Linear Algebra and its Applications 710 (2025) 448–457

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Additive mappings preserving orthogonality between complex inner product spaces

Lei Li [a], Siyu Liu [a], Antonio M. Peralta [b,*]

[a] *School of Mathematical Sciences and LPMC, Nankai University, 300071 Tianjin, China*

[b] *Instituto de Matemáticas de la Universidad de Granada (IMAG), Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*

A R T I C L E I N F O

Article history:
Received 10 October 2024
Received in revised form 14 October 2024
Accepted 31 January 2025
Available online 3 February 2025
Submitted by C.-K. Li

MSC:
primary 46C99
secondary 46T99, 39B55

Keywords:
Birkhoff orthogonality
Euclidean orthogonality
Orthogonality preserving additive mappings
Inner product spaces

A B S T R A C T

Let H and K be two complex inner product spaces with $\dim(H) \geq 2$. We prove that for each non-zero mapping $A : H \to K$ with dense image the following statements are equivalent:

- (a) A is (complex) linear or conjugate-linear mapping and there exists $\gamma > 0$ such that $\|A(x)\| = \gamma\|x\|$, for all $x \in H$, that is, A is a positive scalar multiple of a linear or a conjugate-linear isometry;
- (b) There exists $\gamma_1 > 0$ such that one of the next properties holds for all $x, y \in H$:

  $(b.1)$ $\langle A(x)|A(y)\rangle = \gamma_1\langle x|y\rangle$,
  $(b.2)$ $\langle A(x)|A(y)\rangle = \gamma_1\langle y|x\rangle$;

- (c) A is linear or conjugate-linear and preserves orthogonality;
- (d) A is additive and preserves orthogonality in both directions;
- (e) A is additive and preserves orthogonality.

* Corresponding author.
E-mail addresses: leilee@nankai.edu.cn (L. Li), 760659676@qq.com (S. Liu), aperalta@ugr.es (A.M. Peralta).

https://doi.org/10.1016/j.laa.2025.01.042

This extends to the complex setting a recent generalization of the Koldobsky–Blanco–Turnšek theorem obtained by Wójcik for real normed spaces.

1. Introduction

Elements x, y in a real or complex inner product space $(H, \langle\cdot|\cdot\rangle)$ are called *orthogonal* in the Euclidean sense ($x \perp_2 y$ in short) if $\langle x|y\rangle = 0$. It is an intriguing question to determine how much information about H is preserved by knowing all orthogonal pairs of elements in H. We are led to the following problem on preservers: If K is another real or complex inner product space, a mapping $\Delta : H \to K$ *preserves (Euclidean) orthogonality* if $\forall x, y \in H,\ x \perp_2 y \Rightarrow \Delta(x) \perp_2 \Delta(y)$. If the implication "$\Rightarrow$" is replaced with the equivalence "$\Leftrightarrow$" we say that Δ preserves (Euclidean) orthogonality in both directions. When can we conclude that such a mapping Δ is linear or conjugate linear? Are the inner product spaces H and K isometrically isomorphic?

We should begin by observing that it is simply hopeless to characterize additive surjective maps preserving orthogonality in both directions on the one-dimensional complex Hilbert space $H = \mathbb{C}$, since every additive surjective mapping on $\mathbb{C}$ automatically preserves orthogonality in both directions. If we take a non-continuous bijective additive mapping $f : \mathbb{R} \to \mathbb{R}$, the natural extension $\tilde{f} : \mathbb{C} \to \mathbb{C}$, $\tilde{f}(\alpha + i\beta) = f(\alpha) + if(\beta)$ is an additive bijection preserving orthogonality in both directions. We shall see next that this counterexample can only occur when the inner product space in the domain is one–dimensional.

Concerning the above questions, a result by J. Chmieliński (see [2, Theorem 1]) assures that for each non-zero mapping T between two (real or complex) Hilbert spaces H and K the following statements are equivalent:

(a) T is linear and there exists $\gamma > 0$ such that $\|T(x)\| = \gamma\|x\|$ for all $x \in H$;
(b) There exists $\gamma_1 > 0$ such that $\langle T(x)|T(y)\rangle = \gamma_1\langle x|y\rangle$, for all $x, y \in H$;
(c) T is linear and preserves orthogonality in both directions;
(d) T is linear and orthogonality preserving.

Chmieliński also gave examples of non-linear (actually non-additive) and discontinuous maps preserving orthogonality on the 2-dimensional real Hilbert space ℓ_2^2.

