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Juggling Balls and Mathematics: An Ethnomathematical Exploration

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Abstract: Ethnomathematics, as a field of study, promotes recognizing the diversity in ways of thinking and doing mathematics, challenging the hierarchies and exclusions typical of traditional mathematics education. This research explores the practice of juggling, specifically analyzing three-ball juggling sequences to uncover the mathematical structures and patterns embedded in this ancient art form. In a social association during a workshop, two jugglers and seven juggling learners interact with one of the researchers, a mathematics educator, to co-construct a shared model establishing a symmetrical dialogue based on the Alangui's principles of "mutual interrogation" between the practice of juggling and the domain of mathematics. The knowledge exchange process is envisioned as a "barter" where both the mathematics educator and the jugglers contribute their unique perspectives to generate new and hybrid understandings. With a qualitative approach, from the analysis of the data collected during the ethnographic field work (notes, audiovisual recordings) emerges how the initial model, created by mathematicians and jugglers, was reinterpreted to better align with the cultural community's practice. The research revealed that juggling serves as a concrete context for exploring abstract mathematical concepts and that mathematical analysis of juggling sequences helps jugglers gain a deeper understanding of underlying structures, enhancing their creativity. The hybrid model developed in this study offers a promising resource to integrating ethnomathematical perspectives into formal mathematics education, fostering a more situated and engaging learning experience for students.

Keywords: model; juggling; ethnomathematics; local knowledge

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1. Introduction

The reform of the mathematics curriculum is a central topic in the international debate on mathematics education. The 2024 report by the OECD's Future of Education and Skills 2030 project presents an international analysis of mathematics curricula, outlining 12 policy implications for reform: focus, rigor, coherence, transferability, interdisciplinarity, choice, authenticity, flexibility, alignment, engagement, student agency, and teacher agency (OECD, 2024).

Additionally, the 24th ICMI (International Commission on Mathematical Instruction) study highlights that mathematics curriculum reform is influenced by various educational movements, such as realistic mathematics education, critical mathematics education, and ethnomathematics. These movements, along with discussions on social justice and equity, have shaped changes in teaching methods. These influences are evident, both explicitly and implicitly, in recent reforms (Shimizu & Vithal, 2023).

Ethnomathematics promotes the recognition of diversity in mathematical thinking and practice, challenging traditional hierarchies and exclusions in mathematics education. It focuses on how different cultural communities develop and use mathematical ideas, examining the reasons and methods behind the creation and application of mathematical concepts. Among its educational implications, ethnomathematics can enhance cultural awareness in the classroom, fostering respect and appreciation for diverse cultural knowledge. Research in ethnomathematics aids teachers in identifying mathematical learning opportunities within students' cultural backgrounds, experiences, and interests (Albanese et al., 2017). This equips educators with the tools to develop authentic, culturally contextualized tasks, enabling students to connect mathematics with real-life situations (Chavarría-Arroyo & Albanese, 2023). Such an approach promotes a broader, more democratic, and socially engaging perspective on mathematical thinking. Juggling, an ancient art with roots in diverse cultures, provides a rich context for exploring ethnomathematics. While often associated with the circus, juggling has deep connections to mathematics and physics.

Siteswap notation, as a representation of juggling tricks, began as a mathematical curiosity and found interest among juggling mathematicians. However, its use is not yet widespread within the juggling community. Therefore, our goal is to bridge the gap between these two worlds. The use of juggling in mathematics education has been explored through projects like the NSF (National Science Foundation)-funded Engaged Learning through Creativity in Mathematics and Science. The project aimed to use juggling to teach mathematical concepts and engage students in a new way of learning, with the goal of promoting creativity by showing the presence of mathematics "everywhere" and encouraging students to discover links among juggling, math, and other subjects on their own (Monahan et al., 2020).

Our research, instead, focuses on the mathematical aspects that interest both jugglers and mathematics educators. This approach aims to build a model with input from a group of non-mathematical jugglers and a mathematics educator.

This paper discusses the potential of viewing juggling from a mathematical point of view and of viewing mathematics from a juggler's point of view, focusing on three-ball juggling sequences.

The research questions are the following:

How can interaction, dialogue, and the exchange of ideas between the juggling community and mathematics educators lead to new understandings and hybrid models of practices?

How can the hybrid model co-constructed by mathematicians and jugglers bridge the gap between abstract mathematical concepts and practical, culturally rooted practices?

What impact does mathematical analysis of juggling have on the creativity and understanding of jugglers regarding their own practice?

What opportunities does juggling offer for mathematics education?

Specifically, the objective of the study is to co-construct a shared model between the community and academia, establishing a symmetrical dialogue between the practice of juggling and the domain of mathematics.

Through the lens of ethnomathematics, this research seeks to shed light on the mathematical structures embedded in the art of juggling and how this understanding can enhance both juggling practice and mathematics education.

1.1. Ethnomathematics

Ethnomathematics focuses on how humans develop and use mathematics, examining the reasons and methods behind creating and applying mathematical concepts. It promotes a creative exploration of how different cultural communities utilize mathematical ideas (Rosa & Orey, 2011).

Cultural groups are entities marked by common objects and traditions within the group. They might encompass urban and rural communities, labor unions, professional classes, children of a particular age, indigenous societies, and others (D'Ambrosio, 1985).

The focus of this perspective is essentially a critical analysis of knowledge generation and production (creativity) and forms an intellectual process for its production, the social mechanisms of institutionalization of knowledge (academic world), and its transmission (education).

Both Meaney and Parra recognize the importance of decolonizing math education and promoting a more inclusive view of knowledge (Meaney et al., 2021; Parra, 2017). Both argue that traditional approaches to ethnomathematics, which focus on identifying mathematical aspects within cultural practices, can perpetuate the view of Western mathematics as superior.

Parra criticizes the "intersectional" approach to ethnomathematics, which tries to find the intersection between mathematics and culture, arguing that this approach leads to a false dilemma: cultural practices either are seen as a form of primitive mathematics or are dismissed as non-mathematical. Instead, Parra proposes an "interactional" approach, which focuses on creating connections between mathematics and culture through dialogue and the exchange of ideas. This approach recognizes that both mathematics and culture are dynamic, evolving systems and that their meeting can lead to the creation of new knowledge.

