{m_0}}(x_{n_0}) \subset B_{\frac{\varepsilon}{2}}(x_{n_0}) \subset B_\varepsilon(b)$, which implies that $B_{n_0,m_0} \subset B + \varepsilon B_X$. Therefore, we can state that $\overline{\operatorname{conv}}\big(B + \varepsilon B_X\big) \supset A$, and the arbitrariness of ε gives us that $\overline{\operatorname{conv}}(B) \supset A$. ☐

We may now state the promised stability result.

THEOREM 10.3.15. *Let X be a Banach space with a subspace Z such that Z and $Y = X/Z$ are SCD spaces. Then, X is also an SCD space.*

PROOF. We denote by $q \colon X \to Y = X/Z$ the quotient map. Let us show that every *open* convex bounded subset $A \subset X$ is SCD, and then Lemma 10.3.14 will imply that X is SCD. To do so, as X is separable since Y and Z are, and separability is a three-space property (see [**77**, Theorem 2.4.h]), we only need to find, for every point $a \in A$, a sequence of weakly open subsets such that whenever $B \subset A$ intersects every member of the sequence, then $a \in \overline{\operatorname{conv}}(B)$ (see Remark 10.1.9). We fix some $a \in A$ and denote $A_a = \{x \in A \colon q(x) = q(a)\}$. Then, A_a is affine isomorphic to an open convex bounded subset of Z which is an SCD space (indeed, $A_a = (Z + a) \cap A$). It follows that there is a determining family $\{S_n \colon n \in \mathbb{N}\}$ of slices of A_a. Let $\{\widetilde{S}_n \colon n \in \mathbb{N}\}$ be their extensions to A. For every $n \in \mathbb{N}$, consider $q(\widetilde{S}_n) \subset Y$, which is open bounded and convex (its openness is a consequence of the Open Mapping Theorem). Now, as long as Y is SCD, we may find a determining sequence $(S_{n,m})_{m\in\mathbb{N}}$ of slices of $q(\widetilde{S}_n)$. Let $V_{n,m} = \widetilde{S}_n \cap q^{-1}(S_{n,m})$ for every $n, m \in \mathbb{N}$. It is easy to see that $V_{n,m}$ are relatively weakly open. We will now prove that they are the sets we need.

Let $B \subset A$ be convex and such that $B \cap V_{n,m} \neq \emptyset$ for all $n, m \in \mathbb{N}$. Fix some $\varepsilon > 0$, and denote $B_\varepsilon = \{x \in A\colon \operatorname{dist}(x, B) < \varepsilon\}$. Evidently, B_ε is an open convex set intersecting all the $V_{n,m}$. Fixed $n \in \mathbb{N}$, we have that

$$B_\varepsilon \cap V_{n,m} = B_\varepsilon \cap \widetilde{S}_n \cap q^{-1}(S_{n,m}) \neq \emptyset,$$

so

$$q\big(B_\varepsilon \cap \widetilde{S}_n\big) \cap S_{n,m} \neq \emptyset$$

and the choice of $S_{n,m}$ allows us to get that

$$\overline{\operatorname{conv}}\big(q(B_\varepsilon \cap \widetilde{S}_n)\big) = \overline{q\big(B_\varepsilon \cap \widetilde{S}_n\big)} \supset q(\widetilde{S}_n).$$

Notice that $B_\varepsilon \cap \widetilde{S}_n$ is open and convex, hence, so is $q(B_\varepsilon \cap \widetilde{S}_n)$. This implies that the interior of the set $\overline{q(B_\varepsilon \cap \widetilde{S}_n)}$ coincides with $q(B_\varepsilon \cap \widetilde{S}_n)$. Now, using that $q(\widetilde{S}_n)$ is open, we get that

$$q\big(B_\varepsilon \cap \widetilde{S}_n\big) \supset q(\widetilde{S}_n)$$

and, in particular, $q(B_\varepsilon \cap \widetilde{S}_n) \ni q(a)$. This means that there exists $x_n \in B_\varepsilon \cap \widetilde{S}_n$ such that $q(x_n) = q(a)$, i.e., that $x_n \in B_\varepsilon \cap S_n$. Since $B_\varepsilon \subset A$ and $\{S_n\colon n \in \mathbb{N}\}$ is a determining family for A_a, we get that $B_\varepsilon \supset A_a$. Finally, the arbitrariness of ε implies that $\overline{B} \supset A_a \ni a$. □

Let us state two immediate consequences of this result.

Corollary 10.3.16. *Let X be a separable Banach space which is not SCD.*

(a) *X contains copies of ℓ_1, and the quotient of X over any copy of ℓ_1 also contains ℓ_1.*

(b) *Consequently, for every ℓ_1-subspace Y_1 of X, there is another ℓ_1-subspace Y_2 such that Y_1 and Y_2 are mutually complemented in the closure of $Y_1 + Y_2$ (i.e., $\overline{Y_1 + Y_2} = Y_1 + Y_2 = Y_1 \oplus Y_2$). In particular, $Y_1 \cap Y_2 = 0$.*

Proof. (a) is immediate from the above theorem and Theorem 10.1.21. (b) follows from (a) and the "lifting" property of ℓ_1 (Theorem 2.3.8). □

One may wonder whether item (b) of the above corollary can actually be a characterisation of those separable Banach spaces which are not SCD. This is not the case as the following remark shows.

Remark 10.3.17. *The space $X = \ell_2(\ell_1)$ (which is an SCD space, even more, it has the RNP) has the following property: it contains isomorphic copies of ℓ_1 and for every ℓ_1-subspace $Y \subset X$, there is another ℓ_1-subspace $Z \subset X$ such that Z and Y are mutually complemented in the closure of $Y + Z$.*

Proof. Let $(X_n)_{n=1}^\infty$ be a sequence of isometric copies of ℓ_1. Then, X is isometric to the ℓ_2 direct sum of the spaces X_n, $\big(\bigoplus_{n\in\mathbb{N}} X_n\big)_{\ell_2}$. Fix an ℓ_1-subspace $Y \subset X$ and let us prove that some of the X_n can be taken as Z. Assume to the contrary that for every $n \in \mathbb{N}$

$$\inf\{\|y - x\|\colon y \in S_Y,\ x \in X_n\} = 0.$$

Then, for every $n \in \mathbb{N}$ there are $y_n \in S_Y$ and $x_n \in X_n$ with $\|y_n - x_n\| < 10^{-n}$. Since (x_n) forms a bounded sequence of disjoint elements, $(x_n) \to 0$ in the weak topology. But then $(y_n) \to 0$ in the weak topology as well, which is impossible since $(y_n) \subset S_Y$ and Y has the Schur property. □

10.4. SCD operators, HSCD operators, and HSCD-dominated operators

Definition 10.4.1. Let X and Y be Banach spaces. A bounded linear operator $T\colon X \to Y$ is said to be an *SCD-operator* if $T(B_X)$ is an SCD set, and it is said to be a *hereditary-SCD-operator* if $T(B_X)$ is a hereditarily SCD set.

By just recalling the examples given in the previous sections, we get the main examples of SCD- and HSCD-operators.

Examples 10.4.2. Let X and Y be Banach spaces and let $T\colon X \to Y$ be a bounded linear operator such that $T(X)$ is separable.

(a) *If $\overline{T(B_X)}$ has small combinations of slices, then T is an SCD-operator.*
(b) *If $\overline{T(B_X)}$ is a strongly regular set (in this case the operator T is also called strongly regular) or $\overline{T(B_X)}$ is a Radon-Nikodým set, (i.e., if T is a strong Radon-Nikodým operator), then T is an HSCD-operator.*
(c) *If $T(B_X)$ does not contain ℓ_1-sequences, then T is an HSCD-operator.*
(d) In particular, *if T does not fix copies of ℓ_1, then T is an HSCD-operator.* Indeed, if $T(B_X)$ contains an ℓ_1-sequence $(Te_n)_{n\in\mathbb{N}}$ with $e_n \in B_X$ $(n \in \mathbb{N})$, then by Lemma 2.3.7, $Y = \overline{\operatorname{lin}}\{e_n\colon n \in \mathbb{N}\}$ is a copy of ℓ_1 and $T|_Y$ is an isomorphic embedding, a contradiction (see [**295**, Proposition 1]).

The aim of this section is to show that SCD- and HSCD-operators behave in a very good way with respect to the Daugavet equation.

First, we need a reformulation of Daugavet centres in terms of behaviour of extreme points and slices of the dual ball.

Remark 10.4.3. *Every weak*-slice of B_{Y^*} intersects the set of extreme points of B_{Y^*}. In other words,* $\operatorname{Slice}(B_{Y^*}, y_0, \varepsilon_0) \cap \operatorname{ext}(B_{Y^*}) \neq \emptyset$ *for every $y_0 \in Y$ and* $\varepsilon_0 > 0$. This follows immediately from the Krein-Milman Theorem 2.6.14.

Definition 10.4.4. For $x^* \in X^*$ and $\varepsilon > 0$, we write

$$S'(x^*, \varepsilon) := \{x \in B_X\colon \operatorname{Re} x^*(x) > 1 - \varepsilon\}.$$

When not empty, $S'(x^*, \varepsilon)$ is a slice of B_X. For Banach spaces X and Y, an operator $G \in S_{L(X,Y)}$, $\varepsilon > 0$, and a slice S of B_X, we denote

$$A(G, S, \varepsilon) := \{y^* \in \operatorname{ext}(B_{Y^*})\colon S \cap S'(G^*y^*, \varepsilon) \neq \emptyset\}.$$

Remark 10.4.5. *The set $A(G, S, \varepsilon)$ can be rewritten as follows:*

$$\begin{aligned} A(G, S, \varepsilon) &= \{y^* \in \operatorname{ext}(B_{Y^*})\colon \exists x \in S \text{ such that } \operatorname{Re} G^*y^*(x) > 1 - \varepsilon\} \\ &= \operatorname{ext}(B_{Y^*}) \cap \bigcup_{x\in S} \{y^* \in Y^*\colon \operatorname{Re} y^*(Gx) > 1 - \varepsilon\}. \end{aligned}$$

This formula demonstrates that $A(G, S, \varepsilon)$ is relatively weak*-open in $\operatorname{ext}(B_{Y^*})$.

Lemma 10.4.6. *For an operator $G \in S_{L(X,Y)}$ the following assertions are equivalent:*

(i) *G is a Daugavet centre.*
(ii) *For every $\varepsilon > 0$, every $x_0^* \in S_{X^*}$ and every weak*-slice* $\operatorname{Slice}(B_{Y^*}, y_0, \varepsilon)$, *$y_0 \in S_Y$ of B_{Y^*} there is a weak*-slice* $\operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1) \subset \operatorname{Slice}(B_{Y^*}, y_0, \varepsilon)$ *such that every $y^* \in \operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1)$ satisfies $\|G^*y^* + x_0^*\| > 2 - \varepsilon$.*

(iii) *For every* $x_0^* \in S_{X^*}$ *and every weak** *slice* $\operatorname{Slice}(B_{Y^*}, y_0, \varepsilon_0)$ *there is* $y^* \in \operatorname{Slice}(B_{Y^*}, y_0, \varepsilon_0)$ *which satisfies the inequality* $\|G^*y^* + x_0^*\| > 2 - \varepsilon_0$.

(iv) *For every* $\varepsilon > 0$*, every* $y \in S_Y$ *and every slice* S *of* B_X *there is* $y^* \in A(G, S, \varepsilon)$ *such that* $y \in \operatorname{Slice}(B_Y, y^*, \varepsilon)$.

(v) *For every* $\varepsilon > 0$ *and every slice* S *of* B_X *the set* $A(G, S, \varepsilon)$ *is weak***-dense in* $\operatorname{ext}(B_{Y^*})$.

(vi) *For every* $\varepsilon > 0$ *and every sequence* $(S_n)_{n\in\mathbb{N}}$ *of slices of* B_X*, the set* $\bigcap_{n\in\mathbb{N}} A(G, S_n, \varepsilon)$ *is weak***-dense in* $\operatorname{ext}(B_{Y^*})$.

Proof. The reformulation (ii) and (iii) are dual versions of characterisations given in Theorem 5.1.2. In order to entertain the reader we will not repeat the previous demonstration of (ii) in a dual form, but will give an alternative proof.

So, let us prove that (i) $\Rightarrow$ (ii). Let $\varepsilon \in (0, 1)$, $x_0^* \in S_{X^*}$ and $\operatorname{Slice}(B_{Y^*}, y_0, \varepsilon)$, $y_0 \in S_Y$ be from the conditions of (ii). Define $T \in L(X, Y)$ by $Tx = x_0^*(x)y_0$. Then $\|G + T\| = 2$, so there exists an element $x_1 \in S_X$ such that $\|Gx_1 + Tx_1\| > 2 - \varepsilon$ and $x_0^*(x_1) > 0$. Put

$$y_1 = \frac{Gx_1 + Tx_1}{\|Gx_1 + Tx_1\|}, \quad \varepsilon_1 = 1 - \frac{2 - \varepsilon}{\|Gx_1 + Tx_1\|}.$$

Then for every $y^* \in \operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1)$ we have

$$\operatorname{Re}\langle y^*, Gx_1 + Tx_1\rangle > (1 - \varepsilon_1)\|Gx_1 + Tx_1\| = 2 - \varepsilon. \tag{10.4.1}$$

Hence

$$1 + \operatorname{Re} y^*(y_0) \geqslant \operatorname{Re} y^*(Gx_1) + \operatorname{Re} y^*(y_0)x_0^*(x_1) = \operatorname{Re}\langle y^*, Gx_1 + Tx_1\rangle > 2 - \varepsilon,$$

which implies that $\operatorname{Re} y^*(y_0) > 1 - \varepsilon$, i.e., $y^* \in \operatorname{Slice}(B_{Y^*}, y_0, \varepsilon)$. So the inclusion $\operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1) \subset \operatorname{Slice}(B_{Y^*}, y_0, \varepsilon)$ is proved. It remains to show that $\|G^*y^* + x_0^*\| > 2 - \varepsilon$. Indeed, the same inequality (10.4.1) implies that

$$\begin{aligned} 2 - \varepsilon &< \operatorname{Re} y^*(Gx_1) + \operatorname{Re} y^*(y_0)x_0^*(x_1) \\ &\leqslant \operatorname{Re} y^*(Gx_1) + x_0^*(x_1) \\ &= \operatorname{Re}\langle G^*y^* + x_0^*, x_1\rangle \leqslant \|G^*y^* + x_0^*\|. \end{aligned}$$

The implication (ii) $\Rightarrow$ (iii) is evident. Let us prove (iii) $\Rightarrow$ (i). Let $T = x_0^* \otimes y_0 \in L(X, Y)$ be a rank-one operator of norm $\|T\| = 1$, $y_0 \in S_Y$, $x_0^* \in S_{X^*}$. Fix a sequence of numbers $\varepsilon_n > 0$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and let $y_n^* \in \operatorname{Slice}(B_{Y^*}, y_0, \varepsilon_n)$ be the corresponding elements from (iii). Then

$$\begin{aligned} 2 &\leqslant \|G^*y_n^* + x_0^*\| + \varepsilon_n \\ &\leqslant \|G^*y_n^* + y_n^*(y_0)x_0^*\| + |1 - y_n^*(y_0)| + \varepsilon_n \\ &= \|(G^* + T^*)y_n^*\| + |1 - y_n^*(y_0)| + \varepsilon_n \\ &\leqslant \|G + T\| + |1 - y_n^*(y_0)| + \varepsilon_n \xrightarrow[n\to\infty]{} \|G + T\|. \end{aligned}$$

(ii) $\Rightarrow$ (iv). Pick $\varepsilon > 0$, $y \in S_Y$ and $S = \operatorname{Slice}(B_X, x_0^*, \delta)$ with $x_0^* \in S_{X^*}$ and $\delta > 0$. Denote $\varepsilon_0 := \min\{\varepsilon, \delta\}$.

According to (ii), there is a weak*-slice $\operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1) \subset \operatorname{Slice}(B_{Y^*}, y, \varepsilon)$ such that every $y^* \in \operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1)$ satisfies $\|G^*y^* + x_0^*\| > 2 - \varepsilon_0/2$.

Proposition 10.4.3 enables us to pick a $y^* \in \operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1) \cap \operatorname{ext}(B_{Y^*})$. Consider the following slice S_1 of B_X:

$$S_1 = \{x \in B_X\colon \operatorname{Re} G^*y^*(x) + \operatorname{Re} x_0^*(x) > \|G^*y^* + x_0^*\| - \varepsilon_0/2\}.$$

Then every $x \in S_1$ fulfills $\operatorname{Re} G^* y^*(x) + \operatorname{Re} x_0^*(x) > 2 - \varepsilon_0$. But $\operatorname{Re} G^* y^*(x) \leqslant 1$ and $\operatorname{Re} x_0^*(x) \leqslant 1$, hence we have

$$\operatorname{Re} G^* y^*(x) > 1 - \varepsilon_0 \geqslant 1 - \varepsilon \quad \text{and} \quad \operatorname{Re} x_0^*(x) > 1 - \varepsilon_0 \geqslant 1 - \delta.$$

This means that $x \in S \cap S'(G^* y^*, \varepsilon)$. Consequently, $y^* \in A(G, S, \varepsilon)$. And, since $y^* \in \operatorname{Slice}(B_{Y^*}, y_1, \varepsilon_1) \subset \operatorname{Slice}(B_{Y^*}, y, \varepsilon)$, then $\operatorname{Re} y^*(y) > 1 - \varepsilon$, hence $y \in \operatorname{Slice}(B_Y, y^*, \varepsilon)$.

(iv) $\Rightarrow$ (i). Pick $\varepsilon > 0$, $y \in S_Y$ and $x^* \in S_{X^*}$. Then there is $y^* \in A(G, \operatorname{Slice}(B_X, x^*, \varepsilon), \varepsilon/2)$ such that $y \in \operatorname{Slice}(B_Y, y^*, \varepsilon/2)$. By the definition of the set $A(G, \operatorname{Slice}(B_X, x^*, \varepsilon), \varepsilon/2)$, there exist an $x \in \operatorname{Slice}(B_X, x^*, \varepsilon)$ such that $y^*(Gx) = (G^* y^*)(x) > 1 - \varepsilon/2$ and so

$$\|Gx + y\| \geqslant |y^*(Gx) + y^*(y)| > 1 - \varepsilon/2 + 1 - \varepsilon/2 = 2 - \varepsilon.$$

Then, according to the basic characterisation given in (iii) of Theorem 5.1.2, G is a Daugavet centre.

(iv) $\Rightarrow$ (v). We need to show that the weak* closure of $A(G, S, \varepsilon)$ contains every extreme point y^* of B_{Y^*}. Since weak*-slices form a base of neighbourhoods of extreme points in B_{Y^*}, it is sufficient to demonstrate that every weak*-slice $\operatorname{Slice}(B_{Y^*}, y, \delta)$ of B_{Y^*} with $\delta \in (0, \varepsilon)$ intersects $A(G, S, \varepsilon)$, i.e., that there is a point $y^* \in A(G, S, \varepsilon)$ such that $y^* \in \operatorname{Slice}(B_{Y^*}, y, \delta)$. But we know that there is a point $y^* \in A(G, S, \delta) \subset A(G, S, \varepsilon)$ such that $y \in \operatorname{Slice}(B_Y, y^*, \delta)$ which means that $y^* \in \operatorname{Slice}(B_{Y^*}, y, \delta)$.

(v) $\Rightarrow$ (iv). If $A(G, S, \varepsilon)$ is weak*-dense in $\operatorname{ext}(B_{Y^*})$, then by Proposition 10.4.3 for every $y \in S_Y$ the weak*-slice $\operatorname{Slice}(B_{Y^*}, y, \varepsilon)$ intersects $A(G, S, \varepsilon)$. Therefore there is $y^* \in A(G, S, \varepsilon)$ such that $y \in \operatorname{Slice}(B_Y, y^*, \varepsilon)$.

Since the $A(G, S_n, \varepsilon)$ are weak*-dense and weak*-open, the remaining equivalence (v) $\Leftrightarrow$ (vi) follows from the Baire property of $(\operatorname{ext}(B_{Y^*}), w^*)$ (see Lemma 2.6.17). □

THEOREM 10.4.7. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, E be a Banach space, and let $T\colon X \to E$ be an SCD-operator. Then T is a G-strong Daugavet operator.*

PROOF. Since T is an SCD-operator, we may find a sequence $(S_n)_{n\in\mathbb{N}}$ of slices of $T(B_X)$ such that $T(B_X) \subset \overline{\operatorname{conv}}(B)$ whenever $B \subset T(B_X)$ intersects all the S_n's. Notice that the sets $\hat{S}_n := T^{-1}(S_n) \cap B_X$ are slices of B_X (Proposition 2.6.5). We are going to demonstrate that $T \in \mathcal{SD}_G(X)$ using Definition 6.2.1. So, fix arbitrary $\varepsilon > 0$, $x \in S_X$, and $y \in S_Y$. Since G is a Daugavet centre, Lemma 10.4.6 gives us that $\bigcap_{n\in\mathbb{N}} A(G, \hat{S}_n, \varepsilon/2)$ is weak*-dense in $\operatorname{ext}(B_{Y^*})$. Proposition 10.4.3 implies that we may find $y^* \in \bigcap_{n\in\mathbb{N}} A(G, \hat{S}_n, \varepsilon/2)$ such that

$$y \in \operatorname{Slice}(B_Y, y^*, \varepsilon/2). \tag{10.4.2}$$

Then, by the definition of $A(G, \hat{S}_n, \varepsilon/2)$, we have that $S'(G^* y^*, \varepsilon/2) \cap T^{-1}(S_n) \neq \emptyset$ for every $n \in \mathbb{N}$. Thus,

$$T(S'(G^* y^*, \varepsilon/2)) \cap S_n \neq \emptyset$$

for every $n \in \mathbb{N}$. Then

$$T(B_X) \subset \overline{\operatorname{conv}}\big(T(S'(G^* y^*, \varepsilon/2))\big) = \overline{T(S'(G^* y^*, \varepsilon/2))}.$$

In particular, $Tx \in \overline{T(S'(G^* y^*, \varepsilon/2))}$, which implies that there is $z \in S'(G^* y^*, \varepsilon/2)$ such that

$$\|Tx - Tz\| < \varepsilon.$$

We have $\operatorname{Re} y^*(Gz) > 1 - \varepsilon/2$. By (10.4.2) we also have $\operatorname{Re} y^*(y) > 1 - \varepsilon/2$. Therefore

$$\|y + Gz\| \geqslant \operatorname{Re} y^*(y) + \operatorname{Re} y^*(Gz) > 1 - \varepsilon/2 + 1 - \varepsilon/2 = 2 - \varepsilon.$$

Thus T is a G-strong Daugavet operator by Definition 6.2.1. □

COROLLARY 10.4.8. *Let $G\colon X \to Y$ be a Daugavet centre. If $T\colon X \to Y$ is an SCD-operator, then $\|G + T\| = \|G\| + \|T\|$.*

PROPOSITION 10.4.9. *Let X be a Banach space and $T_i\colon X \to Y_i$, $i = 1, 2$, be hereditarily SCD operators. Then $T_1 \tilde{+} T_2$ is also a hereditarily SCD operator.*

PROOF. We know that the sets $T_i(B_X)$, $i = 1, 2$, are hereditarily SCD. Then according to Theorem 10.2.8, $T_1(B_X) \oplus T_2(B_X)$ is also hereditarily SCD. All that is left now is to notice that $T_1 \tilde{+} T_2\colon X \to Y_1 \oplus_1 Y_2$ acts by the rule $(T_1 \tilde{+} T_2)(x) = (T_1(x), T_2(x))$; so $(T_1 \tilde{+} T_2)(B_X) \subset T_1(B_X) \oplus T_2(B_X)$. Since every subset of an HSCD set is also HSCD, this gives the desired result. □

The previous Proposition implies in particular that for every HSCD operator $T\colon X \to Y$ and every $x^* \in X^*$ the operator $T \tilde{+} x^*$ is an HSCD-operator. Combining this with Theorem 10.4.7 and with the definition of G-narrow operators we deduce the following:

THEOREM 10.4.10. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, E be a Banach space, and let $T\colon X \to E$ be an HSCD-operator. Then T is G-narrow.*

The following proposition enables us, using the separable determination Theorem 6.2.21 for G-narrow operators, to relax the separability restriction that appears implicitly in the previous theorem.

COROLLARY 10.4.11. *Let $G \in S_{L(X,Y)}$ be a Daugavet centre, E a Banach space, and let $T \in L(X, E)$ be an operator such that every separable convex subset of $T(B_X)$ is SCD. Then T is G-narrow.*

COROLLARY 10.4.12. *For a Daugavet centre $G \in S_{L(X,Y)}$, all strongly regular operators, weakly compact operators, operators not fixing a copy of ℓ_1, and strong Radon-Nikodým operators on X are G-narrow.*

REMARKS 10.4.13.

(a) *The class of hereditary-SCD-operators is a right operator ideal.* Indeed, if $T\colon X_1 \to X_2$ is an arbitrary operator and $S\colon X_2 \to X_3$ is a hereditary-SCD-operator, then $[ST](B_{X_1}) \subset S(\|T\| B_{X_2})$, so ST is a hereditary-SCD-operator.

(b) *The class of hereditary-SCD-operators is not a left operator ideal.* Indeed, let us consider a subspace $X \subset L_1[0,1]$ which is 1-complemented and isometric to ℓ_1 (for example, such X may be obtained as the linear span of the set $\{\mathbb{1}_{(\frac{1}{n+1}, \frac{1}{n})\colon n \in \mathbb{N}}\}$). The corresponding norm-one projection $T\colon L_1[0,1] \to X \cong \ell_1$ is a hereditary-SCD-operator since ℓ_1 has the RNP. Consider also a quotient map $S\colon X \to X/Y \cong L_1[0,1]$ (by just using the quotient universality of ℓ_1, Theorem 2.5.9). Then, $ST(B_{L_1[0,1]}) = B_{L_1[0,1]}$ so ST is not even an SCD-operator.

(c) As a consequence, *there are narrow operators which are not SCD-operators.* Indeed, since the set of narrow operators is clearly a left operator ideal, the operator ST above is narrow.

(d) *In Theorem 10.6.1 below we will show that the set of HSCD operators between two Banach spaces need not be a linear subspace.*

The following concept of HSCD-dominated operators helps to fix some drawbacks of HSCD operators mentioned above.

DEFINITION 10.4.14. An operator T on a Banach space X is said to be *HSCD-dominated* if there exists a hereditarily SCD operator $\tilde{T}$ on X such that T is dominated by $\tilde{T}$, i.e., $\|T(x)\| \leqslant \|\tilde{T}(x)\|$ for every $x \in X$.

REMARK 10.4.15. *The class of HSCD-dominated operators, of course, contains all hereditarily SCD operators and, in particular, in the separable case it contains the classes of strongly regular operators, strong Radon-Nikodým operators, and the class of operators which do not fix a copy of ℓ_1.*

REMARK 10.4.16. *If $G \in S_{L(X,Y)}$ is a Daugavet centre, then the class of HSCD-dominated operators on X is contained in the class of G-narrow operators on X. This follows from Theorem 10.4.10 and from the evident observation that an operator dominated by a G-narrow operator is itself G-narrow.*

Now, we are ready to prove the following theorem.

THEOREM 10.4.17. *The class of HSCD-dominated operators is an order ideal (in the sense of Definition 6.1.2) which is a $\tilde{+}$-semigroup and a two-sided operator ideal.*

PROOF. The order ideal property follows immediately from the definition. Let us prove that this class is stable under $\tilde{+}$. Let T_1 and T_2 be two operators from this class. Then there exist hereditarily SCD operators $\widetilde{T_1}$ and $\widetilde{T_2}$ such that T_i is dominated by $\widetilde{T_i}$, $i = 1, 2$. Proposition 10.4.9 gives us that $\widetilde{T_1} \tilde{+} \widetilde{T_2}$ is also a hereditarily SCD operator. Now, $T_1 \tilde{+} T_2$ is dominated by $\widetilde{T_1} \tilde{+} \widetilde{T_2}$, and the job is done.

The stability under the ordinary sum is a simple consequence of the $\tilde{+}$-stability and the order ideal property, because $T_1 + T_2$ is clearly dominated by $T_1 \tilde{+} T_2$.

Now, we'll prove that it is a two-sided operator ideal. Let $T\colon X \to Y$ be an HSCD-dominated operator and $U\colon Y \to Z$, $V\colon F \to X$ be two arbitrary bounded operators. We want to prove that $U \circ T$ and $T \circ V$ are HSCD-dominated operators. For this we need to find hereditarily SCD operators T_1 and T_2 such that T_1 dominates $U \circ T$ and T_2 dominates $T \circ V$. Let $\widetilde{T}$ be a hereditarily SCD operator that dominates T. Then we can take $T_1 = \|U\|\widetilde{T}$ and $T_2 = \widetilde{T} \circ V$. These operators are hereditarily SCD and it is easy to see that T_1 dominates $U \circ T$ and T_2 dominates $T \circ V$. □

The above theorem combined with a separable determination argument enables us to extend Corollary 10.4.12 to linear combinations of the operators considered there.

COROLLARY 10.4.18. *For a Daugavet centre $G \in S_{L(X,Y)}$, all strongly regular operators, weakly compact operators, operators not fixing a copy of ℓ_1, and strong Radon-Nikodým operators on X, as well as their linear combinations, are G-narrow.*

REMARK 10.4.19. *The part of Corollary 10.4.18 that deals with strong Radon-Nikodým operators and operators not fixing a copy of ℓ_1 can be deduced from Theorems 6.2.18 and 6.3.5 together with the properties of $\operatorname{cp}(\mathcal{NAR}_G(X))$. We don't* *know whether, for a Daugavet centre $G \in S_{L(X,Y)}$, all the HSCD-dominated operators on X belong to* $\operatorname{cp}(\mathcal{NAR}_G(X))$*. In particular, to the best of our knowledge, nobody considered strongly regular operators in the* $\operatorname{cp}(\mathcal{NAR}_G(X))$ *setting.*

10.5. Sets with a countable π-base of the relative weak topology

It was shown in Proposition 10.1.20 that a convex bounded subset A of a Banach space X is SCD if $\sigma_A(X, X^*)$ (the restriction of $\sigma(X, X^*)$ to A) has a countable π-base. We do not know whether these two properties of convex bounded sets are equivalent (see Question (10.1) in Section 10.9). The aim of this section is to discuss this possible equivalence. We already know that the class of convex bounded sets that have a countable π-base of the weak topology contains those sets which do not have ℓ_1-sequences (Theorem 10.1.21). Below, in Examples 10.5.1, we list without proofs a series of results from [**27**, Section 6] which can be summarised as follows: this class of sets contains the main examples of SCD sets mentioned in this book. After that we present some characterisations of SCD sets which remind us of the property we are dealing with.

EXAMPLES 10.5.1.

(1) Let X be a separable Banach space with a LUR norm. Then, B_X has a countable π-base of the weak topology.
(2) As a consequence, every separable Banach space X admits an equivalent norm $|\cdot|$ such that $B_{(X,|\cdot|)}$ has a countable π-base of the weak topology.
(3) Let X be a Banach space and let A be a separable closed convex bounded subset of X with the CPCP. Then, A has a countable π-base for the weak topology.
(4) Every bounded convex subset A of the space $c_0(\ell_1)$ has a countable π-base of the weak topology.
(5) Every bounded convex subset A of the space $\ell_1(c_0)$ has a countable π-base of the weak topology.

With the above result, most of the types of SCD sets presented in Section 10.1 have a countable π-base of the weak topology. The only exception is the family of strongly regular sets which are not CPCP. There are two main examples of sets of this kind, but in both cases, the sets have a countable π-base of the weak topology.

EXAMPLES 10.5.2.

(a) The set constructed by S. Argyros, E. Odell, and H. P. Rosenthal [**25**], which is strongly regular but does not have the CPCP, is a subset of c_0, so it has a countable π-base of the weak topology since it does not have ℓ_1-sequences.
(b) The set constructed by W. Schachermayer [**281**], which is a subset C of a Banach space Z which does not have the CPCP, but Z^{**} is strongly regular (so Z is strongly regular). But then, $(C, \sigma(X, X^*))$ has a countable π-base of the weak topology since Z does not contain ℓ_1.

Our next goal is to establish some characterisations of SCD sets which remind us of countable π-bases of the weak topology. The first one deals with convex combinations of slices.

THEOREM 10.5.3. *A bounded convex subset A of a Banach space X is an SCD set if and only if there is a countable family $\{V_n \colon n \in \mathbb{N}\}$ of convex combinations of* *slices of A such that every relatively weakly open subset of A contains some of the V_n.*

PROOF. The "if" part is direct consequence of Propositions 10.1.2 and 10.1.16.

Conversely, assume that A is an SCD set and suppose without loss of generality that $A \subset B_X$. Let $S_n = \mathrm{Slice}(A, x_n^*, \varepsilon_n)$, for $n \in \mathbb{N}$, be a determining sequence of slices for A. Let us show that the convex combinations of the S_n's with rational coefficients form the countable collection of convex combinations of slices that we need. Indeed, let U be a relatively weakly open subset of A. Select another relatively weakly open subset $V \subset U$ such that $\alpha = \mathrm{dist}(V, A \setminus U) > 0$. Due to Bourgain's lemma (Lemma 2.6.19), there is a convex combination of slices $\sum_{j=1}^m \lambda_j G_j \subset V$. According to Proposition 10.1.2, for every $j = 1, 2, \ldots, m$ there is $n(j) \in \mathbb{N}$ such that $S_{n(j)} \subset G_j$. Then, $\sum_{j=1}^m \lambda_j S_{n(j)} \subset V$. What remains is to find rationals $\mu_j > 0$ with $\sum_{j=1}^m \mu_j = 1$ and $|\mu_j - \lambda_j| < \alpha$. Then, the Hausdorff distance between $\sum_{j=1}^m \mu_j S_{n(j)}$ and $\sum_{j=1}^m \lambda_j S_{n(j)}$ is smaller than α, so

$$\sum_{j=1}^{m} \mu_j S_{n(j)} \subset V + \alpha B_X \subset U. \qquad \square$$

Let us recall that convex combinations of slices may have relative empty interior. Actually, there are convex combinations of slices of the unit ball of infinite-dimensional spaces which do not intersect the unit sphere. We refer the reader to [**210**] for a discussion of this.

The second result gives a reformulation of an SCD set in terms of topological properties of the set of extreme points of its weak* closure in the bidual. For a convex bounded subset A of a Banach space X, denote $\overline{A}^{**}$ the weak-star closure of A in X^{**}.

