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Results in Mathematics

A computational Approach to the Study of Finite-Complement Submonids of an Affine Cone

J. C. Rosales, R. Tapia-Ramos, and A. Vigneron-Tenorio

Abstract. Let $\mathcal{C} \subseteq \mathbb{N}^p$ be an integer cone. A $\mathcal{C}$-semigroup $S \subseteq \mathcal{C}$ is an affine semigroup such that the set $\mathcal{C}\setminus S$ is finite. Such $\mathcal{C}$-semigroups are central to our study. We develop new algorithms for computing $\mathcal{C}$-semigroups with specified invariants, including genus, Frobenius element, and their combinations, among other invariants. To achieve this, we introduce a new class of $\mathcal{C}$-semigroups, termed $\mathcal{B}$-semigroups. By fixing the degree lexicographic order, we also research the embedding dimension for both ordinary and mult-embedded $\mathbb{N}^2$-semigroups. These results are applied to test some generalizations of Wilf's conjecture.

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Introduction

Let $\mathbb{N}$ be the set of natural numbers. We consider an affine semigroup S to be a finitely generated commutative additive submonoid of $\mathbb{N}^p$ (for a positive integer p) such that the zero element belongs to S. For convenience, we use 0 instead of $(0, 0, \ldots, 0) \in \mathbb{N}^p$ whenever it is unambiguous. It is well known that any affine semigroup S admits a unique minimal system of generators, denoted by $\mathrm{msg}(S)$ (see [22]), and the cardinality of the minimal generating set, called the embedding dimension, is represented by $\mathrm{e}(S)$. Let $\mathcal{C} \subseteq \mathbb{N}^p$ be an affine (integer) cone. A submonoid $S \subseteq \mathcal{C}$ is a $\mathcal{C}$-semigroup if the set $\mathcal{C}\setminus S$

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is finite; this structure was introduced in [17]. When $\mathcal{C} = \mathbb{N}^p$, S is referred to as a generalized numerical semigroup, first defined in [14]. In the special case where $p = 1$, the $\mathcal{C}$-semigroup S is known as a numerical semigroup. Note that the structures of $\mathcal{C}$-semigroups and generalized numerical semigroups naturally extend the notion of numerical semigroups to higher dimensions.

Most of the invariants analysed in the study of numerical semigroups can be generalized to $\mathcal{C}$-semigroups. In addition to the embedding dimension, the set $\mathcal{H}(S) = \mathcal{C}\backslash S$ is called the set of gaps of S, and the genus, $\mathrm{g}(S)$, is the cardinality of $\mathcal{H}(S)$. Following the notation given in [19], let $\{\tau_1, \ldots, \tau_t\}$ be the set of extremal rays of $\mathcal{C}$. For each $i \in \{1, \ldots, t\}$, the i-multiplicity of S, denoted by $\mathrm{mult}_i(S)$, is the minimum element in $\tau_i \cap S$ under the componentwise partial order in $\mathbb{N}^p$. To extend certain invariants to $\mathcal{C}$-semigroups, it is necessary to define a total order on $\mathbb{N}^p$, which is an order relation $\preceq$ on $\mathbb{N}^p$ that is compatible with addition and satisfies $0 \preceq x$ for any $x \in \mathbb{N}^p$ (see [10]). Once a total order $\preceq$ is fixed on $\mathbb{N}^p$, for instance, the Frobenius element of S, $\mathrm{Fb}(S)$, is defined as $\max_{\preceq}(\mathcal{C}\backslash S)$. By convention, if $S = \mathcal{C}$, then $\mathrm{Fb}(S) = (-1, -1, \ldots, -1)$. The conductor of S, denoted by $\mathrm{c}(S)$, is the minimum element $x \in S$ such that $\mathrm{Fb}(S) \prec x$. An element s of S is said to be a small element if $s \prec \mathrm{Fb}(S)$. The set of all small elements is denoted by $\mathrm{N}(S)$, and its cardinality by $\mathrm{n}(S)$. Additionally, the smallest non-zero element of S with respect to the total order $\preceq$ is called the multiplicity of S, denoted by $m(S)$. For any element f of $\mathcal{C}$, let $\mathcal{N}(f)$ the cardinality of the set $\{x \in \mathcal{C}\backslash\{0\} \mid x \preceq f\}$. In particular, when f is the Frobenius element of S, $\mathcal{N}(\mathrm{Fb}(S))$ is referred to as the Frobenius number of S. In the case of numerical semigroups, note that $\mathcal{N}(\mathrm{Fb}(S)) = \mathrm{Fb}(S)$.

Although the study of $\mathcal{C}$-semigroups and generalized numerical semigroup is relatively recent, much research has focused on examining these structures through their invariants. For instance, [7] and [14] include algorithms for computing all possible $\mathbb{N}^p$-semigroups with a fixed genus. A recent study on the unbounded behaviour of certain invariants, such as the conductor, in $\mathcal{C}$-semigroups can be found in [1]. Moreover, [16] provides a method to compute the set of all $\mathcal{C}$-semigroups with a fixed Frobenius element, defined as $\mathfrak{C}(Fb = f) = \{S \text{ is a } \mathcal{C}\text{-semigroup} \mid \mathrm{Fb}(S) = f\}$, we provide an alternative procedure for computing it.

In this work, we contribute to this ongoing research by developing and describing several algorithms to compute all possible $\mathcal{C}$-semigroups with specified invariants, including the small elements and the genus, the Frobenius element, the genus, and the combination thereof, i.e., we focus on $\mathfrak{C}(\mathrm{gen} = g, \mathrm{se} = n)$ which corresponds to $\mathcal{C}$-semigroups with a fixed genus and a number of small elements, $\mathfrak{C}(Fb = f, \mathrm{gen} = g)$ for a fixed Frobenius element and genus, and $\mathfrak{C}(\mathrm{gen} = g)$ denotes the set of all $\mathcal{C}$-semigroups with genus g. Additionally, we develop a new class of $\mathcal{C}$-semigroups based on their invariants, which we call $\mathcal{B}$-semigroups. We provide a graphical classification of these semigroups and show how the study of $\mathcal{B}$-semigroups is a tool for computing the set $\mathfrak{C}(Fb = f)$.

In 1978, Wilf conjectured that for any numerical semigroup S, the inequality $\mathrm{e}(S)\mathrm{n}(S) \geq \mathrm{Fb}(S) + 1$ holds (see [25]). Although the general case of this conjecture remains unsolved, specific cases have been addressed (see, for instance, [12,13] and [15]). A detailed discussion of this topic can be found in [11] and the references therein. The suggestion to extend Wilf's conjecture to higher dimensional structures was proposed in [14], leading to several contributions, such as the Generalized Wilf Conjecture (see [6]) and the Extended Wilf Conjecture (see [17]).

Following the analysis of invariants, another objective of this work is to discuss the embedding dimension and the set of minimal generators of two specific classes of $\mathbb{N}^2$-semigroups: ordinary $\mathbb{N}^2$-semigroups and mult-embedded $\mathbb{N}^2$-semigroups. Given a total order $\preceq$, an ordinary $\mathcal{C}$-semigroup is defined as a semigroup $S_c = \{0\} \cup \{x \in \mathcal{C} \mid x \succeq c\}$ for some $c \in \mathcal{C}$. Note that our definition of an ordinary semigroup differs from the one given in [5]. In the context of numerical semigroups, it is also known as *half-lines* (see [23]). In contrast, a mult-embedded $\mathcal{C}$-semigroup is defined as $S = \{m, 2\,m, \ldots, (k-1)m\} \sqcup S_{km}$, where m is a non-zero element of $\mathcal{C}$. For $\mathcal{C} = \mathbb{N}^2$, studying the embedding dimension of these classes of semigroups ordered by the graded lexicographic order allows us to test the Generalized and the Extended Wilf Conjecture for $\mathbb{N}^2$-semigroups.

The content of this work is organized as follows: In Sect. 1, we study the sets $\mathfrak{C}(Fb = f, gen = g)$ and $\mathfrak{C}(gen = g)$. We show some bounds for computing the mentioned invariants. Section 2 is devoted to study $\mathcal{B}$-semigroups. Besides, some procedures are given to compute the sets of all $\mathcal{B}$-semigroups with a fixed genus, a fixed Frobenius element, and both fixed genus and Frobenius element. In Sect. 3, we provide an algorithm to compute the set $\mathfrak{C}(Fb = f)$ by using $\mathcal{B}$-semigroups. In the last sections (Sects. 4 and 5), we prove that ordinary and mult-embedded $\mathbb{N}^2$-semigroups ordered by the graded lexicographic order satisfy the Generalized and the Extended Wilf Conjecture. The results introduced throughout are not only theoretical but also provide computational methods, which are illustrated through examples. To this aim, we have used third-party software (`Normaliz` [3]) and some libraries developed by the authors in `Mathematica` [26].

1. $\mathcal{C}$-Semigroups with Fixed Frobenius Element and Genus, and Fixed Genus

Given A a non-empty subset of $\mathbb{R}^p_\geq$, with $\mathbb{R}_\geq$ denoting the set of non-negative real numbers, the real cone determined by A is

$$L(A) = \left\{ \sum_{i=1}^{h} \lambda_i a_i \mid h \in \mathbb{N},\, \lambda_1, \ldots, \lambda_h \in \mathbb{R}_\geq,\, a_1, \ldots, a_h \in A \right\},$$

and the integer cone determined by A is the set $L(A)\cap\mathbb{N}^p$. In general, a non-degenerated real (or integer) cone is the set of real (or integer) points belonging to the convex hull of finitely many half lines in $\mathbb{R}^p_{\geq}$ emanating from the origin.

An integer cone $\mathcal{C}\subseteq\mathbb{N}^p$ is affine, that is, it is finitely generated, if there is a finite subset A of $\mathcal{C}$ such that $\mathcal{C}=L(A)\cap\mathbb{N}^p$. In [2, Chapter 2] is proved that a cone $\mathcal{C}\subseteq\mathbb{N}^p$ is affine if and only if it has a rational point in each of its extremal rays. Moreover, any subsemigroup of $\mathcal{C}$ is finitely generated if and only if there exists an element in the subsemigroup in each extremal ray of $\mathcal{C}$. We assume that any integer cone considered in this work is affine.

Fix an integer cone $\mathcal{C}$, and a total order $\preceq$ on $\mathbb{N}^p$. Recall that

$$\mathfrak{C}(Fb=f, gen=g)=\{S \text{ is a } \mathcal{C}\text{-semigroup } \mid \mathrm{Fb}(S)=f, \mathrm{g}(S)=g\}.$$

In this section, we have two main goals. The objectives are to describe an algorithm for the computation of $\mathfrak{C}(Fb=f, gen=g)$ and to provide a procedure to compute those $\mathcal{C}$-semigroups with a fixed genus. We consider some bounds to the genus and the Frobenius number of the semigroups to achieve these.

Using the terminology from [18], a $\mathcal{C}$-semigroup is called irreducible if it cannot be expressed as the intersection of two $\mathcal{C}$-semigroups that properly contain it. For any $x,y\in L\subseteq\mathbb{N}^p$, consider the partial order $x\leq_L y$ if $y-x\in L$. The following proposition establishes an irreducible $\mathcal{C}$-semigroup from an existing $\mathcal{C}$-semigroup, we provide the most general case applying any total order.

Proposition 1 [16, Lemma 12]. *Let $\mathcal{C}$ be an integer cone and let $f\in\mathcal{C}\backslash\{0\}$. Then, the set*

$$\Delta(\mathcal{C},f)=\Big(\mathcal{C}\backslash\{f\}\Big)\backslash\left\{x\in\mathcal{C}\backslash\{0\}\mid x\leq_{\mathcal{C}} f,\ \text{and } x\preceq\frac{f}{2}\right\}$$

is an irreducible $\mathcal{C}$-semigroup, with Frobenius element f.

