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Review

Exploring unresolved inquiries regarding the meaning of Reynolds averaging and decomposition: A review

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ABSTRACT

In the late 19th century, Osborne Reynolds published two papers whose impact on atmospheric turbulence studies can hardly be overstated. The first, Reynolds (1883) established both his eponymous, dimensionless number and his reputation as the father of turbulence science, which is beyond doubt. However, his second famous paper (Reynolds, 1895) sowed seeds of confusion regarding the mathematical separation of average (mean) and fluctuating (turbulent) components of a fluid flow. Here, we revisit both the prehistory and after-effects of Reynolds's second famous article, which seems to have been published largely thanks to his already entrenched reputation.

We show that successions of authors have misrepresented Reynolds's innovations – now known as Reynolds averaging and decomposition (RAAD) –, putting his name to methodologies that he never intended. We attribute this, in part, to Reynolds's predilection for long, inscrutable sentences, as well as his self-contradiction regarding the methodology for averaging the normal stress (or pressure). We examine two additional issues that are intimately related to using RAAD to define turbulent fluxes, namely its application to intensive versus extensive variables and the appearance of "Leonard terms" in the averaged equation of motion, neither of which is completely resolved. Throughout the manuscript, we identify a set of unanswered questions concerning RAAD and conclude that a complete mathematical description of turbulence is unlikely to emerge without addressing these issues, including the original inconsistency that was introduced by Osborne Reynolds himself.

1. Challenges posed by turbulence and Reynolds averaging

In *Hydrodynamics* – perhaps the most highly revered of fluid dynamics textbooks – Lamb (1906) called turbulence "the chief outstanding difficulty of our subject". Many decades later, Nobel Laureate Richard Feynman identified turbulence as the most important unsolved problem of classical physics (Moin and Kim, 1997). A consensual mathematical specification of turbulence remains elusive to this day, and this is true in particular regarding turbulent mass transport (of carbon dioxide, for example; Kowalski et al., 2021), whose relevance reaches far.

Since the 1980s, assessments of turbulent flux densities (or "fluxes") of carbon dioxide and other atmospheric constituents have expanded dramatically around the world. Turbulent flux measurements quantify greenhouse gas exchanges at the useful ecosystem scale, aligning with the requirements identified by the Kyoto and Paris climate agreements, and making them highly desirable. Once limited to few pioneering experiments by micrometeorologists with expertise in fluid dynamics, publications based on "flux towers" now exceed one hundred per year (Baldocchi et al., 2024), many by interdisciplinary scientists with little experience in micrometeorology. Hand in hand with the proliferation of flux-tower research, there has emerged an imperative to standardize methodologies across networks and continents (Sabbatini et al., 2018). Meanwhile, the much-cited failure to close the surface energy balance (Wilson et al., 2002), and thereby demonstrate conservation of energy, casts doubt (Twine et al., 2000) on the very methods that are being standardised across networks of scientists who apply the increasingly common eddy covariance technique.

Fundamental to a mathematical description of turbulence is the methodology of Reynolds averaging and decomposition (RAAD) – introduced by Reynolds (1895) – that is essential to the derivation of turbulent transport in the atmospheric boundary layer. This methodology allows defining the Reynolds stress and the friction velocity, and so has outstanding relevance for "boundary-layer meteorology". For

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example, in the October 2022 issue of the journal of that name, every paper published mentioned either Reynolds averaging or the Reynolds stress.

Here we show that, despite the widespread use and familiarity of RAAD, its meaning is still quite unclear. In fact, most authors define averaging in a manner that is inconsistent with the definitions of Reynolds (1895), and many credit Reynolds while employing methods that are at odds with what he published, while others reject Reynolds's methods as somehow incompatible with mainstream micrometeorology (e.g., Foken, 2017). We present a condensed history of RAAD, starting before its acceptance for publication, and including a review of literature extending well over a century afterward. Furthermore, we identify in Reynolds's own specifications both ambiguity regarding the RAAD of intensive-versus-extensive variables and inconsistency regarding the appropriate technique for averaging certain variables, with consequences that relate to the so-called "Leonard terms" in the averaged equation of motion. Recognizing these issues helps to explain the history of RAAD in the literature, and resolving them may greatly improve our understanding of this key problem of classical physics.

