Boundary Controllability for Degenerate/Singular Hyperbolic Equations in Nondivergence Form with Drift

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Abstract

We study the null controllability for a degenerate/singular wave equation with drift in non divergence form. In particular, considering a control localized on the non degenerate boundary point, we provide some conditions for the boundary controllability via energy methods and boundary observability.

Keywords Boundary controllability · Degenerate hyperbolic equations · Drift term · Singular potentials

Mathematics Subject Classification 35L10 · 35L80 · 93B05 · 93B07 · 93D15

1 Introduction

The aim of this paper is to study the null controllability for a degenerate hyperbolic equation with drift in presence of a singular term, with singularity at the same point where the leading coefficient degenerates. To be more precise, we consider the

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following control problem:

$$\begin{cases} u_{tt} - a(x)u_{xx} - \dfrac{\lambda}{d(x)}u - b(x)u_x = 0, & (t, x) \in Q_T, \\ u(t, 0) = 0, \quad u(t, 1) = f(t), & t > 0, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

where $Q = (0, +\infty) \times (0, 1)$, $\lambda \in \mathbb{R}$ and u_0, u_1 are the initial values. The control function f, which is used to steer the solution to its equilibrium state 0 at a sufficiently large time $T > 0$, acts on the non degenerate and non singular boundary point. The interest in this kind of equation comes from the standard linear theory for transverse waves in a string of length L under tension $\mathcal{T}$. Actually, if $u(t, x)$ denotes the vertical displacement of the string from the x axis at position $x \in (0, L)$ and time $t > 0$, then the classical wave equation can be rewritten as

$$\frac{\partial^2 u}{\partial t^2}(t, x) = a(x)\frac{\partial^2 u(t, x)}{\partial x^2} + b(x)\frac{\partial u}{\partial x}(t, x),$$

where $a(x) := \mathcal{T}(x)\rho^{-1}(x)$, $b(x) := \mathcal{T}'(x)\rho^{-1}(x)$. Here $\rho(x)$ is the mass density of the string at position x, while $\mathcal{T}(x)$ denotes the tension in the string at position x and time t. If the string is made of a non homogeneous material, it can happen that at some point, say $x = 0$, the density is extremely large or the tension is extremely small; then the previous equation *degenerates* at $x = 0$, in the sense that we can consider $a(0) = 0$. The remainder term is a drift one, and the resulting equation is thus a *degenerate equation with drift*. For such a situation, in [8] the null controllability for (1.1) with $\lambda = 0$ is considered. For the same problem in divergence form we refer to [30] for the prototype case ($a(x) = x^\alpha$, $\alpha \in (0, 1)$) and to [1] for a general function a. (See also the arxiv version of 2015).

On the other hand, wave equations with singularities appear in some natural contexts, for instance when a gravitational-like potential is given (see the recent [28] and [29]). Actually, the sign of λ determines different physical interpretation and behaviour of the system: indeed, if $\lambda < 0$ the singular potential acts as a repulsive one, causing the wave function to be pushed away from $x = 0$ (where the potential concentrates), while if $\lambda > 0$ it acts as an attractive potential. If the string has the physical structures assumed above, we are in the simultaneous presence of a degeneracy and a singularity, the object of the investigation of this paper.

Let us recall that null controllability for the one dimensional *nondegenerate* wave equation can be attacked in several ways: for instance, consider

$$\begin{cases} u_{tt} - u_{xx} = f_\omega(t, x), & (t, x) \in Q_t, \\ u(t, 0) = 0, \quad u(t, 1) = f(t), & t > 0, \\ u(0, x) = u_0(x) \in H_0^1(0, 1), \quad u_t(0, x) = u_1(x) \in L^2(0, 1), & x \in (0, 1). \end{cases} \quad (1.2)$$

Here u is the state, while f_ω and f are the controls: one may have $f = 0$ and f_ω acting as a control localized in the subset ω of $[0, 1]$, or $f_\omega = 0$ and f acting as a

boundary control. In any case, one looks for conditions in order to drive the solution to equilibrium at a given time T, i.e. given the initial data (u_0, u_1) in a suitable space, we look for a control (f or f_ω) such that

$$u(T, x) = u_t(T, x) = 0, \quad \text{for all } x \in (0, 1). \tag{1.3}$$

Clearly, due to the finite speed of propagation of solutions of the wave equation, we cannot expect to have null controllability at any final time T (as in the parabolic case), but we need T to be sufficiently large: for equation (1.2) it is well known that null controllability holds if $T > 2$, see [35, Chapter 4]. Moreover, the Hilbert Uniqueness Method (HUM) permits to characterize such a control in terms of minimum of a certain functional. A related approach for (1.2) with $f = 0$ is showed in [34]. We also mention the recent paper [36], where the author studies (1.2) with two linearly moving endpoints, establishing observability results in a sharp time and deriving exact boundary controllability results.

However, in recent years great attention is given to controllability issues for parabolic problems not only with a degenerate terms, but also with a singular term. Indeed, many problems coming from Physics (see [3, 32, 38]), Biology (see [6, 7, 14, 18, 19, 27]) or Mathematical Finance (see [31]) are described by degenerate parabolic equations with a singular term. A common strategy in showing controllability is to prove global Carleman estimates for the operator which is the adjoint of the given one. In this framework, new Carleman estimates and null controllability properties have been established in [5, 9, 10, 20, 33] for regular degenerate coefficients, in [7, 21, 24] for non smooth degenerate coefficients and in [15–17, 22, 23, 25, 37] for degenerate and singular coefficients.

Null controllability for wave equations with degeneracy and singularity in presence of pure powers and in divergence form has been recently tacled in [2]. As far as we know, this is the first paper to consider a *degenerate* hyperbolic equation in *non divergence form* with drift, where both the degeneracy and the singularity are described by more general functions. Clearly the presence of the drift term and of the singular term leads to use different spaces with respect to the ones in [1, 2, 8] and give rise to some new difficulties. As a consequence, the tools used in those papers cannot be simply adapted and a different functional setting is needed, together with suitable assumptions on b. For this, following [1] or [8], we consider two types of degeneracy for a and b, which we introduce for a general function g.

Definition 1.1 A function $g : [0, 1] \to \mathbb{R}$ is weakly degenerate at 0, (WD) for short, if $g \in C^0[0, 1] \cap C^1(0, 1]$ is such that $g(0) = 0$, $g > 0$ on $(0, 1]$ and, if

$$\sup_{x \in (0,1]} \frac{x \, |g'(x)|}{g(x)} := K_g, \tag{1.4}$$

then $K_g \in (0, 1)$.

Definition 1.2 A function $g : [0, 1] \to \mathbb{R}$ is strongly degenerate at 0, (SD) for short, if $g \in C^1[0, 1]$ is such that $g(0) = 0$, $g > 0$ on $(0, 1]$ and in (1.4) we have $K_g \in [1, 2)$.

Roughly speaking, when $g(x) \sim x^K$, it is (WD) if $K \in (0, 1)$ and (SD) if $K \in [1, 2)$. Notice that the case $K \geq 2$ is not considered, since controllability cannot hold, not even when $\lambda = 0$, see [8].

Remark 1 Although the prototypes we have in mind are a pure power degeneracy x^{K_a} and a pure power singularity x^{K_d}, in spite of the additional technicalities due to the presence of general functions a and d (and b), we prefer to handle the general case, in order to cover not only ideal situations. Moreover, this choice makes the paper more general, not only a nondivergence version with drift of [2].

The paper is organized as follows: in Sect. 2 we study the well posedness of the associated problem with Dirichlet boundary conditions, which we need to study the adjoint problem of (1.1); in Sect. 3 we consider such an adjoint problem and we prove that for this kind of problem the associated (weighted) energy is constant in time. Moreover, we prove some estimates of the energy and, thanks to them, we will prove in Sect. 4 boundary observabilities and null controllability for (1.1). The paper ends with the Appendix where we present some technical results, which we postpone for the readers' convenience.

Concerning the controllability result given in Theorem 4.1 - the main result of this paper -, we recall that it is a consequence of suitable observability inequalities and an application of the HUM method. As for the classical case, such a result can hold provided that the controllability time is sufficiently large. However, the value of the optimal control time is not an issue for us. Indeed, even in the prototype case of pure power functions for a, b and d, it is not easy to determine the minimum time for controllability, since we don't know the velocity of the wave; thus, for general coefficients, the calculation of the optimal time seems to be impossible.

2 Well Posedness for the Problem with Homogeneous Dirichlet Boundary Conditions

Consider the degenerate/singular hyperbolic problem with Dirichlet boundary conditions

$$\begin{cases} y_{tt} - ay_{xx} - \dfrac{\lambda}{d}y - by_x = 0, & (t, x) \in (0, +\infty) \times (0, 1), \\ y(t, 1) = y(t, 0) = 0, & t \in (0, +\infty), \\ y(0, x) = y_T^0(x), & x \in (0, 1), \\ y_t(0, x) = y_T^1(x), & x \in (0, 1), \end{cases} \tag{2.1}$$

where the functional setting and the motivation for the notation of the initial data will be made clear below.

We assume the following hypothesis.

Hypothesis 1 The functions $a, b, d \in C^0[0, 1]$ are such that

1. $\dfrac{b}{a} \in L^1(0, 1)$,

2. $a(0) = d(0) = 0$, $a, d > 0$ on $(0, 1]$,
3. there exist $K_1, K_2 \in (0, 2)$ such that $K_1 + K_2 \le 2$ and the functions

$$x \longmapsto \frac{x^{K_1}}{a(x)} \tag{2.2}$$

and

$$x \longmapsto \frac{x^{K_2}}{d(x)} \tag{2.3}$$

are nondecreasing in a right neighborhood of $x = 0$.

Remark 2 We notice that our assumptions involve the coefficients $a = \mathcal{T}\rho^{-1}$, and $b = \mathcal{T}'\rho^{-1}$; this implies conditions on the string, and, in particular, on its physical structure. Indeed, the request $b/a \in L^1(0, 1)$ reads as $\mathcal{T}'/\mathcal{T} \in L^1(0, 1)$: in a vibrating string the tension $\mathcal{T}$ is never 0, so the previous mathematical condition implies that $\log \mathcal{T}(1) - \log \mathcal{T}(0) \in \mathbb{R}$, which is obvious also from a physical point of view.

It is clear that, if Hypothesis 1 holds, then

$$\lim_{x \to 0} \frac{x^\gamma}{a(x)} = 0, \tag{2.4}$$

for all $\gamma > K_1$ and

$$\lim_{x \to 0} \frac{x^\gamma}{d(x)} = 0, \tag{2.5}$$

for all $\gamma > K_2$.

