Results Math (2025) 80:62

1422-6383/25/020001-22
published online February 28, 2025
https://doi.org/10.1007/s00025-025-02376-8

Results in Mathematics

Translators of the Mean Curvature Flow in the Special Linear Group $\mathrm{SL}(2,\mathbb{R})$

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Abstract. Translators in the special linear group $\mathrm{SL}(2,\mathbb{R})$ are surfaces whose mean curvature H and unit normal vector N satisfy $H=\langle N,X\rangle$, where X is a fixed Killing vector field. In this paper we study and classify those translators that are invariant by a one-parameter group of isometries. By the Iwasawa decomposition, there are three types of such groups. The dimension of the Killing vector fields is 4 and an exhaustive discussion is done for each one of the Killing vector fields and each of the invariant surfaces. In some cases, explicit parametrizations of translators are obtained.

Mathematics Subject Classification. Primary 53A10; Secondary 53C44, 53C21, 53C42.

Keywords. mean curvature flow, translator, Iwasawa decomposition, invariant surface.

1. Introduction and Preliminaries

In the theory of mean curvature flow (MCF for short) in Euclidean space $\mathbb{R}^3$, translators of the MCF are surfaces that evolve purely by translations of $\mathbb{R}^3$. A translator $\Sigma\subset\mathbb{R}^3$ is characterized by

$$H=\langle N,\mathbf{v}\rangle, \tag{1}$$

where H and N are the mean curvature and unit normal of Σ respectively and $\mathbf{v}$ is the direction of translations of $\mathbb{R}^3$. The role of translators is important because they are, after rescaling, a type of singularities of the MCF according to Huisken and Sinestrari [10]. The simplest example is any plane parallel to **v**. Other examples of translators are those that are invariant by a one-parameter group of isometries of $\mathbb{R}^3$. In $\mathbb{R}^3$ there are three types of such groups: translations, rotations about an axis and helicoidal motions about an axis. If the translator is invariant by translations, then it is a plane parallel to **v**, the grim reaper or a tilted grim reaper. On the other hand, if the translator is a surface of revolution, then the rotation axis is parallel to **v**. In such a case there are two types of translators depending whether the surface intersects the rotation axis (bowl soliton) or not (wing-like rotational translators) [1,7]. If the translator is invariant by helicoidal motions, existence and properties have been obtained in [9,13]. Recently, there is a great interest in extending the notion of translators and solitons in general of the MCF in other homogenous spaces. Without to be complete, we refer: hyperbolic space [4,5,17,19]; the product $\mathbb{H}^2\times\mathbb{R}$ [2,3,6,16]; the product $\mathbb{S}^2\times\mathbb{R}$ [18]; the Sol space [21]; and the Heisenberg group [22].

In this paper we study translators of the MCF in the special linear group $\mathrm{SL}(2,\mathbb{R})$ motivated by the Euclidean setting. For this, we first need to give the definition of translator replacing **v** in (1) by a Killing vector field of $\mathrm{SL}(2,\mathbb{R})$. In a second step, we need to consider surfaces invariant by some one-parameter subgroups of $\mathrm{SL}(2,\mathbb{R})$. For this, we need to recall the NAK decomposition in $\mathrm{SL}(2,\mathbb{R})$ that generates some symmetries for surfaces in $\mathrm{SL}(2,\mathbb{R})$. Once we have the definition of translators and the surfaces invariant by one-parameter subgroups of $\mathrm{SL}(2,\mathbb{R})$, we will describe each type of translator invariant by each such a one-parameter subgroups of symmetries.

The space $\mathrm{SL}(2,\mathbb{R})$ is viewed as a homogeneous space equipped with a canonical left-invariant Riemannian metric whose group of isometries is of dimension 4. For the definition of translator in $\mathrm{SL}(2,\mathbb{R})$, we extend the notion in (1) replacing **v** by a Killing vector field of $\mathrm{SL}(2,\mathbb{R})$.

Definition 1.1. *Let $X\in\mathfrak{X}(\mathrm{SL}(2,\mathbb{R}))$ be a Killing vector field. A surface Σ in $\mathrm{SL}(2,\mathbb{R})$ is said to be a X-translator if its mean curvature H and unit normal vector N satisfy*

$$H=\langle N,X\rangle. \tag{2}$$

Let

$$\mathrm{SL}(2,\mathbb{R})=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in\mathbb{R}, ad-bc=1\right\}.$$

For the definition of the Riemannian metric, we recall the so-called Iwasawa decomposition [12]. There are three one-dimensional subgroups of $\mathrm{SL}(2,\mathbb{R})$

$$\mathcal{N}=\left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix} : x\in\mathbb{R}\right\},$$
$$\mathcal{A}=\left\{\begin{pmatrix} \sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix} : y\in\mathbb{R}_+\right\},$$

$$\mathcal{K} = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

that generate the whole group $\mathrm{SL}(2,\mathbb{R})$. More precisely, by the Iwasawa decomposition, denoted by NAK, for every $A \in \mathrm{SL}(2,\mathbb{R})$ there is a unique representation of A given by $A = nak$, where $n \in \mathcal{N}$, $a \in \mathcal{A}$ and $k \in \mathcal{K}$. This allows to give global coordinates (x, y, θ) in $\mathrm{SL}(2,\mathbb{R})$ by means of

$$(x,y,\theta) \in \mathbb{R}^3 \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{3}$$

With respect to these coordinates, let $\{\partial_x, \partial_y, \partial_\theta\}$ be the canonical basis of $\mathfrak{X}(\mathrm{SL}(2,\mathbb{R}))$. Notice the isomorphisms $\mathcal{N} = (\mathbb{R}, +)$, $\mathcal{A} = (\mathbb{R}_+, \cdot)$ and $\mathcal{K} = (\mathbb{S}^1, \cdot)$.

Remark 1.2. *If $A \in \mathrm{SL}(2,\mathbb{R})$, then the characteristic polynomial of A is $\lambda^2 - trace(A)\lambda + 1$. This distinguishes the matrices of $\mathrm{SL}(2,\mathbb{R})$ in three types depending on the number of roots of this polynomial. This classification is equivalent to the NAK decomposition. Indeed, if $|trace(A)| = 2$, there is a unique double eigenvalue; if $|trace(A)| > 2$, there are two distinct real eigenvalues and if $|trace(A)| < 2$, there are no real eigenvalues. Examples of such matrices are, respectively, that of $\mathcal{N}$, $\mathcal{A}$ and $\mathcal{K}$. In the literature, the elements of the subgroups $\mathcal{N}$, $\mathcal{A}$ and $\mathcal{K}$ are also called parabolic, hyperbolic and elliptic matrices, respectively [14].*

Let $\mathbb{H}^2(-4)$ be the hyperbolic plane of constant curvature -4 and its upper half plane model

$$\mathbb{H}^2(-4) = \left(\mathbb{R}^2_+, \frac{dx^2 + dy^2}{4y^2} \right).$$

The special linear group $\mathrm{SL}(2,\mathbb{R})$ acts transitively and isometrically on $\mathbb{H}^2(-4)$ by the linear fractional transformation

$$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (x,y) = x + iy = z \right) \mapsto A \cdot z = \frac{az + b}{cz + d}.$$

The isotropy subgroup of $\mathrm{SL}(2,\mathbb{R})$ at $i = (0,1)$ is the subgroup $\mathcal{K}$. In terms of the NAK decomposition, the natural projection is

$$\pi : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})/\mathcal{K} = \mathbb{H}^2(-4), \quad \pi(x,y,\theta) = (x,y). \tag{4}$$

The mapping

$$\psi : \mathbb{H}^2(-4) \times \mathbb{S}^1 \to \mathrm{SL}(2,\mathbb{R}),$$

$$\psi(x,y,\theta) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is a diffeomorphism. In particular, the space $\mathrm{SL}(2,\mathbb{R})$ is topologically the open solid torus $\mathbb{D}\times\mathbb{S}^1$. If we endow $\mathrm{SL}(2,\mathbb{R})$ by the metric $\langle\ ,\ \rangle$ which makes ψ an isometry, the expression of $\langle,\rangle$ is

$$\langle\ ,\ \rangle=\frac{dx^2+dy^2}{4y^2}+\left(d\theta+\frac{dx}{2y}\right)^2.$$

With this metric, the projection π defined in (4) becomes a Riemannian submersion.

