Bull. Malays. Math. Sci. Soc. (2025) 48:62
https://doi.org/10.1007/s40840-025-01847-z

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Slice Diameter Two Property in Ultrapowers

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Received: 5 September 2024 / Revised: 23 February 2025 / Accepted: 24 February 2025

Abstract

In this note we study the inheritance of the slice diameter two property by ultrapower spaces. Given a Banach space X, we give a characterisation of when $(X)_{\mathcal{U}}$, the ultrapower of X through a free ultrafilter $\mathcal{U}$, has the slice diameter two property obtaining that this is the case for many Banach spaces which are known to enjoy the slice diameter two property. We also provide, for every $\eta > 0$, an example of a Banach space X with the Daugavet property such that the unit ball of $(X)_{\mathcal{U}}$ contains a slice of diameter smaller than η for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. This proves, in particular, that the slice diameter two property is not in general inherited by taking ultrapower spaces.

Keywords Slice-diameter two property · Ultraproducts · Daugavet property

Mathematics Subject Classification 46B04 · 46B08 · 46B20 · 46M07

1 Introduction

Ultrapowers of Banach spaces have been intensively studied in the literature as they have proved to be a useful tool in order to study local theory of Banach spaces (as a matter of fact, ultraproducts are used in [3, Chapter 11] in order to prove that ℓ_1 is finitely representable in X if, and only if, X fails to have type $p > 1$).

Various topological and geometrical properties have been studied in ultrapowers of Banach spaces. For topological properties, e.g. reflexivity [14] and weak compactness of sets [12, 29] have been explored. From the geometrical perspective, properties like being an L_1-predual [16], an almost square Banach space [15], an extreme point or a strongly exposed point of the unit ball [10, 28], have been investigated

Communicated by Mohammad Sal Moslehian.

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Published online: 11 March 2025

A classical result about ultrapower spaces is the following: given a Banach space X and a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, it follows that $((X)_{\mathcal{U}})^* = (X^*)_{\mathcal{U}}$ if, and only if, X is superreflexive. Moreover, if X is not superreflexive, there is not a good description of the topological dual of $(X)_{\mathcal{U}}$. Because of this reason, informally speaking, properties of Banach spaces which are described using elements of the topological dual may be difficult to analyse in ultrapower spaces. This is the case, for instance, for properties which deal with the behaviour of the slices of the unit ball (see Sect. 2 for details), like the slice diameter two property.

A Banach space X is said to have the *slice diameter two property (slice-D2P)* if every slice of B_X has diameter exactly two. We refer the interested reader to [2, 5, 25] and references therein for background on the topic.

This property has been widely studied during the last 25 years but, as far as the author knows, little is known about when the slice-D2P passes on from a Banach space to its ultrapowers. Let us point out that, from the study of stronger properties of Banach spaces, some ultrapower spaces are known to enjoy the slice-D2P. For instance, in [15] it is shown that $(X)_{\mathcal{U}}$ has the slice-D2P whenever the space X is *locally almost square*, a property which is strictly stronger than the slice-D2P (see Example 3.10 for details). Moreover, other examples of ultrapowers with the slice-D2P come from ultrapowers actually satisfying the *Daugavet property*.

Let us formally introduce the Daugavet property. We say that a Banach space X has the *Daugavet property* if, for every slice S of B_X, every $x \in S_X$ and every $\varepsilon > 0$, there exists $y \in S$ satisfying

$$\|x - y\| > 2 - \varepsilon.$$

Observe that the above is an equivalent formulation of the original one which has to do with the equation $\|\mathrm{Id} + T\| = 1 + \|T\|$ for the class of rank-one operators. We refer the reader to [17, 18, 26, 31] and references therein for background. It is clear from the definition that Banach spaces with the Daugavet property enjoy the slice-D2P.

The study of the Daugavet property implies to deal with slices of the unit ball (and consequently with elements of X^*) so, at a first glance, one could expect a big difficulty in the analysis of the Daugavet property in an ultrapower space. However, a complete characterisation of when an ultrapower space has the Daugavet property was obtained in [7].

The key idea was to make use of a characterisation of the Daugavet property which avoids the use of slices: a Hahn-Banach separation argument implies that X has the Daugavet property if, and only if, $B_X = \overline{\mathrm{conv}}\{y \in B_X : \|x - y\| > 2 - \varepsilon\}$ holds for every $x \in S_X$ and every $\varepsilon > 0$ (c.f. e.g. [31, Lemma 2.3]).

With this idea in mind, the authors of [7] considered a uniform version of the Daugavet property, the so called uniform Daugavet property (see [7, p. 59]), and they characterised those Banach spaces X for which $(X)_{\mathcal{U}}$ has the Daugavet property. They also showed that all the classical examples of Banach spaces with the Daugavet property actually satisfy its uniform version. In [19], however, the authors constructed a Banach space X with the Daugavet and the Schur properties such that $(X)_{\mathcal{U}}$ fails the Daugavet property for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$.

In this note our starting point will be a characterisation of the slice-D2P in the spirit of the above mentioned [31, Lemma 2.3] coming from [13]: a Banach space X has the slice-D2P if, and only if, $B_X = \overline{\text{conv}}\{\frac{x+y}{2} : x, y \in B_X, \|x - y\| > 2 - \varepsilon\}$ holds for every $\varepsilon > 0$.

Using the above, in Theorems 3.1 and 3.2 we completely characterise when, given a sequence $(X_n)_{n\in\mathbb{N}}$ of Banach spaces and a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, the ultraproduct $(X_n)_{\mathcal{U}}$ has the slice-D2P in terms of requiring that all X_n have the slice-D2P "in a uniform way". This motivates us to introduce the *uniform slice diameter two property* in Definition 3.5, showing that this property is enjoyed by most of the classical spaces which are known to have the slice-D2P. All this is discussed in Sect. 3.

In Sect. 4 we will have a look at the involved construction from [19] of a Daugavet space whose ultrapowers fail the Daugavet property. We will make use of the above example in order to construct, for every $\eta > 0$, a Banach space with the Daugavet property such that the unit ball of $(X)_{\mathcal{U}}$ contains slices of diameter smaller than η for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. This will show, in particular, that the slice-D2P is not in general inherited by taking ultrapower spaces.

2 Notation and Preliminary Results

We will consider Banach spaces over the scalar field $\mathbb{R}$ or $\mathbb{C}$.

Given a Banach space X then B_X (respectively S_X) stands for the closed unit ball (respectively the unit sphere) of X. We will denote by X^* the topological dual of X. Given a subset C of X, we will denote by $\text{conv}(C)$ the convex hull of C and by $\text{span}(C)$ the linear span of C. We also denote, given $n \in \mathbb{N}$, the set

$$\text{conv}_n(C) := \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_1, \ldots, \lambda_n \in [0, 1], \sum_{i=1}^{n} \lambda_i = 1, x_1, \ldots, x_n \in C \right\}.$$

In other words, $\text{conv}_n(C)$ stands for the set of all convex combinations of at most n elements of C.

If C is a bounded set, by a *slice* of C we will mean a set of the following form

$$S(C, f, \alpha) := \{x \in C : \text{Re} f(x) > \sup \text{Re} f(C) - \alpha\}$$

where $f \in X^*$ and $\alpha > 0$. Notice that a slice is nothing but the nonempty intersection of a half-space with the bounded (and not necessarily convex) set C.

In [13, Lemma 1] it is proved that a Banach space X has the slice-D2P if, and only if, $B_X := \overline{\text{conv}}\{\frac{x+y}{2} : \|x - y\| > 2 - \varepsilon\}$ holds for every $\varepsilon > 0$. Indeed, we state here for future reference the following more general version, which was already observed in [22, Section 5]. Since the above mentioned [22] deals only with real Banach spaces, we include a complete proof of the following proposition to cover the complex case too and for the sake of completeness.