THEOREM 10.5.4. *Let X be a Banach space and let A be a convex bounded subset of X. Put $W = \Big(\mathrm{ext}\big(\overline{A}^{**}\big), \sigma(X^{**}, X^*)\Big)$. Then, the following are equivalent:*

(i) *A is an SCD set.*
(ii) *W has a countable π-base.*

PROOF. (i) $\Rightarrow$ (ii). We take a sequence of slices $S_n = \mathrm{Slice}(A, x_n^*, \varepsilon_n)$, $n \in \mathbb{N}$, which is determining for A and we write

$$S_n^{**} = \mathrm{Slice}\big(\overline{A}^{**}, x_n^*, \varepsilon_n\big) \subset \overline{A}^{**}$$

for the natural extensions of S_n to slices of $\overline{A}^{**}$. Then, the family $U_n = S_n^{**} \cap W$ for $n \in \mathbb{N}$ forms a π-base of W. Indeed, we consider a relatively weak*-open subset U of W. Due to Choquet's lemma 2.6.16, there is a slice $S^{**} = \mathrm{Slice}\big(\overline{A}^{**}, x^*, \varepsilon\big)$ of $\overline{A}^{**}$ generated by some $x^* \in X^*$ and $\varepsilon > 0$ such that $U \supset S^{**} \cap W \neq \emptyset$. Now, according to Proposition 10.1.2, there is $n \in \mathbb{N}$ such that

$$S_n \subset \mathrm{Slice}(A, x^*, \varepsilon/2) \subset S\big(\overline{A}^{**}, x^*, \varepsilon/2\big).$$

Then, S_n^{**} is contained in the relative weak*-closure of $\mathrm{Slice}\big(\overline{A}^{**}, x^*, \varepsilon/2\big)$ in $\overline{A}^{**}$, so $S_n^{**} \subset S^{**}$ and

$$U_n = S_n^{**} \cap W \subset S^{**} \cap W \subset U.$$

(ii) $\Rightarrow$ (i). We consider a countable π-base $\{U_n \colon n \in \mathbb{N}\}$ of W consisting of relatively weak*-star open subsets. Again by Choquet's lemma, there are $x_n^* \in X^*$

and $\varepsilon_n > 0$ such that

$$U_n \supset \widetilde{U_n} = \mathrm{Slice}\big(\overline{A}^{**}, x_n^*, \varepsilon_n\big) \cap W \neq \emptyset.$$

Let us prove that the slices $S_{n,m} = \mathrm{Slice}(A, x_n^*, 1/m)$, with $n, m \in \mathbb{N}$, form a determining sequence for A. Indeed, we denote $S_{n,m}^{**}$ the *closed* slices of $\overline{A}^{**}$ generated by x_n^* and $1/m$. For every slice $S = \mathrm{Slice}(A, x^*, \varepsilon)$ of A, since $\{\widetilde{U_n}\colon n \in \mathbb{N}\}$ is a π-base of W, there is $n \in \mathbb{N}$ such that

$$S^{**} \cap W \supset \widetilde{U_n} \qquad \text{where } S^{**} = \mathrm{Slice}\big(\overline{A}^{**}, x^*, \varepsilon_n\big),$$

so for $m \in \mathbb{N}$ big enough we have

$$S^{**} \cap W \supset S_{n,m}^{**} \cap W.$$

Then, taking into account that, for every $n \in \mathbb{N}$,

$$G_n = \bigcap_{m \in \mathbb{N}} S_{n,m}^{**}$$

is a closed face of $\overline{A}^{**}$, the Krein-Milman theorem gives us that

$$G_n = \overline{\mathrm{conv}\big(G_n \cap W\big)}^{\sigma(X^{**}, X^*)}.$$

Therefore,

$$S^{**} \supset \overline{\mathrm{conv}\big(S^{**} \cap W\big)}^{\sigma(X^{**}, X^*)} \supset \overline{\mathrm{conv}\left(\bigcap_{m \in \mathbb{N}} S_{n,m}^{**} \cap W\right)}^{\sigma(X^{**}, X^*)} = G_n.$$

This means that the intersection of the decreasing sequence of $\sigma(X^{**}, X^*)$ compact sets $(S_{n,m}^{**})_{m \in \mathbb{N}}$ is contained in S^{**}. But S^{**} is a relatively $\sigma(X^{**}, X^*)$ open set in $\overline{A}^{**}$, so for sufficiently big $m \in \mathbb{N}$, all the $S_{n,m}^{**}$ are subsets of S^{**}. For these m, we have

$$S = S^{**} \cap A \supset S_{n,m}^{**} \cap A \supset S_{n,m}.$$

Finally, we use the characterisation of SCD sets from Proposition 10.1.2. □

The following is an easy consequence of the above result.

COROLLARY 10.5.5. *Let X be a Banach space and let A be a bounded convex subset of X. If A is SCD, then $\Big(\mathrm{ext}\big(\overline{A}^{**}\big), \sigma(X^{**}, X^*)\Big)$ is separable.*

10.6. Sums of SCD operators and sets

In Proposition 10.4.9 we proved that the class of hereditarily SCD operators is stable under the operation $\tilde{+}$ and in Proposition 10.4.13(d) we promised to demonstrate later that the result is not valid for the ordinary sum. Now, it is time to fulfill that promise.

THEOREM 10.6.1. *There exist Banach spaces X, Y and two hereditarily SCD operators $T_1, T_2 \in L(X, Y)$ whose sum $T_1 + T_2$ is not even an SCD operator.*

PROOF. In the required example below, we use $X = \ell_1$ and $Y = \ell_1 \oplus_1 C[0,1]$.

We know that if a Banach space has the Daugavet property, then its unit ball is not SCD. In particular $B_{C[0,1]}$ is not an SCD set. Let $T\colon \ell_1 \to C[0,1]$ be a bounded linear operator of norm 1 such that $\overline{T(B_{\ell_1})} = B_{C[0,1]}$ (such a T exists because of the separability of $C[0,1]$ and the quotient universality of ℓ_1, see Theorem 2.5.9).

We then take $T_i\colon \ell_1 \to \ell_1 \oplus_1 C[0,1]$, $i = 1,2$, such that $T_1(x) = (x, \frac{T(x)}{2})$ and $T_2(x) = (-x, \frac{T(x)}{2})$ for all $x \in \ell_1$. Both of these operators are continuous and bounded from below. Indeed, for every $x \in \ell_1$

$$\frac{\|x\|}{2} \leqslant \|x\| - \frac{\|T(x)\|}{2} \leqslant \left\| \left(x, \frac{T(x)}{2} \right) \right\| = \|T_1(x)\| \leqslant \frac{3}{2}\|x\|$$

and analogously for T_2. Thus T_i is an isomorphism between ℓ_1 and $T_i(\ell_1)$, $i = 1,2$. But we know that ℓ_1 is an SCD space and that the SCD-property is stable under isomorphisms, so we can deduce that the spaces $T_1(\ell_1)$ and $T_2(\ell_1)$ are SCD. From this it evidently follows that T_1 and T_2 are hereditarily SCD operators. Now, consider their sum: $(T_1 + T_2)(x) = (0, T(x))$. But then the closure of $(T_1 + T_2)(B_{\ell_1}) = (0, T(B_{\ell_1}))$ is $\{0\} \times B_{C[0,1]}$ by the choice of T and hence $T_1 + T_2$ is not an SCD operator. □

REMARK 10.6.2. *Note that the same construction works also to give an example of two strong Radon-Nikodým operators whose sum doesn't have this property, because ℓ_1 is a Radon-Nikodým space and $C[0,1]$ is not.*

Also from our example we deduce the following corollary.

COROLLARY 10.6.3. *The sum of two hereditarily SCD sets need not be an HSCD set.*

PROOF. As an example we can take the sets $T_1(B_{\ell_1})$ and $T_2(B_{\ell_1})$ from the previous theorem. These sets are hereditarily SCD but their sum contains a set isometric to the unit ball of $C[0,1]$, which is not SCD, which means that this sum is not HSCD. □

10.7. More operations with SCD sets

The content of this section is a kind of warning that one has to be careful when dealing with SCD sets. Namely, we demonstrate that the class of SCD sets is not stable with respect to the main elementary operations like the union, intersection and the Minkowski sum.

The section consists of three subsections. At the beginning of Subsection 10.7.1 we construct a set A whose properties will be the base of all the remaining examples (the letter A will be fixed afterwards for that special set). Then, we present the promised examples for the intersection of SCD sets (which will be A and $-A$). After that, in Subsection 10.7.2 we give the instability examples for the Minkowski sum (which will again be A and $-A$), and for the union (some shifts of A and $-A$). In fact, we demonstrate the existence of such examples in every space with the Daugavet property. The examples constructed in the initial subsections are not centrally symmetric, which is not entirely satisfactory, because in all the applications sets symmetric with respect to zero appear. Subsection 10.7.3 is devoted to the symmetrisation of our examples, after which one can see that the operations of Minkowski sum, union and intersection do not preserve the property SCD even if the sets in question are unit balls of some equivalent norms.

Since in the definition of a slice and, consequently, in the definition of an SCD space only real scalars are used, below, if the contrary is not stated explicitly, we will consider only *real* Banach spaces.

10.7.1. The intersection of SCD sets. The examples which we are going to present in this section will be constructed in an arbitrary Banach space X with the Daugavet property. According to Theorem 4.1.7, X contains a separable subspace with the Daugavet property, so without loss of generality we assume that X itself is separable. Fix a 1-codimensional closed subspace $E \subset X$. E, being a rich subspace, also has the Daugavet property, so B_E is not SCD. The aim of the construction below is to include B_E into an SCD set $A \subset X$ in such a way that B_E lies in the boundary of A. This construction will be used in all the examples presented in this section.

The separability of E gives the existence of an equivalent LUR norm $\varphi\colon E \to [0, +\infty)$ such that $\frac{1}{2}\|x\| \leqslant \varphi(x) \leqslant \|x\|$ for all $x \in E$ (Theorem 2.10.4). Then for every $t > 0$ the formula $\|x\|_t = \sqrt{\|x\|^2 + t^2\varphi(x)^2}$, $x \in E$, defines an equivalent LUR norm on E [**94**, Chapter 2, p. 52, beginning of Section 4] satisfying that

$$\|x\| \leqslant \|x\|_t \leqslant \sqrt{1+t^2}\,\|x\|. \tag{10.7.1}$$

In particular, every point of the unit sphere $S_{(E,\|\cdot\|_t)}$ is strongly exposed (Proposition 2.10.3). If $t = 0$, then we get the original norm on E, i.e., $\|x\|_0 = \|x\|$. We are going to use the notation $\|\cdot\|_t^*$ for the norm of $(E, \|\cdot\|_t)^*$. In the case of $t = 0$, where $\|\cdot\|_0$ is just the original norm $\|\cdot\|$, we will write $\|y^*\|_0^* = \|y^*\|$.

We now construct the set which plays the fundamental role in all the promised counterexamples. Let $e_0 \in X \setminus E$ be a fixed element with $\|e_0\| = 1$. Then $X = E \oplus \operatorname{lin} e_0$. In the sequel we will use the notation $x \oplus t$ in order to denote an element of the form $x + te_0$, where $x \in E$, $t \in \mathbb{R}$. We will also consider the following equivalent norm on X: $\|x \oplus t\|_\infty = \max\{\|x\|, |t|\}$. Remark that the dual space to our $X = E \oplus \operatorname{lin} e_0$ can be represented as the set of formal expressions $y^* \oplus \lambda$, $y^* \in E^*$, $\lambda \in \mathbb{R}$, that act on elements of X by the natural rule $\langle y^* \oplus \lambda, x \oplus t\rangle = y^*(x) + \lambda t$.

Proposition 10.7.1. *The subset*

$$A = \{x \oplus t \in X\colon \|x\|_t^2 + 3t^2 \leqslant 1,\ t \geqslant 0\} \subset X \tag{10.7.2}$$

has the following properties:

(a) *Every element $x \oplus t \in A$ satisfies $t \in \left[0, \frac{1}{\sqrt{3}}\right]$ and $\|x\| \leqslant \sqrt{1-3t^2}$, in particular A is bounded.*
(b) *Every element $x \oplus t \in X$ satisfying $t \in \left[0, \frac{1}{\sqrt{3}}\right]$ and $\|x\| \leqslant \sqrt{\frac{1-3t^2}{1+t^2}}$ belongs to A.*
(c) *A is closed.*
(d) *A is convex.*
(e) *A is SCD.*

Proof. Conditions (a) and (b) follow immediately from (10.7.2) and (10.7.1). (c) follows from the continuity of the map $x \oplus t \mapsto \|x\|_t$. To check (d), that A is convex, note that the set can be rewritten as

$$A = \{x \oplus t \in X\colon H(\|x\|, \varphi(x), t) \leqslant 1\} \cap \{x \oplus t \in X\colon t \geqslant 0\}$$

where $H(r, s, t) := r^2 + t^2 s^2 + 3t^2$. H is a convex function on $[0, 1]^3$, indeed its Hessian matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2t^2 & 4ts \\ 0 & 4ts & 6+2s^2 \end{pmatrix}$$

is positive definite on $(0,1)^3$, since the determinants of its principal minors are all positive on this domain: $\Delta_1 = 2$, $\Delta_2 = 4t^2$ and $\Delta_3 = 12t^2(1-s^2)$. Furthermore, H is nondecreasing in each variable when considered defined on $[0,1]^3$, so for $x_i \oplus t_i \in A$ $(i = 1,2)$ and $0 \leqslant \lambda \leqslant 1$ we have that

$$\begin{aligned}
H\big(&\|\lambda x_1 + (1-\lambda)x_2\|, \varphi(\lambda x_1 + (1-\lambda)x_2), \lambda t_1 + (1-\lambda)t_2\big) \\
&\leqslant H\big(\lambda\|x_1\| + (1-\lambda)\|x_2\|, \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2), \lambda t_1 + (1-\lambda)t_2\big) \\
&\leqslant \lambda\, H\,(\|x_1\|, \varphi(x_1), t_1) + (1-\lambda)\, H\,(\|x_2\|, \varphi(x_2), t_2) \leqslant 1 - \lambda + \lambda = 1.
\end{aligned}$$

Therefore, $\lambda(x_1 \oplus t_1) + (1-\lambda)(x_2 \oplus t_2) \in A$.

We finally prove (e), that A is an SCD set, by showing that it is huskable (see Definition 10.1.17, Theorem 10.1.18 and the short paragraph between them). To this end, denote

$$\tilde{A} = \{x \oplus t \in A\colon 0 < t < 1/\sqrt{3},\ \|x\|_t^2 = 1 - 3t^2\}.$$

If we demonstrate that each element of $\tilde{A}$ is a point of continuity of A, then by the evident identity $\overline{\operatorname{conv}}(\tilde{A}) = A$ we obtain that A is huskable. So, it remains to demonstrate the following claim:

Claim. For every $\varepsilon > 0$ and every $x_0 \oplus t_0 \in \tilde{A}$ there is a relatively weakly open subset of A containing $x_0 \oplus t_0$ with $\|\cdot\|_\infty$-diameter not greater than 4ε.

Let us do this. Denote $r_0 := (1 - 3t_0^2)^{1/2} = \|x_0\|_{t_0}$. Since x_0 is a strongly exposed point of $r_0 B_{(E,\|\cdot\|_{t_0})}$, there exist $x_0^* \in S_{(E^*,\|\cdot\|_{t_0}^*)}$ and $\beta_0 \in (0,1)$ satisfying:

(i) $x_0^*(x_0) = r_0$.
(ii) If $x \in r_0 B_{(E,\|\cdot\|_{t_0})}$ and $x_0^*(x) > \beta_0 r_0$, then $\|x - x_0\| < \varepsilon$.

Take $\delta > 0$ small enough so that

$$x_0^*(x_0) > \beta_0(r_0 + 2\delta) \quad \text{and} \quad \frac{2\delta}{2\delta + r_0} + \frac{\delta^2}{2} < \varepsilon. \tag{10.7.3}$$

We will show that

$$W := \{x \oplus t \in A\colon x_0^*(x) > \beta_0(r_0 + 2\delta),\ |t - t_0| < \delta^2/2\}$$

is the relatively weakly open subset of A we are looking for. It is immediate that $x_0 \oplus t_0 \in W$. Furthermore, given $x \oplus t \in W$ we have that $|t^2 - t_0^2| < \delta^2$ and hence

$$\begin{aligned}
\|x\|_{t_0} &= \big(\|x\|^2 + t_0^2\varphi(x)\big)^{1/2} = \big(\|x\|_t^2 + (t_0^2 - t^2)\varphi(x)\big)^{1/2} \\
&\leqslant \big(\|x\|_t^2 + |t_0^2 - t^2|\big)^{1/2} \leqslant \big(1 - 3t^2 + |t_0^2 - t^2|\big)^{1/2} \\
&\leqslant \big(1 - 3t_0^2 + 4\delta^2\big)^{1/2} \leqslant r_0 + 2\delta.
\end{aligned}$$

The last inequality together with (10.7.3) gives

$$\left\|\frac{r_0\, x}{r_0 + 2\delta}\right\|_{t_0} \leqslant r_0 \quad \text{and} \quad x_0^*\left(\frac{r_0\, x}{r_0 + 2\delta}\right) > \beta_0 r_0.$$

By (ii) it follows that

$$\varepsilon > \left\|\frac{r_0\, x}{r_0 + 2\delta} - x_0\right\| \geqslant \|x - x_0\| - \|x\|\frac{2\delta}{r_0 + 2\delta} \geqslant \|x - x_0\| - \frac{2\delta}{r_0 + 2\delta}$$

and, therefore,

$$\|x \oplus t - x_0 \oplus t_0\|_\infty = \max\{\|x - x_0\|, |t - t_0|\}$$
$$< \max\left\{\varepsilon + \frac{2\delta}{2\delta + r_0}, \frac{\delta^2}{2}\right\} < 2\varepsilon.$$

Then the diameter of W does not exceed 4ε, which completes the proof of the claim above. □

Remark also that A in the above Proposition has two more evident properties: it has non-empty interior, and for every $x \oplus t \in A$ also $(-x) \oplus t \in A$.

THEOREM 10.7.2. *In every Banach space X with the Daugavet property there are closed convex bounded SCD sets $A, D \subset X$ whose intersection $A \cap D$ is not SCD.*

PROOF. Let A and E be as in Proposition 10.7.1, and let $D = -A$. Both sets are SCD by Proposition 10.7.1 although $A \cap D = B_E$ is not. □

10.7.2. Sum and union of SCD sets. In Corollary 10.6.3 we demonstrated that the Minkowski sum $U+V$ of two hereditarily SCD sets need not be hereditarily SCD. One may check that $\overline{U+V}$ in the example from Corollary 10.6.3 is the closed convex hull of its strongly exposed points, so $U+V$ is SCD. In this subsection we complement the above result, demonstrating that the Minkowski sum of two SCD sets need not be SCD. We also give an analogous negative result about unions of SCD sets.

At first, remark the following easy properties:

LEMMA 10.7.3. *Let $B_1, B_2 \subset X$ be non-empty bounded sets and let $x^* \in X^*$, $\varepsilon > 0$. We then have the following properties:*

(i) $\mathrm{Slice}(B_1, x^*, \varepsilon/2) + \mathrm{Slice}(B_2, x^*, \varepsilon/2) \subset \mathrm{Slice}(B_1 + B_2, x^*, \varepsilon)$.
(ii) *If $b_1 \in B_1$, $b_2 \in B$ satisfy that $b_1 + b_2 \in \mathrm{Slice}(B_1 + B_2, x^*, \varepsilon)$, then $b_1 \in \mathrm{Slice}(B_1, x^*, \varepsilon)$ and $b_2 \in \mathrm{Slice}(B_2, x^*, \varepsilon)$.*

The above lemma and Proposition 10.1.2 imply the following result.

LEMMA 10.7.4. *Let $B_1, B_2 \neq \emptyset$ be bounded subsets of a Banach space X. Then the following assertions are equivalent:*

(a) *$B_1 + B_2$ is SCD.*
(b) *There exists a countable family $(x_n^*, \varepsilon_n) \in X^* \times (0, +\infty)$ satisfying that for every $(x^*, \varepsilon) \in X^* \times (0, +\infty)$ there is $m \in \mathbb{N}$ such that*

$$\mathrm{Slice}(B_j, x_m^*, \varepsilon_m) \subset \mathrm{Slice}(B_j, x^*, \varepsilon) \tag{10.7.4}$$

for both $j = 1, 2$.

PROOF. (a) ⇒ (b). Let $S_n = \mathrm{Slice}(B_1 + B_2, x_n^*, 2\varepsilon_n)$ with $(x_n^*, \varepsilon_n) \in X^* \times (0, +\infty)$, $n \in \mathbb{N}$, be slices of $B_1 + B_2$ which form a determining sequence. Let us demonstrate that (x_n^*, ε_n) form the sequence we need for (b). Indeed, according to Proposition 10.1.2 for every $(x^*, \varepsilon) \in X^* \times (0, +\infty)$ there is $m \in \mathbb{N}$ such that $S_m \subset \mathrm{Slice}(B_1 + B_2, x^*, \varepsilon)$, and by (i) of Lemma 10.7.3 also

$$\mathrm{Slice}(B_1, x_m^*, \varepsilon_m) + \mathrm{Slice}(B_2, x_m^*, \varepsilon_m) \subset \mathrm{Slice}(B_1 + B_2, x^*, \varepsilon).$$

An application of (ii) of Lemma 10.7.3 gives us the desired inclusions (10.7.4).

(b) $\Rightarrow$ (a). Assume $(x_n^*, \varepsilon_n) \in X^* \times (0, +\infty)$ are from (b), and let us demonstrate that the slices $S_n = \mathrm{Slice}(B_1 + B_2, x_n^*, \varepsilon_n)$, $n \in \mathbb{N}$, form a determining sequence of slices for $B_1 + B_2$. Fix a slice $\mathrm{Slice}(B_1 + B_2, x^*, 2\varepsilon)$ with $x^* \in X^* \setminus \{0\}$, $\varepsilon > 0$ and, using (b), select m for which (10.7.4) is valid. We are going to demonstrate that $\mathrm{Slice}(B_1 + B_2, x_m^*, \varepsilon_m) \subset \mathrm{Slice}(B_1 + B_2, x^*, 2\varepsilon)$. Indeed, let $x \in \mathrm{Slice}(B_1 + B_2, x_m^*, \varepsilon_m)$ be an arbitrary element. Then it is of the form $x = b_1 + b_2$, $b_1 \in B_1$, $b_2 \in B_2$, and, by (ii) of Lemma 10.7.3, $b_1 \in \mathrm{Slice}(B_1, x_m^*, \varepsilon_m)$, $b_2 \in \mathrm{Slice}(B_2, x_m^*, \varepsilon_m)$. It remains to apply (i) of Lemma 10.7.3:

$$\begin{aligned} x = b_1 + b_2 &\subset \mathrm{Slice}(B_1, x_m^*, \varepsilon_m) + \mathrm{Slice}(B_2, x_m^*, \varepsilon_m) \\ &\subset \mathrm{Slice}(B_1, x^*, \varepsilon) + \mathrm{Slice}(B_2, x^*, \varepsilon) \\ &\subset \mathrm{Slice}(B_1 + B_2, x^*, 2\varepsilon). \end{aligned}$$ □

The above lemma leads to the following result.

THEOREM 10.7.5. *Let B_1, B_2 be non-empty bounded subsets of a Banach space X such that $B_1 + B_2$ is SCD. Then, B_1 (and so also B_2) is SCD.*

PROOF. Let $(x_n^*, \varepsilon_n) \in X^* \times (0, +\infty)$ be the family from (b) of Lemma 10.7.4, then the slices $\mathrm{Slice}(B_1, x_m^*, \varepsilon_m)$, $m \in \mathbb{N}$, form a determining sequence for B_1. □

The next proposition explains some difficulties that arise when one has to demonstrate that a non-convex set is SCD.

PROPOSITION 10.7.6. *There are non-convex non-SCD sets containing a determining sequence of relatively weakly open subsets. Such examples exist in every Banach space with the Daugavet property.*

PROOF. Let X be a space with the Daugavet property (as before it can be assumed separable), E be a 1-codimensional closed subspace. Then X is isomorphic to $E \oplus_\infty \mathbb{R}$. Take a sequence (x_n) in the unit ball of E such that both subsequences $(x_{2n})_{n\in\mathbb{N}}$ and $(x_{2n-1})_{n\in\mathbb{N}}$ are dense and a sequence $(t_n) \subset (0, 1)$ such that $t_{2n} \to 0$ and $t_{2n+1} \to 1$. The set in question will be the following subset of $E \oplus_\infty \mathbb{R}$:

$$U = \{x_n \oplus t_n \colon n \in \mathbb{N}\}.$$

This set is easily seen to be discrete in the weak topology, so $(\{x_n \oplus t_n\})_{n\in\mathbb{N}}$ is the requested determining sequence of relatively weakly open subsets. On the other hand, the closed convex hull of U equals $B_E \oplus [0, 1]$ which, according to Theorem 10.7.5, is not SCD because the unit ball of E is not SCD. □

Now, we are ready for the first main result of the subsection demonstrating that the converse to Theorem 10.7.5 is not true.

THEOREM 10.7.7. *In every Banach space X with the Daugavet property there are closed convex bounded SCD sets $A, D \subset X$ whose sum $A + D$ is not SCD.*

PROOF. We will use the same sets $A, D \subset X$ as in Theorem 10.7.2:

$$\begin{aligned} A &= \{x \oplus t \colon \|x\|_t \leqslant 1 - 3t^2,\ t \geqslant 0\}, \\ D = -A &= \{x \oplus t \colon \|x\|_t \leqslant 1 - 3t^2,\ t \leqslant 0\}, \end{aligned}$$

whose intersection is B_E. It has already been shown in Proposition 10.7.1 that A and D are SCD.

To see that the sum $A + D$ is not SCD we will argue by contradiction. If we assume that $A + D$ is SCD then we could find a countable family $(x_n^*, \varepsilon_n) \in S_{X^*} \times$

$(0,1)$ as in Lemma 10.7.4. Notice that we can write $x_n^* = y_n^* \oplus \lambda_n \in X^* = E^* \oplus \mathbb{R}$. Since B_E is not SCD we can find $y^* \in S_{E^*}$ and $\delta \in (0,1)$ such that for every $n \in \mathbb{N}$

$$\operatorname{Slice}(B_E, y_n^*, \varepsilon_n) \not\subset \operatorname{Slice}(B_E, y^*, \delta). \tag{10.7.5}$$

Considering the element $x^* = y^* \oplus 0 \in B_{X^*}$ we have that there is $k \in \mathbb{N}$ satisfying

$$\operatorname{Slice}(A, x_k^*, \varepsilon_k) \subset \operatorname{Slice}(A, x^*, \delta) \quad \text{and} \quad \operatorname{Slice}(D, x_k^*, \varepsilon_k) \subset \operatorname{Slice}(D, x^*, \delta)$$

from which it easily follows that

$$\operatorname{Slice}(A, x_k^*, \varepsilon_k) \cup \operatorname{Slice}(D, x_k^*, \varepsilon_k) \subset \{x \oplus t \in X \colon x \in \operatorname{Slice}(B_E, y^*, \delta)\}. \tag{10.7.6}$$

We now claim that

$$\operatorname{Slice}(B_E, y_k^*, \varepsilon_k) \subset \operatorname{Slice}(A, x_k^*, \varepsilon_k) \cup \operatorname{Slice}(D, x_k^*, \varepsilon_k) \tag{10.7.7}$$

which together with (10.7.6) leads to

$$\operatorname{Slice}(B_E, y_k^*, \varepsilon_k) \subset \operatorname{Slice}(B_E, y^*, \delta),$$

contradicting (10.7.5) and finishing the proof. To show the validity of the claim we distinguish two cases. Assuming that $\lambda_k \leqslant 0$ we get that

$$\sup\{x_k^*(a) \colon a \in A\} = \sup\{y_k^*(x) \colon x \in B_E\} = \|y_k^*\|.$$

As a consequence, $\operatorname{Slice}(B_E, y_k^*, \varepsilon_k) \subset \operatorname{Slice}(A, x_k^*, \varepsilon_k)$. On the other hand, if $\lambda_k \geqslant 0$ then analogously $\sup\{x_k^*(d) \colon d \in D\} = \sup\{y_k^*(x) \colon x \in B_E\}$ and therefore $\operatorname{Slice}(B_E, y_k^*, \varepsilon_k) \subset \operatorname{Slice}(D, x_k^*, \varepsilon_k)$. □

Nevertheless, for the direct sum of SCD sets the situation remains simple (for hereditarily SCD sets that was proved earlier in Theorem 10.2.8).

THEOREM 10.7.8. *Let $B_1 \subset X_1$, $B_2 \subset X_2$ be bounded subsets of a Banach space $X = X_1 \oplus X_2$, and suppose that B_1, B_2 are SCD. Then $B_1 + B_2$ is SCD.*

PROOF. Let $\operatorname{Slice}(B_i, y_{n,i}^*, \delta_{n,i})$, $y_{n,i}^* \in X_i^*$, $n \in \mathbb{N}$, form determining sequences of slices for B_i, $i = 1, 2$. Then, the collection of functionals $x_{n,m}^* = y_{n,1}^* \oplus y_{m,2}^*$ and corresponding $\varepsilon_{n,m} = \min\{\delta_{n,1}, \delta_{m,2}\}$ will be a countable family that satisfies condition (b) of Lemma 10.7.4. □

And now for the last of the promised main examples of the subsection.

THEOREM 10.7.9. *In every Banach space X with the Daugavet property there is an SCD set B such that $B \cup (-B)$ is not SCD.*

PROOF. We follow the notation of Proposition 10.7.1. Let $\alpha := 1/(2\sqrt{3})$ and $B := A - \alpha e_0 = \{x \oplus (t - \alpha) \colon x \oplus t \in A\}$, where A is the set in (10.7.2). We claim that

$$\operatorname{conv}(B \cup (-B)) = B_E \oplus [-\alpha, \alpha].$$

Indeed, it is clear that B is contained in $B_E \oplus [-\alpha, \alpha]$, and so is $-B$. For the converse, use simply that $B_E - \alpha e_0 \subset B$, $B_E + \alpha e_0 \subset -B$, and consequently

$$B_E \oplus [-\alpha, \alpha] = \operatorname{conv}((B_E - \alpha e_0) \cup (B_E + \alpha e_0)) \subset \operatorname{conv}(B \cup (-B)).$$

Finally, if $B \cup (-B)$ were SCD, then $B_E \oplus [-\alpha, \alpha]$ would be SCD by Lemma 10.1.7. But it has already been remarked above that this is never the case by Theorem 10.7.5, as B_E is not SCD because of the Daugavet property of E. □

10.7.3. Symmetrisation of the examples. In the most important applications of SCD sets, the sets which appear are balls and images of balls under the action of linear operators. So, it would be natural to ask whether examples demonstrating the non-stability of the property SCD can be constructed to be balls of some equivalent norms, that is, to be closed convex bounded symmetric bodies. The keyword here is "symmetric" because the examples that we have constructed above possess all the remaining properties of being closed convex bounded, and to have non-empty interior. In this subsection we apply a natural symmetrisation procedure which helps to obtain symmetric examples from non-symmetric ones.

Let U be a bounded non-empty subset of a Banach space X. By the *symmetrisation of U* we will mean the following subset $\mathrm{Sym}(U)$ of $X \oplus_\infty \mathbb{R}$:

$$\mathrm{Sym}(U) = \mathrm{aconv}(U \oplus 1).$$

LEMMA 10.7.10. *Let $U, V \neq \emptyset$ be bounded subsets such that U is contained in a closed hyperplane H_0, and V lies on one side of H_0 at a positive distance from H_0. Then, every slice of U is at the same time a slice of $U \cup V$.*

PROOF. Without loss of generality we can assume that $0 \in U$, and that $U, V \subset B_X$ (this can be done by shifting and scaling). Then there are an $x_0^* \in S_{X^*}$ and $\varepsilon_0 > 0$ such that $H_0 = \ker x_0^* \supset U$, and $V \subset \{x \in X \colon x_0^*(x) < -\varepsilon_0\}$. Let $x^* \in S_{X^*}$ and let $S = \mathrm{Slice}(U, x^*, \varepsilon)$ be a slice of U. Denote $r = \sup_{x \in U} x^*(x) \in [-1, 1]$ and consider for every $t > 0$ the functional $x_t^* = x^* + t x_0^*$. Since on U the values of x_t^* and of x^* are the same, $S = \mathrm{Slice}(U, x_t^*, \varepsilon)$ for all $t > 0$. We are going to demonstrate that for some values of $t > 0$ the slice $S_t = \mathrm{Slice}(U \cup V, x_t^*, \varepsilon)$ of $U \cup V$ is also equal to S, which will complete our proof.

So our goal is to show that there is $t > 0$ such that $S_t \cap V = \emptyset$. Assume to the contrary that for every $t > 0$ there is an element $v_t \in V \cap S_t$. Then

$$\begin{aligned} 1 - t\varepsilon_0 &\geqslant x^*(v_t) + t x_0^*(v_t) = x_t^*(v_t) > \sup_{x \in U \cup V} x_t^*(x) - \varepsilon \\ &= \max\Big\{\sup_{x \in U} x^*(x), \sup_{x \in V} x^*(x) + t x_0^*(x)\Big\} - \varepsilon \\ &\geqslant \max\{r, r - t\varepsilon_0\} - \varepsilon = r - \varepsilon, \end{aligned}$$

which means that $t < \frac{1+\varepsilon-r}{\varepsilon_0}$. This is a contradiction. □

LEMMA 10.7.11. *If, under the conditions of Lemma 10.7.10, $U \cup V$ is SCD, then U is also an SCD set.*

PROOF. Let $\{V_n \colon n \in \mathbb{N}\}$ be a determining family of slices of $U \cup V$. Denote $N_1 = \{n \in \mathbb{N} \colon V_n \cap U \neq \emptyset\}$. Then $S_n := V_n \cap U$, $n \in N_1$, are slices of U. We are going to demonstrate that the collection $\{S_n \colon n \in N_1\}$ is determining for U, which will do the job. Let us use Proposition 10.1.2. Consider a slice S of U. Then, by Lemma 10.7.10, S is at the same time a slice of $U \cup V$. So, there is $n \in \mathbb{N}$ such that $V_n \subset S$, but this n automatically belongs to N_1. □

LEMMA 10.7.12. *The following conditions for a bounded non-empty subset $U \subset X$ are equivalent:*

(i) *U is SCD,*
(ii) *$(U \oplus 1) \cup -(U \oplus 1)$ is SCD,*
(iii) *$\mathrm{Sym}(U)$ is SCD.*

Proof. Taking into account that $\mathrm{Sym}(U) = \mathrm{conv}\left((U \oplus 1) \cup -(U \oplus 1)\right)$ the equivalence (ii) $\Leftrightarrow$ (iii) follows from Lemma 10.1.7.