Let us remark that if a is (WD) or (SD), then (1.4) implies that the function

$$x \mapsto \frac{x^\gamma}{a(x)} \text{ is nondecreasing in } (0, 1] \text{ for all } \gamma \ge K_a. \tag{2.6}$$

In particular, the monotonicity condition of the map in (2.2) holds and

$$\left| \frac{x^\gamma b(x)}{a(x)} \right| \le \frac{1}{a(1)} \|b\|_{L^\infty(0,1)}, \tag{2.7}$$

for all $\gamma \ge K_a$. For further purposes, let us introduce

$$M := \frac{\|b\|_{L^\infty(0,1)}}{a(1)}, \tag{2.8}$$

noticing that if a and b are both (WD) or (SD), Hypothesis 1 reduces to require $\frac{b}{a} \in L^1(0, 1)$ and $K_a + K_d \le 2$.

Let us also remark that for the prototypes $a(x) = x^K$ and $b(x) = x^h$ with $K \in (0, 1]$ and $h \geq 0$, the condition $\frac{b}{a} \in L^1(0, 1)$ is clearly satisfied when $h > K - 1$.

In order to study the well-posedness of (2.1), we consider as in [8] the well-known absolutely continuous weight function

$$\eta(x) := \exp\left\{\int_{\frac{1}{2}}^x \frac{b(s)}{a(s)}ds\right\}, \quad x \in [0, 1],$$

introduced by Feller several years ago. Since $\frac{b}{a} \in L^1(0, 1)$, we find that $\eta \in C^0[0, 1] \cap C^1(0, 1]$ is a strictly positive function. When b degenerates at 0 not slower than a, for instance if $a(x) = x^K$ and $b(x) = x^h$, $K \leq h$, then η can be extended to a function of class $C^1[0, 1]$. Now set

$$\sigma(x) := a(x)\eta^{-1}(x),$$

and introduce, as in [12], the following Hilbert spaces with the related inner products and norms

$$L^2_{\frac{1}{\sigma}}(0, 1) := \left\{u \in L^2(0, 1) \mid \|u\|_{\frac{1}{\sigma}} < \infty\right\}, \ \langle u, v\rangle_{\frac{1}{\sigma}} := \int_0^1 uv\frac{1}{\sigma}dx, \ \|u\|^2_{\frac{1}{\sigma}} = \int_0^1 \frac{u^2}{\sigma}dx$$

and

$$H^1_{\frac{1}{\sigma}}(0, 1) := L^2_{\frac{1}{\sigma}}(0, 1) \cap H^1_0(0, 1),$$

$$\langle u, v\rangle_{1, \frac{1}{\sigma}} := \langle u, v\rangle_{\frac{1}{\sigma}} + \int_0^1 \eta u'v'dx, \ \|u\|^2_{1, \frac{1}{\sigma}} = \|u\|^2_{\frac{1}{\sigma}} + \int_0^1 \eta(u')^2dx.$$

Notice that the presence of the weight η in the integrals above is non-essential, but it is useful to have easier calculations in some steps below.

The following result holds:

Proposition 2.1 *Assume Hypothesis 1. If $u \in H^1_{\frac{1}{\sigma}}(0, 1)$, then $\frac{u}{\sqrt{\sigma d}} \in L^2(0, 1)$ and there exists a positive constant $C > 0$ such that*

$$\int_0^1 \frac{u^2}{\sigma d}dx \leq C \int_0^1 \eta(u')^2dx. \tag{2.9}$$

Proof Taking $u \in H^1_{\frac{1}{\sigma}}(0, 1)$, recalling the monotonicity condition in Hypothesis 1, by Hardy's inequality one has

$$\int_0^1 \frac{u^2}{\sigma d}dx \leq \max_{[0,1]} \eta \int_0^1 \frac{u^2}{ad}dx \leq C \int_0^1 \frac{u^2}{x^{K_1+K_2}}dx$$

$$\leq C \int_0^1 \frac{u^2}{x^2}dx \leq \tilde{C} \int_0^1 \eta(u')^2dx$$

for some $C, \tilde{C} > 0$.   □

Let

$$C_{HP} \text{ be the best constant of (2.9)}, \qquad (2.10)$$

and let us remark that, if the monotonicity conditions in Hypothesis 1 are global, for instance if a and d are (WD) or (SD), then the estimate in the proof of Proposition 2.1 shows that

$$C_{HP} \le \frac{4 \max_{[0,1]} \eta}{a(1)d(1) \min_{[0,1]} \eta}.$$

Another assumption we will use is the next one:

Hypothesis 2 The constant $\lambda \in \mathbb{R}$ is such that

$$\lambda < \frac{1}{C_{HP}}. \qquad (2.11)$$

Under Hypotheses 1 and 2, one can consider on $H^1_{\frac{1}{\sigma}}(0, 1)$ also the inner product

$$\langle u, v \rangle_1 := \langle u, v \rangle_{\frac{1}{\sigma}} + \int_0^1 \eta u' v' dx - \lambda \int_0^1 \frac{uv}{\sigma d} dx,$$

which induces the norm

$$\|u\|_1^2 := \|u\|_{\frac{1}{\sigma}}^2 + \int_0^1 \eta(u')^2 dx - \lambda \int_0^1 \frac{u^2}{\sigma d} dx.$$

For all $u \in H^1_{\frac{1}{\sigma}}(0, 1)$, consider also the two norms

$$\|u\|_{1,\circ}^2 := \int_0^1 (u')^2 dx \quad \text{and} \quad \|u\|_{1,\bullet}^2 := \int_0^1 \eta(u')^2 dx.$$

By Proposition 2.1 and the classical Poincaré inequality, it is straightforward to prove the equivalence below.

Corollary 2.1 *Assume Hypotheses 1 and 2. Then the norms $\| \cdot \|_{1,\frac{1}{\sigma}}^2$, $\| \cdot \|_1^2$, $\| \cdot \|_{1,\circ}^2$ and $\| \cdot \|_{1,\bullet}^2$ are equivalent in $H^1_{\frac{1}{\sigma}}(0, 1)$.*

Moreover, from (2.9), one can also deduce that there exists $C > 0$ such that

$$\int_0^1 \frac{v^2}{\sigma} dx \le C \int_0^1 (v')^2 dx \quad \forall v \in H^1_{\frac{1}{\sigma}}(0, 1). \qquad (2.12)$$

Thus the spaces $H_0^1(0, 1)$ and $H_{\frac{1}{\sigma}}^1(0, 1)$ algebraically coincide.

Now define the operator

$$Ay := ay_{xx} + by_x = \sigma(\eta y_x)_x, \tag{2.13}$$

for all $y \in D(A)$, where $D(A)$ is the Hilbert space

$$H_{\frac{1}{\sigma}}^2(0, 1) := \left\{ u \in H_{\frac{1}{\sigma}}^1(0, 1) \mid Au \in L_{\frac{1}{\sigma}}^2(0, 1) \right\},$$

with inner products

$$\langle u, v \rangle_{2, \frac{1}{\sigma}} := \langle u, v \rangle_{1, \frac{1}{\sigma}} + \langle Au, Av \rangle_{\frac{1}{\sigma}}$$

or

$$\langle u, v \rangle_2 := \langle u, v \rangle_1 + \langle Au, Av \rangle_{\frac{1}{\sigma}}.$$

The following integration by parts holds.

Lemma 2.1 (see [7], Lemma 2.1) *Assume Hypothesis 1. If $u \in H_{\frac{1}{\sigma}}^2(0, 1)$ and $v \in H_{\frac{1}{\sigma}}^1(0, 1)$, then*

$$\langle Au, v \rangle_{\frac{1}{\sigma}} = - \int_0^1 \eta u' v' dx. \tag{2.14}$$

In order to study the well posedness of problem (2.1), we introduce the operator

$$A_\lambda y := Ay + \frac{\lambda}{d} y, \quad \forall y \in D(A_\lambda),$$

where

$$D(A_\lambda) := \left\{ u \in H_{\frac{1}{\sigma}}^1(0, 1) \mid A_\lambda u \in L_{\frac{1}{\sigma}}^2(0, 1) \right\}. \tag{2.15}$$

Remark 3 Notice that if $u \in H_{\frac{1}{\sigma}}^1(0, 1)$ and $K_a + 2K_d \le 2$, reasoning as in the proof of Proposition 2.1, we immediately find that $\frac{u}{d} \in L_{\frac{1}{\sigma}}^2(0, 1)$, and so $u \in D(A_\lambda)$ if and only if $u \in H_{\frac{1}{\sigma}}^2(0, 1)$, that is $D(A_\lambda) = H_{\frac{1}{\sigma}}^2(0, 1)$.

If $K_a + 2K_d > 2$, the definition of $D(A_\lambda)$ is not sufficient to ensure that its members are smooth enough to justify the integration by parts given in (2.14): indeed, if $u \in D(A_\lambda)$, the request that $\sigma(\eta u')' + \lambda \frac{u}{d} \in L_{\frac{1}{\sigma}}^2$ doesn't permit to take u in $H_{\frac{1}{\sigma}}^2$, since $\frac{u^2}{d^2\sigma} \sim \frac{u^2}{x^{2K_d + K_a}}$ near 0, and so, in general, $\frac{u^2}{d^2\sigma} \notin L^1(0, 1)$.

In view of the previous remark, we assume the following condition, which will assume also for the final controllability result, see Theorem 4.1.

Hypothesis 3 Hypothesis 1 holds with $K_1 + 2K_2 \leq 2$.

Remark 4 We notice that Hypothesis 3 excludes the case of d being (SD). On the other hand, a can be (SD), but in such a case d must be (WD) with small K_d.

Now, let us consider the matrix operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H}_0 \to \mathcal{H}_0$ given by

$$\mathcal{A} := \begin{pmatrix} 0 & Id \\ A_\lambda & 0 \end{pmatrix}, \quad D(\mathcal{A}) := D(A_\lambda) \times H^1_{\frac{1}{\sigma}}(0, 1), \tag{2.16}$$

and

$$\mathcal{H}_0 := H^1_{\frac{1}{\sigma}}(0, 1) \times L^2_{\frac{1}{\sigma}}(0, 1) \tag{2.17}$$

is endowed with scalar product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle_{\mathcal{H}_0} = \langle u, w \rangle_1 + \langle v, z \rangle_{L^2_{\frac{1}{\sigma}}(0,1)}.$$

In this way, we can rewrite (2.1) as the Cauchy problem

$$\begin{cases} \dot{\mathcal{Y}}(t) = \mathcal{A}\mathcal{Y}(t), & t \geq 0, \\ \mathcal{Y}(0) = \mathcal{Y}_0, \end{cases} \tag{2.18}$$

with

$$\mathcal{Y}(t) := \begin{pmatrix} y \\ y_t \end{pmatrix} \quad \text{and} \quad \mathcal{Y}_0 := \begin{pmatrix} y^0_T \\ y^1_T \end{pmatrix}.$$

Theorem 2.1 *Assume Hypotheses 2 and 3. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is non positive with dense domain and generates a contraction semigroup $(S(t))_{t \geq 0}$.*

For the proof of the previous theorem we use the next result.