Proposition 2.1 *Let X be a Banach space. The following are equivalent:*

1. *Every slice of B_X has diameter at least α.*

2. $B_X = \overline{\mathrm{conv}}\left\{\frac{x+y}{2} : x, y \in B_X, \|x-y\| \geqslant \alpha - \varepsilon\right\}$ *holds for every* $\varepsilon > 0$.

Proof (1)⇒(2). Assume that (2) does not hold. Then there exists $\varepsilon > 0$ and $x_0 \in B_X$ such that $x_0 \notin \overline{\mathrm{conv}}\left\{\frac{x+y}{2} : x, y \in B_X, \|x-y\| \geqslant \alpha - \varepsilon\right\}$. Call $A := \left\{\frac{x+y}{2} : x, y \in B_X, \|x-y\| \geqslant \alpha - \varepsilon\right\}$. By the Hahn-Banach theorem we can find a slice S of B_X such that $x_0 \in S$ and $S \cap A = \emptyset$. We claim that if $u, v \in S$ it follows that $\|u-v\| < \alpha - \varepsilon$. Indeed, if there existed $u, v \in S$ with $\|u-v\| \geqslant \alpha - \varepsilon$, then $\frac{u+v}{2}$ would belong to S by the convexity of S. Since clearly $\frac{u+v}{2} \in A$ we would get that $S \cap A \neq \emptyset$, which is impossible. This proves that $\|u-v\| \leqslant \alpha - \varepsilon$ holds for every $u, v \in S$, which proves the negation of (1).

(2)⇒(1). Take a slice $S := S(B_X, x^*, \beta)$, where $x^* \in S_{X^*}$ and $\beta > 0$, and let $\varepsilon > 0$, and let us prove that there are $u, v \in S$ such that $\|u-v\| \geqslant \alpha - \varepsilon$. The arbitrariness of ε will imply (1). In order to do so, consider the slice $S(B_X, x^*, \frac{\beta}{2})$. Since $\overline{\mathrm{conv}}\left\{\frac{x+y}{2} : x, y \in B_X, \|x-y\| \geqslant \alpha - \varepsilon\right\} = B_X$ we infer that $S(B_X, x^*, \frac{\beta}{2}) \cap \left\{\frac{x+y}{2} : x, y \in B_X, \|x-y\| \geqslant \alpha - \varepsilon\right\} \neq \emptyset$ (since the complement in B_X of slices are clearly convex sets). Consequently, we can find $u, v \in B_X$ with $\|u-v\| \geqslant \alpha - \varepsilon$ and such that $\frac{u+v}{2} \in S\left(B_X, x^*, \frac{\beta}{2}\right)$. In order to finish the proof, let us prove that both $u, v \in S = S(B_X, x^*, \beta)$ which means, by definition, that $\mathrm{Re}\, x^*(u) > 1-\beta$ and $\mathrm{Re}\, x^*(v) > 1-\beta$. To this end observe that, $\frac{u+v}{2} \in S(B_X, x^*, \frac{\beta}{2})$ means $\mathrm{Re}\, x^*\left(\frac{u+v}{2}\right) > 1 - \frac{\beta}{2}$. Now

$$1 - \frac{\beta}{2} < \frac{\mathrm{Re}\, x^*(u) + \mathrm{Re}\, x^*(v)}{2} \leqslant \frac{\mathrm{Re}\, x^*(u) + \|x^*\|}{2} = \frac{\mathrm{Re}\, x^*(u) + 1}{2}.$$

This implies $\mathrm{Re}\, x^*(u) + 1 > 2 - \beta$, from where $\mathrm{Re}\, x^*(u) > 1 - \beta$. In a similar way, it is proved that $\mathrm{Re}\, x^*(v) > 1 - \beta$, which means $u, v \in S$, as desired. □

The above result motivates us to introduce the following notation, which will be useful throughout the text. Given a Banach space X and $\alpha > 0$, define

$$S^\alpha(X) := \left\{\frac{x+y}{2} : x, y \in B_X, \|x-y\| \geqslant \alpha\right\}.$$

Given $n \in \mathbb{N}$ we denote

$$S_n^\alpha(X) := \mathrm{conv}_n(S^\alpha(X)).$$

Finally, given $n \in \mathbb{N}$ and $\alpha > 0$, we define

$$C_n^\alpha(X) := \sup_{x \in S_X} d(x, S_n^\alpha(X)) = \sup_{x \in S_X} \inf_{y \in S_n^\alpha(X)} \|x-y\|.$$

From the very definition of $C_n^\alpha(X)$ the following two properties follow:

1. Given $0 < \alpha < \beta$ then $C_n^\alpha \geqslant C_n^\beta$ and,
2. given two natural numbers $n \geqslant m$ then $C_n^\alpha(X) \leqslant C_m^\alpha(X)$.

Given a sequence of Banach spaces $\{X_n : n \in \mathbb{N}\}$ we denote

$$\ell_\infty(\mathbb{N}, (X_n)) := \left\{ f : \mathbb{N} \longrightarrow \prod_{n\in\mathbb{N}} X_n : f(n) \in X_n \ \forall n \text{ and } \sup_{n\in\mathbb{N}} \|f(n)\| < \infty \right\}.$$

Given a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, consider $c_{0,\mathcal{U}}(\mathbb{N}, (X_n)) := \{f \in \ell_\infty(\mathbb{N}, (X_n)) : \lim_{\mathcal{U}} \|f(n)\| = 0\}$. The *ultraproduct of* $\{X_n : n \in \mathbb{N}\}$ *with respect to* $\mathcal{U}$ is the Banach space

$$(X_n)_{\mathcal{U}} := \ell_\infty(\mathbb{N}, (X_n))/c_{0,\mathcal{U}}(\mathbb{N}, (X_n)).$$

We will naturally identify a bounded function $f : \mathbb{N} \longrightarrow \prod_{n\in\mathbb{N}} X_n$ with the element $(f(n))_{n\in\mathbb{N}}$. In this way, we denote by $(x_n)_{\mathcal{U}}$ or simply by (x_n), if no confusion is possible, the coset in $(X_n)_{\mathcal{U}}$ given by $(x_n)_{n\in\mathbb{N}} + c_{0,\mathcal{U}}(\mathbb{N}, (X_n))$.

From the definition of the quotient norm, it is not difficult to prove that $\|(x_n)\| = \lim_{\mathcal{U}} \|x_n\|$ holds for every $(x_n) \in (X_n)_{\mathcal{U}}$.

3 Uniform slice-D2P

Let us start by looking for necessary conditions for an ultraproduct space to enjoy the slice-D2P. In order to do so, as announced before, we will make use of Proposition 2.1.

Theorem 3.1 *Let (X_n) be a sequence of Banach spaces, $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$ and $\alpha > 0$. Set $X := (X_n)_{\mathcal{U}}$ and assume that every slice of B_X has diameter at least α. Then, for every $\delta > 0$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : C_n^{\alpha-\varepsilon}(X_k) < \delta\} \in \mathcal{U}.$$

Proof Assume that there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ such that, for every $n \in \mathbb{N}$, we get

$$\{k \in \mathbb{N} : C_n^{\alpha-\varepsilon_0}(X_k) \geqslant \delta_0\} \in \mathcal{U}.$$

We can select, for every $n \geqslant 2$, a set $A_n \subseteq \{k \in \mathbb{N} : C_n^{\alpha-\varepsilon_0}(X_k) \geqslant \delta_0\}$ such that $A_n \in \mathcal{U}$ holds for every $n \in \mathbb{N}$, $\bigcap_{n\geqslant 2} A_n = \emptyset$ and $A_{n+1} \subseteq A_n$ holds for $n \geqslant 2$. Take $A_1 = \mathbb{N}$. Observe that $\{A_n \setminus A_{n+1} : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$. Moreover, for every $n \geqslant 2$, for every $p \in A_n \setminus A_{n+1}$ we can find $x_p \in S_{X_p}$ satisfying that $d(x_p, S_n^{\alpha-\varepsilon_0}(X_p)) \geqslant \frac{\delta_0}{2}$. For $p \in A_1 \setminus A_2$ select any $x_p \in S_{X_p}$.