The conclusion in the just quoted result is closely related to the celebrated Koldobsky–Blanco–Turnšek theorem. It is known that elements x, y in a Hilbert space H are orthogonal if, and only if, they are *Birkhoff or Birkhoff-James orthogonal* ($x \perp_B y$ in short), that is, for all $\alpha \in \mathbb{K}$ we have $\|x\| \leq \|x + \alpha y\|$. Birkhoff orthogonality makes sense for normed spaces. Let us observe that the underlying field $\mathbb{K}$ is crucial in the above definitions. Consider, for example, the complex Hilbert space $H = \ell_2^2$ with inner product $\langle(\lambda_1, \lambda_2)|(\mu_1, \mu_2)\rangle = \sum_i \lambda_i \overline{\mu_i}$, and the underlying real Hilbert space $H_{\mathbb{R}}$ with respect to the inner product $(\cdot|\cdot) = \Re\mathrm{e}\langle\cdot|\cdot\rangle$. It is easy to check that $((i,1)|(1,-i)) = \Re\mathrm{e}(2i) = 0$, that is $(i,1) \perp_2 (1,-i)$ in $H_{\mathbb{R}}$, while $(i,1) \not\perp_2 (1,-i)$ in H.

The celebrated *Koldobsky–Blanco–Turnšek theorem* asserts that a non-zero linear mapping T between two real or complex normed spaces X and Y, preserves Birkhoff orthogonality if, and only if, there exists $\gamma > 0$ such that $\|T(x)\| = \gamma\|x\|$, for all $x \in X$ (see [1] for the case of real normed spaces and [5] for complex normed spaces). In a more recent reference (cf. [8, Theorem 3.1]), P. Wójcik established a generalization of the Koldobsky–Blanco–Turnšek theorem by showing that for each non-zero additive mapping A between two normed real spaces X and Y with $\dim(X) \geq 2$, the following conditions are equivalent:

(a) A preserves Birkhoff orthogonality;
(b) A is a linear mapping and there exists $\gamma > 0$ such that $\|T(x)\| = \gamma\|x\|$, for all $x \in X$.

Obviously, Wójcik's theorem holds when X and Y are real inner product spaces, where it is more natural to speak about preservers of (Euclidean) orthogonality. However, the strong dependence on the base field of Birkhoff orthogonality and (Euclidean) orthogonality makes impossible to apply Wójcik's result for additive orthogonality preserver between complex Hilbert spaces (even more for additive preservers of Birkhoff orthogonality between complex normed spaces). It seems interesting to fill the existing gap and characterize all additive maps preserving orthogonality between complex Hilbert spaces. This short note is aimed to provide a full characterization of additive orthogonality preserving maps between complex inner product spaces, in this setting we show that, as it can be naturally expected, positive scalar multiples of conjugate-linear isometries are also possible. In our main result we prove that if $A : H \to K$ is a non-zero additive mapping with dense image between complex inner product spaces, then A is a positive scalar multiple of a (complex) linear or conjugate linear isometry if and only if it preserves (Euclidean) orthogonality (see Theorem 2.1). We also prove that if X and Y are complex normed spaces with $\dim(X) \geq 2$ and X admits a conjugation τ (i.e., a period-2 conjugate linear isometry), then every additive mapping $A : X \to Y$ preserving Birkhoff orthogonality is real-linear. Furthermore, if A is surjective and preserves Birkhoff orthogonality in both directions, then A is a real-linear isomorphism and the underlying real normed spaces $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ are isomorphic (see Proposition 2.2).

2. The results

Before stating the desired characterization we include a technical lemma.

Lemma 2.1. *Let $A : H \to K$ be a real-linear mapping with dense image between two complex inner product spaces. Suppose additionally that A preserves orthogonality. Then for each norm-one element $x_0 \in H$ we have $A(\mathbb{C}x_0) \subseteq \mathbb{C}A(x_0)$.*

Proof. We can suppose that $A(ix_0) \neq 0$, otherwise $A(\mathbb{C}x_0) = A(\mathbb{R}x_0) = \mathbb{R}A(x_0) \subseteq \mathbb{C}A(x_0)$.

Let us take $z \in K = \overline{A(H)}$ with $z \perp_2 A(x_0)$. By hypothesis, we can find a sequence $(x_n)_n$ in H such that $(A(x_n))_n \to z$. Since $H = (\mathbb{R}x_0 + \mathbb{R}(ix_0)) \oplus \{x_0\}^\perp = (\mathbb{R}x_0 + \mathbb{R}(ix_0)) \oplus \{ix_0\}^\perp$, there exist sequences $(\alpha_n)_n$, $(\beta_n)_n$ in $\mathbb{R}$, and a sequence $(y_n)_n \subseteq \{ix_0\}^\perp = \{x_0\}^\perp$ satisfying $\alpha_n x_0 + \beta_n ix_0 + y_n = x_n$ for all n, and hence $\alpha_n A(x_0) + \beta_n A(ix_0) + A(y_n) = A(\alpha_n x_0 + \beta_n ix_0 + y_n) \to z$. By hypotheses, $(\alpha_n A(x_0) + \beta_n A(ix_0)) \perp_2 A(y_n)$ for all natural n, and hence the sequences $(\alpha_n A(x_0) + \beta_n A(ix_0))_n$ and $(A(y_n))_n$ must be bounded.

