UNIVERSIDAD DE GRANADA

Doctoral Thesis

Analysis of Partial Differential Equations Arising from Mechanical and Biochemical Interactions in Cell Dynamics and Stochastic Particle Systems

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Summary

This thesis focuses on the study of partial differential equations (PDEs) arising in biological frameworks, from two distinct perspectives. The first explores various models that describe interactions between cell populations, with particular attention to pattern formation. We will examine how pressure influences the dynamics of these interacting populations, analyzing models where two cell populations interact and evolve over time. A key aspect of this investigation is the existence of monotonic traveling waves, which describe the propagation of these interactions. Additionally, we will extend this analysis to biochemical interaction models, such as the flux-saturated Keller-Segel model, aiming to establish the existence of traveling pulse or soliton-type solutions.

The second perspective involves deriving PDEs applicable to biological problems from systems of interacting particles, whose dynamics are governed by stochastic equations, with interactions defined through graphs. In this framework, we aim to study the mean-field limit of these particles to obtain a Vlasov-Fokker-Planck equation. This approach bridges microscopic particle dynamics and macroscopic PDE behavior, providing insight into the emergent phenomena in biological systems.

The thesis is organized as follows:

- **Chapter 1:** This chapter introduces the topics covered in the thesis, establishing the theoretical framework and related literature. The introduction is divided into three parts.

  First, we will introduce the biological model characterized by mechanical interactions and present various results obtained, as well as how other researchers have approached this problem from different perspectives.

  The second part will present the Keller-Segel models, introducing the saturated flux Keller-Segel models, discussing the significance of this model, and motivating the study and analysis of soliton-type patterns.

  Finally, we will introduce the systems of interacting particles, detailing how the mean-field limits for stochastic interacting particles have been addressed, emphasizing the use of diagraph measures.

- **Chapter 2:** This chapter corresponds to the paper [40]. We focused on proving the existence of traveling wave solutions in a model proposed by Joanny et al., which examines the dynamics of interfaces between two cell populations during tumor growth. The model includes non-local and strongly nonlinear advection terms representing biomechanical interactions. We establish upper and lower bounds for the wave propagation

speed across different biological parameters by employing various techniques from dynamical systems, geometric singular perturbation theory, and degree theory.

- **Chapter 3:** This chapter corresponds to an ongoing collaboration with R. Granero. We analyze the existence of solutions for the model by Joanny et al., where the Laplacian is replaced by a fractional Laplacian. The existence of solutions is derived using a combination of energy and pointwise methods.

- **Chapter 4:** This chapter corresponds to the paper [41], which is published in Mathematical Models and Methods in Applied Sciences and is a collaboration with M. Veruete. The objective of this study is to clarify the existence of patterns in Keller-Segel type models that manifest as traveling pulse solutions. The research explores transport mechanisms that describe these compact support waves, focusing on nonlinear diffusion via saturated flux mechanisms for cell movement. Additionally, various transport operators for the chemoattractant are analyzed. The methodologies employed integrate phase diagram analysis in dynamical systems with partial differential equations, utilizing the concept of entropic solutions and admissible jump conditions of the Rankine-Hugoniot type. The study identifies two types of traveling pulse waves that align with experimental observations.

- **Chapter 5:** This chapter corresponds to the paper [94], a collaboration with C. Kuehn. This paper focuses on particle systems modeled by stochastic differential equations (SDEs), where the mean field limit converges to a Vlasov-Fokker-Planck-type equation. It departs from traditional stochastic analysis by examining particle network connectivity through diagraph measures (DGMs), which capture various network interactions beyond classical approaches like graphons. The aim is to encompass a broad range of mean-field limits, employing measure-theoretic arguments combined with moment estimates to ensure approximation results for the mean field.

- **Chapter 6:** This chapter is dedicated to presenting various ongoing research projects related to this thesis. We will outline some conclusions from previous studies and discuss future research directions that remain open as a result of this work.

Resumen

Esta tesis se centra en el estudio de ecuaciones en derivadas parciales (EDPs), originadas en biología celular, desde dos perspectivas distintas. La primera explora varios modelos que describen interacciones entre poblaciones celulares, con especial atención a la formación de patrones. Examinaremos cómo la presión influye en la dinámica de estas poblaciones interactivas, analizando modelos donde dos poblaciones celulares interactúan y evolucionan a lo largo del tiempo. Un aspecto clave de esta investigación es la existencia de ondas viajeras monótonas, que describen la propagación de estas interacciones. Además, extenderemos este análisis a modelos de interacción bioquímica, como el modelo de Keller-Segel con flujo saturado, con el objetivo de establecer la existencia de soluciones tipo pulso viajero o solitón.

La segunda perspectiva implica derivar EDPs aplicables a problemas biológicos a partir de sistemas de partículas interactivas, cuya dinámica está gobernada por ecuaciones estocásticas, con interacciones definidas a través de grafos. En este marco, buscamos estudiar el límite de campo medio de estas partículas para obtener una ecuación de Vlasov-Fokker-Planck. Este enfoque une la dinámica microscópica de partículas y el comportamiento macroscópico de las EDPs, proporcionando información sobre los fenómenos emergentes en sistemas biológicos.

La tesis está organizada de la siguiente manera:

- **Capítulo 1:** Este capítulo introduce los temas tratados en la tesis, estableciendo el marco teórico y la literatura relacionada. La introducción se divide en tres partes.

  Primero, presentaremos el modelo biológico caracterizado por interacciones mecánicas y mostraremos varios resultados obtenidos, así como la forma en que otros investigadores han abordado este problema desde distintas perspectivas.

  La segunda parte presentará los modelos de Keller-Segel, introduciendo los modelos de flujo saturado de Keller-Segel, discutiendo la importancia de este modelo y motivando el estudio y análisis de patrones tipo solitón.

  Por último, introduciremos los sistemas de partículas interactivas, detallando cómo se han abordado los límites de campo medio para partículas estocásticas interactivas, haciendo hincapié en el uso de medidas de diagrama.

- **Capítulo 2:** Este capítulo corresponde al artículo [40]. Nos centramos en probar la existencia de soluciones de ondas viajeras en un modelo

propuesto por Joanny et al., que examina la dinámica de interfaces entre dos poblaciones celulares durante el crecimiento tumoral. El modelo incluye términos de advección no locales y fuertemente no lineales que representan interacciones biomecánicas. Establecemos límites superiores e inferiores para la velocidad de propagación de la onda a través de diferentes parámetros biológicos empleando diversas técnicas de sistemas dinámicos, teoría de perturbaciones singulares geométricas y teoría de grado.

- **Capítulo 3:** Este capítulo corresponde a una colaboración en curso con R. Granero. Analizamos la existencia de soluciones para el modelo de Joanny et al., donde el laplaciano es reemplazado por un laplaciano fraccionario. La existencia de soluciones se deriva mediante una combinación de métodos de energía y métodos puntuales.

- **Capítulo 4:** Este capítulo corresponde al artículo [41], que se publica en Mathematical Models and Methods in Applied Sciences y es una colaboración con M. Veruete. El objetivo de este estudio es aclarar la existencia de patrones en modelos de tipo Keller-Segel que se manifiestan como soluciones tipo pulso viajero. La investigación explora los mecanismos de transporte que describen estas ondas de soporte compacto, centrándose en la difusión no lineal a través de mecanismos de flujo saturado para el movimiento celular. Además, se analizan varios operadores de transporte para el chemoatrayente. Las metodologías empleadas integran el análisis del diagrama de fases en sistemas dinámicos con ecuaciones en derivadas parciales, utilizando el concepto de soluciones entrópicas y condiciones de salto admisibles del tipo Rankine-Hugoniot. El estudio identifica dos tipos de ondas de pulso viajero que se alinean con las observaciones experimentales.

- **Capítulo 5:** Este capítulo corresponde al artículo [94], una colaboración con C. Kuehn. Este artículo se centra en sistemas de partículas modelados por ecuaciones diferenciales estocásticas (EDEs), donde el límite de campo medio converge a una ecuación del tipo Vlasov-Fokker-Planck. Se aparta del análisis estocástico tradicional al examinar la conectividad de la red de partículas a través de medidas de diagrama (DGMs), que capturan diversas interacciones de red más allá de enfoques clásicos como los graphons. El objetivo es abarcar una amplia gama de límites de campo medio, empleando argumentos teóricos de medida combinados con estimaciones de momento para asegurar resultados de aproximación para el campo medio.

- **Capítulo 6:** Este capítulo está dedicado a presentar varios proyectos de investigación en curso relacionados con esta tesis. Resumiremos algunas conclusiones de estudios previos y discutiremos direcciones de investigación futuras que permanecen abiertas como resultado de este trabajo.

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Part I

INTRODUCTION AND BACKGROUND

Chapter 1

Introduction

In recent years, there has been a true revolution in the mathematical modeling of problems arising in biology and biomedicine, particularly in the field of developmental biology. This new approach stems from the ability to manage large amounts of real-world data, made possible by significant advancements in microscopy, various imaging techniques that offer sharper images, post-processing methods that enhance image quality, and major developments in sequencing, visualizing the dynamics, and understanding of signaling mechanisms (proteins, cytokines, cells, etc.) involved in cell communication. These advances have led to deeper insights into processes such as growth, migration, and competition, among others.

Historically, mathematical models tended to focus narrowly on either the biochemical interactions driving these processes or on the mechanical interactions (such as pressure, stiffness, and deformations) they produced. One of the key insights that modern data has revealed is that these two aspects -biochemical and biomechanical- are deeply intertwined. For instance, tumors modify the extracellular matrix by producing metalloproteases (biochemical) that reorganize its fibers, adjusting stiffness and porosity (biomechanical) continuously to suit their needs. Similarly, during cell growth, mutual pressure between cells alters membrane stiffness, which, in turn, influences biochemical interactions.

Furthermore, significant progress in the theoretical and numerical treatment of non-local and nonlinear equations, along with advances in linking problems across various scales (micro and macro), has paved the way for a new approach to these challenges. Our thesis is positioned within this evolving context, aiming to shed light on some of these issues from a mathematical perspective while being firmly inspired by real-world problems in biology and biomedicine.