Theorem 2.2 ([13], Corollary 3.20) *Let $(\mathcal{A}, D(\mathcal{A}))$ be a dissipative operator on a reflexive Banach space such that $\mu I - \mathcal{A}$ is surjective for some $\mu > 0$. Then $\mathcal{A}$ is densely defined and generates a contraction semigroup.*

Proof of Theorem 2.1 According to the previous theorem, it is sufficient to prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}_0$ is dissipative and that $I - \mathcal{A}$ is surjective.

$\underline{\mathcal{A} \text{ is dissipative:}}$ take $(u, v) \in D(\mathcal{A})$. Then $(u, v) \in H^2_{\frac{1}{\sigma}}(0, 1) \times H^1_{\frac{1}{\sigma}}(0, 1)$ and so Lemma 2.1 holds. Hence,

$$\langle \mathcal{A}(u, v), (u, v) \rangle_{\mathcal{H}_0} = \langle (v, A_\lambda u), (u, v) \rangle_{\mathcal{H}_0}$$
$$= \int_0^1 \eta u' v' dx - \lambda \int_0^1 \frac{uv}{\sigma d} dx + \int_0^1 v A_\lambda u \frac{1}{\sigma} dx$$
$$= \int_0^1 \eta u' v' dx - \lambda \int_0^1 \frac{uv}{\sigma d} dx - \int_0^1 \eta u' v' dx + \lambda \int_0^1 \frac{uv}{\sigma d} dx = 0.$$

$\underline{I - \mathcal{A} \text{ is surjective:}}$ take $(f, g) \in \mathcal{H}_0 = H^1_{\frac{1}{\sigma}}(0, 1) \times L^2_{\frac{1}{\sigma}}(0, 1)$. We have to prove that there exists $(u, v) \in D(\mathcal{A})$ such that

$$(I - \mathcal{A}) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \iff \begin{cases} v = u - f, \\ -A_\lambda u + u = f + g. \end{cases} \tag{2.19}$$

Thus, define $F : H^1_{\frac{1}{\sigma}}(0, 1) \to \mathbb{R}$ as

$$F(z) = \int_0^1 (f + g) z \frac{1}{\sigma} dx.$$

Obviously, $F \in H^{-1}_{\frac{1}{\sigma}}(0, 1)$, the dual space of $H^1_{\frac{1}{\sigma}}(0, 1)$ with respect to the pivot space $L^2_{\frac{1}{\sigma}}(0, 1)$. Now, introduce the bilinear form $L : H^1_{\frac{1}{\sigma}}(0, 1) \times H^1_{\frac{1}{\sigma}}(0, 1) \to \mathbb{R}$ given by

$$L(u, z) := \int_0^1 uz \frac{1}{\sigma} dx + \int_0^1 \eta u' z' dx - \lambda \int_0^1 \frac{uz}{\sigma d} dx,$$

for all $u, z \in H^1_{\frac{1}{\sigma}}(0, 1)$, which is well defined by Proposition 2.1. Clearly, $L(u, z)$ is coercive: indeed, for all $u \in H^1_{\frac{1}{\sigma}}(0, 1)$, when $\lambda \le 0$ we have

$$L(u, u) = \int_0^1 \frac{u^2}{\sigma} dx + \int_0^1 \eta (u')^2 dx - \lambda \int_0^1 \frac{u^2}{\sigma d} dx$$
$$\ge \int_0^1 \frac{u^2}{\sigma} dx + \int_0^1 \eta (u')^2 dx = \|u\|^2_{1, \frac{1}{\sigma}}.$$

On the other hand, if $\lambda \in (0, \frac{1}{C_{HP}})$, then

$$L(u, u) \ge \int_0^1 \frac{u^2}{\sigma} dx + (1 - \lambda C_{HP}) \int_0^1 \eta (u')^2 dx \ge (1 - \lambda C_{HP}) \|u\|^2_{1, \frac{1}{\sigma}},$$

by (2.9). Moreover $L(u, z)$ is continuous: indeed, one has

$$|L(u, z)| \le \|u\|_{L^2_{\frac{1}{\sigma}}(0,1)} \|z\|_{L^2_{\frac{1}{\sigma}}(0,1)} + (\|\eta\|_{L^\infty(0,1)} + |\lambda|) \|u'\|_{L^2(0,1)} \|z'\|_{L^2(0,1)},$$

for all $u, z \in H^1_{\frac{1}{\sigma}}(0, 1)$, and the conclusion follows by (2.12).

As a consequence, by the Lax-Milgram Theorem, there exists a unique solution $u \in H^1_{\frac{1}{\sigma}}(0, 1)$ of

$$L(u, z) = F(z) \text{ for all } z \in H^1_{\frac{1}{\sigma}}(0, 1),$$

namely

$$\int_0^1 \frac{uz}{\sigma} dx + \int_0^1 \eta u' z' dx - \lambda \int_0^1 \frac{uz}{\sigma d} dx = \int_0^1 (f + g) z \frac{1}{\sigma} dx \qquad (2.20)$$

for all $z \in H^1_{\frac{1}{\sigma}}(0, 1)$.

Now, take $v := u - f$; then $v \in H^1_{\frac{1}{\sigma}}(0, 1)$. We will prove that $(u, v) \in D(\mathcal{A})$ and solves (2.19). To begin with, (2.20) holds for every $z \in C^\infty_c(0, 1)$. Thus we have

$$\int_0^1 \eta u' z' dx = \int_0^1 \left(f + g - u + \lambda \frac{u}{d}\right) z \frac{1}{\sigma} dx$$

for every $z \in C^\infty_c(0, 1)$. Hence $-(\eta u')' = \left(f + g - u + \lambda \frac{u}{d}\right) \frac{1}{\sigma}$ in the distributional sense and a.e. in $(0, 1)$; in particular, $-\sigma (\eta u')' = \left(f + g - u + \lambda \frac{u}{d}\right)$ a.e. in $(0, 1)$, that is $A_\lambda u = \sigma (\eta u')' + \lambda \frac{u}{d} = -(f + g - u) \in L^2_{\frac{1}{\sigma}}(0, 1)$; thus $u \in D(A)$ and (2.19) holds. $\qquad \square$

As usual in semigroup theory, the mild solution of (2.18) obtained by using Theorem 2.1 can be more regular: if $\mathcal{Y}_0 \in D(\mathcal{A})$, then the solution is classical, in the sense that $\mathcal{Y} \in C^1([0, +\infty); \mathcal{H}_0) \cap C([0, +\infty); D(\mathcal{A}))$ and the equation in (2.1) holds for all $t \ge 0$. Hence, as in [1, Corollary 4.2] or in [4, Proposition 3.15], one can deduce the next result.

Theorem 2.3 *Hypotheses 2 and 3 hold. If $\left(y_T^0, y_T^1\right) \in H^1_{\frac{1}{\sigma}}(0, 1) \times L^2_{\frac{1}{\sigma}}(0, 1)$, then there exists a unique mild solution*

$$y \in C^1\left([0, +\infty); L^2_{\frac{1}{\sigma}}(0, 1)\right) \cap C\left([0, +\infty); H^1_{\frac{1}{\sigma}}(0, 1)\right)$$

of (2.1) which depends continuously on the initial data. Moreover, there exists $C > 0$ such that

$$\int_0^1 \frac{y_t(t, x)^2}{\sigma} dx + \int_0^1 \eta y_x^2(t, x) dx \le C\left(\int_0^1 \frac{(y_T^1)^2}{\sigma} dx + \int_0^1 \eta (y_T^0)_x^2 dx\right) \qquad (2.21)$$

for every $t > 0$. Moreover, if $(y_T^0, y_T^1) \in D(A_\lambda) \times H^1_{\frac{1}{\sigma}}(0, 1)$, then the solution y is classical, in the sense that

$$y \in C^2\left([0, +\infty); L^2_{\frac{1}{\sigma}}(0, 1)\right) \cap C^1\left([0, +\infty); H^1_{\frac{1}{\sigma}}(0, 1)\right) \cap C\left([0, +\infty); D(A_\lambda)\right)$$

and the equation in (2.1) holds for all $t \geq 0$.

Proof We only prove (2.21), the rest being standard. First, assume that y is a classical solution for which (2.14) holds, for instance when $(y_T^0, y_T^1) \in C^\infty_c(0, 1) \times C^\infty_c(0, 1) \subset H^2_{\frac{1}{\sigma}}(0, 1) \times H^1_{\frac{1}{\sigma}}(0, 1)$. Then, taken any $t > 0$, multiply the equation by $\frac{y_t}{\sigma}$ and integrate over $[0, 1] \times [0, t]$, obtaining

$$\frac{1}{2}\int_0^t \int_0^1 \frac{1}{\sigma}\frac{d}{d\tau}(y_\tau(\tau, x))^2 dx d\tau + \frac{1}{2}\int_0^t \int_0^1 \eta\frac{d}{d\tau}(y_x(\tau, x)^2) dx d\tau$$
$$= \frac{\lambda}{2}\int_0^t \int_0^1 \frac{1}{\sigma d}\frac{d}{d\tau}y^2(\tau, x) dx d\tau.$$

As a consequence,[1]

$$\int_0^1 \frac{y_t^2(t, x))}{\sigma}dx + \int_0^1 \eta y_x^2(t, x)dx = \int_0^1 \frac{(y_T^1)^2}{\sigma}dx + \int_0^1 \eta(y_T^0)_x^2 dx$$
$$+ \lambda\int_0^1 \frac{y^2(t, x)}{\sigma d}dx - \lambda\int_0^1 \frac{(y_T^0)^2}{\sigma d}dx. \tag{2.22}$$

Now, if $\lambda \leq 0$, the previous inequality, together with Proposition 2.1, immediately give (2.21). On the other hand, if $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$, there exists $\varepsilon > 0$ such that $\lambda C_{HP} = 1 - \varepsilon$. In this way, by Proposition 2.1 we find

$$\lambda\int_0^1 \frac{y^2(t, x)}{\sigma d}dx \leq (1 - \varepsilon)\int_0^1 \eta y_x^2 dx,$$

which, together with (2.22), gives the desired statement.