If $A(x_0) = 0$, we have $A(H) = \mathbb{R}A(ix_0) \oplus^\perp A\left(\{x_0\}^\perp\right)$, and thus

$$\begin{aligned} K = \overline{A(H)} &= \overline{\mathbb{R}A(ix_0) \oplus^\perp A\left(\{x_0\}^\perp\right)} = \mathbb{R}A(ix_0) \oplus^\perp \overline{A\left(\{x_0\}^\perp\right)} \\ &= \mathbb{R}A(ix_0) \oplus^\perp \overline{A\left(\{ix_0\}^\perp\right)} \subseteq \mathbb{R}A(ix_0) \oplus^\perp \{A(ix_0)\}^\perp \subseteq K, \end{aligned}$$

which is impossible. Therefore $A(x_0) \neq 0$.

If $A(x_0)$ and $A(ix_0)$ are $\mathbb{R}$-linearly dependent we can write $A(ix_0) = tA(x_0)$ for some real t, and thus $A(ix_0) \in \mathbb{C}A(x_0)$.

We can now deal with the case that $A(x_0)$ and $A(ix_0)$ are $\mathbb{R}$-linearly independent. Since $(\alpha_n A(x_0) + \beta_n A(ix_0))_n$ is bounded, by basic theory of normed spaces the sequences, $(\alpha_n)_n$ and $(\beta_n)_n$ must be bounded. Up to taking appropriate subsequences, we can assume that $(\alpha_n)_n \to \alpha_0 \in \mathbb{R}$ and $(\beta_n)_n \to \beta_0 \in \mathbb{R}$, and hence $(A(y_n))_n \to z_1 \in \overline{A(\{ix_0\}^\perp)} \subseteq \{A(ix_0)\}^\perp \cap \{A(x_0)\}^\perp$. We therefore have $z = \alpha_0 A(x_0) + \beta_0 A(ix_0) + z_1$.

We shall next show that $\beta_0 = 0$. Arguing by contradiction, we assume $\beta_0 \neq 0$. By applying that $z \perp_2 A(x_0)$, we deduce that

$$\begin{aligned} 0 = \langle z | A(x_0)\rangle &= \alpha_0 \|A(x_0)\|^2 + \beta_0 \langle A(ix_0) | A(x_0)\rangle + \langle z_1 | A(x_0)\rangle \\ &= \alpha_0 \|A(x_0)\|^2 + \beta_0 \langle A(ix_0) | A(x_0)\rangle, \end{aligned} \tag{1}$$

which implies that $\operatorname{Im}\langle A(ix_0) | A(x_0)\rangle = 0$, equivalently, $\langle A(ix_0) | A(x_0)\rangle \in \mathbb{R}$. This assures that

$$A(ix_0) = sA(x_0) + z_2, \text{ with } s \in \mathbb{R}, \text{ and } z_2 \perp_2 A(x_0).$$

We therefore arrive to

$$A(H) = (\mathbb{R}A(x_0) + \mathbb{R}A(ix_0)) \oplus^\perp A\left(\{x_0\}^\perp\right) \subseteq \mathbb{R}A(x_0) \oplus^\perp \{A(x_0)\}^\perp,$$

and thus $K = \overline{A(H)} \subseteq \mathbb{R}A(x_0) \oplus^\perp \{A(x_0)\}^\perp \subseteq K$, which is also impossible, and hence $\beta_0 = 0$, and by (1), $\alpha_0 = 0$.

All the previous conclusions lead to $z = z_1 \in \{A(ix_0)\}^\perp \cap \{A(x_0)\}^\perp$. We have then proved that $A(ix_0) \perp_2 z$ for every $z \perp_2 A(x_0)$, and then $A(ix_0) \in \mathbb{C}A(x_0)$. $\square$

We can now state the desired extension of the Blanco–Turnšek and Wójcik theorems.