2. Peer review of Reynolds (1895)

The bulk of the historical information in this section is based upon the research of Jackson and Launder (2007) and Launder (2014) following the Royal Society's release of previously confidential documents.

The Reynolds (1895) paper was published after a review process involving four scientists, all among the who's who of 19th century physicists. The author, obviously, was Osborne Reynolds. The editor, Lord Rayleigh, went on to win the Nobel Prize for Physics. The two reviewers he consulted, no longer anonymous, are now known as Sir George Stokes – of Stokes's law and the Navier-Stokes equations – and Sir Horace Lamb, author of the celebrated textbook *Hydrodynamics*. (Names and titles are hereinafter abbreviated with respect.) The objectivity of the review process might reasonably be questioned, since Lamb was Reynolds's colleague – the other senior fluid dynamics expert at Owens College, Manchester – while Rayleigh and Stokes had acted as reviewers for the already famous Reynolds (1883) paper that had established both Reynolds's reputation and what we now call the Reynolds number.

The review process included irregularities beyond Rayleigh's selection as a reviewer of Reynolds's contemporary at Owens College. The first reviewer, Stokes, neither understood the paper nor appreciated its importance, as he readily admitted. He neither endorsed nor rejected its claims, but concluded that if the deservedly reputed Reynolds felt it was important, then perhaps it should be published once its meaning was clarified. The second reviewer, Lamb was similarly conflicted; he recommended publication yet acknowledged that "much of it is obscure and there are some fundamental points which are not clearly established." Dissatisfied with the reviewers' vague ambivalence, Rayleigh persuaded Stokes to make a franker appraisal, which turned out to be even less favourable ("the paper is very obscure. In its present state it would hardly be understood"). Ultimately, it seems the editor coerced the two reviewers to write a collaborative assessment of the manuscript.

The key results of this coupled review, regarding a manuscript of which Stokes and Lamb were clearly quite weary, were first that they "found great difficulty in following the arguments of this paper", and second that they suggested shortening the introduction. In revision, beyond some minor changes, Reynolds *expanded* the introduction by a diffuse, four-page insertion (on February 18th, 1895) intended to clarify the meaning of what we now call RAAD. Although the editor ultimately accepted the paper, the subsequent history summarised below strongly suggests both that this attempted clarification failed calamitously and that Stokes's prediction regarding reading comprehension was prophetic.

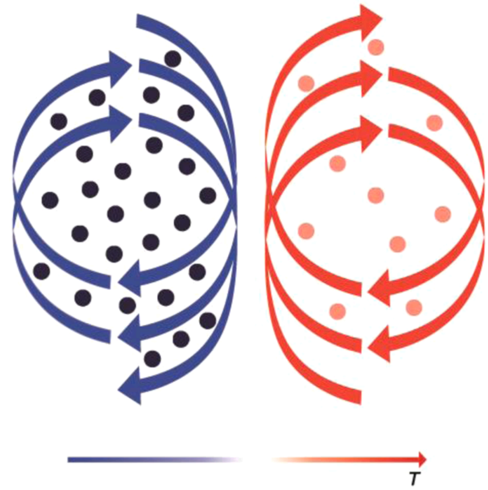

Fig. 1. Depiction of two eddies of equal size, with a downdraft (cold, blue) containing more air molecules than an updraft (warm, red). Illustration by Esther Cardell.