Now, let $(y_T^0, y_T^1) \in H^1_{\frac{1}{\sigma}}(0, 1) \times L^2_{\frac{1}{\sigma}}(0, 1)$ and take $(y_{T,n}^0, y_{T,n}^1) \in C^\infty_c(0, 1) \times C^\infty_c(0, 1)$ be such that $(y_{T,n}^0, y_{T,n}^1) \rightarrow (y_T^0, y_T^1)$ in $H^1_{\frac{1}{\sigma}}(0, 1) \times L^2_{\frac{1}{\sigma}}(0, 1)$ - recall that $C^\infty_c(0, 1)$ is dense in $H^1_{\frac{1}{\sigma}}(0, 1)$, and so in $L^2_{\frac{1}{\sigma}}(0, 1)$, see [11, Corollary 2.2]. In this way, the mild solution y_n to problem (2.1) with y_T^0 replaced by $y_{T,n}^0$ and y_T^1 replaced by $y_{T,n}^1$ is also classical, and thus (2.21) holds for y_n. Moreover, by linearity of the equation, it is readily seen that (2.21) implies that $\frac{(y_n)_t}{\sqrt{\sigma}} + \sqrt{\eta}(y_n)_x$ is a Cauchy sequence in $L^2(0, 1)$. This permits to pass to the limit and show that (2.21) holds also for mild solutions. □

---

[1] Notice that this equality is an obvious consequence of Theorem 3.1 below.

Remark 5 Due to the reversibility in time of the equation, solutions exist with the same regularity for $t < 0$. Indeed, the associated matrix operator is skew-adjoint and it generates a C_0 group of unitary operators on the associated reference space $\mathcal{H}_0$ by Stone's Theorem. We will use this fact in the proof of the controllability result, by considering a backward problem whose final time data will be transformed in initial ones: this is the reason for the notation of the initial data in problem (2.1).

3 Energy Estimates

In this section we prove estimates of the energy associated to the solution of (2.1) from below and from above. The former will be used in the next section to prove a controllability result, while the latter is used here to prove a boundary observability inequality.

To begin with, we give the following lemma which will be crucial for the rest of the paper. Some points are similar to the ones given in [8, Lemma 3.2] or in [26, Lemma 2.2]; however, due to the different domain of the operator A, some computations are more complicate that those in [8, Lemma 3.2] or in [26, Lemma 2.2], for this reason we will give the proof of the next lemma in the Appendix.

Lemma 3.1 *Assume Hypothesis 1.*

1. *If $y \in D(A)$ and $u \in H^1_{\frac{1}{\sigma}}(0,1)$, then $\lim_{x \to 0} u(x)y'(x) = 0$.*
2. *If $u \in H^1_{\frac{1}{\sigma}}(0,1)$, then $\lim_{x \to 0} \dfrac{x}{a} u^2(x) = 0$.*
3. *If $u \in H^1_{\frac{1}{\sigma}}(0,1)$ and $K_1 + K_2 < 2$, then $\lim_{x \to 0} \dfrac{x}{ad} u^2(x) = 0$.*
4. *If $u \in H^1_{\frac{1}{\sigma}}(0,1)$, then $\lim_{x \to 0} \dfrac{x^2}{ad} u^2(x) = 0$.*
5. *If $u \in D(A)$, then $\lim_{x \to 0} x^2(u'(x))^2 = 0$.*
6. *If $u \in D(A)$, $K_1 \le 1$, then $\lim_{x \to 0} x(u'(x))^2 = 0$.*
7. *If $u \in D(A)$, $K_1 > 1$ and $\dfrac{xb}{a} \in L^\infty(0,1)$, then $\lim_{x \to 0} x(u'(x))^2 = 0$.*

Now, let us introduce the definition of the energy.

Definition 3.1 For a mild solution y of (2.1) we define its energy as the continuous function

$$E_y(t) = \frac{1}{2} \int_0^1 \left[\frac{1}{\sigma} y_t^2(t,x) + \eta y_x^2(t,x) - \frac{\lambda}{\sigma d} y^2(t,x) \right] dx, \quad \forall t \ge 0. \tag{3.1}$$

The definition above guarantees that the classical conservation of the energy still holds true also in the degenerate/singular situation with a drift term, see Theorem 3.1 below.

Theorem 3.1 *Assume Hypothesis 1 and let y be a mild solution of (2.1). Then*

$$E_y(t) = E_y(0), \quad \forall \, t \geq 0. \tag{3.2}$$

Proof First, suppose that y is a classical solution. Then multiply the equation by $\frac{y_t}{\sigma}$, integrate over $(0, 1)$ and use the boundary conditions to get

$$0 = \frac{1}{2} \int_0^1 \frac{d}{dt} \left(\frac{y_t^2}{\sigma} \right) dx - [\eta y_x y_t]_{x=0}^{x=1} + \int_0^1 \eta y_x y_{tx} dx - \int_0^1 \frac{\lambda}{\sigma d} y y_t \, dx.$$

Now, observe that, thanks to the boundary condition at 1, $(\eta y_x y_t)(t, 1) = 0$. Moreover, thanks to Lemma 3.1.1 applied with $u = y_t$, one has

$$\lim_{x \to 0} \eta(x) y_t(t, x) y_x(t, x) = 0.$$

Thus

$$\begin{aligned}
0 &= \frac{1}{2} \int_0^1 \frac{d}{dt} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 \right) dx - \frac{1}{2} \int_0^1 \frac{d}{dt} \left(\frac{\lambda y^2}{\sigma d} \right) dx \\
&= \frac{1}{2} \int_0^1 \frac{d}{dt} \left(\frac{y_t^2}{\sigma} + \eta y_x^2 - \frac{\lambda y^2}{\sigma d} \right) dx = \frac{1}{2} \frac{d}{dt} E_y(t).
\end{aligned}$$

If y is a mild solution we can proceed with a standard approximation method with more regular initial data, as in the proof of Theorem 2.3. $\qquad\square$

Now we prove an inequality for the energy which we will use in the next section to prove the controllability result. For that, we start proving the following result.

Moreover, we introduce the next assumption, which collects all the needed requirements, in particular Hypothesis 3.

In order to face the original problem, we introduce the next assumption, which collects all the needed requirements, in particular Hypothesis 3.

Hypothesis 4 Hypothesis 2 holds. Moreover, functions $a, b, d \in C^0[0, 1]$ are such that

- $\dfrac{b}{a} \in L^1(0, 1)$;
- a and d are (WD) or (SD) with $K_a + 2K_d \leq 2$.

Theorem 3.2 *Assume a and d (WD) or (SD) with $K_a + 2K_d \leq 2$ and $\frac{b}{a} \in L^1(0, 1)$. If y is a classical solution of (2.1), then for any $T > 0$ we have*

$$\begin{aligned}
\frac{1}{2} \eta(1) \int_0^T y_x^2(t, 1) dt &= \int_0^1 \left[\frac{x^2 y_x y_t}{\sigma} \right]_{t=0}^{t=T} dx + \frac{1}{2} \int_{Q_T} \left(2 - \frac{x(a' - b)}{a} \right) x \frac{y_t^2}{\sigma} dx dt \\
&+ \int_{Q_T} x \eta y_x^2 dx dt - \frac{1}{2} \int_{Q_T} x^2 \eta y_x^2 \frac{b}{a} dx dt + \frac{\lambda}{2} \int_{Q_T} \frac{y^2 x}{\sigma d} \left(2 - \frac{x(a' - b)}{a} - \frac{x d'}{d} \right) dx dt.
\end{aligned} \tag{3.3}$$

Here $Q_t := (0, T) \times (0, 1)$

As a consequence, if Hypothesis 4 holds and if y is a mild solution of (2.1), then $y_x(\cdot, 1) \in L^2(0, T)$ for every $T > 0$ and there exist two constants $C_i > 0$, $i = 1, 2$, such that

$$\eta(1) \int_0^T y_x^2(t, 1) dt \le C_1 E_y(0) + C_2 T E_y(0). \tag{3.4}$$

Proof First, assume that y is a classical solution of (2.1). Then, multiplying the equation in (2.1) by $\dfrac{x^2 y_x}{\sigma}$ and integrating by parts over Q_T we have

$$
\begin{aligned}
0 = &\int_0^1 \left[\frac{x^2 y_x y_t}{\sigma}\right]_{t=0}^{t=T} dx - \frac{1}{2} \int_0^T \left[\frac{x^2}{\sigma} y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_T} \left(\frac{x^2}{\sigma}\right)_x y_t^2 dx dt \\
&- \int_{Q_T} Ay \frac{x^2 y_x}{\sigma} dx dt - \lambda \int_0^1 \frac{x^2 y y_x}{\sigma d} dx.
\end{aligned}
\tag{3.5}
$$

Observe that by assumptions $Au \in L^2_{\frac{1}{\sigma}}(0, 1)$, $y_x \in L^2(0, 1)$ and $\dfrac{x^2}{a}$ is bounded, thus $\displaystyle\int_{Q_T} Ay \frac{x^2 y_x}{\sigma} dx dt$ is well defined by Hölder's inequality. Moreover, also

$$\int_0^1 \frac{x^2 y y_x}{\sigma d} dx \in \mathbb{R}.$$

Indeed, by Proposition 2.1 there exists $c > 0$ such that

$$\left| \int_0^1 \frac{x^2 y y_x}{\sigma d} dx \right| \le \max_{[0,1]} \sqrt{\eta} \int_0^1 \frac{|y|}{\sqrt{\sigma d}} |y_x| \frac{x^{K_1 + K_2}}{\sqrt{ad}} dx \le c \int_0^1 \frac{|y|}{\sqrt{\sigma d}} |y_x| dx^2,$$

and Hölder's inequality permits to conclude. Hence, by (3.5),

$$
\begin{aligned}
0 = &\int_0^1 \left[\frac{x^2 y_x y_t}{\sigma}\right]_{t=0}^{t=T} dx - \frac{1}{2} \int_0^T \left[\frac{x^2}{\sigma} y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_T} \left(\frac{x^2}{\sigma}\right)_x y_t^2 dx dt \\
&- \int_{Q_T} (\eta y_x)_x x^2 y_x dx dt - \lambda \int_{Q_T} \frac{x^2 y y_x}{\sigma d} dx dt \\
= &\int_0^1 \left[\frac{x^2 y_x y_t}{\sigma}\right]_{t=0}^{t=T} dx - \frac{1}{2} \int_0^T \left[\frac{x^2}{\sigma} y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_T} \left(2 - \frac{x(a' - b)}{a}\right) x \frac{y_t^2}{\sigma} dx dt \\
&- \int_0^T \left[x^2 \eta y_x^2\right]_{x=0}^{x=1} dt + \int_{Q_T} \eta y_x (x^2 y_x)_x dx dt - \frac{\lambda}{2} \int_{Q_T} \frac{x^2}{\sigma d} (y^2)_x dx dt \\
= &\int_0^1 \left[\frac{x^2 y_x y_t}{\sigma}\right]_{t=0}^{t=T} dx - \frac{1}{2} \int_0^T \left[\frac{x^2}{\sigma} y_t^2\right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_T} \left(2 - \frac{x(a' - b)}{a}\right) x \frac{y_t^2}{\sigma} dx dt
\end{aligned}
$$