Parra's interactional approach is closely linked to Meaney's cultural symmetry model, which emphasizes the need to balance mathematical knowledge and cultural knowledge in the educational process. The first step of Meaney's model, which focuses on the intrinsic value of cultural practices, is particularly important in this context; it ensures that cultural practices are not simply seen as a means to teach Western mathematics but are valued for their own merits.

Both Parra's interactional approach and Meaney's cultural symmetry model recognize the importance of dialogue and collaboration between mathematicians and cultural professionals. This dialogue is essential for creating meaningful connections between mathematics and culture and for the creation of new knowledge. The approach of reciprocal interrogation (N. A. Adam, 2010) provides a useful framework for structuring this dialogue. Cross-questioning, with its emphasis on critical confrontation between different systems of knowledge, can help to challenge hypotheses and promote a deeper understanding of both mathematics and culture.

Parra uses the barter metaphor to describe this process of exchanging ideas and creating new knowledge. Just like in a barter, both parties bring something of value and both benefit from it. Western mathematics can provide new tools and perspectives for understanding cultural practices, while cultural practices can challenge existing assumptions about mathematics and inspire new ideas.

Finally, it is worth noting that both Parra and Meaney view ethnomathematics as a political project. The aim is not simply to identify mathematics in other cultures, but to decolonize mathematical education and promote a more just and fair point of view of knowledge.

Mutual interrogation is a methodology for ethnomathematical investigation that sets two systems of knowledge in parallel to explore their similarities and differences, enhancing and transforming each other. It involves a critical dialogue between a cultural practice and mathematics, drawing parallels and questioning assumptions about mathematics while exploring alternative conceptions. The process involves examination, struggle, perceptual shift, and transformation, but is not linear. Uncertainty is integral to the process, enabling shifts in perception about mathematics and alternative conceptions. Ideally, practitioners from both knowledge domains have equal opportunities to interrogate each other through the researcher, gaining an enhanced understanding of the other's practice. The ultimate goal is for practitioners from both domains to have their views altered in significant ways (A. Adam et al., 2014).

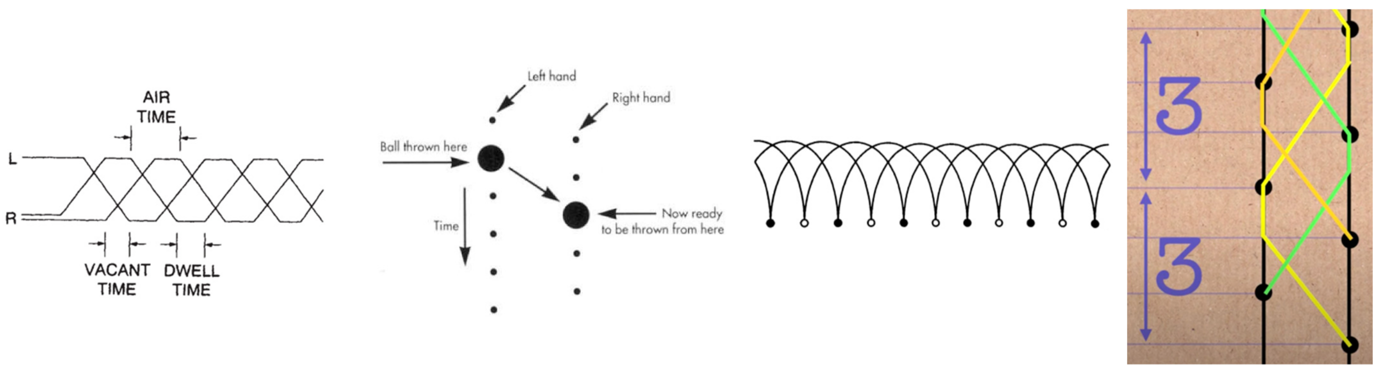
1.2. Juggling and Its (Mathematical) Representation

Juggling dates to around 2000 B.C., with evidence found across various cultures, including the Pacific Islands and the Aztec Empire. While often associated with circus performances, juggling also has deep connections to mathematics and physics. Mathematical notation can be used to describe juggling patterns, with new patterns discovered through mathematical research (Naylor, 2012).

Juggling also has applications in fields like graph theory (Lundmark, 2004), combinatorics (Buhler & Graham, 2004), and modular mathematics (Butler, 2010). In the early 1980s, detailed juggling diagrams and mathematical principles began emerging from computer science research (Shannon, 1980; Walker & Francisco, 1982).

A space–time diagram serves as a two-dimensional visual depiction of a juggling sequence. When a juggler moves forward while juggling, a specific pattern can be observed from an overhead perspective. Time, conceptualized as beats of a metronome, plays a fundamental role in this pattern, known as the time–space diagram. Claude Shannon and Jeff Walker were the first individuals known to employ diagrams for notating juggling patterns, doing so in the early 1980s (Shannon, 1980; Walker & Francisco, 1982).

In Figure 1, different models are shown: (a), (b), and (d) are named ladder diagrams for their typical shape. In (a), the time axis runs from the left to the right, and is a common model of time in mathematical and physics functions (Lewbel, 1994), with the ladder having a horizontal orientation. In (b), the time axis runs from the top to the bottom (mainly the first models from the mid-1980s to the early 1990s), while in (d), there is a vertical orientation from the bottom to the top. Finally, in (c), hands move alternately at each beat; the black (white) circle is the right (left) hand and the empty circle is the left (right) one. The time axis run from the left to the right.



(a) (b) (c) (d)

Figure 1. Different space–time diagrams to represent juggling: (**a**) source (Shannon, 1980); (**b**) source (Tiemann & Magnusson, 1991); (**c**) source (Polster, 2002); (**d**) source (Wright, 2017).

During this research, the (d) ladder diagram was chosen by R1 as the initial model, and the Numberphile video (Wright, 2017)[1] was used as a clarifying resource during the field work.

In this diagram,

- The vertical axis symbolizes time, with the "rungs" of the ladder delineating evenly spaced beats within the pattern.
- Lines in the diagram represent the paths that props take through time and space.
- The number of paths that intersect any horizontal line on the diagram represents the number of props in the pattern.
- The sides of the diagram indicate whether throws and catches are made with the right or left hand.