Theorem 2.1. *Let $A : H \to K$ be a non-zero mapping between two complex inner product spaces with $dim(H) \geq 2$. Suppose that A has dense image. Then the following statements are equivalent:*

(a) *A is a (complex) linear or a conjugate-linear mapping and there exists $\gamma > 0$ such that $\|A(x)\| = \gamma\|x\|$, for all $x \in H$, that is, A is a positive scalar multiple of a linear or a conjugate-linear isometry;*
(b) *There exists $\gamma_1 > 0$ such that one of the next properties holds for all $x, y \in H$:*

  ($b.1$) $\langle A(x)|A(y)\rangle = \gamma_1\langle x|y\rangle$,
  ($b.2$) $\langle A(x)|A(y)\rangle = \gamma_1\langle y|x\rangle$;

(c) *A is linear or conjugate-linear and preserves orthogonality in both directions;*
(d) *A is linear or conjugate-linear and preserves orthogonality;*
(e) *A is additive and preserves orthogonality in both directions;*
(f) *A is additive and preserves orthogonality.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ is known, actually $(b.1)$ (respectively, $(b.2)$) holds if, and only if, A is linear (respectively, conjugate-linear). The implications $(b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (f)$, $(c) \Rightarrow (e)$ and $(e) \Rightarrow (f)$ are clear.

$(f) \Rightarrow (a)$ Suppose that A is additive and preserves orthogonality. We shall first prove that A is real-linear. One is first tempted to apply Wójcik's theorem [8, Theorem 3.1] to the additive mapping $A : H_{\mathbb{R}} \to K_{\mathbb{R}}$ regarded as a map between the underlying real inner product spaces $H_{\mathbb{R}}$ and $K_{\mathbb{R}}$ when equipped with the inner product $(x|y) = \Re\text{e}\langle x|y\rangle$. However, in such a case we can find points $x, y \in H$ with $(x|y) = 0$ but $\langle x|y\rangle \neq 0$. We need a subtle argument to avoid the problem.

Our goal is to show that $A(\alpha x_1) = \alpha A(x_1)$ for all $\alpha \in \mathbb{R}$ and $x_1 \in H\backslash\{0\}$. By hypothesis, we can find $x_1 \in H$ such that $A(x_1) \neq 0$. Since $\dim(H) \geq 2$, there exists $x_2 \in H\backslash\{0\}$ such that $x_1 \perp_2 x_2$ in H. Let H_1 be the real subspace of H generated by $\{x_1, x_2\}$, that is, $H_1 = \mathbb{R}x_1 \oplus \mathbb{R}x_2$. Since the elements in H_1 are real-linear combinations of x_1 and x_2, we can easily see that $\langle x|y\rangle = \Re\text{e}\langle x|y\rangle = (x|y)$ for all $x, y \in H_1$, and thus elements in H_1 are orthogonal in $(H, \langle\cdot|\cdot\rangle)$ if, and only if, they are orthogonal in $(H_1, \Re\text{e}\langle\cdot|\cdot\rangle)$. We can therefore conclude that the mapping $A|_{H_1} : H_1 \to K_{\mathbb{R}}$ is additive and orthogonality preserving. So, by Wójcik's theorem [8, Theorem 3.1], $A|_{H_1}$ is real-linear (and there exists a positive γ satisfying $\|A(x)\| = \gamma\|x\|$ for all $x \in H_1$). We have also shown that $A(x_2) \neq 0$ for all non-zero x_2 with $\langle x_2|x_1\rangle = 0$. Replacing the roles of x_1 and x_2 we get $A(\lambda x_1) \neq 0$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. If x is a non-zero vector in H we can find $\lambda \in \mathbb{C}$ and $x_2 \perp_2 x_1$ such that $x = \lambda x_1 + x_2$ and $\|x\|^2 = \|\lambda x_1\|^2 + \|x_2\|^2$, since $A(x)$ is the orthogonal sum of $A(\lambda x_1)$ and $A(x_2)$, it follows from the previous conclusions that $A(x) \neq 0$. Consequently, in the above arguments, x_1 can be replaced with any non-zero vector. The arbitrariness of x_1 allows us to conclude that A is real-linear (and preserves orthogonality by hypothesis).

We shall next prove that A is (complex) linear or conjugate-linear. Observe first that, by Lemma 2.1, $A(\mathbb{C}x_1) \subseteq \mathbb{C}A(x_1)$ for all norm-one element $x_1 \in H$. Fix a norm-one element $x_1 \in H$. As before, since $\dim(H) \geq 2$, we can find an orthonormal system of the form $\{x_1, x_2\}$. A new application of Lemma 2.1 proves that $A(\mathbb{C}x_2) \subseteq \mathbb{C}A(x_2)$. Observe that, by hypotheses, each element in the set $\{A(x_1), A(ix_1)\}$ is orthogonal to every element in the set $\{A(x_2), A(ix_2)\}$.