---

[2] Let us remark that, if a and d are (WD) or (SD), then a good constant in such an inequality is $\dfrac{\max_{[0,1]} \sqrt{\eta}}{\sqrt{a(1) d(1)}}$.

$$- \int_0^T \left[x^2 \eta y_x^2 \right]_{x=0}^{x=1} dt + \int_{Q_T} \eta y_x (2xy_x + x^2 y_{xx}) dx dt - \frac{\lambda}{2} \int_0^T \left[\frac{x^2 y^2}{\sigma d} \right]_{x=0}^{x=1} dt$$

$$+ \frac{\lambda}{2} \int_{Q_T} \left(\frac{x^2}{\sigma d} \right)_x y^2 dx dt$$

$$= \int_0^1 \left[\frac{x^2 y_x y_t}{\sigma} \right]_{t=0}^{t=T} dx - \frac{1}{2} \int_0^T \left[\frac{x^2}{\sigma} y_t^2 \right]_{x=0}^{x=1} dt - \frac{1}{2} \int_0^T \left[x^2 \eta y_x^2 \right]_{x=0}^{x=1} dt$$

$$- \frac{\lambda}{2} \int_0^T \left[\frac{x^2 y^2}{\sigma d} \right]_{x=0}^{x=1} dt + \frac{1}{2} \int_{Q_T} \left(2 - \frac{x(a'-b)}{a} \right) x \frac{y_t^2}{\sigma} dx dt + \int_{Q_T} x \eta y_x^2 dx dt$$

$$- \frac{1}{2} \int_{Q_T} x^2 \eta y_x^2 \frac{b}{a} dx dt + \frac{\lambda}{2} \int_{Q_T} \frac{y^2 x}{\sigma d} \left(2 - \frac{x(a'-b)}{a} - \frac{xd'}{d} \right) dx dt. \tag{3.6}$$

As in [8], using Lemma 3.1, we can prove

$$\int_0^T \left[\frac{x^2}{\sigma} y_t^2 \right]_{x=0}^{x=1} dt = 0;$$

in addition, thanks to Lemma 3.1, $\lim_{x \to 0} x^2 y_x^2(t, x) = 0$ and $\lim_{x \to 0} y^2(t, x) \frac{x^2}{ad} = 0$, so that

$$\int_0^T \left[x^2 \eta y_x^2 \right]_{x=0}^{x=1} dt = \eta(1) \int_0^T y_x^2(t, 1) dt$$

and

$$\lambda \int_0^T \left[y^2 \frac{x^2}{\sigma d} \right]_{x=0}^{x=1} dt = 0,$$

so that (3.3) holds true.

Now assume Hypothesis 4 and let us prove that inequality (3.4) holds. First, suppose that y is a classical solution. As in [8, Theorem 3.4], one can prove that

$$\int_0^1 \left[\frac{x^2 y_x y_t}{\sigma} \right]_{t=0}^{t=T} dx \le 2 \max \left\{ \frac{1}{a(1)}, 1 \right\} E_y(0).$$

Then, we estimate the distributed terms in the previous equation. Using (2.6), one has

$$\frac{1}{2} \left| \int_{Q_T} \eta \frac{bx^2}{a} y_x^2 dx dt \right| \le \frac{M}{2} \int_{Q_T} \eta y_x^2 dx dt,$$

where M is defined in (2.8); moreover

$$\left| \frac{1}{2} \int_{Q_T} \frac{x y_t^2}{\sigma} \left(2 - \frac{x(a'-b)}{a} \right) dx dt \right| \le \frac{2 + K_a + M}{2} \int_{Q_T} \frac{y_t^2}{\sigma} dx dt$$

and, by Proposition 2.1,

$$\left| \frac{\lambda}{2} \int_{Q_T} \frac{y^2 x}{\sigma d} \left(2 - \frac{x(a'-b)}{a} - \frac{xd'}{d} \right) dxdt \right| \le \frac{|\lambda|}{2} \int_{Q_T} \frac{y^2}{\sigma d} (2 + K_a + K_d + M) \, dxdt$$

$$\le \frac{|\lambda|(2 + K_a + K_d + M)C_{HP}}{2} \int_{Q_T} \eta y_x^2 dxdt.$$

Hence, by (3.3) and the previous computations, thanks to Theorem 3.1, we get

$$\frac{1}{2}\eta(1) \int_0^T y_x^2(t, 1)dt \le 2 \max \left\{ \frac{1}{a(1)}, 1 \right\} E_y(0) + \frac{(2 + K_a + M)}{2} \int_{Q_T} \frac{y_t^2}{\sigma} dxdt$$

$$+ \left(\frac{|\lambda|(2 + K_a + K_d + M)C_{HP}}{2} + \frac{M}{2} + 1 \right) \int_{Q_T} \eta y_x^2 dxdt$$

$$\le C_1 E_y(0) + C_2 T E_y(0),$$

where

$$C_1 = 2 \max \left\{ \frac{1}{a(1)}, 1 \right\}$$

and

$$C_2 = \max \{ |\lambda|(2 + K_a + K_d + M)C_{HP} + M + 2, 2 + K_a + M \}.$$

Now, let y be the mild solution associated to the initial data $(y_T^0, y_T^1) \in \mathcal{H}_0$, see (2.17), and consider a sequence $\left\{ \left(y_{T,n}^0, y_{T,n}^1 \right) \right\}_{n \in \mathbb{N}} \subset D(\mathcal{A}) = H_{\frac{1}{\sigma}}^2(0, 1) \times H_{\frac{1}{\sigma}}^1(0, 1)$ - see (2.16) - that approximates (y_T^0, y_T^1) and let y_n be the classical solution of (2.1) associated to $\left(y_{T,n}^0, y_{T,n}^1 \right)$. Clearly y_n satisfies (3.4) and by linearity $(y_n)_x$ is a Cauchy sequence in $L^2(0, T)$ and we can conclude by passing to the limit. □

We shall also use an estimate from above for the energy. The next preliminary result holds.

Theorem 3.3 *Assume a and d (WD) or (SD) with $K_a + 2K_d \le 2$ and $\frac{b}{a} \in L^1(0, 1)$. If y is a classical solution of (2.1), then for any $T > 0$ the following identity holds:*

$$\frac{1}{2}\eta(1) \int_0^T y_x^2(t, 1)dt = \int_0^1 \left[\frac{xy_x y_t}{\sigma} \right]_{t=0}^{t=T} dx - \frac{1}{2} \int_{Q_T} x\eta \frac{b}{a} y_x^2 dxdt + \frac{1}{2} \int_{Q_T} \eta y_x^2 dxdt$$

$$+ \frac{1}{2} \int_{Q_T} \left(1 - \frac{x(a'-b)}{a} \right) \frac{1}{\sigma} y_t^2 dxdt + \frac{\lambda}{2} \int_{Q_T} \frac{y^2}{\sigma d} \left(1 - x \left(\frac{d'}{d} + \frac{a'-b}{a} \right) \right) dxdt.$$

$$(3.7)$$

Proof Multiplying the equation in (2.1) by $\dfrac{xy_x}{\sigma}$, integrating by parts over Q_T and recalling (2.13) we have, as in [8, Theorem 3.6],

$$\frac{1}{2}\int_0^T \left[x\eta y_x^2\right]_{x=0}^{x=1} dt = \int_0^1 \left[\frac{xy_x y_t}{\sigma}\right]_{t=0}^{t=T} dx - \frac{1}{2}\int_{Q_T} x\eta\frac{b}{a}y_x^2\,dxdt$$
$$+ \frac{1}{2}\int_{Q_T}\left(1 - \frac{x(a'-b)}{a}\right)\frac{1}{\sigma}y_t^2\,dxdt + \frac{1}{2}\int_{Q_T}\eta y_x^2\,dxdt$$
$$- \lambda\int_{Q_T}\frac{xyy_x}{d\sigma}\,dxdt.$$

$$(3.8)$$

Now, consider the last term in the previous equality:

$$-\lambda\int_{Q_T}\frac{xyy_x}{d\sigma}\,dxdt = -\frac{\lambda}{2}\int_{Q_T}\frac{x\left(y^2\right)_x}{\sigma d}\,dxdt$$
$$= -\frac{\lambda}{2}\int_0^T\left[\frac{xy^2}{\sigma d}\right]_{x=0}^{x=1}dt + \frac{\lambda}{2}\int_{Q_T}\frac{y^2}{\sigma d}\left(1 - x\frac{\sigma d' + \sigma'd}{\sigma d}\right)dxdt \quad (3.9)$$
$$= -\frac{\lambda}{2}\int_0^T\left[\frac{xy^2}{\sigma d}\right]_{x=0}^{x=1}dt + \frac{\lambda}{2}\int_{Q_T}\frac{y^2}{\sigma d}\left[1 - x\left(\frac{d'}{d} + \frac{a'-b}{a}\right)\right]dxdt.$$

Clearly $\left(\dfrac{xy^2}{\sigma d}\right)(t,1) = 0$. Moreover, the fact that $K_a + 2K_d \le 2$ implies that $K_a + K_d < 2$, so by Lemma 3.1.3 we find

$$\lim_{x\to 0}\left(\frac{xy^2}{\sigma d}\right)(t,x) = 0$$

and the conclusion follows.   □

In order to conclude the estimate which leads to the observability inequality, we distinguish between two cases; the first case is when a is (WD) or (SD) with $K_a = 1$, while the second one is for a (SD) and $K_a \ne 1$.

Hypothesis 5 Hypothesis 4 holds with $K_a \le 1$ and $K_a < 2 - 2M$, Where M is defined in (2.8)

Remark 6 1. The assumption $K_a < 2 - 2M$ implies that $M < 1$.

2. The assumption $K_a < 2 - 2M$ is clearly satisfied if $\|b\|_{L^\infty(0,1)} < \dfrac{a(1)}{2}$. Indeed in this case $2 - 2M > 1$ and $K_a \le 1$.