The Siteswap notation, which originated in the late 1980s, is a numerical system used to represent juggling patterns, with each number indicating the timing of throws. Siteswap was independently developed by groups from Caltech, UC Santa Cruz, and the University of Cambridge, and has since been the subject of multiple research papers (Magnusson & Tieman, 1989; Polster, 2002). These developers were mathematicians and computer scientists who also engaged in (sometimes high-level) juggling in their universities' recreational juggling clubs. The term "Siteswap" was used for the first time in a Juggler's World magazine article (Tiemann & Magnusson, 1991).

Despite the mentioned publications, the ladder diagram and the Siteswap notation have remained a curiosity within academic circles, without achieving widespread adoption in the juggling community. For this reason, in this study, the notation/model is regarded as a proposal originating from the academic world.

Barton (1999) introduced the term "QRS" (quantitative, relational, or spatial aspects of human experience) to describe the systems that people use to deal with quantity, relationships, and space, providing a useful framework for redefining what is considered mathematics from the ethnomathematical perspective. QRS systems represent the wide range of mathematical practices found in different cultures, as opposed to NUC (near-universal, conventional mathematics) systems, which refer to conventional academic mathematics. In other words, according to Barton, when you handle quantities, consider space, and/or establish relationships, the constructed system of meanings should be considered mathematical (Barton, 1999).

Siteswap notation implies spatial relationships through the alternation of throws, which can be further clarified by breaking down a juggling diagram like a ladder into the paths of individual objects. Based on this information, it can be inferred how a ladder diagram functions as a QRS system:

- Quantity: A ladder diagram represents quantity through the number of props involved in the pattern, which can be seen by counting the number of lines that intersect any horizontal line on the diagram. The number of rungs crossed by each path also relates to the quantity of time that a prop spends in the air.
- Relationships: The diagram shows relationships through the connections between throws and catches and the order in which these occur in time. The paths of the props show how throws relate to each other, in terms of both timing and which hand is used. The ladder diagram's structure shows the sequential relationships between throws and catches.
- Space: The spatial aspect is shown by the paths of the props, which represent movement through time and space (although the vertical dimension of physical patterns is ignored). The alternation of the throws between the hands and how paths intersect also imply spatial relationships.

Thus, it can be seen that the ladder diagram functions as a QRS system that complements the quantitative, relational, and spatial elements of Siteswap notation within the cultural context of juggling.

The Siteswap notation and the ladder diagram of each pattern highlights several crucial aspects of QRS systems. They are both culture-specific, having been developed by a small group of jugglers, who were also mathematical and computer scientists, to describe and analyze their tricks, reflecting the unique needs and practices of this community. As tools for making sense of the world, juggler mathematicians use them to understand and communicate juggling patterns, enabling them to analyze, create, and share complex sequences.

Moreover, Siteswap has evolved and changed over time; with the advent of new techniques and juggling styles, the notation has expanded to include new concepts. For example, variants have been developed to describe multiplex juggling, where more than one ball is in the hand at the time of the throw, and collaborative juggling, where multiple people throw objects. Even new ladder diagrams can be developed, depending on the context in which they are used.

In conclusion, the ladder diagrams and the Siteswap notation, as systems for quantifying, relating, and implying spatiality within the cultural context of juggling, can be considered concrete examples of a QRS system.

2. Materials and Methods

The research was conducted using an ethnographic, participatory, and symmetrical methodological approach, similar to those employed by ethnomathematicians in studying the relationship between mathematics and cultures (Coppe & Mesquita, 2015; Parra & Caicedo, 2012).

2.1. Context and Participants

The research was carried out in a social association in the context of a juggling workshop. The choice of the association was dictated by previous contacts and the good disposition. As ethnographic research took place in a natural setting, the participants were those teaching and attending the workshop.

The key informants were the juggling workshop director, an expert juggler with 6 years of experience (J1), and a volunteer juggler (J2) with 4 years of experience. The seven workshop participants (six girls and one boy) also provided, through their interventions, ideas for reflection on juggling and pattern modeling (hereafter referred to as P1, . . . P7). The field work was carried out by the first author of this paper (R1), who is a mathematics educator.

2.2. Techniques and Data Collection

The technique of mutual interrogation (A. Adam et al., 2014) facilitates the creation of symmetrical dialogue spaces between researchers and the community. This approach fosters an exchange of knowledge and perspectives, where each form of knowledge questions challenges the other, leading to the emergence of a new, hybrid understanding of both mathematics and cultural practices (Meaney et al., 2021; Parra, 2017).

As expected in a mutual interrogation, R1 immersed herself in the cultural practice of juggling (participant observation), attending the workshop in the beginning as an apprentice, learning its processes, concepts, and limits. This allowed R1 to develop a deep understanding of the practice and to communicate effectively with its practitioners. Then, R1 promoted a critical dialogue with all the participants, bringing the mathematical perspective of the Siteswap notation and allowing the interactions between the different actors and knowledge to create new proposals and perspectives that were gathered as they emerged.

After the workshop, during which nine meetings were held, another five meetings were held between R1 and J1, and sometimes also J2, to further develop shared ideas and build experiences of juggling modeling and checking results in unstructured interviews.

This last step is particularly important to ensure communal validity (Moral, 2006), co-constructed with community participants.

Data collection was carried out through field notes, audiovisual records (video, photo, and audio) of the activities, as well as written productions of the participants.

In particular, the latter were produced in response to R1's proposal of different activities that forced interaction with the mathematical model initially proposed and juggling activities (typically tossing three balls) performed with a mechanical metronome. The different activities are described below.

2.3. Data Analysis

The qualitative analysis of the data included an initial organization of the data into units of analysis composed of the moments of interaction between the participants, the juggling activity, and its mathematical model.

In each of these moments, a recurring and cyclical dynamic phase was observed, determining the categories of analysis: criticalities of the proposed model were found that caused the emergence of changes to the initial model. The criticalities and variations were from time to time recognized by one or another of the actors in the research process and arose during different forms of interaction among participants with juggling practice. Thus, the categorizations are the initial model, the form of interaction and actors participating in it, the criticalities and potentialities that emerged, and the variations proposed.

3. Findings

The presentation of the findings is organized into several cycles. In each cycle, a version of the model is described and then submitted to a briefly presented researcher–community interaction. Then, the critical issues that emerged are described, and the proposed variation introduced into the model is indicated, which is then submitted to a subsequent cycle.