The space of Killing vector fields in $\mathrm{SL}(2,\mathbb{R})$ is of dimension 4 and it is generated by

$$\left\{\partial_x,\partial_\theta,x\partial_x+y\partial_y,\frac{1}{2}(x^2-y^2)\partial_x+xy\partial_y\right\}. \tag{5}$$

In order to give examples of translators of $\mathrm{SL}(2,\mathbb{R})$, and following the motivation from the Euclidean case, we will assume that the surface is invariant by some one-parameter subgroup of symmetries. In $\mathrm{SL}(2,\mathbb{R})$ there are three fundamental one-parameter subgroups of symmetries, as we have previously seen, obtained from the NAK decomposition of $\mathrm{SL}(2,\mathbb{R})$. This allows to give the following definitions.

Definition 1.3. *Let Σ be an immersed surface in* $\mathrm{SL}(2,\mathbb{R})$. *We say that Σ is $\mathcal{N}$-invariant (resp. $\mathcal{A}$-invariant, $\mathcal{K}$-invariant) if Σ is invariant under the left translations of the subgroup $\mathcal{N}$ (resp. $\mathcal{A}$, $\mathcal{K}$). Moreover, $\mathcal{K}$-invariant surfaces are also known as rotational surfaces, while $\mathcal{A}$-invariant surfaces are often called conoids.*

Invariant surfaces in $\mathrm{SL}(2,\mathbb{R})$ with constant mean curvature or constant Gauss curvature have been studied in [8,11,15,20,23].

Once we have established the definition of an invariant surface, the work ahead is the classification of invariant X-translators depending of the Killing vector field X of (5). The paper is organized in sections according to the Killing vector field X. In each section, namely, Sects. 3, 4, 5 and 6, we will study and classify the X-translators that are invariant by each of the three subgroups $\mathcal{N}$, $\mathcal{A}$ and $\mathcal{K}$. Previously, in Sect. 2, we compute the unit normal vector N and the mean curvature H of the invariant surfaces. These computations are needed to study the translator equation (2).

By the variety of vector fields and invariant surfaces, we summarize in Table 1 the results of classification obtained in this paper. In the table, by explicit parametrization, we mean that we obtain a parametrization of the surface by known functions. Other surfaces that we obtain are those where one of the coordinates x, y or θ in the Iwasawa decomposition are constant. Let Σ_{x_0}, Σ_{y_0} and Σ_{θ_0} be the corresponding surfaces, respectively. Finally, in the case of $\mathcal{K}$-invariant translator, by ODE we mean that we obtain the differential equation that describes the generating curve of the surface. In general, this equation is difficult to be studied in all its generality.

TABLE 1. Classification of the invariant X-translators

	$\mathcal{N}$-surfaces	$\mathcal{A}$-surfaces	$\mathcal{K}$-surfaces
∂_x	Explicit parametrization	Σ_{θ_0}	Description
∂_θ	Σ_{θ_0}, explicit parametrization	Σ_{x_0}	Minimal surface
$x\partial_x + y\partial_y$	Explicit parametrization	$\Sigma_{x_0=0}$, Σ_{θ_0}	ODE
$\frac{1}{2}(x^2 - y^2)\partial_x + xy\partial_y$	Σ_{θ_0}	Σ_{θ_0}	ODE

We obtain two direct consequences.

Corollary 1.4. *All $\mathcal{N}$-invariant translators have explicit parametrizations by known functions.*

Corollary 1.5. *The only $\mathcal{A}$-invariant translators are of type Σ_{x_0} or Σ_{θ_0} depending on the case.*

2. The Mean Curvature of Invariant Surfaces

In this section, we compute the unit normal vector N and the mean curvature H of surfaces invariant by each one of the three subgroups $\mathcal{N}$, $\mathcal{A}$ and $\mathcal{K}$ of $\mathrm{SL}(2,\mathbb{R})$. Part of the computations of this section have appeared in [15,20]. By completeness of the paper and for the subsequent study of the X-translators, we recall them. Consider in $\mathrm{SL}(2,\mathbb{R})$ the orthonormal frame $B = \{e_1, e_2, e_3\}$ defined by

$$e_1 = 2y\partial_x - \partial_\theta,\ e_2 = 2y\partial_y,\ e_3 = \partial_\theta. \tag{6}$$

The Levi-Civita connection ∇ of the metric $\langle\ ,\ \rangle$ of $\mathrm{SL}(2,\mathbb{R})$ is given by the relations

$$\begin{array}{lll} \nabla_{e_1}e_1 = 2e_2, & \nabla_{e_1}e_2 = -2e_1 - e_3, & \nabla_{e_1}e_3 = e_2, \\ \nabla_{e_2}e_1 = e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = -e_1, \\ \nabla_{e_3}e_1 = e_2, & \nabla_{e_3}e_2 = -e_1, & \nabla_{e_3}e_3 = 0. \end{array}$$

The expressions of Killing vector fields (5) in terms of the above basis are

$$\begin{aligned} \partial_x &= \frac{1}{2y}(e_1 + e_3), \\ \partial_\theta &= e_3, \\ x\partial_x + y\partial_y &= \frac{1}{2y}(xe_1 + ye_2 + xe_3), \\ \frac{1}{2}(x^2 - y^2)\partial_x + xy\partial_y &= \frac{1}{2y}\left(\frac{1}{2}(x^2 - y^2)e_1 + xye_2 + \frac{1}{2}(x^2 - y^2)e_3\right). \end{aligned}$$

We now give parametrizations of the invariant surfaces in order to compute N and H. To have a consistent notation, if $\Psi = \Psi(s,t)$ is the parametrization in local coordinates on the invariant surface, the parameter s will be assigned for the generating curve of the surface, whereas t will denote the parameter of the group.