(i) $\Rightarrow$ (ii). Let $\{S_n\colon n \in \mathbb{N}\}$ be a determining family of slices of U. Then, the $V_n := S_n \oplus 1$, $n \in \mathbb{N}$, form a determining sequence of slices of $U \oplus 1$ and the $-V_n$ form a determining sequence of slices of $-(U \oplus 1)$. By Lemma 10.7.10, $\pm V_n$ are also slices of $(U \oplus 1) \cup -(U \oplus 1)$. But then the countable collection $\{\pm V_n\colon n \in \mathbb{N}\}$ forms a determining family of slices of $(U \oplus 1) \cup -(U \oplus 1)$. Indeed, let $V \subset X \oplus_\infty \mathbb{R}$ intersect all $\pm V_n$, $n \in \mathbb{N}$. Then, since $\{V_n\colon n \in \mathbb{N}\}$ is determining for $U \oplus 1$, we have $\overline{\mathrm{conv}}(V) \supset U \oplus 1$ and since $(-V_n)_{n \in \mathbb{N}}$ form a determining sequence of slices of $-(U \oplus 1)$, we also have $\overline{\mathrm{conv}}(V) \supset -(U \oplus 1)$, which completes the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Applying Lemma 10.7.11 we obtain that $U \oplus 1$ is SCD, but $U \oplus 1$ is a shift of U, so U is also SCD. $\square$

The next example is based on the elementary fact that the convex hull of the union of symmetrised sets $\mathrm{Sym}(U_1) \cup \mathrm{Sym}(U_2)$ is equal to the symmetrised union $\mathrm{Sym}(U_1 \cup U_2)$:

$$\mathrm{conv}\left(\mathrm{Sym}(U_1) \cup \mathrm{Sym}(U_2)\right) = \mathrm{Sym}(U_1 \cup U_2).$$

Indeed, the left hand side is convex and symmetric, contains $(U_1 \cup U_2) \oplus 1$, so contains $\mathrm{Sym}(U_1 \cup U_2)$. Conversely, the right hand side is convex, contains $\mathrm{Sym}(U_1)$ and $\mathrm{Sym}(U_2)$, so it contains the convex hull $\mathrm{conv}(\mathrm{Sym}(U_1) \cup \mathrm{Sym}(U_2))$.

Theorem 10.7.13. *In every Banach space Y with the Daugavet property there are closed convex bounded symmetric sets $\tilde{B}_1, \tilde{B}_2 \subset Y$ which are SCD sets, but whose union $\tilde{B}_1 \cup \tilde{B}_2$ is not SCD. If, additionally, Y is separable, then these $\tilde{B}_1, \tilde{B}_2 \subset Y$ can be chosen to have non-empty interior.*

Proof. It is sufficient to consider the case of separable Y (otherwise, substitute it by a separable subspace with the Daugavet property). Let X be a 1-codimensional closed subspace of Y. Then X also has the Daugavet property. Our Y is isomorphic to $X \oplus_\infty \mathbb{R}$, so it is sufficient to construct the requested example in $X \oplus_\infty \mathbb{R}$. Let B and $-B$ be SCD sets from Theorem 10.7.9, and take $B_1 = \mathrm{Sym}(B)$, $B_2 = \mathrm{Sym}(-B)$. Since B, $-B$ are convex, bounded and have non-empty interior, B_1 and B_2 are convex bounded symmetric bodies which are SCD by the previous Lemma 10.7.12. Also, $\mathrm{conv}(B_1 \cup B_2) = \mathrm{Sym}(B \cup (-B))$, so by the same Lemma 10.7.12 $\mathrm{conv}(B_1 \cup B_2)$ is not SCD, and consequently $B_1 \cup B_2$ is not SCD. To finish the proof define $\tilde{B}_1$ and $\tilde{B}_2$ to be the closures of B_1 and B_2 and apply Lemma 10.1.7. $\square$

In order to proceed with the symmetrisation of the example about the sum of SCD sets, we first need a natural lemma.

Lemma 10.7.14. *Let $U_1, U_2 \subset X$ be bounded not empty subsets. Then,*

$$\overline{\mathrm{conv}}(U_1 + U_2) = \overline{\mathrm{conv}(U_1) + \mathrm{conv}(U_2)}.$$

Proof. Both the right hand side and the left hand side of the equality in question are closed convex sets, so (see [**156**, Section 18.1.2, Lemma 6]) in order to prove the equality it is sufficient to demonstrate that for every $x^* \in X^*$

$$\sup x^*(\overline{\mathrm{conv}}(U_1 + U_2)) = \sup x^*\left(\overline{\mathrm{conv}(U_1) + \mathrm{conv}(U_2)}\right).$$

This equality is easily seen to be true, because its right hand side and left hand side are both equal to $\sup x^*(U_1) + \sup x^*(U_2)$. $\square$

THEOREM 10.7.15. *In every Banach space Y with the Daugavet property there are closed convex bounded symmetric SCD sets (which in the separable case can be chosen to be bodies) $C_1, C_2 \subset Y$ whose sum $C_1 + C_2$ is not SCD.*

PROOF. As before, we can reduce the situation to a separable space of the form $X \oplus_\infty \mathbb{R}$, where X has the Daugavet property. Let $B_1 := A$, $B_2 := D$ be SCD subsets of X from Theorem 10.7.7 such that $B_1 + B_2$ is not SCD, and take $C_1 = \overline{\mathrm{Sym}(B_1)}$, $C_2 = \overline{\mathrm{Sym}(B_2)}$, which are closed convex bounded symmetric SCD bodies. It remains to demonstrate that $C_1 + C_2$ is not SCD. Using Lemma 10.7.14 we can see that

$$\begin{aligned}\overline{C_1 + C_2} &= \overline{\mathrm{conv}\big((B_1 \oplus 1) \cup -(B_1 \oplus 1)\big) + \mathrm{conv}\big((B_2 \oplus 1) \cup -(B_2 \oplus 1)\big)} \\ &= \overline{\mathrm{conv}}\big(((B_1 \oplus 1) \cup -(B_1 \oplus 1)) + ((B_2 \oplus 1) \cup -(B_2 \oplus 1))\big).\end{aligned}$$

According to Lemma 10.1.7, it is sufficient to show that the set

$$\begin{aligned}&((B_1 \oplus 1) \cup -(B_1 \oplus 1)) + ((B_2 \oplus 1) \cup -(B_2 \oplus 1)) \\ &\qquad = ((B_1 + B_2) \oplus 2) \cup ((B_1 - B_2) \oplus 0) \\ &\qquad\qquad \cup ((B_2 - B_1) \oplus 0) \cup (-(B_1 + B_2) \oplus (-2))\end{aligned}$$

is not SCD. With the help of Lemma 10.7.10 this can be deduced from the fact that $B_1 + B_2$ is not SCD exactly the same way as in the implication (ii) $\Rightarrow$ (i) of Lemma 10.7.12, because $(B_1 + B_2) \oplus 2$ lies in the hyperplane of those elements whose second coordinate equals 2, and the rest of the set lies at a distance at least 2 from that hyperplane. $\square$

Before coming to the symmetrisation of the non-SCD intersection example, one more easy remark.

LEMMA 10.7.16. *Let $U_0, U_1 \subset X$ be non-empty subsets with $U_0 \subset U_1$, and let U_1 be convex. Then, $U_\lambda := \lambda U_1 + (1-\lambda)U_0$ increases when $\lambda \in [0,1]$ increases.*

PROOF. Let $0 \leqslant \lambda \leqslant \mu \leqslant 1$. Then

$$\begin{aligned}U_\mu = \mu U_1 + (1-\mu)U_0 &= \lambda U_1 + (\mu - \lambda)U_1 + (1-\mu)U_0 \\ &\supset \lambda U_1 + (\mu-\lambda)U_0 + (1-\mu)U_0 \\ &\supset \lambda U_1 + (1-\lambda)U_0 = U_\lambda.\end{aligned}$$ $\square$

Remark that if $U \subset X$ is convex, then $\mathrm{Sym}(U) \subset X \oplus_\infty \mathbb{R}$ can be written as

$$\mathrm{Sym}(U) = \{(tu - (1-t)v) \oplus (2t-1) \colon u, v \in U,\ t \in [0,1]\}.$$

In other words,

$$\mathrm{Sym}(U) = \bigcup_{t \in [0,1]} (tU - (1-t)U) \oplus (2t-1).$$

This implies the following formula for the intersection of $\mathrm{Sym}(U_1) \cap \mathrm{Sym}(U_2)$ in the case of convex $U_1, U_2 \subset X$:

$$\mathrm{Sym}(U_1) \cap \mathrm{Sym}(U_2) = \bigcup_{t \in [0,1]} \big((tU_1 - (1-t)U_1) \cap (tU_2 - (1-t)U_2)\big) \oplus (2t-1).$$

THEOREM 10.7.17. *In every (separable) Banach space Y with the Daugavet property there are closed convex bounded symmetric sets (bodies) which are SCD sets, but whose intersection is not SCD.*

PROOF. Again, it is sufficient to consider a separable space of the form $X \oplus_\infty \mathbb{R}$, where X has the Daugavet property. Let $A \subset X$ be as in Proposition 10.7.1. Denote $U_1 = A$, $U_2 = -A$. We are going to demonstrate that $\mathrm{Sym}(U_1), \mathrm{Sym}(U_2) \subset X \oplus_\infty \mathbb{R}$ are the requested non-empty bounded convex symmetric SCD bodies such that $W := \mathrm{Sym}(U_1) \cap \mathrm{Sym}(U_2)$ is not SCD.

Each element of X is of the form $e + te_0$, $e \in E$, $t \in \mathbb{R}$, and in order to avoid misunderstandings we will not use the expression $e \oplus t$ for $e + te_0$ in the current proof. The notation $x \oplus t$ is reserved for elements of $X \oplus_\infty \mathbb{R}$, and $x^* \oplus \tau$ for elements of $(X \oplus_\infty \mathbb{R})^* = X^* \oplus_1 \mathbb{R}$.

For every $t \in [0,1]$ denote $A_t = (tA - (1-t)A) \cap ((1-t)A - tA)$. Then,

$$W = \bigcup_{t\in[0,1]} \left(A_t \oplus (2t-1)\right). \tag{10.7.8}$$

Geometrically this means that the lowest level section (with $t = 0$) of W is the set $(A \cap -A) \oplus (-1) = B_E \oplus (-1)$, when we move to higher levels the section transforms up to $\frac{A-A}{2} \oplus 0$ when $t = \frac{1}{2}$, and then transforms back until $(A \cap -A) \oplus 1 = B_E \oplus 1$ when $t = 1$. The set W is not only centrally symmetric with respect to zero, but also doubly mirror-symmetric in the following sense: for every $e \in E$, $a, b \in \mathbb{R}$, if $(e + ae_0) \oplus b \in W$, then $(\pm e \pm ae_0) \oplus (\pm b) \in W$ for all choices of $\pm$.

Let us assume to the contrary that W is SCD. From this assumption we are going to deduce that B_E is SCD, which will be the desired contradiction. Let $S_n = \mathrm{Slice}(W, w_n^*, \varepsilon_n)$ form a determining sequence of slices of W, $w_n^* = x_n^* \oplus \tau_n$. Denote also by $e_n^* \in E^*$ and $s_n \in \mathbb{R}$ those elements that represent the corresponding x_n^*, i.e., $x_n^*(e + te_0) = e_n^*(e) + s_n t$ for all $e \in E$, $t \in \mathbb{R}$. By the Bishop-Phelps theorem the set of functionals that attain their supremum on W is norm-dense in the dual space, consequently, by a small perturbation argument, we may assume that each w_n^* attains its supremum R_n on W at some point $w_n = x_n \oplus b_n = (e_n + a_n e_0) \oplus b_n \in W$, $b_n = 2t_n - 1$, that is,

$$R_n := \sup_{w\in W} w_n^*(w) = x_n^*(x_n) + \tau_n b_n = e_n^*(e_n) + s_n a_n + \tau_n b_n.$$

We are going to show that $\tilde{S}_n = \mathrm{Slice}(B_E, e_n^*, \varepsilon_n)$, $n \in \mathbb{N}$, form a determining sequence of slices of B_E. Fix an arbitrary $e^* \in S_{E^*}$ and $\varepsilon \in (0,1)$. According to Proposition 10.1.2, our task is to find $n \in \mathbb{N}$ such that $\tilde{S}_n \subset \mathrm{Slice}(B_E, e^*, \varepsilon)$. Let us extend e^* to the whole $X \oplus_\infty \mathbb{R}$ by the natural rule $e^*((e + t_1 e_0) \oplus t_2) := e^*(e)$ and consider the corresponding slice $\mathrm{Slice}(W, e^*, \frac{\varepsilon}{2})$. Due to the same Proposition 10.1.2 there is $m \in \mathbb{N}$ such that $S_m \subset \mathrm{Slice}(W, e^*, \frac{\varepsilon}{2})$. Remark that the corresponding e_m^* is non-zero, otherwise with every point $(e + ce_0) \oplus d$ the slice S_m would contain also $(ce_0) \oplus d$, thus contradicting the inclusion $S_m \subset \mathrm{Slice}(W, e^*, \frac{\varepsilon}{2})$. Without loss of generality we may assume that $s_m, \tau_m \geqslant 0$ (here we use the symmetry of W and of $\mathrm{Slice}(W, e^*, \frac{\varepsilon}{2})$ with respect to corresponding changes of signs). Then we can also assume $a_m, b_m \geqslant 0$ and consequently $t_m \geqslant \frac{1}{2}$.

By the definition, $x_m^*(x_m) = \sup x_m^*(A_{t_m})$. We claim that, in fact ,

$$\begin{aligned} x_m^*(x_m) &= \sup x_m^*\big((1-t_m)A + t_m B_E\big) \\ &= (1-t_m) \sup x_m^*(A) + t_m \|e_m^*\|. \end{aligned} \tag{10.7.9}$$

Indeed, $A_{t_m} = (t_m A - (1-t_m)A) \cap ((1-t_m)A - t_m A) \subset (1-t_m)A - t_m A$, so $x_m \in A_{t_m}$ has a representation of the form $x_m = (1-t_m)y - t_m z$ with $y, z \in A$. Consequently,

$$\begin{aligned} x_m^*(x_m) &= (1-t_m)x_m^*(y) + t_m x_m^*(-z) \\ &\leqslant (1-t_m)\sup x_m^*(A) + t_m \sup x_m^*(-A) \\ &= (1-t_m)\sup x_m^*(A) + t_m\|e_m^*\|, \end{aligned}$$

where we used the positivity of s_m in the last step. For the reverse inequality in (10.7.9) we can use the inclusion $A \supset B_E$, the inequality $t_m \geqslant 1 - t_m$ and Lemma 10.7.16 which together give us the inclusion

$$(1-t_m)A + t_m B_E \subset t_m A + (1-t_m)B_E.$$

This implies that

$$\begin{aligned} A_{t_m} &= (t_m A - (1-t_m)A) \cap ((1-t_m)A - t_m A) \\ &\supset (t_m A - (1-t_m)B_E) \cap ((1-t_m)A - t_m B_E) \\ &= (t_m A + (1-t_m)B_E) \cap ((1-t_m)A + t_m B_E) \\ &\supset (1-t_m)A + t_m B_E, \end{aligned} \tag{10.7.10}$$

so

$$x_m^*(x_m) = \sup x_m^*\left(A_{t_m}\right) \geqslant \sup x_m^*\left((1-t_m)A + t_m B_E\right).$$

Thus, the formula (10.7.9) is proved. It remains to prove that $\tilde{S}_m \subset \operatorname{Slice}(B_E, e^*, \varepsilon)$, or in other words that $\tilde{S}_m \setminus \operatorname{Slice}(B_E, e^*, \varepsilon) = \emptyset$. Assume that this set is not empty, and pick an arbitrary $e \in \tilde{S}_m \setminus \operatorname{Slice}(B_E, e^*, \varepsilon)$. Then $e \in B_E$ and e satisfies simultaneously two inequalities:

$$e_m^*(e) > \|e_m^*\| - \varepsilon_m, \qquad \text{and} \qquad e^*(e) \leqslant 1 - \varepsilon. \tag{10.7.11}$$

Take an arbitrary $g \in A$ with $x_m^*(g) > \sup x_m^*(A) - \varepsilon_m$. According to (10.7.10), $(1-t_m)g + t_m e \in A_{t_m}$, so

$$((1-t_m)g + t_m e) \oplus b_m = ((1-t_m)g + t_m e) \oplus (2t_m - 1) \in W.$$

Then, the following inequality

$$\begin{aligned} w_m^*(((1-t_m)g + t_m e) \oplus b_m) &= x_m^*((1-t_m)g + t_m e) + \tau_m b_m \\ &> (1-t_m)(\sup x_m^*(A) - \varepsilon_m) \\ &\qquad + t_m(\|e_m^*\| - \varepsilon_m) + \tau_m b_m \\ &= x_m^*(x_m) + \tau_m b_m - \varepsilon_m = R_m - \varepsilon_m \end{aligned}$$

implies that $((1-t_m)g + t_m e) \oplus b_m \in S_m$, and consequently $((1-t_m)g + t_m e) \oplus b_m \in \operatorname{Slice}(B_E, e^*, \frac{\varepsilon}{2})$. This means that

$$(1-t_m)e^*(g) + t_m e^*(e) = e^*(((1-t_m)g + t_m e) \oplus b_m) > 1 - \frac{\varepsilon}{2}.$$

Together with the second condition from (10.7.11), this gives

$$1 - \frac{\varepsilon}{2} < (1-t_m)e^*(g) + t_m e^*(e) < (1-t_m) + t_m(1-\varepsilon) = 1 - t_m\varepsilon \leqslant 1 - \frac{\varepsilon}{2}.$$

This contradiction proves that $\tilde{S}_m \setminus \operatorname{Slice}(B_E, e^*, \varepsilon) = \emptyset$. $\square$

10.8. Notes and remarks

Section 10.1. The concept of SCD sets was introduced for convex sets in [**26, 27**] ([**26**] is a short announcement of the results from [**27**]), and the main results of this section are taken from those papers. The generalisation to the non-convex setting together with Lemma 10.1.7 are taken from [**166**].

Section 10.2. This section is based on [**167**].

Section 10.3. This section is based on [**26, 27**]. Theorem 10.3.12 is from [**199**], where a pointwise version of slice countable determination is introduced. Namely, given a Banach space X and a bounded and convex subset $A \subset X$, we say that a point $a \in A$ is a *slicely countably determined point* of A (*SCD point* of A) if there exists a sequence of slices $(S_n)_{n\in\mathbb{N}}$ such that $a \in \overline{\mathrm{conv}}(B)$ whenever $B \subset A$ intersects all the slices S_n. The set of all SCD points of A is denoted by $\mathrm{SCD}(A)$ and it is convex and closed (relative to A). Strongly regular points (in particular, PCP points or even denting points) of a closed convex set A are SCD points of A.

In [**199**], this concept is deeply studied and there are some interesting consequences. For instance, it is shown using this tool that a separable Banach space X contains an isomorphic copy of ℓ_1 and its unit ball does not contain strongly regular points if every convex series of slices of B_X intersects the unit sphere [**199**, Theorem 6.3]. Besides, if a Banach space X has the Daugavet property, then every operator $T \in L(X)$ for which there are elements in $\mathrm{SCD}(T(B_X))$ with norm arbitrarily closed to $\|T\|$ satisfies the Daugavet equation [**199**, Theorem 6.5], giving a pointwise version of results like Corollary 10.4.8 above. Let us mention in addition that this paper also contains Theorem 11.2.11 from the next chapter that characterises those Lipschitz-free spaces whose unit balls are SCD.

Section 10.4. The first version of Lemma 10.4.6, parts (iv)–(vi), appeared in [**27**] for the Daugavet property, its extension to Daugavet centres was done in [**60**]; [**147**] is also relevant in this regard. The current version is a bit simpler: the Baire property of $\mathrm{ext}(B_{X^*})$, which the authors of [**27**] and [**60**] were not aware of, enables us to avoid the weak-star closure of $\mathrm{ext}(B_{X^*})$ in the statement and in the proof, and thus do not care of the subtleties related to the fact that some elements of the weak-star closure of $\mathrm{ext}(B_{X^*})$ may have norm smaller than 1. The idea of this simplification comes from [**162**], where it was used in similar questions related to the alternative Daugavet property.

HSCD operators were introduced in [**26, 27**] and HSCD-majorised operators were introduced in [**167**], in particular Proposition 10.4.9 and Theorem 10.4.17 originate from [**167**].

Section 10.5 has its origin in [**27**].

Section 10.6. The results are taken from [**167**]. Remark that the construction in Theorem 10.6.1, although it was discovered independently, is ideologically close to Schachermayer's construction of two Radon-Nikodým sets whose sum is not a Radon-Nikodým set [**280**].

Section 10.7. The results in this section come from [**166**].

10.9. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

(10.1) Let X be a Banach space and let A be a convex bounded subset of X. If A is SCD, does A have a countable π-base for the weak topology?

(10.2) Let X be an SCD space. Does every convex bounded subset of X have a countable π-base for the weak topology?

(10.3) Let L be a compact subset of a locally convex space and let K be its closed convex hull. If L has a countable π-base, does it imply that K also has a countable π-base? What if $L = \mathrm{ext}(K)$?

Let us explain why this question is related to the above two. Observe that if D is a dense subspace of a topological space E and $\mathcal{B}$ is a π-base for E, then $\{B \cap D\colon B \in \mathcal{B}\}$ is a π-base for D. In particular, if $(\overline{A}^{**}, \sigma(X^{**}, X^*))$ has a countable π-base, then so does $(A, \sigma(X, X^*))$. Thus, a positive answer to the preceding question combined with Theorem 10.5.4 would imply a positive answer to the previous two questions.

(10.4) Is it true that every space that contains a separable bounded convex non-SCD subset is isomorphic to a space with the Daugavet property?

(10.5) Is it true that every bounded convex subset of a Banach space with an unconditional basis must be SCD?

(10.6) Let X be a space with the Daugavet property. Is it true that every bounded linear operator $T\colon X \to X$ with $T(B_X)$ an SCD set, is necessarily narrow?

Remark that in Theorem 10.4.10 a similar statement is demonstrated under the stronger assumption that all subsets of $T(B_X)$ are SCD.

It is known that every operator in a space with the Daugavet property that is dominated by an HSCD operator is narrow (Remark 10.4.16). This class of HSCD dominated operators includes all the basic examples of narrow operators and is a two-sided operator ideal (Theorem 10.4.17). Hence the following question is reasonable.

(10.7) Is it true that the class of HSCD dominated operators forms the biggest two-sided operator ideal in the class of narrow operators? What is the description of HSCD dominated operators in the classical $C(K)$ and L_1 spaces?

(10.8) Given two Banach spaces X and Y such that B_X and B_Y are SCD, is $B_{X\widehat{\otimes}_\pi Y}$ SCD?

(10.9) Does every separable Banach space that is not SCD possess the Daugavet property in some equivalent norm?

(10.10) Does there exist a pair U_1, U_2 of hereditarily SCD subsets of a Banach space such that $U_1 + U_2$ is not SCD?

(10.11) Is every space with an unconditional basis an SCD space?

(10.12) Is it true that for every convex weakly compact set $W \subset X$ and every convex SCD (or HSCD) subset $A \subset X$ the sum $W + A$ is SCD (HSCD, respectively)?

It looks like nobody has ever tried to prove this. An analogous fact for the RNP sets can be found in [**288**, p. 46], for a more general result see [**280**, Proposition 1.6(a)]. The same question can be asked for a set W that does not have ℓ_1-sequences.

A kind of converse question is the following one.

(10.13) Let $W \subset C[0,1]$ be a closed convex set such that $W + A$ is SCD (HSCD) for every convex SCD subset $A \subset C[0,1]$ (for every convex HSCD subset $A \subset C[0,1]$, respectively). Must this W be weakly compact? Can such a W contain an ℓ_1-sequence?

(10.14) Must the union of two hereditarily SCD subsets of a Banach space be an SCD set?

Concerning the last problem, remark that $\operatorname{conv}(U_1 \cup U_2)$ need not be hereditarily SCD when U_1, U_2 are hereditarily SCD. Indeed, if U_1, U_2 are the hereditarily SCD sets from [**167**, Corollary 2.2] whose Minkowski sum is not hereditarily SCD, then $\operatorname{conv}(U_1 \cup U_2) \supset \frac{1}{2}(U_1 + U_2)$, so $\operatorname{conv}(U_1 \cup U_2)$ is not hereditarily SCD either.

CHAPTER 11

Spaces of Lipschitz functions, Lipschitz maps, and the Daugavet equation

In this chapter we consider only *real* spaces. In order to apply the results about Daugavet centres to complex spaces, one needs to consider the same spaces as real ones. This is not a problem, thanks to Corollary 5.1.3.

11.1. Preliminaries

Given a metric space (M, d) and a point $x \in M$, we will denote by $B(x, r)$ the closed ball centred at x with radius r. Let M be a metric space with a distinguished point that we will call 0 for convenience; thus $0 \in M$. The couple $(M, 0)$ is commonly called a *pointed metric space*. By an abuse of language we will only say "let M be a pointed metric space" and similarly in other sentences. The vector space of Lipschitz functions from M to $\mathbb{R}$ will be denoted by $\mathrm{Lip}(M)$. Given a Lipschitz function $f \in \mathrm{Lip}(M)$, we denote its Lipschitz constant by

$$\|f\|_L = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} \colon x, y \in M,\ x \neq y\right\}.$$

This is a seminorm on $\mathrm{Lip}(M)$ which is clearly a Banach space norm on the space $\mathrm{Lip}_0(M) \subset \mathrm{Lip}(M)$ of Lipschitz functions on M vanishing at 0. (An alternative way of defining $\mathrm{Lip}_0(M)$ is considering the quotient of the space $\mathrm{Lip}(M)$ over all the constant functions. This quotient turns out to be isometrically isomorphic to $\mathrm{Lip}_0(M)$.) For the norm of $f \in \mathrm{Lip}_0(M)$ we use either $\|f\|_L$, or the generic normed space theory notation $\|f\|$ when it is clear which normed space is considered. The choice of the base point is immaterial since $\mathrm{Lip}_0(M, p_1)$ and $\mathrm{Lip}_0(M, p_2)$ are isometrically isomorphic for each selection of base points p_1, p_2. It is well known that $\mathrm{Lip}_0(M)$ is a dual Banach space whose canonical predual is the *Lipschitz free space*

$$\mathcal{F}(M) := \overline{\mathrm{lin}}\{\delta_x \colon x \in M\} \subset \mathrm{Lip}_0(M)^*$$

where $\delta_x(f) := f(x)$ for every $x \in M$ and $f \in \mathrm{Lip}(M)$ (see [**120, 122, 294**]). We use the name *molecule* for those elements of $\mathcal{F}(M)$ of the form

$$m_{x,y} := \frac{\delta_x - \delta_y}{d(x, y)}$$

for $x, y \in M$ such that $x \neq y$, and we will denote by $\mathrm{Mol}(M)$ the set of all the molecules of $\mathcal{F}(M)$.

Remark 11.1.1. *It is clear that*

$$\|f\|_L = \sup\{\langle m_{x,y}, f\rangle \colon x, y \in M,\ x \neq y\} \text{ for every } f \in \mathrm{Lip}_0(M),$$

so every slice S of $B_{\mathcal{F}(M)}$ intersects $\mathrm{Mol}(M)$. *Then Lemma 2.6.7 implies that* $\overline{\mathrm{conv}}(\mathrm{Mol}(M)) = B_{\mathcal{F}(M)}$.

If N is a dense subset of M, then $\mathcal{F}(N)$ and $\mathcal{F}(M)$ are isometrically isomorphic Banach spaces as every Lipschitz function on N extends uniquely to a Lipschitz function on M with the same Lipschitz constant. Thus the results about $\mathcal{F}(M)$ or $\mathrm{Lip}_0(M)$ can be stated for *complete* M without any loss of generality.

Every Banach space is a pointed metric space (with the canonical choice of the zero vector as base point – hence the notation in the general case), so the above notation is applicable in particular to Banach spaces M. Apart from Lipschitz functions, we are also going to consider Banach space valued Lipschitz maps.

Let M be a pointed metric space and let Y be Banach space. The space $\mathrm{Lip}_0(M,Y)$ of all Lipschitz maps $F\colon X \to Y$ that vanish at 0 will be equipped with the norm

$$\|F\|_L = \sup\left\{ \frac{\|F(m_1) - F(m_2)\|}{d(m_1, m_2)} \colon m_1 \neq m_2 \in M \right\}. \tag{11.1.1}$$

Remark that $\mathrm{Lip}_0(M) = \mathrm{Lip}_0(M, \mathbb{R})$.

The most interesting case occurs when $M = X$ is a Banach space. Clearly, in this case, for a bounded linear operator from X to Y, the newly defined Lipschitz norm coincides with the standard operator norm.

The map $\delta_X\colon x \mapsto \delta_x$ establishes an isometric non-linear embedding $X \to \mathcal{F}(X)$ since $\|\delta_x - \delta_y\|_{\mathcal{F}(X)} = \|x - y\|_X$ for all $x, y \in X$. Some features of the Lipschitz-free space which we are going to use below are contained in the following easy to check result that may be found in [**122**] or Section 2.2 of the book [**294**].

LEMMA 11.1.2. *Let X, Y be Banach spaces.*

(a) *For every $F \in \mathrm{Lip}_0(X,Y)$, there exists a unique bounded linear operator $\widehat{F}$: $\mathcal{F}(X) \to Y$ such that $\widehat{F} \circ \delta_X = F$ and $\|\widehat{F}\| = \|F\|_L$. Moreover, the map $F \mapsto \widehat{F}$ is an isometric isomorphism from $\mathrm{Lip}_0(X,Y)$ onto $L(\mathcal{F}(X), Y)$. The map $\widehat{F}$ may be constructed using extensions by linearity and continuity: it is defined as*
$$\widehat{F}\left(\sum_{k=1}^{n} a_k \delta_{x_k}\right) = \sum_{k=1}^{n} a_k F(x_k),$$
on finite linear combinations of elements of the form δ_x and is extended afterwards to the whole $\mathcal{F}(X)$ by continuity and density.

(b) *There exists a norm-one surjective linear map $\beta_X\colon \mathcal{F}(X) \to X$ which is a left inverse of δ_X, that is, $\beta_X \circ \delta_X = \mathrm{Id}_X$. It is called the* barycentre map *in* [**122**]*, and is defined, using the previous item, as $\beta_X := \widehat{\mathrm{Id}_X}$. In particular,*
$$\beta_X\left(\sum_{k=1}^{n} a_k \delta_{x_k}\right) = \sum_{k=1}^{n} a_k x_k,$$
on finite linear combinations of elements of the form δ_x.

(c) *From the uniqueness in item (a), it follows that $\widehat{F} = F \circ \beta_X$ for every $F \in L(X,Y)$.* □

11.2. The Daugavet equation in spaces of Lipschitz functions and Lipschitz-free spaces

In this section we address the following question: given a metric space (M, d), when does $\mathrm{Lip}_0(M)$ enjoy the Daugavet property? A characterisation in terms of the metric space M will be given at the very end of the section.

Let us start by finding a necessary condition for $\mathrm{Lip}_0(M)$ to enjoy the Daugavet property. By the general theory, it is clear that for $\mathrm{Lip}_0(M) \in \mathrm{DPr}$, necessarily its predual $\mathcal{F}(M)$ has to enjoy the Daugavet property. Now, the fact that $\mathrm{Mol}(M)$ is norming for $\mathrm{Lip}_0(M)$ allows us to obtain the following necessary condition.

THEOREM 11.2.1. *Let M be a metric space and assume that $\mathcal{F}(M)$ has the Daugavet property. Then, for every $x, y \in M$ and every function $f \in S_{\mathrm{Lip}_0(M)}$ such that $f(x) - f(y) > (1-\varepsilon)d(x,y)$ there exist $u, v \in M$ such that*

$$f(u) - f(v) > (1-\varepsilon)d(u,v) \quad \text{and} \quad d(u,v) < \frac{\varepsilon}{(1-\varepsilon)^2} d(x,y).$$

In order to prove Theorem 11.2.1, let us introduce some notation. Let M be a pointed metric space. The *metric segment* in M that connects $x, y \in M$ is the set

$$[x,y] := \{z \in M \colon d(x,z) + d(z,y) = d(x,y)\}.$$

Now, we will consider for every $x, y \in M$, $x \neq y$, the function

$$f_{xy}(t) := \frac{d(x,y)}{2} \frac{d(t,y) - d(t,x)}{d(t,y) + d(t,x)}.$$

The properties collected in the next lemma have been proved in [**146**]. They make f_{xy} a useful tool for studying the geometry of $B_{\mathcal{F}(M)}$ because the increasing quotients of f_{xy} measures how far the points are from the metric segment $[x,y]$ (see (c) below).

LEMMA 11.2.2. *Let $x, y \in M$ with $x \neq y$. We have*

(a) *For all $u \neq v \in M$,*

$$\frac{f_{xy}(u) - f_{xy}(v)}{d(u,v)} \leqslant \frac{d(x,y)}{\max\{d(x,u) + d(u,y), d(x,v) + d(v,y)\}}.$$

(b) *f_{xy} is Lipschitz and $\|f_{xy}\|_L \leqslant 1$.*

(c) *Let $u \neq v \in M$ and $\varepsilon > 0$ be such that $\frac{f_{xy}(u) - f_{xy}(v)}{d(u,v)} > 1 - \varepsilon$. Then*

$$(1-\varepsilon)\max\{d(x,v) + d(y,v), d(x,u) + d(y,u)\} < d(x,y).$$

(d) *If $u \neq v \in M$ and $\frac{f_{xy}(u) - f_{xy}(v)}{d(u,v)} = 1$, then $u, v \in [x,y]$.*

PROOF. Statement (a) follows from the next (easily proved) fact: for arbitrary positive numbers u_1, v_1, u_2, v_2 the function q defined by $q(u,v) = (u-v)/(u+v)$ satisfies the condition

$$|q(u_1,v_1) - q(u_2,v_2)| \leqslant 2\frac{\max\{|u_1 - u_2|, |v_1 - v_2|\}}{\max\{|u_1 + v_1|, |u_2 + v_2|\}}. \tag{11.2.1}$$

Indeed,

$$\begin{aligned} |q(u_1,v_1) - q(u_2,v_2)| &= \left| \frac{2u_1v_2 - 2u_2v_1}{(u_1+v_1)(u_2+v_2)} \right| \\ &= 2\left| \frac{(u_1-u_2)v_2 + u_2(v_2-v_1)}{(u_1+v_1)(u_2+v_2)} \right| \\ &\leqslant 2\frac{\max\{|u_1-u_2|, |v_1-v_2|\}}{u_1+v_1}. \end{aligned}$$

Changing the roles of the pairs (u_1, v_1) and (u_2, v_2), one obtains

$$|q(u_1, v_1) - q(u_2, v_2)| \leqslant 2\frac{\max\{|u_1 - u_2|, |v_1 - v_2|\}}{u_2 + v_2}$$

which, together with the previous inequality, gives (11.2.1).