3. Uncertainties regarding the meaning of Reynolds (1895)

Our examination of decades of publications that confused the issue of RAAD focuses on three particular aspects of the methodology. The first and foremost dominates this section, including Subsections 3.1 through 3.4, and concerns how an average is defined mathematically. This may seem obvious: simply sum *N* observations and divide by *N*. However, while this is what many subsequent researchers employed and described as RAAD, such "arithmetic averaging" is neither universally appropriate nor what Reynolds intended regarding velocities. The second aspect is the issue of which variables – or which type of variable – to decompose; in Subsection 3.5 we distinguish in particular between variables of extensive versus intensive natures. Finally, the third aspect addressed in Subsection 3.6 relates to the so-called "Leonard terms" (Galmarini et al. 2000) that arise in the equation of motion whenever non-arithmetic averaging procedures are employed.

3.1. Defining averaging mathematically

Reynolds (1895) persistently defined the average velocity via the momentum-to-mass ratio, using an expression equivalent to

$$\widetilde{u} = \frac{\frac{1}{N}\sum_{i=1}^{N}\rho_i u_i}{\frac{1}{N}\sum_{i=1}^{N}\rho_i}, \tag{1}$$

as in his Eqs. 4 and (12), and also an un-numbered equation between (8a) and (8b). In this expression, u is the velocity component in the x-direction and ρ the fluid density. The numerator and denominator each represent the arithmetic averages of the density of a function; the numerator is the average momentum density, and the denominator the average mass density (or simply, density). Analogous equations for the y- and z-directions define $\widetilde{v}$ and $\widetilde{w}$. Eq. (1) equivalently specifies $\widetilde{u}$ as the density-weighted average of u (Kowalski, 2012; beware the change in notation: Reynolds denoted this density-weighted average as $\overline{u}$, but since modern scientists thusly specify arithmetic averages, the notation has been changed to avoid confusion.)

The impact of such weighting is significant only where ρ varies, as in the atmosphere but not in the incompressible case that Reynolds (1895) examined. Nevertheless, his calculations accounted for such "density effects". In meteorology, comparing warm and cold eddies of equal size (volume) and at equal pressure, denser cold eddies contain more air and therefore are more heavily weighted when calculating statistical moments of the velocity, including the average (Fig. 1). Reynolds's rationale for incorporating such weighting when treating the incompressible case remains unclear, but likely simply respects the Newtonian definition of velocity as the ratio of the conserved quantities momentum and mass. Nonetheless, very few atmospheric researchers seem to have taken notice of this definition, perhaps due to Reynolds's cumbersome writing style (see Appendix A).

Influential practitioners of fluid dynamics that succeeded Reynolds (1895), some of whom cited his work, directly defined the average velocity using arithmetic averaging as

$$\overline{u} = \frac{1}{\mathrm{N}} \sum_{i=1}^{\mathrm{N}} u_i, \tag{2}$$

or an integral version thereof with infinitesimal summation (e.g., Taylor, 1915; Bernstein, 1966). Reynolds (1895) never wrote any such equation. Perhaps first among the prominent scientists who mis-interpreted Reynolds (1895) was Lamb, whose *Hydrodynamics* had a tremendous influence on studies of fluid dynamics during decades. This classic textbook ran to six editions and is still in print today. In the third edition, published while Reynolds still lived although in retirement, Lamb (1906) introduced what we now call RAAD, describing Reynolds (1895) as a "remarkable paper" but mischaracterising the mean values of velocities according to Eq. (2), rather than Eq. (1) as Reynolds had insisted.

Another prominent early influence on atmospheric turbulence studies was Taylor (1915), who averaged vertical velocities over a horizontal area of atmosphere (see the last double-integral on page 3), also without the density weighting that momentum conservation demands. Taylor cited not Reynolds but Lamb, although not specifically in the context of averaging. Regarding averaging, a great number of successive scientists applied arithmetic averaging to velocities – many citing Reynolds despite not heeding his equations – a fact that might be ascribed to Reynolds indecipherable writing style; compare his "long rambling sentences" with the "crisply stated" style of Lamb (Jackson and Launder, 2007). We surmise that more atmospheric scientists read about "Reynolds averaging" via descriptions by Lamb – or his disciples – than struggled through the Reynolds (1895) paper.