1.1 Patterns formation on Biomechanical Models

The study of biomechanical properties, particularly the influence of stress and pressure on cell behavior, presents a fascinating challenge, especially in biological tissues composed of various cell populations. This research aims to understand how pressure affects the behavior of the separation interface between these populations, which continuously interacts with biochemical effects. Numerous experiments have demonstrated that pressure exerted on a cell population significantly impacts its dynamics, particularly its growth [55, 82, 85]. Consequently, several models have recently been developed to investigate how these populations grow as a function of internal pressures within a population, as well as the pressures exerted by different populations on each other and by the surrounding environment [102, 106, 112, 113, 116].

In Chapter 1 we will study certain patterns, particularly traveling waves, underlying the following nonlinear and nonlocal model that arises in the interface of cell populations:

$$\begin{aligned} \partial_T \phi + V \partial_X \phi &= \partial_X^2 \phi + \phi(1-\phi), \\ \Lambda^2 \partial_X^2 V - V &= 2\Lambda V_0 \partial_X \left(\phi + \beta(\partial_X \phi)^2\right), \end{aligned} \tag{1.1}$$

where ϕ represents the interface and V is the average propagation velocity of the system. The model was proposed by J-F Joanny *et al.* [113], and it describes the evolution of interfaces between two distinct cell populations in the presence of cell division and cell death. The focus is on the effects of mechanical coupling between the populations and its impact on cell growth and pressure. Additionally, the model considers how the surrounding environment exerts pressure on the system, thereby influencing the growth dynamics of the populations.

The equation governing the movement of the interface arises from the hypothesis that the evolution of the two populations resembles the behavior of two fluids, whose mutual pressure connects their dynamics [113]. Studying the interface between these two populations enables us to predict their interactions and potential invasion patterns. In this context, the analysis and characterization of possible traveling waves, i.e., solutions of the form $\phi(x-\sigma t)$, $V(x-\sigma t)$, as a function of the biological parameters of the system: V_0, β, Λ, and the wave speed σ, will shed light on the dynamics of the interface and its consequences. From a mathematical perspective, this analysis will require of a combination of techniques from partial differential equations, dynamical systems, and degree theory, depending on the various constants of the system.

Let us briefly outline the derivation of the model. The initial idea is to consider that the dynamics of each cell population are governed by the evolution equation:

$$\partial_t n_i + \partial_x (n_i v_x^i) = n_i K_i(P), \quad i = 1, 2; \tag{1.2}$$

where n_i is the density of population i, v_i is the velocity field of the population i, and K_i is the net cell division rate of the population i. The hypothesis that allows for the development of the model is that the cell growth of each population will depend on the pressure exerted on it, through a pressure threshold (Homeostatic pressure) that the cell can withstand. Thus, if the pressures are higher than this threshold, cell death is favored and if they are lower, growth is favored. Close to this equilibrium state, the net cell division rate can be approximated by a linear expansion $K_i \approx \kappa(P - P_i^h)$, where κ_i is a constant that describes the sensitivity of the rate to changes in pressure.

We assume that the system formed by bothg populations satisfies the hypothesis of incompressibility:

$$n_1\Omega_1 + n_2\Omega_2 = 1,$$

where Ω_i is the constant volumen of each cell i. Setting $\phi = n_1\Omega_1$ and defining the average velocity as:

$$v = v_1\phi + v_2(1-\phi),$$

we can deduce, under certain considerations, that ϕ satisfies the equation:

$$\partial_t\phi + v_x\partial_x\phi = -\partial_x J\phi + \phi(1-\phi)\kappa(P_1^h - P_2^h). \tag{1.3}$$

where J is the relative flux, and as an important assumption of the problem is to consider that this flux is of the form $J = -D\partial_x\phi$, obtaining the following equation:

$$\partial_t\phi + v_x\partial_x\phi = -\partial_x J\phi + \phi(1-\phi)\kappa(P_1^h - P_2^h). \tag{1.4}$$

On the other hand, it remains to determine the equation satisfied by the average velocity. To achieve this, the system is considered as a viscous fluid. From its deformation tensor ($\sigma_{\alpha\beta}$) and considering that the fluid moves through the medium with friction ($\partial_\alpha\sigma_{\alpha\beta} = -\gamma v$), the following equation is obtained:

$$-\Delta P^h\partial_x\phi + \partial_x^2 v_x\left(\frac{1}{\kappa} + \frac{4}{3}\eta\right) - \frac{4}{3}B\partial_x^2\phi\partial_x\phi = \gamma v_x. \tag{1.5}$$

It is important to emphasize the fact that the reaction logistic term in (1.4) and the structure of the velocity equation (1.5) are obtained as a consequence of the concept of pressure that we considered, along with the consideration of homeostatic pressure.

By adimensionalizing the equation, we obtain the dimensionless form (1.1) for further study. The equation for the average propagation velocity V follows a Helmholtz equation. However, in contrast to other models for fluid interfaces, where the propagation velocity is typically related to the pressure through Darcy's law or the Brinkman law, and the pressure is expressed as a power of the population densities ([102],[112],[70],[65]) We have models like the one presented by Perthame *et al.* [70]

$$\begin{cases} \dfrac{\partial n_i}{\partial t} - \nabla\cdot(n_i\nabla V) = n_i G_i(p) \\ \qquad -\nu\Delta V + V = p \end{cases} \tag{1.6}$$

where p denotes the joint population pressure generated by the two species, $p = \frac{\gamma}{\gamma-1}(n_1+n_2)^{\gamma-1}$ and the velocity field V, is related with the joint population pressure through so-called Brinkman's law. And the function $G(P)$ takes into account the homeostatic pressure as a regulatory mechanism of growth.

Or the Hele-Shaw problem for tumor growth, presented by Perthame, Quirós and Vazquez [112], whose model has later been used in [65, 102], in which they rely on the use of Darcy's law ($V = -\nabla P$) to relate pressure with the velocity.

On the other hand, our model exhibits a more intricate relationship, involving the gradient of the density and the variation of its modulus.

As we mentioned earlier, the aim of the paper is to analyze the existence of traveling waves solutions of (1.1). One of the main challenges of the model is how to manage the non-local advection term that arises, given that the velocity can be expressed as:

$$V = \Gamma * \left(2\Lambda V_0 \partial_X \left(\phi + \beta(\partial_X \phi)^2\right)\right), \tag{1.7}$$

where Γ is the kernel associated with the Helmholtz operator $\Lambda^2 \partial^2_{XX} V - V = \delta$. The objective is to determine how the different parameters of the system influence the existence or absence of traveling waves. By analyzing the role of these parameters, we aim to understand the conditions under which traveling waves can form and propagate, as well as the characteristics of these waves. This analysis will provide insights into the dynamic behavior of the system and its response to various internal and external factors.

The existence of this type of solutions for problems in which nonlocal terms appear, either in the reaction term or in the advection term, have been analyzed by different authors, and always considering that the nonlocality had the form $K * \phi$, where K could be a kernel in some L^p space ([84],[21],[86]), or the Helmholtz kernel ([59],[70]), which in our case corresponds to (1.7), or some sort of non local diffusion kernel ([1],[59],[103]), like in the fractional Laplacian case.

Seeking traveling wave profiles $\phi(t, x) = \phi(x - \sigma t)$ that solve the partial differential equation (1.1), we derive the following system of second-order differential equations:

$$\begin{aligned} -\sigma\phi' + V\phi' &= \phi'' + \phi(1 - \phi), \\ \Lambda^2 V'' - V &= 2\Lambda V_0 \phi'(1 + \beta\phi''), \end{aligned} \tag{1.8}$$

where $\sigma > 0$ is the wave speed. The solutions to this differential equation represent the stationary profiles of traveling waves moving at a velocity σ. Our goal is to find decreasing profiles that satisfy the boundary conditions $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$.

Redefining the parameters as $a = 2\Lambda V_0$ and $b = 2\Lambda V_0 \beta$, the boundary problem to be studied then takes the form:

$$\begin{aligned} -\sigma\phi' + V\phi' &= \phi'' + \phi(1 - \phi), \\ \Lambda^2 V'' - V &= a\phi' + b\phi'\phi'', \\ \phi(-\infty) &= 1, \quad \phi(+\infty) = 0. \end{aligned} \tag{1.9}$$

The existence of such traveling wave patterns will depend on the different values taken by the parameters a, b, and Λ, as well as their relationship with the wave speed σ.

Chapter 2 presents three main theorems addressing different scenarios:

Theorem 2.1.1 focuses on the case where $\Lambda = 0$, simplifying the system to a local nonlinear velocity behavior. Using dynamical systems theory, the theorem establishes the existence of unique traveling wave solutions for wave speeds σ exceeding a threshold $\sigma^*(a, b)$, which depends on the parameters a and b. The theorem provides explicit bounds for σ^* under various conditions on a and b.

Theorem 2.1.2 extends the analysis to small positive values of Λ. By applying perturbative methods, it demonstrates that the wave speed thresholds identified in Theorem 2.1.1 persist when Λ is slightly greater than zero, ensuring the existence of solutions in this perturbed regime.

Theorem 2.1.3 addresses the general case with $\Lambda > 0$. Utilizing the fundamental solution of the Helmholtz equation and employing Leray-Schauder topological degree techniques, the theorem confirms the existence of traveling wave solutions provided that the ratio $\frac{b}{2\Lambda^2} < 1$. It also establishes properties of the solutions, including monotonicity of φ and upper bounds on the wave speed σ.

1.1.1 Existence of solutions in fractional Laplacian biomechanical model

As previously mentioned, one of the assumptions established in Joanny's model was that the relative flow was approximated by the gradient of the density, resulting in a Laplacian operator. However, what happens if, instead of having a Laplacian, we utilize a different operator?