3.1. First Cycle: Ladder Diagram

During the workshops, R1 introduced two different models of juggling: the ladder diagram and Siteswap notation, with the latter being numerical sequences of natural numbers. The ladder diagram, a graphical model leading to Siteswap notation, will be adapted based on the exchange of knowledge among the actors of the research. In contrast, Siteswap notation will be presented in a simplified form for instrumental use, without undergoing modifications during the research process.

The interaction was intentionally provoked by R1 and involved all participants, being an activity representing a juggling trick with the ladder diagram, then with the Siteswap notation, and finally vice versa. The R1's choice of the diagram among the ones from the literature has already been explained in Section 1.2.

3.1.1. The Ladder Diagram and Siteswap Notation

In order to introduce the transition from the ladder diagram to the Siteswap notation, some clarifications must be made.

In the Siteswap notation, throws are depicted through a sequence of natural numbers, with each number indicating the future beats at which each object is thrown again. Siteswap's underlying idea is to accurately adjust a metronome to monitor the number of beats taking place for each ball from the time it is thrown until it is caught. The sequence of natural numbers is henceforth treated as the mathematical model for this.

Its simplest form is often called "vanilla Siteswap," and the sequences have the following properties:

1. Both hands throw alternate throws at equally spaced time intervals.
2. Both hands throw at the same rhythm.
3. One prop can be thrown from one hand at a time, or caught by one hand at a time.

As examples, two juggling patterns are described.

The simplest trick (which is what the jugglers call a pattern) is the so-called cascade. In cascade juggling, one hand operates in a clockwise direction while the other moves counterclockwise. Specifically, in this pattern, the balls are thrown from within the ellipses and caught on their outer sides (see Figure 2).

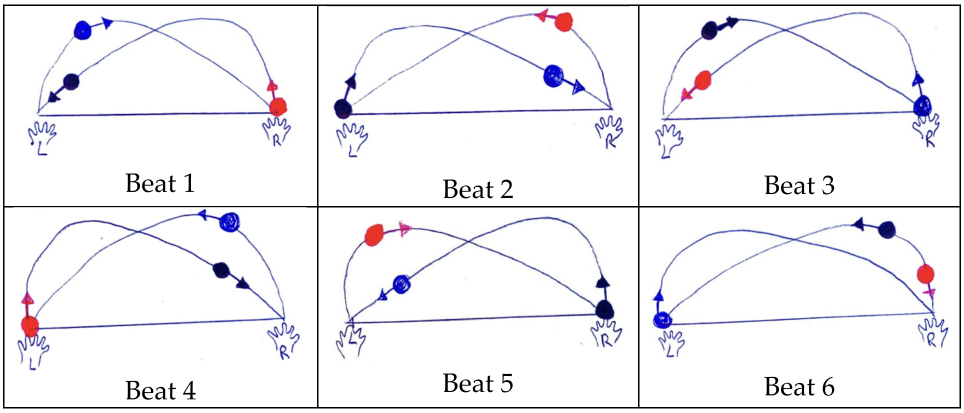


Figure 2. Schematic model of the three-ball cascade juggling pattern.

The Siteswap notation for this trick with three balls is 333. . .(that is, the number of beats each ball takes to land on the opposite hand).

In the pattern referred to as 423 (see Figure 3), the initial ball is tossed and caught by the right hand, which alternates between making a four-throw and a two-throw. The second ball is similarly thrown and caught by the left hand, alternating between a four-throw and a two-throw. The third ball, on the other hand, is consistently passed back and forth between the hands, always executing a three-throw.

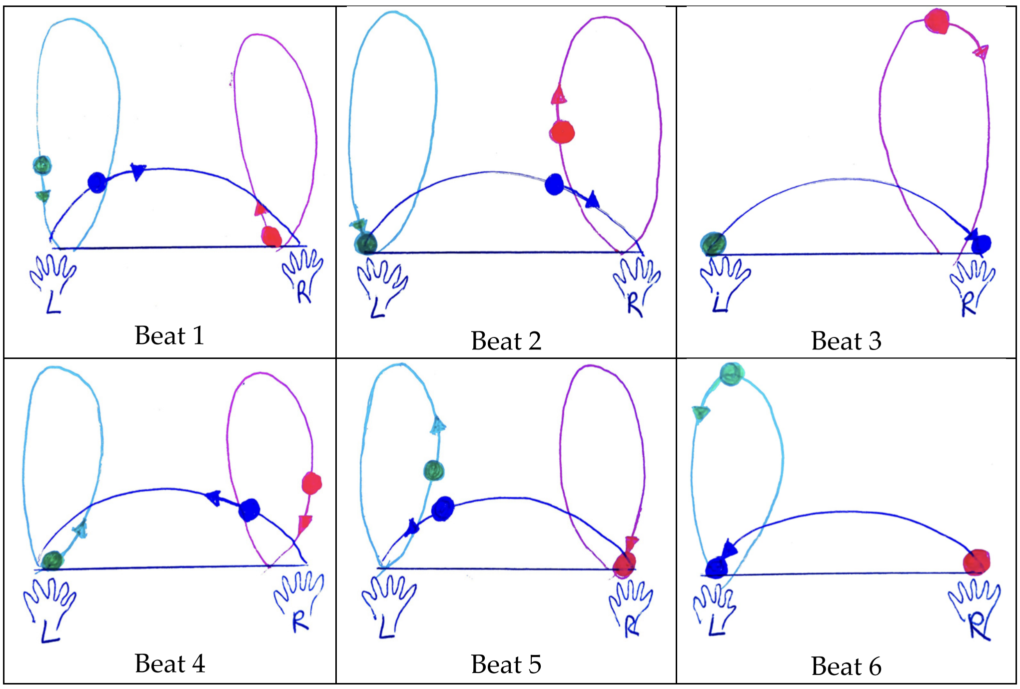


Figure 3. Schematic model of the three-ball 423 juggling pattern.