3.2. The legacy of Reynolds (1895)

Whether or not this guess is correct, the fact is that many 20th and 21st century authors of turbulence textbooks describe RAAD ignoring the density-weighted averaging procedure that Reynolds repeatedly defined. Many specify RAAD as arithmetic (Sutton, 1953; Tennekes and Lumley, 1972; Stull, 1988; Arya, 2004; Wyngaard, 2010; Ciofalo, 2022), or even eschew density weighting as inconsistent with mainstream theory (Foken, 2017) or inconvenient because ρ fluctuations are not readily mensurable (Doolan and Moreau, 2022). Sometimes the method of averaging is not defined explicitly, but arithmetic averaging can be inferred mathematically from the treatment of RAAD (e.g., Holton, 2004).

Many research publications treated the Reynolds (1895) paper similarly, although the more common practice seems to have been to cite more recent papers rather than crediting the original. Various authors (Priestley and Sheppard, 1952; Cramer and Record, 1953; Charnock, 1957) attributed arithmetic averaging of velocities to Reynolds (1895), including Kampé de Fériet (1951) who called it "naïve". Likewise, 21st century researchers continue to justify arithmetic averaging either citing Reynolds (1895) explicitly (Finnigan and Shaw, 2008; Treviño and Andreas, 2008) or invoking "Reynolds averaging" (Moncrieff et al., 2004). Elsewhere, citation chains can be identified that stretch back decades, employing arithmetic averaging without necessarily citing Reynolds (1895). This includes, for example, the current methodology for the Integrated Carbon Observation System (ICOS; Sabbatini et al., 2018 → Lee et al., 2004 → Webb et al., 1980→ Calder, 1949 → Priestley and Swinbank, 1947→ Taylor, 1915), a pan-European protocol whose influence is practically global. A minority of scientists who did use density weighting when defining eddy transport did not credit Reynolds (e.g., Montgomery, 1948; Swinbank, 1951), possibly due to the already entrenched misinterpretation of Reynolds' original work. Some authors even explicitly contrast "Reynolds averaging" – which they define as arithmetic – with density-weighted averaging for which they credit others like Hesselberg (Herbert 1995; Kramm et al., 1995; Foken, 2017), or Favre (Zhang et al. 2022; Klein et al. 2022; Doolan and Moreau, 2022). All of this bibliographical history suggests that the meaning of RAAD as expressed in Reynolds (1895) has not been broadly, or even narrowly in many cases, understood.

3.3. How did this situation arise?

How were so many scientists able to mis-represent Reynolds's seminal ideas regarding RAAD? As exemplified in Appendix A, Reynolds (1895) is challenging to read and understand. The work of Jackson and Lauder (2007) shows that even his eminent contemporaries Stokes and Lamb, the latter labouring within walking distance of Reynolds, struggled to digest the paper. Apparently, Reynolds became aware of this situation and attempted to clarify it by publishing a compendium of papers (Reynolds, 1903). This also seems to have failed. For example, Brunt (1934) specifically cites p51 of Reynolds (1903) – where Reynolds allows for arithmetic averaging to define the mean of "the density of any function" (such as ρ or ρu) – and almost immediately applies arithmetic averaging to velocities, in his Eq. (4). However, Reynolds never applied arithmetic averaging to a velocity, as it is not the density of any function. It seems unthinkable that such disregard for Reynolds's methodologies could be intentional, and far more likely that access to Reynolds's writings for Brunt and others was limited or indirect – perhaps influenced by Lamb or some other author – obstructing appreciation of Reynolds's meaning. (We have obtained electronic copies of every paper cited herein, and will be happy to share them with requesters.)