The aim of chapter 3 is to study the well-posedness of (1.1), where the Laplacian has been replaced by a fractional Laplacian, resulting in the following set of coupled differential equations:

$$\begin{aligned} &\partial_t\phi = -(-\Delta)^\alpha\phi + \partial_x u\partial_x\phi + \phi(1-\phi) \\ &-\Lambda^2\partial_x^2 u + u = a\phi + \frac{b}{2}\left(\partial_x\phi\right)^2, \\ &\phi(0,x) = \phi_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1.10}$$

where the system parameters are related to the biochemical and biomechanical variables of the system, with $\alpha \in (0,1)$, $\Lambda > 0$, $a \geq 0$, and $b \geq 0$. The term $(-\Delta)^\alpha\phi$ represents the fractional Laplacian, defined as follows:

$$(-\Delta)^\alpha\phi = C(\alpha)\,\mathrm{P.V.}\int_{\mathbb{R}} \frac{\phi(x)-\phi(y)}{|x-y|^{1+2\alpha}}\,\mathrm{d}y.$$

We observe that, since u satisfies the Helmholtz equation, we can express it as

$$u = \Gamma * \left(a\phi + \frac{b}{2}(\partial_x\phi)^2\right), \tag{1.11}$$

where $\Gamma = \frac{1}{2\Lambda}e^{-\frac{|x|}{\Lambda}}$.

The previous equation (3.1) can be interpreted as an *active scalar* equation, where the scalar ϕ is advected by the velocity $\partial_x u$, which is itself determined from ϕ by solving an elliptic equation. Research on active scalar equations dates back to the study of the two-dimensional incompressible Euler equations in vorticity form, widely regarded as one of the most significant examples of active scalar systems. In their seminal work, Constantin, Majda, and Tabak [61] proposed the two-dimensional surface quasi-geostrophic equation as a model for the three-dimensional incompressible Euler equations, sparking significant interest. Since then, studies on various active scalar equations, such as the incompressible porous medium equation (e.g., [63]), the Stokes-transport equations [10], the magnetogeostrophic equations [74], and other related problems (see [117] and references therein), have continued to expand, attracting substantial attention from researchers in mathematical fluid dynamics.

Although active scalar systems are widely studied in fluid dynamics, they naturally arise in other fields as well. For instance, the well-known parabolic-elliptic Keller-Segel model of chemotaxis is also an example of an active scalar system (see, e.g., [32] and the references therein). In fact, the one-dimensional version of the parabolic-elliptic Keller-Segel model bears a striking resemblance to equation (3.1) (see, for instance, [30] and the references therein).

Of particular significance in mathematical fluid mechanics is the study of one-dimensional toy models that simplify complex fluid equations. A pioneering contribution in this area is the work of Constantin, Lax, and Majda [60]. Since then, various one-dimensional active scalar models with fractional dissipation have emerged in the literature (see, for example, [108, 62, 49]). These simplified models often capture key dynamics of more complex systems, providing valuable insights into phenomena such as formation and blow-up.

It is important to highlight that a common feature of the previously discussed active scalar equations is the presence of nonlocality, introduced through the elliptic equation governing the velocity, as well as the crucial role that fractional dissipation plays in shaping the qualitative behavior of the solutions (see, for instance, [31]). Additionally, it is worth noting that in all the aforementioned active scalar systems, the velocity is typically recovered from the scalar through a linear operator. In contrast, in (3.1), the velocity depends nonlinearly on the advected scalar, which, to the best of our knowledge, is a unique characteristic of this specific active scalar equation.

Likely due to the connection between early active scalar equations and the three-dimensional incompressible Euler equations, one of the central questions in this area of study involves the dichotomy between global existence of classical solutions and finite-time singularity formation. Numerous works explore different levels of diffusion and various types of nonlocal velocities, leading to both outcomes. For a comprehensive overview, the interested reader is referred to [48, 11] and the references therein.

The well-posedness of the problem (3.1) will be established in Chapter 3 through **Theorem** 3.1.1. This theorem states that given an initial condition ϕ_0 in the space $H^3(\mathbb{R})$ and a parameter α in the interval $(0,1]$, the problem defined by the equation (3.1) has a unique local solution ϕ. This solution belongs to the functional spaces $C([0,T];H^3(\mathbb{R}))$ and $L^2([0,T],H^{3+\alpha/2}(\mathbb{R}))$. Furthermore, if the initial condition is non-negative ($0 \leq \phi_0$), the solution extends globally in time and remains uniformly bounded in the space L^∞.

1.2 Biochemical Interactions: Patterns on Keller-Segel Type Models

The collective behavior of species and how dynamic patterns emerge (defense, invasion, resilience , ...) is one of the most important topics in current research that requires a multidisciplinary approach to be addressed. In addition to the intrinsic value of studying the dynamics of a population of birds, fish, ants or sheep, these models could provide foundations for understanding other, more microscopic problems, such as morphogen-cell interaction or the evolution of tumors. However, the impressive evolution in microscopy and in antibody concentration morphogenesis cell signaling have allowed the study of collective behavior at the subcellular and cellular level to be analyzed and modeled directly [2]. This provides a new impetus in which the models initially developed by Keller and Segel (KS) [93, 92, 110] for chemotaxis processes (the movement of biological entities in response to chemical gradients) take on a new dimension. In addition, in recent years various applications have been developed in exotic contexts [17] beyond cell signaling mechanisms that have provided more flexible and diverse variants of KS-type models.

The classical KS model consists in a reaction–diffusion system of two coupled parabolic equations

$$\begin{aligned} &\partial_t U = \mathrm{div}_x(D_U \nabla_x U - \chi\, U\, \nabla_x Q) + H(U,Q), \quad x \in \mathbb{R}^N,\ t > 0, \\ &\tau \partial_t Q = D_Q \Delta Q + K(U,Q), \quad x \in \mathbb{R}^N,\ t > 0. \end{aligned} \tag{1.12}$$

In the biological context of cell dynamics, U represents the cell-density and Q the chemoattractant concentration. The positive definite terms D_Q and D_U are respectively the diffusivity of the chemoattractant and of the cells, $\chi \geq 0$ is the chemotactic sensitivity and the functions $H(U,Q)$ and $K(U,Q)$ in (1.12) model the interaction (production and degradation) between the cell density and the chemical substance. In most simplified models and in the original Keller–Segel system, these terms are modeled as $K(U,Q) = U - Q$ and $H(U,Q) = 0$. The hyperbolic limit (high–field limit) and some special parabolic limit (low–field limit) have been derived from kinetic equations describing the run and tumble process for bacterial motion [15, 13, 14, 16, 17, 53, 52, 87, 109]. The parameter $\tau \geq 0$ is introduced to distinguish if there is an adjustment of the chemoattacker during the evolution of the process, it is standard that it only takes values only 0 or 1 giving rise to different models that may not be dynamically equivalents.

We will consider two variants of the flux–saturated Keller–Segel (FSKS) model:

$$\begin{aligned} &\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - a\, U\, \frac{\partial Q}{\partial x}\right) \\ &\mathcal{T}(Q) = U, \end{aligned} \tag{1.13}$$

where $\mathcal{T} = \mathcal{T}(Q)$ is one of the following linear differential operators

$$\mathcal{T}(Q) = \frac{\partial^2 Q}{\partial t \partial x} - \nu \frac{\partial^2 Q}{\partial x^2}, \tag{1.14}$$

or

$$\mathcal{T}(Q) = \tau \frac{\partial Q}{\partial t} + \alpha \frac{\partial Q}{\partial x} - \nu \frac{\partial^2 Q}{\partial x^2}. \tag{1.15}$$

The parameters $\alpha \geq 0$ and $\nu \geq 0$ stands for the transport and diffusion coefficients respectively. The flux function $\Phi = \Phi(s)$ is a bounded, regular, increasing and odd function. The value $c > 0$ is defined as

$$c = \lim_{s \to \infty} \Phi(s),$$

and it is finite. Also $\Phi \in C^1(\mathbb{R})$ in order to have uniquenees of the initial value poblems. The value $\mu = \Phi'(0)$ is the kinematic viscosity for small velocities and near $u_x = 0$ the flow means

$$U^m \Phi \left(U^{-1} \frac{\partial U}{\partial x} \right) \sim \mu U^{m-1} U_x,$$

being $m \geq 1$ a parameter that measures the porosity of the medium. In some sense, we have a flux–saturated combined with a porous media operator [36]. Different proposals to ours to use flux–saturetad operators as an alternative to linear diffusion for the Keller-Segel model use hyperbolic, fractional diffusion or porous medium type approximations, see for example [26, 115, 54] and the references therein.

One of the main objectives of our study is to consider the so-called relativistic-heat case

$$\Phi(s) = \mu \frac{s}{\sqrt{1 + \frac{\mu^2}{c^2} s^2}}$$

that leads to

$$\frac{\partial U}{\partial t} = \mu \frac{\partial}{\partial x} \left(\frac{U^m \frac{\partial U}{\partial x}}{\sqrt{U^2 + \frac{\mu^2}{c^2} \left| \frac{\partial U}{\partial x} \right|^2}} - a\, U \, \frac{\partial Q}{\partial x} \right).$$

Other examples of great interest are $\Phi(s) = \mu \frac{s}{1+\frac{\mu}{c}|s|}$ usually referred as Wilson operator [104], the Larson operator [36] $\Phi(s) = \mu \frac{s}{\sqrt[p]{1+\frac{\mu^p}{c^p} s^p}}$ that include the relativistic case, and $\Phi(s) = c \tanh\left(\frac{\mu s}{c}\right)$ usually referred as the hyperbolic tangent operator, see for instance [97].

If $m = 1$, which is the case of the relativistic heat equation, then c is the speed at which the solution support moves [5]. In the general choice of Φ, c represents the maximum speed at which the solution support can move [36]. Therefore, c is a parameter that can be taken from the biological experimental data [119]. Note that for any flux-saturated Φ, if c tends to infinity the heat equation or the porous media equation are recovered for the different values of $m \geq 1$ [47].

The aim of Chapter 4 is to find, in the context of Keller–Segel models, soliton-type patterns with compact support, which represent collective models of cell invasion, propagation or behavior in which the interface with the medium is singular. This type of patterns usually appears with diverse geometry in the experimental data, and cannot be captured with linear diffusion terms in the classical KS model.