Proposition 2.1. *An $\mathcal{N}$-invariant surface of* $\mathrm{SL}(2,\mathbb{R})$ *can be parametrized by*

$$\Psi(s,t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y(s)} & 0 \\ 0 & 1/\sqrt{y(s)} \end{pmatrix} \begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{pmatrix}, \tag{7}$$

where $t \in \mathbb{R}$, $s \in I \subset \mathbb{R}$. The generating curve is $\alpha(s) = (y(s), \theta(s))$. The unit normal N and the mean curvature H are

$$\begin{aligned} N &= \frac{y'}{\sqrt{2}\Phi}(e_1 - e_3) + \frac{\sqrt{2}y\theta'}{\Phi} e_2, \\ H &= \frac{\sqrt{2}y^2}{\Phi^3}\left(\theta' y'' - y'\theta'' + 2y\theta'^3\right), \end{aligned} \tag{8}$$

where $\Phi = \sqrt{y'^2 + 2y^2\theta'^2}$. If we take $\alpha(s)$ such that

$$\begin{aligned} y'(s) &= \sqrt{2}y(s)\cos\varphi(s), \\ \theta'(s) &= \sin\varphi(s), \end{aligned} \tag{9}$$

then

$$\begin{aligned} N &= \frac{1}{\sqrt{2}}\cos\varphi(e_1 - e_3) + \sin\varphi e_2, \\ H &= -\frac{\varphi'}{\sqrt{2}} + \sin\varphi. \end{aligned} \tag{10}$$

Proposition 2.2. *An $\mathcal{A}$-invariant surface of* $\mathrm{SL}(2,\mathbb{R})$ *can be parametrized by*

$$\Psi(s,t) = \begin{pmatrix} 1 & x(s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix} \begin{pmatrix} \cos\theta(s) & \sin\theta(s) \\ -\sin\theta(s) & \cos\theta(s) \end{pmatrix}, \tag{11}$$

where $t \in \mathbb{R}^+$, $s \in \mathbb{R}$. The generating curve is $\alpha(s) = (x(s), \theta(s))$. The unit normal N and the mean curvature H are

$$\begin{aligned} N &= \frac{1}{\Phi}\left(-\left(x' + 2t\theta'\right)e_1 + x' e_3\right) \\ H &= \frac{2t^2}{\Phi^3}(x'\theta'' - \theta' x''), \end{aligned} \tag{12}$$

where

$$\Phi = \sqrt{(x' + 2t\theta')^2 + x'^2}.$$

Proposition 2.3. *A rotational surface of* $\mathrm{SL}(2,\mathbb{R})$ *can be parametrized by*

$$\Psi(s,t) = \begin{pmatrix} 1 & x(s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y(s)} & 0 \\ 0 & 1/\sqrt{y(s)} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \tag{13}$$

where $s \in I \subset \mathbb{R}$, $t \in \mathbb{R}$. The generating curve is $\alpha(s) = (x(s), y(s)) \in \mathbb{H}^2(-4)$. The unit normal N and the mean curvature H are

$$\begin{aligned} N &= \frac{1}{\Phi}\left(-y'e_1 + x'e_2\right) \\ H &= \frac{1}{\Phi^3}(y(x'y'' - x''y') + x'\Phi^2), \end{aligned} \tag{14}$$

where $\Phi = \sqrt{x'^2 + y'^2}$. If α is parametrized by arclength, then there is $\varphi = \varphi(s)$ such that

$$\begin{aligned} x'(s) &= 2y(s)\cos\varphi(s), \\ y'(s) &= 2y(s)\sin\varphi(s), \end{aligned} \tag{15}$$

which implies

$$\begin{aligned} N &= -\sin\varphi(s)e_1 + \cos\varphi(s)e_2 \\ H &= \frac{\varphi'(s)}{2} + \cos\varphi(s). \end{aligned} \tag{16}$$

We emphasize three particular examples of invariant surfaces which are defined by fixing one of the three coordinates x, y or θ in $\mathrm{SL}(2,\mathbb{R})$.

Example 2.4. *Let Σ_{x_0} be the surface in $\mathrm{SL}(2,\mathbb{R})$ defined by*

$$\Psi(s,t) = \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}. \tag{17}$$

The surface Σ_{x_0} is a Hopf cylinder over a geodesic in the hyperbolic plane $\mathbb{H}^2(-4)$, hence it is both minimal and flat. The induced metric is $g_{x_0} = ds^2 + \frac{dt^2}{4t^2}$. The surface Σ_{x_0} is both $\mathcal{A}$-invariant and $\mathcal{K}$-invariant. As the unit normal is $N = -e_1$, the surface Σ_{x_0} is a translator with respect to ∂_θ (for any x_0) and the surface $\Sigma_{x_0=0}$ is a translator with respect to $x\partial_x + y\partial_y$, too.

Example 2.5. *Let Σ_{y_0} be the surface in $\mathrm{SL}(2,\mathbb{R})$ defined by*

$$\Psi(s,t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y_0} & 0 \\ 0 & 1/\sqrt{y_0} \end{pmatrix} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}. \tag{18}$$

The surface Σ_{y_0} is a Hopf cylinder over a Riemannian circle in the hyperbolic plane $\mathbb{H}^2(-4)$, hence it is flat and of constant mean curvature $H = 1$. The induced metric is $g_{y_0} = \frac{dt^2}{4y_0^2} + (ds + \frac{dt}{2y_0})^2$. The surface Σ_{y_0} is both $\mathcal{N}$-invariant and $\mathcal{K}$-invariant. The unit normal is $N = e_2$ and $H = 1$. The product of N by the first two vector fields in (5) is 0, and by the last two ones is $\frac{1}{2}$. Thus Σ_{y_0} is not a translator.

Example 2.6. *Let Σ_{θ_0} be the surface in $\mathrm{SL}(2,\mathbb{R})$ defined by*

$$\Psi(s,t) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{t} & 0 \\ 0 & 1/\sqrt{t} \end{pmatrix} \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}. \tag{19}$$

The surface Σ_{θ_0} is both $\mathcal{N}$-invariant and $\mathcal{A}$-invariant surface. The induced metric is $g_{\theta_0} = \frac{2ds^2+dt^2}{4t^2}$ and its curvature is constant -4, that is Σ_{θ_0} is the *hyperbolic plane* $\mathbb{H}^2(-4)$*. Moreover,* Σ_{θ_0} *is a minimal surface in* $\mathrm{SL}(2,\mathbb{R})$ *by* (8)*. The unit normal is* $N = \frac{1}{\sqrt{2}}(e_1 - e_3)$*. Therefore,* Σ_{θ_0} *is a translator with respect to* ∂_x*,* $x\partial_x + y\partial_y$ *and* $\frac{1}{2}(x^2 - y^2)\partial_x + xy\partial_y$*, respectively.*

We end this section with a particular example of rotational surface.

Example 2.7. *Let* Σ *be a rotational surface whose generating curve* α *is a straight-line. Then* α *is parametrized by* (15) *where the function* φ *is constant.*

(1) Case $\sin\varphi = 0$*. Without loss of generality, we can suppose* $\varphi(s) = 0$*. An integration of* (15) *gives* $\alpha(s) = (2c_2 s + c_1, c_2)$*,* $c_1, c_2 \in \mathbb{R}$*,* $c_2 > 0$*. Then* $H = 1$ *and it is not difficult to check that* Σ *is not a* X*-translator for any vector field* X *of* (5)*.*

(2) Case $\sin\varphi \neq 0$*. The solution of* (15) *is* $\alpha(s) = c_2 e^{2s\sin\varphi}(\cot\varphi, 1) + (c_1, 0)$*,* $c_1, c_2 \in \mathbb{R}$*,* $c_2 > 0$*. Now* $H = \cos\varphi$ *and* Σ *is not a* X*-translator with the first and fourth vector fields of* (5)*. For the vector fields* ∂_θ *and* $x\partial_x + y\partial_y$*, we have* $\langle N, \partial_\theta\rangle = \langle N, x\partial_x + y\partial_y\rangle = 0$*. Thus* Σ *is a* ∂_θ*-translator and a* $(x\partial_x + y\partial_y)$*-translator if and only if* $\cos\varphi = 0$*, that is,* $\varphi(s) = \pi/2$*.*