3.4. What Reynolds actually meant remains unclear

Rigorous examination of Reynolds's equations makes quite clear that he made a distinction between variables whose means could be determined by arithmetic averaging, versus those for which density weighting is necessary. Reynolds used his overbar notation ("barred symbols") exclusively for the latter. Thus, in Reynolds (1903) we find that

a. Equation (78) defines the notation for including density weighting, with barred symbols, when determining displacements (whose time derivatives yield velocities), consistent with Reynolds (1895);
b. Equation (92) uses neither density weighting nor barred symbols when allowing simple arithmetic averaging to define the mean of a "density function"; and
c. Equation (94) lists the densities of conserved, extensive quantities (mass, momentum, and energy) as those to which it applies, requiring neither density-weighted averaging nor barred symbols.

Regarding the third point, we can highlight an inherent inconsistency of RAAD introduced by Reynolds (1895; 1903). We note that "energy density" (J m^{-3}) is another way of saying pressure (Pa), as the units of the Ideal Gas Law ($p = nRT/V$) make clear. Since pressures sum linearly, as attested by Dalton's law, there is no need for weighting factors when summing or averaging pressures (Kowalski, 2012). Nevertheless, Reynolds (1895) put a bar over the pressure (p) in his Equation (14) – i.e., he averaged it with density weighting – and called this the mean value of the stress (or its normal component). Although this arose from applying the same mathematical operator to every term in the equation of motion, there is no justification for interpreting this as the mean of the variable p. Such an interpretation is inconsistent with Equations (92) and (94) of Reynolds (1903). Thus, quite apart from how subsequent researchers may have interpreted Reynolds's derivation, it seems possible that even Osborne Reynolds did not fully appreciate the meaning of RAAD. This is especially true concerning the average pressure whose gradient is the driving force for most fluid motions.

To summarise the mathematical definitions of an "average" in Reynolds (1903), some variables (including velocities) have means defined via density weighting as in Equation (78) while others (like density and pressure) require no density weighting as in Equation (92). The difference between averages computed arithmetically versus using density weighting are hardly trivial for certain variables, as has been shown for the vertical velocity (Kowalski et al., 2021). Since the equation of motion contains variables of each type, the viability of "averaging" the entire equation – a key step in RAAD – seems anything but certain and Reynolds's own publications raise a fundamental question: how should each term of the equation be averaged?

3.5. Which (type of) variables to decompose?

Since Reynolds (1895) focused exclusively on the constant-density case – as his very title indicates – it is a curious fact that he insisted upon density weighting when averaging velocities. There are no consequences when applying a non-varying weighting factor. Therefore, one wonders why he bothered to do so, unless it was simply that he wished to express explicit conformity with Newton's laws in general, and perhaps momentum conservation in particular. In contrast, however, he did not include ρ among the flow and state variables to which he applied RAAD. This means that Reynolds made no specific pronouncement concerning a subtle distinction that varies among atmospheric researchers regarding which type of variables to decompose, whether intensive or extensive.

Let us recall that the difference between these two types involves whether a variable depends or not on the amount of fluid present. This is most easily appreciated via examples. A system composed of elements A and B has a mass ($m = m_A + m_B$) that is the sum of the elements' masses, because mass is extensive and conserved. The same is true about momentum and energy, but not about intensive variables. For example, if two airmasses, each with a mass fraction (f) for oxygen of 230 g kg^{-1}, are combined, the resulting aggregate also has 230 g kg^{-1} because f is intensive (and because mass is conserved). The same is true regarding intensive velocity and temperature (with momentum and energy conserved). If this seems simple, we note that disagreement exists regarding the nature of density, which some textbooks claim is intensive (e.g., Lewis and Randall, 1961) but this does not extend to the compressible case for an Eulerian reference frame. Considering a rigid gas bottle with high pressure content, if a valve is opened and half of the content escapes, then the density will be reduced by half. Thus, gas density is not universally intensive. All of this has relevance because, in the history of RAAD, there has been divergence among researchers regarding the decomposition of intensive versus extensive state and flow variables.