Finally, the statements (b), (c) (respectively, (d)) are straightforward consequences of (a) (respectively, (c)). ☐

The following result gives a characterisation of when two molecules are far apart.

LEMMA 11.2.3. *Let M be a metric space and $x, y, u, v \in M$ so that $x \neq y$ and $u \neq v$. Assume that $\|m_{x,y} \pm m_{u,v}\| > 2 - \varepsilon$. Then*

$$(1 - \varepsilon)(d(x, y) + d(u, v)) \leqslant \min\{d(x, u) + d(y, v), d(x, v) + d(y, u)\}.$$

PROOF. Since $\|m_{x,y} + m_{u,v}\| > 2 - \varepsilon$, pick a function $f \in S_{\mathrm{Lip}_0(M)}$ such that $\langle f, m_{xy} + m_{uv}\rangle > 2 - \varepsilon$. This implies that $\langle f, m_{x,y}\rangle = \frac{f(x)-f(y)}{d(x,y)} > 1 - \varepsilon$ and, similarly, $\frac{f(u)-f(v)}{d(u,v)} > 1 - \varepsilon$. Consequently

$$\begin{aligned} 1 \geqslant \frac{f(x) - f(v)}{d(x, v)} &= \frac{f(x) - f(y) + f(u) - f(v) + f(y) - f(u)}{d(x, v)} \\ &\geqslant \frac{(1 - \varepsilon)d(x, y) + (1 - \varepsilon)d(u, v) - d(y, u)}{d(x, v)}. \end{aligned}$$

Therefore, the inequality $(1 - \varepsilon)(d(x, y) + d(u, v)) \leqslant d(x, v) + d(y, u)$ holds. Using the condition $\|m_{x,y} - m_{u,v}\| > 2 - \varepsilon$, one can see that the other inequality holds, too. ☐

Now, we are ready to prove Theorem 11.2.1.

PROOF OF THEOREM 11.2.1. Let us consider the following functions:

$$f_1 = f, \quad f_2(t) = d(y, t), \quad f_3(t) = -d(x, t), \quad f_4(t) = f_{xy}(t)$$

We have $f_1(x) - f_1(y) > (1 - \varepsilon)d(x, y)$ and $f_i(x) - f_i(y) = d(x, y)$ for $i = 2, 3, 4$. Moreover, clearly $\|f_i\|_L = 1$ for $i = 1, 2, 3$, and $\|f_4\|_L = 1$ as a consequence of Lemma 11.2.2. Consider the function $g = \frac{1}{4}\sum_{i=1}^4 f_i$. First notice that

$$1 \geqslant \|g\|_L \geqslant \frac{1}{4}\sum_{i=1}^{4} \frac{f_i(x) - f_i(y)}{d(x, y)} > 1 - \frac{\varepsilon}{4}.$$

Now, by a combination of Lemmata 3.1.10 and 2.6.6, there exist $u \neq v \in M$ so that $g(u) - g(v) > (1 - \frac{\varepsilon}{4})d(u, v)$, that is,

$$\frac{1}{4}\sum_{i=1}^{4}(f_i(u) - f_i(v)) > \left(1 - \frac{\varepsilon}{4}\right) d(u, v),$$

and so that $\|m_{x,y} - m_{u,v}\| > 2 - \frac{\varepsilon}{4}$. Since $2 - \frac{\varepsilon}{4} \leqslant \langle g, m_{xy} + m_{uv}\rangle \leqslant \|m_{xy} + m_{uv}\|$, we get from Lemma 11.2.3 that

$$(1 - \varepsilon)(d(x, y) + d(u, v)) \leqslant \min\{d(x, u) + d(y, v), d(x, v) + d(y, u)\}. \tag{11.2.2}$$

Notice that each of these summands is less than or equal to $d(u, v)$. Thus, we get

$$\min\{f_i(u) - f_i(v) \colon i \in \{1, 2, 3, 4\}\} > (1 - \varepsilon)d(u, v).$$

The case $i = 1$ gives us $f(u) - f(v) > (1 - \varepsilon)d(u, v)$. Moreover, the cases $i = 2, 3$ yield

$$\min\{d(y, u) - d(y, v), d(x, v) - d(x, u)\} > (1 - \varepsilon)d(u, v). \tag{11.2.3}$$

By Lemma 11.2.2 and the case $i = 4$, we have

$$(1 - \varepsilon) \max\{d(x, v) + d(y, v), d(x, u) + d(y, u)\} < d(x, y). \tag{11.2.4}$$

The above inequalities yield

$$\begin{aligned} \frac{d(x, y)}{1 - \varepsilon} &\overset{(11.2.4)}{>} d(x, u) + d(y, u) \\ &\overset{(11.2.3)}{>} d(x, u) + d(y, v) + (1 - \varepsilon)d(u, v) \\ &\overset{(11.2.2)}{\geqslant} (1 - \varepsilon)(d(x, y) + d(u, v)) + (1 - \varepsilon)d(u, v) \end{aligned}$$

and so,

$$2(1 - \varepsilon)d(u, v) < \left(\frac{1}{1 - \varepsilon} - (1 - \varepsilon)\right) d(x, y) = \frac{\varepsilon(2 - \varepsilon)}{1 - \varepsilon} d(x, y) < \frac{2\varepsilon}{1 - \varepsilon} d(x, y)$$

as desired. □

Note that the previous lemma says that if $\mathcal{F}(M)$ has the Daugavet property, then every Lipschitz function $f\colon M \to \mathbb{R}$ approximates its Lipschitz norm at arbitrarily close points. This motivates us to introduce the following concept of local metric space.

Definition 11.2.4. Let M be a metric space. We say that M is:

(1) *local* if, for every $f \in \mathrm{Lip}_0(M)$ and every $\varepsilon > 0$, there are two distinct points $u, v \in M$ with $d(u, v) < \varepsilon$ and $\frac{f(u)-f(v)}{d(u,v)} > \|f\|_L - \varepsilon$;

(2) *spreadingly local* if for every $\varepsilon > 0$ and every Lipschitz function $f\colon M \to \mathbb{R}$ the set

$$\left\{x \in M\colon \inf_{\delta>0} \left\| f|_{B(x,\delta)} \right\|_L > \|f\|_L - \varepsilon \right\}$$

is infinite;

(3) a *length space* if, for every pair of points $x, y \in M$, the distance $d(x, y)$ is equal to the infimum of the lengths of rectifiable curves joining them;

(4) *geodesic* if, for every $x, y \in M$, there exists a curve $\alpha\colon [0, d(x, y)] \to M$ whose length equals $d(x, y)$ and so that $\alpha(0) = y$ and $\alpha(d(x, y)) = x$.

Bearing in mind the previous definition, one can formulate Theorem 11.2.1 as follows: if $\mathcal{F}(M)$ has the Daugavet property, then M is local.

Let us now focus on the properties of Definition 11.2.4. The notions of length spaces and geodesic spaces are classical and widely studied in the literature of metric spaces (see e.g. [**71**] for examples and background); for the moment we just point out that convex subsets of Banach spaces are geodesic and hence are length spaces. Let us mention here a characterisation of length spaces and geodesic spaces among *complete* metric spaces.

Remark 11.2.5. *A* complete *metric space M is a length space (respectively, geodesic) if, and only if, for every pair of different points $x, y \in M$ and for every $\varepsilon > 0$ (respectively, $\varepsilon = 0$), the set $B(x, (1 + \varepsilon)\frac{d(x,y)}{2}) \cap B(y, (1 + \varepsilon)\frac{d(x,y)}{2})$ is non-empty* [**71**, Theorem 2.4.16].

It is clear that if M is geodesic, then M is a length space. Making use of the previous characterisation, it is clear that if M is compact, the converse holds true. However, removing the assumption of compactness, there are complete length metric spaces with are not geodesic.

EXAMPLE 11.2.6. *There is a length metric space M which is not geodesic.* Let $M := \{0,1\} \cup \{x^n_t \colon n \in \mathbb{N}, t \in [0, 1+1/n]\}$ with $x^n_0 = 0$ and $x^n_{1+\frac{1}{n}} = 1$ for every $n \in \mathbb{N}$. We define the following distance: $d(0,1) = 1$, $d(x^n_t, x^n_s) = |t-s|$ for every $n \in \mathbb{N}$ and $t \in [0, 1+\frac{1}{n}]$ and

$$\begin{aligned} d(x^n_t, x^m_s) &= \min\{d(x^n_t, 0) + d(x^m_s, 0), d(x^n_t, 1) + d(x^m_s, 1)\} \\ &= \min\Big\{t+s, 2 - \frac{1}{n} - \frac{1}{m} - s - t\Big\}. \end{aligned}$$

It is not difficult to see that d defines a distance on M which is complete. Indeed, given a Cauchy sequence in M then, either there exists a tail of the sequence contained in $\{x^n_t \colon t \in [0, 1+\frac{1}{n}]\}$ for some n (in that case the convergence is immediate) or, in the other case, the definition of the distance and the Cauchy condition forces the sequence to converge either to 0 or to 1, from where the completeness follows. (The completeness of M can be also checked by identifying it with an appropriate quotient space.)

To see that M is a length space, notice that it is immediate that, given $x, y \in M$ with $x \neq y$ and $(x,y) \neq (0,1)$, then $B(x, \frac{d(x,y)}{2}) \cap B(y, \frac{d(x,y)}{2}) \neq \emptyset$. Moreover, given $\varepsilon > 0$, select $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$; then $x^n_\alpha \in B(0, \frac{1+\varepsilon}{2}) \cap B(1, \frac{1+\varepsilon}{2})$, where $\alpha = \frac{1+\frac{1}{n}}{2}$. This proves that M is a length space.

However, M is not geodesic since $B(0, \frac{1}{2}) \cap B(1, \frac{1}{2}) = \emptyset$.

The remaining properties in Definition 11.2.4 turn out to be equivalent when M is complete.

PROPOSITION 11.2.7. *Let M be a complete metric space. The following are equivalent:*

(1) *M is a length space.*
(2) *M is spreadingly local.*
(3) *M is local.*

PROOF. (1) $\Rightarrow$ (2). Pick a non-zero Lipschitz function $f \colon M \to \mathbb{R}$. We can assume, with no loss of generality, that $\|f\|_L = 1$. Pick $\varepsilon > 0$ and take $x, y \in M$ with $x \neq y$ so that $\frac{f(x)-f(y)}{d(x,y)} > 1 - \varepsilon^2$. Since M is a length space we can take a curve α: $[0, (1+\varepsilon^2)d(x,y)] \to M$ with $\alpha(0) = y$, $\alpha((1+\varepsilon^2)d(x,y)) = x$ having the arclength as parameter (and in particular α is 1-Lipschitz). Now, $f \circ \alpha$: $[0, (1+\varepsilon)d(x,y)] \to \mathbb{R}$ is 1-Lipschitz and

$$\begin{aligned} &(1-\varepsilon^2)d(x,y) < f(x) - f(y) \\ &\qquad\qquad = (f \circ \alpha)((1+\varepsilon^2)d(x,y)) - (f \circ \alpha)(0) = \int_0^{(1+\varepsilon^2)d(x,y)} (f \circ \alpha)'. \end{aligned}$$

(At this point we are using the classical fact that a Lipschitz function on an interval is differentiable almost everywhere and it is the integral of its (a.e. existing) derivative; see [**294**, Theorem 1.36].)

It is not difficult to prove that the measure of the set

$$A := \{t \in [0, (1+\varepsilon)d(x,y)] \colon (f \circ \alpha)'(t) > 1-\varepsilon\}$$

is bigger than $(1-\varepsilon)d(x,y)$, and then it follows quickly that $\alpha(A) \subset M$ is infinite. In order to finish the proof, it only remains to prove that, for every $t_0 \in A$, the inequality

$$\inf_{\delta>0} \left\| f|_{B(\alpha(t_0),\delta)} \right\|_L > 1-\varepsilon$$

holds. To this end, pick $\delta > 0$ and, since $(f \circ \alpha)'(t_0) > 1-\varepsilon$, we can find $t \neq t_0$ so that $|t-t_0| < \delta$ and $\frac{f(\alpha(t))-f(\alpha(t_0))}{|t-t_0|} > 1-\varepsilon$. Now, since $d(\alpha(t), \alpha(t_0)) \leqslant |t-t_0| < \delta$ we get that

$$1-\varepsilon < \frac{f(\alpha(t))-f(\alpha(t_0))}{|t-t_0|} \leqslant \frac{f(\alpha(t))-f(\alpha(t_0))}{d(\alpha(t),\alpha(t_0))} \leqslant \inf_{\delta>0} \|f|_{B(\alpha(t_0),\delta)}\|_L,$$

as required.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Assume that M is not a length space. Then there exist $x, y \in M$ and $\delta > 0$ such that $B(x, \frac{1+\delta}{2}d(x,y) \cap B(y, \frac{1+\delta}{2}d(x,y))) = \emptyset$. Let us denote $r := \frac{d(x,y)}{2}$. Notice in passing that

$$\operatorname{dist}(B(x,(1+\delta)r), B(y,(1+\delta)r)) \geqslant \delta r.$$

Let $f_i \colon M \to \mathbb{R}$ be defined by

$$f_1(t) = \max\Big\{r - \frac{1}{1+\delta}d(x,t), 0\Big\} \text{ and } f_2(t) = \min\Big\{-r + \frac{1}{1+\delta}d(y,t), 0\Big\}.$$

Clearly $\|f_i\|_L \leqslant \frac{1}{1+\delta}$, so $f = f_1 + f_2$ is a Lipschitz function. Since $f(x) - f(y) = d(x,y)$, we have that $\|f\|_L \geqslant 1$. Moreover, we have that

$$\{z \colon f_1(z) \neq 0\} \subset B(x,(1+\delta)r) \text{ and } \{z \colon f_2(z) \neq 0\} \subset B(y,(1+\delta)r).$$

It follows that if $\frac{f(u)-f(v)}{d(u,v)} > \frac{1}{1+\delta}$, then $u \in B(x,(1+\delta)r)$ and $v \in B(y,(1+\delta)r)$. But then $d(u,v) \geqslant \delta r$ and so M is not local. $\square$

THEOREM 11.2.8. *Let M be a spreadingly local metric space. Then* $\operatorname{Lip}_0(M)$ *has the Daugavet property.*

PROOF. In order to prove that $\operatorname{Lip}_0(M)$ has the Daugavet property we will prove, invoking Remark 3.1.12, that, for each $f, g \in S_{\operatorname{Lip}_0(M)}$ and every $\varepsilon > 0$, we have that

$$g \in \overline{\operatorname{conv}}\left(\left\{u \in (1+\varepsilon)B_{\operatorname{Lip}_0(M)} \colon \|f+u\| > 2-\varepsilon\right\}\right).$$

Fix $n \in \mathbb{N}$. Since M is spreadingly local we can find $r > 0$ and $\delta_0 > 0$ such that, for every $0 < \delta < \delta_0$, there are $x_1, y_1, \ldots, x_n, y_n \in M$ such that $d(x_i, y_i) < \delta$, $\frac{f(x_i)-f(y_i)}{d(x_i,y_i)} > 1-\varepsilon$ holds for each i and such that $B(x_i, r) \cap B(x_j, r) = \emptyset$ for all $i \neq j$. Now, for every $i \in \{1, \ldots, n\}$ and for δ small enough, we can define a $(1+\varepsilon)$-Lipschitz function $f_i \colon M \to \mathbb{R}$ such that $f_i = g$ in $M \setminus B(x_i, r) \cup \{x_i\}$ and $f_i(y_i) := f_i(x_i) - d(x_i, y_i)$. Indeed, up to an application of the McShane extension theorem [**294**, Th. 1.33] it suffices to show that f_i defined as above is $(1+\varepsilon)$-Lipschitz when restricted to $M \setminus B(x_i, r) \cup \{x_i, y_i\}$. To do so, notice that given

$z \in M \setminus B(x_i, r)$ we get that

$$\begin{aligned}\frac{|f_i(y_i) - f(z)|}{d(y_i, z)} &= \frac{|g(x_i) - g(z) - d(x_i, y_i)|}{d(y_i, z)} \\ &\leqslant \frac{d(x_i, z) + d(x_i, y_i)}{d(y_i, z)} \leqslant \frac{d(y_i, z) + 2d(x_i, y_i)}{d(y_i, z)} \\ &\leqslant 1 + \frac{2d(x_i, y_i)}{r - d(x_i, y_i)} \leqslant 1 + \frac{2\delta}{r - \delta},\end{aligned}$$

and the previous quantity is smaller than $1 + \varepsilon$ as soon as δ is chosen so small that $\frac{2\delta}{r-\delta} < \varepsilon$.

Since $f_i(x_i) - f_i(y_i) = d(x_i, y_i)$ for every i, we deduce that the inclusion

$$f_i \in \left\{u \in (1+\varepsilon)B_{\mathrm{Lip}_0(M)} \colon \|f + u\| > 2 - \varepsilon\right\}$$

holds for every $i \in \{1, \ldots, n\}$. On the other hand, notice that, given $x \in M$, the set $\{i \in \{1, \ldots, n\} \colon f_i(x) \neq g(x)\}$ is, at most, a singleton. From the definition of the Lipschitz norm, we deduce that

$$\left\| g - \frac{1}{n} \sum_{i=1}^{n} f_i \right\|_L \leqslant \frac{4 + 2\varepsilon}{n}.$$

Since n was arbitrary, we can conclude that

$$g \in \overline{\mathrm{conv}} \left(\left\{u \in (1+\varepsilon)B_{\mathrm{Lip}_0(M)} \colon \|f + u\| > 2 - \varepsilon\right\}\right). \qquad \square$$

REMARK 11.2.9. The above proof actually shows that $\mathrm{Lip}_0(M)$ *satisfies the uniform Daugavet property.* Indeed, what we have proved is that, given $f, g \in S_{\mathrm{Lip}_0(M)}$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$\mathrm{dist}\left(g, \mathrm{conv}_n\left(\{h \colon \|h\|_L \leqslant 1 + \varepsilon,\ \|f + h\|_L \geqslant 2 - \varepsilon\}\right)\right) \leqslant \frac{4 + 2\varepsilon}{n}$$

which goes to 0 as $n \to \infty$. As a consequence, we get that $\mathrm{Lip}_0(M)$ has the Daugavet property if and only if the ultrapower $\mathrm{Lip}_0(M)_{\mathfrak{U}}$ has the Daugavet property for every free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$.

Gathering together Theorem 11.2.1 in the already remarked equivalent form "if $\mathcal{F}(M)$ has the Daugavet property, then M is local", Proposition 11.2.7, Theorem 11.2.8, and Remark 11.2.9, we are ready to establish the metric characterisation of when $\mathrm{Lip}_0(M)$ enjoys the Daugavet property.

THEOREM 11.2.10. *Let M be a complete metric space. Then, the following are equivalent:*

(1) $\mathrm{Lip}_0(M)$ *has the uniform Daugavet property.*
(2) $\mathrm{Lip}_0(M)$ *has the Daugavet property.*
(3) $\mathcal{F}(M)$ *has the Daugavet property.*
(4) M *is a length space.*
(5) M *is local.*

We will finish the section proving a characterisation of when the unit ball of a Lipschitz-free space $\mathcal{F}(M)$ is an SCD set; cf. Definition 10.1.3 for this notion.

THEOREM 11.2.11. *Let M be a complete separable metric space (for instance, if M is compact). The following are equivalent:*

(i) $B_{\mathcal{F}(M)}$ *is an SCD set;*

(ii) $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}\big(\mathrm{dent}(B_{\mathcal{F}(M)})\big)$.

We need a preliminary result which is interesting in itself. In accordance with Definition 11.2.4, we say that a Lipschitz function $f \in \mathrm{Lip}_0(M)$ is *local* if, for every $\varepsilon > 0$, there exist $u, v \in M$ with $0 < d(u,v) < \varepsilon$ and $\frac{f(u)-f(v)}{d(u,v)} > \|f\|_L - \varepsilon$. In short, a local Lipschitz function is a function whose Lipschitz norm can be approximated by pairs of points which are arbitrarily close. The next lemma shows that only non-local functions are needed to construct a sequence of determining slices for the unit ball of a Lipschitz-free space.

LEMMA 11.2.12. *Let M be a metric space. Assume that a sequence of slices $S_n = \mathrm{Slice}(B_{\mathcal{F}(M)}, f_n, \alpha_n)$, $n \in \mathbb{N}$, is determining for $B_{\mathcal{F}(M)}$. Set*

$$I := \big\{n \in \mathbb{N}\colon f_n \textit{ is not local}\big\}.$$

Then $\{S_n\colon n \in I\}$ is determining for $B_{\mathcal{F}(M)}$.

PROOF. Assume, by contradiction, that $\{S_n\colon n \in I\}$ is not determining for $B_{\mathcal{F}(M)}$. Consequently, for every $n \in I$ there exist $x_n \in S_n$ and there exists $\mu \in S_{\mathcal{F}(M)}$ satisfying that $\mu \notin \overline{\mathrm{conv}}(\{x_n\colon n \in I\})$. By the Hahn-Banach theorem, there exist $f \in S_{\mathrm{Lip}_0(M)}$ and $\alpha > 0$ such that

$$f(\mu) > \alpha > \sup\big\{f(z)\colon z \in \overline{\mathrm{conv}}(\{x_n\colon n \in I\})\big\}.$$

Furthermore, we can find $0 < \beta < \alpha$ satisfying

$$f(\mu) > \alpha > \beta > f(x_n)$$

for every $n \in I$. Let us also find $\varepsilon, \eta > 0$ small enough so that

$$(1-\beta)(\eta + 2\varepsilon) + \eta < \alpha - \beta. \tag{11.2.5}$$

Now, set $J = \mathbb{N} \setminus I = \{n \in \mathbb{N}\colon f_n \text{ is local}\}$ and write $J = \{k_n\colon n \in \mathbb{N}\}$ (admitting that (k_n) may be eventually constant if J is finite). Choose a sequence (ε_n) of positive real numbers such that $1 - \varepsilon < \prod_{n=1}^{\infty}(1-\varepsilon_n)$. Our aim is to construct, by induction, a sequence $(x_{k_n}) \subset S_{k_n}$ with the property that

$$\left\|\mu + \sum_{i=1}^{n} \lambda_i x_{k_i}\right\| > \Big[\prod_{i=1}^{n}(1-\varepsilon_i)\Big]\Big(1 + \sum_{i=1}^{n}|\lambda_i|\Big) \tag{11.2.6}$$

for every $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Let us construct x_{k_1}. Since f_{k_1} is local, we can find a sequence of points $u_j, v_j \in M$ with $0 < d(u_j, v_j) \to 0$ such that $f_{k_1}(m_{u_j,v_j}) = \frac{f_{k_1}(u_j) - f_{k_1}(v_j)}{d(u_j,v_j)} > 1 - \alpha_{k_1}$ or, in other words, that $m_{u_j,v_j} \in S_{k_1}$ for every $j \in \mathbb{N}$. Since $d(u_j, v_j) \to 0$, [**152**, Theorem 2.6] implies that

$$\|\nu + m_{u_j,v_j}\| \to 1 + \|\nu\|$$

holds for every $\nu \in \mathcal{F}(M)$. Consequently, we can find $j \in \mathbb{N}$ big enough so that $x_{k_1} := m_{u_j,v_j}$ satisfies

$$\|\mu \pm x_{k_1}\| > 2 - \frac{\varepsilon_1}{3}.$$

Notice that an application of Lemma 2.8.7 ensures that

$$\|\mu + \lambda x_{k_1}\| > (1-\varepsilon_1)(1 + |\lambda|)$$

holds for every $\lambda \in \mathbb{R}$, hence equation (11.2.6) is satisfied for this choice of x_{k_1}.

Now, assume for the inductive step that $x_{k_1}, \ldots x_{k_n}$ have been constructed with the desired property, and let us construct $x_{k_{n+1}}$. In order to do so, let $Y :=$

$\mathrm{lin}\{\mu, x_{k_1}, \ldots, x_{k_n}\}$, which is a finite-dimensional subspace of $\mathcal{F}(M)$. Since S_Y is compact as Y is finite-dimensional, we can select a finite set $F \subset S_Y$ which is an $\frac{\varepsilon_{n+1}}{3}$-net for S_Y. Once again the condition that $f_{k_{n+1}}$ is local allows us to guarantee the existence of a sequence $m_{u_j,v_j} \in S_{k_{n+1}}$ such that $d(u_j, v_j) \to 0$. Since

$$\|\nu + m_{u_j,v_j}\| \to 2$$

for every $\nu \in F$, again by [**152**, Theorem 2.6], we can find $j \in \mathbb{N}$ large enough so that, if we select $x_{k_{n+1}} := m_{u_j,v_j}$, we have $\|\nu \pm x_{k_{n+1}}\| > 2 - \frac{\varepsilon_{n+1}}{3}$ for every $\nu \in F$ (since F is finite). As F is an $\varepsilon_{n+1}/2$-net, a new appeal to Lemma 2.8.7 implies that

$$\|\nu \pm x_{k_{n+1}}\| > 2 - \varepsilon_{n+1}$$

holds for every $\nu \in S_Y$. From here, it can be proved that

$$\|\nu + \lambda x_{k_{n+1}}\| > (1 - \varepsilon_{n+1})(\|\nu\| + |\lambda|)$$

holds for every $\nu \in Y$ and every $\lambda \in \mathbb{R}$. Let us prove that $x_{k_1}, \ldots, x_{k_{n+1}}$ satisfy the desired condition. In order to do so, select $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$. Observe that, since $\mu + \sum_{i=1}^n \lambda_i x_{k_i} \in Y$, we obtain

$$\left\|\mu + \sum_{i=1}^{n+1} \lambda_i x_{k_i}\right\| \geqslant (1 - \varepsilon_{n+1}) \left(\left\|\mu + \sum_{i=1}^{n} \lambda_i x_{k_i}\right\| + |\lambda_{n+1}|\right).$$

Now, the inductive step implies $\left\|\mu + \sum_{i=1}^n \lambda_i x_{k_i}\right\| \geqslant \left[\prod_{i=1}^n (1-\varepsilon_i)\right]\left(1 + \sum_{i=1}^n |\lambda_i|\right)$, so

$$\begin{aligned}
\left\|\mu + \sum_{i=1}^{n+1} \lambda_i x_{k_i}\right\| &\geqslant (1 - \varepsilon_{n+1}) \left(\left[\prod_{i=1}^n (1-\varepsilon_i)\right]\left(1 + \sum_{i=1}^n |\lambda_i|\right) + |\lambda_{n+1}|\right) \\
&\geqslant (1 - \varepsilon_{n+1}) \left(\left[\prod_{i=1}^n (1-\varepsilon_i)\right]\left(1 + \sum_{i=1}^n |\lambda_i|\right) + \prod_{i=1}^n (1-\varepsilon_i)|\lambda_{n+1}|\right) \\
&= \left[\prod_{i=1}^{n+1} (1-\varepsilon_i)\right]\left(1 + \sum_{i=1}^{n+1} |\lambda_i|\right)
\end{aligned}$$

which finishes the proof of the construction of $x_{k_{n+1}}$.

As we have $x_n \in S_n$ for every $n \in \mathbb{N}$ and $\{S_n \colon n \in \mathbb{N}\}$ is determining for μ, we conclude $\mu \in \overline{\mathrm{conv}}(\{x_n \colon n \in \mathbb{N}\})$. Consequently, we can find $(\lambda_n) \subset [0, 1]$ with $\sum_{n=1}^\infty \lambda_n = 1$ and only finitely many λ_n being non-zero, satisfying

$$\left\|\mu - \sum_{n=1}^\infty \lambda_n x_n\right\| < \eta. \tag{11.2.7}$$

If we evaluate at f, we obtain

$$\begin{aligned}
\eta > f\left(\mu - \sum_{n=1}^\infty \lambda_n x_n\right) &= f(\mu) - \sum_{n=1}^\infty \lambda_n f(x_n) \\
&> \alpha - \sum_{n \in I} \lambda_n f(x_n) - \sum_{n \in J} \lambda_n f(x_n) \geqslant \alpha - \beta \sum_{n \in I} \lambda_n - \sum_{n \in J} \lambda_n \\
&= \alpha - \beta\left(1 - \sum_{n \in J} \lambda_n\right) - \sum_{n \in J} \lambda_n = \alpha - \beta - (1 - \beta) \sum_{n \in J} \lambda_n,
\end{aligned}$$

hence

$$\sum_{n\in J}\lambda_n > \frac{\alpha-\beta-\eta}{1-\beta}. \tag{11.2.8}$$

On the other hand, by the construction of x_n, $n \in J$, we have from (11.2.6) that

$$\left\|\mu-\sum_{n\in J}\lambda_n x_n\right\| > (1-\varepsilon)\Big(1+\sum_{n\in J}\lambda_n\Big).$$

These estimations imply that

$$\begin{aligned}
\eta > \left\|\mu-\sum_{n=1}^{\infty}\lambda_n x_n\right\| &\geqslant \left\|\mu-\sum_{n\in J}\lambda_n x_n\right\| - \left\|\sum_{n\in I}\lambda_n x_n\right\| \\
&> (1-\varepsilon)\Big(1+\sum_{n\in J}\lambda_n\Big) - \sum_{n\in I}\lambda_n \\
&= (1-\varepsilon)\Big(1+\sum_{n\in J}\lambda_n\Big) - \Big(1-\sum_{n\in J}\lambda_n\Big) \\
&= 2\sum_{n\in J}\lambda_n - \varepsilon\Big(1+\sum_{n\in J}\lambda_n\Big) \geqslant 2\sum_{n\in J}\lambda_n - 2\varepsilon \\
&> 2\frac{\alpha-\beta-\eta}{1-\beta} - 2\varepsilon > \frac{\alpha-\beta-\eta}{1-\beta} - 2\varepsilon.
\end{aligned}$$

This, in turn, yields that $\alpha-\beta < (1-\beta)(\eta+2\varepsilon)+\eta$, which disagrees with (11.2.5). This contradiction finishes the proof. □

We are now ready to present the pending proof.

Proof of Theorem 11.2.11. (ii) ⇒ (i) follows from Proposition 10.1.10.

For the proof of (i) ⇒ (ii), take a determining sequences of slices $S_n = \operatorname{Slice}(B_{\mathcal{F}(M)}, f_n, \alpha_n)$, $n \in \mathbb{N}$, for $B_{\mathcal{F}(M)}$. By Lemma 11.2.12 we can assume without loss of generality that $f_n \in S_{\mathrm{Lip}_0(M)}$ is non-local for every $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, since M is complete and f_n is non-local, [**293**, Proposition 2.7] implies that there exists a denting point x_n of $B_{\mathcal{F}(M)}$ with $x_n \in S_n$ for every $n \in \mathbb{N}$. Since $\{S_n\colon n \in \mathbb{N}\}$ is determining, we get that $B_{\mathcal{F}(M)} \subset \overline{\operatorname{conv}}(\{x_n\colon n \in \mathbb{N}\}) \subset \overline{\operatorname{conv}}(\operatorname{dent}(B_{\mathcal{F}(M)})$, as requested. □

11.3. The Daugavet property in spaces of vector-valued Lipschitz functions

In the above section we have given a characterisation of when a space $\mathrm{Lip}_0(M)$ has the Daugavet property in terms of the property that M is a length space. A natural question at this point is: if M is a length space, does $\mathrm{Lip}_0(M,X)$ have the Daugavet property regardless of X? This would be analogous to the Daugavet property of $L_\infty(\mu,X)$ for nonatomic μ.

Observe that the proof of Theorem 11.2.8 cannot be performed in the vector-valued setting because the McShane extension theorem is applied at a decisive juncture, and the McShane extension theorem is false in the vector-valued setting (cf. e.g. [**51**, Theorem 2.11]).

Because of that, a new approach is needed to prove the Daugavet property for a vector-valued Lipschitz function space. The main aim of this section is to prove the following result.

THEOREM 11.3.1. *Let M be a length metric space and let X be a Banach space. Then* $\mathrm{Lip}_0(M,X)$ *has the Daugavet property.*

In the proof of this result we will need a couple of lemmata in order to find this new approach. Let us start with the following result, which will be very useful in the future. Recall that by segment in a Banach space we mean the convex hull of two (distinct) points.

LEMMA 11.3.2. *Let X, Y be Banach spaces, $\lambda \geqslant 0$ and $f\colon X \to Y$. If f is λ-Lipschitz when restricted to segments, then f is λ-Lipschitz.*

LEMMA 11.3.3. *Let X be any Banach space, $a < b < c$ in $\mathbb{R}$ and $f\colon [a,c] \to X$. If $f|_{[a,b]}$ and $f|_{[b,c]}$ are λ-Lipschitz for some $\lambda \geqslant 0$, then f is λ-Lipschitz.*

PROOF. Let $t \neq s \in [a,c]$. We may assume that $t \in [a,b]$ and $s \in [b,c]$. Then,

$$\begin{aligned}\|f(t)-f(s)\| &\leqslant \|f(t)-f(b)\| + \|f(b)-f(s)\| \\ &\leqslant \lambda(|t-b|+|b-s|) = \lambda|t-s|.\end{aligned}$$

□

LEMMA 11.3.4. *Let X be a Banach space, let $0 < r < R$, and consider a function $f\colon X \to X$. Assume that there are $A, B, C > 0$ such that*

$$\sup_{x,y\in B(0,r),x\neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|} \leqslant A,$$

$$\sup_{r\leqslant\|x\|,\|y\|\leqslant R,x\neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|} \leqslant B,$$

$$\sup_{x,y\in X\setminus B(0,R),x\neq y} \frac{\|f(x)-f(y)\|}{\|x-y\|} \leqslant C.$$

Then f is Lipschitz and $\|f\|_L \leqslant \max\{A,B,C\}$.

PROOF. A combination of Lemma 11.3.2 and Lemma 11.3.3 does the trick. □

The above lemma allows us to ease the proof of the following proposition.

PROPOSITION 11.3.5. *Let X be a Banach space and let $0 < a < b$. Then the function $f\colon X \to X$ defined by*

$$f(x) := \begin{cases} 0 & \text{if } \|x\| \leqslant a, \\ \dfrac{b}{b-a}\left(1-\dfrac{a}{\|x\|}\right)x & \text{if } a \leqslant \|x\| \leqslant b, \\ x & \text{if } b \leqslant \|x\|, \end{cases}$$

is Lipschitz with $\|f\|_L \leqslant \frac{b}{b-a}$.