By applying that A is real-linear we deduce that

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \Re\mathrm{e}(\lambda_1)A(x_1) + \mathrm{Im}(\lambda_1)A(ix_1) + \Re\mathrm{e}(\lambda_2)A(x_2) + \mathrm{Im}(\lambda_2)A(ix_2), \quad (2)$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}$. Observe that $ix_1 + ix_2$ and $ix_1 - ix_2$ are orthogonal vectors in H, and thus, by the hypothesis on A, we must have

$$\|A(ix_1)\|^2 - \|A(ix_2)\|^2 = \langle A(ix_1 + ix_2)|A(ix_1 - ix_2)\rangle = 0. \quad (3)$$

On the other hand, for every $\lambda_2 = \alpha + i\beta \in \mathbb{C}\backslash\{0\}$ and $\lambda_1 = s + it \in \mathbb{C}$ the vectors $x_1 + \lambda_2 x_2$ and $\lambda_1 x_1 - \lambda_1 \overline{\lambda_2^{-1}} x_2$ are orthogonal in H, we can therefore conclude from the assumptions on A that $A(x_1 + \lambda_2 x_2) \perp_2 A(\lambda_1 x_1 - \lambda_1 \overline{\lambda_2^{-1}} x_2)$, equivalently, $\langle A(x_1 + \lambda_2 x_2)|A(\lambda_1 x_1 - \lambda_1 \overline{\lambda_2^{-1}} x_2)\rangle = 0$, which expanded gives

$$\begin{aligned}0 =& \Re\mathrm{e}(\lambda_1)\|A(x_1)\|^2 + \mathrm{Im}(\lambda_1)\langle A(x_1)|A(ix_1)\rangle + \Re\mathrm{e}(\lambda_2)\Re\mathrm{e}(-\lambda_1\overline{\lambda_2^{-1}})\|A(x_2)\|^2 \\ &+ \Re\mathrm{e}(\lambda_2)\,\mathrm{Im}(-\lambda_1\overline{\lambda_2^{-1}})\langle A(x_2)|A(ix_2)\rangle + \mathrm{Im}(\lambda_2)\Re\mathrm{e}(-\lambda_1\overline{\lambda_2^{-1}})\langle A(ix_2)|A(x_2)\rangle \\ &+ \mathrm{Im}(\lambda_2)\,\mathrm{Im}(-\lambda_1\overline{\lambda_2^{-1}})\|A(ix_2)\|^2.\end{aligned}$$

If we rewrite the previous identity in terms of real and imaginary parts of λ_2 and λ_1 we arrive to

$$\begin{aligned}0 =& s\|A(x_1)\|^2 + t\langle A(x_1)|A(ix_1)\rangle + \alpha\frac{\beta t - \alpha s}{\alpha^2 + \beta^2}\|A(x_2)\|^2 \\ &- \alpha\frac{\beta s + \alpha t}{\alpha^2 + \beta^2}\langle A(x_2)|A(ix_2)\rangle + \beta\frac{\beta t - \alpha s}{\alpha^2 + \beta^2}\langle A(ix_2)|A(x_2)\rangle - \beta\frac{\beta s + \alpha t}{\alpha^2 + \beta^2}\|A(ix_2)\|^2,\end{aligned} \quad (4)$$

for all $\alpha, \beta, s, t \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$. Taking a quadruple of the form $(\alpha, \beta, s, t) = (1, 0, -1, 0)$ in (4) we get

$$0 = -\|A(x_1)\|^2 + \|A(x_2)\|^2. \quad (5)$$

By applying the latter equality to simplify (4) we obtain

$$\begin{aligned}0 =& \beta\frac{\beta s + \alpha t}{\alpha^2 + \beta^2}\|A(x_1)\|^2 + t\langle A(x_1)|A(ix_1)\rangle - \alpha\frac{\beta s + \alpha t}{\alpha^2 + \beta^2}\langle A(x_2)|A(ix_2)\rangle \\ &+ \beta\frac{\beta t - \alpha s}{\alpha^2 + \beta^2}\langle A(ix_2)|A(x_2)\rangle - \beta\frac{\beta s + \alpha t}{\alpha^2 + \beta^2}\|A(ix_2)\|^2,\end{aligned} \quad (6)$$

for all $\alpha, \beta, s, t \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$. Take now $(\alpha, \beta, s, t) = (1, 0, 0, 1)$ to obtain

$$\langle A(x_1)|A(ix_1)\rangle - \langle A(x_2)|A(ix_2)\rangle = 0. \tag{7}$$

By combining (7) and (6) we arrive to

$$\begin{aligned} 0 = \beta\frac{\beta s + \alpha t}{\alpha^2+\beta^2}\|A(x_1)\|^2 + \beta\frac{\beta t - \alpha s}{\alpha^2+\beta^2}\langle A(x_1)|A(ix_1)\rangle \\ + \beta\frac{\beta t - \alpha s}{\alpha^2+\beta^2}\langle A(ix_2)|A(x_2)\rangle - \beta\frac{\beta s + \alpha t}{\alpha^2+\beta^2}\|A(ix_2)\|^2, \end{aligned}$$

which leads to

$$0 = (\beta s + \alpha t)\left(\|A(x_1)\|^2 - \|A(ix_2)\|^2\right) + 2(\beta t - \alpha s)\Re\mathrm{e}\langle A(x_1)|A(ix_1)\rangle, \tag{8}$$

for all $\alpha, \beta, s, t \in \mathbb{R}$ with $\beta \neq 0$. In the cases $(\alpha, \beta, s, t) = (0, 1, 0, 1)$ and $(\alpha, \beta, s, t) = (0, 1, 1, 0)$ we get

$$\Re\mathrm{e}\langle A(x_1)|A(ix_1)\rangle = 0, \text{ and } \|A(x_1)\|^2 = \|A(ix_2)\|^2, \text{ respectively.}$$