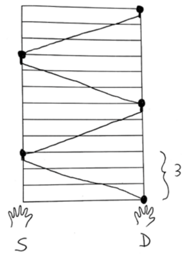
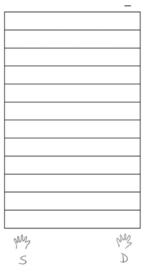
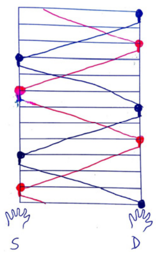
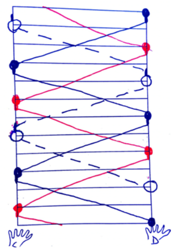
The transition between the action of performing the juggling tricks and the concept of expressing them in terms of sequences of numbers is mediated by the introduction of the ladder notation. Once a juggling pattern is described graphically, it is easier to complete the transition to a more symbolic way of describing it.

3.1.2. Idea and Interaction in First Cycle

The dialogue of knowledge in this cycle took place among J1, R1, and the workshop participants. The interaction took place through two tasks, proposed by R1 to the workshop participants, in order to find out if the proposed model was suitable to be used in the context of a jugglers community.

The first task was meant to introduce the "use" of the ladder model, pointing out the potentialities and criticalities that emerged (Table 1).

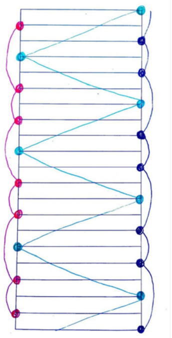
Table 1. Task 1 description.

Task 1 Formulation	Task 1 Solution
A juggling pattern or juggling trick is a specific manipulation of props during the practice of toss juggling. Each juggling trick involving the throwing of objects (in our case, 3 balls of different colors) can be described graphically by a mathematical model called a ladder (see the picture below).	A 
 The model looks like a ladder and each rung represents a beat, counting from the bottom to the top. The right ladder rail represents the right hand and the left ladder rail the left hand. Each ball is represented by a circle of the same color as the ball used during the trick. The path that each ball makes is represented by an arc of the same color as the thrown ball: • Represents the 1-ball cascade (in black); • Represents the 2-ball cascade (in red); • Represents the 3-ball cascade (in blue).	B  C 

The second task's purpose was to experiment with the transition from the Siteswap notation to the ladder model and to bring out the difference between odd-number throws and even-number throws (Table 2).

A juggling simulator helped to test the correctness of possible patterns (Figure 4b). The most suitable juggling simulator for the purpose of our research was Juggling Lab[2] online, 1.6.5.

Table 2. Task 2 description.

Task 2 Formulation	Task 2 Solution
This is the model of the trick that jugglers call "423" with the ladder model. What can you state about the even-number throws and the odd-number throws? 	Throws represented by even numbers are caught by the same hand, while the ones represented by odd numbers are caught by the other hand.

(**a**) (**b**)

Figure 4. Workshops. (**a**) Participants solving tasks 1 and 2. (**b**) Using the juggling simulator.

Observing the participants while completing the tasks and checking their products allowed R1 and J1 to modify the initial model (the proposed ladder) into something more suitable for the jugglers community.

The participants solved the tasks not only with paper and pencil but also by juggling the cascade while R1 assisted them using the metronome. The juggler's intervention, simulating the cascade hand movement without the balls, was crucial in clarifying where to place the second ball on the ladder, drawing the participants' attention to the asynchronous movement (see rule 1 above: "both hands throw alternate throws at equally spaced time intervals"). Figure 5 shows some of the participants' solutions to task 1.

P1 successfully overlapped the real movement of the balls in space with the model of the pattern on the ladder, P2 solved task 1 correctly, P3 did not complete task 1, and P4's solution indicated that both rule 1 and the correspondence between the number of beats and the model on the ladder were unclear. P5 gave up, stating that she was tired and confused, and P6 and P7 were absent.

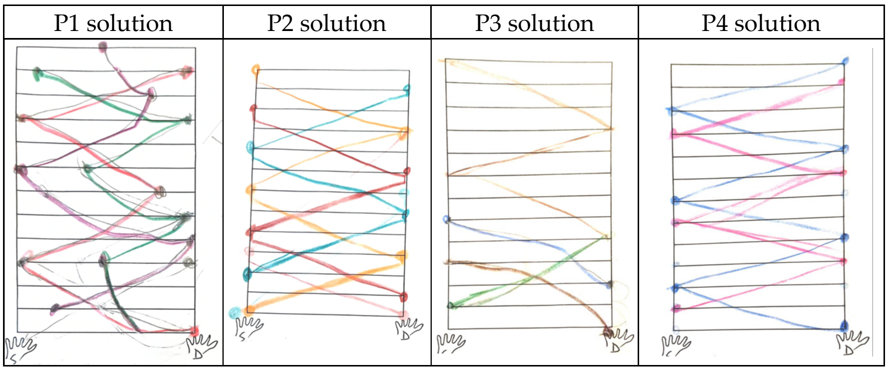


Figure 5. Participants' task 1 solution.

Only P2 completed task 2 (Figure 6), while the other participants gave up.

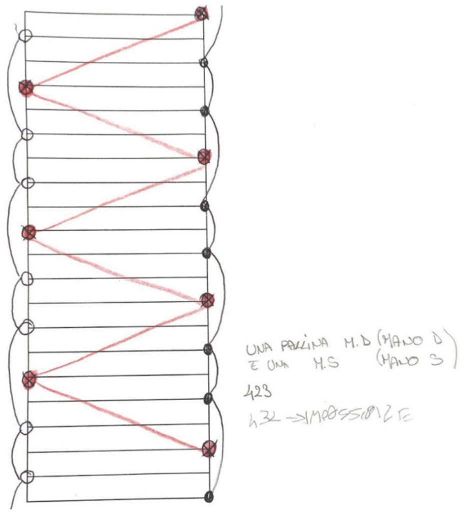


Figure 6. P2's task 2 solution.

3.1.3. Critical Points and Potentialities in First Cycle

The correct placement of the balls on the appropriate step (P4 solution), along with J1's intervention of juggling the cascade while the metronome was running slower than usual and naming aloud the corresponding beat number, helped the participants resolve the issue.

> J1: It was natural to me to help them showing the three-ball cascade while they were solving the task.
>
> R1: Actually, the juggling exercise and task 1 resolution should be performed simultaneously to be effective. (From field note, March 2024)

The overlap of the actual movement of the balls in space with the pattern model using the ladder (as shown in Figure 4) was only successful in one case.