In particular, for every $x_0 \in X$ and every $R, \varepsilon > 0$, there exist $\delta > 0$ and a Lipschitz-mapping $\psi\colon X \to X$ such that $\psi(x) = x$ holds for every $x \in X \setminus B(x_0,R)$, $\|\psi\|_L \leqslant 1+\varepsilon$ and $\psi(z) = x_0$ holds for every $z \in B(x_0,\delta)$.

PROOF. Let $x, y \in X$ with $x \neq y$. By virtue of Lemma 11.3.4, and since f is clearly Lipschitz on $B(0,a)$ and on $X \setminus B(0,b)$, let us assume that $a \leqslant \|x\|, \|y\| \leqslant b$.

Now, we estimate $\|f(x)-f(y)\|$:

$$\begin{aligned}\|f(x)-f(y)\| &= \frac{b}{b-a}\left\|y-x+\frac{a}{\|x\|}x-\frac{a}{\|y\|}y\right\| \\ &= \frac{b}{b-a}\left\|y-x+\frac{a}{\|x\|}x-\frac{a}{\|y\|}y+\frac{a}{\|x\|}y-\frac{a}{\|x\|}y\right\| \\ &\leqslant \frac{b}{b-a}\left(\left(1-\frac{a}{\|x\|}\right)\|x-y\|+\left|\frac{a}{\|x\|}-\frac{a}{\|y\|}\right|\|y\|\right) \\ &= \frac{b}{b-a}\left(\left(1-\frac{a}{\|x\|}\right)\|x-y\|+\frac{a|\|x\|-\|y\||}{\|x\|\|y\|}\|y\|\right) \\ &\leqslant \frac{b}{b-a}\left(\left(1-\frac{a}{\|x\|}\right)\|x-y\|+\frac{a}{\|x\|}\|x-y\|\right)=\frac{b}{b-a}\|x-y\|,\end{aligned}$$

as desired.

For the second part of the theorem, given $x_0 \in X$ and $R, \varepsilon > 0$, take $\delta > 0$ such that $\frac{R}{R-\delta} < 1+\varepsilon$, and consider the function f above with the parameters $a = \delta$ and $b = R$. Then the function $\psi\colon X \to X$, $\psi(x) := x_0 + f(x - x_0)$, does the trick. $\square$

The following lemma shows that the set of Lipschitz functions which are injective on a given separated sequence is norm-dense.

LEMMA 11.3.6. *Let M be a metric space and let X be a Banach space. Consider a sequence $\big(B(x_n, r_n)\big)_{n\in\mathbb{N}}$ of pairwise disjoint balls in M. Then, for every Lipschitz function $F\colon M \to X$ and $\varepsilon > 0$, there exists a Lipschitz function $G\colon M \to X$ with the following properties:*

(1) $\|F - G\|_L < \varepsilon$ *and,*
(2) $G(x_n) \neq G(x_m)$ *holds for every* $n \neq m$.

PROOF. Given $n \in \mathbb{N}$ we can take, by the McShane extension theorem, a Lipschitz function $\varphi_n\colon M \to \mathbb{R}$ such that $\varphi_n(x_n) \neq 0$ for every $n \in \mathbb{N}$, $\|\varphi_n\|_L = 1$ and $\varphi_n = 0$ on $M \setminus B(x_n, r_n)$. Let F and ε as in the hypothesis, and consider a sequence (ε_n) of strictly positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$. Take also $x \in S_X$.

We will construct by induction a sequence (δ_n) of positive numbers such that, for every $n \in \mathbb{N}$, the following conditions hold:

(1) $\delta_n \leqslant \varepsilon_n$ for every $n \in \mathbb{N}$, and
(2) $F(x_i) + \delta_i\varphi(x_i)x \neq F(x_n) + \delta_n\varphi(x_n)x$ for $1 \leqslant i \leqslant n-1$.

For $n = 1$ take $\delta_1 = \varepsilon_1$. Now, assume $\delta_1, \ldots, \delta_n$ have been constructed and let us construct δ_{n+1}. In order to do so, observe that the set

$$\{F(x_{n+1}) + \delta\varphi_{n+1}(x_{n+1})x\colon 0 < \delta < \varepsilon_{n+1}\}$$

is infinite since x is a non-zero vector. Since the set

$$\{F(x_i) + \delta_i\varphi_i(x_i)x\colon 1 \leqslant i \leqslant n\}$$

is finite, we can find $0 < \delta_{n+1} < \varepsilon_{n+1}$ such that

$$F(x_{n+1}) + \delta_{n+1}\varphi_{n+1}(x_{n+1})x \notin \{F(x_i) + \delta_i\varphi_i(x_i)x\colon 1 \leqslant i \leqslant n\}.$$

Then $G = F + \sum_{n=1}^{\infty} \delta_n\varphi_n \otimes x$ satisfies our requirements. To begin with, the inequality $\|F - G\|_L \leqslant \sum_{n=1}^{\infty} \delta_n\|\varphi_n\|_L\|x\| \leqslant \sum_{n=1}^{\infty} \varepsilon_n < \varepsilon$ holds. On the other hand, the construction of δ_n, together with the fact that $\operatorname{supp}(\varphi_n) \subset B(x_n, r_n)$, implies that $G(x_n) = F(x_n) + \delta_n \varphi_n(x_n)x$, from where the proof follows. $\square$

Now, we introduce the following result, whose proof is easy, and which will save us a lot of notation in the following. The proof is straightforward, but let us include it for the sake of completeness.

LEMMA 11.3.7. *Let M be a complete metric space and X be a Banach space. Let $f, g\colon M \to X$ be two Lipschitz functions. Assume that there exist $m \in M$ and $0 < \delta < R$ so that*

(1) *g is constant on $B(m, R)$,*
(2) *$f(x) = f(m)$ holds for every $x \in M \setminus B(m, \delta)$.*

Then $\|f + g\| \leqslant \max\{\|f\|, \|g\|\}(1 + \frac{2\delta}{R-\delta})$.

PROOF. Let $C := \max\{\|f\|, \|g\|\}$, and let $x, y \in M$ with $x \neq y$. Let us estimate

$$A := \frac{\|(f(x) + g(x)) - (f(y) + g(y))\|}{d(x, y)} = \frac{\|f(x) - f(y) + g(x) - g(y)\|}{d(x, y)}.$$

We observe that if $f(x) - f(y) = 0$ or $g(x) - g(y) = 0$, then $A \leqslant C$.

The unique possibility for the previous condition not to hold is that, up to relabelling, $x \notin B(m, R)$ and $y \in B(m, \delta)$. In that case, $f(x) = f(m)$ and $g(y) = g(m)$. Consequently

$$A \leqslant \frac{\|f(y) - f(m)\| + \|g(x) - g(m)\|}{d(x, y)} \leqslant C\frac{d(y, m) + d(x, m)}{d(x, y)}.$$

Since $d(x, m) \leqslant d(x, y) + d(y, m)$, the above inequality yields

$$A \leqslant C\frac{d(x, y) + 2d(y, m)}{d(x, y)} = C\left(1 + \frac{2d(y, m)}{d(x, y)}\right).$$

Now, using $d(x, y) \geqslant d(x, m) - d(y, m) \geqslant R - \delta$, we get

$$A \leqslant C\left(1 + \frac{2\delta}{R - \delta}\right),$$

as desired. $\square$

Let us end our preliminaries by recalling the following criterion of weakly null sequences in $\operatorname{Lip}_0(M, X)$ from [**75**].

LEMMA 11.3.8. *Let M be a pointed metric space, let X be a Banach space, and let (f_n) be a sequence of functions in the unit ball of $\operatorname{Lip}_0(M, X)$. For each $n \in \mathbb{N}$, we write $U_n := \{x \in M\colon f_n(x) \neq 0\}$. If $U_n \cap U_m = \emptyset$ for every $n \neq m$, then the sequence (f_n) is weakly null.*

PROOF. We shall establish the inequality

$$\Big\|\sum_{j=1}^{n} a_j f_j\Big\|_L \leqslant 2\max_j |a_j| \qquad (a_1, \ldots, a_n \in \mathbb{R},\ n \in \mathbb{N}).$$

The inequality implies that the map $e_n \mapsto f_n$ extends to a bounded linear operator $T\colon c_0 \to \operatorname{Lip}_0(M, X)$; hence $f_n = T(e_n) \to 0$ weakly since $e_n \to 0$ weakly in c_0.

To prove the above inequality, fix $a_1, \ldots, a_n$ and write $f = \sum_{j=1}^{n} a_j f_j$. We have to estimate

$$\Big\|\frac{f(x) - f(y)}{d(x, y)}\Big\| \leqslant 2\max_j |a_j|$$

for $x \neq y \in M$. If x and y do not belong to two different of the sets U_j, we get

$$\left\| \frac{f(x)-f(y)}{d(x,y)} \right\| \leqslant |a_k|$$

for a certain k since $\|f_j\|_L \leqslant 1$. Otherwise, we have $x \in U_k$ and $y \in U_l$, say, (with $k \neq l$) and

$$\begin{aligned}\left\| \frac{f(x)-f(y)}{d(x,y)} \right\| &= |a_k| \left\| \frac{f_k(x)-f_k(y)}{d(x,y)} \right\| + |a_l| \left\| \frac{f_l(x)-f_l(y)}{d(x,y)} \right\| \\ &\leqslant |a_k| + |a_l| \leqslant 2 \max_j |a_j|.\end{aligned}$$

(Actually, one can show that (f_n) is even equivalent to the unit vector basis of c_0 if $\inf \|f\|_L > 0$.) $\square$

We already have all the needed ingredients to provide the pending proof.

PROOF OF THEOREM 11.3.1. Let $f, g \in S_{\mathrm{Lip}_0(M,X)}$. In order to prove that $\mathrm{Lip}_0(M,X)$ has the Daugavet property, let us prove that for every $\varepsilon > 0$ there exists a sequence $(g_n) \subset (1+\varepsilon)B_{\mathrm{Lip}_0(M,X)}$ such that $g_n \to g$ weakly and $\|f+g_n\|_L \geqslant 2-\varepsilon$ holds for every $n \in \mathbb{N}$. This is enough by Lemma 3.1.19, for instance.

Since $\|f\|_L = 1$, we can find $y^* \in S_{X^*}$ such that $y^* \circ f\colon M \to \mathbb{R}$, given by $y^* \circ f(m) := y^*(f(m))$, satisfies $\|y^* \circ f\|_L > 1 - \frac{\varepsilon}{4}$.

Since M is a length space, it is spreadingly local (see Proposition 11.2.7), that is, the set

$$A = \left\{ m \in M\colon \inf_{r>0} \|y^* \circ f|_{B(m,r)}\| > 1 - \frac{\varepsilon}{2} \right\}$$

is infinite. Hence we can take a sequence of pairwise distinct points $(x_n) \subset A$. After passing to a subsequence we may also assume that (x_n) satisfies the assumptions of Lemma 11.3.6; cf. Lemma 3.5.24. An application of Lemma 11.3.6 allows us to assume, up to a norm-perturbation argument, that $g(x_n) \neq g(x_k)$ if $n \neq k$. As above, after passing to another subsequence we can find, for every $n \in \mathbb{N}$, some $\alpha_n > 0$ such that the $B(g(x_n), \alpha_n)$, $n \in \mathbb{N}$, are pairwise disjoint (observe that since $\|g\|_L \leqslant 1$ it is clear that then the $B(x_n, \alpha_n)$ are also pairwise disjoint).

Consider $0 < \beta_n < \alpha_n$ for every $n \in \mathbb{N}$ such that $\frac{\alpha_n}{\alpha_n - \beta_n} \to 1$ and consider, by virtue of Proposition 11.3.5, Lipschitz functions $\varphi_n\colon X \to X$ such that $\|\varphi_n\|_L \leqslant \frac{\alpha_n}{\alpha_n - \beta_n}$ for every $n \in \mathbb{N}$, that $\varphi_n(x) = x$ for every $x \in X \setminus B(g(x_n), \alpha_n)$ and $\varphi_n(x) = g(x_n)$ for every $x \in B(g(x_n), \beta_n)$.

For every $n \in \mathbb{N}$ write $h_n := \varphi_n \circ g\colon M \to X$. It follows that $\limsup \|h_n\|_L \leqslant 1$ (since $\|h_n\|_L \leqslant \|\varphi_n\|_L \|g\|_L \leqslant \frac{\alpha_n}{\alpha_n - \beta_n}$).

Now, we claim that $(h_n - g)$ is a sequence of mappings with pairwise disjoint supports. Indeed, given $n \in \mathbb{N}$, it follows that $h_n(x) - g(x) \neq 0$ implies $g(x) \in B(g(x_n), \alpha_n)$ which, in other words, means that $\mathrm{supp}(h_n - g) \subset g^{-1}(B(g(x_n), \alpha_n))$ for every $n \in \mathbb{N}$. The fact that the $\mathrm{supp}(h_n - g)$ are pairwise disjoint is now immediate since the balls $B(g(x_n), \alpha_n)$ are pairwise disjoint. Consequently, $(h_n - g)$ is weakly null in view of Lemma 11.3.8 or, equivalently, $h_n \to g$ weakly.

On the other hand, observe that $h_n = \varphi_n \circ g$ takes the value $g(x_n)$ on $B(x_n, \beta_n)$. Indeed, given $z \in B(x_n, \beta_n)$ it follows that $\|g(z) - g(x_n)\| \leqslant \|g\|_L d(z, x_n) \leqslant \beta_n$, which implies $\varphi_n(g(z)) = g(x_n)$ by the very definition of φ_n.

Now, consider a sequence (r_n) of strictly positive numbers such that $2r_n < \beta_n$ and $\frac{r_n}{\beta_n - 2r_n} \to 0$. Since $\{x_n : n \in \mathbb{N}\} \subset A$, we can find $y_n \in M$ with $0 < d(x_n, y_n) < r_n$ such that

$$\frac{y^*(f(y_n)) - y^*(f(x_n))}{d(y_n, x_n)} > 1 - \frac{\varepsilon}{2}.$$

Define a function $s_n : M \to \mathbb{R}$ with $\|s_n\| \leqslant 1$, $s_n(z) = 0$ if $z \in M \setminus B(x_n, 2r_n)$, $s_n(x_n) = 0$ and $s_n(y_n) = d(x_n, y_n)$. This function may be constructed as the McShane extension to M of the function

$$\widetilde{s}_n : (M \setminus B(x_n, 2r_n)) \cup \{x_n, y_n\} \to \mathbb{R}$$

defined as it has just been stated, which is easily seen to be 1-Lipschitz.

Since $\|y^*\| = 1$, one can find $y \in S_X$ such that $y^*(y) > 1 - \frac{\varepsilon}{2}$. Consider $S_n := s_n \otimes y : M \to X$. Observe that $\|S_n\| \leqslant 1$. Moreover, (S_n) is a sequence of Lipschitz functions with pairwise disjoint supports, since the support of S_n is contained in $B(x_n, 2r_n) \subset B(x_n, \alpha_n)$. Consequently, we get $S_n \to 0$ weakly by Lemma 11.3.8.

Define $g_n := h_n + S_n$, and we claim that the sequence (g_n) does the trick. On the one hand, the convergence conditions on h_n and S_n imply that $g_n \to g$ weakly. On the other hand, given $n \in \mathbb{N}$, an appeal to Lemma 11.3.7 for $m = x_n$, $R = \beta_n$ and $\delta = 2r_n$, implies

$$\|g_n\| \leqslant \frac{\alpha_n}{\alpha_n - \beta_n}\left(1 + \frac{4r_n}{\beta_n - 2r_n}\right) \to 1.$$

Consequently, there exists $k \in \mathbb{N}$ such that $g_n \in (1+\varepsilon)B_{\mathrm{Lip}_0(M,X)}$ for $n \geqslant k$. Finally, given $n \in \mathbb{N}$, taking into account that $h_n(x_n) = h_n(y_n)$ since $y_n \in B(x_n, \beta_n)$, we have that

$$\begin{aligned}
\|f + g_n\|_L &\geqslant y^*\left(\frac{(f + g_n)(y_n) - (f + g_n)(x_n)}{d(y_n, x_n)}\right) \\
&= \frac{y^*(f(y_n)) - y^*(f(x_n))}{d(y_n, x_n)} + \frac{y^*(S_n(y_n)) - y^*(S_n(x_n))}{d(y_n, x_n)} \\
&> 1 - \frac{\varepsilon}{2} + \frac{s_n(y_n) - s_n(x_n)}{d(y_n, x_n)} y^*(y) > 1 - \frac{\varepsilon}{2} + 1 - \frac{\varepsilon}{2} = 2 - \varepsilon. \qquad \square
\end{aligned}$$

As a particular consequence, we have the following corollary.

COROLLARY 11.3.9. *Let M be a metric space. The following are equivalent:*

(1) $\mathrm{Lip}_0(M, X)$ *has the Daugavet property for every Banach space* $X \neq \{0\}$.
(2) $\mathcal{F}(M) \mathbin{\widehat{\otimes}_\pi} X$ *has the Daugavet property for every Banach space* $X \neq \{0\}$.
(3) $\mathrm{Lip}_0(M)$ *has the Daugavet property.*
(4) $\mathcal{F}(M)$ *has the Daugavet property.*
(5) M *is a length space.*

PROOF. We know from Theorem 11.2.10 that (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5), and Theorem 11.3.1 says that (5) $\Rightarrow$ (1). It remains to argue that (1) $\Rightarrow$ (2).

Let X and Y be Banach spaces. By the universal property of the Lipschitz free space, a Lipschitz function $F : M \to Y$ gets linearised to a linear operator $T_F : \mathcal{F}(M) \to Y$ of the same norm, and vice versa. In other words, $\mathrm{Lip}_0(M, Y)$ is isometrically isomorphic to $L(\mathcal{F}(M), Y)$. Now, suppose that $Y = X^*$; then $L(\mathcal{F}(M), Y)$ is isometrically isomorphic to $(\mathcal{F}(M) \mathbin{\widehat{\otimes}_\pi} X)^*$. As a result, we have that $(\mathcal{F}(M) \widehat{\otimes}_\pi X)^* = \mathrm{Lip}_0(M, X^*)$. Since a Banach space has the Daugavet property once its dual has, the implication (1) ⇒ (2) follows. □

11.4. The Daugavet equation for Lipschitz maps

Our goal in this section is to transfer some known results for linear operators on spaces with the Daugavet property to Lipschitz maps.

Remark that in the case of $X \in \mathrm{DPr}$, the Daugavet equation in the Lipschitz norm (11.1.1) for a non-linear Lipschitz map $T\colon X \to X$ reduces to an analogous equation for the linearisation $\widehat{T}$, but this linearisation acts from the Lipschitz-free space $\mathcal{F}(X)$ to X. Hence, in order to use this technique, we need to pass from the Daugavet property of a space to the Daugavet centre $\beta_X\colon \mathcal{F}(X) \to X$. On the one hand, this shows how Daugavet centres between two different spaces appear in a natural way when one studies the ordinary Daugavet property and, on the other hand, explains why the general setting of Daugavet centres is the natural framework for the theory we build below.

11.4.1. Lipschitz slices. To deal with Lipschitz maps, we will use the following geometric notion which will play the role of the usual slices in the linear case.

DEFINITION 11.4.1. Let X be a Banach space. A *Lip-slice* of S_X is a non-empty set of the form

$$\left\{\frac{x_1 - x_2}{\|x_1 - x_2\|}\colon x_1 \neq x_2,\ \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > \alpha\right\},$$

where $f \in \mathrm{Lip}_0(X)$ is non-zero and $\alpha \in \mathbb{R}$. The following notation will be useful: for $f \in \mathrm{Lip}_0(X) \setminus \{0\}$ and $\varepsilon > 0$, we write

$$\mathrm{LipSlice}(S_X, f, \varepsilon) := \left\{\frac{x_1 - x_2}{\|x_1 - x_2\|}\colon x_1 \neq x_2,\ \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > \|f\|_L - \varepsilon\right\}$$

and observe that this set is never empty and so, it is a Lip-slice of S_X; conversely, every Lip-slice of S_X can be written in this form.

Remark that for a real-linear functional $f = x^* \in X^*$, the above definition agrees with the formula for slices, generated by linear functionals

$$\mathrm{Slice}(S_X, x^*, \varepsilon) = \{x \in S_X\colon x^*(x) > \|x^*\| - \varepsilon\}.$$

Another relation with ordinary "linear" slices comes from the linearisation procedure explained in Lemma 11.1.2. As we have that $\mathrm{Mol}(X) \subset \mathcal{F}(X)$ and $\mathcal{F}(X)^* = \mathrm{Lip}_0(X)$, every slice of $\mathrm{Mol}(X)$ is of the form

$$\mathrm{Slice}(\mathrm{Mol}(X), f, \alpha) = \left\{\frac{\delta_{x_1} - \delta_{x_2}}{\|x_1 - x_2\|}\colon x_1 \neq x_2 \in X,\ \left\langle f, \frac{\delta_{x_1} - \delta_{x_2}}{\|x_1 - x_2\|}\right\rangle > \|f\|_L - \alpha\right\},$$

where $f \in \mathrm{Lip}_0(X)$ and α is a positive real number. Then

$$\beta_X\big(\mathrm{Slice}(\mathrm{Mol}(X), f, \alpha)\big) = \mathrm{LipSlice}(S_X, f, \alpha).$$

Even though there are Lip-slices which are not slices, the next result ensures that Lip-slices always contain ordinary "linear" slices. In order to show this, we need to introduce a bit of notation. Let X be a Banach space and $f\colon X \to \mathbb{R}$ a Lipschitz function. According to [85], the *generalised derivative* of f at a point $x \in X$ in the direction $v \in X$ is defined by

$$f^\circ(x,v) := \limsup_{y\to x, t\searrow 0} \frac{f(y+tv)-f(y)}{t}.$$

This limsup always exists thanks to the Lipschitz condition. Moreover, it is a sublinear and positively homogeneous function in the variable v [85, Proposition 2.1.1].

In addition, the generalised gradient of f at x is defined as follows

$$\partial f(x) := \{x^* \in X^* \colon f^\circ(x,v) \geqslant x^*(v)\ \forall v \in X\}.$$

Given $v \in X$, it follows that

$$f^\circ(x,v) = \max_{x^*\in\partial f(x)} x^*(v) \quad \forall x \in X$$

(see [85, Proposition 2.1.2]). The previous equality will be the key to proving the promised lemma.

LEMMA 11.4.2. *Let X be a Banach space and S be a Lip-slice of S_X. Then, for each $x \in S$ there exists a slice T of S_X such that*

$$x \in T \subset S.$$

PROOF. Assume that $S := S(S_X, f, \varepsilon)$. Consider $x, y \in X$, $x \neq y$ such that $\frac{y-x}{\|y-x\|} \in S$, i.e.,

$$f(y) - f(x) > (1-\varepsilon)\|y-x\|.$$

Define $\phi\colon [0,1] \to [x,y]$ by $\phi(t) := \lambda y + (1-\lambda)x$, $t \in [0,1]$, and $F := f\circ\phi\colon [0,1] \to \mathbb{R}$. As F is a Lipschitz function, it is differentiable a.e. Moreover,

$$(1-\varepsilon)\|y-x\| < f(y) - f(x) = F(1) - F(0) = \int_0^1 F'(t)\,dt.$$

From here we can find t such that $F'(t)$ exists and is bigger than $(1-\varepsilon)\|y-x\|$. Thus, given $z = \phi(t)$, it follows that

$$\limsup_{h\to 0} \frac{f(z+h(y-x))-f(z)}{h} > (1-\varepsilon)\|y-x\|.$$

Indeed, given $h > 0$ small enough, one has

$$\phi(t+h) = (t+h)y + (1-(t+h))x = ty + (1-t)x + h(y-x).$$

As $ty + (1-t)x = \phi(t) = z$, we conclude that

$$\frac{F(t+h)-F(t)}{h} = \frac{f(z+h(y-x))-f(z)}{h}.$$

As the limits

$$\lim_{h\to 0} \frac{F(t+h)-F(t)}{h} = \lim_{h\to 0} \frac{f(z+h(y-x))-f(z)}{h}$$

exist, we conclude that

$$\lim_{h\to 0} \frac{f(z+h(y-x))-f(z)}{h} = \limsup_{h\to 0} \frac{f(z+h(y-x))-f(z)}{h} \leqslant f^\circ(z, y-x).$$

As the generalised derivative is positively homogeneous, we conclude that $f^\circ(z, \frac{y-x}{\|y-x\|}) > 1-\varepsilon$. Hence, there exists $\varphi \in \partial f(x)$ such that $\varphi\big(\frac{y-x}{\|y-x\|}\big) > 1-\varepsilon$ by [85, Proposition 2.1.2]. Consequently,

$$\frac{y-x}{\|y-x\|} \in T := \{v \in S_X \colon \varphi(v) > 1-\varepsilon\}.$$

We shall prove that $T \subset S$. Towards this aim, pick $v \in T$, so $\varphi(v) > 1-\varepsilon$. As

$$\varphi(v) \leqslant f^\circ(z,v) = \limsup_{y\to z, t\searrow 0} \frac{f(y+tv)-f(y)}{t},$$

we can find $t>0$ and y close enough to z such that $1-\varepsilon < \frac{f(y+tv)-f(y)}{t}$. From the definition of S, one has

$$S \ni \frac{y+tv-y}{\|y+tv-y\|} = \frac{tv}{\|tv\|} = v$$

so, $\frac{y-x}{\|y-x\|} \in T \subset S$. □

11.4.2. Daugavet centres and SCD Lipschitz maps.

At first, we extend the basic description of Daugavet centres (Theorem 5.1.2(iii)) to Lipschitz slices instead of the ordinary ones.

LEMMA 11.4.3. *Let X, Y be Banach spaces, $G \in L(X,Y)$ be a Daugavet centre, $\|G\| = 1$. Then, for every $y \in S_Y$, $f \in S_{\mathrm{Lip}_0(X)}$, and $\varepsilon > 0$, there is $x \in \mathrm{LipSlice}(S_X, f, \varepsilon)$ such that $\|Gx+y\| > 2-\varepsilon$.*

PROOF. This is an immediate consequence of the fact that every Lip-slice contains a slice by Lemma 11.4.2 and of the definition of a Daugavet centre. □

The main result for Daugavet centres is the following one.

THEOREM 11.4.4. *Let X, Y be Banach spaces and let $G \in L(X,Y)$ be a norm-one operator. Then the following assertions are equivalent:*

1. *G is a Daugavet centre;*
2. *$\widehat{G} \colon \mathcal{F}(X) \to Y$ is a Daugavet centre.*

PROOF. (2) ⇒ (1). Consider the rank-one operator $T\colon X \to Y$ given by $Tx = x^*(x)y_0$, where $x^* \in X^*$ and $y_0 \in Y$. Then, $\widehat{T} \colon \mathcal{F}(X) \to Y$ acts by the rule described in Lemma 11.1.2(a):

$$\widehat{T}\left(\sum_{k=1}^n a_k \delta_{x_k}\right) = \sum_{k=1}^n a_k T(x_k) = x^*\left(\sum_{k=1}^n a_k x_k\right) y_0,$$

(extended afterwards to the whole space $\mathcal{F}(X)$ by continuity). So, $\widehat{T}$ is also a rank-one operator, and condition (2), together with the properties from Lemma 11.1.2(a), gives us the desired Daugavet equation for T with respect to G:

$$\|G+T\| = \|\widehat{G+T}\| = \|\widehat{G}+\widehat{T}\| = 1 + \|\widehat{T}\| = 1 + \|T\|.$$

(1) ⇒ (2). Fix $y_0 \in S_Y$, $f \in S_{\mathrm{Lip}_0(X)}$ (that is, $f \in S_{\mathcal{F}(X)^*}$) and $\varepsilon > 0$. Our goal is to demonstrate that there is $u \in \mathrm{Slice}(S_{\mathcal{F}(X)}, f, \varepsilon)$ such that $\|\widehat{G}u + y_0\| > 2-\varepsilon$.

According to Lemma 11.4.3, there is $x \in \mathrm{LipSlice}(S_X, f, \varepsilon)$ such that $\|Gx + y_0\| > 2 - \varepsilon$. By the definition of $\mathrm{LipSlice}(S_X, f, \varepsilon)$, this x is of the form

$$x = \frac{x_1 - x_2}{\|x_1 - x_2\|}, \text{ where } x_1, x_2 \in X,\ x_1 \neq x_2, \text{ and } \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > 1 - \varepsilon.$$

Let us show that the corresponding molecule

$$u = m_{x_1,x_2} = \frac{\delta_{x_1} - \delta_{x_1}}{\|x_1 - x_2\|}$$

is the element that we need. Indeed, $\|u\| = 1$, $\langle f, u\rangle = \frac{f(x_1)-f(x_2)}{\|x_1-x_2\|} > 1 - \varepsilon$, which shows that $u \in \mathrm{Slice}(S_{\mathcal{F}(X)}, f, \varepsilon)$; and

$$\|\widehat{G}u + y_0\| = \left\| \frac{Gx_1 - Gx_2}{\|x_1 - x_2\|} + y_0 \right\| = \|Gx + y_0\| > 2 - \varepsilon. \qquad \square$$

For a map $F \in \mathrm{Lip}_0(X, Y)$ we define the *slope* of F by

$$\mathrm{slope}(F) := \left\{ \frac{F(x_1) - F(x_2)}{\|x_1 - x_2\|} \colon x_1 \neq x_2 \in X \right\}.$$

Observe that if $F \in L(X, Y)$, then $\mathrm{slope}(F) = F(S_X)$. The motivation for considering $\mathrm{slope}(F)$ comes from the following equivalent definition:

$$\mathrm{slope}(F) = \left\{ \widehat{F}(m_{x_1,x_2}) \colon x_1 \neq x_2 \in X \right\} = \widehat{F}(\mathrm{Mol}(X)). \tag{11.4.1}$$

According to Remark 11.1.1, $\overline{\mathrm{conv}}(\mathrm{Mol}(X)) = B_{\mathcal{F}(X)}$, so equation (11.4.1) implies

$$\overline{\widehat{F}(B_{\mathcal{F}(X)})} = \overline{\mathrm{conv}}(\mathrm{slope}(F)). \tag{11.4.2}$$

Corollary 11.4.5. *Let X, Y be Banach spaces and let $G \in \mathrm{Lip}_0(X, Y)$ be a Daugavet centre. If $F \in \mathrm{Lip}_0(X, Y)$ satisfies that $\mathrm{slope}(F)$ is SCD, then $\|G+F\|_L = 1 + \|F\|_L$.*

Proof. If G is a Daugavet centre, then $\widehat{G} \colon \mathcal{F}(X) \to Y$ is also a Daugavet centre by Theorem 11.4.4. Now, if $F \in \mathrm{Lip}_0(X, Y)$ satisfies that $\mathrm{slope}(F)$ is SCD, then formula (11.4.2) together with Lemma 10.1.7 imply that $\widehat{F}(B_{\mathcal{F}(X)})$ is an SCD set, so $\widehat{F}$ is an SCD operator. Therefore, $\|\widehat{G} + \widehat{F}\| = 1 + \|\widehat{F}\|$ by Corollary 10.4.8. Finally, this is equivalent to $\|G + F\|_L = 1 + \|F\|_L$ by Lemma 11.1.2. $\square$

Corollary 11.4.6. *Let X, Y be Banach spaces and let $G \in L(X, Y)$ be a Daugavet centre. If $F \in \mathrm{Lip}_0(X, Y)$ satisfies that $\overline{\mathrm{conv}}(\mathrm{slope}(F))$ has the Radon-Nikodým property, the convex point of continuity property or if it is an Asplund set, or if it does not contain ℓ_1-sequences, then $\|G + F\|_L = 1 + \|F\|_L$.*

Proof. As in the proof of Corollary 11.4.5 we consider $\widehat{G}, \widehat{F} \colon \mathcal{F}(X) \to Y$. Our assumptions imply that every separable subset of $\widehat{F}(B_{\mathcal{F}(X)})$ is an SCD set, so we may apply Corollary 10.4.11 and get that $\widehat{F}$ is $\widehat{G}$-narrow. So, $\|\widehat{G} + \widehat{F}\| = 1 + \|\widehat{F}\|$ and $\|G + F\|_L = 1 + \|F\|_L$. $\square$

Remark 11.4.7. *Changing in Corollary 11.4.5 the assumption that $\mathrm{slope}(F)$ is SCD to $\overline{\mathrm{conv}}(\mathrm{slope}(F))$ is HSCD, one can get that $\widehat{F}$ is $\widehat{G}$-narrow, but it is unclear what advantages this gives for the original F. Also, it is not quite clear what a natural analogue of HSCD-dominated operator for the non-linear Lipschitz case is.*

This can be a motivation to introduce a concept of Lipschitz G-narrow maps and to develop the corresponding theory. By now, such a theory does not yet exist.

11.5. Notes and remarks

Section 11.1. References for this section are [**120**], [**122**], [**294**]. The Lipschitz free space also goes by the name *Arens-Eells space* or *transportation cost space*.

Section 11.2. The first (published) work in which the Daugavet property is studied in spaces of Lipschitz functions is [**145**], motivated by the question of whether the space of Lipschitz functions on the unit square has or does not have the Daugavet property, explicitly asked in [**300**, Question (1)]. (The Daugavet property for $\mathrm{Lip}_0(M)$, with M a compact convex subset of a Banach space, can already be found in the unpublished diploma thesis of D. Pokorný (Charles University Prague; 2005) [**253**].) Most of the material of the section comes from the papers [**115, 145**]. The proof of Theorem 11.2.1 is from [**115**, Lemma 3.7], where the ideas of [**145**, Lemma 3.2] were extended. The Proposition 11.2.7 is from [**115**]. Finally, Theorem 11.2.8 is from [**145**]. Actually, some proofs in [**145**] were, in the process of writing the paper, simplified to the extent that they became invalid; this warranted a corrigendum to this paper [**146**] with correct proofs.

All the properties considered in Definition 11.2.4 are present in [**145**] (there the terminology *metrically convex* (respectively, *almost metrically convex*) was used instead of geodesic (respectively, length space)). Also, in [**145**] the following definition is considered: a metric space M is said to have *property (Z)* if for every $x, y \in M$ with $x \neq y$ and every $\varepsilon > 0$, there exists $z \in M \setminus \{x, y\}$ so that

$$d(x,z) + d(y,z) \leqslant d(x,y) + \varepsilon \min\{d(x,z), d(y,z)\}.$$

This property was introduced in [**145**] as a (metric) characterisation of the Daugavet property of $\mathrm{Lip}_0(M)$ valid for compact metric spaces M. Later, it was proved in [**115**] that a metric space M has property (Z) if, and only if, $B_{\mathcal{F}(M)}$ does not have any strongly exposed point [**115**, Theorem 5.4]. In view of the above mentioned two results, it was conjectured in [**115**] that property (Z) and being a length space are equivalent properties for complete metric spaces. Soon afterwards, making use of transfinite methods, A. Avilés and G. Martínez-Cervantes gave a positive answer to this conjecture in [**28**, Main Theorem]. As a consequence of all this collective effort, the final characterisation of the Daugavet property in $\mathrm{Lip}_0(M)$ can be stated as follows[1].