The next proposition combines Corollaries 8 and 9 in [16]. We denote by $\mathcal{B}(f)$ the set $\{x\in\mathcal{C}\mid x\leq_{\mathcal{C}} f\}$ with $f\in\mathcal{C}\backslash\{0\}$. From now on, $\lceil\cdot\rceil$ denotes the ceiling function, which rounds up to the nearest integer, and for any set A, the symbol $\sharp$ denotes the cardinality of the set A. Besides, using the notation from [20], for any two natural numbers a and b, with $a\leq b$, the set $[\![a,b]\!]=\{r\in\mathbb{N}\mid a\leq r\leq b\}$. If $a=0$, instead of $[\![0,b]\!]$, we use $[\![b]\!]$ for short.

Proposition 2. *Let S be a $\mathcal{C}$-semigroup with Frobenius element f. Then, S is irreducible if and only if $g(S)=\lceil\frac{\sharp\mathcal{B}(f)}{2}\rceil$.*

In the specific context of $\mathbb{N}^p$-semigroups, the value of the cardinality of $\mathcal{B}(f)$ can be determined explicitly as $\sharp\mathcal{B}(f)=\prod_{i\in[\![p]\!]}(f_i+1)$ for $f=(f_1,\dots,f_p)\in\mathcal{C}\backslash\{0\}$.

Corollary 3 [8, Theorem 5.6 and Theorem 5.7]. *Let S be a $\mathbb{N}^p$-semigroup with Frobenius element $f=(f_1,\dots,f_p)$. Then, S is irreducible if and only if $g(S)=\lceil\frac{\prod_{i\in[\![p]\!]}(f_i+1)}{2}\rceil$.*

We gather the previous results to obtain the announced result. For any element $f \in \mathcal{C}$, recall that $\mathcal{N}(f)$ is the cardinality of the set $\{x \in \mathcal{C}\backslash\{0\} \mid x \preceq f\}$. For this section, we need a total order $\preceq$ such that $\mathcal{N}(f)$ is finite. So, we assume that the fixed total order $\preceq$ satisfies that property. For example, a graded order can be used (see [10]).

Theorem 4. *For any $f \in \mathcal{C}\backslash\{0\}$ and any positive integer g, the set $\mathfrak{C}(Fb = f, gen = g)$ is non-empty if and only if*

$$\left\lceil \frac{\sharp\mathcal{B}(f)}{2} \right\rceil \leq g \leq \mathcal{N}(f).$$

Proof. Assume that $\mathfrak{C}(Fb = f, gen = g)$ is non-empty, and let $S \in \mathfrak{C}(Fb = f, gen = g)$. Trivially, $\mathrm{g}(S) \leq \mathcal{N}(f)$. To the other inequality, consider $X = \mathcal{B}(f) \cap S$ and $Y = \mathcal{B}(f)\backslash S$. Define the injective map $\varphi : X \longrightarrow Y$ via $\varphi(x) = f - x$. Note that φ is well-defined, since $f - x \notin S$, otherwise $f \in S$, which it is not possible. Therefore, $\sharp X \leq \sharp Y$, and since $\mathcal{B}(f)$ equals the disjoint union of X and Y, then $\mathrm{g}(S) \geq \sharp Y \geq \left\lceil \frac{\sharp\mathcal{B}(f)}{2} \right\rceil$.

Conversely, by combining Propositions 1 and 2, we obtain that $\Delta(\mathcal{C}, f) = (\mathcal{C}\backslash\{f\})\backslash\{x \in \mathcal{C}\backslash\{0\} \mid x \leq_{\mathcal{C}} f, \text{ and } x \preceq \frac{f}{2}\}$ belongs to $\mathfrak{C}\Big(Fb = f, gen = \left\lceil \frac{\sharp\mathcal{B}(f)}{2} \right\rceil\Big)$. Hence, we can define the following sequence: $T_0 = \Delta(\mathcal{C}, f)$, and $T_{i+1} = T_i\backslash\{\mathrm{m}(T_i)\}$ if $\mathrm{m}(T_i) \prec f$, otherwise $T_{i+1} = T_i$, for every positive integer i. Note that there exists some natural number i_0 such that $T_{i_0} = T_{i_0+1}$, and $\cup_{i=0}^{i_0} \mathrm{g}(T_i) = [\![\left\lceil \frac{\sharp\mathcal{B}(f)}{2} \right\rceil, \mathcal{N}(f)]\!]$. □

Since $\sharp\mathcal{B}(f)$ equals $\prod_{i\in[\![p]\!]}(f_i + 1)$ when $\mathcal{C} = \mathbb{N}^p$, the previous result can be specialized to $\mathbb{N}^p$-semigroups.

Corollary 5. *For any $f \in \mathbb{N}^p\backslash\{0\}$ and any positive integer g, the set $\mathfrak{C}(Fb = f, gen = g)$ is non-empty if and only if*

$$\left\lceil \frac{\prod_{i\in[\![p]\!]}(f_i + 1)}{2} \right\rceil \leq g \leq \mathcal{N}(f).$$

The last theorem provides an algorithm (Algorithm 1) to compute the set of all $\mathcal{C}$-semigroups fixed the Frobenius element and the genus. Before presenting the algorithm, we introduce two definitions. We denote by $\Delta(f)$ the set $\{x \in \mathcal{C} \mid x \succ f\} \cup \{0\}$. Note that $\Delta(f)$ is an ordinary $\mathcal{C}$-semigroup with Frobenius element f. For any $\mathcal{C}$-semigroup S, we say that $x \in \mathcal{H}(S)$ is a special gap of S if $x + S\backslash\{0\} \subset S$, and $2x \in S$. The set of all special gaps of S is denoted by $SG(S)$.

From the argument given in the proof of Theorem 4, it follows that for any positive integer i, if $\mathrm{m}(T_i) \prec f$ then $T_{i+1} \cup \{\mathrm{m}(T_i)\} = T_i$. Since T_i is a $\mathcal{C}$-semigroup, it follows that $\mathrm{m}(T_i) \in SG(T_{i+1})$. This fact, combined with the definition of $\Delta(f)$, ensures that the set B in the following algorithm is non-empty.

Algorithm 1: Computing the set $\mathfrak{C}(Fb = f, gen = g)$.

Input: Let $f \in \mathcal{C}\setminus\{0\}$ and a positive integer g.
Output: The set $\mathfrak{C}(Fb = f, gen = g)$.

if $g \notin [\![\lceil \frac{\sharp\mathcal{B}(f)}{2} \rceil, \mathcal{N}(f)]\!]$ **then**
    return $\emptyset$
$A \leftarrow \{\Delta(f)\}$;
for $i \in [0, \mathcal{N}(f) - g)$ **do**
    $Y \leftarrow \emptyset$;
    while $A \neq \emptyset$ **do**
        $T \leftarrow \mathrm{First}(A)$;
        $B \leftarrow \{x \in SG(T)\setminus\{f\} \mid x \prec \mathrm{m}(T)\}$;
        $Y \leftarrow Y \cup \{T \cup \{x\} \mid x \in B\}$;
        $A \leftarrow A\setminus\{T\}$;
    $A \leftarrow Y$;
return A

Our work aims not to perform a computational comparison between existing algorithms and the alternatives we propose. We focus on providing alternative algorithms that offer distinct approaches to the problem at hand. We illustrate Algorithm 1 with the following example.

Example 6. Consider $\mathcal{C} = \mathbb{N}^2$, the degree lexicographic order, and let $f = (2, 2)$ and $g = 5$. The $\mathcal{C}$-semigroup $\Delta(f)$ is shown in Fig. 1, where the empty circles are the gaps of $\Delta(f)$, the blue squares are the minimal generators of $\Delta(f)$, and the red circles are elements of $\Delta(f)$.

By applying Algorithm 1, we obtain that the set $\mathfrak{C}(Fb = (2, 2), gen = 5)$ is

$$\begin{aligned}\{S_1 &= \{(0,1),(1,2),(2,3),(3,0),(4,0),(5,0)\},\\ S_2 &= \{(0,3),(0,4),(0,5),(1,0),(2,1),(3,2)\},\\ S_3 &= \{(0,2),(0,3),(1,2),(1,3),(2,1),(3,0),(3,1),(4,0),(4,1),(5,0)\},\\ S_4 &= \{(0,3),(0,4),(0,5),(1,2),(1,3),(1,4),(2,0),(2,1),(3,0),(3,1)\}\},\end{aligned}$$

where S_i is the minimal generating set of each $\mathbb{N}^2$-semigroup.

From Theorem 4 (or Corollary 5), we know that fixed the Frobenius element $(2, 2)$ there exist at least one $\mathbb{N}^2$-semigroup with genus belonging to $[\![\lceil \frac{\sharp\mathcal{B}(f)}{2} \rceil, \mathcal{N}(f)]\!] = [\![5, 12]\!]$. Table 1 shows the cardinality of $\mathfrak{C}(Fb = (2, 2), gen = g)$ for g in $[\![5, 12]\!]$.

Motivated by their relationship between the genus and the Frobenius number, precisely, $\mathrm{n}(S) + \mathrm{g}(S) = \mathcal{N}(\mathrm{Fb}(S)) + 1$, we turn our attention to

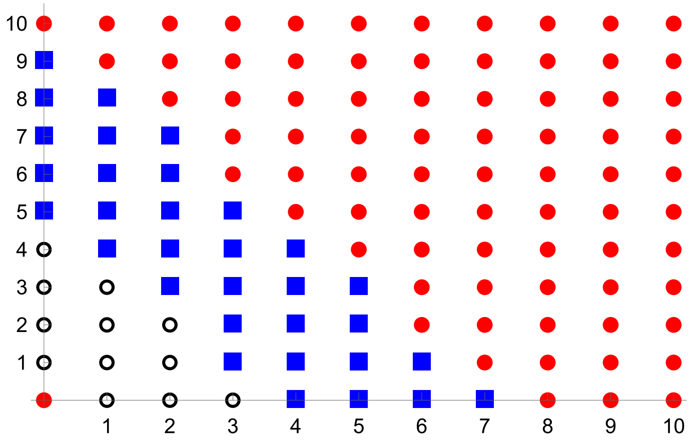

FIGURE 1. $\mathcal{C}$-semigroup $\Delta(f)$

TABLE 1. For $\mathcal{C} = \mathbb{N}^2$, $\sharp\mathfrak{C}(Fb = (2,2), gen = g)$ for all $g \in [\![5, 12]\!]$

Genus	5	6	7	8	9	10	11	12
Cardinality	4	17	37	49	41	22	7	1

provide a method for computing the set of $\mathcal{C}$-semigroups with a fixed genus and a fixed number of small elements.

Proposition 7. *Let S be a $\mathcal{C}$-semigroup, $f \in \mathcal{C}\backslash\{0\}$, and let g be a positive integer such that $\lceil \frac{\sharp\mathcal{B}(f)}{2} \rceil \leq g \leq \mathcal{N}(f)$. Then, $S \in \mathfrak{C}(Fb = f, gen = g)$ if and only if $n(S) = \mathcal{N}(f) + 1 - g$ and $g(S) = g$.*

If S is a numerical semigroup, then $\mathrm{n}(S) \leq \mathrm{g}(S)$. This inequality holds because for each small element s, $\mathrm{Fb}(S) - s$ is a gap of S; otherwise, $\mathrm{Fb}(S) = x+s$ for some $x \in S\backslash\{0\}$, which contradicts the definition of $\mathrm{Fb}(S)$. In contrast, for $\mathcal{C}$-semigroups, this inequality does not necessarily hold, as illustrated by the following counterexample.