Now $x := (x_p) \in S_X$. We claim that $d((x_p), \operatorname{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))) \geqslant \frac{\delta_0}{2}$. Once this is proved, Proposition 2.1 implies that there exists a slice in $(X_n)_{\mathcal{U}}$ of diameter smaller than α, which will finish the proof of the theorem. In order to do so, take $z \in \operatorname{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))$, so there is $q \in \mathbb{N}$ such that $z \in \operatorname{conv}_q(S^{\alpha-\frac{\varepsilon_0}{2}}(X))$.

By definition we can find $\lambda_1, \ldots, \lambda_q \in [0,1]$ with $\sum_{i=1}^q \lambda_i = 1$ and $(u_n^i), (v_n^i) \in S_X$ with $\|(u_n^i) - (v_n^i)\| \geqslant \alpha - \frac{\varepsilon_0}{2}$ and $z = \sum_{i=1}^q \lambda_i \frac{(u_n^i)+(v_n^i)}{2}$. Let $\eta > 0$. Since $\|(x_n) - (z_n)\| = \lim_{\mathcal{U}} \|x_n - z_n\|$, the set

$$B := \{n \in \mathbb{N} : |\|x_n - z_n\| - \|x - z\|| < \eta\} \in \mathcal{U}.$$

On the other hand, given $1 \leqslant i \leqslant q$ it follows that $\lim_{\mathcal{U}} \|u_n^i - v_n^i\| \geqslant \alpha - \frac{\varepsilon_0}{2} > \alpha - \varepsilon_0$. This implies that the set

$$C := \bigcap_{i=1}^{q} \left\{ n \in \mathbb{N} : \|u_n^i - v_n^i\| > \alpha - \varepsilon_0 \right\} \in \mathcal{U}.$$

Select any $k \in A_q \cap B \cap C$. Then, since $k \in B$, we have

$$\|(x_n) - (z_n)\| \geqslant \|x_k - z_k\| - \eta.$$

On the other hand, $z_k = \sum_{i=1}^q \lambda_i \frac{u_k^i + v_k^i}{2}$ with $\|u_k^i - v_k^i\| \geqslant \alpha - \varepsilon_0$ since $k \in C$. Hence, $z_k \in \operatorname{conv}_q(S^{\alpha-\varepsilon_0}(X_k))$. Finally, since $k \in A_q$ we conclude by the choice of x_k that $\|x_k - z_k\| \geqslant \frac{\delta_0}{2}$, so

$$\|(x_n) - (z_n)\| \geqslant \frac{\delta_0}{2} - \eta.$$

The arbitrariness of $\eta > 0$ and $(z_k) \in \operatorname{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))$ implies that $d((x_n), \operatorname{conv}(S^{\alpha-\frac{\varepsilon_0}{2}}(X))) \geqslant \frac{\delta_0}{2}$, as desired. □

In the following result we establish the converse.

Theorem 3.2 *Let (X_n) be a sequence of Banach spaces, $0 < \alpha < 2$ and let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Assume that for every $\delta > 0$ there exists $n \in \mathbb{N}$ such that*

$$\{k \in \mathbb{N} : C_n^\alpha(X_k) < \delta\} \in \mathcal{U}.$$

Then, every slice of $(X_n)_{\mathcal{U}}$ has diameter at least α.

Proof Let $(x_n) \in S_{(X_n)_{\mathcal{U}}}$ and let us prove, in view of Proposition 2.1, that

$$(x_n) \in \overline{\operatorname{conv}}\left(\left\{\frac{(u_n)+(v_n)}{2} : (u_n), (v_n) \in B_{(X_n)_{\mathcal{U}}}, \|(u_n) - (v_n)\| \geqslant \alpha\right\}\right).$$

In order to do so take $\delta > 0$. By the assumption there exists $n \in \mathbb{N}$ such that

$$A := \{k \in \mathbb{N} : C_n^\alpha(X_k) < \delta\} \in \mathcal{U}.$$

Consequently, for every $k \in A$ there exists $\sum_{i=1}^n \lambda_i^k \frac{u_k^i+v_k^i}{2}$, where $\lambda_i^k \in [0,1]$ satisfy $\sum_{i=1}^n \lambda_i^k = 1$, such that

$$\left\| x_k - \sum_{i=1}^n \lambda_i^k \frac{u_k^i + v_k^i}{2} \right\| < \delta$$

and

$$\|u_k^i - v_k^i\| \geqslant \alpha$$

holds for every $1 \leqslant i \leqslant n$.

Define $\lambda_i^k = 0$ if $k \notin A$ for every $1 \leqslant i \leqslant n$. Since $\lambda_i^k \in [0,1]$ we can consider $\lambda_i := \lim_{k,\mathcal{U}} \lambda_i^k \in [0,1]$. It is not difficult to prove that $\sum_{i=1}^n \lambda_i = 1$. Now, given $1 \leqslant i \leqslant n$ define

$$u_k^i = v_k^i = 0 \ \forall k \notin A.$$

It is immediate that $(u_k^i), (v_k^i) \in B_{(X_n)_\mathcal{U}}$. Let us start by proving that $\|(u_k^i) - (v_k^i)\| = \lim_\mathcal{U} \|u_k^i - v_k^i\| \geqslant \alpha$ holds for $1 \leqslant i \leqslant n$. In order to do so, fix $\eta > 0$ and $1 \leqslant i \leqslant n$. By definition of limit through $\mathcal{U}$ and the definition of the norm of ultraproducts the set

$$B_\eta := \{p \in \mathbb{N} : |\|(u_k^i) - (v_k^i)\| - \|u_p^i - v_p^i\|| < \eta\} \in \mathcal{U}.$$

Consequently, given $p \in B_\eta \cap A$ we obtain by the choice of u_i^p and v_i^p that

$$\|(u_k^i) - (u_k^i)\| \overset{p\in B_\eta}{\geqslant} \|u_p^i - v_p^i\| - \eta \overset{p\in A}{\geqslant} \alpha - \eta.$$

The arbitrariness of $\eta > 0$ implies $\|(u_k^i) - (v_k^i)\| \geqslant \alpha$.

Now it is time to prove that

$$\left\| (x_k) - \sum_{i=1}^n \lambda_i \frac{(u_k^i) + (v_k^i)}{2} \right\| \leqslant \delta.$$

In order to do so, take $\nu > 0$. Set

$$C_\nu := \left\{ p \in \mathbb{N} : \left| \left\| (x_k) - \sum_{i=1}^n \lambda_i \frac{(u_k^i) + (v_k^i)}{2} \right\| - \left\| x_p - \sum_{i=1}^n \lambda_i \frac{u_p^i + v_p^i}{2} \right\| \right| < \nu \right\} \in \mathcal{U}.$$

On the other hand set

$$D := \bigcap_{i=1}^n \left\{ p \in \mathbb{N} : \left|\lambda_i^p - \lambda_i\right| < \frac{\nu}{n} \right\} \in \mathcal{U}.$$

Now given $p \in C_\nu \cap D \cap A$ we get

$$\begin{aligned}
\left\| (x_k) - \sum_{i=1}^{n} \lambda_i \frac{(u_k^i) + (v_k^i)}{2} \right\| & \overset{p\in C_\nu}{\leqslant} \nu + \left\| x_p - \sum_{i=1}^{n} \lambda_i \frac{u_p^i + v_p^i}{2} \right\| \\
& \leqslant \nu + \left\| x_p - \sum_{i=1}^{n} \lambda_i^p \frac{u_p^i + v_p^i}{2} \right\| + \sum_{i=1}^{n} |\lambda_i - \lambda_i^p| \\
& \overset{p\in A}{\leqslant} \nu + \delta + \sum_{i=1}^{n} |\lambda_i^p - \lambda_i| \\
& \overset{p\in D}{\leqslant} 2\nu + \delta.
\end{aligned}$$

The arbitrariness of $\nu > 0$ proves $\left\| (x_n) - \sum_{i=1}^n \lambda_i \frac{(u_k^i)+(v_k^i)}{2} \right\| \leqslant \delta$, which finishes the proof. □

Remark 3.3 Observe that, given a Banach space X and any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, if we consider $X_1 = X_2 = \ldots, = X$, the quantity $C_n^\alpha(X_k)$ is independent of k, and therefore the condition $\{k \in \mathbb{N} : C_n^\alpha(X_k) < \delta\} \in \mathcal{U}$ simply means $C_n^\alpha(X) < \delta$.