Theorem 11.5.1. *Let M be a complete metric space. The following assertions are equivalent:*

(1) $\mathrm{Lip}_0(M)$ *has the uniform Daugavet property.*
(2) $\mathrm{Lip}_0(M)$ *has the Daugavet property.*
(3) $\mathcal{F}(M)$ *has the Daugavet property.*
(4) *The unit ball of $\mathcal{F}(M)$ does not have any strongly exposed point.*
(5) *M is a length space.*
(6) *M has the property (Z).*

[1] Since 2022, outside mathematics, the symbol Z has become the hallmark of the supporters of the Russian invasion of and aggression against Ukraine. Therefore, we are especially glad that we can avoid this symbol in the main body of the text.

The idea behind Lemma 11.2.3 comes from [**256**, Theorem 3.1], where it is characterised when the norm of $\mathcal{F}(M)$ is octahedral (see Definition 12.2.10). This idea was elaborated further and applied in [**21**] in order to characterise those elements $\mu \in \mathcal{F}(M)$ which are limits of convex series of molecules.

Theorem 11.2.11 is from [**199**] where the notion of SCD points is used; with this, separability assumptions are not needed and it is proved that $\mathrm{SCD}(B_{\mathcal{F}(M)}) = \overline{\mathrm{conv}}\big(\mathrm{dent}(B_{\mathcal{F}(M)})\big)$ for every complete pointed metric space M (see the Notes and Remarks on Section 10.3 in Chapter 10 for the notion of an SCD point).

Let us also point out that the notion of local Lipschitz function was already considered in the literature earlier. For instance, it appeared in [**79**], where it was proved that if M is a compact metric space and $f\colon M \to \mathbb{R}$ is a non-local Lipschitz function, then f attains its norm (as a functional on $\mathcal{F}(M)$) on a strongly exposed point of $B_{\mathcal{F}(M)}$, a result which found interesting applications to the study of *strongly norm attaining Lipschitz maps* (see [**79**, Section 3] for details).

Companion results for the Daugavet property of spaces of C^1-functions on $\mathbb{R}^d$ have been obtained in the above-mentioned diploma thesis of Pokorný [**253**] and in the likewise unpublished diploma thesis of D. Dubray (FU Berlin; 2009) [**102**].

Section 11.3. The results of this section come from the recent preprint [**230**]. The motivation was the question [**115**, Question 4.1], where it was asked whether $\mathrm{Lip}_0(M, X)$ has the Daugavet property for every non-zero Banach space X if M is a length space.

Let us point out that, before the definitive solution from [**230**], a step forward towards the solution was made in [**269**], where partial positive results were obtained making use of the contraction-extension property [**51**, Definition 2.10] proving the Daugavet property for $\mathrm{Lip}_0(M, X)$, for instance, when M is a convex subset of a Hilbert space.

Let us also mention the story behind Theorem 11.3.1. A first version appeared on `arXiv` on May 10, 2023, where it was proved that $\mathcal{F}(M) \mathbin{\widehat{\otimes}_\pi} X$ has the Daugavet property for any length metric space M and any Banach lattice X such that X^* has the metric approximation property.

Later, during the celebration of the congress "Æasy to define, hard to analyse: First conference on Lipschitz free spaces" held in Besançon (France) in September 2023, the third author of this book presented the above mentioned result, leaving as an open question whether a weaker version of the "In particular" part of Lemma 11.3.5 holds in general, with the corresponding implications on the Daugavet property in $\mathcal{F}(M) \mathbin{\widehat{\otimes}_\pi} X$. After this conference, Richard Smith (University College Dublin) kindly left the authors Lemma 11.3.5 together with its complete proof. With this new lemma, the authors developed different methods which resulted in the current version of Theorem 11.3.1 which is more general in the hypothesis and stronger in its conclusion, appearing in v2 of the preprint, uploaded to `arXiv` on October 26, 2023.

This altruist gesture is a sample of how science progresses when it is based on cooperation instead of on competition.

Section 11.4. The idea of Lipschitz slices comes from [**164**], where the main resuts were given for the identity operator and the ordinary Daugavet equation. Those results were extended to Daugavet centres in [**162**, pp. 112–113]. An alternative form of such an extension can be found in [**83**].

The proof of Lemma 11.4.2 is from [**42**, Lemma 2.3], where the authors proved that, given a Banach space X, the topology generated by Lipschitz slices on S_X agrees with the restriction of the weak topology to S_X.

In order to prove Lemma 11.4.3 in [**164**], the key is the following result: if $S = \mathrm{LipSlice}(S_X, f, \alpha)$ is a Lipschitz slice and $\big(\overline{\mathrm{conv}}(A)\big) \cap S \neq \emptyset$, then $A \cap S \neq \emptyset$ [**164**, Lemma 2.4]. This lemma was shown to be equivalent to Lemma 11.4.2 in the paragraph after [**162**, Lemma 7.2].

11.6. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

(11.1) How to generalise the concept of narrow operator to Lipschitz mappings? Are there natural analogs for characterisations of narrow operators in classical spaces $C(K)$ and $L_1(\mu)$?

(11.2) For a metric space M embedded in a Banach space Y by an isometry $J\colon M \to Y$ we can consider the equation $\|J + T\| = 1 + \|T\|$ for Lipschitz mappings $T\colon M \to Y$. For what natural pairs $M \subset Y$ is this generalised Daugavet equation valid for all T of rank one? For example, does this work for Y being a space with the Daugavet property and M being a convex subset of Y? In particular, for $M = B_Y$?

(11.3) What kind of theory can be built for such pairs $M \subset Y$? Is there some kind of "rich subsets" theory that generalises the theory of rich subspaces?

(11.4) Are the Daugavet property and the WODP (see Definition 4.2.3) equivalent on $\mathcal{F}(M)$ spaces or on $\mathrm{Lip}_0(M)$ spaces?

CHAPTER 12

Geometric properties related to the Daugavet property

This final chapter will touch upon variants of the Daugavet property. The first section looks at general norm equations of the form $\|g(T)\| = f(\|T\|)$. It will turn out that quite often the study of such an equation can be reduced to the Daugavet equation, but we will encounter differences between the real and the complex case. In the second section we will survey a farrago of diameter two properties, and the theme of the third section is the alternative Daugavet property and the numerical index of a Banach space. Finally, the anti-Daugavet property, when only the smallest collection possible of operators satisfies the Daugavet equation, is presented in the last section.

12.1. Norm equalities for operators on Banach spaces

The Daugavet equation

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

implies, by its very nature, that for such operators the norm of $\mathrm{Id} + T$ is a function of $\|T\|$, viz., $f_0(\|T\|)$ for $f_0(t) = 1 + t$. One might wonder whether there are other functions f that lead to other interesting Daugavet type properties. However, we shall see in Proposition 12.1.1 that f_0 is basically the only sensible choice.

PROPOSITION 12.1.1. *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an arbitrary function. Suppose that there exist $a, b \in \mathbb{K}$ and a non-null Banach space X over $\mathbb{K}$ such that the norm equality*

$$\|a\,\mathrm{Id} + b\,T\| = f(\|T\|)$$

holds for every rank-one operator $T \in L(X)$. Then, $f(t) = |a| + |b|t$ for every $t \in \mathbb{R}_0^+$. In particular, if $a \neq 0$ and $b \neq 0$, then X has the Daugavet property.

PROOF. If $ab = 0$ we are trivially done; so we may assume that $a \neq 0$, $b \neq 0$ and we write $\omega_0 = \frac{\overline{b}}{|b|}\frac{a}{|a|} \in \mathbb{T}$. Now, we fix $x_0 \in S_X$, $x_0^* \in S_{X^*}$ such that $x_0^*(x_0) = \omega_0$ and, for each $t \in \mathbb{R}_0^+$, we consider the rank-one operator $T_t = t\,x_0^* \otimes x_0 \in L(X)$. Observe that $\|T_t\| = t$, so we have

$$f(t) = \|a\,\mathrm{Id} + b\,T_t\| \qquad (t \in \mathbb{R}_0^+).$$

Then, it follows that

$$\begin{aligned} |a| + |b|\,t \geqslant f(t) &= \|a\,\mathrm{Id} + b\,T_t\| \geqslant \big\|[a\,\mathrm{Id} + b\,T_t](x_0)\big\| \\ &= \|ax_0 + b\omega_0 t x_0\| = |a + b\omega_0 t|\,\|x_0\| \\ &= \left|a + b\frac{\overline{b}}{|b|}\frac{a}{|a|}\,t\right| = \left|\frac{|a|}{a}a + b\frac{\overline{b}}{|b|}t\right| = |a| + |b|t. \end{aligned}$$

Finally, if the norm equality

$$\|a\,\mathrm{Id} + b\,T\| = |a| + |b|\|T\|$$

holds for every rank-one operator on X, then X has the Daugavet property (we just have to use Remark 2.6.2 as $ab \neq 0$). $\square$

With the above Proposition in mind, we have to look for Daugavet-type norm equalities in which $\mathrm{Id} + T$ is replaced by another function of T. If we want such a function to carry operators to operators and to be applied to arbitrary rank-one operators on arbitrary Banach spaces, it is natural to consider power series with infinite radius of convergence. Let us introduce some notation. We say that $g\colon \mathbb{K} \to \mathbb{K}$ is an *entire function* if g is represented by an everywhere convergent Taylor series; in other words, when $\mathbb{K} = \mathbb{C}$ this is the usual definition of an entire function, but when $\mathbb{K} = \mathbb{R}$, g is the restriction to $\mathbb{R}$ of a complex entire function which carries the real line into itself. Given an entire function g, for each operator $T \in L(X)$ we define

$$g(T) = \sum_{k=0}^{\infty} a_k\,T^k,$$

where $g(\zeta) = \sum_{k=0}^{\infty} a_k\,\zeta^k$ is the power series expansion of g. The following easy result shows how to calculate $g(T)$ when T is a rank-one operator.

LEMMA 12.1.2. *Let $g\colon \mathbb{K} \to \mathbb{K}$ be an entire function with power series expansion*

$$g(\zeta) = \sum_{k=0}^{\infty} a_k\,\zeta^k \qquad (\zeta \in \mathbb{K}),$$

and let X be a Banach space over $\mathbb{K}$. For $x^* \in X^*$ and $x \in X$, we write $T = x^* \otimes x$ and $\alpha = x^*(x)$. Then, for each $\lambda \in \mathbb{K}$,

$$g(\lambda T) = \begin{cases} a_0\mathrm{Id} + a_1\lambda T & \text{if } \alpha = 0 \\ a_0\mathrm{Id} + \frac{\widetilde{g}(\alpha\lambda)}{\alpha} T & \text{if } \alpha \neq 0, \end{cases}$$

where

$$\widetilde{g}(\zeta) = g(\zeta) - a_0 \qquad \big(\zeta \in \mathbb{K}\big).$$

PROOF. Given $\lambda \in \mathbb{K}$, it is immediate to check that

$$(\lambda T)^k = \alpha^{k-1}\lambda^k\,T \qquad (k \in \mathbb{N}).$$

Now, if $\alpha = 0$, then $T^2 = 0$ and the result is clear. Otherwise, we have

$$\begin{aligned} g(\lambda T) &= a_0\mathrm{Id} + \sum_{k=1}^{\infty} a_k\,\alpha^{k-1}\,\lambda^k\,T \\ &= a_0\mathrm{Id} + \left(\frac{1}{\alpha}\sum_{k=1}^{\infty} a_k\,\alpha^k\lambda^k\right) T = a_0\mathrm{Id} + \frac{\widetilde{g}(\alpha\lambda)}{\alpha} T. \end{aligned}$$ $\square$

We would now like to study norm equalities for operators of the form

$$\|g(T)\| = f(\|T\|), \tag{12.1.1}$$

where $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an arbitrary function and $g\colon \mathbb{K} \to \mathbb{K}$ is an entire function.

Our goal is to show that again the Daugavet property is the only non-trivial property that one can obtain from this approach. More precisely, if one requires all rank-one operators on a Banach space X of dimension greater than one to satisfy a norm equality of the form (12.1.1), then X has the Daugavet property.

We start by proving that g has to be a polynomial of degree $\leqslant 1$, and then we will deduce the result from Proposition 12.1.1.

THEOREM 12.1.3. *Let* $g\colon \mathbb{K} \to \mathbb{K}$ *be an entire function and* $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *an arbitrary function. Suppose that there is a Banach space* X *over* $\mathbb{K}$ *with* $\dim(X) \geqslant 2$ *such that the norm equality*

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator T *on* X*. Then, there are* $a, b \in \mathbb{K}$ *such that*

$$g(\zeta) = a + b\zeta \qquad \big(\zeta \in \mathbb{K}\big).$$

PROOF. Let $g(\zeta) = \sum_{k=0}^{\infty} a_k\, \zeta^k$ be the power series expansion of g and let $\widetilde{g} = g - a_0$. Given $\alpha \in \mathbb{D}$, the set of scalars of modulus $\leqslant 1$, we take $x_\alpha^* \in S_{X^*}$ and $x_\alpha \in S_X$ such that $x_\alpha^*(x_\alpha) = \alpha$ (we can do this since $\dim(X) \geqslant 2$), and we write $T_\alpha = x_\alpha^* \otimes x_\alpha$, which satisfies $\|T_\alpha\| = 1$. Using Lemma 12.1.2, we obtain for each $\lambda \in \mathbb{K}$ that

$$g(\lambda T_0) = a_0 \mathrm{Id} + a_1 \lambda\, T_0$$

and

$$g(\lambda T_\alpha) = a_0 \mathrm{Id} + \frac{1}{\alpha} \widetilde{g}(\lambda\alpha)\, T_\alpha \qquad (\alpha \neq 0).$$

Now, for fixed $\lambda \in \mathbb{K}$, we have

$$f(|\lambda|) = \|g(\lambda T_0)\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\|,$$

and

$$f(|\lambda|) = \|g(\lambda T_\alpha)\| = \left\| a_0 \mathrm{Id} + \frac{1}{\alpha} \widetilde{g}(\lambda\alpha) T_\alpha \right\|.$$

Therefore, we have the equality

$$\left\| a_0 \mathrm{Id} + \frac{1}{\alpha} \widetilde{g}(\lambda\alpha) T_\alpha \right\| = \|a_0 \mathrm{Id} + a_1 \lambda T_0\| \qquad (\lambda \in \mathbb{K},\ 0 < |\alpha| \leqslant 1). \tag{12.1.2}$$

In the complex case, it is enough to consider the above equality for $\alpha = 1$ and to use the triangle inequality to get that

$$|\widetilde{g}(\lambda)| \leqslant 2|a_0| + |a_1|\, |\lambda| \qquad (\lambda \in \mathbb{C}). \tag{12.1.3}$$

From this, it follows by just using Cauchy's estimates, that $\widetilde{g}$ is a polynomial of degree $\leqslant 1$ (see [**125**, Theorem 3.4.4], for instance), and we are done.

In the real case, it is not possible to deduce from inequality (12.1.3) that $\widetilde{g}$ is a polynomial, so we have to return to (12.1.2). From this equality, we can deduce by applying the triangle inequality that

$$\left| \frac{\widetilde{g}(\lambda\alpha)}{\alpha} \right| - |a_0| \leqslant |a_0| + |a_1|\, |\lambda| \qquad \text{and} \qquad |a_1|\, |\lambda| - |a_0| \leqslant \left| \frac{\widetilde{g}(\lambda\alpha)}{\alpha} \right| + |a_0|$$

for every $\lambda \in \mathbb{R}$ and every $\alpha \in [-1, 1] \setminus \{0\}$. It follows that

$$\left| \left| \frac{\widetilde{g}(\lambda\alpha)}{\alpha} \right| - |a_1|\, |\lambda| \right| \leqslant 2|a_0| \qquad \big(\lambda \in \mathbb{R},\ \alpha \in [-1, 1] \setminus \{0\}\big). \tag{12.1.4}$$

Next, for $t \in (1, +\infty)$ and $k \in \mathbb{N}$, we use (12.1.4) with $\lambda = t^k$ and $\alpha = \frac{1}{t^{k-1}}$ to obtain that

$$\big|\, |\widetilde{g}(t)| - |a_1|\, t \big| \leqslant \frac{2|a_0|}{t^{k-1}},$$

and so, letting $k \to \infty$, we get that

$$|\widetilde{g}(t)| = |a_1|\, t \qquad \big(t \in (1, +\infty)\big).$$

Finally, an obvious continuity argument allows us to deduce from the above equality that $\widetilde{g}$ coincides with a polynomial of degree $\leqslant 1$ in the interval $(1, +\infty)$, thus the same is true on the whole line $\mathbb{R}$ by analyticity. □

We summarise the information given in Proposition 12.1.1 and Theorem 12.1.3.

COROLLARY 12.1.4. *Let $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be an arbitrary function and $g\colon \mathbb{K} \to \mathbb{K}$ an entire function. Suppose that there is a Banach space X over $\mathbb{K}$ with* $\dim(X) \geqslant 2$ *such that the norm equality*

$$\|g(T)\| = f(\|T\|)$$

holds for every rank-one operator T on X. Then, only three possibilities may happen:

(a) *g is a constant function (trivial case).*
(b) *There is a non-null $b \in \mathbb{K}$ such that $g(\zeta) = b\,\zeta$ for every $\zeta \in \mathbb{K}$ (trivial case).*
(c) *There are non-null $a, b \in \mathbb{K}$ such that $g(\zeta) = a + b\,\zeta$ for every $\zeta \in \mathbb{K}$, and X has the Daugavet property.*

In addition to the previous study the paper [**160**] also considers

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|) \tag{12.1.5}$$

for a continuous function f on $[0, \infty)$ and a nonconstant entire function g on $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Here one obtains slightly different results in the real and complex case. Let us start with the latter.

THEOREM 12.1.5 (Complex case). *Let $g\colon \mathbb{C} \to \mathbb{C}$ be an entire function and $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a continuous function. Suppose that there is a complex Banach space X with* $\dim(X) \geqslant 2$ *such that the norm equality*

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator T on X. Then:

(a) *If* $\mathrm{Re}\, g(0) \neq -\frac{1}{2}$*, then X has the Daugavet property.*
(b) *If* $\mathrm{Re}\, g(0) = -\frac{1}{2}$*, then there is $\omega \in \mathbb{T} \setminus \{1\}$ such that* $\|\mathrm{Id} + \omega T\| = \|\mathrm{Id} + T\|$ *for every rank-one operator T.*

Let us comment that in the case (b), X need not have the Daugavet property (a counterexample being $X = C[0,1] \oplus_2 C[0,1]$ [**284**]). If ω is not a root of unity, then even $\|\mathrm{Id} + \xi T\| = \|\mathrm{Id} + T\|$ for every $\xi \in \mathbb{T}$, but also this does not guarantee that $X \in \mathrm{DPr}$ (the same counterexample works).

In the real case, one obtains the same results under the additional assumption that $g\colon \mathbb{R} \to \mathbb{R}$ is surjective.

THEOREM 12.1.6 (Real case). *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a surjective entire function and $f\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ a continuous function. Suppose that there is a real Banach space X with $\dim(X) \geqslant 2$ such that the norm equality*

$$\|\mathrm{Id} + g(T)\| = f(\|g(T)\|)$$

holds for every rank-one operator T on X. Then:

(a) *If $g(0) \neq -\frac{1}{2}$, then X has the Daugavet property.*
(b) *If $g(0) = -\frac{1}{2}$, then $\|\mathrm{Id} - T\| = \|\mathrm{Id} + T\|$ for every rank-one operator T.*

The exceptional norm equality in the real case is then the following one:

$$\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|. \tag{12.1.6}$$

As in the complex case, the validity of this equation for all rank-one operators does not imply the Daugavet property (again $C[0,1] \oplus_2 C[0,1]$ is a counterexample [**284**]). Therefore, we shall now study this equation.

DEFINITION 12.1.7. We say that a Banach space X has the *plus-minus property* and write $X \in D_\pm$ if (12.1.6) is satisfied for all finite-rank operators on X.

Incidentally, we do not know whether it is sufficient for the plus-minus property that all rank-one operators satisfy (12.1.6); see Question (12.1) in Section 12.6.

We record one easy consequence of this definition. Consider a Banach space X such that (12.1.6) holds true for every rank-one operator $T \in L(X)$. If we apply this equality to an operator P which is a rank-one projection, we get

$$\|\mathrm{Id} - P\| \geqslant 2,$$

i.e., every 1-codimensional projection in $L(X)$ is at least of norm 2. Such spaces were introduced in [**144**] and called "spaces with bad projections"; so every $X \in D_\pm$ is a space with bad projections.

In much of this monograph it has become clear that working with the geometrical characterisations of the Daugavet property is much more convenient than working with the original definition. We shall now introduce a geometric analogue of the plus-minus property called strong plus-minus property, which indeed implies the plus-minus property (see Theorem 12.1.11 below); but it remains open whether the two properties are actually equivalent. (See Question (12.2) in Section 12.6.)

DEFINITION 12.1.8. We say that a Banach space X has the *strong plus-minus property* and write $X \in SD_\pm$ if for every relatively weakly open set U in B_X and every element $y \in X$

$$\sup_{x\in U} \|x + y\| = \sup_{x\in U} \|x - y\|.$$

We remark that a finite-dimensional space cannot have the strong plus-minus property because in such a space there are weakly open sets of arbitrarily small diameter. The same argument rules out reflexive spaces, RNP spaces, and even CPCP spaces.

We will now give a characterisation of the strong plus-minus property which is more appropriate to work with.

LEMMA 12.1.9. *For a Banach space X the following conditions are equivalent:*

(i) *X has the strong plus-minus property;*
(ii) *for every $x \in S_X$, every $y \in X$, every relative weak neighbourhood U of x in B_X, and every $\varepsilon > 0$, there exists $z \in U$ such that $\|z - y\| \geqslant \|x + y\| - \varepsilon$.*

Proof. First let X be a space with the strong plus-minus property. Consider arbitrary $x \in S_X$, $y \in X$, a relative weak neighbourhood U of x in B_X, and $\varepsilon > 0$. Since $X \in SD_\pm$, we have that $\|x+y\| \leqslant \sup_{u\in U} \|u+y\| = \sup_{u\in U} \|u-y\|$. So there exists $z \in U$ such that $\|z-y\| \geqslant \sup_{u\in U} \|u-y\| - \varepsilon \geqslant \|x+y\| - \varepsilon$, and we are done.

Conversely, let X satisfy (ii). Take an arbitrary relatively weakly open set U in B_X, $y \in X$ and $\varepsilon > 0$. We first show that we can choose $x \in U \cap S_X$ such that $\|x+y\| > \sup_{u\in U} \|u+y\| - \varepsilon$. Let $x_0 \in U$ be such that $\|x_0+y\| > \sup_{u\in U} \|u+y\| - \varepsilon$. If $x_0 \in S_X$ we are done; so suppose that $\|x_0\| < 1$. U is a weak neighbourhood of x_0 in B_X and so $U = V \cap B_X$ for a weakly open subset $V \subset X$. Now, it follows from our observation following Definition 12.1.8 that X is infinite-dimensional and thus there exists a straight line l in X such that $x_0 \in l$ and $l \subset V$. Since $\|x_0\| < 1$ and $x_0 \in l$, we have that l has two points of intersection with S_X – say x_1 and x_2. Then $x_1, x_2 \in U$ and $x_0 \in [x_1, x_2]$. Now, because the function $f(u) = \|u+y\|$ is convex, we get that $\max_{i=1,2} \|x_i + y\| \geqslant \|x_0 + y\| > \sup_{u\in U} \|u+y\| - \varepsilon$. This means that taking $x = x_i$ for an appropriate i we'll obtain $x \in U \cap S_X$ such that $\|x+y\| > \sup_{u\in U} \|u+y\| - \varepsilon$. Now, we can use (ii) to find $z \in U$ such that $\|z-y\| \geqslant \|x+y\| - \varepsilon$. Then we get that

$$\sup_{u\in U} \|u-y\| \geqslant \|z-y\| \geqslant \|x+y\| - \varepsilon > \sup_{u\in U} \|u+y\| - 2\varepsilon,$$

and since ε was arbitrary this proves that $\sup_{u\in U} \|u-y\| \geqslant \sup_{u\in U} \|u+y\|$. The converse inequality can be obtained by taking $(-y)$ instead of y. □

Remark 12.1.10. The above characterisation allows us to prove the following: *given a Banach space X with the strong plus-minus property then, given $x \in S_X$, given any non-empty weakly open set U of B_X with $x \in U$ and $\varepsilon > 0$, there exists $z \in U$ with $\|z-x\| > 2-\varepsilon$* (simply apply the above lemma to $y = x$). This implies that *if X has the strong plus-minus property then X has the diametral diameter two property* (see Definition 12.2.1), a property formally stronger than the property of X having bad projections (see Subsection 12.2 for more information).

Using this characterisation we can prove that the strong plus-minus property implies the plus-minus property.

Theorem 12.1.11. *Let X be a Banach space and $X \in SD_\pm$. Then every strong Radon-Nikodým operator $T\colon X \to X$ satisfies the equality $\|\mathrm{Id}+T\| = \|\mathrm{Id}-T\|$. In particular, every finite-rank operator satisfies this equality and so $X \in SD_\pm$ implies $X \in D_\pm$.*

Proof. We first prove that $\|\mathrm{Id}-T\| \geqslant \|\mathrm{Id}+T\|$. Fix $\varepsilon > 0$ and consider

$$A = \big\{x \in B_X\colon \|(\mathrm{Id}+T)x\| > \|\mathrm{Id}+T\| - \varepsilon\big\}.$$

Fix some $x_0 \in A$ and choose $x^* \in S_{X^*}$ such that $x^*((\mathrm{Id}+T)x_0) = \|(\mathrm{Id}+T)x_0\|$, i.e., x^* is a support functional for the element $(\mathrm{Id}+T)x_0$. Then the set

$$V = \big\{x \in B_X\colon x^*(\mathrm{Id}+T)x > \|\mathrm{Id}+T\| - \varepsilon\big\}$$

is a slice of B_X and $V \subset A$.

Let us consider $T(V)$. Since T is a strong Radon-Nikodým operator we can find a slice W of $T(V)$ with $\operatorname{diam} W < \varepsilon$. Take $U = V \cap T^{-1}(W)$. We have that U is an intersection of two slices and is therefore a relatively weakly open subset of B_X. Choose an arbitrary element $x \in U \cap S_X$. Then U is a weak neighbourhood of x in

B_X and, since $X \in SD_\pm$, we can apply Lemma 12.1.9 for x, U, and $y = Tx$ to get an element $z \in U$ such that $\|z - y\| \geqslant \|x + y\| - \varepsilon$. Since $x, z \in U = V \cap T^{-1}(W)$, we have that $Tx, Tz \in W$. But $\operatorname{diam} W < \varepsilon$ and thus $\|Tz - y\| = \|Tz - Tx\| < \varepsilon$. Also, since $x \in U \subset V \subset A$, we have that $\|x + Tx\| > \|\mathrm{Id} + T\| - \varepsilon$.

Now, we can make the following estimates:

$$\begin{aligned}
\|\mathrm{Id} - T\| &\geqslant \|z - Tz\| = \|(z - y) - (Tz - y)\| \\
&\geqslant \|z - y\| - \|Tz - y\| > \|z - y\| - \varepsilon \\
&\geqslant \|x + y\| - 2\varepsilon = \|x + Tx\| - 2\varepsilon \geqslant \|\mathrm{Id} + T\| - 3\varepsilon.
\end{aligned}$$

So we have finally proved that $\|\mathrm{Id} - T\| > \|\mathrm{Id} + T\| - 3\varepsilon$ and, because of the arbitrariness of ε, we get that $\|\mathrm{Id} - T\| \geqslant \|\mathrm{Id} + T\|$.

The converse inequality is just the same inequality for the strong Radon-Nikodým operator $(-T)$. □

The next proposition will give us a lot of examples of spaces with the strong plus-minus property.

Proposition 12.1.12. *The Daugavet property implies $SD_\pm$.*

Proof. If a space X possesses the Daugavet property, then from the characterisation of this property from Lemma 3.1.15 it easily follows that for every $x \in S_X$, every $y \in X$, every weak neighbourhood U of x, and every $\varepsilon > 0$ there exists $z \in U$ such that $\|z - y\| = \|z + (-y)\| \geqslant 1 + \|y\| - \varepsilon \geqslant \|x + y\| - \varepsilon$ and so $X \in SD_\pm$ by Lemma 12.1.9. □

One can show that $C[0,1] \oplus_2 C[0,1]$ is an example of a Banach space with the strong plus-minus property that does not have the Daugavet property [**284**].

We next provide more examples of Banach spaces with the strong plus-minus property.

Proposition 12.1.13. *The strong plus-minus property is inherited by finite-codimensional subspaces.*

Proof. Let X be a Banach space, $X \in SD_\pm$, and Y be a subspace of X of finite co-dimension. We want to prove that then $Y \in SD_\pm$. First we note that the "ε-neighbourhood" $Y_\varepsilon = \{x \in X\colon \operatorname{dist}(x, Y) < \varepsilon\}$ of Y is weakly open in X for every $\varepsilon > 0$. Indeed, Y_ε is the pre-image of the open ball B_ε of radius ε under the quotient map $q\colon X \to X/Y$, and B_ε is weakly open in X/Y since this space is finite-dimensional.

Now, let $x \in S_Y$, U be a weak neighbourhood of x in B_Y, $y \in Y$ and $\varepsilon > 0$. According to Lemma 12.1.9, it is enough to find $z \in U$ such that $\|z - y\| \geqslant \|x + y\| - \varepsilon$. Without loss of generality, we can assume that

$$U = \big\{u \in B_Y\colon |f_i(u - x)| < \delta,\ i \in \overline{1, n}\big\}$$

for some $f_1, \dots, f_n \in S_{Y^*}$ and $\delta > 0$. Let $\hat{f}_i \in S_{X^*}$ be a Hahn-Banach extension of f_i for $i \in \overline{1, n}$. Then

$$V = \big\{u \in B_X\colon |\hat{f}_i(u - x)| < \delta/2,\ i \in \overline{1, n}\big\}$$

is a relatively weakly open set in X. Take $\varepsilon_0 = \min\{\varepsilon, \delta\}/4$ and consider $U_0 = V \cap Y_{\varepsilon_0}$. Since Y_{ε_0} is weakly open, U_0 is a weak neighbourhood of $x \in S_Y \subset S_X$ in B_X. So we can use that $X \in SD_\pm$ to find $z_0 \in U_0$ such that $\|z_0 - y\| \geqslant \|x + y\| - \varepsilon/2$. Since $z_0 \in U_0 \subset Y_{\varepsilon_0}$ there is $\hat{z} \in Y$ such that $\|\hat{z} - z_0\| < \varepsilon_0$.

The problem now is that this $\hat{z}$ does not necessarily belong to B_Y. So there are two possibilities: $\|\hat{z}\| \leqslant 1$ and $\|\hat{z}\| > 1$. In the first case, we take $z = \hat{z}$ and so we'll automatically have that $\|z - z_0\| < \varepsilon_0$. In the second case, we take $z = \hat{z}/\|\hat{z}\|$ and prove that $\|z - z_0\| < 2\varepsilon_0$. Since

$$\|z - z_0\| \leqslant \|z - \hat{z}\| + \|\hat{z} - z_0\| < \|z - \hat{z}\| + \varepsilon_0,$$

it is enough to prove that $\|z - \hat{z}\| < \varepsilon_0$. We have

$$\begin{aligned}\|z - \hat{z}\| = \left\| \frac{\hat{z}}{\|\hat{z}\|} - \hat{z} \right\| &= |1 - \|\hat{z}\|| = \|\hat{z}\| - 1 \\ &\leqslant \|\hat{z} - z_0\| + \|z_0\| - 1 < \varepsilon_0,\end{aligned}$$

because $z_0 \in U_0 \subset B_X$ and so $\|z_0\| \leqslant 1$. So, in both cases we have got some $z \in B_Y$ such that $\|z - z_0\| < 2\varepsilon_0$.

We'll prove that this z meets all our requirements, i.e., that $z \in U$ and $\|z-y\| \geqslant \|x + y\| - \varepsilon$. To prove that $z \in U$, it is enough to show that $|f_i(z - x)| < \delta$ for all $i \in \overline{1, n}$ because we already know that $z \in B_Y$. Since $z_0 \in U_0$, we have that $|\hat{f}_i(z_0 - x)| < \delta/2$ and so,

$$\begin{aligned}|f_i(z - x)| = |\hat{f}_i(z - x)| &\leqslant |\hat{f}_i(z - z_0)| + |\hat{f}_i(z_0 - x)| \\ &\leqslant \|\hat{f}_i\| \cdot \|z - z_0\| + \delta/2 < 2\varepsilon_0 + \delta/2 \leqslant \delta\end{aligned}$$

because $\varepsilon_0 \leqslant \delta/4$ by its definition. Thus, we have that indeed $z \in U$. Finally,

$$\|z - y\| \geqslant \|z_0 - y\| - \|z_0 - z\| > \|x + y\| - \varepsilon/2 - 2\varepsilon_0 \geqslant \|x + y\| - \varepsilon$$

since $\varepsilon_0 \leqslant \varepsilon/4$, which completes the proof. □

Let us finish the section with another possible norm equation for operators. Let us first comment that it is not known if Theorem 12.1.6 remains valid when one removes the surjectivity hypothesis on g. A testing case could be $g(t) = t^2$ for every $t \in \mathbb{R}$ (observe that for $X = \mathbb{R}$, $|1 + t^2| = 1 + |t|^2$ for every $t \in L(\mathbb{R}) = \mathbb{R}$). Discussing these topics in 2005, Gilles Godefroy asked the authors of [**160**] whether there could exist a real Banach space X of dimension greater than one satisfying

$$\|\mathrm{Id} + T^2\| = 1 + \|T^2\| \qquad \text{for } \textbf{all } T \in L(X). \tag{12.1.7}$$

Some comments are needed. Suppose the real space X admits a complex structure, i.e., there is a real-linear $J \in L(X)$ with $J^2 = -\mathrm{Id}$. (Such a J allows one to consider X as a vector space over $\mathbb{C}$ by means of $(a+ib)x := ax + bJx$; J simulates multiplication by i.) Then (12.1.7) is clearly violated. For example, if X is the "square" of another Banach space Y, meaning X is isomorphic to $Y \oplus Y$, then $J(y_1, y_2) = (-y_2, y_1)$ defines a complex structure on X. Therefore, a real Banach space in which (12.1.7) is valid is as remote from a complex space as possible and hence is called *extremely non-complex.*

It seems that Dieudonné in 1952 was one of the first authors who found an example of an infinite-dimensional real space X admitting no complex structure (X being the James space). The famous Gowers-Maurey hereditarily indecomposable space of 1993 provides another example of a Banach space without a complex structure. We refer the reader to the recent paper [**235**] for a discussion of differences between real and complex Banach spaces and to [**23, 110**] as a sample of papers dealing with complex structures.