Example 8. Let S be the $\mathbb{N}^2$-semigroup graphically represented in Fig. 2, where the empty circles are the gaps of S, the blue squares are the minimal generators of S, and the red circles are elements of S. In this example, we fix the degree lexicographic order. Note that $\mathrm{g}(S) = 9$, $\mathrm{Fb}(S) = (5, 1)$, and $\mathrm{n}(S) = 18$. Hence, $\mathrm{n}(S) > \mathrm{g}(S)$.

To achieve the announced method, for any two natural numbers n and g, recall that $\mathfrak{C}(\mathrm{gen} = g, \mathrm{se} = n)$ denotes the set of the $\mathcal{C}$-semigroups with genus g and n small elements. Since the genus and the number of small elements of any

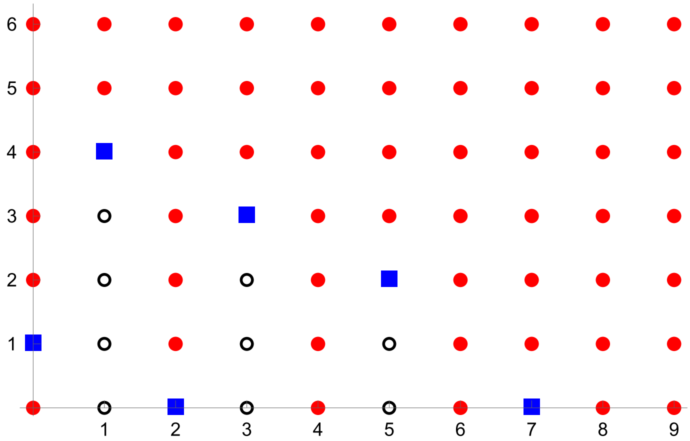

FIGURE 2. A $\mathbb{N}^2$-semigroup S such that $\mathrm{n}(S) > \mathrm{g}(S)$

$\mathcal{C}$-semigroups depend on the choice of total order, the set $\mathfrak{C}(\mathrm{gen} = g, \mathrm{se} = n)$ does as well. We deduce the following result as a consequence of Proposition 7.

Corollary 9. *If g and n are two positive integers, then*

$$\mathfrak{C}(gen = g, se = n) = \mathfrak{C}(Fb = f, gen = g),$$

where $f \in \mathcal{C}$ satisfies $\mathcal{N}(f) = g + n - 1$.

Based on Corollary 9, a method to explicitly determine $\mathfrak{C}(\mathrm{gen} = g, \mathrm{se} = n)$ is obtained. The first step of this algorithm is to look for the element f such that $\mathcal{N}(f) = g+n-1$, and the second one is to apply Algorithm 1 to get $\mathfrak{C}(Fb = f, \mathrm{gen} = g)$. For example, taking $\mathcal{C} = \mathbb{N}^2$, consider the set $\mathfrak{C}(\mathrm{gen} = 5, \mathrm{se} = 8)$. For $g = 5$, the element $f = (2, 2)$ satisfies that $\mathcal{N}(f) = 12 = g+n-1 = 5+8-1$ (Fig. 1). Thus, $\mathfrak{C}(\mathrm{gen} = 5, \mathrm{se} = 8)$ corresponds with $\mathfrak{C}(Fb = (2, 2), \mathrm{gen} = 5)$, which has already been computed (see Example 6).

Recall that $\mathfrak{C}(gen = g)$ is the set formed by all the $\mathcal{C}$-semigroups with genus equals g. We focus on introducing an algorithm to compute $\mathfrak{C}(gen = g)$. Note that this set is non-empty since the ordinary $\mathcal{C}$-semigroup $\Delta(f) \in \mathfrak{C}(gen = g)$, where $f \in \mathcal{C}\setminus\{0\}$ satisfies $\mathcal{N}(f) = g$.

Given g a positive integer, we define $\mathcal{F}(g) = \{\mathrm{Fb}(S) \mid S \in \mathfrak{C}(gen = g)\}$. Clearly, $\mathfrak{C}(gen = g) = \cup_{f\in\mathcal{F}(g)}\mathfrak{C}(Fb = f, gen = g)$. Consequently, to compute $\mathfrak{C}(gen = g)$, we proceed as follows. First, we compute the set $\mathcal{F}(g)$. Second, for each $f \in \mathcal{F}(g)$, we compute $\mathfrak{C}(Fb = f, gen = g)$. Since Algorithm 1 addresses the second step, we develop a method to compute $\mathcal{F}(g)$ without computing the set $\mathfrak{C}(gen = g)$.

Proposition 10. *If g is a positive integer, then*

$$\mathcal{F}(g) = \left\{ f \in \mathcal{C} \mid \left\lceil \frac{\sharp\mathcal{B}(f)}{2} \right\rceil \leq g \leq \mathcal{N}(f) \right\}.$$

Proof. Given $f \in \mathcal{F}(g)$, by applying Theorem 4 there exists at least a $\mathcal{C}$-semigroup with genus g and Frobenius element f. $\square$

Example 11. Consider $\mathcal{C} = \mathbb{N}^2$. Hence, the set $\mathcal{F}(5)$ is

$$\{(0,3),(0,4),(0,5),(0,6),(0,7),(0,8),(0,9),(1,2),(1,3),$$
$$(1,4),(2,0),(2,1),(2,2),(3,0),(3,1),(4,0),(4,1),(5,0),(6,0),(7,0),(8,0),(9,0)\}.$$

It is well known that $\sharp\mathfrak{C}(gen = 5) = 210$ (see [17, Table 3]).

2. $\mathcal{B}$-Semigroups

Fixed $f \in \mathcal{C}\backslash\{0\}$, and a total order $\preceq$ on $\mathbb{N}^p$, for any $\mathcal{C}$-semigroup S, we consider $\mathcal{O}(S) = S \cup (\mathcal{C}\backslash\mathcal{B}(f))$, recall that $\mathcal{B}(f) = \{x \in \mathcal{C} \mid x \leq_{\mathcal{C}} f\}$ with $f \in \mathcal{C}\backslash\{0\}$. It is straightforward from the definition that $\mathcal{O}(S)$ is a $\mathcal{C}$-semigroup. In particular, if $\mathcal{O}(S) = S$ we say that S is a $\mathcal{B}$-semigroup. The set of all $\mathcal{B}$-semigroups with Frobenius element f is denoted by $\mathfrak{B}(Fb = f)$.

This section is devoted to discuss $\mathcal{B}$-semigroups. This study is mainly structured as in Sect. 1. Firstly, we compute the set of $\mathfrak{B}(Fb = f)$. To achieve this, we provide an algorithm which additionally allows us to introduce its associative tree, whose vertex set is $\mathfrak{B}(Fb = f)$. Secondly, we compute the set $\mathfrak{B}(Fb = f, gen = g)$, that is, the set of all $\mathcal{B}$-semigroups with Frobenius element f and genus g. And finally, we compute the set of all $\mathcal{B}$-semigroups with genus g, denoted by $\mathfrak{B}(gen = g)$.

For any $\mathcal{C}$-semigroup S, we consider

$$\alpha(S) = \min_{\preceq}\{x \in S\backslash\{0\} \mid x \in \mathcal{B}(\mathrm{Fb}(S))\}.$$

This element can be interpreted as the multiplicity in $\mathcal{B}(\mathrm{Fb}(S))$ of S. If $S \cap \mathcal{B}(f) = \{0\}$, we consider that $\alpha(S) = f$.

In this context, for any $f \in \mathcal{C}\backslash\{0\}$, we define the graph $G(\mathfrak{B}(Fb = f))$, whose vertex set is $\mathfrak{B}(Fb = f)$ and the pair $(S, T) \in \mathfrak{B}^2(Fb = f)$ is an edge if and only if $T = S\backslash\{\alpha(S)\}$. If (S, T) is an edge, we say that S is a child of T. A path connecting the vertices S and T of any directed graph is a sequence of distinct edges of the form $(S_0, S_1), (S_1, S_2), \ldots, (S_{n-1}, S_n)$ where $S_0 = S$ and $S_n = T$.

Theorem 12. *If $f \in \mathcal{C}\backslash\{0\}$, then $G(\mathfrak{B}(Fb = f))$ is a tree with root $(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}$. Furthermore, the set of children of any $T \in \mathfrak{B}(Fb = f)$ is the set $\{T \cup \{x\} \mid x \in SG(T)$ and $x \prec \alpha(T)\}$.*

Proof. Let $S \in \mathfrak{B}(Fb = f)$. We define recursively the following sequence:

$$S_0 = S,$$
$$S_{i+1} = \begin{cases} S_i \backslash \{\alpha(S_i)\} & \text{if } S_i \neq (\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}, \\ S_i & \text{otherwise.} \end{cases}$$

Since the set $\mathcal{B}(f)$ is finite, the above sequence becomes stationary. Thus, any $S \in \mathfrak{B}(Fb = f)$ is connected by a path to $(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}$, and the uniqueness of the path is deduced from the uniqueness of $\alpha(S)$. If S is a child of T, then $T = S\backslash\{\alpha(S)\}$, and therefore $S = T \cup \{\alpha(S)\}$ is a $\mathcal{C}$-semigroup, which implies that $\alpha(S) \in SG(T)$ and $\alpha(S) \prec \alpha(T)$. Conversely, if $x \in SG(T)$ and $x \prec \alpha(T)$, then $S = T \cup \{x\}$ which ensures S is a $\mathcal{C}$-semigroup with $\alpha(S) = x$. Thus, S is indeed a child of T. □

With the theoretical foundation established, we introduce Algorithm 2 for computing $\mathfrak{B}(Fb = f)$. Example 13 illustrates it and shows the graph $G(\mathfrak{B}(Fb = f))$ obtained.

Algorithm 2: Computing the set $\mathfrak{B}(Fb = f)$.

Input: Let $f \in \mathcal{C}\backslash\{0\}$.
Output: The set $\mathfrak{B}(Fb = f)$.

$A \leftarrow \{(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}\}$;
$X \leftarrow A$;
while $A \neq \emptyset$ **do**
    $Y \leftarrow \emptyset$;
    $B \leftarrow A$;
    while $B \neq \emptyset$ **do**
        $T \leftarrow \mathrm{First}(B)$;
        $C \leftarrow \{x \in SG(T) \mid x \prec \alpha(T)\}$;
        if $C \neq \emptyset$ **then**
            $Y \leftarrow Y \cup \{T \cup \{x\} \mid x \in C\}$;
        $B \leftarrow B\backslash\{T\}$;
    $X \leftarrow X \cup Y$;
    $A \leftarrow Y$;
return X

Example 13. Consider $\mathcal{C} = \mathbb{N}^2$. Fixed $f = (2, 2) \in \mathcal{C}$ and the degree lexicographic order, the $\mathcal{C}$-semigroup $(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}$ is shown in Fig. 3. The blue squares are the minimal generators, specifically,

$$\begin{aligned}\mathrm{msg}\,((\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}) = \{&(0,3),(0,4),(0,5),(1,3),(1,4),(1,5),(2,3),\\ &(2,4),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(5,0),(5,1),(2,5),(5,2)\}.\end{aligned}$$

The empty circles represent the set $\mathcal{B}(f)$ and the red circles are elements of $\mathcal{C}\backslash\mathcal{B}(f)$.