Going back to the slice diameter two property in ultrapower spaces, given a Banach space X we have that, a combination of Theorems 3.1 and 3.2 together with the fact that $(C_n^\alpha(X))_{n\in\mathbb{N}}$ is a decreasing sequence, yield the following corollary.

Corollary 3.4 *Let X be a Banach space. The following are equivalent:*

1. *$(X)_\mathcal{U}$ has the slice-D2P for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$.*
2. *For every $0 < \alpha < 2$, $\lim_{n\to\infty} C_n^\alpha(X) = 0$.*

Proof (2)⇒(1). Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$ and $0 < \alpha < 2$. Let us prove that every slice of the unit ball of $B_{(X)_\mathcal{U}}$ has diameter at least α, for which we will make use of Theorem 3.2. In order to do so, let $\delta > 0$ and, thanks to Remark 3.3, it is enough to find $n \in \mathbb{N}$ such that $C_n^\alpha(X) < \delta$. But this is immediate by the assumption that $\lim_{n\to\infty} C_n^\alpha(X) = 0$. The arbitrariness of $0 < \alpha < 2$ yields the conclusion.

(1)⇒(2). Take $\alpha > 0$. In order to prove that $\lim_{n\to\infty} C_n^\alpha(X) = 0$ select $\delta > 0$ and let us find $m \in \mathbb{N}$ such that $C_n^\alpha(X) < \delta$ holds for every $n \geqslant m$. To do so, select any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. Since every slice of the unit ball of $(X)_\mathcal{U}$ has diameter at least α, Theorem 3.1 and Remark 3.3 imply that there exists some $m \in \mathbb{N}$ such that $C_m^\alpha(X) < \delta$. Since $(C_n^\alpha(X))$ is a decreasing sequence we get that $C_n^\alpha(X) \leqslant C_m^\alpha(X) < \delta$ holds for every $n \geqslant m$. The above condition together with the clear fact that $C_n^\alpha(X) \geqslant 0$ holds for every $n \in \mathbb{N}$ imply that $\lim_{n\to\infty} C_n^\alpha(X) = 0$, which finishes the proof. □

Corollary 3.6 motivates the following definition.

Definition 3.5 Let X be a Banach space. We say that X has the uniform slice diameter two property (uniform slice-D2P) if, for every $0 < \alpha < 2$,

$$\lim_n C_n^\alpha(X) = 0.$$

With the above definition, Corollary 3.4 can be re-written in the following language.

Corollary 3.6 *Let X be a Banach space. Then $(X)_{\mathcal{U}}$ has the slice-D2P for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$ if, and only if, X has the uniform slice-D2P.*

The rest of this section will be devoted to providing examples of Banach spaces with the uniform slice-D2P.

Example 3.7 Let $X = L_1(\mu)$. It follows that X has the slice-D2P if, and only if, μ contains no atom (c.f. e.g. [4, Theorem 2.13 (ii)]). But if μ is an atomless measure it follows that $(X)_{\mathcal{U}}$ has the Daugavet property for every free ultrafilter $\mathcal{U}$ [7, Lemma 6.6 and Theorem 6.2]. In particular, $(X)_{\mathcal{U}}$ has the slice-D2P for every free ultrafilter $\mathcal{U}$.

Consequently, an L_1 space has the slice-D2P if, and only if, it satisfies the uniform slice-D2P.

More examples of spaces enjoying the uniform slice-D2P come from ultrapower spaces with the slice-D2P.

Example 3.8 Let X be a Banach space with the uniform slice-D2P and let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. We claim that $(X)_{\mathcal{U}}$ has the uniform slice-D2P. In order to prove this it is enough to prove that, given any ultrafilter $\mathcal{V}$ over $\mathbb{N}$ then $((X)_{\mathcal{U}})_{\mathcal{V}}$ has the slice-D2P. However, this result follows since X has the uniform slice-D2P and $((X)_{\mathcal{U}})_{\mathcal{V}}$ is isometrically isomorphic to $(X)_{\mathcal{W}}$ where $\mathcal{W}$ is a free ultrafilter over $\mathbb{N}$. Indeed, $\mathcal{W} = \mathcal{U} \times \mathcal{V}$ (see [27, Proposition 2.1]).

Another class where the slice-D2P and its uniform version are equivalent is the one of L_1-preduals.

Example 3.9 Let X be an L_1 predual. Observe that X has the slice-D2P if, and only if, X is infinite-dimensional (c.f. e.g. [4, Corollary 2.9]). Since the ultrapower of any L_1 predual is again an L_1 predual by [16, Proposition 2.1], it follows that $(X)_{\mathcal{U}}$ has the slice-D2P for every free ultrafilter $\mathcal{U}$ as soon as X has the slice-D2P, from where the uniform slice-D2P follows on X.

The following examples will come from [15], for which we need to introduce a bit of notation. According to [1], a Banach space X is

1. *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$.
2. *weakly almost square* (WASQ) if for every $x \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly.
3. *almost square* (ASQ) if for every $x_1, \ldots, x_k \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \ldots, k\}$.

We refer the reader to [1, 11, 24] and references therein for examples of LASQ, WASQ and ASQ Banach spaces.

Example 3.10 If X is LASQ then X has the uniform slice-D2P. Indeed, if X is LASQ then $(X)_{\mathcal{U}}$ is LASQ for every free ultrafilter over $\mathbb{N}$ by [15, Proposition 4.2]. The result follows since LASQ spaces have the slice-D2P (c.f. e.g. [20, Proposition 2.5]).

The next result shows that the uniform slice-D2P is inherited by the ℓ_∞-sum of spaces.

Proposition 3.11 *Let X be a Banach space with the uniform slice-D2P. Then, for any non-zero Banach space Y, the space $X \oplus_\infty Y$ has the uniform slice-D2P.*

Proof It is known that $(X \oplus_\infty Y)_{\mathcal{U}} = (X)_{\mathcal{U}} \oplus_\infty Y_{\mathcal{U}}$. The result follows from the fact that slice-D2P is inherited by the ℓ_∞-sum if one of the factors has the slice-D2P (c.f. e.g. [21, Lemma 2.1]). □

For the ℓ_p-sum we have the following result.

Proposition 3.12 *Given $\alpha > 0$ and $n \in \mathbb{N}$, the following inequality holds*

$$C^\alpha_{n^2}(X \oplus_p Y) \leqslant \left(C^\alpha_n(X)^p + C^\alpha_n(Y)^p\right)^{\frac{1}{p}}.$$

In particular, if X and Y have the uniform slice-D2P, then so does $X \oplus_p Y$.