3. The assumption $K_a < 2 - 2M$ is not surprising since it is the same assumption made in [8] when $\lambda = 0$.

The observability inequality we get in this case is the following:

Theorem 3.4 *Assume Hypothesis 5. If y is a mild solution of (2.1), then there exists a positive constant C_3 such that for any $T > 0$ we have*

$$\eta(1) \int_0^T y_x^2(t, 1)dt \geq 2T \left(1 - \frac{K_a}{2} - M - |\lambda| C_{HP}(2 + K_a + K_d) \right) E_y(0)$$
$$- \left\{ 4 \max \left\{ 1, \frac{1}{a(1)} \right\} + K_a \max \left\{ 1, C_{HP} \max_{[0,1]} d \right\} \right\} E_y(0),$$

if $\lambda \leq 0$, and

$$\eta(1) \int_0^T y_x^2(t, 1)dt \geq 2T \left(1 - \frac{K_a}{2} - M \right) E_y(0) + \lambda(2 - K_a - K_d - M) \int_{Q_T} \frac{y^2}{\sigma d} dxdt$$
$$- \frac{1}{\epsilon} \left(4 \max \left\{ \frac{1}{a(1)}, 1 \right\} + K_a \max \left\{ 1, C_{HP} \max_{[0,1]} d \right\} \right) E_y(0)$$

if $\lambda \in \left(0, \frac{1}{C_{HP}} \right)$.
  Here $\epsilon \in (0, 1)$ is such that

$$\lambda = \frac{1}{C_{HP}} - \frac{\epsilon}{C_{HP}} > 0. \tag{3.10}$$

Proof As a first step, assume that y is a classical solution. By multiplying the equation in (2.1) by $\dfrac{-K_a y}{2\sigma}$ and integrating by parts on Q_T, we have

$$0 = -\frac{K_a}{2} \int_0^1 \left[\frac{yy_t}{\sigma} \right]_{t=0}^{t=T} dx + \frac{K_a}{2} \int_{Q_T} \frac{y_t^2}{\sigma} dxdt - \frac{K_a}{2} \int_{Q_T} \eta y_x^2 dxdt + \frac{\lambda K_a}{2} \int_{Q_T} \frac{y^2}{\sigma d} dxdt.$$

Summing the previous equality to (3.7) multiplied by 2, we have

$$\eta(1) \int_0^T y_x^2(t, 1)dt = 2 \int_0^1 \left[\frac{xy_x y_t}{\sigma} \right]_{t=0}^{t=T} dx - \frac{K_a}{2} \int_0^1 \left[\frac{yy_t}{\sigma} \right]_{t=0}^{t=T} dx + \frac{K_a}{2} \int_{Q_T} \frac{y_t^2}{\sigma} dxdt$$
$$- \frac{K_a}{2} \int_{Q_T} \eta y_x^2 dxdt + \int_{Q_T} \frac{y_t^2}{\sigma} dxdt + \int_{Q_T} \eta y_x^2 dxdt - \int_{Q_T} x\frac{b}{a} y_x^2 dxdt$$
$$- \int_{Q_T} \frac{x(a'-b)}{a} \frac{y_t^2}{\sigma} dxdt + \lambda \int_{Q_T} \frac{y^2}{\sigma d} \left(1 - x\left(\frac{d'}{d} + \frac{a'-b}{a} \right) \right) dxdt + \frac{\lambda K_a}{2} \int_{Q_T} \frac{y^2}{\sigma d} dxdt.$$

Analogously to what done in the proof of [8, Theorem 3.7] for $\lambda = 0$, we find

$$
\begin{aligned}
\eta(1) \int_0^T y_x^2(t,1)dt \geq{} & 2 \int_0^1 \left[\frac{x y_x y_t}{\sigma}\right]_{t=0}^{t=T} dx - \frac{K_a}{2} \int_0^1 \left[\frac{y_t y}{\sigma}\right]_{t=0}^{t=T} dx \\
& + \left(1 - \frac{K_a}{2} - M\right) \int_{Q_T} \eta y_x^2 dx dt \\
& + \left(1 - \frac{K_a}{2} - M\right) \int_{Q_T} \frac{y_t^2}{\sigma} dx dt \\
& + \lambda \int_{Q_T} \frac{y^2}{\sigma d} \left(1 + \frac{K_a}{2} - x\left(\frac{d'}{d} + \frac{a'-b}{a}\right)\right) dx dt.
\end{aligned}
\tag{3.11}
$$

First, we concentrate on the boundary terms. By the Schwartz inequality, for all $\tau \in [0,T]$ we have

$$
\begin{aligned}
\left|2\int_0^1 \frac{x y_x(\tau,x) y_t(\tau,x)}{\sigma(x)} dx\right| &\leq \int_0^1 \frac{x^2}{\sigma(x)} y_x^2(\tau,x) dx + \int_0^1 \frac{y_t^2(\tau,x)}{\sigma(x)} dx \\
&\leq \frac{1}{a(1)} \int_0^1 (\eta y_x^2)(\tau,x) dx + \int_0^1 \frac{y_t^2(\tau,x)}{\sigma(x)} dx \\
&\leq \max\left\{\frac{1}{a(1)}, 1\right\} \left(\int_0^1 \eta y_x^2(\tau,x) dx + \int_0^1 \frac{y_t^2(\tau,x)}{\sigma} dx\right),
\end{aligned}
\tag{3.12}
$$

and by Proposition 2.1

$$
\begin{aligned}
\frac{K_a}{2} \left|\int_0^1 \frac{y y_t}{\sigma}(\tau,x) dx\right| &\leq \frac{K_a}{4} \int_0^1 \frac{y_t^2}{\sigma}(\tau,x) dx + \frac{K_a C_{HP}}{4} \max_{[0,1]} d \int_0^1 \eta y_x^2(\tau,x) dx \\
&\leq \frac{K_a}{4} \max\left\{1, C_{HP} \max_{[0,1]} d\right\} \left(\int_0^1 \eta y_x^2(\tau,x) dx + \int_0^1 \frac{y_t^2(\tau,x)}{\sigma} dx\right)
\end{aligned}
\tag{3.13}
$$

for all $\tau \in [0,T]$.

Now, we proceed in different ways, according to the sign of λ.

Case $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$. In this case, by (3.10) and by (2.9),

$$
\int_0^1 \eta y_x^2 dx - \lambda \int_0^1 \frac{y^2}{\sigma d} dx \geq \int_0^1 \eta y_x^2 dx - (1-\epsilon) \int_0^1 \eta y_x^2 dx = \epsilon \int_0^1 \eta y_x^2 dx.
$$

Thus, by Theorem 3.1, for all $\tau \in [0,T]$

$$
2E_y(0) = 2E_y(\tau) \geq \epsilon \int_0^1 (\eta y_x^2)(\tau,x) dx + \int_0^1 \frac{y_t^2(\tau,x)}{\sigma(x)} dx
$$

$$\geq \epsilon \left(\int_0^1 (\eta y_x^2)(\tau, x) dx + \int_0^1 \frac{y_t^2(\tau, x)}{\sigma(x)} dx \right)$$

so that

$$\int_0^1 (\eta y_x^2)(\tau, x) dx + \int_0^1 \frac{y_t^2(\tau, x)}{\sigma(x)} dx \leq \frac{2}{\epsilon} E_y(0), \tag{3.14}$$

for all $\tau \in [0, T]$.

Thus, by (3.12) and (3.14) we get for any $\tau \in [0, T]$

$$2 \left| \int_0^1 \frac{x y_x(\tau, x) y_t(\tau, x)}{\sigma(x)} dx \right| \leq \frac{2}{\epsilon} \max \left\{ \frac{1}{a(1)}, 1 \right\} E_y(0), \tag{3.15}$$

while from (3.13) and (3.14)

$$\frac{K_a}{2} \left| \int_0^1 \frac{y y_t}{\sigma}(\tau, x) dx \right| \leq \frac{K_a}{2\epsilon} \max \left\{ 1, C_{HP} \max_{[0,1]} d \right\} E_y(0). \tag{3.16}$$

Moreover, since $\lambda > 0$, by Theorem 3.1 we have

$$\left(1 - \frac{K_a}{2} - M \right) \int_{Q_T} \left(\eta y_x^2 + \frac{y_t^2}{\sigma} \right) dx dt$$
$$+ \lambda \int_{Q_T} \frac{y^2}{\sigma d} \left(1 + \frac{K_a}{2} - x \left(\frac{d'}{d} + \frac{a' - b}{a} \right) \right) dx dt$$
$$= \left(1 - \frac{K_a}{2} - M \right) \int_{Q_T} \left(\eta y_x^2 + \frac{y_t^2}{\sigma} - \lambda \frac{y^2}{\sigma d} \right) dx dt \tag{3.17}$$
$$+ \lambda \int_{Q_T} \frac{y^2}{\sigma d} \left(\left(1 - \frac{K_a}{2} - M \right) + 1 + \frac{K_a}{2} - x \left(\frac{d'}{d} + \frac{a' - b}{a} \right) \right) dx dt$$
$$\geq 2T \left(1 - \frac{K_a}{2} - M \right) E_y(0) + \lambda (2 - K_a - K_d - M) \int_{Q_T} \frac{y^2}{\sigma d} dx dt.$$

Hence, from (3.11), (3.15), (3.16) and (3.17) we find the desired inequality.
Case $\lambda \leq 0$. In this case, in (3.12) and (3.13) we can estimate the sum of the two integrals in right-hand-side with $E_y(\tau) = E_y(0)$ (by Theorem 3.1).

Moreover, as in (3.17) we find

$$
\left(1 - \frac{K_a}{2} - M\right) \int_{Q_T} \left(\eta y_x^2 + \frac{y_t^2}{\sigma}\right) dx dt
$$

$$
+ \lambda \int_{Q_T} \frac{y^2}{\sigma d} \left(1 + \frac{K_a}{2} - x\left(\frac{d'}{d} + \frac{a' - b}{a}\right)\right) dx dt
$$

$$
\geq 2T \left(1 - \frac{K_a}{2} - M\right) E_y(0) + \lambda(2 + K_a + K_d) \int_{Q_T} \frac{y^2}{\sigma d} dx dt
$$

$$
\geq 2T \left(1 - \frac{K_a}{2} - M\right) E_y(0) - |\lambda| 2(2 + K_a + K_d) C_{HP} T E_y(0),
$$

and as before the conclusion follows.