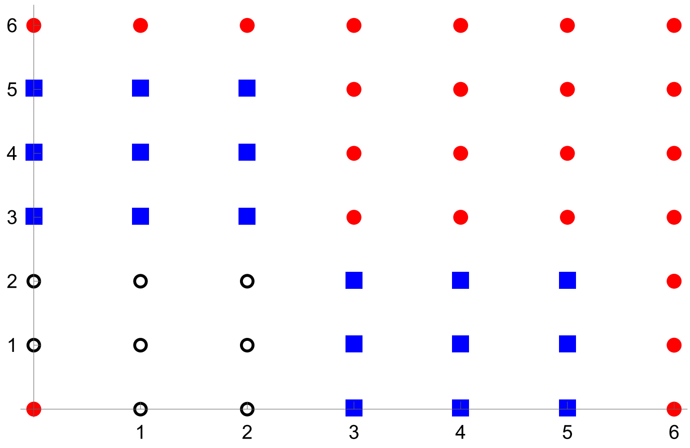

FIGURE 3. Example of $(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}$

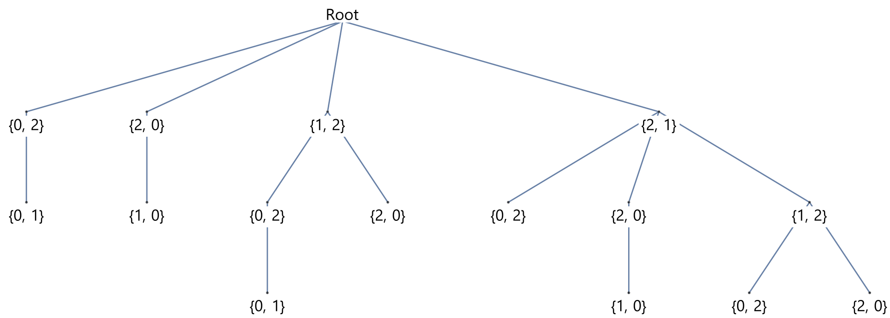

FIGURE 4. $G(\mathfrak{B}(Fb = (2, 2)))$ with the degree lexicographic order

For this example, Fig. 4 illustrates the 4-level tree $G(\mathfrak{B}(Fb = f))$ defined in Theorem 12. Its root is the $\mathcal{C}$-semigroup $(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}$, and each node represents a $\mathcal{C}$-semigroup in $\mathfrak{B}(Fb = f)$. For example, the rightmost node $\{2, 0\}$ in the last level is the $\mathcal{C}$-semigroup $\big(\mathcal{C}\backslash\mathcal{B}(f)\big) \cup \{(2, 1), (1, 2), (2, 0), (0, 0)\}$. In each level, some special gaps of each node are joined to obtain its children (step 10 in Algorithm 2).

Recall that the tree $G(\mathfrak{B}(Fb = f))$ depends on the fixed total order. For example, when the chosen total order is defined as $(a, b) \prec (c, d)$ if $2a + b < 2c + d$, or, in case $2a + b = 2c + d$, $a < c$, the obtained tree is shown in Fig. 5.

We can provide a result equivalent to Theorem 4 for the set $\mathfrak{B}(Fb = f, gen = g)$.

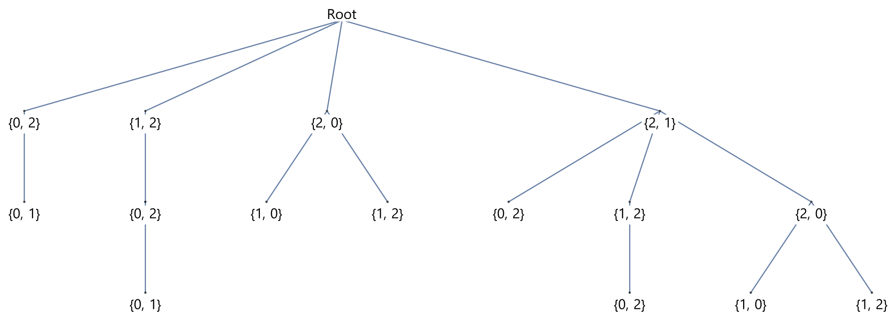

FIGURE 5. $G(\mathfrak{B}(Fb = (2,2)))$ with the total order $(a,b) \prec (c,d)$ iff $2a+b < 2c+d$, or, in case $2a+b = 2c+d$, $a < c$

Proposition 14. *For any $f \in \mathcal{C}\backslash\{0\}$ and any positive integer g, the set $\mathfrak{B}(Fb = f, gen = g)$ is non-empty if and only if*

$$\left\lceil \frac{\sharp\mathcal{B}(f)}{2} \right\rceil \leq g \leq \sharp\mathcal{B}(f) - 1.$$

Proof. Consider that $\mathfrak{B}(Fb = f, gen = g)$ is non-empty, and let $S \in \mathfrak{B}(Fb = f, gen = g)$, then $\mathrm{g}(S) \leq \sharp\mathcal{B}(f) - 1$. Analogously to the proof of Theorem 4, $\mathrm{g}(S) \geq \lceil \frac{\sharp\mathcal{B}(f)}{2} \rceil$.

Conversely, since $\Delta(\mathcal{C}, f) \in \mathfrak{B}\left(Fb = f, gen = \lceil \frac{\sharp\mathcal{B}(f)}{2} \rceil\right)$, this set is connected with the root $(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\}$ in the tree $G(\mathfrak{B}(Fb = f))$ by removing an element in each level, and $(\mathcal{C}\backslash\mathcal{B}(f)) \cup \{0\} \in \mathfrak{B}(Fb = f, gen = \sharp\mathcal{B}(f) - 1)$, we can conclude that $\mathfrak{B}(Fb = f, gen = g)$ is non-empty for any $g \in [\![\lceil \frac{\sharp\mathcal{B}(f)}{2} \rceil, \sharp\mathcal{B}(f) - 1]\!]$. □

Assuming $\mathcal{C} = \mathbb{N}^p$, we reformulated Corollary 5 as follows.

Corollary 15. *For any $f \in \mathbb{N}^p\backslash\{0\}$ and any positive integer g, the set $\mathfrak{B}(Fb = f, gen = g)$ is non-empty if and only if*

$$\left\lceil \frac{\prod_{i \in [\![p]\!]}(f_i + 1)}{2} \right\rceil \leq g < \prod_{i \in [\![p]\!]} (f_i + 1).$$

From the previous results, it is established that, for a fixed Frobenius element, the genus belongs to a bounded interval. Using Algorithm 2 as a basis, we determine the set $\mathfrak{B}(Fb = f, \mathrm{gen} = g)$ by focusing specifically on the first $\sharp\mathcal{B}(f) - g$ steps. Continuing with the framework established in Sect. 1, we now address the computation of $\mathfrak{B}(\mathrm{gen} = g)$. Rather than calculating $\mathfrak{B}(\mathrm{gen} = g)$ directly, according to Proposition 14, we compute the set

$$\mathcal{F}_{\mathfrak{B}}(g) = \left\{ f \in \mathcal{C} \mid \left\lceil \frac{\sharp\mathcal{B}(f)}{2} \right\rceil \leq g \leq \sharp\mathcal{B}(f) - 1 \right\}.$$

TABLE 2. For $\mathcal{C} = \mathbb{N}^2$, $\sharp\mathfrak{B}(\text{gen} = g)$ for some $g \in \mathbb{N}$

g	1	2	3	4	5	6	7	8	9	10
$\sharp\mathfrak{B}(\text{gen} = g)$	2	6	15	30	58	137	240	457	900	1784

For each $f \in \mathcal{F}_{\mathfrak{B}}(g)$, we determine $\mathfrak{B}(Fb = f, \text{gen} = g)$ as mentioned before. After repeating this procedure for each $f \in \mathcal{F}_{\mathfrak{B}}(g)$, we get the set $\mathfrak{B}(\text{gen} = g)$. The following example provides some computational results of this procedure.

Example 16. Let $\mathcal{C} = \mathbb{N}^2$, the set $\mathcal{F}_{\mathfrak{B}}(2)$ is $\{(0,2),(0,3),(1,0),(1,1),(2,0),(3,0)\}$, and $\mathfrak{B}(\text{gen} = 2)$ has also six elements, the $\mathcal{B}$-semigroups with gaps sets $\{(0,1),(0,2)\}$, $\{(0,1),(0,3)\}$, $\{(1,0),(1,1)\}$, $\{(0,1),(1,1)\}$, $\{(1,0),(2,0)\}$, and $\{(1,0),(3,0)\}$. For genus five,

$$\begin{aligned}\mathcal{F}_{\mathfrak{B}}(5) = \{&(0,5),(0,6),(0,7),(0,8),(0,9),(1,2),(1,3),(1,4),(2,1),\\ &(2,2),(3,1),(4,1),(5,0),(6,0),(7,0),(8,0),(9,0)\},\end{aligned}$$

but only 58 out of 210 elements in $\mathfrak{C}(\text{gen} = 5)$ are also $\mathcal{B}$-semigroups.

In general, the computation on $\mathcal{C}$-semigroups is very hard, and few examples can be constructed. For $\mathcal{C} = \mathbb{N}^2$, Table 2 collects the number of $\mathcal{B}$-semigroups for some genus g.

Computational results seem to suggest that the following conjecture could be true.

Conjecture 17. *For any integer cone $\mathcal{C} \subseteq \mathbb{N}^p$, and any non-zero $g \in \mathbb{N}$,*

$$\sharp\mathfrak{B}(gen = g) < \sharp\mathfrak{B}(gen = g + 1).$$

3. A Partition of $\mathfrak{C}(Fb = f)$

Let us start by introducing some notations. Again, an integer cone $\mathcal{C} \subseteq \mathbb{N}^p$ and a total order $\preceq$ on $\mathbb{N}^p$ are fixed. Given $f \in \mathcal{C}\backslash\{0\}$ and $S, T \in \mathfrak{C}(Fb = f)$, we define an equivalence relation $\sim$ such that $S \sim T$ if and only if $\mathcal{O}(S) = \mathcal{O}(T)$, recall that $\mathcal{O}(S) = S \cup (\mathcal{C}\backslash\mathcal{B}(f))$. Equivalently, $S \sim T$ if and only if $S \cap \mathcal{B}(f) = T \cap \mathcal{B}(f)$. For $S \in \mathfrak{C}(Fb = f)$, the equivalence class of S modulo $\sim$, called its $\sim$-class, is defined as $[S] = \{T \in \mathfrak{C}(Fb = f) \mid S \sim T\}$. The collection of all $\sim$-classes forms a partition of $\mathfrak{C}(Fb = f)$, denoted by $\mathfrak{C}(Fb = f)/_\sim$.

This section aims to compute $\mathfrak{C}(Fb = f)$. To this end, we study the partition $\mathfrak{C}(Fb = f)/_\sim$ as a tool for the desired computation. We show how the set $[S]$ can be arranged in a tree for any $S \in \mathfrak{C}(Fb = f)$.

The following proposition establishes the relationship between the partition $\mathfrak{C}(Fb = f)/_\sim$ and the set $\mathfrak{B}(Fb = f)$, proving that the set $\mathfrak{C}(Fb = f)/_\sim$ can be determined from the $\mathcal{B}$-semigroups with Frobenius element f.

Proposition 18. *If $f \in \mathcal{C}\backslash\{0\}$, then $\mathfrak{C}(Fb = f)/_\sim = \{[R] \mid R \in \mathfrak{B}(Fb = f)\}$. Moreover, if $R_1, R_2 \in \mathfrak{B}(Fb = f)$ such that $R_1 \neq R_2$, then $[R_1] \cap [R_2] = \emptyset$.*

Proof. To prove the first statement, consider $S \in \mathfrak{C}(Fb = f)$, by definition, $[S] = [R]$, where $R = \mathcal{O}(S) \in \mathfrak{B}(Fb = f)$. For the second statement, assume $R_1, R_2 \in \mathfrak{B}(Fb = f)$ such that $R_1 \neq R_2$, this implies that there is at least one gap x of R_i such that $x \in R_j$ with $i, j \in \{1, 2\}$ and $i \neq j$. Thus $R_1 \not\sim R_2$, hence $[R_1] \cap [R_2] = \emptyset$. □

Let S be a $\mathcal{C}$-semigroup with Frobenius element $\mathrm{Fb}(S) = f$, we define $\lambda(S)$ as the maximum element in $(\mathcal{C}\backslash\mathcal{B}(f)) \cap \mathcal{H}(S)$ with respect to the fixed total order on $\mathbb{N}^p$. If $(\mathcal{C}\backslash\mathcal{B}(f)) \cap \mathcal{H}(S) = \emptyset$, by convention $\lambda(S) = 0$. We deduce the following result from the maximality of $\lambda(S)$. Let $R \in \mathfrak{B}(Fb = f)$.