Proof Let $(x, y) \in S_{X\oplus_p Y}$ and let $r > 0$. We can assume up to a density argument that both $x \neq 0$ and $y \neq 0$. Since $x \in B_X$, by definition of $C^\alpha_n(X)$, we can find $u := \sum_{i=1}^n \lambda_i \frac{u_i+v_i}{2}$ with $\left\| \frac{x}{\|x\|} - u \right\| < d\left(\frac{x}{\|x\|}, S^\alpha_n(X)\right) + r \leqslant C^\alpha_n(X) + r$, where $u_i, v_i \in B_X$ satisfy $\|u_i - v_i\| \geqslant \alpha$ for every $1 \leqslant i \leqslant n$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ are such that $\sum_{i=1}^n \lambda_i = 1$.

Similarly, since $y \in B_Y$ we can find $v := \sum_{i=1}^n \mu_i \frac{a_i+b_i}{2}$ with $\left\| \frac{y}{\|y\|} - v \right\| < C^\alpha_n(Y) + r$, where $a_i, b_i \in B_X$ satisfy $\|a_i - b_i\| \geqslant \alpha$ for every $1 \leqslant i \leqslant n$ and $\mu_1, \ldots, \mu_n \in [0, 1]$ are such that $\sum_{i=1}^n \mu_i = 1$.

Now $(\tilde{u}, \tilde{v}) = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \mu_j \frac{(\|x\|u_i, \|y\|a_j)+(\|x\|v_i, \|y\|b_j)}{2} \in S^\alpha_{n^2}(X \oplus_p Y)$. Indeed, given $i, j \in \{1, \ldots, n\}$ we have

$$\|(\|x\|u_i, \|y\|a_j)\|^p = \|x\|^p\|u_i\|^p + \|y\|^p\|a_j\|^p \leqslant \|x\|^p + \|y\|^p = \|(x, y)\|^p = 1.$$

In a similar way we obtain that $(\|x\|v_i, \|y\|b_j) \in B_{X\oplus_p Y}$. On the other hand we have

$$\begin{aligned}\|(\|x\|u_i, \|y\|a_j) - (\|x\|v_i, \|y\|b_j)\|^p &= \|x\|^p\|u_i - v_i\|^p + \|y\|^p\|a_j - b_j\|^p \\ &\geqslant \alpha^p(\|x\|^p + \|y\|^p) = \alpha^p.\end{aligned}$$

Consequently $(\tilde{u}, \tilde{v}) \in S^\alpha_{n^2}(X \oplus_p Y)$.

Finally, in order to estimate $\|(x, y) - (\tilde{u}, \tilde{v})\|$ observe that $\tilde{u} = \|x\|u$. Indeed

$$\begin{aligned}\tilde{u} = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \mu_j \frac{\|x\|u_i + \|x\|v_i}{2} &= \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^n \mu_j\right) \frac{\|x\|u_i + \|x\|v_i}{2} \\ &= \sum_{i=1}^n \lambda_i \|x\| \frac{u_i + v_i}{2} \\ &= \|x\| \sum_{i=1}^n \lambda_i \frac{u_i + v_i}{2} = \|x\|u.\end{aligned}$$

With a similar argument it follows that $\tilde{v} = \|y\|v$.

This implies

$$\begin{aligned}(C_n^\alpha(X)+r)^p + (C_n^\alpha(Y)+r)^p &\geqslant \left\|\frac{x}{\|x\|} - u\right\|^p + \left\|\frac{y}{\|y\|} - v\right\|^p \\ &\geqslant \left\|\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) - \left(\frac{\tilde{u}}{\|x\|}, \frac{\tilde{v}}{\|y\|}\right)\right\|^p \\ &= \frac{\|x-\tilde{u}\|^p}{\|x\|^p} + \frac{\|y-\tilde{v}\|^p}{\|y\|^p} \\ &\geqslant \|x-\tilde{u}\|^p + \|y-\tilde{v}\|^p = \|(x,y)-(\tilde{u},\tilde{v})\|^p\end{aligned}$$

since $0 < \|x\|^p < 1$ and $0 < \|y\|^p < 1$. The arbitrariness of $r > 0$ and $(x, y) \in B_{X\oplus_p Y}$ proves the result. □

Let us continue with an example coming from [13] in the context of Lipschitz function spaces.

Example 3.13 Let M be a metric space with a distinguished point $0 \in M$ and let $\mathrm{Lip}_0(M)$ be the space of Lipschitz functions $f : M \longrightarrow \mathbb{R}$ which vanish at 0 endowed with the standard Lipschitz norm (see [30] for background).

From the results of [13, Section 2] it follows that if either $\inf\{d(x, y) : x, y \in M, x \neq y\} = 0$ or if M is unbounded, then $\mathrm{Lip}_0(M)$ has the uniform slice-D2P.

Indeed, in [13, Theorems 1 and 2] it is proved that in both the above cases $\mathrm{Lip}_0(M)$ satisfies the hypothesis of [13, Lemma 2]. Moreover, in the proof of the above mentioned [13, Lemma 2] it is proved that, given any $\varepsilon > 0$ and $f \in B_{\mathrm{Lip}_0(M)}$ then, for every $n \in \mathbb{N}$ there are Lipschitz functions $x_1, y_1, \ldots, x_n, y_n \in (1+\varepsilon)B_{\mathrm{Lip}_0(M)}$ such that $\|x_k - y_k\| \geqslant 2$ and

$$\left\| f - \frac{1}{n}\sum_{k=1}^{n} \frac{x_k + y_k}{2} \right\| < \frac{4}{n}.$$

If we define $\tilde{x}_k := \frac{x_k}{1+\varepsilon}$ and $\tilde{y}_k := \frac{y_k}{1+\varepsilon}$ then $\|\tilde{x}_k - \tilde{y}_k\| \geqslant \frac{2}{1+\varepsilon}$ and

$$\left\| f - \frac{1}{n}\sum_{k=1}^{n} \frac{\tilde{x}_k + \tilde{y}_k}{2} \right\| < \frac{4}{n} + \varepsilon.$$

The arbitrariness of $f \in B_{\mathrm{Lip}_0(M)}$ reveals that

$$C_n^{\frac{2}{1+\varepsilon}}(\mathrm{Lip}_0(M)) \leqslant \frac{4}{n} + \varepsilon.$$

From here the uniform slice-D2P of $\mathrm{Lip}_0(M)$ follows. Indeed, given $0 < \alpha < 2$ and $\delta > 0$, find $m \in \mathbb{N}$ such that $\frac{5}{n} < \delta$ holds for every $n \geqslant m$. Furthermore, we can find $\varepsilon > 0$ small enough to guarantee $\frac{2}{1+\varepsilon} > \alpha$ (consequently $C_n^\alpha(\mathrm{Lip}_0(M)) \leqslant C_n^{\frac{2}{1+\varepsilon}}(\mathrm{Lip}_0(M))$ holds for every $n \in \mathbb{N}$) and $\varepsilon < \frac{1}{n}$. Now, given $n \geqslant m$, we get

$$C_n^\alpha(\mathrm{Lip}_0(M)) \leqslant C_n^{\frac{2}{1+\varepsilon}}(\mathrm{Lip}_0(M)) \leqslant \frac{4}{n} + \varepsilon < \frac{5}{m} < \delta.$$

Summarising we have proved that, given any $0 < \alpha < 2$ and any $\delta > 0$ there exists $m \in \mathbb{N}$ such that $C_n^\alpha(\mathrm{Lip}_0(M)) < \delta$ holds for every $n \geqslant m$. Consequently, $\mathrm{Lip}_0(M)$ has the uniform slice-D2P.

Remark 3.14 We want to point out that, in the paper [13], the author considers the Banach space quotient $\mathrm{Lip}(M)$ resulting from considering the space of all the Lipschitz functions over M when endowed with the classical seminorm

$$L(f) := \sup_{x,y \in M; x \neq y} \frac{f(x) - f(y)}{d(x,y)}.$$

However, it is well known that $\mathrm{Lip}(M)$ and $\mathrm{Lip}_0(M)$ are isometrically isomorphic Banach spaces regardless the choice of distinguished point $0 \in M$ (c.f. e.g. [30, p. 36]).