3. Translators by the Vector Field ∂_x

Consider the Killing vector field ∂_x. We know that with respect to B, this vector field is

$$\partial_x = \frac{1}{2y}(e_1 + e_3).$$

Theorem 3.1. *(1) Let* Σ *be an* $\mathcal{N}$*-invariant* ∂_x*-translator. Then* Σ *is either a surface of type* Σ_{θ_0} *given by* (19)*, or a minimal surface parametrized by* (7) *with* $\theta(s) = s$ *and* $y(s) = c_1\cos(\sqrt{2}s) + c_2\sin(\sqrt{2}s)$*, where* c_1 *and* c_2 *are real constants. The interval for* s *is such that* $y(s) > 0$*.*

(2) The only $\mathcal{A}$*-invariant* ∂_x*-translators are the surfaces of type* Σ_{θ_0}*.*

(3) Let Σ *be a rotational surface whose generating curve is parametrized by* (15)*. If* Σ *is a* ∂_x*-translator, then*

$$\varphi' = -\frac{\sin\varphi + 2y\cos\varphi}{y}. \tag{20}$$

The generating curve α *is a bi-graph over the line* $y = 0$ *and converges to it as* $s \to \infty$*. See Fig. 1.*

Proof. *(1) From* (8)*, we have* $\langle N, \partial_x\rangle = 0$*, hence the surface is minimal. If* $\theta = \theta_0$ *is a constant function* θ_0 *then* $\Sigma = \Sigma_{\theta_0}$ *described in Ex. 2.6. Otherwise, that is, if* $\theta' \neq 0$ *then we can take* $\theta(s) = s$*. If this is the case, the equation* $H = 0$ *implies* $y''(s) + 2y(s) = 0$*. The solution is* $y(s) = c_1\cos(\sqrt{2}s) + c_2\sin(\sqrt{2}s)$*,* $c_1, c_2 \in \mathbb{R}$*.*

(2) *From* (12), *we have* $\langle N, \partial_x\rangle = -\dfrac{\theta'}{\Phi}$. *Thus an* $\mathcal{A}$*-invariant* ∂_x*-translator must satisfy*

$$\frac{2t^2}{\Phi^3}(x'\theta'' - \theta' x'') = -\frac{\theta'}{\Phi}.$$

This equation is equivalent to

$$2t^2(x'\theta'' - \theta' x'') + \theta' x'^2 + \theta'(x' + 2t\theta')^2 = 0.$$

Writing this equation as a polynomial equation on the variable t*, we have*

$$t^2(x'\theta'' - \theta' x'' + 2\theta'^3) + 2x'\theta'^2 t + x'^2\theta' = 0.$$

Then all coefficients must vanish. First, $x'^2\theta' = 0$*. This implies that* $\theta'(s) = 0$ *for all* s *or* $x'(s) = 0$ *for all* s*. However,* x' *cannot be identically* 0*, otherwise* $\theta'(s)$ *also vanishes for all* s *and this is not allowed by regularity. Thus* θ *is a constant function, obtaining the surface* Σ_{θ_0} *of Ex. 2.6.*

(3) *Using* (16), *we have* $\langle N, \partial_x\rangle = -\frac{\sin\varphi}{2y}$, *obtaining Eq.* (20). *Then the generating curve* $\alpha(s) = (x(s), y(s))$ *is given by the ordinary system formed by the two equations* (15) *together with* (20). *Since the first equation of* (15) *can be obtained if we solve the second equation and* (20), *then it is enough to consider the autonomous system*

$$\begin{cases} y' = 2y\sin\varphi \\ \varphi' = -\dfrac{\sin\varphi}{y} - 2\cos\varphi. \end{cases} \tag{21}$$

The phase portrait is shown in Fig. 1. The phase plane is $A = \{(y,\varphi) : y > 0, \varphi \in (-\pi,\pi)\}$*. Up to a factor, the ODE system* (21) *is the same that appeared in [6, Th. 5.1] in the study of p-grim reapers in the space* $\mathbb{H}^2 \times \mathbb{R}$*. We refer there for details. The trajectories in the* (y,φ)*-plane repeat along the* φ*-axis at distance* π*. Multiplying by* y *in* (21), *the equilibrium points are* $(0, n\pi)$, $n \in \mathbb{Z}$*. If* n *is even, the eigenvalues of the linearized system are* 0 *and* -1*, while if* n *is odd, the eigenvalues are* 0 *and* 1*. Therefore the trajectories start at points of type* $(0, 2n\pi)$ *along the unstable manifolds and ends at the points* $(0, 2(n \pm 1)\pi)$ *along the stable manifold. See Fig. 1, left. Since in each trajectory the value of* y *lies between* 0 *and a maximum, the corresponding solution* α *is not a graph on the* x*-axis (because the value* $\varphi = \frac{2n\pm1}{2}\pi$ *is always attained) and the function* $y(s)$ *goes from* 0 *to* 0 *in each branch of* α*. The function* $x'(s) = 2y(s)\cos\varphi(s)$ *only vanishes at* $\varphi = \frac{2n\pm1}{2}\pi$*. Moreover, in that points* s*, we have* $x''(s) = -2y(s)\varphi'(s)\sin\varphi(s) = 2(\sin\varphi(s))^2 = 2 > 0$*, proving that* x *has a minimum. This proves the shape of* α*.*

□

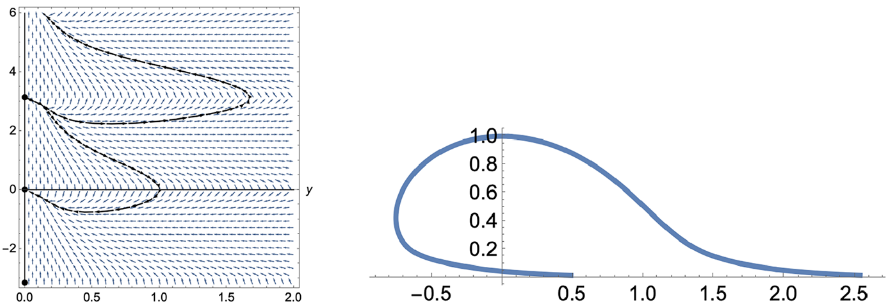

FIGURE 1. Left: the phase portrait of the nonlinear autonomous system (21). Here we have indicated two trajectories going from $(0, \pi)$ to $(0, 0)$ and $(0, 2\pi)$. Right: a solution of (20) for initial conditions $\alpha(0) = (0, 1)$

4. Translators by the Vector Field ∂_θ

Consider the Killing vector field ∂_θ. This vector field coincides with e_3.

Theorem 4.1. *(1) Let Σ be an $\mathcal{N}$-invariant surface whose generating curve is parametrized by (9). If Σ is a ∂_θ-translator, then either $\alpha(s) = (c_1 e^{2s/\sqrt{3}}, -\frac{s}{\sqrt{3}} + c_2)$, where $c_1 > 0$, $c_2 \in \mathbb{R}$, or $\alpha(s) = (y(s), \theta(s))$, with*

$$\begin{aligned} y(s) &= c_1 \exp\left(\frac{2\sqrt{2}}{3}\Lambda(s) + \frac{2}{3}\psi(s)\right) \\ \theta(s) &= \frac{2\sqrt{2}}{3}\Lambda(s) - \frac{1}{3}\psi(s) + c_2, \end{aligned} \tag{22}$$

and $\Lambda(s) = \arctan\left(\tanh\frac{s\sqrt{3}}{2}\right)$, $\psi(s) = \log\cosh(s\sqrt{3})$, where $c_1 > 0$, $c_2 \in \mathbb{R}$.