Now, having in mind that $A(ix_1) \in \mathbb{C}A(x_1)$ and both are non-zero, $\|A(ix_1)\| = \|A(ix_2)\|$, and $\|A(x_1)\| = \|A(x_2)\|$ (cf. (3) and (5)), it can be easily deduced that $A(ix_1) \in \{\pm iA(x_1)\}$. The roles of x_1 and x_2 are clearly interchangeable, so $A(ix_2) \in \{\pm iA(x_2)\}$. Let us write $A(ix_j) = \sigma_j(i)A(x_j)$ for $j \in \{1, 2\}$ and $\sigma_j(i) \in \{\pm i\}$. Furthermore, it follows from the equality $\langle A(x_1)|A(ix_1)\rangle = \langle A(x_2)|A(ix_2)\rangle$ (cf. (7)) that $\sigma_1(i) = \sigma_2(i)$. Back to (2) we easily check that A restricted to the linear complex span of x_1 and x_2 must be complex-linear or conjugate-linear.

We have therefore shown that if $\{x_1, x_2\}$ is an orthonormal system in H, the restriction of A to the complex-linear span of $\{x_1, x_2\}$ must be complex-linear or conjugate-linear.

Let us pick a norm-one vector z_0 in H. It follows from the above that $A(iz_0) = iA(z_0)$ or $A(iz_0) = -iA(z_0)$. If the first case holds, for each norm-one element x in H, there exists another norm-one element x_2 in H such that $\{z_0, x_2\}$ is an orthonormal system and x belongs to its linear span. Since $A(iz_0) = iA(z_0)$, the previous conclusion proves that the restriction of A to the linear span of $\{z_0, x_2\}$ is complex-linear, and thus $A(ix) = iA(x)$, which implies that A is complex-linear. If $A(iz_0) = -iA(z_0)$, similar arguments show that A is conjugate-linear.

We have implicitly shown that A is a positive scalar multiple of an isometry. Namely, the conclusions in the first part of the proof guarantee that for each non-zero $x_1 \in H$, there exists a positive $\gamma_1 \in \mathbb{R}$ satisfying $\|A(x_1)\| = \gamma_1\|x_1\|$. If x_2 is another non-zero vector in H with $x_1 \perp_2 x_2$ (we can clearly assume that $\|x_1\| = \|x_2\| = 1$), by considering the subspace $H_1 = \mathbb{R}x_1 \oplus \mathbb{R}x_2$, as in the second part of the proof, we deduce from (5) that $\gamma_1 = \gamma_1\|x_1\| = \|A(x_1)\| = \|A(x_2)\| = \gamma_2\|x_2\| = \gamma_2$. We have therefore shown that if $\{x_1, x_2\}$ is an orthonormal system in H, we have $\|A(x_1)\| = \|A(x_2)\|$. Furthermore, since A is linear or conjugate-linear, for each $x \in \mathbb{C}x_1 \oplus^{\ell_2} \mathbb{C}x_2$ we also have

$$\begin{aligned}\|A(x)\| =& \|A(\lambda_1 x_1 + \lambda_2 x_2)\| \in \{\|\lambda_1 A(x_1) + \lambda_2 A(x_2)\|, \|\overline{\lambda_1} A(x_1) + \overline{\lambda_2} A(x_2)\|\} \\ &= \left\{\sqrt{|\lambda_1|^2\|A(x_1)\|^2 + |\lambda_2|^2\|A(x_2)\|^2}, \sqrt{|\overline{\lambda_1}|^2\|A(x_1)\|^2 + |\overline{\lambda_2}|^2\|A(x_2)\|^2}\right\} \\ &= \|A(x_1)\|\sqrt{|\lambda_1|^2 + |\lambda_2|^2} = \|A(x_1)\|\|\lambda_1 x_1 + \lambda_2 x_2\| = \|A(x_1)\|\|x\|.\end{aligned}$$

Finally, a standard argument, like the one employed in the previous paragraph, assures that $\|A(x)\| = \|A(x_1)\|$ for every couple of norm-one vectors $x, x_1 \in H$, which finishes the proof.