We end the section by exhibiting another example with the uniform slice-D2P. Throughout the rest of the section we will consider uniform algebras over the scalar field $\mathbb{C}$. Let us introduce some notation used in [23]. Recall that a *uniform algebra over a compact Hausdorff topological space* K is a closed subalgebra $X \subseteq \mathcal{C}(K)$, where $\mathcal{C}(K)$ is the space of all the continuous functions $f : K \longrightarrow \mathbb{K}$, which separates the points of K and contains the constant functions.

Given a uniform algebra on a compact space K, a point $x \in K$ is said to be a *strong boundary point* if, for every neighbourhood V of x and every $\delta > 0$, there exists $f \in S_X$ such that $f(x) = 1$ and $|f| < \delta$ on $K \setminus V$. The *Silov boundary* of X, denoted by ∂_X following the notation of [9], is the closure of the set of all strong boundary points. A fundamental result of the theory of uniform algebras is that X can be indentified with a uniform algebra on its Silov boundary (see [23]). This fact allows us to assume, with no loss of generality, that the set of strong boundary points of X is dense in K.

Now we get the following example.

Example 3.15 Let X be an infinite-dimensional uniform algebra. Then X has the uniform slice-D2P.

Observe that in the proof of [23, Theorem 1] the following is proved: given $f \in B_X$, a strong boundary point $x_0 \in K$, an open neighbourhood V of x_0 in K and $\delta > 0$ there exists $g \in B_X$ and $\varphi \in S_X$ such that

1. $|g(t)| < \delta$ holds for every $t \in K \setminus V$.
2. $h := f(1-g)$ satisfies $\|h\| \leqslant 1 + 3\delta$.
3. $\|h \pm \varphi\| \leqslant 1 + 4\delta$.

Let us concluye the uniform slice-D2P from the above construction. In order to do so, let $f \in B_X$.

Since X is infinite-dimensional we conclude that K is infinite, so we can take a sequence of pairwise disjoint open sets $\{V_n\}$ in K. By the density of the the set of strong boundary points we can take one such point $t_n \in V_n$ for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $\delta > 0$. Given $1 \leqslant k \leqslant n$ consider g_k, h_k, φ_k (associated to the strong boundary x_k and the open set V_k) as exposed above and define

$$a_k := \frac{h_k + \varphi_k}{1+4\delta}; \, b_k := \frac{h_k - \varphi_k}{1+4\delta}.$$

It is clear (by (3)) that $a_k, b_k \in B_X$ and, moreover,

$$\|a_k - b_k\| = \frac{2\|\varphi_k\|}{1+4\delta} = \frac{2}{1+4\delta}.$$

Hence $z := \frac{1}{n}\sum_{k=1}^{n} \frac{a_k+b_k}{2} = \frac{1}{n}\sum_{k=1}^{n} \frac{h_k}{1+4\delta} = \frac{\frac{1}{n}\sum_{k=1}^{n} h_k}{1+4\delta} \in S_n^{\frac{2}{1+4\delta}}(X)$. Let us estimate $\|f - z\|$, for which we will estimate first $\|f - (1+4\delta)z\|$. Observe that

$$f - (1+4\delta)z = f - \frac{1}{n}\sum_{k=1}^{n} h_k = \frac{1}{n}\sum_{k=1}^{n} f - h_k = \frac{1}{n}\sum_{k=1}^{n} f - f(1-g_k) = \frac{1}{n}\sum_{k=1}^{n} f g_k.$$

In order to estimate $\|f - (1+4\delta)z\|$ select $t \in K$. Since $V_i \cap V_j = \emptyset$ if $i \neq j$ we get that $t \notin V_k$ for all $k \in \mathbb{N}$ except, at most, for one k_0. Anyway, for every $k \neq k_0$ we get $t \notin V_k$, which in turn implies $|g_k(t)| < \delta$ (by (1)). Consequently

$$\begin{aligned} |f(t) - (1+4\delta)z(t)| &= \left| \frac{1}{n}\sum_{k=1}^{n} f(t)g_k(t) \right| \leqslant \frac{1}{n}\sum_{k=1}^{n} |f(t)||g_k(t)| \\ &\leqslant \frac{1}{n}\sum_{k=1}^{n} |g_k(t)| = \frac{1}{n}\left(|g_{k_0}(t)| + \sum_{k \neq k_0} |g_k(t)| \right) \\ &< \frac{1}{n}(1 + (n-1)\delta)) \leqslant \frac{1}{n} + \delta. \end{aligned}$$

The arbitrariness of $t \in K$ implies that $\|f - (1+4\delta z)\| \leqslant \frac{1}{n} + \delta$, so $\|f - z\| \leqslant \frac{1}{n} + 5\delta$. The arbitrariness of $f \in B_X$ implies that

$$C_n^{\frac{2}{1+4\delta}}(X) \leqslant \frac{1}{n} + 5\delta.$$

A reasoning similar to that of the end of Example 3.13 concludes that X has the uniform slice-D2P, as desired.

4 A Daugavet space failing the uniform slice-D2P

The aim of this section is to construct a Banach space X with the Daugavet property satisfying that $(X)_{\mathcal{U}}$ fails the slice-D2P for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. In order to do so, we will follow the construction of a Banach space X with the Daugavet property satisfying that $(X)_{\mathcal{U}}$ fails the Daugavet property from [19]. Our example will be a particular case of this example by a suitable choice of scalar sequence (see below). The above mentioned construction of [19] is in turn based on a construction of [8] of a space failing the Radon-Nikodym property but where every uniformly bounded dyadic martingale converges. We will consider in this section the scalar field $\mathbb{R}$.

In the sequel we will follow word-by-word the construction of [19, Section 2]. We denote by $L_1 := L_1(\Omega, \Sigma, \mu)$ over a separable non-atomic measure space, and we will denote by $\|\cdot\|$ the canonical norm on L_1 throughout the section. We also consider the topology of convergence in measure, which is the one generated by the metric

$$d_m(f, g) := \inf \{\varepsilon > 0 : \mu\{t : |f(t) - g(t)| \geqslant \varepsilon\} \leqslant \varepsilon\}.$$

Observe that, given $f, g \in L_1$ it is immediate that $d_m(f, g) = d_m(f - g, 0)$. Consequently,

$$\begin{aligned} d_m(f + g, 0) \leqslant d_m(f + g, g) + d_m(g, 0) &= d_m(f + g - g, 0) + d_m(g, 0) \\ &= d_m(f, 0) + d_m(g, 0) \end{aligned}$$

and, inductively, $d\left(\sum_{i=1}^n f_i, 0\right) \leqslant \sum_{i=1}^n d_m(f_i, 0)$ holds for every $f_1, \ldots, f_n \in L_1$. It is also easy to prove that given $f \in L_1$ and given $\lambda \in [0, 1]$ it follows that $d_m(\lambda f, 0) \leqslant d_m(f, 0)$.

The following result, based on an argument of disjointness of supports of functions in L_1, will be used in the future. For a complete proof we refer to [19, Lemma 2.1].

Lemma 4.1 *Let H be a uniformly integrable subset of L_1 and $\varepsilon > 0$. Then there exists $\delta > 0$ such that, if $g \in H$ and $f \in L_1$ satisfies $d_m(f, 0) < \delta$ then*

$$\|f + g\| \geqslant \|f\| + \|g\| - \varepsilon.$$

The following lemma, whose proof can be found in [6, Lemma 5.26], is essential in the future construction.