Lemma 19. *If $S \in [R]$, then $S \cup \{\lambda(S)\} \in [R]$.*

We define the directed graph $G([R])$ whose vertex set is the $\sim$-class $[R]$, and $(S, T) \in [R]^2$ is directed edge if and only if $T = S \cup \{\lambda(S)\}$. In particular, as mentioned before, if (S, T) is a directed edge, S is usually known as a child of T.

Theorem 20. *The graph $G([R])$ is a tree with root $\mathcal{O}(R)$. Furthermore, the set of children of any $T \in [R]$ is the set*

$$\{T\backslash\{x\} \mid x \in msg(T),\ \lambda(T) \prec x \prec f \text{ and } x \notin \mathcal{B}(f)\}\}.$$

Proof. Let $S \in [R]$ such that $S \neq \mathcal{O}(R)$. We construct the sequence $\{S_i\}_{i \in \mathbb{N}} \subseteq [R]$ defined by

$$S_0 = S$$
$$S_{i+1} = \begin{cases} S_i \cup \{\lambda(S_i)\} & \text{if } S_i \neq \mathcal{O}(R), \\ S_i & \text{otherwise.} \end{cases}$$

Since $\lambda(S_i) \in \mathcal{H}(S_i)$, and each time we add $\lambda(S_i)$ to S_i, the set of remaining possible λ-values decreases, the above sequence becomes stationary, and thus the sequence $\{S_i\}_{i \in \mathbb{N}}$ defines a path from S to $\mathcal{O}(R)$. By the maximality property of $\lambda(S)$, we deduce the uniqueness of the path from S to $\mathcal{O}(R)$. Consider $S = T\backslash\{x\}$ for some $x \in msg(T)\backslash\mathcal{B}(f)$ such that $\lambda(T) \prec x \prec f$. Therefore, $T = S \cup \{x\}$, and by the properties of x, it follows that $\lambda(S) = \max_{\preceq}\big((\mathcal{C}\backslash\mathcal{B}(f)) \cap (\mathcal{H}(T) \cup \{x\})\big) = x$, which proves that S is a child of T. Now, let S be a child of T, then $T = S \cup \{\lambda(S)\}$, which implies $\lambda(S)$ is a minimal generator of T. Finally, note that $\lambda\left(T\backslash\{\lambda(S)\}\right) = \lambda(S)$, which completes the proof. □

The above results allow us to present Algorithm 3 for computing the $\sim$-class $[S]$ for any $S \in \mathfrak{C}(Fb = f)$.

Example 21. As in Example 13, consider $\mathcal{C} = \mathbb{N}^2$, $f = (2, 2) \in \mathcal{C}$ and the degree lexicographic order. Let $S \in \mathfrak{C}(Fb = f)$ be the semigroup minimally generated by

Algorithm 3: Computing the $\sim$-class [S].

Input: A $\mathcal{C}$-semigroup S with Frobenius element $f \in \mathcal{C}\setminus\{0\}$.
Output: The $\sim$-class $[S]$.

$A \leftarrow \{\mathcal{O}(S)\}$;
$B \leftarrow A$;
while $A \neq \emptyset$ **do**
  $T \leftarrow \mathrm{First}(A)$;
  $C \leftarrow \{x \in msg(T) \mid \lambda(T) \prec x \prec f \text{ and } x \notin \mathcal{B}(f)\}$;
  $B \leftarrow B \cup \{T\setminus\{x\} \mid x \in C\}$;
  $A \leftarrow (A\setminus\{T\}) \cup \{T\setminus\{x\} \mid x \in C\}$;
return B

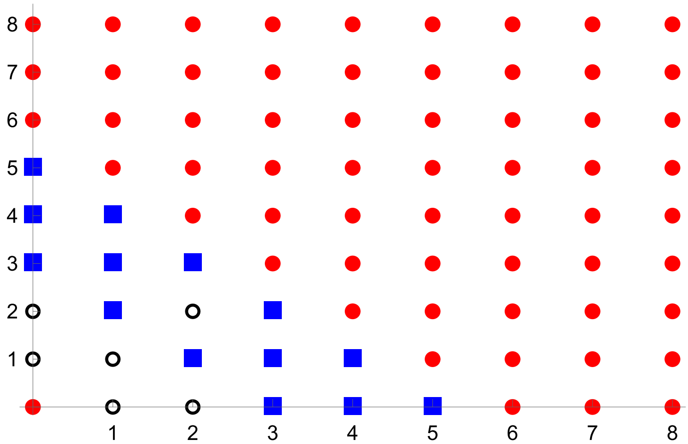

FIGURE 6. Example of $\mathcal{O}(S)$

$$\{(0,4),(0,5),(1,2),(1,4),(2,1),(2,3),(3,0),(3,1),(3,2),(4,0),\\ (4,1),(5,0),(0,6),(0,7),(1,5)\},$$

with $\mathcal{H}(S) = \{(0,1),(0,2),(0,3),(1,0),(1,1),(1,3),(2,0),(2,2)\}$. The $\mathcal{B}$-semigroup $\mathcal{O}(S)$ is represented in Fig. 6, where the empty circles are the gaps of $\mathcal{O}(S)$, the blue squares are the minimal generators of $\mathcal{O}(S)$, and the red circles are elements of $\mathcal{O}(S)$.

From $\mathcal{O}(S)$, Fig. 7 shows the tree $G([S])$ containing all the elements in the $\sim$-class of S. Its root is $\mathcal{O}(S)$, and each node represents a $\mathcal{C}$-semigroup in $\mathfrak{C}(Fb = f)$. To ensure more clarity in the figure, each tree vertex is labelled with the element removed to reach its parent node. For example, the leftmost node labelled $\{1,3\}$ in the last level is the $\mathcal{C}$-semigroup $\mathcal{O}(S)\setminus\{(0,3),(3,0),(0,4),$

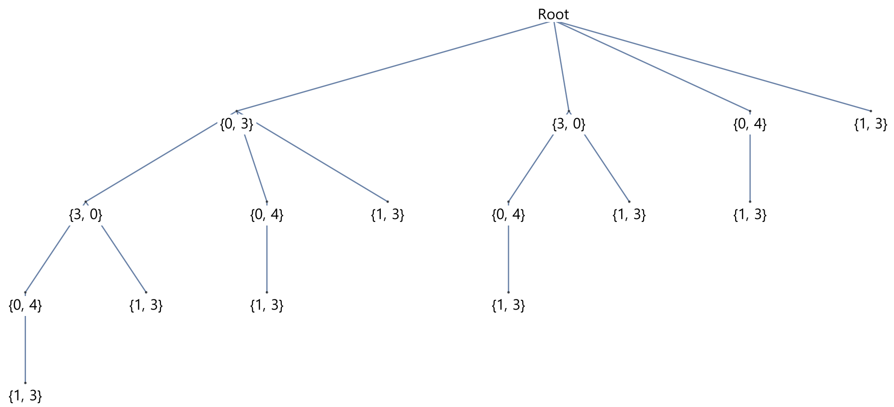

Figure 7. $G([S])$ with the degree lexicographic order

$(1,3)\}$. In each level, some minimal generators of each node are removed to obtain its children (step 5 in Algorithm 3).

We have already developed all the necessary background for computing $\mathfrak{C}(Fb=f)$: Proposition 18 states that $\mathfrak{C}(Fb=f)$ can be expressed as the union $\bigcup_{S\in\mathfrak{B}(Fb=f)}[S]$, and Algorithm 3 computes $[S]$. Thus, by combining these two ideas, we achieve a procedure for computing all $\mathfrak{C}(Fb=f)$.

Example 22. Fix the degree lexicographic order and let $\mathcal{C}=\mathbb{N}^2$. Figure 4 in Example 13 contains all the elements in $\mathfrak{B}(Fb=(2,2))$, that is, all the elements needed to construct the union $\bigcup_{S\in\mathfrak{B}(Fb=(2,2))}[S]$. After to compute $[S]$ (Algorithm 3) for every $S\in\mathfrak{B}(Fb=(2,2))$, we concluded that there exists 202 $\mathbb{N}^2$-semigroups with Frobenius element $(2,2)$ respect to the degree lexicographic order. However, if we consider the total order used to construct Fig. 5, then there are 500 elements in $\mathfrak{C}(Fb=f)$. Recall that all these sets are highly dependent on the total order considered.

4. Ordinary $\mathbb{N}^2$-Semigroups

Fixed a total order $\preceq$, we say that a $\mathcal{C}$-semigroup S is ordinary if $S=\{0\}\cup\{x\in\mathcal{C}\mid x\succeq c\}$ for some $c\in\mathcal{C}$, and it is denoted by S_c which depends on the choice of the order. Equivalently, S is ordinary if the conductor of S equals the multiplicity of S. Note that an ordinary $\mathcal{C}$-semigroup is a $\mathcal{C}$-semigroup containing all the $\mathcal{C}$-semigroups determined by the given total order and the multiplicity.

Our goal in this section is to prove that any ordinary $\mathbb{N}^2$-semigroup ordered by the graded lexicographic order satisfies the Generalized Wilf Conjecture and the Extended Wilf Conjecture. We study the minimal generating set of any ordinary $\mathcal{C}$-semigroup to achieve this. First, we provide a lower bound for the embedding dimension of any ordinary $\mathcal{C}$-semigroup.

Proposition 23. *Let S be an ordinary $\mathcal{C}$-semigroup. Then $g(S) < e(S)$.*

Proof. We fix a total order $\preceq$ and assume that S is an ordinary $\mathcal{C}$-semigroup. Trivially, if $S = \mathcal{C}$, then $\mathrm{e}(S) > 0$. We proceed by induction on $\mathrm{g}(S)$. Suppose that $\mathrm{g}(S) < \mathrm{e}(S)$, and we show that $\mathrm{g}(S\backslash\{m\}) < \mathrm{e}(S\backslash\{m\})$ where $m = \mathrm{m}(S)$. Note that $\mathrm{g}(S\backslash\{m\}) = \mathrm{g}(S) + 1$. Consider $r = \min_{\preceq}(S\backslash\{0, m\})$, and $t = \min_{\preceq}(S\backslash\{0, m, r\})$ the second and third minimum elements of $S\backslash\{0\}$, respectively. Let $z \in \{2\,m, m + r, m + t\}$, if z is not a minimal generator of $S\backslash\{m\}$, then $z = a + b$, where a and b are two non-zero elements of $S\backslash\{m\}$. Without loss of generality, we deduce $a \prec m$, which leads to a contradiction, since m is the multiplicity of S. So, $2m, m + r, m + t$ are minimal generators of $S\backslash\{m\}$. From [17, Lemma 3] follows $\mathrm{msg}(S)\backslash\{m\} \subset \mathrm{msg}(S\backslash\{m\})$, hence $\mathrm{e}(S\backslash\{m\}) \geq \mathrm{e}(S) - 1 + 3$. By applying the induction hypothesis, $\mathrm{g}(S\backslash\{m\}) = \mathrm{g}(S) + 1 < \mathrm{e}(S) + 1 < \mathrm{e}(S\backslash\{m\})$, which completes the proof. □

The following definitions will be needed throughout the remainder of the work. Let $w = (w_1, \ldots, w_p) \in \mathbb{R}^p_{\geq}$ be a vector, and consider the map $\pi_w : \mathbb{N}^p \to \mathbb{N}$ defined via $\pi_w(x) = w \cdot x$, where $\cdot$ denotes the inner product. For any $x, y \in \mathbb{N}^p$, we define $x \preceq_w y$ if and only if $\pi_w(x) \leq \pi_w(y)$. We refer to $\preceq_w$ as the weight order determined by w.