Although the reasoning in the above lines is self-contained, and almost explicit from what we proved before, there is also another method to deduce that A is a positive scalar multiple of a linear or conjugate-linear isometry. We already deduced that A is linear or conjugate-linear (and preserves orthogonality). In the first case, we can apply Chmieliński' theorem [2, Theorem 1] to A and we get the desired conclusion. In the second case, let $\overline{K}$ be the complex inner product space obtained from K by replacing the complex structure with the conjugate one, that is, $\lambda \odot x := \overline{\lambda} x$ ($x \in \overline{K}$, $\lambda \in \mathbb{C}$) and inner product $\langle x|y\rangle^{rev} = \langle y|x\rangle$ ($x, y \in \overline{K}$). The mapping $A^{rev} : H \to \overline{K}$, $A^{rev}(x) = A(x)$ is a linear mapping preserving orthogonality, and hence Chmieliński' theorem proves that A is positive scalar multiple of a conjugate-linear isometry. $\square$

The problem of determining when a real-linear mapping between complex Banach spaces is actually complex-linear or conjugate-linear is a topic studied in several contributions, for example Dang established in [3, Proposition 2.6] that every real-linear surjective isometry between Cartan factors with rank ≥ 2 must be either (complex) linear or conjugate-linear. In the case of rank-one Cartan factors (i.e. complex Hilbert spaces) the conclusion does not hold. For example, the mapping $R : \ell_2 \to \ell_2$, $R((\lambda_n)_n) = \left(\frac{1+(-1)^n}{2}\lambda_n + \frac{1-(-1)^n}{2}\overline{\lambda_n}\right)_n$ is a surjective real-linear which is not complex-linear nor conjugate-linear. In our result, the Hilbert spaces are rank-one Cartan factors and the mapping A is not assumed to be surjective nor isometric, however, the hypothesis of being an orthogonality preserving additive mapping forces A to be complex-linear or conjugate-linear. Similarly, the conclusions around Tingley's problem in the case of Hilbert spaces assert that every isometric mapping from the unit sphere of a Hilbert space H "into" the unit sphere of another Hilbert space K can be extended to a real-linear isometric mapping from H into K (see [4, Theorem 2.2 and Corollary 2]), but nothing can be concluded about the (complex) linearity or conjugate-linearity of the extension.

It is natural to ask whether a generalization of Wójcik theorem holds for complex normed spaces and Birkhoff orthogonality. Concerning this question, we can present some partial answer in the case that the domain is a complex normed space admitting a conjugation. We recall that a conjugation on a complex normed space X is a period-2 conjugate-linear isometry $\tau : X \to X$. Define a conjugation $\tau^\sharp : X^* \to X^*$ given by $\tau^\sharp(\phi)(x) := \overline{\phi(\tau(x))}$ ($\phi \in X^*$, $x \in X$). The sets $X^\tau = \{x \in X : \tau(x) = x\}$ and $(X^*)^{\tau^\sharp} = \{\phi \in X^* : \tau^\sharp(\phi) = \phi\}$ are real-linear subspaces of X and X^*, respectively. By construction $\phi(X^\tau) \subseteq \mathbb{R}$ for all $\phi \in (X^*)^{\tau^\sharp}$, and $X = X^\tau \oplus iX^\tau$. It is also known that the mapping $(X^*)^{\tau^\sharp} \ni \phi \mapsto \phi|_{X^\tau} = \Re\mathrm{e}\phi|_{X^\tau}$ is a surjective linear isometry from $(X^*)^{\tau^\sharp}$ onto $(X^\tau)^*$.

Proposition 2.2. *Let X and Y be complex normed spaces with $dim(X) \geq 2$, and assume that X admits a conjugation τ. Let $A : X \to Y$ be an additive mapping preserving Birkhoff orthogonality. Then A is real-linear. Furthermore, if A is surjective and preserves Birkhoff orthogonality in both directions, then A is a real-linear isomorphism and the underlying real normed spaces $X_\mathbb{R}$ and $Y_\mathbb{R}$ are isomorphic.*