Lemma 4.2 *Let $0 < \varepsilon < 1$. Then there exists a function $f \in L_1([0, 1])$ such that*

1. *$f \geqslant 0$, $\|f\| = 1$ and $\|f - \mathbf{1}\| \geqslant 2 - \varepsilon$.*
2. *Let $\{f_j\}$ be a sequence of independent random variables with the same distribution as f. If $g \in \overline{\text{span}}\{f_j\}$ with $\|g\| \leqslant 1$ then there exists a constant function c with $d_m(g, c) \leqslant \varepsilon$.*
3. *$\left\| \frac{1}{n} \sum_{j=1}^n f_j - \mathbf{1} \right\| \to 0$ as $n \to \infty$.*

In the lemma and in the construction below we consider (Ω, Σ, μ) as the product of countably many copies of the measure space $[0, 1]$.

We say that a subspace X of L_1 *depends on finitely many coordinates* if all $f \in X$ are functions depending on a finite common set of coordinates.

Now we consider the following lemma, which appears in [19] (see [19, Lemma 2.4] for a proof).

Lemma 4.3 *Let G be a finite dimensional subspace of L_1 that depends on finitely many coordinates. Let $\{u_k\}_{k=1}^m \subseteq S_G$ and $\varepsilon > 0$. Then there exists a finite dimensional subspace F of L_1 containing G and depending on finitely many coordinates and there exist $n \in \mathbb{N}$ and functions $\{v_{k,j}\}_{k \leqslant m, j \leqslant n}$ such that:*

1. *$\|u + v_{k,j}\| \geqslant 2 - \varepsilon$ holds for every $u \in S_G$ and all $k \leqslant m$ and $j \leqslant n$,*
2. *$\left\| u_k - \frac{1}{n} \sum_{j=1}^n v_{k,j} \right\| \leqslant \varepsilon$ for every k,*
3. *For every $\varphi \in B_F$ there exists $\psi \in B_G$ with $d_m(\varphi, \psi) \leqslant \varepsilon$.*

Now we will make the construction of the space. Fix a decreasing sequence (ε_N) of positive numbers with $\sum_{j=N+1}^{\infty} \varepsilon_j < \varepsilon_N$ for all $N \in \mathbb{N}$ and select inductively finite-dimensional subspaces of L_1,

$$\operatorname{span} \mathbf{1} = E_1 \subset E_2 \subset E_3 \subset \ldots,$$

each of them depending on finitely many coordinates, ε_N-nets $\{u_k^N\}_{k=1}^{m(N)}$ of S_{E_N} and collections of elements $\{v_{k,j}^N\}_{k \leqslant m(N), j \leqslant n(N)}$ in such a way that the conclusion of Lemma 4.3 holds with $\varepsilon = \varepsilon_N$, $G = E_N$, $F = E_{N+1}$, $\{u_k\}_{k=1}^m = \{u_k^N\}_{k=1}^{m(N)}$, $\{v_{k,j}\}_{k \leqslant m, j \leqslant n} = \{v_{k,j}^N\}_{k \leqslant m(N), j \leqslant n(N)}$. Denote $E := \overline{\bigcup_{N=1}^{\infty} E_N}$.

The above space E satisfies the following properties, obtained from [19, Theorem 2.5].

Theorem 4.4 *The space E constructed as above satisfies the following properties:*

1. *E has the Daugavet property,*
2. *For every $f \in B_E$ and every $N \in \mathbb{N}$ there exists $g \in B_{E_N}$ satisfying that $d_m(f, g) < \varepsilon_N$,*
3. *E has the Schur property.*

In [19, Theorem 3.3] the authors make use of the above space in order to construct a Banach space X with the Daugavet property such that $(X)_{\mathcal{U}}$ fails the Daugavet property for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. In the following, we will make use of many of their ideas in order to prove the following theorem.

Theorem 4.5 *Let $n \in \mathbb{N}$ and $\eta > 0$. There exists a Banach space X with the Daugavet property such that*

$$C_n^{2\eta}(X) \geqslant \frac{\eta}{8}.$$

Proof Select $\delta > 0$ small enough so that

$$5\delta < \frac{\eta}{2}.$$

Let X be the space of Theorem 4.4 with $\varepsilon_1 > 0$ small enough to satisfy that given any constant function $g \in [-2, 2]$ (i.e. $g \in E_1$) and $f \in L_1$, the condition $d_m(f, 0) < 2n\varepsilon_1$ implies

$$\|f + g\| \geqslant \|f\| + \|g\| - \delta. \tag{4.1}$$

Our aim is to prove that

$$d\left(\mathbf{1}, S_n^{2\eta}(X)\right) \geqslant \frac{\eta}{8}. \tag{4.2}$$

In order to do so take $z \in S_n^{2\eta}(X)$. Then $z = \sum_{k=1}^n \lambda_k z_k$ with $z_k \in S^{2\eta}(X)$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum_{k=1}^n \lambda_k = 1$. Moreover, since $z_k \in S^{2\eta}(X)$ it follows that $z_k = \frac{u_k + v_k}{2}$ with $u_k, v_k \in B_X$ satisfying that $\|u_k - v_k\| \geqslant 2\eta$ holds for every $1 \leqslant k \leqslant n$. Now given k, the triangle inequality implies

$$2\eta \leqslant \|u_k - \mathbf{1} + \mathbf{1} - v_k\| \leqslant \|\mathbf{1} - u_k\| + \|\mathbf{1} - v_k\|.$$

The above inequality implies that either $\|\mathbf{1} - u_k\| \geqslant \eta$ or $\|\mathbf{1} - v_k\| \geqslant \eta$. Assume, up to a relabeling, that $\|\mathbf{1} - u_k\| \geqslant \eta$ holds for every $1 \leqslant k \leqslant n$.

Now, given $1 \leqslant k \leqslant n$ apply (2) of Theorem 4.4 (applied to $f = u_k$ and v_k respectively and $N = 1$) to find constant functions $\alpha_k, \beta_k \in [-1, 1]$ satisfying $d_m(u_k, \alpha_k) < \varepsilon_1$ and $d_m(v_k, \beta_k) < \varepsilon_1$.

Now, given $1 \leqslant k \leqslant n$, we have

$$1 \geqslant \|u_k\| = \|\alpha_k + (u_k - \alpha_k)\| \geqslant |\alpha_k| + \|u_k - \alpha_k\| - \delta$$

since α_k is a constant function and $d_m(u_k - \alpha_k, 0) = d_m(u_k, \alpha_k) < \varepsilon_1 < 2n\varepsilon_1$, so the inequality (4.1) holds. Now

$$\begin{aligned} 1 \geqslant |\alpha_k| + \|u_k - \mathbf{1} + \mathbf{1} - \alpha_k\| - \delta &\geqslant |\alpha_k| + \|\mathbf{1} - u_k\| - |1 - \alpha_k| - \delta \\ &= |\alpha_k| + \|\mathbf{1} - u_k\| - (1 - \alpha_k) - \delta, \end{aligned}$$

where the last equality follows since $\alpha_k \leqslant 1$. Taking into account that $\|\mathbf{1} - u_k\| \geqslant \eta$ the above inequality implies

$$1 \geqslant |\alpha_k| + \eta - (1 - \alpha_k) - \delta = |\alpha_k| + \alpha_k + \eta - 1 - \delta \geqslant 2\alpha_k - 1 + \eta - \delta.$$

Consequently

$$2\alpha_k \leqslant 2 - \eta + \delta \Rightarrow \alpha_k \leqslant \frac{2 - \eta}{2} + \frac{\delta}{2}.$$

Since $\beta_k \in [-1, 1]$ holds for every k we get

$$\sum_{k=1}^n \lambda_k \frac{\alpha_k + \beta_k}{2} \leqslant \frac{\frac{2-\eta}{2} + \frac{\delta}{2} + 1}{2} = \frac{4 - \eta + \delta}{4}. \tag{4.3}$$

Now

$$d_m\left(z-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2},0\right)=d_m\left(\sum_{k=1}^{n}\frac{\lambda_k}{2}(u_k-\alpha_k+v_k-\beta_k),0\right)$$
$$\leqslant\sum_{k=1}^{n}d_m(u_k-\alpha_k,0)+d_m(v_k-\beta_k,0)<2n\varepsilon_1.$$