> J1: The detected problem indicates a misunderstanding of the task.
>
> R1: I think that the transition from reality to an adequate model did not occur in this specific instance. (From field note, March 2024)

Only two of the participants (specifically P1 and P2) could work independently, highlighting the need for assistance in carrying out the tasks.

> R1: Just P1 and P2 worked on their own.
>
> J1: They usually work together as a group during their workshop.

J1: Besides, I guess that the proposed tasks were too "artificial" compared to the juggling practice during the workshops.

R1: This could be the reason why the initial enthusiasm was followed by tiredness and someone gave up. (From field note, March 2024)

Working independently in solving task 2, P2 noted, "One ball always stays in the right hand and the other in the left hand," indicating a correct understanding of the model.

It is worth noting that the ladder model allowed P2 to identify whether a sequence is feasible or not. Indeed, she made a relevant comment: "The trick 432 is impossible," as shown in Figure 5, and she was right, as all three balls collide on the fourth beat. She fully grasped the initial model and achieved a result on her own.

The concept of periodicity encountered in the repetition of numerical sequences of each juggling trick can be used as a starting point for future opportunities in the field of formal education within an activity on modular mathematics.

Finally, the possibility of swapping numbers within a sequence to check whether it is playable or not represents an opportunity to introduce the idea of permutations and more generally combinatorics in formal education.

3.1.4. Variation in First Cycle

J1: Why don't we create an exercise involving physical movement? I am used to working in this way! We can use a real ladder, like this one, [indicating the one suspended from the ceiling].

R1: I guess it could be quite tricky.

J1: And even dangerous for the guys.

R1: We could easily build a horizontal ladder, a paper model to be laid out on the floor. (From field note, April 2024)

J1 agreed that this new and more dynamic version of the model could help participants connect the model with the reality of juggling.

3.2. Second Cycle: Floor Ladder

The floor ladder was made by pasting four sheets of wrapping paper (1 m $\times$ 1.5 m), with the steps drawn with a marker (Figure 7).

Figure 7. Participants dealing with the 441 pattern.

3.2.1. Idea and Interaction in Second Cycle

Initially, to represent the 333 pattern, different colored balls were to be used, connecting the "real" juggling balls with threads.

P7: Why don't we use colored rings instead? It is easier to join them with the threads! (From field note, April 2024)

The balls were thus represented by colored rings, and the throws by threads connecting circles of the same color.

> J1: I want to add some dynamism to the model...we usually use our body during our workshops! They [the participants] could throw the balls into the corresponding rings while walking! (From field note, April 2024)

The paper ladder was laid out on the floor.

The participants began by placing the rings on the corresponding rungs of the ladder, following the same scheme they used in Task 1. They then linked rings of the same color (representing the same ball) with threads of the same color, discussing the pattern as they built it.

P7, standing on the first step of the paper ladder, took three balls. He threw the first ball onto the fourth rung, into the corresponding ring on the left rail. He then walked to the second rung and threw the second ball onto the fifth rung on the right rail. Finally, he walked to the third rung and threw the third ball onto the sixth rung on the left rail.

P7, along with P1 and P2, successfully created the paper ladder pattern with the 441 trick, as shown in Figure 6, despite it being their first time attempting this trick.

> R1: This new model could be used to generate new patterns from existing ones by swapping the landing times of any two "sites" (hence the name Siteswap). Actually, this is where the word Siteswap comes from! (From field note, April 2024)

Using the floor ladder, R1 demonstrated the transition from 333 to 342.

Once the process of creating new sequences from existing ones was shown on the floor ladder, J1 and the participants managed to create new and more elaborate juggling patterns, starting from the cascade and swapping two rings without causing collisions (i.e., two balls landing in the same hand at the same beat) among the props. Finally, J1 successfully performed the 441 trick[3] for the first time ever. The dialogue of knowledge took place between J1, R1, and the participants.

3.2.2. Critical Points and Potentialities in Second Cycle

The first issue was that J1's idea of throwing the balls while walking the ladder step by step was not very effective. As soon as the balls fell on the floor, the "action" stopped, and someone else had to pick up the balls to throw them again, disrupting continuity.

The second issue was that the floor model was impractical due to its size and not suitable for a deeper reflection on the process of creating new sequences and, consequently, new juggling tricks.

On the other hand, the model allowed R1, J1, J2, and the participants to easily use it during workshops in the chapiteau to test transitions between sequences without causing collisions. Notably, the transition between 432 and 441 was tested on the ladder by J1 and the participants.

3.2.3. Variation in Second Cycle

Some changes were introduced to address the first critical point.

> R1: While the initial model was interactive and effective, it lacked dynamism.

> J1: One possible solution could be to adjust the distance between the rungs of the ladder to match my step length.

J2: I guess we have to modify the metronome rhythm while he juggles the cascade. (From field note, April 2024)

This was the starting idea for the next cycle.

3.3. Loop: Small-Scale Model

For the second critical point, R1 needed a handier model in order to gain insight into the sequence transition. Specifically, R1 needed to test how the swapping of the prop landing worked and how the transition could be expressed numerically.

3.3.1. Idea and Interaction in the Loop Cycle

For this purpose, R1 created a scaled-down model of the ladder, using stones and washers instead of rings, to test the transition from 333 to 342 and study the numerical aspect of the process, as shown in Figure 8.

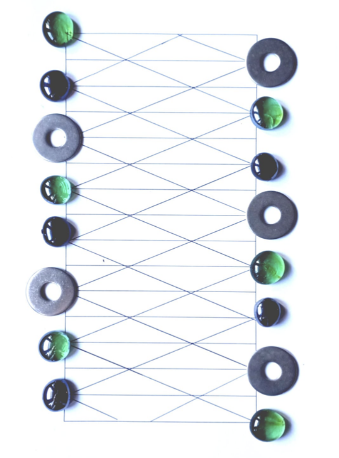
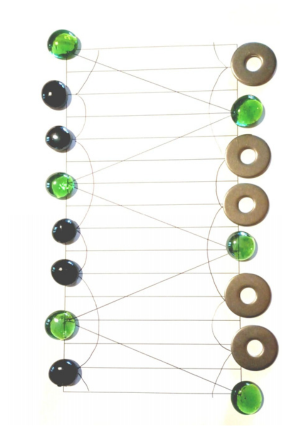
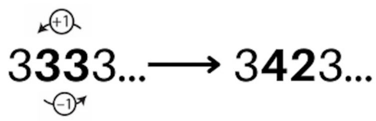
Initial situation: the initial sequence is the cascade model, which is 333.	Final situation after having swapped 2 sites: the blue on the left swaps with the ring on the right (the blue prop lands a beat after—instead of a 3-throw it is a 4-throw, and the ring throw becomes 2 instead of the initial 3. The green prop remains the same. The final sequence is 3423423423.
	