Experimental data show that the movement of cells affected by a chemo-attractant occurs through a pulse or soliton type solution [66, 119]. Moreover, from the point of view of modeling, the Keller–Segel model combines a system of partial differential equations that represents the evolution of the cell density and the chemotractant concentration. However, the classical KS system, although it admits regular traveling waves with a birth term of either a Fisher–KPP term-type, does not seem to admit soliton type solutions. The modification of the transport terms, especially preventing free diffusion, allows to build solutions that better reflect the experimental results. This fact was rigorously proved in the case of a flux–saturated as an alternative to linear diffusion in cell density in [7]. A great effort has been devoted in recent years to study the properties of the evolution by flux–saturated mechanisms, in particular the existence of traveling waves, see for instance [5, 6, 19, 18, 37, 35, 38, 39, 42, 43, 46, 44, 45, 57, 56, 75, 96, 114] and the references therein.

In the case where the time evolution of the chemoattractant is negligible, the resulting model produces a self-generated potential in terms of cell density. This has been the most studied approach in the context of the Keller–Segel models [23, 68, 69, 88, 87].

The main reason to modify the linear diffusion by a non-linear one is that this reproduces more faithfully the experimental data. In this context, the FSKS is a macroscopic model describing cell motion by chemotaxis, in which saturation of the velocity is taken into account. The FSKS model also has the advantage that traveling pulse or soliton–type solutions with compact support emerge as a prototype of pattern under this system [7]. The existence of this type of solutions is relevant for biological applications since, from a modeling perspective, the compactly supported property is well suited.

The main results of Chapter 4 are the following:

- In Section 4.3, we study the case where $\mathcal{T}(Q)$ is given by the following expression:

$$\mathcal{T}(Q) = \frac{\partial^2 Q}{\partial t \partial x} - \nu \frac{\partial^2 Q}{\partial x^2}.$$

  We prove that for any parametric configuration and for all $\sigma > 0$, we can find block-type solutions, as shown in **Theorem** 4.3.1.

- In Section 4.4, $\mathcal{T}(Q)$ is given by the following expression:

$$\mathcal{T}(Q) = \tau \frac{\partial Q}{\partial t} + \alpha \frac{\partial Q}{\partial x} - \nu \frac{\partial^2 Q}{\partial x^2}.$$

  Depending on the values of $\alpha \geq 0$ and $\nu \geq 0$, we obtain different results.

  Theorem 4.4.1 studies the case of α and ν, establishing that for values of $m \in (1, 2)$, there exists a block-type solution for all positive σ. Moreover, if $m \geq 2$, there exists a $\sigma^*(\alpha)$ such that there are block-type solutions for all $\sigma > \sigma^*(\alpha)$. Additionally, **Theorem** 4.4.2 provides conditions for the non-existence of solutions.

  For the case $\alpha = 0$ and $\nu > 0$, Corollary 4.4.1 shows that block-type solution exists, for all $\sigma > 0$.

  Finally, in Subsection 4.4.4, we address the case $\alpha > 0$ and $\nu = 0$. It is shown that under certain parametric configurations, we can find block-type solutions, which differ from the previous ones as they have a decreasing profile, unlike the others which are concave with a change in monotonicity.

1.3 Stochastics Particle Systems: Mean-field limit and Digraph Measures

Consider a physical system composed of N particles that interact with each other through a network/graph of connections, where information about the connectivity between particles can be represented by a graph G. The vertices of this graph represent each particle, and the edges of the graph model pairwise interactions between them.

The study of these systems, particularly when N is a very large number, has been approached in various ways, depending on different mathematical treatments used to represent the graph. To set the context, let us begin by introducing the equations that determine the system's dynamics. Let $X_i(t) \in \mathbb{R}^d$ denote the state of particle i, and consider the following stochastic differential equations (SDEs) that define the evolution of all particles in the system.

$$\begin{aligned} &\mathrm{d}X_i(t) = f(X_i(t))\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N} A_{ij}^N g(X_i, X_j)\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N} \hat{A}_{ij}^N h(X_i, X_j)\mathrm{d}B_t^i, \\ &X_i(0) = X_i^0, \quad i = 1, \ldots, N. \end{aligned} \tag{1.16}$$

Here, $A_{i,j}^N$ is the adjacency matrix of the system, representing the basic deterministic network interaction between particles i and j, and $\hat{A}_{ij}^N$ serves as an adjacency matrix to represent, how interactions between particles i and j influence the noise term. A_{ij}^N and $\hat{A}_{ij}^N$ are not necessarily identical, which already provides a useful generalization to existing mean-fields that aim to include network coupling. The functions f, g and h are Lipschitz and bounded. The noise term contains a sequence $\{B_t^i\}$, for $1 \leq i \leq N$, of independent and identically distributed Brownian motions in $\mathbb{X} \subseteq \mathbb{R}^d$. The initial conditions $\{X_i^0\}_{i=1,..,N}$ form a sample of independently and identically distributed random variables with a probability distribution $\mu^0 \in \mathcal{P}(\mathbb{X})$. $X_i(t)$ represents, at the microscopic level, the trajectory of particle i. As the number of particles in the system becomes very large, the study of the system's dynamics becomes increasingly complex. Therefore, to analyze these issues, various techniques have been introduced, focusing on the study of the density of a typical particle and its evolution [27, 67, 107, 118, 78]. These techniques were classically developed on an all-to-all coupled graphs.

The main objective is to determine the mean-field limit for the interacting particle system (5.1). The main challenge is that we have to consider also a suitable notion of graph limits and incorporate the graph limit object in our analysis. For dense graphs, it is understood that graphons provide a suitable tool, and one can stay within a familiar setting of functions and norms as graphons are just functions themselves. Yet, to capture sparse and intermediate density graph limits, it is necessary to work with graph limits that are only measures. In this work we show that mean-fields can be obtained on a type of graph representation called digraph measure (DGM).

The study of the mean-field limit for equations similar to (5.1) has been approached in different ways, depending on how the adjacency matrices have been handled in the analysis.

First, in cases where the stochastic terms absent, numerous studies explore the mean-field limit by considering $A_{ij} = 1$, which are considered mean-field limits for systems of completely exchangeable particles, as studied in e.g. [90, 27, 67, 107]. On the other hand, when considering systems of non-exchangeable particles and accounting for graph properties, the mean-field limit has been derived in various frameworks, e.g., in [58, 101, 77, 95, 89, 8, 111]. Each one employs different graph representation techniques such as graphons [89, 101, 58], digraph measures [95], or graphops [77]. In all these cases, it is demonstrated that in the limit, the system satisfies a Vlasov equation.

Second, when considering the stochastic terms, different perspectives have been studied. For example, [29, 12, 22] have addressed various treatments of the matrix, ranging from $A_{ij} = 1$ to considering the graph as a graphon. Here we focus on studying the graph as a digraph measure (DGM) and derive the resulting Vlasov-Fokker-Plank equation in the stochastic case. To approach this, we will concentrate on the coupling method [50, 51], and utilize the techniques of Sznitmann [118], along with those used in [12, 22], to study the mean-field limit of the particle system (5.1).

As our goal is to obtain a mean-field limit, we aim to determine a measure of the form $\mu(t, x)$ representing the mesoscopic/typical particle evolution in time and space of the original microscopic model. Since the SDEs for the particles lead to a stochastic process, the density μ must be a probability measure. However, as we are working with DGMs, this measure will also depend on an additional variable representing the heterogeneity of the graph, which we denote as $u \in I$, where I is the set of all possible values that the graph variable can take. Our probability measure will take the form $\mu(t, x, u)$, and we will define $\bar{\mu}(t, x) = \int_I \mu(t, x, u)\, \mathrm{d}u$. For simplicity, we will also use the following notation: $\mu(t, x) = \mu_t(x)$ and $\mu(t, x, u) = \mu_{u,t}(x)$.

Our objective is, therefore, to establish, from (5.1), a probability measure whose limit converges in a suitable metric space to a (weak) solution of some Vlasov-Fokker-Planck equation. To achieve this, we will represent the law of the entire system through the empirical measure, which will play a central role. The empirical measure is defined as follows:

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i(t)}, \text{ for } t \in [0, T].$$

The next step, once the empirical measure is defined, is to identify a problem with sufficiently good properties, whose solution can be compared with ours. This is the basis of coupling methods. We seek a problem where the dynamics of each particle is suitably independent of the others, and we are going to prove the existence of a solution to this problem. Once the existence results is established, we are then going to show that as the number of particles tends to infinity, both problems exhibit similar behavior. Mathematically, this is expressed by proving that the distances between the empirical measure and a suitable limiting measure tend to zero, as $N \to \infty$, in some metric space on probability measures that we have to select.

To start, let us introduce the system of independent processes that we are going to consider:

$$
\begin{aligned}
X_u(t) =& X_u(0) + \int_0^t f(X_u(s))\ \mathrm{d}s + \int_0^t \int_I \int_{\mathbb{X}} g(X_u(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\
&+ \int_0^t \int_I \int_{\mathbb{X}} h(X_u(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B_s^u,
\end{aligned}
\tag{1.17}
$$

where $\mu_{u,t} = \mathcal{L}(X(t)|U = u)$, and U is a uniform random variable on I. The initial conditions $X_u(0) = X_u^0$, for $u \in I$, form a sample of independently and identically distributed random variables with a probability distribution $\bar{\mu}^0 \in \mathcal{P}(\mathbb{X})$. Note that two new measures appear, which are finite positive Borel measures $\eta, \hat{\eta} \in \mathcal{B}(I, \mathcal{M}_+(I))$. The measures $\eta, \hat{\eta}$ are the fiber measures associated to the adjacency matrices of the graphs $A_{i,j}^N$ and $\hat{A}_{i,j}^N$. The fiber measures are defined on a vertex space I. In this way, these measures represents the edges between different u and the rest of the vertices of I. This representation of graphs by a measure is known as a digraph measure (DGM); the concept is a step towards potentially covering even broader classes of graph limits, e.g., those given by graphops (graph operators). For background on graph limits and DGMs we refer to [9, 95]. The Brownian motion satisfies the same properties as before.