Such divergence can be seen in the history of publications by individual scientists. The textbook by Kaimal and Finnigan (2008) defined turbulent fluxes in terms of covariances between intensive variables

$$\tau = -\rho\overline{u'w'}, \tag{3}$$

$$H = \rho c_p \overline{w'\theta'}, \text{ and} \tag{4}$$

$$E = \rho\overline{w'q'}\,. \tag{5}$$

In these expressions, τ is the momentum flux, u and w the streamwise and vertical velocities, H the heat flux, c_p the specific heat at constant pressure, θ the potential temperature, E the water vapour flux, and q the mass fraction of water vapour, or specific humidity. Following modern tradition, over-bars denote arithmetic averages and primes indicate deviations therefrom. Consistent with the above conclusion, the extensive variable ρ multiplies "kinematic fluxes" to yield the flux. The authors called these intensive-variable covariance terms "unambiguous measures of the fluxes". By contrast, one of these authors employed and decomposed scalars defined as "any absolute fluid property such as density of water vapour, ρ_w or heat content, $\rho\, c_p\, \theta$ where θ is temperature". Such application of RAAD to extensive variables appeared in publications both prior to (Finnigan et al., 2003) and after (Finnigan, 2009) the textbook from which Eqs. (3-5) are taken. Experience over decades (Lee and Massman, 2011) in applying "density corrections" (Webb et al., 1980) to turbulent mass fluxes has demonstrated that such distinctions can matter greatly in the atmospheric boundary layer.

Similar divergence can be found throughout the history of micrometeorology, and not only concerning state variables. Although most micrometeorology textbooks coincide with Kaimal and Finnigan (2008) in defining kinematic fluxes via covariances between intensive variables (Sutton, 1953; Stull; 1988; Arya, 2004; Foken, 2017), this is not universal. For example, Oke (1987) is inconsistent in this regard, using the intensive mass fraction in Equation (A2.9) but the extensive density in Equation (A2.10d) on the next page. Even regarding flow variables, RAAD is usually applied to velocities (intensive), as done by Priestley and Swinbank (1947). However, Priestley (1959) defined vertical eddy fluxes via decomposed momentum (extensive) and "some property whose measure per unit mass of air is s" (intensive), as

$$\overline{\rho w s} = \overline{\rho w}\,\bar{s} + \overline{(\rho w)'\,s'}. \tag{6}$$

Swinbank (1951) likewise defined the heat flux via decomposed momentum (extensive) and temperature (intensive).

The preceding highlights uncertainty regarding the question: to which variables should RAAD be applied when defining turbulent transport in the atmosphere?

3.6. "Leonard terms"

It is common to find reference in the literature to Reynolds's rules of averaging that must be obeyed or satisfied (Rannik and Vesala, 1999; Galmarini et al., 2000; Mahrt et al. 2001; Finnigan, 2004; Moncrieff et al. 2004; Pekour et al. 2006; Charuchittipan et al., 2014) in order to enable the expression of a flux as an eddy covariance. Although Reynolds neither enunciated nor obeyed such rules, they are often treated as inviolable and, for our purposes, their consequences may be described as follows. Applying arithmetic averaging to the kinematic flux yields an expression such as

$$\overline{ws} = \overline{w}\,\bar{s} + \overline{\overline{w}s'} + \overline{w'\overline{s}} + \overline{w's'}\,, \tag{7}$$

which simplifies to the final, covariance term based on two key assumptions. First, $\overline{w}$ is assumed to be zero based on the idea that winds flow parallel to – but neither into nor out of – the surface. This is often artificially forced via coordinate rotation schemes (McMillen, 1988; Wilczak et al. 2001) to account for tilt of either landscape or anemometer. Second the so-called Leonard terms $\overline{\overline{w}s'}$ and $\overline{w'\,\overline{s}}$ (Galmarini et al. 2000) are exactly zero if the over-bar denotes arithmetic averaging. However, if the over-bar represents an average determined by detrending or some other type of filtering process, the Leonard terms would not be equal to zero (Mahrt et al., 2001; Pekour et al., 2006). Practitioners of detrending and filtering – as alternative methods of defining averages – recognize non-zero or sizeable Leonard terms as a disadvantage of such flow decomposition to specify turbulence (Moncrieff et al., 2004; Finnigan, 2004).