Proof. A characterization of the Birkhoff orthogonality via Hahn-Banach theorem assures that elements x, y in a real or complex normed space Z are Birkhoff orthogonal if and only if there exists a norm-one functional $\phi \in Z^*$ satisfying $\phi(x) = \|x\|$ and $\phi(y) = 0$. Consider the real subspace X^τ. If $x \perp_B y$ in X^τ there exists a norm-one functional $\phi \in (X^\tau)^* \equiv (X^*)^{\tau^\sharp}$ satisfying $\phi(x) = \|x\|$ and $\phi(y) = \overline{\phi(\tau(y))} = \Re\mathrm{e}\phi(y) = 0$. In particular, ϕ is a norm-one functional in X^* (with $\tau^\sharp(\phi) = \phi$), and hence $x \perp_B y$ in X. Therefore, $x \perp_B y$ in X^τ if, and only if, $x \perp_B y$ in X. This implies that $A|_{X^\tau} : X^\tau \to Y_\mathbb{R}$ is an additive mapping preserving Birkhoff orthogonality between two normed real spaces. So, Wójcik's theorem [8, Theorem 3.1] implies that $A|_{X^\tau}$ is real-linear and there exists a positive constant γ_1 satisfying $\|A(x)\| = \gamma_1\|x\|$ for all $x \in X^\tau$. By applying a similar argument to the real subspace iX^τ, whose dual space can be identified with $i(X^*)^{\tau^\sharp} = \{\phi \in X^* : \tau^\sharp(\phi) = -\phi\}$ (or by simply replacing τ with $\tau_1 = -\tau$, and apply the above argument to $X^{\tau_1} = iX^\tau$), we deduce via Wójcik's theorem [8, Theorem 3.1] that $A|_{iX^\tau}$ is real-linear too, and there exists a positive constant γ_2 satisfying $\|A(iy)\| = \gamma_2\|iy\| = \|y\|$ for all $y \in X^\tau$. Finally, since $X = X^\tau \oplus iX^\tau$, the mapping A must be real linear. Furthermore,

$$\|A(x + iy)\| \leq \|A(x)\| + \|A(iy)\| = \gamma_1\|x\| + \gamma_2\|y\| \leq (\gamma_1 + \gamma_2)\|x + iy\|,$$

for all $x + iy \in X^\tau \oplus iX^\tau = X$.

Assume now that A is surjective and preserves Birkhoff orthogonality in both directions (and, of course, real-linear). We first observe that A is injective. Namely, if $A(x) = 0$, it follows that $A(x) \perp_B A(z)$ for all $z \in X$, and hence $x \perp_B z$ for all $z \in X$, which clearly gives $x = 0$. Therefore, A is a continuous real linear bijection. By replacing A with A^{-1} we deduce that A is a real linear isomorphism of normed spaces. □

Paraphrasing R. Tanaka [7], the conclusion in our Theorem 2.1 assures that the vector addition and the relationship of orthogonality determine the entire structure of a complex inner product space, while by Proposition 2.2 vector addition and the relationship of Birkhoff orthogonality determine isomorphically the entire structure of a complex normed space admitting a conjugation. It can be added that in [6], Tanaka studies the problem whether the existence of a (possibly non-additive) bijection Δ preserving Birkhoff–James orthogonality in both directions between two real Banach spaces X and Y assures the existence of a linear isomorphism Φ between X and Y, however no conclusion is obtained on the mapping Δ itself. Positive answers are known when X is finite-dimensional, or when X and Y are reflexive and smooth, or when X is a Hilbert space with $\dim(X) \geq 3$ (cf. [6]).

Declaration of competing interest

All authors declare that they have no conflicts of interest to disclose.

Acknowledgements

L. Li was supported by National Natural Science Foundation of China (Grant No. 12171251). A.M. Peralta supported by grant PID2021-122126NB-C31 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU, by Junta de Andalucía grant FQM375, IMAG–María de Maeztu grant CEX2020-001105-M/AEI/10.13039/501100011033 and (MOST) Ministry of Science and Technology of China grant G2023125007L.

Part of this work was completed during a visit of A.M. Peralta to Nankai University and the Chern Institute of Mathematics, which he thanks for the hospitality.

We thank B. Kuzma for addressing our attention to Wójcik's paper [8].

Data availability

No data was used for the research described in the article.

References

[1] A. Blanco, A. Turnšek, On maps that preserve orthogonality in normed spaces, Proc. R. Soc. Edinb., Sect. A 136 (2006) 709–716.
[2] J. Chmieliński, Linear mappings approximately preserving orthogonality, J. Math. Anal. Appl. 304 (1) (2005) 158–169.
[3] T. Dang, Real isometries between JB*-triples, Proc. Am. Math. Soc. 114 (4) (1992) 971–980.
[4] G. Ding, The 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real-linear isometry of the whole space, Sci. China Ser. A 45 (4) (2002) 479–483.
[5] A. Koldobsky, Operators preserving orthogonality are isometries, Proc. R. Soc. Edinb., Sect. A 123 (1993) 835–837.
[6] R. Tanaka, Nonlinear equivalence of Banach spaces based on Birkhoff-James orthogonality, J. Math. Anal. Appl. 505 (1) (2022) 125444.
[7] R. Tanaka, On Birkhoff-James orthogonality preservers between real non-isometric Banach spaces, Indag. Math., New Ser. 33 (6) (2022) 1125–1136.
[8] P. Wójcik, Mappings preserving B-orthogonality, Indag. Math., New Ser. 30 (1) (2019) 197–200.