(2) The only $\mathcal{A}$-invariant ∂_θ-translators are the surfaces of type Σ_{x_0}.

(3) The only rotational ∂_θ-translators are minimal surfaces.

Proof. *(1) From (10), Eq. (2) is*

$$\varphi' = \cos\varphi + \sqrt{2}\sin\varphi.$$

A first solution appears when φ is a constant function. This occurs when $\tan\varphi = -1/\sqrt{2}$. If we consider $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ it is immediate from (9) that $y(s) = c_1 e^{2s/\sqrt{3}}$ and $\theta(s) = -\frac{s}{\sqrt{3}} + c_2$, where the constants c_1 and c_2 are obtained from the initial condition. By (10) the surface Σ is of constant mean curvature $H = -1/\sqrt{3}$.

In case that φ is not constant let us fix the initial conditions for φ (equivalently to a translation in the parameter s that does not affect *the assumption for the parametrization of* α*)* $\varphi(0)=\varphi_0=\arctan\sqrt{2}\in(0,\frac{\pi}{2})$*. By separation of variables, after integrating, we obtain*

$$\frac{1+\sin(\varphi-\varphi_0)}{\cos(\varphi-\varphi_0)}=e^{s\sqrt{3}}.$$

We obtain

$$\varphi(s)=\arctan\sqrt{2}+2\arctan\left(\tanh\frac{s\sqrt{3}}{2}\right).$$

With this value of φ*, it is possible to integrate* (9) *obtaining* (22)*. See Fig. 2, with the generating curve* $\alpha(s)$ *in the* $y\theta$*-plane.*

(2) From (12)*, we have* $\langle N,\partial_\theta\rangle=\frac{x'}{\Phi}$*. Then Eq.* (2) *becomes*

$$t^2(x'\theta''-\theta'x'')=x'((x'+t\theta')^2+t^2\theta'^2).$$

This is equivalent to

$$t^2(x'\theta''-\theta'x''-2x'\theta'^2)-2x'^2\theta't-x'^3=0.$$

Because the arbitrariness of t*, we must have* $x'=0$ *identically, which implies that* $\Sigma=\Sigma_{x_0}$ *for a certain constant* x_0*.*

(3) From (14)*, we have* $\langle N,e_3\rangle=0$*. Thus* (2) *implies that the surface is minimal. For a detailed study of constant mean curvature rotational surfaces in* SL(2, ℝ) *see [15, §4].* □

5. Translators by the Vector Field $x\partial_x+y\partial_y$

Consider the Killing vector field

$$V=x\partial_x+y\partial_y.$$

In terms of the basis B, we have $V=\frac{1}{2y}(xe_1+ye_2+xe_3)$.

Theorem 5.1. *(1) The only* $\mathcal{N}$*-invariant* V*-translators are the surfaces of type* Σ_{θ_0} *or they are parametrized by* (7)*, where* $\theta(s)=s$ *and* $\alpha(s)=\big(c(1+\cos(\sqrt{2}(s-s_0)),s\big)$*, with* $c>0$ *and* $s_0\in\mathbb{R}$*.*

(2) The only $\mathcal{A}$*-invariant* V*-translators are the surfaces of type* Σ_{θ_0} *or of type* Σ_{x_0} *with* $x_0=0$*.*

(3) Let Σ *be a rotational surface whose generating curve is parametrized by* (15)*. If* Σ *is a* V*-translator, then*

$$\varphi'=-\frac{y\cos\varphi+x\sin\varphi}{y}. \tag{23}$$

Proof. *(1) Obviously,* Σ_{θ_0} *is a solution of the problem. Indeed, the normal to* Σ_{θ_0} *is* $N=\frac{1}{\sqrt{2}}(e_1-e_3)$ *which is orthogonal to* V*. On the other hand,* Σ_{θ_0} *is minimal.*

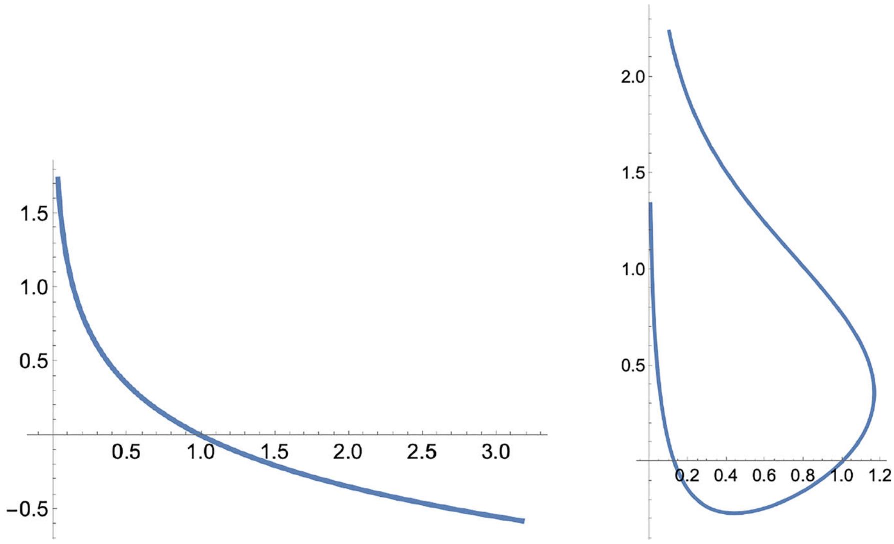

Figure 2. The generating curve α: φ is constant (left) and for the initial conditions $\alpha(0) = (1, 0)$ and $\alpha'(0) = (0, 1)$ (right)

Suppose that $\theta' \neq 0$, hence we can choose $\theta(s) = s$. The equation (2) becomes

$$y'^2 = 2y(y'' + y).$$

With the change of variable $f(s) = \frac{y'(s)}{y(s)}$, this equation writes then as

$$2f' + f^2 + 2 = 0.$$

The solution of this equation is $f(s) = -\sqrt{2}\tan\left(\frac{s-s_0}{\sqrt{2}}\right)$, for a certain constant s_0 obtained from the initial conditions. Then we have proved

$$\frac{y'}{y} = -\sqrt{2}\tan\left(\frac{s-s_0}{\sqrt{2}}\right).$$

Consequently, we find

$$y(s) = c\left(1 + \cos(\sqrt{2}(s - s_0))\right), \quad c > 0.$$

The domain of s is such that $\frac{s-s_0}{\sqrt{2}} \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(2) By using the parametrization (11), we obtain $\langle N, V\rangle = -\frac{x\theta'}{\Phi}$. Equation (2) becomes

$$2t^2(x'\theta'' - \theta' x'') = -x\theta'((x' + 2t\theta')^2 + x'^2).$$

This equation writes as

$$t^2(x'\theta''-\theta'x''+2x\theta'^3)+2txx'\theta'^2+x\theta'x'^2=0.$$

As t is arbitrary, we must have either $\theta'(s)=0$ for all s, that is $\Sigma=\Sigma_{\theta_0}$, or $\theta'\neq 0$ and this implies $x=0$, that is $\Sigma=\Sigma_{x_0}$ with $x_0=0$.