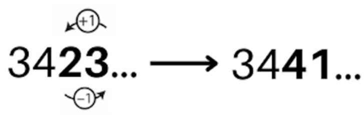
Figure 8. A schematic model of the transition from 333 to 342.

Starting with sequence 333, the first ball landed on the fourth beat (assuming it started on the first rung of the ladder), the second ball on the fifth beat, and the third ball on the sixth beat. By swapping the landing times of the second and third balls, the new sequence of 342 was generated. In this sequence, the timing and path of the first ball remained unchanged (landing on the fourth beat), the second ball landed on the sixth beat (starting on the second beat and taking four beats to land), and the third ball landed on the fifth beat (starting on the third beat and taking two beats to land).

Numerically, the transformation worked in this way:



Repeating the Siteswap on the last sequence, the transition from 423 and 441 happened:



It is straightforward to demonstrate that applying a Siteswap transformation to a pattern preserves both the number of balls and the average throw number. The "Siteswap averaging theorem" states that for a vanilla Siteswap pattern, the average throw number will always equal the number of balls (props) being juggled. This theorem provides a useful constraint and check when analyzing or creating juggling patterns (Polster, 2002).

3.3.2. Critical Points and Potentialities in the Loop Cycle

Obviously, the model lacked physical interaction, as it was not walkable and not shareable among a group of people but proved to be very helpful for individual reflection.

Finally, it is important to remark that the search for an even more effective representation to quickly find valid Siteswap juggling patterns and transitions between different Siteswap patterns can be used in formal education within activities on graph theory.

3.3.3. Variation in the Loop Cycle

No variation had to be made; it was a standalone loop.

3.4. Third Cycle: Improved Floor Ladder

The initial idea was to draw the steps on the paper ladder according to the juggler's actual steps. J2 and R1 assisted J1 by tossing him the balls and catching the balls he threw while juggling and walking (a simplified form of multiplexing in motion). The metronome was set to a slower rhythm.

3.4.1. Idea and Interaction in the Third Cycle

R1, J1, and J2 realized that the distance between the steps on the previous ladder was appropriate for J1's stride. The issue was not the step distance but the speed at which the exercise was performed.

3.4.2. Critical Points and Potentialities in the Third Cycle

Despite the metronome playing at a slower rhythm, and R1 and J2 managing to throw balls to J1 and retrieve those he launched, albeit with difficulty, the exercise was not feasible. The "cascade" trick required a rhythm that was too fast to maintain continuity.

3.4.3. Variation in the Third Cycle

> J2: Why don't we try with the hats? In this way we can decrease the level of difficulty of the exercise by using hats instead of taking out the throwing of the balls, eliminating the need to throw objects without. (From field note, May 2024)

R1 and J1 agreed it could be the key point for a new exercise.

3.5. Fourth Cycle: Hat Ladder

The exercise was performed by working with the colored rings in place of the hats, setting the metronome to the rhythm needed for the exercise, and it was recorded.[4]

3.5.1. Idea and Interaction in the Fourth Cycle

J1: We could even create a new circus number to be proposed during a show: we can tie the rings with three colored threads attached to the opposite end at a fixed point and play the 333 and 432 to see what kind of braids are formed.

J2: We could even use hair extensions to be tied to a girl's head.

R1: Mathematically speaking, this exercise is very suitable for representing the movement of objects on a plane parallel to the direction of motion! (From field note, June 2024).

3.5.2. Critical Points and Potentialities in the Fourth Cycle

The exercise proposed by J1 and J2 proved not to be very suitable for use during a performance, since after some rehearsal the final effect was of a poor braid, as it must be possible, while performing a braid, to tighten the strings from time to time.

R1: Could you repeat the exercise by removing the time constraint played by the metronome and tightening the braid one at a time? (From field note, August 2024).

The exercise would not be usable during a circus show, but it would be useful in a different context—for example, that of formal education, as it does not require high-level juggling skills—while retaining its value in the application of a model.

J1: I can try attaching the balls with a heavy rope I normally use to open the curtain of the chapiteau. (From field note, August 2024).

J1 proposed testing the exercise with balls attached to a heavy rope; on the one hand, handling was complicated by the weight of the rope, but on the other hand, the braid was more easily formed precisely thanks to the type of rope used. Again, there was no metronome constraint.

3.5.3. Variation in the Fourth Cycle

As shown in Figure 9, the exercise was performed with balls and rope of the same color to verify its feasibility initially, but it could have been proposed using different colors for both the balls and the ropes.[5]

(**a**) (**b**)

Figure 9. Creating braids with the cascade. (**a**) J1 is going to juggle the balls tied with the rope, and (**b**) the braid made as a result of juggling the cascade.

4. Discussion

Ethnomathematics establishes connections between diverse knowledge systems, such as institutional disciplinary mathematics and local ways of knowing. These connections can be traced from cultural knowledge systems to the realm of mathematics or vice versa. The process involves engaging in debates on the plausibility, relevance, and utility of these connections, leading to new forms of learning and re-elaborations. This approach emphasizes the importance of collective learning and the creation of new, organic communal knowledge that integrates both local and academic perspectives (Parra, 2024).

One of the research fields in ethnomathematics focuses on the formulation of ethnomodels, which are community models rewritten in mathematical terms (Orey & Rosa, 2021). In this research, a different approach was proposed, starting from a model created by mathematician jugglers that was validated by a scientific community composed of mathematicians, computer scientists, and physicists. This model was then reinterpreted, rewritten, and adapted to better align with the understanding, conception, and practices of the involved cultural community.

This is a symmetric participatory methodological research experiment. In the first cycle, the initial model (ladder diagram) was proposed by R1. In subsequent cycles, the models emerged from the exchange of knowledge between R1 and the juggling community involved in the research (Meaney et al., 2021; Parra, 2017).