Hence, to prove the existence of extremely non-complex spaces is quite a nontrivial matter, and this was first accomplished in [**189**], building on work by Koszmider [**188**] and Plebanek [**251**]. The starting point is the following observation.

LEMMA 12.1.14. *Let K be a perfect compact Hausdorff space and consider the real space $C(K)$. Let $h \in C(K)$, and let M_h denote the multiplication operator $f \mapsto hf$. Further, let $S \in L(X)$ be weakly compact and suppose that $h \geqslant 0$. Then $M_h + S$ satisfies the Daugavet equation.*

PROOF. We will rely on the method suggested in Section 1.4 and use the notation from there. If $(\mu_s)_{s\in K}$ denotes the kernel of S, then $(h(s)\delta_s + \mu_s)_{s\in K}$ is the kernel of $M_h + S$. By Lemma 1.4.1, it is enough to show (cf. (1.4.1))

$$\sup_{s\in U}\big(h(s) + \mu_s(\{s\})\big) \geqslant 0 \qquad \text{for all nonvoid open sets } U \subset K.$$

But this is so by Lemma 1.4.2 and since $h \geqslant 0$. □

Now, suppose $T = M_g + S \in L(C(K))$; then $T^2 = M_{g^2} + (M_g S + SM_g + S^2)$, and Lemma 12.1.14 tells us that T^2 satisfies the Daugavet equation if K is perfect and S is weakly compact. The punchline here is that there exist perfect compact Hausdorff spaces K (even connected ones) such that all $T \in L(C(K))$ have the form $T = M_g + S$ with S weakly compact ([**251**]; $C(K)$ has "few operators" in the parlance of [**188**]). Consequently, for these $C(K)$-spaces (12.1.7) is valid. Hence:

THEOREM 12.1.15. *There exist extremely non-complex Banach spaces; in fact, there exist extremely non-complex $C(K)$-spaces.*

The papers [**189**] and [**190**] contain more results along these lines; in particular, in the latter paper the following result is proved.

THEOREM 12.1.16 ([**190**, Theorem 6.2]). *Given a separable real Banach space E, there exists an extremely non-complex Banach space X whose dual contains E^* as an L-summand.*

The space in the theorem above is actually constructed with the additional property that the only surjective isometries on X are $\pm\mathrm{Id}$ [**190**, Theorem 6.2]. As E^* is an L-summand of X^*, it clearly follows that every surjective isometry on E^* extends to a surjective isometry on X^*. Therefore, using the result for $E = \ell_2$, we obtain the following very surprising example.

EXAMPLE 12.1.17 ([**190**, Example 6.3]). *There is a (extremely non-complex) Banach space X whose group of surjective isometries reduces to* $\{\mathrm{Id}, -\mathrm{Id}\}$, *while the group of surjective isometries of X^* contains the group of surjective isometries of ℓ_2 (which are the unitary operators) as a subgroup.*

12.2. Big slices and big weak neighbourhoods

Observe that Lemma 3.1.15 yields that in Banach spaces with the Daugavet property slices, non-empty relatively weakly open subsets of B_X, and convex combination of slices are big in many senses (in diameter, in Chebyshev radius, etc.). In recent years, these geometric properties of slices, weakly open subsets and convex combinations of slices have generated a research topic of its own with a vast literature. In this section we pursue to present a humble survey on these properties. Because of the amount of results, we decided not to include any proofs. In addition to the long list of references given throughout the section, we refer the reader interested in the phenomenon of big slices to the following PhD theses [**195**, **227**, **237**, **250**, **266**]

12.2.1. Diametral diameter two properties. We begin with the properties that are probably closest to the Daugavet property.

DEFINITION 12.2.1. Let X be a Banach space. We say that X has:

(1) the *diametral local diameter two property* (DLD2P) if, for every slice S of B_X, every $x \in S \cap S_X$ and every $\varepsilon > 0$ there exists $y \in S$ so that $\|x - y\| > 2 - \varepsilon$;
(2) the *diametral diameter two property* (DD2P) if, for every non-empty relatively weakly open subset U of B_X, every $x \in U \cap S_X$ and every $\varepsilon > 0$ there exists $y \in U$ so that $\|x - y\| > 2 - \varepsilon$;
(3) the *diametral strong diameter two property* (DSD2P) if given a convex combination of slices C of B_X, $x \in C$ and $\varepsilon \in \mathbb{R}^+$ there exists $y \in C$ such that

$$\|x - y\| > 1 + \|x\| - \varepsilon. \tag{12.2.1}$$

In the case of a dual Banach space X, the *weak-star diametral local diameter two property* (w^*-*DLD2P*), the *weak-star diametral diameter two property* (w^*-*DD2P*), and the *weak-star strong diameter two property* (w^*-*DSD2P*) are defined in a natural way just replacing the concept of slices and weakly open set by its corresponding weak-star version.

Before establishing the main results on the diametral diameter two properties, let us take a break to explain where these properties have originated and to recall the different terminologies under which these properties have been known. One of the first places where this property appeared is [**64**], where the authors exhibit a Schur subspace of $L_1[0,1]$ with the following property: For every $x \in S_X$ and every $\varepsilon > 0$, it follows that $x \in \overline{\text{conv}}\big(\{y \in B_X\colon \|y - x\| > 2 - \varepsilon\}\big)$ (this property is clearly equivalent to the diametral local diameter two property by a standard Hahn-Banach separation argument).

Later, in [**144**, Theorem 3.2], it was proved that this property, under the name of *spaces with bad projections*, is inherited by 1-unconditional sums (in contrast to what happens with the Daugavet property by Corollary 7.5.7). This terminology was motivated by the following result.

THEOREM 12.2.2. [**144**, Theorem 1.4]. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has the DLD2P.*
(2) *For every rank-one projection $P : X \to X$ the following inequality holds*

$$\|\text{Id} - P\| \geqslant 2.$$

This allowed these authors to prove that the DLD2P is actually different from the Daugavet property.

As far as we know, this property remained unnoticed until the paper [**2**]. In that paper, the authors aimed to construct a midpoint locally uniformly rotund (MLUR) Banach space where every slice of the unit ball has diameter two. In [**2**, Section 2] they constructed a renorming of $C[0,1]$ which is MLUR with a "rather strong form of the local diameter two property", which they called LD2P+.

Finally, the terminology exposed here (and which continued later on in the diameter two properties community) is from [**43**], introducing the versions for weakly open sets and convex combinations of slices. The epithet "diametral" is added because the DLD2P (respectively, the DD2P) means exactly that every slice (respectively, every non-empty relatively weakly open subset of B_X) has diameter two and every point of its intersection with the unit sphere is *diametral*.

Let us also indicate that the variant of convex combination of slices was defined in such a way because, in general, a convex combination of slices of B_X does not have to intersect the unit sphere. This fact is easy to believe if we deal with spaces where convex combinations of slices of small diameter exist. However, it turns out that even in the case that every convex combination of slices has diameter two, it is possible that there are convex combinations of slices which do not intersect the unit sphere (see [**210**, Section 3]).

Let us consider the following diagram

$$\begin{tikzcd}
\text{DP} \arrow[r, Rightarrow, "(1)"] & \text{DSD2P} \arrow[r, Rightarrow, "(2)"] \arrow[d, Rightarrow, "(4)"'] & \text{DD2P} \arrow[r, Rightarrow, "(3)"] \arrow[d, Rightarrow, "(5)"'] & \text{DLD2P} \arrow[d, Rightarrow, "(6)"'] \\
& w^*\text{-DSD2P} \arrow[r, Rightarrow, "(7)"] & w^*\text{-DD2P} \arrow[r, Rightarrow, "(8)"] & w^*\text{-DLD2P}
\end{tikzcd}$$

FIGURE 12.1. Relations between diametral diameter two properties

It is known that none of the vertical implications can be reversed. For instance, observe that the Daugavet property implies the DSD2P and, for a dual space X, if the predual has the Daugavet property then X enjoys the w^*-DSD2P. Consequently, $C[0,1]^*$ is an example having the w^*-DSD2P but containing slices of arbitrarily small diameter (the Dirac measures δ_t are strongly exposed points of $B_{C[0,1]^*}$), so in particular fails the DLD2P.

For the horizontal arrows, the following results were proved in [**43**] for two Banach spaces X and Y.

- Given $1 \leqslant p \leqslant \infty$, the space $X \oplus_p Y$ has the DD2P if, and only if, X and Y have the DD2P.
- $X \oplus_\infty Y$ has the DLD2P if, and only if, X and Y have the DLD2P. Moreover, $X \oplus_p Y$ fails the DLD2P for every $1 < p < \infty$.

This shows that neither implication (2) nor implication (7) can be reversed. In recent years, a big effort has been done in different ways (by studying stability results with respect to absolute norms [**130**], by studying inheritance of the diametral properties by subspaces [**250**, Section 2.3], or by studying intermediate properties [**4**]) in order to know if the converse of (1) holds. Finally, the solution was proved: yes, it does.

THEOREM 12.2.3. [**157**, Theorem 2.3] *Let X be a Banach space with the DSD2P. Then X has the Daugavet property.*

12.2.2. Diameter two properties. Observe that, in the language of Chebyshev radius (see the paragraph preceding Definition 8.6.7), a Banach space X has the DLD2P (respectively, the DD2P) if given any slice S of B_X (respectively, any non-empty relatively weakly open subset U of B_X) it follows that $r_S(S) = 2$ (respectively, $r_U(U) = 2$). It is natural to study the weaker condition that the previous objects have diameter two rather than radius two. In this spirit, we consider the following definition.

DEFINITION 12.2.4. Let X be a Banach space.

(1) X has the *slice diameter two property* (*slice-D2P*) if every slice of B_X has diameter two.
(2) X has the *diameter two property* (*D2P*) if every non-empty relatively weakly open subset of B_X has diameter two.
(3) X has the *strong diameter two property* (*SD2P*) if every convex combination of slices of B_X has diameter two.

In dual Banach spaces, the *weak-star slice diameter two property* (w^*-slice-D2P) (respectively the *weak-star diameter two property* (w^*-D2P), *weak-star strong diameter two property* (w^*-SD2P)) are defined in the natural way just replacing slices and weakly open sets with their weak-star versions.

As in Subsection 12.2.1, let us start with a brief historical summary to explain where these properties came from and the different terminologies for these properties.

Probably the first appearance of the diameter two properties was in duality with other geometric conditions related to a non-differentiability condition on the norm like *rough norms* [**149**], *average rough norms* [**93**] and *octahedral norms* [**93, 119**]. Moreover, there is a big presence of the diameter two conditions in all the characterisations of the Daugavet property that appeared in the first few years of the present century.

However, it is fair to set the starting point of the diameter two properties as a research topic of its own in the geometry of Banach spaces in the paper [**238**]. This is because the main theorem of that paper states that every non-void relatively weakly open subset of the unit ball of an infinite-dimensional uniform algebra has diameter two.

This motivated a series of papers [**15, 33, 34, 35, 209**] where these properties (actually the slice-D2P and the D2P) were studied in concrete classes of Banach spaces without giving any name to these properties.

In his 2006 paper [**143**], Y. Ivakhno defined the *r-big slice property* if $r_S(S) \geqslant 1$ and the *d-big slice property* if every slice of B_X has diameter two (i.e., the slice diameter two property). He simply called the slice-D2P the *big slice property* in [**142**].

Finally, to the best of our knowledge, the diameter two properties were formally defined in [**7**] with a small difference; namely in that paper the slice-D2P is called the *local diameter two property*. As far as we are aware, the term slice diameter two property appeared first in the paper [**16**] (submitted in 2012). Since then, both notations have survived, and we have maintained the use of slice-D2P since, in our opinion, it is more descriptive.

Let us start with giving some examples. Clearly, one list of examples of spaces with the SD2P comes from the Banach spaces with the Daugavet property thanks to Lemma 3.1.15. Here is another example.

EXAMPLE 12.2.5. c_0 has the SD2P. Indeed, let $C := \sum_{i=1}^n \lambda_i S_i$ be a convex combination of slices. Find an element $c := \sum_{i=1}^n \lambda_i x_i \in C$, where each $x_i \in S_i$ can be assumed, by a density argument, to have finite support. Consequently, we can find $m \in \mathbb{N}$ so that $x_i(k) = 0$ holds for every $k \geqslant m$ and every $1 \leqslant i \leqslant n$. This implies $x_i \pm e_k \in S_{c_0}$ for every $1 \leqslant i \leqslant n$ and every $k \geqslant m$. Moreover, observe that since $e_k \to 0$ weakly, $x_i \pm e_k \to x_i$ weakly. Since each S_i is weakly open there is a sufficiently big $k_0 \in \mathbb{N}$ so that $k_0 \geqslant m$ and so that $x_i \pm e_k \in S_i$ holds for every $k \geqslant k_0$ and $1 \leqslant i \leqslant n$. In particular, given $k \geqslant k_0$, it follows that $c \pm e_k = \sum_{i=1}^n \lambda_i(x_i \pm e_k) \in C$. Consequently

$$\operatorname{diam} C \geqslant \|(c + e_k) - (c - e_k)\| = 2\|e_k\| = 2.$$

We also observe that every $x \in S_{c_0}$ is reasonable (see Definition 8.6.7). Indeed, pick $k \in \mathbb{N}$ such that $|x_k| = 1$, say $x_k = 1$ without loss of generality. For the k^{th} unit vectors $e_k \in S_{c_0}$ and $e_k^* \in S_{\ell_1}$ we have $e_k^*(x) = 1$, and for $z = (z_1, z_2, \dots) \in \operatorname{Slice}(e_k^*, \varepsilon)$ it follows $z_k > 1 - \varepsilon$ so that $\|z - e_k\| \leqslant 1$.

In general, it is known (from [**143**]) that if X has the slice-D2P then $r_S(S) \geqslant 1$ for every slice S of B_X. The previous example shows that we cannot, in general, improve the constant 1 above. On the other hand, Ivakhno [**143**, p. 102] posed the question whether the converse holds, i.e., if a Banach space X such that $r_S(S) \geqslant 1$ for every slice S of B_X necessarily has the slice-D2P. It has recently been answered in the negative in [**128**, Theorem 3.7], where it is shown that taking X as the predual of the James tree space JT_∞, then $r_S(S) \geqslant 1$ holds for every slice S of B_X, but there exists $\varphi \in S_{X^*}$ satisfying

$$\inf_{\alpha > 0} \operatorname{diam} \operatorname{Slice}(B_X, \varphi, \alpha) \leqslant \sqrt{2}.$$

This example was improved in [**271**], where it was proved that there exists a Banach space X such that $r_S(S) \geqslant 1$ holds for every slice S but

$$\inf\{\operatorname{diam}(S)\colon S \text{ slice of } B_X\} = 1.$$

We have proved in Example 12.2.5 that c_0 is an example of Banach space with the SD2P but failing even the DLD2P. Before giving a more exhaustive list of examples, let us consider the general diagram of implications for the diameter two properties:

The same example as for their diametral versions proves that none of the vertical implications reverse. For the horizontal arrows, observe that a Banach space X has one of the diameter two properties if, and only if, X^{**} has the corresponding weak-star version. This fact is an easy consequence of the w^*-density of B_X in $B_{X^{**}}$ and the w^*-lower semicontinuity of the norm of X^{**}.

For that reason, we will focus on (1) and (2) because, if a Banach space X is a counterexample to one of them, then X^{**} is a counterexample for the corresponding w^*-version.

Let us start with (1), for which a counterexample comes from the study of the diameter two properties under ℓ_p-sums. Let X and Y be two Banach spaces. Then:

(1) $X \oplus_\infty Y$ has the slice-D2P (respectively the D2P, SD2P) if, and only if, either X or Y has the slice-D2P (respectively the D2P, the SD2P).

$$\begin{tikzcd}
\text{SD2P} \arrow[r, Rightarrow, "(1)"] \arrow[d, Rightarrow, "(3)"'] & \text{D2P} \arrow[r, Rightarrow, "(2)"] \arrow[d, Rightarrow, "(4)"'] & \text{slice-D2P} \arrow[d, Rightarrow, "(5)"'] \\
w^*\text{-SD2P} \arrow[r, Rightarrow, "(6)"] & w^*\text{-D2P} \arrow[r, Rightarrow, "(7)"] & w^*\text{-slice-D2P}
\end{tikzcd}$$

FIGURE 12.2. Relations between diameter two properties

(2) $X \oplus_1 Y$ has the slice-D2P (respectively the D2P, SD2P) if, and only if, both X and Y has the slice-D2P (respectively the D2P, the SD2P).
(3) If $1 < p < \infty$, then $X \oplus_p Y$ has the slice-D2P (respectively the D2P) if, and only if, X and Y have the slice-D2P (respectively the D2P).
(4) If $1 < p < \infty$ then $X \oplus_p Y$ fails the SD2P.

The cases of $p = 1$ and $p = \infty$ were considered in [**33, 209**]. On the other hand, the "if" part for case (3) was proved in [**7**, Section 3] (the authors also reproved the cases $p = 1$, $p = \infty$). Moreover, the complete study of the diameter two properties under ℓ_p-sums was performed in [**16**]. Observe that (4) was also proved in [**126**] and, later, reproved in [**241**].

Anyway, as a consequence of the results exposed above, $c_0 \oplus_2 c_0$ is an example of Banach space with the D2P but failing the SD2P.

The possible counterexample for the difference between the slice-D2P and the D2P proved to be a harder task. The solution appeared in 2015.

THEOREM 12.2.6. [**37**] *Every Banach space containing c_0 admits an equivalent renorming with the slice-D2P but whose unit ball contains non-empty relatively weakly open subsets of B_X of arbitrarily small diameter.*

REMARK 12.2.7. *Very recently, another example of a Banach space X with the slice-D2P but whose unit ball contains non-empty relatively weakly open subsets of arbitrarily small diameter has been constructed in* [**6**, Section 4].

Observe that the isomorphic nature of this theorem provides a wide range of non-isomorphic counterexamples for the difference between the slice-D2P and the D2P. On the other hand, the extreme difference in the previous theorem between the diameter of slices and the diameter of weakly open sets motivated the question of whether such extreme counterexamples could be constructed for the D2P and the SD2P. After showing that the convex combinations of slices of $c_0 \oplus_p c_0$ have diameter $\geqslant 1$ for $1 < p < \infty$, in [**39**] the following result was obtained.

THEOREM 12.2.8. *Every Banach space containing c_0 admits an equivalent renorming with the D2P but whose unit ball contains convex combinations of slices of arbitrarily small diameter.*

REMARK 12.2.9. *Later, in* [**2**, Section 2]*, a new example of a Banach space with the D2P and whose unit ball contains convex combinations of slices of arbitrarily small diameter was constructed.*

Let us now pass to exhibit a list of concrete examples of Banach spaces enjoying the diameter two properties.

(1) Every Daugavet space enjoys the (D)SD2P and its dual enjoys the w^*-DSD2P.
(2) Every Banach space whose dual norm is 2-rough has the slice-D2P [**149**].
(3) Every infinite-dimensional uniform algebra enjoys the SD2P (implicitly proved in [**238**], but pointed out in [**7**]).
(4) Non-reflexive JB*-triples (in particular, infinite-dimensional C^*-algebras) have the D2P [**34**].
(5) Every non-reflexive M-embedded Banach space X satisfies that X^{**} (and hence X) has the SD2P (proved in [**7**], a previous version for the D2P was proved in [**209**]).
(6) Every Banach space with infinite-dimensional centraliser has the SD2P (proved in [**16**], a previous version for the D2P was proved in [**47**]).
(7) If X and Y are Banach spaces with infinite-dimensional centraliser then $X \widehat{\otimes}_\pi Y$ has the D2P [**17**].

Let us mention that, similarly to the Daugavet property, when dealing with the diameter two properties on X you need to have a good knowledge of the topological dual (in order to determine the shape of slices) and of the norm of X (in order to compute distances and diameters). This double knowledge is, sometimes, difficult to obtain (one can think for instance of projective tensor products, where the expression for the norm is difficult to deal with).

Because of this reason, we will point out here some characterisations of the diameter two properties which only require detailed knowledge of the dual (we can call them "dual characterisations"). Investigating these characterisations turned out to be of importance because, on the one hand, valuable connections with some open problems from the end of the 1980s were thus established and, on the other hand, progress on diameter two properties in new classes of Banach spaces (e.g., in tensor product spaces or in Lipschitz free spaces) could be obtained.

As we pointed out at the beginning of the section, there is a characterisation for the slice-D2P (respectively, the SD2P) in terms of an (average) roughness condition. However, the new properties have turned out to be much more applicable in concrete Banach spaces.

Let us start with a well-known definition.

Definition 12.2.10. Let X be a Banach space. We say that (the norm of) X is *octahedral* if, given any finite-dimensional subspace Y of X and $\varepsilon > 0$, there exists $x \in S_X$ so that

$$\|y + \lambda x\| \geqslant (1-\varepsilon)(\|y\| + |\lambda|)$$

holds for every $y \in Y$ and every $\lambda \in \mathbb{R}$ (in other words, x is $(\varepsilon, 1)$-orthogonal to Y according to Definition 2.8.5).

Remark 12.2.11. *If we compare the definition of an octahedral norm with Lemma 3.1.14, we observe that Daugavet spaces are octahedral. The difference between octahedral spaces and Daugavet spaces is that, for the Daugavet property, we require in the setting of Definition 12.2.10 that, given Y and ε, the set of those $x \in S_X$ witnessing the definition of octahedral norm is weakly dense. Observe also that indeed almost Daugavet spaces are octahedral (Lemma 9.2.4).*

According to [**94**, p. 121], the definition of octahedral norm comes from an unpublished paper of G. Godefroy and B. Maurey from 1987, where it is proved that every separable Banach space X containing an isomorphic copy of ℓ_1 admits an equivalent renorming which is octahedral. This result was improved in [**119**] where the separability assumption was removed and, in the conclusion, even non-zero L-orthogonal elements in the bidual are obtained.

In [**93**] it is proved, using average rough norms, that if X is octahedral then X^* has the w^*-SD2P, leaving as an open problem whether or not the converse holds true. Observe that the validity of the converse is stated in [**119**, Remark II.5], but no proof is given. The first complete published proof for the converse is in [**36**].

Theorem 12.2.12. *Let X be a Banach space. Then:*

(a) *X is octahedral if, and only if, X^* has the w^*-SD2P.*
(b) *X has the SD2P if, and only if, X^* is octahedral.*

Later, in the paper [**129**], different and useful characterisations of octahedral norms were exhibited (e.g., it is proved that, in the definition of octahedrality, one can replace "every finite dimensional subspace Y" with "every finite subset of S_X"). Moreover, the authors of that paper obtained a characterisation of the w^*slice-D2P and the w^*-D2P in terms of weaker versions of octahedrality.

Theorem 12.2.13. *Let X be a Banach space.*

(1) *X^* has the w^*-slice-D2P if, and only if, X is* locally octahedral *(*LOH*), i.e., for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$.*
(2) *X^* has the w^*-D2P if, and only if, X is* weakly octahedral *(*WOH*), i.e., for every $x_1, \ldots, x_n \in S_X$, $x^* \in B_{X^*}$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm ty\| \geqslant (1-\varepsilon)(|x^*(x_i)| + t)$ for all $i \in \{1, \ldots, n\}$ and $t > 0$.*

As we have pointed out before, Theorem 12.2.12 opened the door to great progress in the study of the SD2P and of the octahedrality of the norm in tensor product spaces and the Lipschitz-free spaces. Let us list some of the highlights in this area.

(1) Let X and Y be two Banach spaces. If X and Y have the SD2P, then so does $X \widehat{\otimes}_\pi Y$ [**40**]. However, the result is not true if we remove the SD2P from one factor [**197**]. Analogously, if X and Y are octahedral, then so is $X \widehat{\otimes}_\varepsilon Y$ [**196**]. However, the result is no longer true if we remove octahedrality from one factor [**197**].
(2) Let X be a Banach space with the SD2P. If Y is a Banach space with the metric approximation property and Y is finitely representable in ℓ_1, then $X \widehat{\otimes}_\pi Y$ has the SD2P [**200**, Theorem 3.2].
(3) Let M be a metric space. If the Lipschitz free space $\mathcal{F}(M)$ has the SD2P, then $\mathcal{F}(M) \widehat{\otimes}_\pi X$ has the SD2P [**200**, Theorem 3.3].

12.2.3. Symmetric strong diameter two property. In the proof of the fact that c_0 enjoys the SD2P, it is even proved that c_0 actually enjoys the following stronger property.

Definition 12.2.14. Let X be a Banach space. We say that X has the *symmetric strong diameter two property* (*SSD2P*) if, for every $k \in \mathbb{N}$, every finite family of slices $S_1, \ldots, S_k$ of B_X, and every $\varepsilon > 0$, there are $x_i \in S_i$ and there exists $\varphi \in B_X$ with $\|\varphi\| > 1 - \varepsilon$ such that $x_i \pm \varphi \in S_i$ for every $i \in \{1, \ldots, k\}$.

When X is a dual space, the *weak-star symmetric strong diameter two property* (*w^*-SSD2P*) is defined in the natural way replacing slices with weak-star slices.

Also this property has been implicitly used many times until it was formally defined. Probably one of the first places where this property was implicitly used is [**238**, Theorem 2]. Actually, in [**7**, Lemma 4.1] it is proved, without the language of the SSD2P, that if a Banach space has the SSD2P then it actually has the SD2P. Moreover, it is also pointed out in the same paper that infinite-dimensional uniform algebras actually enjoy the SSD2P.

The first place where this definition was formally exhibited is [**9**, Definition 1.3]. In that paper, the authors prove that certain classes of subspaces of a $C_0(L)$ space, which they call somewhat regular, enjoy the SSD2P. See the introduction of [**127**], where a list of examples of Banach spaces with the SSD2P is presented including infinite-dimensional L_1-predual spaces, Banach spaces with infinite-dimensional centraliser, or Müntz spaces.

In the above mentioned paper [**127**], a big effort is made in order to understand how the SSD2P is preserved by absolute sums of Banach spaces or how this property is inherited by taking subspaces. Among all these results, let us point out that no absolute sum, except the ℓ_∞-sum, inherits the SSD2P from its factors. As a consequence, the authors obtain that $L_1[0,1]$ is an example of Banach space with the SD2P (actually with the Daugavet property) but failing the SSD2P.

At first glance, it seems that this property is nothing but a mere generalisation of the SD2P. Let us point out, however, that this is not true because of the following result proved in [**127**, Theorem 2.1].

THEOREM 12.2.15. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has the SSD2P.*
(2) *For every $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in S_X$ there are nets $(y_\alpha^i) \subset S_X$, $1 \leqslant i \leqslant n$, and $(z_\alpha) \in S_X$ so that $(y_\alpha^i) \to x_i$ weakly, $(z_\alpha) \to 0$ weakly and $\|y_\alpha^i \pm z_\alpha\| \leqslant 1$ for every α.*

As we pointed out before Definition 12.2.10, if one wants to play with the SD2P in a given Banach space X, one requires a good knowledge of the norm of the space and the topological dual (for describing slices). On the other hand, in order to prove that X^* is octahedral only a good access to the dual space is needed.

However, the good point of the SSD2P is that, thanks to Theorem 12.2.15, one only needs to have a good description of the norm of the space and a criterion of weak convergence on the space X. This simple idea has found strong applications in the context of Lipschitz function spaces because, thanks to Lemma 11.3.8, the following result was obtained.

THEOREM 12.2.16. [**200**, Theorem 2.2] *Let M be a metric space. If M is unbounded or not uniformly discrete, then* $\mathrm{Lip}_0(M)$ *has the SSD2P. In particular, the norm of $\mathcal{F}(M)^{**}$ is octahedral.*

Let us point out that previous results on the w^*-SSD2P in spaces of Lipschitz functions were obtained in [**127**, Section 5]. This work was pushed quite a bit further in [**243**], where the author obtained a characterisation of when $\mathrm{Lip}_0(M)$

enjoys the w^*-SSD2P in terms of a geometric condition on M. See also [**244**] for another characterisation in terms of a geometric condition on $\mathcal{F}(M)$.

12.2.4. Almost square Banach spaces. Let us start with the definition of three new properties.

DEFINITION 12.2.17. Let X be a Banach space. We say that X is:

(1) *locally almost square* (*LASQ*) if for every $x \in S_X$ there exists a sequence (y_n) in B_X such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$;
(2) *weakly almost square* (*WASQ*) if for every $x \in S_X$ there exists a sequence (y_n) in B_X such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly;
(3) *almost square* (*ASQ*) if for every $x_1, \ldots, x_k \in S_X$ there exists a sequence (y_n) in B_X such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \ldots, k\}$.

The properties LASQ and WASQ appeared in [**193**], where they are implicitly used to study the slice-D2P and the D2P in Cesàro spaces. The terminology of Definition 12.2.17 is from [**5**], where the authors also introduced the strongest notion (ASQ). Among all these properties, the most studied one in the literature has been that of ASQ spaces.

It was proved in [**5**] that every Banach space which admits an ASQ renorming contains an isomorphic copy of c_0. The converse was proved in [**41**] (note that a version for separable spaces already appeared in [**5**]).

One of the main open problems in this line, posed in [**5**], is whether or not there is any dual ASQ Banach space. This problem was solved in the positive way in [**3**].

The study of ASQ spaces has shown to be useful in the study of the SD2P in projective symmetric tensor products of Banach spaces [**41, 267**]. Furthermore, the study of a stronger variant of the ASQ, the unconditional almost squareness, was used to prove that certain spaces of vector-valued Lipschitz functions are not isometrically isomorphic to dual Banach spaces [**114, 116**].

We finish the section including a list of examples of Banach spaces which are LASQ, WASQ or ASQ:

(1) c_0 is ASQ.
(2) More generally, every non-reflexive M-embedded space is ASQ [**5**].
(3) The (James-) Hagler space JH is ASQ [**38**].
(4) $X \oplus_\infty Y$ is ASQ (respectively, LASQ, WASQ) if either X or Y is ASQ (respectively, LASQ, WASQ). Moreover, LASQ and WASQ is preserved by taking absolute sums [**5**].
(5) $L_1[0,1]$ is WASQ but not ASQ [**5**].
(6) If X is ASQ and Y is a non-zero Banach space, then $X \widehat{\otimes}_\varepsilon Y$ is ASQ [**196**].
(7) The space of little Lipschitz functions over a locally compact and totally disconnected metric space which is not uniformly discrete is ASQ [**116**].

12.3. The alternative Daugavet property

In Proposition 1.4.8 and Remark 1.4.10, we have shown that every operator $T\colon C(K) \to C(K)$, no matter whether K is perfect or not, satisfies the following variant of the Daugavet equation, called the *alternative Daugavet equation*:

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega T\| = 1 + \|T\|. \tag{12.3.1}$$

(In the case of real scalars there are of course only two choices for ω: $\omega = 1$ or $\omega = -1$.)

Before proceeding, let us tie the alternative Daugavet equation to the theory of numerical ranges. The *(spatial) numerical range* of an operator $T \in L(X)$ is the set of scalars given by

$$V(T) = \{x^*(Tx)\colon \|x^*\| = \|x\| = x^*(x) = 1\},$$

and its *numerical radius* is

$$v(T) = \sup\{|\lambda|\colon \lambda \in V(T)\}.$$

It is clear that $v(\cdot)$ is a seminorm in $L(X)$ and that $v(T) \leqslant \|T\|$ for every $T \in L(X)$. The concept of numerical range was introduced by F. Bauer in 1962 and there is a related concept introduced by G. Lumer in 1961. Both are different generalisations of Toeplitz's field of values of an operator (matrix) on a (finite-dimensional) Hilbert space, but they produce the same concept of numerical radius. We refer the reader to the monographs [**56, 57**] from the 1970s for a detailed discussion of this. The following result from [**103**] relates the numerical range with the Daugavet and alternative Daugavet equations.

LEMMA 12.3.1. *For $T \in L(X)$ we have:*

(a) $\|\mathrm{Id} + T\| = 1 + \|T\|$ *if and only if* $\sup \operatorname{Re} V(T) = \|T\|$.

(b) $\max\limits_{|\omega|=1} \|\mathrm{Id} + \omega T\| = 1 + \|T\|$ *if and only if* $v(T) = \|T\|$.

PROOF. It is a well-known and important fact in the theory of numerical ranges that

$$\sup \operatorname{Re} V(T) = \inf_{\alpha>0} \frac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha} = \lim_{\alpha\to 0^+} \frac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha} \tag{12.3.2}$$

(see [**56**, §9]).

Suppose $\sup \operatorname{Re} V(T) = \|T\|$; then it immediately follows from the above that

$$\|T\| = \sup \operatorname{Re} V(T) \leqslant \|\mathrm{Id} + T\| - 1,$$

i.e., T satisfies the Daugavet equation. Conversely, if T satisfies the Daugavet equation, then so does αT for each $\alpha > 0$ (see Remark 3.1.2), and (12.3.2) implies $\sup \operatorname{Re} V(T) = \|T\|$.

To derive (b) from (a), just observe that $v(T) = v(\omega T)$ for each $\omega \in \mathbb{T}$ and that $v(T) = \max_{|\omega|=1} \sup \operatorname{Re} V(\omega T)$. □

The *numerical index*, $n(X)$, of a Banach space X is the best constant $k \in [0, 1]$ in the inequality

$$k\|T\| \leqslant v(T) \qquad \text{for all } T \in L(X).$$

While there are real Banach spaces of numerical index 0 (for example, $\mathbb{R}^2$ with the Euclidean norm or ℓ_2), it is a surprising fact that for complex Banach spaces one always has $n(X) \geqslant 1/e$ [**56**, §4, Theorem 1]. Combining this notion with Lemma 12.3.1, one immediately gets:

COROLLARY 12.3.2. *A Banach space X has numerical index $n(X) = 1$ if and only if every $T \in L(X)$ satisfies the alternative Daugavet equation* (12.3.1).

Thus, Proposition 1.4.8 says that $C(K)$-spaces have numerical index one, as do $L_1(\mu)$-spaces for all measures μ (see Corollary 1.4.9 or Example 12.3.5 below). On the other hand, for an infinite-dimensional Hilbert space H, depending on the scalar field $n(H) = 0$ ($\mathbb{K} = \mathbb{R}$) or $n(H) = \frac{1}{2}$ ($\mathbb{K} = \mathbb{C}$) [**56**, p. 87]. Incidentally, the numerical index of L_p-spaces for $p \neq 1, 2, \infty$ has not been computed yet (but there are successive improvements for the two-dimensional case in the recent papers [**231**], [**234**], and [**232**]) although it is known that it is not zero for any $p \neq 2$ [**222**].