(3) Since $\langle N,V\rangle=\frac{1}{2y}(-x\sin\varphi+y\cos\varphi)$, then (2) writes as in (23). Obviously, $\sin\varphi\neq 0$ on a certain interval. From the relation above we deduce

$$\frac{x}{y}=-\frac{\varphi'+\cos\varphi}{\sin\varphi},\quad \textit{for } \sin\theta\neq 0.$$

□

Let us emphasize two particular situations of Eq. (23).

Corollary 5.2. *Let Σ be a rotational V-translator whose generating curve is parametrized by (15).*

(1) If φ is constant, then Σ is the surface of Ex. 2.7 with $\varphi(s)=\pi/2$.
(2) If Σ has constant mean curvature H, then $H=0$ and the surface is of previous item (1).

Proof. *The first part was proved in Ex. 2.7. Suppose now that the mean curvature H is constant. We must have*

$$\varphi'=2(H-\cos\varphi)\quad \textit{and}\quad \varphi'=-\frac{y\cos\varphi+x\sin\varphi}{y}.$$

It follows that $2H=\cos\varphi-\frac{x}{y}\sin\varphi$. Taking the derivative and after some manipulations we obtain

$$\left(\sin\varphi+\frac{x}{y}\cos\varphi\right)\varphi'=-2\sin\varphi\left(\cos\varphi-\frac{x}{y}\sin\varphi\right).$$

Thus $H\sin\varphi=\frac{x}{y}\left(1-H\cos\varphi\right)$. Multiply by $\sin\varphi$ and replace $\frac{x}{y}\sin\varphi=\cos\varphi-2H$ to obtain

$$3H=(1+2H^2)\cos\varphi.$$

It follows that $H=0$ and φ is constant with $\varphi(s)=\frac{\pi}{2}$. □

We obtain qualitative properties of some solutions of (15). See Fig. 3.

Proposition 5.3. *Let Σ be a rotational surface whose generating curve $\alpha(s)=(x(s),y(s))$ is parametrized by (15). Suppose that Σ is a V-translator and α is a maximal solution of (23).*

(1) If α intersects the y-axis orthogonally, then α is the graph of a function defined in the x-axis. This function is symmetric about the y-axis, it has a unique maximum at $x=0$ and α is asymptotic to the x-axis as $s\to\infty$.
(2) If α does not intersect the y-axis, then α is a bi-graph on the x-half axis and asymptotic to the x-axis.

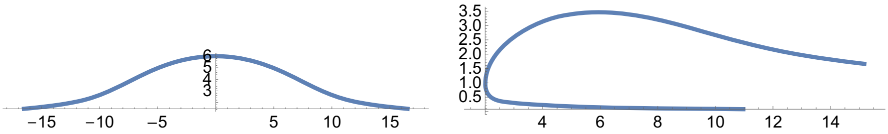

FIGURE 3. V-translators of rotational type, Proposition 5.3. Cases 1 (left) and 2 (right)

Proof. *If α is parametrized by (15), then we know that (x, y, φ) solves the initial value problem on a certain interval I, $0 \in I$,*

$$\begin{cases} x'(s) = 2y(s)\cos\varphi(s), \\ y'(s) = 2y(s)\sin\varphi(s), \\ \varphi'(s) = -\cos\varphi(s) - \dfrac{x(s)}{y(s)}\sin\varphi(s), \\ x(0) = x_0, y(0) = y_0, \varphi(0) = \varphi_0. \end{cases} \tag{24}$$

It is convenient to introduce the polar angle $\theta = \theta(s)$ given by

$$\tan\theta = \frac{y}{x}.$$

A straightforward computation gives

$$\frac{d\theta}{d\varphi} = \frac{\theta'(s)}{\varphi'(s)} = -2(\sin\theta)^2\frac{\sin(\varphi-\theta)}{\sin(\varphi+\theta)}.$$

By using (24), the functions (φ, θ) satisfy the planar ordinary system

$$\begin{cases} \dfrac{d\varphi}{dt} = \sin(\varphi+\theta), \\ \dfrac{d\theta}{dt} = -2(\sin\theta)^2\sin(\varphi-\theta). \end{cases} \tag{25}$$

The equilibrium points (φ, θ) are

$$P_1 = (0,0), \quad P_2 = \left(\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The other equilibrium points are obtained after translations of $(k_1\pi, k_2\pi)$, k_1, $k_2 \in \mathbb{Z}$. The behaviour of the trajectories are studied after the analysis of the linearized system near the equilibrium points. See Fig. 4. For this, the matrix of the linearized system at the points P_1 and P_2 are, respectively

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix}.$$

The eigenvalues corresponding to P_1 are 1 and 0 hence P_1 is a degenerate point. The eigenspace of the zero eigenvalue is $\varphi = -\theta$. The eigenspace of $\lambda = 1$ is $\theta = 0$. See Fig. 4. For each point P_2, the eigenvalues for the system are real numbers being one positive and the other one is negative. Thus P_2 is an unstable saddle point. The point P_2 corresponds with the surface (1) of Cor.

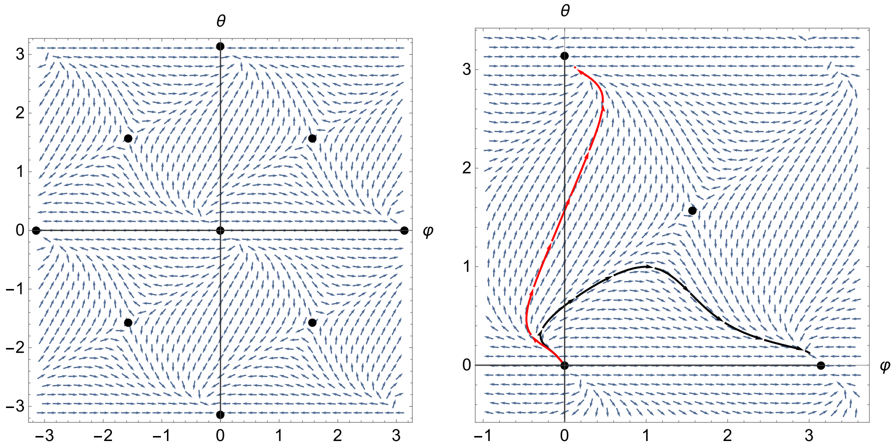

Figure 4. The phase portrait of the nonlinear autonomous system (25). Left: the black points indicate the equilibrium points of the system. Right: we indicate two trajectories. A (red) trajectory acrosses the value $(0, \frac{\pi}{2}$ and it goes from $(0,0)$ to $(0,\pi)$. A (black) trajectory acrosses the value $(\frac{\pi}{2}, \frac{\pi}{4})$ going from $(0,0)$ to $(\pi,0)$ (Color figure online)

5.2 because $\varphi(s) \equiv \pi/2$. The trajectories in the (φ,θ)-plane go from $P_1 = (0,0)$ to the point $(\pi,0)$ or $(0,\pi)$.