If we apply (4.1) to the constant function $\mathbf{1}-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}$ and the function $z-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}$, which is $2n\varepsilon_1$ close to 0 with respect to the distance d_m, we obtain

$$\begin{aligned}\|\mathbf{1}-z\|&=\left\|\left(\mathbf{1}-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}\right)-\left(z-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}\right)\right\|\\&\geqslant\left\|\mathbf{1}-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}\right\|+\left\|z-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}\right\|-\delta\\&\geqslant\left\|\mathbf{1}-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}\right\|-\delta\\&\geqslant 1-\sum_{k=1}^{n}\lambda_k\frac{\alpha_k+\beta_k}{2}-\delta\overset{(4.3)}{\geqslant}1-\frac{4-\eta+\delta}{4}-\delta=\frac{\eta-5\delta}{4}>\frac{\eta}{8}.\end{aligned}$$

Now the result follows by the arbitrariness of $z\in S_n^{2\eta}(X)$. □

Let $\eta>0$ and, for every $n\in\mathbb{N}$, consider X_n as the Banach space claimed in Theorem 4.5, and consider $X=\left(\bigoplus_{n=1}^{\infty}X_n\right)_1$. X has the Daugavet property as it is an ℓ_1-sum of Banach spaces with the Daugavet property [32, Theorem 1]. Let $r>0$ small enough to guarantee $2r<\eta$ and $\frac{r^2}{4}+r<\frac{\eta}{8}$. We claim that, given $n\in\mathbb{N}$, we get that

$$d\left((0,0,0,\ldots,\underbrace{\mathbf{1}}_{n},0,0,\ldots),S_n^{3\eta}(X)\right)\geqslant\frac{r^2}{4}.$$

In order to prove it write $x:=(0,0,0,\ldots,\underbrace{\mathbf{1}}_{n},0,0,\ldots)$ and assume by contradiction that there is $z\in S_n^{3\eta}(X)$ such that $\|x-z\|<\left(\frac{r}{2}\right)^2$. Consequently

$$\|\mathbf{1}-z(n)\|=\|x(n)-z(n)\|\leqslant\sum_{k=1}^{\infty}\|x(k)-z(k)\|=\|x-z\|<\left(\frac{r}{2}\right)^2.$$

If we write $z = \sum_{i=1}^{n} \lambda_i z_i$ for $0 \leqslant \lambda_i \leqslant 1$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $z_i \in S^{3\eta}(X)$, we obtain from the above inequality that $\left\| \sum_{i=1}^{n} \lambda_i z_i(n) \right\| > 1 - \left(\frac{r}{2}\right)^2$. Set

$$G := \left\{ i \in \{1, \ldots, n\} : \|z_i(n)\| > 1 - \frac{r}{2} \right\}.$$

We claim that $\sum_{i \notin G} \lambda_i < \frac{r}{2}$. Indeed,

$$\begin{aligned} 1 - \left(\frac{r}{2}\right)^2 < \sum_{i=1}^{n} \lambda_i \|z_i(n)\| &= \sum_{i \in G} \lambda_i \|z_i(n)\| + \sum_{i \notin G} \lambda_i \|z_i(n)\| \\ &\leqslant \sum_{i \in G} \lambda_i + \sum_{i \notin G} \lambda_i \left(1 - \frac{r}{2}\right) \\ &= 1 - \frac{r}{2} \sum_{i \notin G} \lambda_i, \end{aligned}$$

from where $\sum_{i \notin G} \lambda_i < \frac{r}{2}$ follows.

On the other hand, since $z_i \in S^{3\eta}(X)$ then for $1 \leqslant i \leqslant n$ there are $u_i, v_i \in B_X$ with $z_i = \frac{u_i + v_i}{2}$ and $\|u_i - v_i\| > 3\eta$. Given $i \in G$ we have $\|z_i(n)\| > 1 - \frac{r}{2}$, from where

$$1 - \frac{r}{2} < \frac{\|u_i(n) + v_i(n)\|}{2} \leqslant \frac{\|u_i(n)\| + \|v_i(n)\|}{2},$$

and an easy convexity argument implies $\|u_i(n)\| > 1 - r$ and $\|v_i(n)\| > 1 - r$. Consequently, given $i \in G$ we have

$$1 - r < \|u_i(n)\| \leqslant \|u_i(n)\| + \sum_{k \neq n} \|u_i(k)\| = \|u_i\| \leqslant 1,$$

from where $\sum_{k \neq n} \|u_i(k)\| < r$. Similarly $\sum_{k \neq n} \|v_i(k)\| < r$. Since $\|u_i - v_i\| > 3\eta$ and $2r < \eta$ we obtain

$$3\eta < \|u_i(n) - v_i(n)\| + \sum_{k \neq n} (\|u_i(k)\| + \|v_i(k)\|) \leqslant \|u_i(n) - v_i(n)\| + 2r,$$

so $\|u_i(n) - v_i(n)\| > 3\eta - 2r > 2\eta$ holds for every $i \in G$. Set $\lambda := 1 - \sum_{i \in G} \lambda_i$ and set $z' := \sum_{i \in G} \lambda_i z_i + \lambda z$ where $z = z_{i_0}$ for any $i_0 \in G$. We clearly get that $z'(n) = \sum_{i \in G} \lambda_i \frac{u_i(n) + v_i(n)}{2} + \lambda \frac{u_{i_0}(n) + v_{i_0}(n)}{2}$ where $\|u_i(n) - v_i(n)\| > 2\eta$ holds for every $i \in G$ and $\|u_{i_0}(n) - v_{i_0}(n)\| > 2\eta$. This means $z'(n) \in S_n^{2\eta}(X_n)$. By (4.2) we obtain

$$\|\mathbf{1} - z'(n)\| \geqslant \frac{\eta}{8}.$$

Consequently

$$\begin{aligned}\frac{\eta}{8} &\leqslant \|x(n) - z'(n)\| \leqslant \|x - z'\| \leqslant \|x - z\| + \|z' - z\| \\ &\leqslant \frac{r^2}{4} + \sum_{i \notin G} \lambda_i \left\| z_i - \frac{u_{i_0} + v_{i_0}}{2} \right\| < \frac{r^2}{4} + r < \frac{\eta}{8},\end{aligned}$$

a contradiction.

This proves that for every $n \in \mathbb{N}$ it follows

$$C_n^{3\eta}(X) \geqslant \frac{r^2}{4}.$$

According to Theorem 3.1 we have proved the following result.

Theorem 4.6 *For every $\varepsilon > 0$ there exists a Banach space X with the Daugavet property such that, for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, the space $(X)_{\mathcal{U}}$ has a slice of diameter smaller than or equal to ε.*

Proof Given $\varepsilon > 0$ select $0 < \eta < \frac{\varepsilon}{3}$, and choose $r > 0$ small enough to guarantee $2r < \eta$ and $\frac{r^2}{4} + r < \frac{\eta}{8}$. We have proved that there exists a Banach space X such that $C_n^{3\eta}(X) \geqslant \frac{r^2}{4}$. According to Theorem 3.1 and Remark 3.3 this means that given any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$ there exists a slice of $B_{(X)_{\mathcal{U}}}$ of diameter smaller than 3η. Since $3\eta < \varepsilon$ the conclusion follows. □

Acknowledgements The author is deeply grateful with the anonymous referees whose advice helped to improve by far the exposition of the results. This work was supported by MCIN/AEI/10.13039/501100011033: Grant PID2021-122126NB-C31, Junta de Andalucía: Grant FQM-0185, by Fundación Séneca: ACyT Región de Murcia: Grant 21955/PI/22 and by Generalitat Valenciana: Grant CIGE/2022/97.

Funding Funding for open access publishing: Universidad de Granada/CBUA.

Data Availability Not applicable.

Declaration

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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