In keeping with the approach to the Daugavet equation put forward in Definition 3.1.1, we now introduce the following notion.

Definition 12.3.3. A Banach space X has the *alternative Daugavet property* (*ADP* for short) if every rank-one operator $T\colon X \to X$ satisfies the alternative Daugavet equation, that is,

$$\max_{|\omega|=1} \|\mathrm{Id} + \omega T\| = 1 + \|T\|.$$

Definition 12.3.3 and Corollary 12.3.2 provide the implications

$$\boxed{X \text{ has the Daugavet property}} \implies \boxed{X \text{ has the ADP}} \impliedby \boxed{n(X) = 1};$$

neither of them can be reversed: c_0 has the ADP (even $n(c_0) = 1$), but not the Daugavet property, and $X = C([0,1], \ell_2)$ has the ADP (even the Daugavet property), but its numerical index is $n(X) = n(\ell_2) < 1$ (and $c_0 \oplus_1 X$ serves both directions at once).

One way to rephrase the difference between the Daugavet property and the alternative Daugavet property is that the Daugavet property means, given a rank-one operator T, one has $\|\mathrm{Id} + \omega T\| = 1 + \|T\|$ for *all* $\omega \in \mathbb{T}$ whereas the ADP means $\|\mathrm{Id} + \omega T\| = 1 + \|T\|$ for *some* $\omega \in \mathbb{T}$.

Remark 12.3.4. An obvious duality argument shows that X *has the ADP when* X^* *does* (and likewise $n(X^*) \leqslant n(X)$), in complete analogy with the Daugavet property. However, unlike the Daugavet property (see Corollary 3.1.6) *the ADP depends on the choice of the scalar field*: The simplest example to show this is $\mathbb{C}$, which has the ADP when considered as a one-dimensional complex space, but does not when considered as a two-dimensional real space, viz. $\mathbb{R}^2$ with the Euclidean norm.

Examples 12.3.5.

(a) We have already pointed out at the beginning of this section that $C(K)$-spaces have numerical index one and hence the ADP. The same is true for $C_0(L)$-spaces and (complex) uniform algebras [**299**].

(b) The duality argument in Remark 12.3.4 implies that all L_1-spaces and their isometric preduals have numerical index one and hence, the ADP.

(c) The case of C^*-algebras and, more generally, JB*-triples was studied in [**223**] and [**218**]. We present the most important results on the alternative Daugavet property for C^*- and von Neumann algebras without proof.

  • For a von Neumann algebra A with (unique) predual A_*, the following assertions are equivalent:

  (i) A has the ADP;
  (ii) A_* has the ADP;
  (iii) $|x(x_*)| = 1$ for all $x \in \mathrm{ext}(B_A)$, $x_* \in \mathrm{ext}(B_{A_*})$;

(iv) $A = C \oplus_\infty N$ for a commutative von Neumann algebra C and a nonatomic (= diffuse) von Neumann algebra N.

- For a C^*algebra A, the following assertions are equivalent:
  (i) A has the ADP;
  (ii) $|x^{**}(x^*)| = 1$ for all $x^{**} \in \text{ext}(B_{A^{**}})$, $x^* \in w^*\text{-stexp}(B_{A^*})$, the set of weak* strongly exposed points;
  (iii) there exists a closed two-sided ideal J such that A/J is nonatomic;
  (iv) all atomic (= minimal) projections are central.

Since the ADP is defined in terms of operators of rank one, it can be characterised in terms of slices. This is the content of the following lemma (cf. Theorem 3.1.5 for the Daugavet property).

LEMMA 12.3.6. *For a Banach space X, the following assertions are equivalent:*

(i) *X has the ADP.*

(ii) *For every $x \in S_X$, for every $\varepsilon > 0$ and every slice S of B_X, there are some $y \in S$ and $\omega \in \mathbb{T}$ such that*

$$\|\omega x + y\| > 2 - \varepsilon.$$

PROOF. The proof follows the lines of the proof of Theorem 3.1.5; in the case of the alternative Daugavet property, one simply observes that the rank-one operator $x^* \otimes \omega x$ satisfies the Daugavet equation for some $\omega \in \mathbb{T}$. □

Also the proof of the following result follows the lines of the argument for the Daugavet property, cf. Theorem 3.2.6; hence we omit it.

PROPOSITION 12.3.7. *If X has the ADP, then every strong Radon-Nikodým operator (in particular, every compact or weakly compact operator) satisfies the alternative Daugavet equation.*

In fact, this also holds for operators whose restrictions to separable subspaces are SCD-operators, and this provides the following interesting particular case; see [**27**, §5] for details.

PROPOSITION 12.3.8. *Let X be a Banach space with the ADP. Then every operator which does not fix copies of ℓ_1 satisfies the alternative Daugavet equation.*

From the above two propositions, we get the following important consequence.

COROLLARY 12.3.9. *Let X be a Banach space with the ADP. If X has the RNP or it does not contain copies of ℓ_1, then $n(X) = 1$.*

We now come to a geometric characterisation of the alternative Daugavet property from [**223**]. We write

$$\Gamma := \{(x^*, x^{**})\colon x^* \in \text{ext}(B_{X^*}),\ x^{**} \in \text{ext}(B_{X^{**}}),\ |x^{**}(x^*)| = 1\}.$$

We first formulate a lemma.

LEMMA 12.3.10. *For $T \in L(X)$ we have*

$$v(T) = \sup\{|x^{**}(T^* x^*)|\colon (x^*, x^{**}) \in \Gamma\}.$$

PROOF. One notes that

$$\overline{\mathrm{conv}}(V(T)) = \{\varphi(T)\colon \varphi \in L(X)^*,\ \|\varphi\| = \varphi(\mathrm{Id}) = 1\}$$

(see e.g. [**56**, Theorem 9.4]) and

$$\{\varphi \in L(X)^*\colon \|\varphi\| = \varphi(\mathrm{Id}) = 1\} = \overline{\mathrm{conv}}^{w^*}\{\varphi_\gamma\colon \gamma \in \Gamma\}$$

with $\varphi_\gamma(T) = x^{**}(T^*x^*)$ for $\gamma = (x^*, x^{**})$, by a Hahn-Banach argument. □

PROPOSITION 12.3.11. *A Banach space X has the ADP if and only if*

$$B_{X^*\oplus_\infty X^{**}} = \overline{\mathrm{conv}}^{w^*}(\Gamma). \tag{12.3.3}$$

PROOF. Let us start with the "if" part. We define $Y = X \oplus_1 X^*$ so that $Y^* = X^* \oplus_\infty X^{**}$; then (12.3.3) means that $\Gamma \subset Y^*$ is 1-norming. That is, whenever $y_0 = (x_0, x_0^*) \in Y$, then

$$\|y_0\| = \|x_0\| + \|x_0^*\| = \sup\{|x^*(x_0) + x^{**}(x_0^*)|\colon (x^*, x^{**}) \in \Gamma\}. \tag{12.3.4}$$

Now, consider the rank-one operator $T = x_0^* \otimes x_0$; i.e., $Tx = x_0^*(x)x_0$. Then, for $\varepsilon > 0$ there is some $(\xi^*, \xi^{**}) \in \Gamma$ such that (by (12.3.4))

$$|\xi^*(x_0) + \xi^{**}(x_0^*)| \geqslant \|x_0\| + \|x_0^*\| - \varepsilon.$$

Therefore,

$$|\xi^*(x_0)| \geqslant \|x_0\| - \varepsilon, \quad |\xi^{**}(x_0^*)| \geqslant \|x_0^*\| - \varepsilon.$$

On the other hand, by Lemma 12.3.10,

$$v(T) \geqslant |\xi^{**}(T^*\xi^*)| = |\xi^{**}(x_0^*)||\xi^*(x_0)| \geqslant (\|x_0^*\| - \varepsilon)(\|x_0\| - \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we get $v(T) \geqslant \|T\|$, and Lemma 12.3.1 shows that T satisfies the alternative Daugavet equation. This completes the proof of the "if" part.

Conversely, suppose that X has the ADP; we shall verify (12.3.4). To this end, let $x_0 \in X$, $x_0^* \in X^*$ and consider the rank-one operator $T = x_0^* \otimes x_0$ as above. Let $\varepsilon > 0$. We know from Lemma 12.3.1 that $v(T) = \|T\| = \|x_0\|\|x_0^*\|$, and by Lemma 12.3.10 there exists $(\xi^*, \xi^{**}) \in \Gamma$ such that

$$(1-\varepsilon)\|x_0\|\|x_0^*\| = (1-\varepsilon)v(T) \leqslant |\xi^{**}(T^*\xi^*)| = |\xi^{**}(x_0^*)||\xi^*(x_0)|.$$

For suitable $\omega_1, \omega_2 \in \mathbb{T}$ we get

$$\omega_1\xi^*(x_0) = |\xi^*(x_0)| \geqslant (1-\varepsilon)\|x_0\|,$$
$$\omega_2\xi^{**}(x_0^*) = |\xi^{**}(x_0^*)| \geqslant (1-\varepsilon)\|x_0^*\|.$$

Now, $(\omega_1\xi^*, \omega_2\xi^{**}) \in \Gamma$ and

$$|\omega_1\xi^*(x_0) + \omega_2\xi^{**}(x_0^*)| \geqslant (1-\varepsilon)(\|x_0\| + \|x_0^*\|);$$

since $\varepsilon > 0$ is arbitrary, we have obtained (12.3.4) and thus (12.3.3). □

Incidentally, the same reasoning provides the following characterisation of the Daugavet property.

PROPOSITION 12.3.12. *A Banach space X has the Daugavet property if and only if*

$$B_{X^*\oplus_\infty X^{**}} = \overline{\mathrm{conv}}^{w^*}(\Gamma')$$

where $\Gamma' = \{(x^*, x^{**})\colon x^* \in \mathrm{ext}(B_{X^*}),\ x^{**} \in \mathrm{ext}(B_{X^{**}}),\ x^{**}(x^*) = 1\}$.

In contrast to the Daugavet property, there are Asplund spaces like c_0, RNP spaces like ℓ_1, and certain finite-dimensional spaces (in particular, reflexive spaces) with the ADP. However, we can use Proposition 12.3.11 to give a different proof of a result from the 1999 paper [**208**] (where it is stated for Banach spaces with numerical index one, but only the ADP is used in its proof) on the impossibility of having infinite-dimensional real reflexive spaces with the ADP. First, we need the following consequence of the ADP on the behaviour of denting points and weak-star denting points.

PROPOSITION 12.3.13 ([**208**, Lemma 1]). *Let X be a Banach space with the alternative Daugavet property. Then:*

(a) $|x^*(x)| = 1$ *for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every* $x \in \operatorname{dent}(B_X)$*.*
(b) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every weak-star denting point* x^* *of* B_{X^*}*.*

PROOF. (Sketch using Proposition 12.3.11.) We start with (b). Let us write $Y = X \oplus_1 X^*$ and observe that Proposition 12.3.11 and Milman's Theorem (see Lemma 2.6.17(a)) give that

$$\operatorname{ext}(B_{X^*}) \times \operatorname{ext}(B_{X^{**}}) \cong \operatorname{ext}(B_{Y^*}) \subset \overline{\Gamma}^{w^*}$$

(we have used Remark 2.9.10 in the identification of the first two sets). It follows that given $x_0^{**} \in \operatorname{ext}(B_{X^{**}})$ and a weak-star denting point x_0^* of B_{X^*} (hence $x_0^* \in \operatorname{ext}(B_{X^*})$), one may find nets (x_λ^*) in B_{X^*} and (x_λ^{**}) in $B_{X^{**}}$ which converge in the weak-star topology to x_0^* and x_0^{**}, respectively, and $(x_\lambda^*, x_\lambda^{**}) \in \Gamma$ for every λ. As x_0^* is weak-star denting, it is a point of weak-star norm continuity of the identity map restricted to B_{X^*}, hence $(x_\lambda^*) \to x_0^*$ in norm. Now,

$$\begin{aligned}\left|x_\lambda^{**}(x_\lambda^*) - x_0^{**}(x_0^*)\right| &\leqslant \left|x_\lambda^{**}(x_\lambda^*) - x_\lambda^{**}(x_0^*)\right| + \left|x_\lambda^{**}(x_0^*) - x_0^{**}(x_0^*)\right| \\ &\leqslant \|x_\lambda^* - x_0^*\| + \left|x_\lambda^{**}(x_0^*) - x_0^{**}(x_0^*)\right| \to 0.\end{aligned}$$

Therefore, $|x_0^{**}(x_0^*)| = 1$ as desired.

To prove (a), observe that if $x_0 \in \operatorname{dent}(B_X)$ then $J_X(x_0)$ is a weak-star denting point of $B_{X^{**}}$ and a completely analogous argument gives the proof. □

This result, with the help of Proposition 2.6.18, immediately gives the following preclusive condition for a Banach space to have the alternative Daugavet property from [**208**].

THEOREM 12.3.14 ([**208**]). *Let X be a real Banach space with the ADP.*

(a) *If the set of denting points of B_X is infinite, then X contains a copy of c_0 or X contains a copy of ℓ_1.*
(b) *If the set of weak-star denting points of B_{X^*} is infinite, then X^* contains a copy of ℓ_1.*

In particular, a reflexive real Banach space with the ADP must be finite-dimensional.

Remark 12.3.4 mentions the inequality $n(X^*) \leqslant n(X)$. It was an open problem for many years whether this is in fact an equality. The first counterexample was exhibited in [**67**] where spaces with $n(X) = 1$, but $n(X^*) < 1$ were constructed. The construction is based on the geometric notion of a lush Banach space; a Banach space X is *lush* if for all $x, y \in S_X$ and every $\varepsilon > 0$, there exists $y^* \in S_{X^*}$ such that $y \in S := \mathrm{Slice}(B_X, y^*, \varepsilon)$ and

$$\mathrm{dist}(x, \mathrm{conv}(\mathbb{T}S)) < \varepsilon.$$

The following results from [**67**] are important for the discussion.

REMARKS 12.3.15 ([**67**]).

(a) *Lush spaces have numerical index one.*

(b) *C-rich subspaces of $C(K)$ (Definition 8.4.2) are lush.*

(c) *Let $\mu \in C(K)^*$. Then,* $\ker \mu$ *is C-rich if and only if* $\mathrm{supp}(\mu)$ *does not intersect the set of isolated points of K.*

Using these results, we can now present the already mentioned example of a Banach space whose numerical index does not coincide with the numerical index of its dual.

EXAMPLE 12.3.16 ([**67**, Example 3.1]). *The space*

$$X = \{(x, y, z) \in c \oplus_\infty c \oplus_\infty c\colon\ \lim x + \lim y + \lim z = 0\}$$

is C-rich in $c \oplus_\infty c \oplus_\infty c$ (which is a $C(K)$ space), so it is lush and hence $n(X) = 1$. On the other hand, $n(X^*) = 1/2$ since X^* contains a two-dimensional L-summand whose unit ball is a hexagon. See [**67**, Example 3.1] *for details.*

REMARK 12.3.17. Let us comment that it is possible to make constructions similar to the one above to get *spaces X satisfying that $n(X) = 1$ and $n(X^*) = 0$ in the real case and $n(X^*) = 1/\mathrm{e}$ in the complex case*, see [**67**]. Even more, it is possible to construct *a Banach space X such that X^{**} is a C^*-algebra, $n(X) = 1$, and there is a norm-one $T \in L(X^*)$ with $v(T) = 0$* (hence, not only $n(X^*) = 0$ but also, $v(\cdot)$ is not a norm on $L(X^*)$), see [**219**].

Lush spaces have been shown to be very useful when working with the ADP and numerical index one. Let us present two interesting results in this regard.

First, the fact that no infinite-dimensional real reflexive space satisfies the ADP has been improved in [**27**] using SCD sets and lush spaces as follows.

THEOREM 12.3.18 ([**27**, Corollary 4.9]). *Let X be an infinite-dimensional real Banach space with the ADP. Then X^* contains a copy of ℓ_1.*

The proof of this result follows four steps:

(1) If X has the ADP and does not contain ℓ_1, then it is lush by [**27**, Corollary 4.8] (this improves Corollary 12.3.9). If, otherwise, X contains ℓ_1, then X^* contains ℓ_1 by the lifting property of ℓ_1, see Theorem 2.3.8.

(2) Hence, we may reduce to the case that X is a lush space. Moreover, lushness is separably determined (in the sense that every separable subspace of X is contained in another separable subspace which is lush, see [**66**, Theorem 4.2]), so we may reduce to the case that X is an infinite-dimensional separable lush space.

(3) If X is a separable lush space, there is a subset $A \subset S_{X^*}$ which is norming for X such that $|x^{**}(a^*)| = 1$ for every $x^{**} \in \mathrm{ext}(B_{X^{**}})$ and every $a^* \in A$ by [**161**, Theorem 4.3] (an appropriate use of the Baire category theorem gives the result).

(4) Finally, as X is infinite-dimensional, A has to be infinite; so, in the real case, Proposition 2.6.18 shows that X^* contains c_0 or X^* contains ℓ_1. But a dual Banach space which contains c_0 actually contains ℓ_∞, hence ℓ_1 (see [**97**, Theorem V.10] for instance).

Another application of lush spaces to spaces with numerical index one is the following.

THEOREM 12.3.19 ([**66**, Corollary 3.6]). *If X is a Banach space containing a copy of c_0, then X admits an equivalent lush norm, hence with numerical index one.*

12.4. Alternatively convex or smooth spaces and related properties

It is easy to see that an operator $T \in L(X)$ for which the number $\|T\|$ belongs to the spectrum of T satisfies the Daugavet equation, viz.

$$\|\mathrm{Id} + T\| \geqslant \max\{|\lambda|\colon \lambda \in \sigma(\mathrm{Id} + T)\} = \max\{|\lambda|\colon \lambda \in 1 + \sigma(T)\} \geqslant 1 + \|T\|.$$

This motivated Abramovich, Aliprantis and Burkinshaw [**14**] to introduce the following definition.

DEFINITION 12.4.1. A Banach space X is *anti-Daugavet* if for all $T \in L(X)$ the following equivalence holds true:

$$\|\mathrm{Id} + T\| = 1 + \|T\| \quad \Longleftrightarrow \quad \|T\| \in \sigma(T) \tag{12.4.1}$$

So, in anti-Daugavet spaces the set of operators that satisfy the Daugavet equation is the smallest possible.

In addition to the above definition, we will say that a Banach space X has the *anti-Daugavet property for a class $\mathcal{M} \subset L(X)$ of operators* if the equivalence (12.4.1) holds for $T \in \mathcal{M}$.

The authors of [**14**] demonstrated that all uniformly convex spaces, as well as all uniformly smooth ones, are anti-Daugavet. There is a relation of this result to the theory of numerical radius, remarked by Espid and Alizadeh [**107**].

Recall from the previous section that $v(T)$ denotes the numerical radius of $T \in L(X)$, and that $v(T) = \|T\|$ if and only if there exists $\theta \in \mathbb{T}$ with

$$\|\mathrm{Id} + \theta T\| = 1 + \|T\|$$

(see Lemma 12.3.1). This implies the following result.

PROPOSITION 12.4.2. *Let X be anti-Daugavet for a class $\mathcal{M} \subset L(X)$. Then for every $T \in \mathcal{M}$ with $v(T) = \|T\|$ the spectral radius of T is equal to $\|T\|$ as well, so $\rho(T) = v(T) = \|T\|$. In particular, in uniformly convex spaces and in uniformly smooth ones, this is applicable to all operators with $v(T) = \|T\|$.*

In this section we present the complete geometric description of finite-dimensional anti-Daugavet spaces given in [**173**] and some related results from [**178**].

DEFINITION 12.4.3. A Banach space X is said to be *alternatively convex or smooth* ($X \in$ (acs) for short) if for every $x, y \in S_X$ and $x^* \in S_{X^*}$ the following implication holds

$$x^*(x) = 1,\ \|x + y\| = 2 \quad \Longrightarrow \quad x^*(y) = 1. \tag{12.4.2}$$

Geometrically, the property (acs) means some kind of smoothness at the end-points of every linear segment that lies on the unit sphere. Namely, $X \in$ (acs) if and only if any two quasi-codirected $x, y \in S_X$ are smooth points of $S_{\operatorname{lin}\{x,y\}}$. This means that every strictly convex space as well as every smooth space is (acs).

LEMMA 12.4.4. *Suppose that $X \in$ (acs) and that $T \in L(X)$ is a weakly compact operator with $\|T\| = 1$ and $\|\mathrm{Id} + T\| = 2$. Suppose in addition that $\|x + Tx\| = 2$ for some $x \in S_X$. Then 1 is an eigenvalue of T.*

PROOF. First remark that $\|Tx\| \leqslant \|T\|\|x\| \leqslant 1$ and

$$\|Tx\| \geqslant \|x + Tx\| - \|x\| = 1,$$

so $Tx \in S_X$. Consider a functional $x^* \in S_{X^*}$ such that $x^*(x) = 1$. By the (acs)-property of X, one has $x^*(Tx) = 1$. Therefore $x_1^* := T^*(x^*)$ attains the value 1 at x and hence belongs to S_{X^*}. Again, using (12.4.2) we obtain $x_1^*(Tx) = 1$. Applying the same argument inductively shows that $x^*(T^n x) = 1$ for all $n \in \mathbb{N}$. This implies that

$$K := \overline{\operatorname{conv}}\big(\{T^n x\colon n \in \mathbb{N}\}\big) \subset \{v \in X\colon x^*(v) = 1\} \cap B_X \subset S_X$$

so, in particular $0 \notin K$. Also, K is a weakly compact convex set, since

$$\{T^n x\colon n \geqslant 1\} = T(\{T^n x\colon n \geqslant 0\}),$$

which is relatively weakly compact, and T maps K into K. By the Schauder-Tikhonov fixed point theorem, T has a fixed point in K, which is a non-zero eigenvector for the eigenvalue 1. $\square$

DEFINITION 12.4.5. Let X be a Banach space.

(a) We say that the space X is *locally uniformly alternatively convex or smooth* (*luacs*) if for all $x_n, y \in S_X$ and $x^* \in S_{X^*}$ the implication

$$x^*(x_n) \to 1,\ \|x_n + y\| \to 2 \quad \Longrightarrow \quad x^*(y) = 1 \tag{12.4.3}$$

holds.

(b) We say that the space X is *uniformly alternatively convex or smooth* (*uacs*) if for all $x_n, y_n \in S_X$ and $x_n^* \in S_{X^*}$ the implication

$$x_n^*(x_n) \to 1,\ \|x_n + y_n\| \to 2 \quad \Longrightarrow \quad x_n^*(y_n) \to 1 \tag{12.4.4}$$

holds.

Remark that uniformly convex spaces and uniformly smooth spaces are (uacs), and locally uniformly convex spaces are (luacs). In general

$$X \in (\text{uacs}) \Rightarrow X \in (\text{luacs}) \Rightarrow X \in (\text{acs}),$$

and (acs), (luacs) and (uacs) are equivalent in finite-dimensional spaces by a compactness argument.

THEOREM 12.4.6. *For a Banach space X, the following conditions are equivalent:*

(i) *X has the anti-Daugavet property for compact operators.*

(ii) *X has the anti-Daugavet property for operators of rank-one.*

(iii) *X is* (luacs).

PROOF. (i) $\Rightarrow$ (ii) is evident. For the proof of (ii) $\Rightarrow$ (iii), assume that X fails to be (luacs). Then there is a functional $x^* \in S_{X^*}$ and there are elements $x_n, y \in S_X$ such that $x_n, y \in S_X$ and $\|x_n + y\| \to 2$, $x^*(x_n) \to 1$, but $\operatorname{Re} x^*(y) < 1$. Consider the operator $T\colon X \to X$ defined by $Tv = x^*(v)y$. Then $\|T\| = 1$ and $\|\mathrm{Id} + T\| = 2$, since

$$\begin{aligned} \|\mathrm{Id} + T\| &\geqslant \limsup \|x_n + Tx_n\| \\ &= \limsup \|x_n + x^*(x_n)y\| \\ &= \limsup \|x_n + y\| \;=\; 2. \end{aligned}$$

Thus T satisfies the Daugavet equation, but $1 \notin \sigma(T)$ because of $\operatorname{Re} x^*(y) < 1$; so X fails the anti-Daugavet property for rank-one operators.

(iii) $\Rightarrow$ (i). Let $U \in L(X) \setminus \{0\}$ be a compact operator that satisfies the Daugavet equation. Then, by the usual quasi-directedness argument, for the compact operator $T := \frac{U}{\|U\|}$ we have that $\|T\| = 1$ and $\|\mathrm{Id}+T\| = 2$. Then there is a sequence $(x_n) \subset S_X$ for which $\|Tx_n + x_n\| \to 2$. By compactness of T, we may assume that $Tx_n \to y \in S_X$. Now, consider $x^* \in S_{X^*}$ such that $x^*(y) = 1$; then $x^*(Tx_n) \to 1$. Put $y^* = T^*x^*$; then we have $\|y^*\| \leqslant 1$ and, from $y^*(x_n) = x^*(Tx_n) \to 1$, we deduce that actually $\|y^*\| = 1$. So we have $\|x_n + y\| \to 2$ and $y^*(x_n) \to 1$, and from (12.4.3) we get that $y^*(y) = 1$. But now,

$$\|y + Ty\| \geqslant x^*(y + Ty) = x^*(y) + y^*(y) = 2,$$

so Lemma 12.4.4 implies that $1 \in \sigma(T)$; hence $\|U\| \in \sigma(U)$ and we obtain (i). $\square$

THEOREM 12.4.7. *For a finite-dimensional Banach space X the following assertions are equivalent:*

(i) *X is an anti-Daugavet space;*
(ii) *$X \in$ (acs).*

PROOF. (i) $\Rightarrow$ (ii). Since X has in particular the anti-Daugavet property for operators of rank one, Theorem 12.4.6 implies that $X \in$ (luacs), which is equivalent to (acs) in the finite-dimensional setting.

(ii) $\Rightarrow$ (i). In finite-dimensional spaces (acs) implies (luacs), so X has the anti-Daugavet property for compact operators (Theorem 12.4.6), but in finite-dimensional spaces every operator is compact. $\square$

We now turn to the relation of the (uacs)-property and the anti-Daugavet property; see Definition 2.2.9 for the notion of superreflexivity.

LEMMA 12.4.8. *If X is* (uacs)*, then X is superreflexive.*

PROOF. The (uacs)-property provides a uniform restriction on the structure of 2-dimensional subspaces of X: For all $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\|x + y\| > 2 - \delta$, $x, y \in S_X$, and $x^* \in S_{(\mathrm{lin}\{x,y\})^*}$ with $x^*(x) > 1 - \delta$, then $x^*(y) > 1 - \varepsilon$. Therefore, not every 2-dimensional Banach space is finitely representable in X and so, it has to be superreflexive by [**172**]. (In fact, X is uniformly non-square, which is enough to imply superreflexivity by a theorem due to James; see [**32**, p. 261].) $\square$

The next theorem is another application of the ultrapower technique in our book.

THEOREM 12.4.9. *If $X \in$* (uacs)*, then X has the anti-Daugavet property.*

PROOF. Let $T\colon X \to X$ be an operator of norm 1 such that $\|\mathrm{Id} + T\| = 2$. Consider an ultrapower $X^{\mathfrak{U}}$. Define $T^{\mathfrak{U}}\colon X^{\mathfrak{U}} \to X^{\mathfrak{U}}$ by $T^{\mathfrak{U}}[(x_n)] = [(Tx_n)]$ as in Definition 2.2.6.

The main advantage of considering $T^{\mathfrak{U}}$ is that $\mathrm{Id}^{\mathfrak{U}} + T^{\mathfrak{U}}$ attains its norm. (Indeed, if $x_n^0 \in S_X$ are chosen so that $\|Tx_n^0\| \to 1$, then $x^0 := [(x_n^0)] \in S_{X^{\mathfrak{U}}}$ and $\|T^{\mathfrak{U}}(x^0)\| = 1$.) By Lemma 12.4.8, $X^{\mathfrak{U}}$ is reflexive and thus, $T^{\mathfrak{U}}$ is weakly compact. It is evident from the definition that if X is (uacs) and Y is finitely representable in X, then Y is (uacs), i.e., (uacs) is a superproperty. Thus we conclude that $X^{\mathfrak{U}}$ is (acs), and we may apply Lemma 12.4.4. So there is an eigenvector $x^1 = [(x_n^1)] \in S_{X^{\mathfrak{U}}}$ with $T^{\mathfrak{U}}x^1 = x^1$. This means that $\|Tx_n^1 - x_n^1\|$ tends to zero along $\mathfrak{U}$, and 1 is an approximate eigenvalue of T, in particular $1 \in \sigma(T)$. $\square$

As we said, it is evident that (uacs) is a superproperty. Also, the super anti-Daugavet property coincides with (uacs), but we doubt whether the anti-Daugavet property is a superproperty itself; cf. Question (12.8) in Section 12.6.

We would like to conclude this section with the following remark. There is no mathematical definition of a “property that is opposite to the given one”. What “an opposite property” is depends on the standpoint and peculiarity of the problem under consideration. In particular, apart from anti-Daugavetness, there are other good contenders for the role of something very opposite to the Daugavet property, among them the SCD property from Chapter 10 that is, as to our taste, the best one.

12.5. Notes and remarks

Section 12.1. The discussion of possible norm equalities for operators is taken from the paper [**160**].

The plus-minus property and its strong version were studied in [**284**]. There, it was also shown for a Banach function space E and a space X with the strong plus-minus property that the Bochner space $E(X)$ has the strong plus-minus property as well. Likewise, if $X_1, X_2 \in SD_{\pm}$, then $X_1 \oplus_E X_2 \in SD_{\pm}$ whenever the direct sum is equipped with an absolute norm (see Subsection 2.9.1 for the definition). Note that the Daugavet property is only preserved by ℓ_1- and ℓ_∞-sums, cf. Corollary 7.5.7.

The results on extremely non-complex Banach spaces are taken from [**189, 190**].

Section 12.2. All the relevant references have been given in the text.

Section 12.3. The notion of the alternative Daugavet property was introduced in [**223**]; the first results in Section 12.3 come from that paper. However, the importance of the equality $v(T) = \|T\|$ for rank-one operators was already realised in [**208**].

The quintessential reference for numerical ranges are the Bonsall-Duncan volumes [**56, 57**] from the 1970s; see also the 2006 survey [**165**] and Sections 2.1 and 2.9 of the recent book [**73**]. The relation between the alternative Daugavet equation and the numerical radius in Lemma 12.3.1 is an old result from [**103**].

The contents of the section are mainly taken from (cited chronologically) [**208**] (1999), [**223**] (2004), [**67**] (2007), [**218**] (2008), [**219**] (2008) [**66**] (2009), [**161**] (2009), [**27**] (2010).

The monograph [**162**] contains a wealth of results explaining the interrelations of the ADP, numerical index one and lushness in many classes of Banach spaces. This work also puts forward a generalisation of the alternative Daugavet equation in the spirit of Daugavet centres; namely, an operator $G\colon X \to Y$ is called a *spear operator* if

$$\max_{|\omega|=1} \|G + \omega T\| = 1 + \|T\| \qquad \text{for all } T \in L(X,Y).$$

If the above norm equality holds only for rank-one operators, we say that G is an *ADP operator*. There is also a notion of *lush operator*. We refer to [**162**] for a detailed discussion of these topics.

There is also a notion of *alternative polynomial Daugavet property*, studied in [**72**], [**81**], and [**276**] among others.

Finally, the very recent paper [**198**] introduces the notion of super ADP, which lies strictly between the ADP and the Daugavet property. It is defined using relatively weakly open subsets of the unit ball: a Banach space X has the *super ADP* if for every $x \in S_X$ and every non-empty relatively weakly open subset U of B_X, we have that

$$\sup_{y\in U} \max_{|\omega|=1} \|x + \omega y\| = 2.$$

(We caution the reader that this is not an instance of a superproperty as defined in Definition 2.2.9.) It follows from Corollary 3.1.16 that the Daugavet property implies the super ADP, while Lemma 12.3.6 gives that the super ADP implies the ADP. The three properties are different and this shows that the ADP does not behave like the Daugavet property with respect to relatively weakly open subsets of the unit ball, that is, Shvydkoy's lemma 3.1.15 does not have an ADP analogue. See more details in [**198**]. Incidentally, the property analogous to the super ADP but using convex combinations of slices instead of weakly open sets is actually equivalent to the Daugavet property [**198**, Proposition 4.12].

Section 12.4. As we have already said in the text, the notion of the anti-Daugavet property is due to Abramovich, Aliprantis, and Burkinshaw [**14**], and most of the results of this section come from [**173**] and [**178**]. A detailed study of the properties (acs), (luacs) and (uacs) along with some generalisations and variants, especially in the context of absolut sums and Köthe-Bochner spaces, was performed by Hardtke [**131, 132**]; for the Bochner L_p-spaces see also [**287**].

12.6. Open questions

In this section we collect different open questions and possible future research lines derived from the results of this chapter.

(12.1) Is it enough for a Banach space to satisfy the plus-minus property that $\|\mathrm{Id} + T\| = \|\mathrm{Id} - T\|$ for all operators of rank one?

(12.2) Are the plus-minus property and the strong plus-minus property equivalent?

(12.3) Does every extremely non-complex Banach space have the Daugavet property? Does it have numerical index equal to one?

Some partial results on the question above can be found in [**220**].

(12.4) What is an analogue of the uniform Daugavet property for the ADP?

(12.5) Let X be a complex Banach space whose underlying real space $X_{\mathbb{R}}$ has the ADP. Must X have the Daugavet property?

We don't know about the validity of the complex analogue of Theorem 12.3.14 (here "infinitely many" has to be understood as meaning "infinitely many $\mathbb{C}$-linearly independent", as $\ell_\infty^{(2)}$ contains infinitely many denting points, even up to multiplication by scalars $\omega \in \mathbb{T}$). In particular, the following question remains open to the best of our knowledge.

(12.6) Does there exist an infinite-dimensional complex reflexive Banach space with the ADP?

As for isometric properties, it is proved in [**161**, Theorem 2.1] that the dual of a Banach space with the alternative Daugavet property cannot be smooth nor strictly convex, extending the corresponding result for the Daugavet property (Corollary 3.2.5). Analogously to Questions (3.3) and (3.4), we may ask the following.

(12.7) Does there exist a strictly convex or smooth Banach space with the ADP?

Finally, from the last section of the chapter, we present the following open question.

(12.8) Is the anti-Daugavet property a superproperty? If so, does it coincide with (uacs)?

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