Next, observe in (5.2) that the particle dynamics, for a given u, are independent of the dynamics of the other particles. Therefore, the system represents an independent particle process. Furthermore, the graph's heterogeneity variable is taken as a random variable, uniformly distributed over the interval I. We can define the probability measure associated with the solution X as $\bar{\mu} = \int_I \mu_{u,t}\,\mathrm{d}u$. Since the aim was to compare the two systems as the number of particles becomes sufficiently large, we want to determine whether there is a metric space with metric d such that:

$$
d(\bar{\mu}, \mu_N) \to 0, \text{ as } N \to \infty.
$$

Since we are dealing with graphs, the graphs $A_{i,j}^N$ and $\hat{A}_{i,j}^N$ will also change with the number of particles, and we must also be able to represent their convergence as N tends to infinity. Moreover, as we are comparing it with the independent system, we must employ a technique of representing matrices/operators in the form of measures and determine a space in which we can establish the convergence.

In Chapter 5, we will focus on proving that if our two graphs, each represented by a directed graph measure (DGM), converge towards certain measures, then under specific assumptions (Assumptions H and $\tilde{H}$ below), the empirical measure describing the dynamics of the states of the particles will converge to a probability measure $\mu_{u,t}$. This limit satisfies the following Vlasov-Fokker-Plank equation:

$$
\begin{aligned}
&\partial_t \bar{\mu}_t + \partial_x \left(\bar{\mu}_t f(x) + \int_I \mu_{u,t} \int_I \int_{\mathbb{X}} g(x, y)\ \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}u \right) \\
&+ \frac{1}{2}\partial_x^2 \left(\int_I \mu_{u,t} \left[\int_I \int_{\mathbb{X}} h(x, y)\ \mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v) \right]^2\ \mathrm{d}u \right) = 0,
\end{aligned}
\tag{1.18}
$$

in a distributional sense.

This will be proved in **Theorem** 5.3.1 and 5.3.2, where in each theorem, we will handle the limit measure $\mu_{u,t}$ differently. However, in both cases, we will obtain the same Vlasov-Fokker-Plank equation.

Part II

CONTRIBUTIONS

Chapter 2

Biomechanical Effects On Traveling Waves At The Interface Of Cell Populations

2.1 Introduction:

In this Chapter we investigate traveling wave patterns within a nonlinear and nonlocal mathematical model that describes the interfaces between distinct cell populations. Originating from the work of J-F Joanny et al. [113], the model captures the dynamics of cell interfaces influenced by cell division, death, and mechanical interactions. Specifically, the model comprises a system of partial differential equations (PDEs) where ϕ represents the interface between two cell populations, and V denotes the average propagation velocity. The equations are given by:

$$\begin{aligned}
&\partial_T\phi + V\partial_X\phi = \partial_X^2\phi + \phi(1-\phi),\\
&\Lambda^2\partial_X^2 V - V = 2\Lambda V_0\partial_X\left(\phi + \beta(\partial_X\phi)^2\right),
\end{aligned}$$

Here, Λ is a parameter related to the Helmholtz operator, and V_0 and β are biological parameters affecting the system's dynamics. The model emphasizes the role of mechanical coupling and external pressures in governing cell growth and interpopulation interactions.

The primary objective is to explore the existence and characteristics of traveling wave solutions of the form $\phi(x-\sigma t)$ and $V(x-\sigma t)$, where σ represents the wave speed. Such solutions provide insights into the invasion patterns and interaction dynamics between the cell populations.

Redefining the parameters as $a = 2\Lambda V_0$ and $b = 2\Lambda V_0\beta$, the boundary problem to be studied then takes the form:

$$\begin{aligned}
-\sigma\phi' + V\phi' &= \phi'' + \phi(1-\phi),\\
\Lambda^2 V'' - V &= a\phi' + b\phi'\phi'',\\
\phi(-\infty) = 1&, \quad \phi(+\infty) = 0.
\end{aligned} \tag{2.1}$$

We will look for monotone decreasing solutions of the problem, which existence will depend on the different values taken by the parameters a, b, and Λ, as well as their relationship with the wave speed σ.

Firstly, we focus on the case $\Lambda = 0$ in (2.1), where the system exhibits local, nonlinear velocity behavior. By applying techniques from dynamical systems theory, we demonstrate the existence of solutions under specific parametric conditions, as detailed in the following result.

Theorem 2.1.1. *Let $\Lambda = 0$. There exists $\sigma^* = \sigma^*(a,b)$ such that there is a unique solution, up to translations, to the problem (2.1) , for every $\sigma \geq \sigma^*(a,b)$. Furthermore,*

(T1) *$\sigma^* \geq 2$, and if $\max\{a,b\} \leq 2$, then $\sigma^* = 2$.*

(T2) *If $b = 0$ and $a > 2$, then*

$$\begin{cases} 2 & \text{if } a \leq 3+2\sqrt{2}, \\ \frac{a-1}{\sqrt{a}} & \text{if } a > 3+2\sqrt{2}, \end{cases} \leq \sigma^*(a,0) \leq \begin{cases} \sqrt{\frac{a^2+4}{a}} & \text{if } 2 \leq a < a^*, \\ 2+\frac{a}{8} & \text{if } a^* \leq a < 16, \\ \sqrt{a} & \text{if } 16 \leq a, \end{cases}$$

where a^* is the unique root of $a^3 - 32a^2 + 256a - 256 = 0$ in the interval $]2,16[$.

(T3) *If $b > 0$,*

$$2 \leq \sigma^* \leq \sqrt{\frac{\sqrt{a^2+8a+4b+16}+a+4}{2}}.$$

Moreover, if $a^2 \geq 4b$,

$$\sigma^* \geq \frac{2b-(a-\sqrt{a^2-4b})}{\sqrt{2b\left(a-\sqrt{a^2-4b}\right)}}.$$

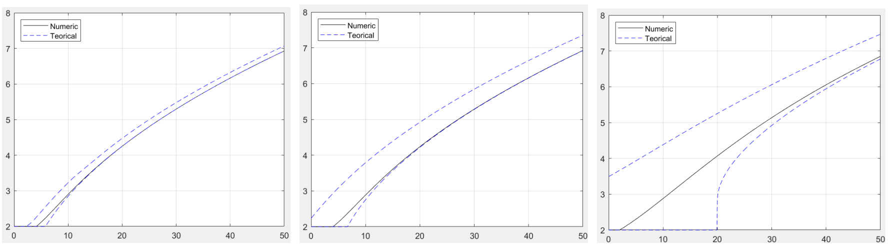


Figure 2.1: This figure represents the theoretical values stated in Theorem 2.1.1 regarding the bounds of σ^* and the numerical approximation of the value of σ^* obtained in Octave. The figures illustrate different scenarios for various values of b: Left-side figure: $b = 0$; Center figure: $b = 5$; and Right-side figure: $b = 40$. These figures provide a visual comparison between the theoretical bounds and numerical approximations of σ^* under varying conditions of b.

The previous theorem enables us to extend our analysis to the case where $\Lambda > 0$, particularly for small values of Λ. We employ perturbative methods to show that the values of σ identified in Theorem 2.1.1, which guaranteed the existence of a solution in the unperturbed case, remain valid when Λ is small but nonzero. This ensures the existence of solutions in this modified setting as well.

Theorem 2.1.2. *For each $\sigma > \sigma^*(a, b)$ as defined in Theorem 2.1.1, there exists a sufficiently small $\Lambda_0 = \Lambda_0(\sigma, a, b) > 0$ such that, for any $\Lambda \in (0, \Lambda_0)$, the problem (2.1) admits a solution.*

Lastly, we will address the general case $\Lambda > 0$. To simplify the two differential equations, we will use the fundamental solution of the Helmholtz equation, allowing us to reduce them to a single non-local differential equation. By applying Leray-Schauder topological degree techniques, we will demonstrate the existence of traveling waves, as outlined in the following result.

Theorem 2.1.3. *If $\frac{b}{2\Lambda^2} < 1$, there exists $\sigma \geq 2$ such that the problem (2.1) has solutions $\phi, V : \mathbb{R} \to \mathbb{R}$. Furthermore, we have $\sigma \leq 2 + \frac{a}{\Lambda} + \frac{b}{4\Lambda^2}\frac{2+\frac{a}{\Lambda}}{1-\frac{b}{2\Lambda^2}}$.*

Remark 2.1.1. *Additionally, it is shown that $\phi'(t) \to 0$ as $|t| \to \infty$ in the three preceding theorems. Furthermore, both V and V' exhibit this same behavior in Theorems 2.1.2 and 2.1.3.*

The Chapter is organized as follows. In Section 2.2, we investigate the existence of traveling waves in the special case where $\Lambda = 0$. This simplifies the problem to a single second-order nonlinear differential equation. By transforming it into a first-order system and employing upper and lower solution techniques, we prove Theorem 2.1.1 and the estimates of the threshold value σ^*. In Section 2.3, we extend the values of σ identified for the case $\Lambda = 0$ to small positive Λ. This extension is accomplished through the use of geometric singular perturbation theory, which enables us to establish the existence of solutions for these small Λ values. In Section 2.4, we address the general case without setting any parameters to zero or consider small values of Λ. We apply a truncation argument, reducing the problem to a boundary value problem on a finite interval, and substitute the Fisher term with a combustion term that converges to the Fisher term in the limit. For this modified problem, we analyze the existence of a solution within a bounded domain using topological degree theory and fixed-point theory. Finally, we take the limit to prove the existence of traveling waves for the original problem.

2.2 Local Advection term

In this section, we will prove Theorem 2.1.1. The case $\Lambda = 0$ simplifies the problem (2.1), reducing it to a second-order differential equation.

$$\begin{aligned} &-\sigma\phi' - a(\phi')^2 = (1 + b(\phi')^2)\phi'' + \phi(1-\phi), \\ &\phi(-\infty) = 1, \quad \phi(+\infty) = 0. \end{aligned} \tag{2.2}$$

We will reduce the study of the existence of solutions of (2.2) to the study of the first-order equation:

$$\begin{aligned} S' &= \frac{\sigma S - aS^2 - \phi(1-\phi)}{S(1+bS^2)}, \\ S(0) &= S(1) = 0,\ S(\phi) > 0\ ,\phi \in (0,1), \end{aligned} \tag{2.3}$$

where $S : [0,1] \to [0,\infty)$ satisfies $S(\phi) = -\phi'$.