To determine a weight order in general, we choose a primary weight vector $w \in \mathbb{R}^p_{\geq}$. A secondary weight vector $u \in \mathbb{R}^p_{\geq}$ is employed to break ties. If ties persist (i.e., when $\pi_w(x) = \pi_w(y)$ and $\pi_u(x) = \pi_u(y)$), a third weight vector is introduced, and so on. Thus, every monomial order $\preceq$ can be obtained through this finite process of applying weight vectors. From now on, we are interested in the first weight vector. Hence, we use $\pi_{\preceq}$ instead of π_w. Graphically, it can be interpreted as the existence of a hyperplane that separates the space into two regions, one containing x and the other containing y. For a detailed treatment of monomial orders and their relation to weight orders, consult [10] and [21]. We assume that the vector w defining the fixed order has non-zero entries.

Let S be a $\mathcal{C}$-semigroup with t extremal rays, and minimally generated by $\mathrm{msg}(S) = E \sqcup A$ with $E = \cup_{i\in[\![t]\!]\backslash\{0\}}\mathrm{mult}_i(S) = \{m_1, \ldots, m_t\}$, and $A = \{m_{t+1}, \ldots, m_r\}$. The next result states that the minimal system of generators of an ordinary $\mathcal{C}$-semigroup is bounded.

Lemma 24. *Let S_c be an ordinary $\mathcal{C}$-semigroup and $x \in msg(S_c)$. Then, $\pi_{\preceq}(x) \leq \pi_{\preceq}(m)$, where $m = \sum_{i\in[\![t]\!]\backslash\{0\}} mult_i(S_c)$.*

Proof. Let $x \in S_c$ such that $\pi_{\preceq}(x) > \pi_{\preceq}(m)$ we have to show that $x \notin \mathrm{msg}(S_c)$. We assume that $x - m_i \notin S_c$ for any $i \in [\![t]\!]\backslash\{0\}$, otherwise it is easily checked that x is not a minimal generator. We distinguish the following cases:

- If there exists $j \in [\![t]\!]\backslash\{0\}$ such that $x - m_j \in \mathcal{C}$, then $x = m_j + h$ for some $h \in \mathcal{H}(S_c)$. Given that $\pi_{\preceq}$ is a linear map, we have $\pi_{\preceq}(x) = \pi_{\preceq}(m_j) + \pi_{\preceq}(h) > \sum_{i\in[\![t]\!]\backslash\{0\}} \pi_{\preceq}(m_i)$. This implies that $\pi_{\preceq}(h) > \sum_{i\in[\![t]\!]\backslash\{0,j\}} \pi_{\preceq}(m_i) > \pi_{\preceq}(c)$. Since $\pi_{\preceq}$ is an increasing map, $h \succ c$, and thus, $h \in S_c$, which it is not possible.
- If $x - m_i \notin \mathcal{C}$ for any $i \in [\![t]\!]\backslash\{0\}$, since x belongs to $S_c \subseteq \mathcal{C}$, there exist some rational numbers $0 \le \lambda_1, \dots, \lambda_t < 1$ such that $x = \sum_{i\in[\![t]\!]\backslash\{0\}} \lambda_i m_i$. So, $\pi_{\preceq}(x) = \sum_{i\in[\![t]\!]\backslash\{0\}} \lambda_i \pi_{\preceq}(m_i) < \sum_{i\in[\![t]\!]} \pi_{\preceq}(m_i) = \pi_{\preceq}(m)$, in contradiction with the hypothesis.

□

Building on this, the following lemma proves a more robust result about the relationship between some elements in $\mathcal{C}$ and the minimal system of generators of S_c. From now on, we assume that the conductor c is non-zero.

Lemma 25. *Let S_c be an ordinary $\mathcal{C}$-semigroup and $x \in S_c$ such that $\pi_{\preceq}(x) < 2\pi_{\preceq}(c)$, then $x \in msg(S_c)$.*

Proof. Let $x \in S_c$ such that $\pi_{\preceq}(x) < 2\pi_{\preceq}(c)$, and suppose that $x = s_1 + s_2$ for some $s_1, s_2 \in S_c$. Since S_c is an ordinary $\mathcal{C}$-semigroup, $\pi_{\preceq}(s_1), \pi_{\preceq}(s_2) \ge \pi_{\preceq}(c)$. Consequently $\pi_{\preceq}(x) = \pi_{\preceq}(s_1) + \pi_{\preceq}(s_2) \ge 2\pi_{\preceq}(c)$, contradicting the initial assumption. □

In particular, if S_c is an ordinary $\mathbb{N}^2$-semigroup, and $\preceq_{glex}$ corresponds to the graded lexicographic order, the set $\mathrm{msg}(S_c)$ can be explicitly determined. Note that in this context, the map $\pi_{\preceq_{glex}} : \mathbb{N}^2 \to \mathbb{N}$ is defined via $\pi_{\preceq_{glex}}(x_1, x_2) = x_1 + x_2$. From now on, we use the symbol π instead of $\pi_{\preceq_{glex}}$ for short.

Proposition 26. *Let S_c be an ordinary $\mathbb{N}^2$-semigroup with conductor $c = (0, c_2)$, ordered by $\preceq_{glex}$. Then, S_c is minimally generated by*

$$\left\{x \in \mathbb{N}^2 \mid \pi(c) \le \pi(x) \le 2\pi(c) - 1\right\}.$$

Therefore, $e(S_c) = \dfrac{c_2(3c_2+1)}{2}$.

Proof. Since $\pi(\mathrm{mult}_i(S_c)) = \pi(c)$ for each $i = 1, 2$, and by applying Lemmas 24 and 25, it suffices to show that any element $x = (x_1, x_2) \in \mathbb{N}^2$ satisfying $\pi(x) \ge 2\pi(c)$ cannot be a minimal generator of S_c. If $x_1, x_2 < c_2$, then $\pi(x) < 2\pi(c)$ which it is impossible by hypothesis. Thus, without loss of generality, we assume $x_1 \ge c_2$. It follows that $x - (c_2, 0) \in \mathbb{N}^2$, and since $\pi(x - (c_2, 0)) \ge \pi(c)$, it can be deduced that $x - (c_2, 0) \in S_c$. Therefore, $x \notin \mathrm{msg}(S_c)$.

The embedding dimension of S_c is determined by the number of solutions $(x_1, x_2) \in \mathbb{N}^2$ of $c_2 \le x_1 + x_2 \le 2c_2 - 1$. Equivalently, it is the number of natural solutions of the equation of $x_1 + x_2 = k$, where $k \in \{c_2, c_2 + 1, \dots, 2c_2 - 1\}$. Note that the above expression involves $2c_2$ equations. Fixed k, it is straightforward to deduce that $x_1 \in \{0, 1, \ldots, k\}$ and consequently, $x_2 = k - x_1$ then, there exist $k+1$ solutions. Thus,

$$\mathrm{e}(S_c) = \sum_{k=c_2}^{2c_2-1} (k+1) = \sum_{i=1}^{c_2} (c_2 + i) = \frac{c_2(3c_2+1)}{2}.$$

□

The above result corresponds to a particular case discussed within the framework of T-stripe generalized numerical semigroups (see [9, Proposition 3.3]). In cases where $c_1 \neq 0$, the determination of $\mathrm{msg}(S_c)$ has not been addressed in the literature, as such semigroups do not fall under the classification of T-stripe $\mathbb{N}^2$-semigroups. Therefore, we propose the following proposition. For any total order $\preceq$, let $a \preceq b$ be two elements in $\mathcal{C}$. We denote the intervals of elements in $\mathcal{C}$ between a and b under the order $\preceq$ as follows: $[a, b]_\preceq, [a, b)_\preceq, (a, b]_\preceq$ and $(a, b)_\preceq$ for the closed interval, left-open interval, right-open interval and open interval, respectively.

Proposition 27. *Let S_c be an ordinary $\mathbb{N}^2$-semigroup with conductor $c = (c_1, c_2)$ such that $c_1 \neq 0$, ordered by $\preceq_{glex}$. Then, S_c is minimally generated by*

$$\Big[c,\ 2c\Big)_{\preceq_{glex}} \sqcup \Big[\big(0, 2\pi(c)+1\big),\ c + \big(0, \pi(c)+1\big)\Big)_{\preceq_{glex}} \tag{1}$$

and $e(S_c) = \dfrac{3\pi^2(c) + \pi(c) + 4c_1}{2}$.

Proof. From direct application of Lemma 24, the set (1) belongs to a system of generators of S_c, since $\mathrm{mult}_1(S_c) + \mathrm{mult}_2(S_c) = 2\pi(c) + 1$. Furthermore, by Lemma 25 it suffices to analyse those $x = (x_1, x_2)$ such that $2\pi(c) \leq \pi(x) \leq 2\pi(c) + 1$. We distinguish two cases: $x \not> c$ and $x > c$.

If $x \not> c$, where $>$ denotes the component-wise order, then $x_2 < c_2$ or $x_1 < c_1$. Suppose that $x_2 < c_2$, by hypothesis $2c_1 + 2c_2 \leq x_1 + x_2 < x_1 + c_2$, hence $\pi(c) + c_1 < x_1$. Consider $y = x - (\pi(c), 0) \in \mathbb{N}^2$. Since $\pi(y) \geq 2\pi(c) - \pi(c) = \pi(c)$, we conclude that $x \notin \mathrm{msg}(S_c)$. Now, assume that $x_1 < c_1$, and suppose that $x = s + t$ for some $s, t \in S_c$. Since $x_1 < c_1$ then $s_1, t_1 < c_1$. By the linearity of π, we obtain that $2\pi(c) \leq \pi(s) + \pi(t) \leq 2\pi(c) + 1$. Without loss of generality, suppose that $\pi(s) = \pi(c)$; otherwise, $\pi(t) = \pi(c)$ and the argument is analogous. By definition of S_c, we have that $s \notin S_c$, which is a contradiction. Therefore, we conclude that the set (1) is contained in $\mathrm{msg}(S_c)$.

If $x > c$, then there exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $x = c + (\lambda_1, \lambda_2)$ and $\lambda_1 + \lambda_2 \neq 0$. We distinguish two cases. When $\pi(x) = 2\pi(c)$, necessarily, $\pi((\lambda_1, \lambda_2)) = \pi(c)$. If $\lambda_1 > c_1$ then $(\lambda_1, \lambda_2) \in S_c$, and thus $x \notin \mathrm{msg}(S_c)$. If $\lambda_1 < c_1$ then $x_1 \leq 2c_1$. Assuming $x = s + t$ for some $s, t \in S_c$, and repeating the mentioned argument, we deduce that $\pi(s) = \pi(t) = \pi(c)$. By definition of S_c, it follows that $s_1, t_1 > c_1$, therefore $x = s_1 + t_1 \geq 2c_1$, which is a contradiction. Hence, $x \in \mathrm{msg}(S_c)$. So, $\Big[c + \big(0, \pi(c)\big), 2c\Big)_{\preceq_{glex}} \subset \mathrm{msg}(S_c)$. Finally, when

$\pi(x) = 2\pi(c) + 1$, then $\pi(c) + 1 = \pi((\lambda_1, \lambda_2))$, and thus $\pi((\lambda_1, \lambda_2)) > \pi(c)$, it follows that $(\lambda_1, \lambda_2) \in S_c$. Hence, $x \notin \mathrm{msg}(S_c)$. Which finishes the first part of the proof.