Since the averaging defined by Reynolds (1895) was not arithmetic, and the Reynolds (1903) publication did little to clarify the original author's intentions, it seems that "Reynolds rules" can be interpreted in different ways. Thus, how should Leonard terms be handled under the density-weighted averages defined by Reynolds (1895)?

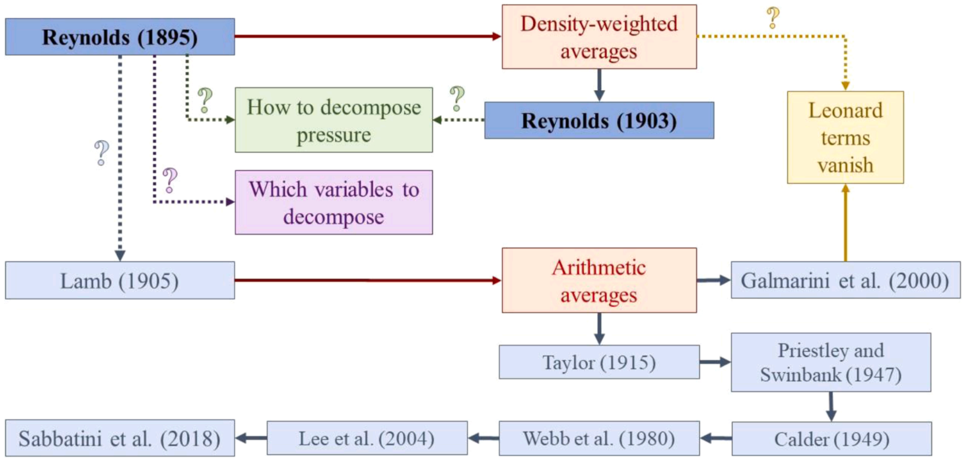

Fig. 2. Flow diagram depicting unresolved inquiries regarding the meaning of Reynolds averaging and decomposition, including associated bibliographical references. Dotted lines with question marks denote uncertain connections. Notably, one path of dubious origin leads to the official, pan-European protocol for processing of eddy covariance data.

4. Practical considerations

Advances in theory and practice generally go hand in hand. Ideally, theory refines practice and practice tests theory, although circumstances are rarely ideal. In the days of Priestley (1959) and Swinbank (1951) fast-response anemometry consisted of hot wires quantifying extensive momentum (not intensive velocity), and this may have influenced their implementation of RAAD. Similarly, the gas abundance index whose covariance defined the turbulent flux evolved with gas-measurement technology (Kowalski et al., 2021). Currently, our ability to quantify methodological differences of the sort defined above wants for instruments to measure ρ or its turbulent fluctuations.

Ideally, manufacturers would provide an instrument to either measure this directly, or by components requiring the dry-air density ρ_d – since water vapour density (ρ_v) is commonly measured – based on mass conservation ($\rho = \rho_d + \rho_v$). This might be achieved effectively via some proxy gas. For example, a greenhouse gas like sulphur hexaflouride (SF_6) could be measured with infrared lasers, and integrated into H_2O and CO_2 gas analyzers. Since SF_6 has negligible natural sources or sinks (Ko et al., 1993), it should maintain a constant fraction of dry air at most field sites, allowing rapid assessment of ρ_d. Another option would be to combine temperature perturbations with fast-response detection of fluctuating pressure (p'; e.g., Burns et al., 2021), and calculate ρ using the Ideal Gas Law. However, p' measurements are particularly challenging (Burns et al., 2021) and rare at flux-tower sites.