Now, let y be the mild solution associated to the initial data $(y_T^0, y_T^1) \in \mathcal{H}_0$. By approximation, as in the proof of Theorem 3.2, we get the same estimates for mild solutions. $\qquad \square$

Now assume the following

Hypothesis 6 Hypothesis 4 holds with $1 < K_a < 2 - 2M$ and $\frac{xb}{a} \in L^\infty(0, 1)$.

Set

$$
M_\infty := \left\|\frac{xb}{a}\right\|_{L^\infty(0,1)}. \tag{3.18}
$$

Under Hypothesis 6, Theorems 3.3 and 3.4 still hold. Indeed, all the calculations made in the proofs can be reformulated verbatim, substituting M with M_∞. We briefly collect this fact in the following

Theorem 3.5 *Assume hypothesis 6; then the statement of Theorem 3.4 holds with M replaced by M_∞.*

4 Boundary Observability and Null Controllability

Following [1], we recall the next definition:

Definition 4.1 Problem (2.1) is said to be *observable in time $T > 0$ via the normal derivative at $x = 1$* if there exists a constant $C > 0$ such that for any $(y_T^0, y_T^1) \in \mathcal{H}_0$ the mild solution y of (2.1) satisfies

$$
C E_y(0) \leq \int_0^T y_x^2(t, 1) dt. \tag{4.1}
$$

Any constant C satisfying (4.1) is called *observability constant* for (2.1) in time T.

Setting

$$C_T := \sup\{C > 0 : C \text{ satisfies } (4.1)\},$$

we have that (2.1) is observable if and only if

$$C_T = \inf_{(y_T^0, y_T^1) \neq (0,0)} \frac{\int_0^T y_x^2(t, 1) dt}{E_y(0)} > 0.$$

The inverse of C_T, $c_t := \dfrac{1}{C_T}$, is called *the cost of observability* (or *the cost of control*) in time T.

Theorems 3.4 and 3.5 admit the following straightforward corollaries.

Corollary 4.1 *Assume Hypothesis 5 and*

$$-\frac{2 - K_a - 2M}{2C_{HP}(2 + K_a + K_d)} < \lambda \leq 0.$$

If

$$T > \frac{C_3}{C_4}, \tag{4.2}$$

then (2.1) is observable in time T. Moreover

$$C_T \geq \frac{C_4 T - C_3}{\eta(1)},$$

where

$$C_3 := 4 \max\left\{1, \frac{1}{a(1)}\right\} + K_a \max\left\{1, C_{HP} \max_{[0,1]} d\right\}$$

and

$$C_4 := 2\left(1 - \frac{K_a}{2} - M - |\lambda| C_{HP}(2 + K_a + K_d)\right).$$

Corollary 4.2 *Assume Hypothesis 5, $\lambda \epsilon \left(0, \frac{1}{C_{HP}}\right)$ and $K_a + K_d \leq 2 - M$. If*

$$T > \frac{C_5}{C_6}, \tag{4.3}$$

then (2.1) is observable in time T. Moreover

$$C_T \geq \frac{C_6 T - C_5}{\eta(1)},$$

where

$$C_5 := \frac{4}{\epsilon} \max\left\{\frac{1}{a(1)}, 1\right\} + K_a \max\left\{1, C_{HP} \max_{[0,1]} d\right\}$$

and

$$C_6 := 1 - \frac{K_a}{2} - M.$$

Remark 7 Corollaries 4.1 and 4.2 still hold true if we replace Hypothesis 5 with Hypothesis 6 and M with M_∞.

We underline that boundary observability is no longer true when $K_a \geq 2$. Indeed if we consider the pure degenerate case, i.e. $\lambda = 0$ and $b \equiv 0$, one can consider the same examples given in [8] with $a(x) = x^K$, $K \geq 2$, which shows the failure of boundary observability.

In the rest of this section we will devote to prove null controllability for (1.1) stated in Theorem 4.1. More precisely, given $(u_0, u_1) \in L^2_{\frac{1}{\sigma}}(0, 1) \times H^{-1}_{\frac{1}{\sigma}}(0, 1)$, we look for a control $f \in L^2(0, T)$ such that the solution of (1.1) satisfies

$$u(T, x) = u_t(T, x) = 0, \quad \text{for all } x \in (0, 1). \tag{4.4}$$

For this last purpose, we first give the definition of a solution for (1.1) by transposition, which permits low regularity on the notion of solution itself: such a definition is formally obtained by re-writing the equation of (1.1) as $u_{tt} - \sigma (\eta u_x)_x - \frac{\lambda}{d} u$ (thanks to (2.13)), multiplying by $\frac{v}{\sigma}$ and integrating by parts. Precisely:

Definition 4.2 Let $f \in L^2_{loc}[0, +\infty)$ and $(u_0, u_1) \in L^2_{\frac{1}{\sigma}}(0, 1) \times H^{-1}_{\frac{1}{\sigma}}(0, 1)$. We say that u is a solution by transposition of (1.1) if

$$u \in C^1\left([0, +\infty); H^{-1}_{\frac{1}{\sigma}}(0, 1)\right) \cap C\left([0, +\infty); L^2_{\frac{1}{\sigma}}(0, 1)\right)$$

and for all $T > 0$

$$\left\langle u_t(T), v^0_T \right\rangle_{H^{-1}_{\frac{1}{\sigma}}(0,1), H^1_{\frac{1}{\sigma}}(0,1)} - \int_0^1 \frac{1}{\sigma} u(T) v^1_T dx = \left\langle u_1, v(0) \right\rangle_{H^{-1}_{\frac{1}{\sigma}}(0,1), H^1_{\frac{1}{\sigma}}(0,1)}$$

$$- \int_0^1 \frac{1}{\sigma} u_0 v_t(0, x) dx + \eta(1) \int_0^T f(t) v_x(t, 1) dt \tag{4.5}$$

for all $(v_T^0, v_T^1) \in H_{\frac{1}{\sigma}}^1(0, 1) \times L_{\frac{1}{\sigma}}^2(0, 1)$, where v solves the backward problem

$$\begin{cases} v_{tt} - av_{xx} - \frac{\lambda}{d}v - bv_x = 0, & (t, x) \in (0, +\infty) \times (0, 1), \\ v(t, 1) = v(t, 0) = 0, & t \in (0, +\infty), \\ v(T, x) = v_T^0(x), & x \in (0, 1), \\ v_t(T, x) = v_T^1(x), & x \in (0, 1). \end{cases} \quad (4.6)$$

Observe that, by Theorem 2.3, there exists a unique mild solution of (4.6) in $[T, +\infty)$. Now, setting $y(t, x) := v(T - t, x)$, one has that y satisfies (2.1) with $y_T^0(x) = v_T^0(x)$ and $y_T^1(x) = -v_T^1(x)$. Hence, thanks to Theorem 2.3 one has that there exists a unique mild solution y of (2.1) in $[0, +\infty)$. In particular, there exists a unique mold solution v of (4.6) in $[0, T]$. Thus, we can conclude that there exists a unique mild solution

$$v \in C^1\left([0, +\infty); L_{\frac{1}{\sigma}}^2(0, 1)\right) \cap C\left([0, +\infty); H_{\frac{1}{\sigma}}^1(0, 1)\right)$$

of (4.6) in $[0, +\infty)$ which depends continuously on the initial data $V_T := (v_T^0, v_T^1) \in \mathcal{H}_0$.

By Theorem 3.1, the energy is preserved in our setting, as well, so that the method of transposition done in [1] continues to hold thanks to (3.4), and so there exists a unique solution by transposition $u \in C^1\left([0, +\infty); H_{\frac{1}{\sigma}}^{-1}(0, 1)\right) \cap C\left([0, +\infty); L_{\frac{1}{\sigma}}^2(0, 1)\right)$ of (1.1), namely a solution of (4.5).

Now, we are ready to pass to null controllability, recalling that by linearity and reversibility of equation (1.1), null controllability for any initial data (u_0, u_1) is equivalent to exact controllability, see [1]. In order to prove that (1.1) is null controllable, let us start with

Hypothesis 7 Assume

- Hypothesis 5 with $-\frac{2 - K_a - 2M}{2C_{HP}(2 + K_a + K_d)} < \lambda \leq 0$ and (4.2), or
- Hypothesis 5 with $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$ and $K_a + K_d \leq 2 - M$ and (4.3), or
- Hypothesis 6 with $-\frac{2 - K_a - 2M_\infty}{2C_{HP}(2 + K_a + K_d)} < \lambda \leq 0$ and (4.2), or
- Hypothesis 6 with $\lambda \in \left(0, \frac{1}{C_{HP}}\right)$ and $K_a + K_d \leq 2 - M_\infty$ and (4.3).

Moreover, we recall the Hilbert space introduced in (2.17)

$$\mathcal{H}_0 := H_{\frac{1}{\sigma}}^1(0, 1) \times L_{\frac{1}{\sigma}}^2(0, 1),$$

endowed with the inner product

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_{\mathcal{H}_0} := \int_0^1 \eta u' \tilde{u}' dx + \int_0^1 v\tilde{v} \frac{1}{\sigma} dx - \lambda \int_0^1 \frac{u\tilde{u}}{\sigma d} dx$$

for every $(u, v), (\tilde{u}, \tilde{v}) \in \mathcal{H}_0$, which induces the equivalent norm

$$\|(u, v)\|_{\mathcal{H}_0}^2 := \int_0^1 \eta(u')^2 dx + \int_0^1 v^2 \frac{1}{\sigma} dx - \lambda \int_0^1 \frac{u^2}{\sigma d} dx.$$

On the product space $\mathcal{H}_0 \times \mathcal{H}_0$, consider the bilinear form Λ defined as

$$\Lambda (V_T, W_T) := \eta(1) \int_0^T v_x(t, 1) w_x(t, 1) dt$$

where v and w are the solutions of (4.6) associated to the final data $V_T := \left(v_T^0, v_T^1\right)$ and $W_T := \left(w_T^0, w_T^1\right)$, respectively. The bilinear form Λ is continuous and coercive (see Lemma 5.1) and thanks to these properties one can prove the null controllability for the original problem (1.1). In particular, defining T_0 as the lower bound found in Corollaries 4.1 and 4.2, which changes according to the different assumptions used therein, one can prove the next result, whose proof is postponed in the Appendix.

Theorem 4.1 *Assume Hypothesis 7. Then, for all $T > T_0$ and for every (u_0, u_1) in $L_{\frac{1}{\sigma}}^2 (0, 1) \times H_{\frac{1}{\sigma}}^{-1}(0, 1)$ there exists a control $f \in L^2(0, T)$ such that the solution of (1.1) satisfies (4.4).*

5 Appendix

For the readers' convenience, in this section we prove some results used throughout the previous sections.