The computation of $\mathrm{e}(S_c)$ relies on the cardinality of the following disjoint sets since the set (1) can be rewritten depending on π as

$$\begin{aligned} msg(S_c) = \{x \in \mathbb{N}^2 \mid \pi(x) = \pi(c) \text{ and } x_1 \geq c_1\} && (2)\\ \sqcup \{x \in \mathbb{N}^2 \mid \pi(c) + 1 \leq \pi(x) \leq 2\pi(c) - 1\} && (3)\\ \sqcup \{x \in \mathbb{N}^2 \mid \pi(x) = 2\pi(c) \text{ and } x_1 < 2c_1\} && (4)\\ \sqcup \{x \in \mathbb{N}^2 \mid \pi(x) = 2\pi(c) + 1 \text{ and } x_1 < c_1\}. && (5) \end{aligned}$$

The cardinality of the set (2) is determined by the number of solutions $(x_1, x_2) \in \mathbb{N}^2$ of $x_1 + x_2 = \pi(c)$ such that $x_1 \geq c_1$, we deduce that $x_1 \in \{c_1, c_1 + 1, \ldots, \pi(c)\}$ and consequently, $x_2 = \pi(c) - x_1$. Therefore, there are $c_2 + 1$ different solutions. For the cardinality of the set (3), by arguing as in the proof of Proposition 26, is determined by

$$\sum_{k=\pi(c)+1}^{2\pi(c)-1} (k+1) = \frac{(\pi(c) - 1)(3\pi(c) + 2)}{2}.$$

Regarding sets (4) and (5), their cardinalities correspond to the number of natural solutions $(x_1, x_2) \in \mathbb{N}^2$ of $x_1 + x_2 = 2\pi(c)$ and $x_1 + x_2 = 2\pi(c) + 1$, respectively, satisfying the conditions $x_1 < 2c_1$, and $x_1 < c_1$ respectively. Therefore, the cardinality of (4) is $2c_1$ and for the set (5) is c_1. Consequently, adding the previous cardinalities, we have $\mathrm{e}(S_c) = \dfrac{3\pi^2(c) + \pi(c) + 4c_1}{2}$. □

Remark 28. From previous results, we observe that the minimal generating set of an ordinary $\mathbb{N}^2$-semigroup (ordered by $\preceq_{glex}$) depends on the first coordinate of its conductor, and that $\dfrac{3\pi^2(c) + \pi(c) + 4c_1}{2}$ equals $\dfrac{c_2(3c_2 + 1)}{2}$ for $c_1 = 0$. This fact simplifies the proof of the Generalized Wilf Conjecture and Extended Wilf Conjecture hold.

Conjecture 29 [6, Conjecture 2.8]. *Let S be a $\mathbb{N}^p$-semigroup. The Generalized Wilf Conjecture is*

$$\nu(S)e(S) \geq p\,\gamma(S), \tag{6}$$

where $\nu(S) = \sharp\{x \in S \mid x \leq_{\mathbb{N}^p} h \text{ for some } h \in \mathcal{H}(S)\}$, and $\gamma(S) = \sharp\{x \in \mathbb{N}^p \mid x \leq_{\mathbb{N}^p} h \text{ for some } h \in \mathcal{H}(S)\}$.

Conjecture 30 [17, Conjecture 14]. *Let S be a $\mathcal{C}$-semigroup. The Extended Wilf Conjecture is*

$$n(S)e(S) \geq \mathcal{N}(\mathrm{Fb}(S)) + 1. \tag{7}$$

The following proposition establishes the relation between the Conjectures 29 and 30.

Proposition 31 [6, Proposition 6.3]. *If $S \subseteq \mathbb{N}^p$ is a generalized numerical semigroup that satisfies the Generalized Wilf Conjecture, then S satisfies the Extended Wilf Conjecture*

As a consequence of Propositions 26 and 27, we obtain the desired result. In [6, Theorem 5.7], the authors prove an equivalent result by using different techniques.

Corollary 32. *Every ordinary $\mathbb{N}^2$-semigroup, ordered by $\preceq_{glex}$, satisfies the Generalized Wilf Conjecture and the Extended Wilf Conjecture.*

Proof. Applying Proposition 31 suffices to prove inequality (6). If $S = \mathbb{N}^2$, by convention $\gamma(S) = 0$, and the inequality (6) is trivial. Let S_c be an ordinary $\mathbb{N}^2$-semigroup with non-null conductor $c = (c_1, c_2)$, ordered by $\preceq_{glex}$. Clearly $\nu(S_c) = 1$, and $\gamma(S) = 1 + 2 + \cdots + c_2 = \frac{c_2(c_2+1)}{2}$ if $c_1 = 0$, thus

$$\frac{e(S)}{2} = \frac{c_2(3c_2+1)}{4} \geq \gamma(S).$$

If $c_1 \neq 0$ then $\gamma(S) = \frac{\big(1+\pi(c)\big)\pi(c)}{2} + c_1$. Therefore, the inequality (6) is equivalent to

$$\frac{e(S)}{2} = \frac{3c_1^2 + 6c_1c_2 + 3c_2^2 + 5c_1 + c_2}{4} \geq \frac{c_1^2 + c_2^2 + 2c_1c_2 + 3c_1 + c_2}{2},$$

which is true for all natural numbers c_1 and c_2. □

To illustrate the results discussed, we provide the following example.

Example 33. Let S_c be an ordinary $\mathbb{N}^2$-semigroup with conductor $c = (7, 3)$, ordered by $\preceq_{glex}$. By Proposition 27, we obtain that S_c is minimally generated by

$$\Big[(7,3),(14,6)\Big)_{\preceq_{glex}} \sqcup \Big[(0,21),(7,14)\Big)_{\preceq_{glex}}$$

and its embedding dimension is $\mathrm{e}(S_c) = 2 \cdot 7 + \frac{10 \cdot 31}{2} = 14 + 155 = 169$. Figure 8 gives a graphical representation of S_c. The empty circles are the gaps of S_c, the blue squares are the minimal generators of S_c, and the red circles are elements of S_c.

5. Mult-Embedded $\mathbb{N}^2$-Semigroups

Given a total order $\preceq$, recall that a $\mathcal{C}$-semigroup S is said to be a mult-embedded $\mathcal{C}$-semigroup if $S = \{m, 2m, \ldots, (k-1)m\} \sqcup S_{km}$ where m is a non-zero element of $\mathcal{C}$. The name *mult-embedded* arises from the fact that given any $\mathcal{C}$-semigroup, we can always find a mult-embedded $\mathcal{C}$-semigroup within it with the same multiplicity. In particular, in the case where $k = 1$, an ordinary $\mathcal{C}$-semigroup is a mult-embedded $\mathcal{C}$-semigroup.

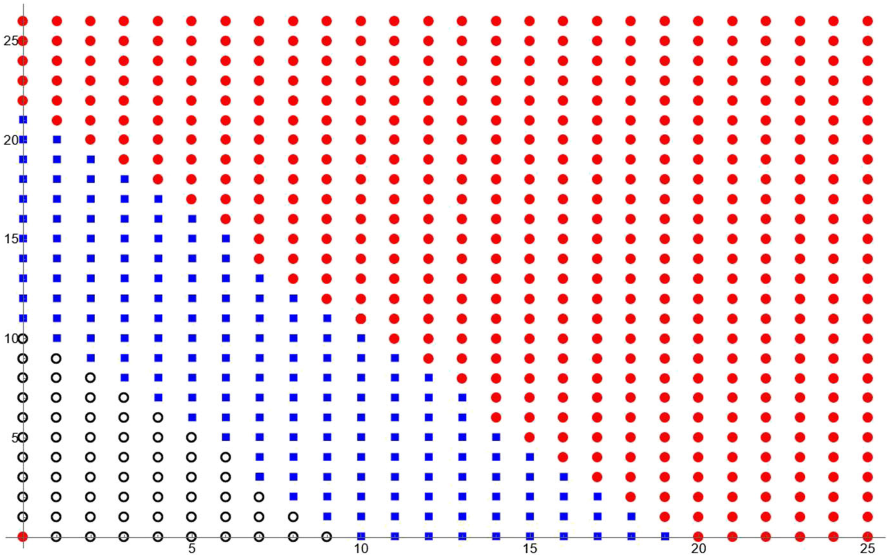

FIGURE 8. Ordinary $\mathbb{N}^2$-semigroup S_c

To emphasize the relevance of the mult-embedded $\mathcal{C}$-semigroups, we introduce two families of $\mathcal{C}$-semigroups: arf semigroups and saturated semigroups, and we prove that any mult-embedded $\mathcal{C}$-semigroup belongs to both families. Recall that for any positive integer b, we denote the set $[\![b]\!] = \{0, 1, 2, \ldots, b\}$.

We say that an affine semigroup S is an arf (affine) semigroup if, for any $x, y, z \in S$ with $x \geq_{\mathcal{C}} y \geq_{\mathcal{C}} z$, then $x + y - z \in S$. We say that S is a saturated (affine) semigroup if $s, s_1, \ldots, s_r \in S$ are such that $s_i \leq_{\mathcal{C}} s$ for all $i \in [\![r]\!]\setminus\{0\}$ and $z_1, \ldots, z_r \in \mathbb{Z}$ are such that $z_1 s_1 + \cdots + z_r s_r \in \mathcal{C}$ then, $s + z_1 s_1 + \cdots + z_r s_r \in S$. In the case of numerical semigroups, both classes of semigroups have been studied in the literature (see, for example, [4] and [24]).

The following lemma generalizes Lemma 3.31 in [23] from numerical semigroups to affine semigroups.

Lemma 34. *Every saturated semigroup is an arf semigroup.*

Proof. Let S be a saturated affine semigroup. Take $x, y, z \in S$ such that $x \geq_{\mathcal{C}} y \geq_{\mathcal{C}} z$, implying $y - z$ belongs to $\mathcal{C}$. By definition of being saturated, we have that $x + y - z \in S$. $\square$

Proposition 35. *Every mult-embedded $\mathcal{C}$-semigroup is a saturated semigroup.*

Proof. Let $S = \{m, 2m, \ldots, (k-1)m\} \sqcup S_{km}$ be a mult-embedded $\mathcal{C}$-semigroup. Suppose $s, s_1, \ldots, s_r \in S$ such that $s_i \leq_{\mathcal{C}} s$ for all $i \in [\![r]\!]\setminus\{0\}$, and $z_1, \ldots, z_r \in \mathbb{Z}$ such that $z_1 s_1 + \cdots + z_r s_r \in \mathcal{C}$. We distinguish two cases. If there exists, at least an element, either s or some s_i, with $i \in [\![r]\!]\setminus\{0\}$ belonging in S_{km}, then by hypothesis, it follows that at least $s \in S_{km}$. Since s is greater than or equal to the conductor, we have $s + t \in S$, where $t = z_1 s_1 + \cdots + z_r s_r \in \mathcal{C}$. If neither s and nor any s_i with $i \in [\![r]\!]\backslash 0$ belongs to S_{km}, then it follows that $s, z_1 s_1 + \cdots + z_r s_r$ are multiple of m. Since the sum of two multiples of m is again a multiple of m, we deduce that S is a saturated semigroup. $\square$

Corollary 36. *Every ordinary $\mathcal{C}$-semigroup is an arf semigroup.*

We continue along the same thread as the previous section. We focus on the minimal generating set of mult-embedded $\mathbb{N}^2$-semigroups ordered by the graded lexicographic order. Another contribution of this section is to provide a formula for its embedding dimension, which allows us to show that the Generalized Wilf Conjecture and the Extended Wilf Conjecture hold.

We can easily rewrite Lemma 24 for mult-embedded $\mathbb{N}^2$-semigroups ordered by the graded lexicographic order. From now on, we assume that the multiplicity m is not the null vector.