Proposition 2.2.1. *There exists a monotone decreasing heterocline solution to the problem (2.2) if and only if there exists $S \in C[0,1] \cap C^1(0,1)$ that satisfies (2.3).*

Proof. If ϕ is a solution of (2.2), we can show that $\phi'(s) < 0$. This allows us to define $S \in C^1(0,1)$. Such reduction principles are frequently employed (see [99]) and their proof is standard. □

First, we will establish an a priori bound on the wave speed σ, showing that if $\sigma < 2$, no monotonically decreasing solutions exist. This is due to the fact that when $\sigma < 2$, the solutions around $\phi = 0$ exhibit oscillatory behavior. This is the central idea behind the following proposition.

Proposition 2.2.2. *If $\sigma < 2$, then there are no solutions to (2.3).*

Proof. Let $S(\phi)$ be a solution of (2.3). According to Proposition 2.2.1, $S(\phi) = -\phi'$ for $\phi \in (0,1)$, and $\phi'(t) \to 0$ as $\phi(t) \to 0$, that is, as $t \to +\infty$.

Let us define the function $r(t) = -\frac{\phi'(t)}{\phi(t)} = \frac{S(\phi(t))}{\phi(t)}$. This function $r(t)$ satisfies the differential equation:

$$r' = \frac{S'\phi'\phi - \phi' S}{\phi^2} = r^2 - S' r.$$

Since S is a solution of (2.3), it follows that

$$r' = r^2 - \left(\frac{\sigma - aS}{1+bS^2}\right) r + \frac{1-\phi}{1+bS^2}.$$

holds. We can observe that r' is expressed as a second-degree polynomial in r. Given the convexity of this polynomial, we obtain

$$r' \geq \frac{1-\phi}{1+bS^2} - \left(\frac{\sigma - aS}{2(1+bS^2)}\right)^2 \to 1 - \frac{\sigma^2}{4},$$

as $t \to +\infty$. Therefore, if $\sigma < 2$, then r is eventually decreasing and has limit. But this leads to a contradiction: If $r(t)$ is bounded and $r(t) \to \bar{r} \in \mathbb{R}$, then by considering a sequence $\{t_n\}_n$ tending to infinity where $r'(t_n) \to 0$, we derive

$$\bar{r}^2 - \sigma \bar{r} + 1 = 0.$$

This implies that $\bar{r}$ would need to be a root of this second-degree polynomial. However, since $\sigma < 2$, the polynomial has no real roots, leading to a contradiction.

On the other hand, if $r(t) \to +\infty$, then $\frac{r'(t)}{r^2(t)} \to 1$, and there exists a constant $C > 0$ such that $\frac{r'(t)}{r^2(t)} \geq C$ for large t. This implies that $r(t)$ must be larger than the solution to the differential equation $y' = Cy^2$. However, the solution to this equation blows up in finite time.

In summary, we have demonstrated that if $\sigma < 2$, then the problem (2.3) admits no solution. □

To establish the existence of a solution, we will use the following result, which relies on the concept of finding appropriate functions (subsolutions) that help control the evolution of solutions to (2.3) and, consequently, demonstrate the existence of a solution.

Lemma 2.2.1. *Let $\bar{S} \in C^1([0,1]) \cap C((0,1))$ satisfying:*

$$\begin{aligned} &\bar{S}'(\phi) < \frac{\sigma \bar{S}(\phi) - a\bar{S}(\phi)^2 - \phi(1-\phi)}{\bar{S}(\phi)(1 + b\bar{S}^2(\phi))}, \ \phi \in (0,1) \\ &\bar{S}(0) = \bar{S}(1) = 0, \ \bar{S}(\phi) > 0 \ \phi \in (0,1). \end{aligned} \tag{2.4}$$

Then, there exists a solution S to the problem (2.3).

Proof. Consider Z satisfying (2.4). Let us construct the sequence of functions $\{S_n\}_{n\geq 1}$, where S_n is the maximal solution of the initial value problem:

$$\begin{aligned} &S' = \tfrac{\sigma S - aS^2 - \phi(1-\phi)}{S(1+bS^2)}, \ \phi \in (0, 1-1/n), \\ &S(1-1/n) = Z(1-1/n). \end{aligned} \tag{2.5}$$

The sequence constructed in this way satisfies $S_{n+1}(\phi) \leq S_n(\phi)$ for all $\phi \in (0, 1-1/n)$ and $n \in \mathbb{N}$. Furthermore, the inequalities $0 < S_n(\phi) < Z(\phi)$ hold, for all $\phi \in (0, 1-1/n)$ and $n \in \mathbb{N}$.

By adapting the proof scheme outlined in [99, Theorem 2.1], we obtain the desired result. This version is already quite clear and concise, so only minor adjustments were made for readability. □

As a consequence of the monotone dependence of equation (2.2) on σ, the solutions exhibit an ordered structure for different values of σ. We can thus establish the following result:

Proposition 2.2.3. *The set of admissible values*

$$\{\sigma > 0 : \textit{such that } (2.3) \textit{ has solution}\} \tag{2.6}$$

forms a closed, upper unbounded interval. The minimum of this interval is denoted by $\sigma^* := \sigma^*(a, b)$.

Proof. First, we will prove that (2.6) is an open, upper unbounded interval. Let us consider a solution S_1 of (2.3) with $\sigma = \sigma_1$. If we take $\sigma_2 \geq \sigma_1$, then S_1 serves as a lower solution to the problem (2.3) with $\sigma = \sigma_2$. According to Lemma 2.2.1, this implies the existence of a solution for σ_2. Hence, if there exists a solution for σ_1, then there exists a solution for all $\sigma \in (\sigma_1, +\infty)$.

This allow us to define $\sigma^* := \sigma^*(a, b)$ as the infimum of this interval. Furthermore, by Proposition 2.2.2, we have $\sigma^* \geq 2$. Now, we need to show that the interval (2.6) is also closed.

Assume $\sigma > \sigma^*$. Then, we know that there exist $\bar{\sigma}$ and $\hat{\sigma}$ such that the problem (2.2) has a solution for $\sigma^* < \hat{\sigma} < \sigma < \bar{\sigma}$. Using Proposition 1, let $\bar{S}$ and $\hat{S}$ be the solutions of problem (2.3) associated with $\bar{\sigma}$ and $\hat{\sigma}$, respectively. Due to the monotonicity of (2.3) with respect to σ, $\bar{S}$ and $\hat{S}$ are subsolution and supersolution of (2.3), respectively.

Let $m = \min\{\bar{S}, \hat{S}\}$ and $M = \max\{\bar{S}, \hat{S}\}$. Consider the sequence of compact intervals $\{A_n\}_n \subset (0,1)$ whose union covers the interval $(0,1)$. For each n, we can find a solution $S_n^1(\phi)$ to the problem (2.3) such that $m(\phi) \leq S_n^1(\phi) \leq M(\phi)$, for $\phi \in A_n$.

Let S_n be the maximal solution of (2.3) in A_n, extended to the entire interval $(0,1)$ by a constant. Then, $S_n \geq S_{n+1}$ in A_n, and the sequence $\{S_n\}_n$ converges to a function S_0 on $(0,1)$ as $n \to \infty$. Moreover, this convergence is uniform on compact sets of $(0,1)$ because S_n' is bounded on $(0,1)$. Let us analyze what happens at the endpoints of the interval $(0,1)$. We know that $0 < m(\phi) \leq S_n^1(\phi) \leq M(\phi)$ for $\phi \in (0,1)$, and that the functions $m(\phi)$ and $M(\phi)$ are continuous on $[0,1]$ and vanish at the boundary. Therefore, $S_0(\phi)$ will also vanish at the boundary.

We have thus proved that S_0 is a solution of (2.3), implying the existence of a solution for (2.2) when $\sigma > \sigma^*$.

Now, we only need to analyze the case $\sigma = \sigma^*$. To handle this, define a decreasing sequence $\{\sigma_n\}_n$ converging to σ^* with $\sigma_n \leq \bar{\sigma}$ for all $n \in \mathbb{N}$. Let $\{S_n\}_n$ be the sequence of solutions to (2.3) for $\sigma = \sigma_n$.

Define $S_0 = \inf_{n\in\mathbb{N}} S_n(\phi)$ for $\phi \in [0,1]$, and verify that $\inf_{\phi\in C} S_0(\phi) > 0$ for every compact subset $C \subset (0,1)$. Assume, by contradiction, that there exists a compact set $C \subset (0,1)$ such that $\inf_{\phi\in C} S_0(\phi) = 0$. In other words, there exists a sequence $\{\phi_n\}_n$ in C converging to $\phi^* \in C$ such that $S_0(\phi_n) < \frac{1}{n}$. Additionally, we can find a sequence $\{k_n\}_n$ such that

$$S_{k_n}(\phi_n) < \frac{1}{n}, \quad n \in \mathbb{N}.$$

Let $I \subset (0,1)$ be a compact interval such that $C \subset \mathrm{int}(I)$. We know that if the solution to problem (2.3) approaches zero, it can only do so in the region where $S' < 0$. Therefore, searching for $S_{k_n}(\phi_n) < \frac{1}{n}$, we have $S_{k_n}'(\phi) < 0$ for all $\phi \in [\phi_n, \max I]$. This is a contradiction since $S_{k_n}(\phi_n) \to 0$, and $\{\phi_n\}_n$ converges to $\phi^* < \max I$, but S_{k_n} is a solution of (2.3) and satisfies $S_{k_n}(\phi) > 0$ for all $\phi \in (0,1)$.

Thanks to the continuity and differentiability of the solutions of (2.3), the sequence $\{S_n\}_n$ is bounded on any compact set $C \subset (0,1)$, implying that it is Lipschitz on any compact set C.