During the intermediate phases (idea and interaction, criticalities and potentialities) of each cycle, the dialogue between R1 and the jugglers enhanced the understanding of mathematics and culture by challenging different knowledge systems (N. A. Adam, 2010).

In the final phase of each cycle (variation), a new model was proposed, which was more effective than the previous one and visually connected the concepts expressed by the theory.

The additional five meetings that took place between R1 and J1, and occasionally with J2, were held to continue refining shared ideas, enhance the practice of juggling modeling, and verify outcomes in unstructured interviews. This final phase, together with the constant dialogue between the academic and jugglers world within the mutual interrogation frame, were crucial for achieving communal validity (Moral, 2006), created in collaboration with community participants.

The main findings have materialized in the creation of new knowledge, benefiting both the mathematics education community and the jugglers community (Meaney et al., 2021; Parra, 2017).

For ethnomathematics and mathematical education:

The research exemplifies a co-construction process with the community to develop a model of practice (Albanese & Herrera-Janques, 2024). The models created will be applicable for future proposals in formal education, incorporating tasks based on local practices (François et al., 2018; Mafra, 2020). These tasks aim to foster creative problem-solving skills by challenging students to create their own juggling sequences, thereby introducing them to mathematical concepts of discrete mathematics in a hands-on and engaging way.

For the jugglers community:

The juggling community has benefited from the introduction of Siteswap notation and the floor ladder as pedagogical tools. The way the juggling community learns, practices, and creates new tricks has been revolutionized. Siteswap notation allowed jugglers to associate each throw with a number, which facilitated the understanding of juggling patterns and encouraged the creation of new tricks. Jugglers gained the ability to understand the movement of props in the air and to predict which hand they will land on, which was previously only possible through imitation.

The floor ladder, meanwhile, provided a spatial and temporal framework for juggling practice, allowing jugglers to work on timing, strength, direction, and awareness of the action of each ball.

This paradigm shifts from imitation to understanding, and conscious creation fostered innovation within the juggling community. The combination of Siteswap notation and the floor ladder has opened up a new range of creative possibilities, such as combining sequences of tricks and creating complex choreography. In addition, the ability to "name" known tricks with numerical sequences and to verify the feasibility of new tricks through the average theorem has empowered jugglers by providing them with tools to analyze and plan their practices.

Ultimately, the introduction of these tools has transformed the way jugglers learn, practice, and create, fostering understanding, creativity, and innovation in the community (Albanese, 2021).

5. Conclusions

The study aimed to co-construct a shared model between the juggling community and academia, establishing a symmetrical dialogue between the practice of juggling and the domain of mathematics. The goal was achieved by building a shared model that allowed for enrichment from both researchers and jugglers.

This investigation involved a close collaboration between a mathematics educator and two experienced jugglers with their apprentices. The mathematics educator brought her expertise in discrete mathematics and pattern analysis to the table along with the baggage of mathematical modeling of juggling that the academic literature in mathematics has developed in the last decades, while the jugglers contributed with their knowledge of juggling techniques, their practical learning style, and the creative process of sequences. The research team explored and tested different representations of juggling, proposing together variations to maintain potentialities and overcome criticalities. The mathematical representation of juggling sequences using different representations allowed for the exploration of the combinatorial possibilities and constraints in creating new juggling sequences.

The research yielded significant results for both the mathematics educator and the jugglers:

- For the mathematician, juggling provided a tangible and engaging context to explore abstract mathematical concepts, which is a valuable resource to develop interdisciplinary mathematical tasks for formal education.
- For the juggler, the mathematical analysis of juggling sequences offered a deeper understanding of the underlying structures and patterns, empowering them to create and innovate new sequences with greater precision and intentionality.

Participatory research usually requires more time and resources than traditional research, due to the need to build trusting relationships, negotiate objectives and methodologies, and ensure the participation of all those involved. The research was conducted within a single community, and it was only possible to engage with participants during the workshop sessions. Additional time would have been beneficial to fully implement the activities. Furthermore, due to personal commitments, the participants were rarely present in full, which led to initial communication challenges and delays in adhering to the planned schedule. This research demonstrates the fruitful interplay between different systems of knowledge. By embracing the principles of mutual interrogation and the metaphor of bartering, this research highlights the potential of ethnomathematics to enrich our understanding of both mathematics and the cultural practices that embody it.

Furthermore, the hybrid model developed in this study offers a promising avenue for integrating ethnomathematical perspectives into formal mathematics education, fostering a more situated and engaging learning experience for students.

6. Recommendations for Future Research

In future research, the model could be proposed in a secondary school context to prepare learning activities on different topics.

Although this research does not focus on mathematical concepts, the periodic nature of the juggling patterns emerges in the construction of the diagrams, models, and notations. Thus, modular arithmetic is a topic that can be enriched by being worked on in this context.

Our findings suggest a successful use of graph theory to model the transitions between different juggling tricks. Educational activities could include constructing graphs of state transitions to study the reachability from one pattern to another, which may enhance students' understanding of dynamic systems and graph-based problem-solving.

A different proposal could be combinatorial analysis to explore the number of feasible minimal juggling patterns. Students can calculate possible patterns for a given number of balls. By framing juggling patterns as combinatorial problems, students can gain a deeper understanding of counting and arrangement in mathematics.

By integrating these mathematical concepts with the hands-on activity of juggling, students can develop a deeper understanding of abstract mathematical principles, enhancing both their engagement and their retention. These recommendations aim to provide a creative and interdisciplinary approach to teaching key mathematical topics in formal education.

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Informed Consent Statement: Written Informed consent was obtained from all the participants involved in the study.

Data Availability Statement: The data presented in this study are available on request from the corresponding author due to privacy restriction.

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Notes

1 Juggling by Numbers (Numberphile): https://www.youtube.com/watch?v=7dwgusHjA0Y (accessed on 1 February 2024)
2 Juggling simulator: https://jugglinglab.org/ (accessed on 2 February 2024)
3 441 trick Video: https://youtu.be/oCm9TQQmy7E (accessed on 2 February 2025)
4 Juggling rings Video: https://youtube.com/shorts/3izO1EE7Ujs?feature=shared (accessed on 15 January 2025)
5 Braid Video: https://www.youtube.com/shorts/2rBYIYSnltc (accessed on 20 January 2025)

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