(1) *Suppose that α intersects orthogonally the y-axis. Without loss of generality, let $x(0) = 0$, $y(0) = y_0 > 0$, $\varphi(0) = 0$. Then the triple $(\tilde{x},\tilde{y},\tilde{\varphi}) = (-x(-s), y(-s), -\varphi(-s))$ is another solution of (24). Using uniqueness of ODEs, we have prove that α is symmetric about the y-axis. Moreover, from (24), we have $y'(0) = 0$ and $\varphi'(0) = -1$. Thus $y''(0) = 2y_0\cos\varphi(0) = -2y_0$. This implies that $y = y(s)$ attains a local maximum at $s = 0$. From now on, we only have to discuss the behaviour of $\alpha(s)$ for $s \geq 0$. The function φ is decreasing at $s = 0$. It is not possible that φ attains the value $-\pi/2$ at finite time $s = s_1$ because in such a case, $\varphi'(s_1) \leq 0$. However, the third equation of (24) yields $\varphi'(s_1) = x(s_1)/y(s_1) > 0$. Also, it is not possible that α attains the x-axis at finite time s_2 during the function φ is decreasing because we know that $-1 < \sin\varphi(s) < 0$ and $\lim_{s\to s_2^-}\varphi(s) = +\infty$. On the other hand, from the second equation of (24), the function y is decreasing. In case that y' vanishes again and s_3 is the first point where this occurs, then $y'(s_3) = 0$ gives $\varphi(s_3) = 0$ with $\varphi'(s_3) \geq 0$. However, we have from the third equation of (24) that $\varphi'(s_3) = -1$, that is $s = s_3$ is a local maximum, a contradiction. This proves that $s = 0$ is the only critical point and, in consequence, a global maximum of the function $y(s)$.*

We now prove that α cannot attain the x-axis at time finite. If α attains the x-axis as $s \to s_2$, then necessarily φ' vanished at least once at $(0, s_2)$. Moreover, we know $\lim_{s\to s_2^-}\varphi(s) = 0$ because otherwise, $\varphi'(s) \nearrow +\infty$ as $s \to s_2$ which it is not possible. The last equation of (24) writes as

$$-y\varphi' = \frac{x'}{2} + \frac{xy'}{2y} = \frac{1}{2y}(xy)'.$$

This gives $-2y^2\varphi' = (xy)'$, or equivalently,

$$8\varphi y^2 \sin\varphi = (xy + 2y^2\varphi)'.$$

Integrating from $s = 0$ to $s = s_2$, we obtain

$$\int_0^{s_2} \varphi(s)\sin\varphi(s) y(s)^2\, ds = 0,$$

which it is not possible since the sign of the integrand is positive. Definitively, the solution of (24) is defined in $\mathbb{R}$ and $\varphi(s) \in (-\pi/2, 0)$ if $s > 0$. This proves that α is a graph on the x-axis. It remains to prove that $y(s) \to 0$ as $s \to \infty$. The trajectory corresponding to the solution α at $s = 0$ takes the value $(\varphi, \theta) = (0, \pi/2)$. This trajectory goes from the point $P_1 = (0, 0)$ to the point $(0, \pi)$ such as it shows Fig. 4, right. Thus the polar angle goes from $\theta = 0$ (the positive direction of the x-axis) until the value $\theta = \pi$ (the negative direction of the x-axis). Moreover, from the starting point $((0, 0))$ until the end point $((0, \pi))$ the function $y(s)$ is close to 0, proving that α has in the x-axis as a horizontal asymptote.

(2) Suppose that α does not intersect the y-axis. Suppose $x(0) = x_0$ which we can assume positive, $x_0 > 0$, and let $\varphi(0) = \pi/2$. The first equation of (24) says that $s = 0$ is a critical point of the function $x(s)$. Since $\varphi'(0) = -x_0/y_0$, then $x''(0) = -2y_0\varphi'(0) = 2x_0 > 0$. This implies that $x(s)$ attains a minimum at $s = 0$. Let us look the trajectory corresponding to this solution α. Now at $s = 0$ we have $(\varphi, \theta) = (\frac{\pi}{2}, \theta_0)$ for some $\theta_0 \in (0, \frac{\pi}{2})$. Notice that $P_2 = (\frac{\pi}{2}, \frac{\pi}{2})$ is another equilibrium point but the initial value θ at $s = 0$ is less than $\pi/2$. Since P_2 is an unstable saddle point, the trajectory does not end at P_2. In fact, and such as it shows Fig. 4, right, the trajectory passing through $(\frac{\pi}{2}, \theta_0)$ starts at $(0, 0)$ and ends at $(\pi, 0)$. This proves that the polar angle varies from 0 to 0 passing by the value $\theta_0 \in (0, \frac{\pi}{2})$ (at $s = 0$). On the other hand, the angle function φ which indicates the angle of the tangent vector of α goes from 0 to π passing by the value $\frac{\pi}{2}$. In consequence, the solution curve α is not a graph on the x-axis. Since $s = 0$ is a local minimum of the function $x(s)$, and as $s \to \pm\infty$, we have $\theta \to 0$, then the curve α tends to the x-axis at infinity. This shows the geometry of the curve α. □

The qualitative properties of the rest of solutions of (24) can be obtained in a similar way from the (φ, θ)-portrait plane. Notice that the trajectories in the (φ, θ)-plane are symmetric with respect to any equilibrium point: Fig. 4, left. Looking on the θ-axis, that is, when $\varphi(0) = 0$, we can restrict to $\{0\} \times [0, \pi)$. For example, the solutions corresponding to the trajectories intersecting $\{0\} \times [0, -\pi)$ are symmetric about the origin of the (x, y)-plane: fixing $\varphi(0) = 0$, a pair (x_0, y_0) determines θ_0 and the value $-\theta_0$ corresponds with the initial condition $(-x_0, -y_0)$.

For example, we can consider in (24) initial conditions $(x_0, y_0) = (0, 1)$, hence $\theta(0) = \pi/2$, but varying the initial angle $\varphi(0)$, say $\varphi(0) \in (0, \frac{\pi}{2})$. We are interesting in the trajectories intersecting the line $\theta = \frac{\pi}{2}$: see red trajectory in Fig. 5. In Fig. 5 we have depicted the horizontal line $\theta = \frac{\pi}{2}$. For $\varphi(0) > 0$ but close to 0, the trajectory is similar to the red one that appears in Fig. 4, right. In that case, the solution α is a graph on the x-axis and the same occurs for the solutions corresponding to $\varphi(0)$. In this case, the solution is not symmetric about a vertical axis as it happened with $\varphi(0) = 0$, but α follows being a graph on the x-axis.

We can also vary the polar angle θ in the initial point. This implies that we are changing the initial conditions (x_0, y_0). Fix $\varphi(0) > 0$ and close to 0. Let, for instance, $x_0 = 1$ and $y_0 > 0$ close to 0. Then y_0 determines the value θ_0 which is now close to 0. In Fig. 5 we show the corresponding trajectory (black) which goes from $(0, 0)$ to $(\pi, 0)$. An argument as in the proof of Prop. 5.3 proves that the solution α has similar properties as (2) of Prop. 5.3. However, if we increase $y_0 \nearrow \infty$ (but fixing $x_0 = 1$), then we have $\theta_0 \nearrow \frac{\pi}{2}$. Then there exists a value θ_0 where the trajectory changes and, instead to end at $(\pi, 0)$, the trajectory ends at $(0, \pi)$: blue trajectory in Fig. 5. The corresponding solution is similar as solutions of (1) of Prop. 5.3, being now α a graph on the x-axis.

6. Translators by the Vector Field $\frac{1}{2}(x^2 - y^2)\partial_x + xy\partial_y$

Consider the Killing vector field

$$W = \frac{1}{2}(x^2 - y^2)\partial_x + xy\partial_y = \frac{1}{2y}\left(\frac{1}{2}(x^2 - y^2)e_1 + xye_2 + \frac{1}{2}(x^2 - y^2)e_3\right).$$

Theorem 6.1. *(1) The only $\mathcal{N}$-invariant W-translators are the surfaces of type Σ_{θ_0}.*

(2) The only $\mathcal{A}$-invariant W-translators are surfaces of type Σ_{θ_0}.