Let $\{C_k\}_k$ be a sequence of increasing compact subsets in $(0,1)$ whose union is a covering of $(0,1)$. For $k = 1$, there exists a subsequence $\left\{S_{k_n^1}\right\}_n$ that converges uniformly to a function S^1 on C_1, which is continuous and positive on C_1. If we now consider $k = 2$, the subsequence $\left\{S_{k_n^1}\right\}_n$ admits a further subsequence $\left\{S_{k_n^2}\right\}_n$ that converges uniformly to a function S^2 on C_2, which is continuous and positive on C_2, coinciding with S^1 on C_1. By continuing this diagonal extraction procedure, we can define a function S^* that is continuous and positive on $(0,1)$, coinciding with S^n on C_n for every $n \in \mathbb{N}$, and S^* is the uniform limit of the sequence of solutions of the equation (2.3) related to the problem (P_{c_n}).

Therefore, taking the limit, S^* is a solution of (2.3) for $\sigma = \sigma^*$ on $(0,1)$. We have $S^*(\phi) > 0$ for $\phi \in (0,1)$, and if we take $\tilde{\sigma} > \sigma^*$, it holds that $S_{\tilde{\sigma}}(\phi) \geq S_{\sigma^*}(\phi)$ for every $\phi \in [0,1]$. Consequently, $S^*(0) = S^*(1) = 0$.

In particular, this demonstrates that σ^* is the infimum of the interval (2.6). □

Let us now conclude the proof of statement (T1) in Theorem 2.1.1. To achieve this, we will utilize Lemma 2.2.1 for a function of the form $S(\phi) = \alpha\phi(1-\phi)$.

Proposition 2.2.4. *If $a < 2$, $b < 2$, and $\sigma = 2$, then there exists a solution to* (2.2).

Proof. We want to verify that $S(\phi) = \alpha\phi(1-\phi)$ satisfies (2.4) for some $\alpha \in \mathbb{R}$. To do this, we need to check that the following condition holds:

$$\frac{\sigma\alpha\phi(1-\phi) - a\alpha^2\phi^2(1-\phi)^2 - \phi(1-\phi)}{\alpha\phi(1-\phi)(1+b\alpha^2\phi^2(1-\phi^2))} > \alpha - 2\alpha\phi,$$

for $\phi \in (0,1)$. Rearranging, we arrive at the following expression

$$2\phi^5 - 5\phi^4 + 4\phi^3 + \frac{a - b\alpha^2}{b\alpha^2}\phi^2 + \frac{(2-a)}{b\alpha^2}\phi + \frac{\sigma\alpha - 1 - \alpha^2}{b\alpha^4} > 0,$$

for $\phi \in (0,1)$. The polynomial $2\phi^5 - 5\phi^4 + 4\phi^3$ is greater than zero, for $\phi \in (0,1)$. It is sufficient to study when

$$\frac{a - b\alpha^2}{b\alpha^2}\phi^2 + \frac{(2-a)}{b\alpha^2}\phi + \frac{\sigma\alpha - 1 - \alpha^2}{b\alpha^4} > 0, \text{ for } \phi \in (0,1).$$

Taking $\alpha = 1$ and $\sigma = 2$, we only need to verify the condition when

$$\phi((a-b)\phi + (2-a))$$

is positive for $\phi \in (0,1)$, This requires $a < 2$ and $b < 2$. Therefore, Lemma 2.2.1 leads to the existence of a solution. □

The next step is to establish the upper bounds for $\sigma^*(a,b)$. We will start by examining the bounds given in statement (T2). Utilizing Lemma 2.2.1 and the solutions from the FKPP model, we will derive the following result.

Proposition 2.2.5. *Let $b = 0$ and*

$$\sigma > \begin{cases} 2 + \frac{a}{8} & si \quad a \leq 16, \\ \sqrt{a} & si \quad a > 16, \end{cases} \tag{2.7}$$

Then, there exists a solution to (2.2).

Proof. Let S_F be a solution of

$$\begin{aligned} &S' = \tfrac{cS - \phi(1-\phi)}{S}, \\ &S(0) = S(1) = 0, \; S(\phi) > 0, \; \phi \in (0,1). \end{aligned} \tag{2.8}$$

It is well known that this first-order equation corresponds to the problem

$$\begin{aligned} -c\phi' &= \phi'' + \phi(1-\phi), \\ \phi(-\infty) &= 1, \quad \phi(+\infty) = 0, \end{aligned} \tag{2.9}$$

whose differential equation is the FKPP equation. It is well established that solutions exist for $c \geq 2$. Moreover, $S_F(\phi) < \frac{1}{4c}$ is satisfied, for $\phi \in (0,1)$.

It can be checked that (2.7) is equivalent to

$$a \leq \begin{cases} 8\sigma - 16 & \text{if } \sigma < 4, \\ \sigma^2 & \text{if } \sigma \geq 4. \end{cases}$$

Let us demonstrate that, under these conditions, there always exists $c \geq 2$ that satisfies the inequality

$$S_F' < \frac{\sigma S_F - a S_F^2 - \phi(1-\phi)}{S_F},$$

for $\phi \in (0,1)$. By using Lemma 2.2.1, the expression (2.8) and the fact that $S_F(\phi) < \frac{1}{4c}$, it is enough to prove that

$$cS < \sigma S - aS^2, \quad \text{for } S \in \left(0, \frac{1}{4c}\right).$$

Or equivalently, if there exists $c \geq 2$ satisfying the above inequality for $S = \frac{1}{4c}$, i.e. , if it verifies

$$c^2 - \sigma c + \frac{a}{4} < 0. \tag{2.10}$$

If $\sigma < 4$, then (2.10) holds, for $c \geq 2$, if

$$\frac{\sigma + \sqrt{\sigma^2 - a}}{2} > 2,$$

which is true for all $a \leq 8\sigma - 16$.

If $\sigma \geq 4$, then (2.10) holds, for $c \geq 2$, if $a < \sigma^2$. □

The other bounds on σ^*, both for $b = 0$ and $b > 0$, rely on finding an $\alpha \in (0, \infty)$ such that $S(\phi) = \alpha\phi(1-\phi)$ satisfies (2.4). We have the following result:

Proposition 2.2.6. *Let us assume that we are in one of the following situations:*

1. $b = 0$, $a > 2$ *and* $\sigma > \sqrt{\frac{4+a^2}{a}}$.
2. $b > 0$ *and* $\sigma > \sqrt{\frac{\sqrt{a^2+8a+4b+16}+a+4}{2}}$.

Then, there exists a solution of (2.2).

Proof. Let $S = \alpha\phi(1-\phi)$, for a certain $\alpha > 0$. We want to determine the values of α that satisfy the following inequality:

$$\frac{\sigma\alpha\phi(1-\phi) - a\alpha^2\phi(1-\phi) - \phi(1-\phi)}{\alpha\phi(1-\phi)(1 + b\alpha^2\phi^2(1-\phi^2))} > \alpha - 2\alpha\phi, \tag{2.11}$$

for $\phi \in (0, 1)$.

Consider $b = 0$. Reordering (2.11), we obtain

$$F(\phi) = a\alpha^2\phi^2 + (2\alpha^2 - a\alpha^2)\phi + (\sigma\alpha - \alpha^2 - 1) > 0, \text{ for } \in (0, 1)$$

We need to determine the conditions under which the quadratic function $F(\phi)$ takes positive values for $\phi \in [0, 1]$.

If $a > 2$, we require that $F(\phi)$ has a negative discriminant, obtaining

$$\alpha^2(a^2 + 4) - 4a\sigma\alpha + 4a < 0.$$

Again, we have a quadratic function in α, then the existence of values of α verifying such equation reduces to the computation of the discriminant:

$$a\sigma^2 - (a^2 + 4) > 0.$$

This yields the condition on σ, which ensures that $F(\phi)$ is positive for $\phi \in [0, 1]$.

Let us now consider $b > 0$. To prove (2.11), it is sufficient to impose

$$\frac{\sigma\alpha - a\alpha^2\phi(1-\phi) - 1}{\alpha(1 + b\alpha^2\phi^2(1-\phi^2))} > \alpha,$$

for $\phi \in]0, 1[$. The expression above can be simplified by rewriting it as a function of $z = \phi(1-\phi)$. After substituting z and performing the necessary simplifications, we obtain:

$$b\alpha^4 z^2 + a\alpha^2 z + 1 - \sigma\alpha + \alpha^2 < 0,$$

for $z \in [0, 1/4]$.

It can be observed that this expression is an increasing function of z. Thus, the only thing to check is the existence of a value $\alpha > 0$ such that

$$\frac{b\alpha^4}{16} + \frac{a\alpha^2}{4} + 1 - \sigma\alpha + \alpha^2 < 0.$$

The expression provided in the Proposition is derived by setting $\alpha = \frac{2}{\sigma}$. By applying Lemma 2.2.1, we establish the existence of a solution. $\square$

To complete our analysis, we now focus on determining the lower bounds of σ^*. For this, we use a concept analogous to that in Lemma 2.2.1, but our aim is to identify a function that facilitates the demonstration of non-existence of solutions. We present the following result:

Proposition 2.2.7. *There is no solution to (2.2) if any of the following conditions hold:*

- $b = 0$, $a \geq 3 + 2\sqrt{2}$ *and* $\sigma < \frac{(a-1)}{\sqrt{a}}$.
- $b > 0$, $a > 2$, $a \geq 2\sqrt{b}$ *and* $\sigma < \frac{2b-(a-\sqrt{a^2-4b})}{\sqrt{2b(a-\sqrt{a^2-4b})}}$.

Proof. Let us consider the problem (2.2) and rewrite it in terms of $\psi = 1 - \phi$.

$$\begin{aligned} &\sigma\psi' - a(\psi')^2 = -(1 + b(\psi')^2)\psi'' + \psi(1-\psi), \\ &\psi(-\infty) = 0, \quad \psi(+\infty) = 1. \end{aligned} \tag{2.12}$$

The corresponding first order equation is

$$J' = \frac{aJ^2 - \sigma J + \psi(1-\psi)}{J(1+bJ^2)}. \tag{2.13}$$

In the vicinity of 0, this equation exhibits two real eigenvalues with opposite signs. Consequently, the unstable manifold corresponds to the eigenvalue $\lambda = \frac{-\sigma + \sqrt{\sigma^2+4}}{2}$.