5.1 Proof of Lemma 3.1

1. We will actually prove an equivalent fact, usually appearing in integrations by parts, namely that

$$\lim_{x \to 0} \eta(x) u(x) y'(x) = 0.$$

For this, consider the function

$$z(x) := \eta(x) u(x) y'(x), \quad x \in (0, 1].$$

As in [8] or in [26] one can prove that $z \in W^{1,1}(0, 1)$. Indeed

$$\int_0^1 |z| dx \leq C \|u\|_{L^2(0,1)} \|y'\|_{L^2(0,1)},$$

thus $z \in L^1(0, 1)$. Moreover

$$z'(x) = (\eta y')' u + \eta y' u'$$

and

$$\int_0^1 \eta |y' u'| dx \le C \|y'\|_{L^2(0,1)} \|u'\|_{L^2(0,1)}.$$

It remains to prove that $(\eta y')' u \in L^1(0, 1)$. To this aim observe that

$$(\eta y')' u dx = \sigma (\eta y')' \frac{u}{\sigma} dx.$$

Thus, by Hölder's inequality and Proposition 2.1,

$$\int_0^1 |(\eta y')' u| dx \le \|Ay\|_{\frac{1}{\sigma}} \|u\|_{\frac{1}{\sigma}},$$

for a positive constant C. Thus z' is summable on $[0, 1]$. Hence $z \in W^{1,1}(0, 1) \hookrightarrow C[0, 1]$ and there exists

$$\lim_{x \to 0} z(x) = \lim_{x \to 0} \eta(x) u(x) y'(x) = L \in \mathbb{R}.$$

We will prove that $L = 0$. If $L \ne 0$ there would exist a neighborhood $\mathcal{I}$ of 0 such that

$$\frac{|L|}{2} \le |\eta y' u|,$$

for all $x \in \mathcal{I}$; but, by Hölder's inequality,

$$|u(x)| \le \int_0^x |u'(t)| dt \le \sqrt{x} \|u'\|_{L^2(0,1)}.$$

Hence

$$\frac{|L|}{2} \le |\eta y' u| \le \|\eta\|_\infty |y'| \sqrt{x} \|u'\|_{L^2(0,1)}$$

for all $x \in \mathcal{I}$. This would imply that

$$|y'| \ge \frac{|L|}{2 \|u'\|_{L^2(0,1)} \|\eta\|_\infty \sqrt{x}}$$

in contrast to the fact that $y' \in L^2(0, 1)$.

Hence $L = 0$ and the conclusion follows.

2. Using the fact that $u(0) = 0$, we have

$$|u(x)| \leq \int_0^x |u'(t)|dt \leq \sqrt{x}\|u'\|_{L^2(0,1)};$$

thus

$$\frac{x}{a(x)}u^2(x) \leq \frac{x^2}{a(x)}\|u'\|^2_{L^2(0,1)}.$$

By (2.4), it follows the thesis.

3. As in the previous point, we know that

$$|u(x)| \leq \sqrt{x}\|u'\|_{L^2(0,1)};$$

thus

$$\frac{x}{a(x)d(x)}u^2(x) \leq \frac{x^2}{a(x)d(x)}\|u'\|^2_{L^2(0,1)}.$$

By assumption we have that

$$\frac{1}{a(x)} \leq \frac{1}{a(1)x^{K_1}} \quad \text{and} \quad \frac{1}{d(x)} \leq \frac{1}{d(1)x^{K_2}}.$$

Hence

$$\frac{x}{a(x)d(x)}u^2(x) \leq \frac{1}{a(1)d(1)}\frac{x^2}{x^{K_1+K_2}}\|u'\|^2_{L^2(0,1)}.$$

Using the fact that $K_1 + K_2 < 2$, the claim follows immediately.

4. If $K_1 + K_2 < 2$, then we can deduce the thesis by the previous point. If $K_1 + K_2 = 2$, then proceeding as before, one has

$$\frac{x^2}{a(x)d(x)}u^2(x) \leq \frac{1}{a(1)d(1)}\frac{x^2}{x^{K_1+K_2}}u^2(x) = \frac{1}{a(1)d(1)}u^2(x)$$

and the thesis follows immediately.

5. It is a straightforward consequence of the first point. Indeed, it is enough to choose any function $u \in H^1_{\frac{1}{\sigma}}(0, 1)$ such that $u(x) = x$ in a right neighborhood of 0, and squaring.

6. Consider the function $z := x\eta(x)(u'(x))^2$. We have

$$z' = \eta(u')^2 + x\eta'(u')^2 + 2x\eta u'u'' = \eta(u')^2 + 2xu'(\eta u')' - x\eta'(u')^2.$$

Since $K_a \le 1$, one has that $x\eta'(u')^2 \in L^1(0, 1)$. Moreover, $2xu'(\eta u')'$ belong to $L^1(0, 1)$; indeed, as before,

$$x(\eta u')'u'dx = x\sigma(\eta u')' \frac{u'}{\sigma}dx = Au\frac{xu'}{\sigma}.$$

Clearly, $Au\dfrac{xu'}{\sigma} \in L^1(0, 1)$; hence $z \in W^{1,1}(0, 1)$ and $\lim_{x\to 0} z(x) = 0$.

7. We can proceed as above, observing that

$$\left| x\eta'(u')^2 \right| = \left| x\eta\frac{b}{a}(u')^2 \right| \le \left\| \frac{xb}{a} \right\|_{L^\infty(0,1)} \|\eta\|_{L^\infty(0,1)} (u')^2.$$

5.2 Proof of Theorem 4.1

The proof of Theorem 4.1 is based on the following properties of Λ.

Lemma 5.1 *Assume Hypothesis 7. Then, the bilinear form Λ is continuous and coercive.*

Proof By Theorem 3.1, E_v and E_w are constant in time and thanks to (3.4), one has that Λ is continuous. Indeed, by Hölder's inequality and Corollary 2.1,

$$
\begin{aligned}
|\Lambda(V_T, W_T)| &\le \eta(1) \int_0^T |v_x(t, 1)w_x(t, 1)| \, dt \\
&\le \left(\eta(1) \int_0^T v_x^2(t, 1)dt \right)^{\frac{1}{2}} \left(\eta(1) \int_0^T w_x^2(t, 1)dt \right)^{\frac{1}{2}} \\
&\le C E_v^{\frac{1}{2}}(T) E_w^{\frac{1}{2}}(T) \\
&\le C \left(\int_0^1 \frac{\left(v_T^1\right)^2(x)}{\sigma}dx + \int_0^1 \eta v_x^2(T, x)dx - \int_0^1 \frac{\lambda}{d}v^2(T, x)dx \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^1 \frac{\left(w_T^1\right)^2(x)}{\sigma}dx + \int_0^1 \eta w_x^2(T, x)dx - \int_0^1 \frac{\lambda}{d}w^2(T, x)dx \right)^{\frac{1}{2}} \\
&= C \left\| (v(T), v_t(T)) \right\|_{\mathcal{H}_0} \left\| (w(T), w_t(T)) \right\|_{\mathcal{H}_0} = C \left\| V_T \right\|_{\mathcal{H}_0} \left\| W_T \right\|_{\mathcal{H}_0},
\end{aligned}
$$

for a positive constant C independent of $(V_T, W_T) \in \mathcal{H}_0 \times \mathcal{H}_0$.

In a similar way, one can prove that Λ is coercive. Indeed, by Theorem 3.4, for all $V_T \in \mathcal{H}_0$, one immediately has

$$\Lambda(V_T, V_T) = \int_0^T \eta(1)v_x^2(t, 1)dt \ge C_T E_v(0) = C_T E_v(T) \ge C \left\| V_T \right\|_{\mathcal{H}_0},$$

for a positive constant C. $\qquad\square$

Proof of Theorem 4.1 Consider the continuous linear map $\mathcal{L} : \mathcal{H}_0 \to \mathbb{R}$ defined as

$$\mathcal{L}(V_T) := \int_0^1 \frac{u_0 v_t(0, x)}{\sigma} dx - \langle u_1, v(0) \rangle_{H_{\frac{1}{\sigma}}^{-1}(0,1), H_{\frac{1}{\sigma}}^1(0,1)},$$

where v is the solution of (4.6) associated to the final data $V_T := (v_T^0, v_T^1) \in \mathcal{H}_0$. Thanks to Lemma 5.1 and by the Lax-Milgram Theorem, there exists a unique $\bar{V}_T \in \mathcal{H}_0$ such that

$$\Lambda(\bar{V}_T, W_T) = \mathcal{L}(W_T) \quad \text{for all } W_T \in \mathcal{H}_0. \tag{5.1}$$

Set $f(t) := \bar{v}_x(t, 1)$, v being the solution of (4.6) with initial data $\bar{V}_T$. Then, by (5.1)

$$\eta(1) \int_0^T f(t) w_x(t, 1) dt = \eta(1) \int_0^T v_x(t, 1) w_x(t, 1) dt = \Lambda(V_T, W_T) = \mathcal{L}(W_T)$$

$$= -\langle u_1, w(0) \rangle_{H_{\frac{1}{\sigma}}^{-1}(0,1), H_{\frac{1}{\sigma}}^1(0,1)} + \int_0^1 \frac{1}{\sigma} u_0 w_t(0, x) dx, \tag{5.2}$$

for all $W_T \in \mathcal{H}_0$. Finally, denote by u the solution by transposition of (1.1)

$$\eta(1) \int_0^T f(t) w_x(t, 1) dt = \left\langle u_t(T), w_T^0 \right\rangle_{H_{\frac{1}{\sigma}}^{-1}(0,1), H_{\frac{1}{\sigma}}^1(0,1)} - \int_0^1 \frac{1}{\sigma} u(T) w_T^1 dx$$

$$- \langle u_1, w(0) \rangle_{H_{\frac{1}{\sigma}}^{-1}(0,1), H_{\frac{1}{\sigma}}^1(0,1)} + \int_0^1 \frac{1}{\sigma} u_0 w_t(0, x) dx. \tag{5.3}$$

By (5.2) and (5.3), it follows that

$$\left\langle u_t(T), w_T^0 \right\rangle_{H_{\frac{1}{\sigma}}^{-1}(0,1), H_{\frac{1}{\sigma}}^1(0,1)} - \int_0^1 \frac{1}{\sigma} u(T) w_T^1 dx = 0,$$

for all $(w_T^0, w_T^1) \in \mathcal{H}_0$. Hence, we have

$$u(T, x) = u_t(T, x) = 0 \quad \text{for all } x \in (0, 1).$$

$\square$

Declarations

Conflict of interest The authors declare no Conflict of interest.

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