Perhaps the most feasible approach for most researchers is to neglect p' in the perturbation ideal gas law, based on scale analysis (Stull, 1988), and estimate ρ' using sonic anemometer and gas analyzer data. Kowalski et al. (2021) used this approach, comparing arithmetic-versus-weighted averaging when applying RAAD to CO_2 exchanges, and found substantial differences in resultant turbulent fluxes that varied seasonally with ecosystem functioning. Their Mediterranean reed wetland had maximum fluxes near the summer solstice, when arithmetic averaging produced turbulent fluxes that exceeded those from weighted averaging by about 10 %, or 2 μmol $m^{-2}\,s^{-1}$. With smaller flux magnitudes near the autumnal equinox, relative differences were greater (about 25 %) while absolute differences were smaller (about 1 μmol $m^{-2}\,s^{-1}$). According to these results, there are consequences regarding the choice of how to apply RAAD, although the issues of extensive-versus-intensive variable decomposition and the Leonard terms remain to be explored.

5. Conclusions

Well into the second century following its publication, and despite decades of authors professing to subscribe to its tenets, Reynolds averaging and decomposition (RAAD) remains nearly as obscure in meaning as it was for the lauded scientists who took part in the peer review process for Reynolds (1895). One participant in the process, Sir Horace Lamb, seems to have contributed greatly to the confusion regarding the meaning of his Owens College faculty colleague Osborne Reynolds, influencing generations of fluid dynamicists in favour of arithmetic averaging of velocities. By contrast, Reynolds (1895) was both explicit and persistent in applying density weighting when averaging velocities.

Throughout this manuscript, scrutiny of the literature has yielded several specific questions regarding the mathematical specification of turbulence that appear to be unresolved. From the beginning, Reynolds's publications were ambivalent regarding whether to apply density weighting when averaging the pressure: Reynolds (1895) says yes, while Reynolds (1903) says no. Given this lack of clarity from Reynolds, how should pressure be averaged, especially within the context of applying RAAD to the equation of motion? Additionally, Reynolds (1895) did nothing to clarify which type of variables – whether extensive or intensive – should be decomposed, and since its publication, micrometeorologists have employed different criteria. Therefore, we ask: to which type of variable should RAAD be applied? Finally, conventional wisdom holds that the Leonard terms that arise when averaging the equation of motion vanish under arithmetic averaging. However, the implication of Reynolds-introduced averaging on these terms remains elusive. What happens to Leonard terms under the density-weighted averaging defined by Reynolds (1895)? These questions and associated bibliographical references are summarized in Fig. 2.

It seems that a redefined methodology may be required to decompose the Navier-Stokes equations and shed light on these unsolved questions. Only then might progress towards a solution be achieved to the tricky problem of turbulence highlighted by Feynman.

CRediT authorship contribution statement

Andrew S. Kowalski: Writing – original draft, Investigation, Conceptualization. **Jesús Abril-Gago:** Writing – review & editing, Visualization, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Reynolds's writing style

A possible explanation for why subsequent authors paid so little attention to Reynolds's writings, as well as why reviewers had such a difficult time reviewing the Reynolds (1895) paper, is that Reynolds wrote sentences that were very long and difficult to understand. As a representative example, we copy here the first sentence of Reynolds (1895):

> The equations of motion of viscous fluid (obtained by grafting on certain terms to the abstract equations of the Eulerian form so as to adapt these equations to the case of fluids subject to stresses depending in some hypothetical manner on the rates of distortion, which equations Navier* seems to have first introduced in 1822, and which were much studied by Cauchy† and Poisson‡) were finally shown by St. Venant§ and Sir Gabriel Stokes,|| in 1845, to involve no other assumption than that the stresses, other than that of pressure uniform in all directions, are linear functions of the rates of distortion with a co-efficient depending on the physical state of the fluid.

This includes the distracting typographical marks found in the original. It occupies about the first quarter of a page in a 42-page treatise, mostly written in the same style and furthermore using bewildering terms like "mean-mean-motion" and "relative-mean-motion" whose meanings are essential to its understanding but seem to have been lost on the reviewers.

Data availability

No data was used for the research described in the article.

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