Lemma 37. *Let k be a positive integer, $m \in \mathbb{N}^2$, and S be the mult-embedded $\mathbb{N}^2$-semigroup ordered by $\preceq_{glex}$, which multiplicity is m and conductor equals km. If $x \in \mathbb{N}^2$ satisfies $\pi(x) > \pi(mult_1(S)) + \pi(mult_2(S))$, then x is not a minimal generator.*

Proof. Suppose $x \in \mathbb{N}^2$ such that $\pi(x) > \pi(\mathrm{mult}_1(S)) + \pi(\mathrm{mult}_2(S))$. By the Lemma 24, since $\mathrm{msg}(S) \subset \mathrm{msg}(S_{km}) \cup \{m\}$, it follows that $x \notin msg(S_{km})$, consequently $x \notin msg(S)$. $\square$

Remark 38. Note that the bound $\pi(\mathrm{mult}_1(S)) + \pi(\mathrm{mult}_2(S))$ introduced in the previous lemma depends on the multiplicity $m = (m_1, m_2)$. For the case $m_1 = 0$, we obtain that $\pi(\mathrm{mult}_1(S)) + \pi(\mathrm{mult}_2(S)) = km_2 + m_2 = (k + 1)\pi(m)$, otherwise $\pi(\mathrm{mult}_1(S)) + \pi(\mathrm{mult}_2(S)) = 2k\pi(m) + 1$. Hence, for any minimal generator x of a mult-embedded $\mathbb{N}^2$-semigroup S, it holds that $\pi(x) \leq 2k\pi(m) + 1$, according to the value of m.

For any $x \in \mathbb{N}^2$, we consider that its coordinates are x_1, and x_2.

Theorem 39. *Let k be a positive integer, $m = (m_1, m_2) \in \mathbb{N}^2$, and S be the mult-embedded $\mathbb{N}^2$-semigroup ordered by $\preceq_{glex}$ with multiplicity m and conductor km. Then, S is minimally generated by*

$$
\begin{aligned}
A = \{m\} \sqcup & \Big(km,\ (k+1)m\Big)_{\preceq_{glex}} \\
& \sqcup \Big\{x \in \mathbb{N}^2 \mid (k+1)\pi(m) \leq \pi(x) \leq 2k\pi(m) - 1 \text{ and } x_2 < m_2\Big\} \\
& \sqcup \Big\{x \in \mathbb{N}^2 \mid (k+1)\pi(m) + 1 \leq \pi(x) \leq 2k\pi(m) + 1 \text{ and } x_1 < m_1\Big\},
\end{aligned}
$$

and $e(S) = \dfrac{(4k-1)\pi^2(m) + \pi(m) + 4m_1}{2}$.

Proof. Trivially, $m \in \mathrm{msg}(S)$. Denote $B = \Big(km,\ (k+1)m\Big)_{\preceq_{glex}}$. First, let us prove that $B \subset \mathrm{msg}(S)$. Suppose $x \in B$ such that $x = s + t$ for some $s, t \in S$. We distinguish the following two cases. Without loss of generality, if we consider $\pi(s) \geq k\pi(m)$, then $\pi(t) < k\pi(m)$, and since, S is a mult-embedded $\mathbb{N}^2$-semigroup, it follows that $t = qm$, for some positive integer q. Then, $\pi(x) = \pi(s)+\pi(qm) \geq k\pi(m)+q\pi(m) \geq (k+1)\pi(m)$, which contradicts the initial hypothesis. So, we assume $\pi(s), \pi(t) < k\pi(m)$, and then we have that s and t are multiples of m. Consequently, x is a multiple of m. So, $x \notin B$ and it leads to a false statement.

Now, let $x \in \mathbb{N}^2$ such that $(k+1)\pi(m) \leq \pi(x) \leq 2k\pi(m)-1$, and $x_2 < m_2$. If $x = s+t$, for some $s = (s_1, s_2)$ and $t = (t_1, t_2)$ belonging to S, then $s_2, t_2 < m_2$. It implies that s and t are not multiples of m, and so $\pi(s), \pi(t) \geq k\pi(m)$. Consequently, $\pi(x) \geq 2k\pi(m)$. Again, we get a contradiction. Therefore, $x \in \mathrm{msg}(S)$.

Assuming that $x \in \mathbb{N}^2$ such that $(k+1)\pi(m)+1 \leq \pi(x) \leq 2k\pi(m)+1$, and $x_1 < m_1$, and using a similar structure as above, if $x = s+t$, for some $s, t \in S$, then $s_1, t_1 < m_1$. So, $\pi(s), \pi(t) \geq k\pi(m) + 1$, and thus, $\pi(x) \geq 2k\pi(m) + 2$, which it is not possible. Hence, $x \in \mathrm{msg}(S)$. Summarizing, we have just proved that A is a subset of $\mathrm{msg}(S)$.

From Remark 38, A is the minimal generating set of S if and only if no minimal generators belong to the set $\{x \in \mathbb{N}^2 \mid \pi(x) \leq 2k\pi(m) + 1\} \backslash A$.

Consider $x \in \mathbb{N}^2$ such that $x \geq m$ and $x \succeq_{glex} (k+1)m$. Trivially, $(k+1)m \notin \mathrm{msg}(S)$. So, we assume that $x \succ_{glex} (k+1)m$. Since $x \geq m$, $x = m + \lambda$, for some $\lambda \in \mathbb{N}^2$. And thus $\pi(m) + \pi(\lambda) \geq (k+1)\pi(m)$, which implies that $\pi(\lambda) \geq k\pi(m)$. We distinguish two cases depending on the value of $\pi(\lambda)$. If $\pi(\lambda) > k\pi(m)$, then $\lambda \in S$. So, we conclude that $x \notin msg(S)$. If $\pi(\lambda) = k\pi(m)$, and as $x \succ_{glex} (k+1)m$, then $x = ((k+1)m_1+i, (k+1)m_2-i)$ for some $i \in [\![km_2]\!]\backslash\{0\}$. Hence, $x - m = (km_1 + i, km_2 - i) \in \mathbb{N}^2$, and $\pi(x - m) = k\pi(m)$. We deduce that $x - m \in S$, so $x \notin msg(S)$.

Finally, suppose $x \in \mathbb{N}^2$, with $x_2 < m_2$ such that $\pi(x) = 2k\pi(m) + i$ where $i \in \{0, 1\}$. Note that $x_1 + m_2 > x_1 + x_2 = \pi(x) \geq 2km_1 + 2km_2$, which implies that $x_1 > 2k\pi(m) - m_2$. Take $s = (k\pi(m), 0) \in S$. Since $x - s = (x_1 - k\pi(m), x_2) \in \mathbb{N}^2$, and $\pi(x - s) = k\pi(m) + i \geq k\pi(m)$, we obtain that $x - s$ belongs to S. Whence, x is not a minimal generator.

By definition, $\mathrm{e}(S_c) = \sharp A$. Similarly to the proof of Lemma 27,

$$\begin{aligned}\sharp A = 1 + km_2 + &\sum_{l=k\pi(m)+1}^{(k+1)\pi(m)-1} (l+1) + (k+1)m_1\\ &+ m_2(k-1)\pi(m) + m_1\big((k-1)\pi(m)+1\big)\\ = &\ \frac{(4k-1)\pi^2(m) + \pi(m) + 4m_1}{2}.\end{aligned}$$

$\square$

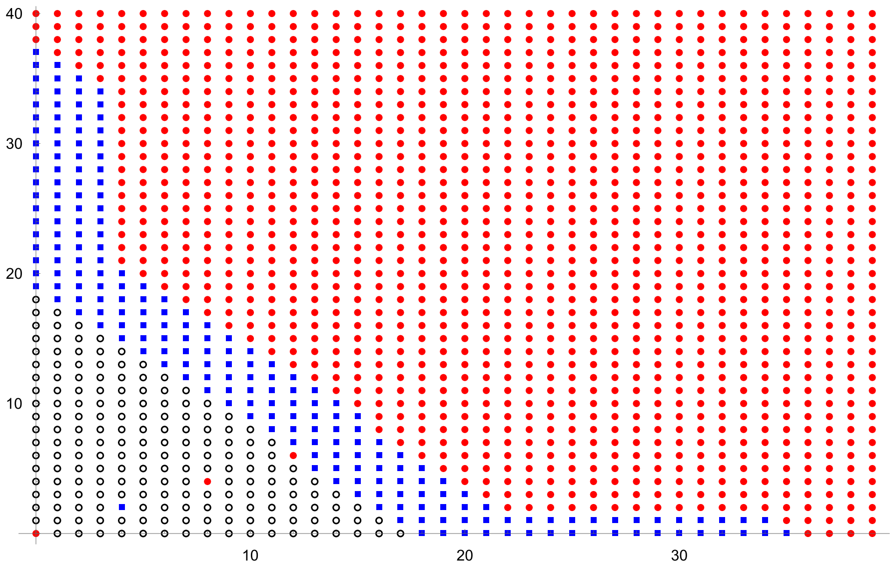

FIGURE 9. Mult-embedded $\mathbb{N}^2$-semigroup

The following corollary is a direct consequence of the Theorem 39.

Corollary 40. *Every mult-embedded $\mathbb{N}^2$-semigroup, ordered by $\preceq_{glex}$, satisfies the Generalized Wilf Conjecture and Extended Wilf Conjecture.*

Proof. Let S be a mult-embedded $\mathbb{N}^2$-semigroup with conductor $km = k(m_1, m_2)$, ordered by $\preceq_{glex}$. Given Proposition 31, it is enough to show that the inequality (6) holds. Trivially, $\nu(S) = k$, and we assume that $k > 1$, otherwise, it has already been proved (Corollary 32). Since $\gamma(S)$ equals the cardinality of $\mathcal{H}(S) \cup \{0, m, 2m \ldots, (k-1)m\}$ we obtain that

$$\gamma(S) = k\frac{k\pi^2(m) + \pi(m) + 2m_1}{2}$$

Thus,

$$ke(S) = k\frac{(4k-1)\pi^2(m) + \pi(m) + 4m_1}{2} \geq k\left(k\pi^2(m) + \pi(m) + 2m_1\right)$$

which proves the result since $(2k-1)\pi^2(m) \geq \pi(m)$. □

The section concludes with an example to illustrate the concepts and results discussed.

Example 41. Let S be a mult-embedded $\mathbb{N}^2$-semigroup ordered by $\preceq_{glex}$, with multiplicity $m = (4, 2)$, and conductor $3m = (12, 6)$, which is shown in Fig. 9. As mentioned earlier, the empty circles are the gaps of S, the blue squares are the minimal generators of S, and the red circles are elements of S.

From Theorem 39, we obtain that $\mathrm{e}(S) = \frac{396+6+16}{2} = 209$, and

$$\mathrm{msg}(S) = \{(4,2)\} \sqcup \Big((12,6),\ (16,8)\Big)_{\preceq_{glex}} \sqcup \bigcup_{i=0}^{11} \Big[(23+i,1),\ (24+i,0)\Big]_{\preceq_{glex}} \sqcup \bigcup_{i=0}^{12} \Big[(0,25+i),\ (3,22+i)\Big)_{\preceq_{glex}}.$$

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Data availability The authors confirm that the data supporting some findings of this study are available within it.

Declarations

Conflict of interest The authors declare no Conflict of interest.

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J. C. Rosales
Departamento de Álgebra
Universidad de Granada
18071 Granada
Spain
e-mail: jrosales@ugr.es

R. Tapia-Ramos
Departamento de Matemáticas
Universidad de Cádiz
11406 Jerez de la Frontera,Cádiz
Spain
e-mail: raquel.tapia@uca.es

A. Vigneron-Tenorio
Departamento de Matemáticas/INDESS
(Instituto Universitario para el Desarrollo Social Sostenible)
Universidad de Cádiz
11406 Jerez de la Frontera,Cádiz
Spain
e-mail: alberto.vigneron@uca.es

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