(3) Let Σ be a rotational surface whose generating curve is parametrized by (15). If Σ is a W-translator, then

$$\varphi' = (x - 2)\cos\varphi - \frac{1}{2y}(x^2 - y^2)\sin\varphi. \tag{26}$$

Proof. *(1) From (8) we have $\langle N, W\rangle = -\frac{t\theta'}{2y\Phi}$, while H depends only on s. This implies $\theta' = 0$ identically and thus $\theta(s) = \theta_0$ is a constant function. This proves that the surface is of type Σ_{θ_0}.*

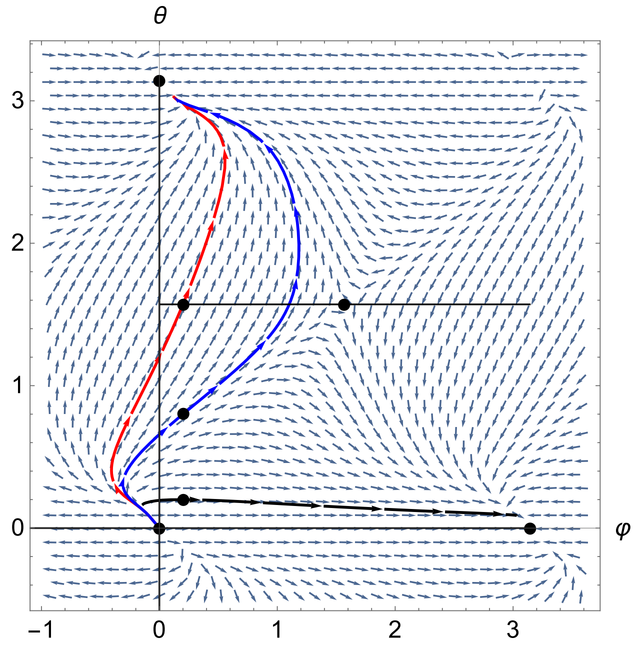

FIGURE 5. The phase portrait of the nonlinear autonomous system (25). Different trajectories for initial values (φ,θ): $(0.2,\frac{\pi}{2})$ (red), $(0.2,0.2)$ (black), $(0.2,0,8)$ (blue) (Color figure online)

(2) We have

$$\langle N,W\rangle=-\frac{x^2-t^2}{2\Phi}\theta'.$$

Consequently, equation (2) *writes as*

$$4t^2(x'\theta''-x''\theta')+(x^2-t^2)\theta'\left((x')^2(x'+2t\theta')^2\right)=0.$$

The arbitrariness of t implies that all coefficients appearing above vanish. Since the coefficient of t^4 is zero, we must have $\theta'=0$. Hence $\Sigma=\Sigma_{\theta_0}$.

(3) We have

$$\langle N,W\rangle=\frac{1}{2y}\left(\frac{1}{2}(y^2-x^2)\sin\varphi+xy\cos\varphi\right), \tag{27}$$

then (2) *becomes*

$$\frac{\varphi'}{2}+\cos\varphi=\frac{y^2-x^2}{4y}\sin\varphi+\frac{x}{2}\cos\varphi. \tag{28}$$

□

Equation (26), together with equations (15) are difficult to solve. A particular case to consider is when the surface has constant mean curvature and it is natural to ask if there exist rotational W-translators with constant mean curvature. The answer is no as we will prove in the next result. This is a consequence that it is possible to find explicit parametrizations of rotational surfaces with constant mean curvature.

Corollary 6.2. *There are no rotational W-translators with constant mean curvature.*

Proof. *Suppose that H is constant which, without loss of generality, we can assume to be non-negative. Then (16) implies $\varphi'(s) = 2H - 2\cos\varphi(s)$. The solution of this ODE is, up to translations in the s-parameter:*

$$\begin{aligned}
\varphi(s) &= -2\tan^{-1}\left(\frac{(1-H)\tanh\left(\sqrt{1-H^2}s\right)}{\sqrt{1-H^2}}\right), \quad (0 \leq H < 1),\\
\varphi(s) &= -2\cot^{-1}(2s), \quad \textit{or} \quad \varphi(s) = 0, \quad (H = 1),\\
\varphi(s) &= 2\tan^{-1}\left(\frac{(H-1)\tan\left(\sqrt{H^2-1}s\right)}{\sqrt{H^2-1}}\right), \quad (H > 1).
\end{aligned}$$

Once we have φ, we can explicitly integrate (15). In the following expressions, c is a constant of integration.

(1) Case $0 \leq H < 1$.

$$\begin{aligned}
x(s) &= -c\frac{\sqrt{1-H^2}\sinh(2\sqrt{1-H^2}s)}{\cosh(2\sqrt{1-H^2}s) + H},\\
y(s) &= c\left(H - \cos\left(2\tan^{-1}\left(\frac{(1-H)\tanh(\sqrt{1-H^2}s)}{\sqrt{1-H^2}}\right)\right)\right).
\end{aligned}$$

(2) Case $H = 1$. If $\varphi(s) = 0$ we obtain the surface Σ_{y_0} which is not a translator. For the other value of φ we get

$$\begin{aligned}
x(s) &= -\frac{1}{2}c\sin(2\cot^{-1}(2s)),\\
y(s) &= \frac{c}{4s^2+1}.
\end{aligned}$$

(3) Case $H > 1$.

$$\begin{aligned}
x(s) &= c\frac{\sqrt{H^2-1}\sin(2\sqrt{H^2-1}s)}{\cos(2\sqrt{H^2-1}s) + H},\\
y(s) &= c\left(H - \cos\left(2\tan^{-1}\left(\frac{(H-1)\tan(\sqrt{H^2-1}s)}{\sqrt{H^2-1}}\right)\right)\right).
\end{aligned}$$

Finally, we compute $\langle N, W\rangle$ *using* (27) *and we check that, indeed,* $H \neq \langle N, W\rangle$*:*

$$\langle N, W\rangle = -\frac{1}{4}c\sqrt{1-H^2}\sinh(2\sqrt{1-H^2}s), \quad (0 \leq H < 1),$$
$$\langle N, W\rangle = 0, \quad (H = 1),$$
$$\langle N, W\rangle = \frac{1}{4}c\sqrt{H^2-1}\sin(2\sqrt{H^2-1}s), \quad (H > 1).$$

□

Acknowledgements

The authors would like to thank the anonymous referees for a careful reading of the manuscript as well as for insightful suggestions, which helped to improve the exposition of the paper.

Author contributions All authors contributed equally in conception, design and preparation of the manuscript.

Funding Funding for open access publishing: Universidad de Granada/CBUA. Rafael López has been partially supported by MINECO/MICINN/ FEDER grant no. PID2023-150727NB-I00, and by the "María de Maeztu" Excellence Unit IMAG, reference CEX2020-001105- M, funded by MCINN/AEI/ 10.13039/ 501100011033/ CEX2020-001105-M. Marian Ioan Munteanu is thankful to Romanian Ministry of Research, Innovation and Digitization, within Program 1—Development of the national RD system, Subprogram 1.2—Institutional Performance—RDI excellence funding projects, Contract no.11PFE/30.12.2021, for financial support.

Data Availability Statement No data sets were generated or analyzed during the current study.

Declarations

Conflict of interest. The author declares no conflict of interest.

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Received: May 23, 2024.
Accepted: February 8, 2025.