Let $J(\psi) = \lambda\psi - \mu\psi^2$, where μ is chosen such that

$$J' < \frac{aJ^2 - \sigma J + \psi(1-\psi)}{J(1+bJ^2)},$$

for $\psi \in (0,1)$. We aim to demonstrate that any solution S of the problem (2.13), initiated from the unstable manifold associated with the eigenvalue λ, will satisfy $S(\psi) \geq J(\psi)$ for $\psi \in (0,1)$ and also $S(1) > \frac{\lambda}{\mu} \geq 1$. Consequently, solutions to (2.13) cannot satisfy $S(0) = S(1) = 0$, indicating that the problem (2.2) has no solution.

Consider S_n as the solution of (2.13) with initial condition $S_n(0) = \frac{1}{n}$. It follows that $S_n(\psi)$ is a decreasing sequence for all $n \geq 1$ due to the uniqueness of the solution.

Define $R_n(\psi) = S_n(\psi) - J(\psi)$. We have $R_n(0) > 0$ and $R_n'(\psi) > 0$, which implies $R_n(\psi) > 0$ for $\psi \geq 0$. Hence, $S_n(\psi) > J(\psi)$ for $\psi \geq 0$.

Consequently, the sequence $\{S_n(\psi)\}_{n\geq 1}$ and its derivative are uniformly bounded because $J(\psi) < S_n(\psi) < S_1(\psi)$ for all $\psi \in (0,1)$ and $n \geq 1$. Therefore, we can extract a convergent subsequence $S_{n_k} \to \bar{S}(\psi)$, which satisfies $\bar{S}(0) = 0$, $\bar{S}(1) > 1$, and $\bar{S}$ is a solution of (2.13) for $\psi \in (0,1)$.

Moreover, due to the uniqueness of the solution from the stable manifold, the only solution emerging from $\psi = 0$ is $\bar{S}(\psi)$. This completes the proof.

We are seeking values of μ that satisfy the following inequality:

$$\frac{a(\lambda\psi - \mu\phi^2)^2 - \sigma(\lambda\psi - \mu\psi^2) + \psi(1-\psi)}{(\lambda\psi - \mu\psi^2)(1 + b(\lambda\psi - \mu\psi^2)^2)} > (\lambda - 2\mu\psi),$$

for all $\psi \in (0,1)$, or equivalently

$$\begin{aligned} &\Big(- a\mu^2\psi^4 + \left(2\mu^2 + 2a\lambda\mu\right)\psi^3 - \left(3\mu\lambda + a\lambda^2 + \sigma\mu - 1\right)\psi^2 + (\lambda^2 + \sigma\lambda - 1)\psi\Big) \\ &+ b\left(2\mu^4\psi^7 - 7\mu^3\lambda\psi^6 + 9\mu^2\lambda^2\psi^5 - 5\lambda^3\mu\psi^4 + \lambda^4\psi^3\right) < 0, \end{aligned} \tag{2.14}$$

for all $\psi \in (0,1)$.

First we analyze the case $b = 0$. We start by noting that λ was defined such a way it satisfies the quadratic equation:

$$\lambda^2 + \sigma\lambda - 1 = 0.$$

This implies that the linear term in ψ is zero. Consequently, the inequality that must be satisfied is:

$$\psi^2\left(-a\mu^2\psi^2 + (2\mu^2 + 2a\lambda\mu)\psi - (3\mu\lambda + a\lambda^2 + \sigma\mu - 1)\right) < 0.$$

To determine whether the inequality is satisfied, we need to evaluate the quadratic function at specific values of ψ. We want to ensure that this function takes negative values at $\psi = 1$ and that the maximum of the function occurs at a value ψ^* where $\psi^* > 1$. Let us check that for $\psi = 1$ this is satisfied. This leads to

$$(2-a)\mu^2 + (2a\lambda - 3\lambda - \sigma)\mu + (1 - a\lambda^2) < 0.$$

If $1 - a\lambda^2 < 0$, then the inequality is satisfied for values of μ close to zero. This condition $1 - a\lambda^2 < 0$ follows from the hypothesis of the Proposition, and it can be verified through a straightforward calculation using $\lambda = \frac{-\sigma+\sqrt{\sigma^2+4}}{2}$.

Let us finally see that ψ^* is greater than 1:

$$\psi^* = \frac{2\mu^2 + 2a\lambda\mu}{2a\mu^2} > 1 \Leftrightarrow \mu < \frac{a\lambda}{a-1},$$

but this is satisfied, since $\mu \leq \lambda$ and $a > 2$ by hypothesis.

In the case $b > 0$, the inequality to examine is

$$\begin{aligned} &\left[-a\mu^2\psi^2 + \left(2\mu^2 + 2a\lambda\mu\right)\psi - \left(3\mu\lambda + a\lambda^2 + \sigma\mu - 1\right)\right] \\ &\quad + b\left[2\mu^4\psi^5 - 7\mu^3\lambda\psi^4 + 9\mu^2\lambda^2\psi^3 - 5\lambda^3\mu\psi^2 + \lambda^4\psi\right] < 0. \end{aligned} \tag{2.15}$$

Moreover, the following inequality

$$b\left[2\mu^4\psi^5 - 7\mu^3\lambda\psi^4 + 9\mu^2\lambda^2\psi^3 - 5\lambda^3\mu\psi^2 + \lambda^4\psi\right] < b\psi\lambda^4$$

holds, for $\psi \in (0,1)$. This can be proven by taking $z = \frac{\mu}{\alpha}\psi$, for $z \in (0, \frac{\mu}{\lambda}) \subset (0,1)$. We can then rewrite the above expression as follows:

$$b\left[2z^5 - 7z^4 + 9z^3 - 5z^2 + z\right] \leq bz,$$

which can be easily verified to hold.

Therefore, it is sufficient to determine for which values of μ the following inequality is satisfied:

$$\left[-a\mu^2\phi^2 + \left(2\mu^2 + 2a\lambda\mu + b\lambda^4\right)\phi - \left(3\mu\lambda + a\lambda^2 + \sigma\mu - 1\right)\right] < 0.$$

It can be checked that the vertex of this parabola is always greater than one, and there always exists a μ in a neighborhood of 0 such that:

$$2(1-a)\mu^2 + 2a\lambda\mu + b\lambda^4 > 0.$$

Also, by evaluating the polynomial at $\psi = 1$, the inequality

$$(2-a)\mu^2 + (2a\lambda - 3\lambda - \sigma)\mu + (b\lambda^4 - a\lambda^2 + 1) < 0$$

is satisfied in a neighborhood of 0, as

$$(b\lambda^4 - a\lambda^2 + 1) < 0$$

holds for σ given by the expression in the statement of the Proposition. $\square$

Once these intermediated results have been presented, we can complete the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. Proposition 2.2.3 establishes the existence of $\sigma^* := \sigma^*(a,b)$ such that, for all $\sigma \geq \sigma^*$, there exists a traveling wave. From Proposition 2.2.2, we know $\sigma^* \geq 2$. Furthermore, Proposition 2.2.4 shows that $\sigma^*(a,b) = 2$ if $\max\{a,b\} \leq 2$. This confirms $(T1)$.

For the case $b = 0$ (i.e., $(T2)$), Proposition 2.2.7 provides that $\sigma^* \geq \frac{a-1}{\sqrt{a}}$, given that $a \geq 3 + 2\sqrt{a}$.

On the other hand, for the study of upper bounds, we have Proposition 2.2.5 and Proposition 2.2.6. Let us determine the optimal upper bound estimates for $\sigma^*(a,0)$. For $a > 16$, the best estimate is clearly $\sigma(a,b) > \sqrt{a}$.

Let us examine the case $a \in (2,16)$. By equating the estimates in this interval, $\sqrt{\frac{a^2+4}{a}} = 2 + \frac{a}{8}$, we obtain the following equation:

$$p(a) = a^3 - 32a^2 + 256a - 256 = 0.$$

From this, we have $p(2) > 0$ and $p(16) < 0$. Therefore, there is a unique root a^* in the interval $(2,16)$, which implies a unique intersection point of the two graphs. Note that for $a \geq 16$, the function $\sqrt{\frac{a^2+4}{a}}$ exceeds $\sqrt{a}$.

We find the following estimates for $\sigma^*(a,0)$:

$$\sigma^*(a,0) \leq \begin{cases} \sqrt{\frac{a^2+4}{a}} & \text{if } 2 \leq a \leq a^*, \\ 2 + \frac{a}{8} & \text{if } a^* \leq a \leq 16, \\ \sqrt{a} & \text{if } a > 16. \end{cases}$$

Finally, let us prove $(T3)$, which studies the case $b > 0$. Proposition 2.2.6 provides the upper bound for σ^*, and Proposition 2.2.7 establishes that if $a^2 \geq 4b$, the lower bound for σ^* is no longer 2, but $\sigma^* \geq \sqrt{\frac{\sqrt{a^2+8a+4b+16}+a+4}{2}}$. $\square$

2.3 Singular Perturbation Theory (Small Λ)

In this section, we will address the case where $\Lambda > 0$ is small. Our approach involves leveraging the results obtained for $\Lambda = 0$ and applying geometric singular perturbation theory, as developed by Fenichel [72], to extend the existence of solutions to small values of Λ. Specifically, if a parameter set (σ, a, b) permits the existence of a subsolution as described in (2.4), then traveling waves will also exist for sufficiently small values of Λ.

Let us rewrite (2.1) as a first-order system in the following form:

$$\begin{aligned} \phi' &= \psi, \\ \psi' &= -\sigma\psi + V\psi - \phi(1-\phi), \\ \Lambda V' &= W, \\ \Lambda W' &= (1 + b\psi^2)V + a\psi - b\sigma\psi^2 - b\psi\phi(1-\phi). \end{aligned} \tag{2.16}$$

By making the change of variable $\xi = \Lambda\eta$, we obtain:

$$\begin{aligned}
\dot{\phi} &= \Lambda\psi, \\
\dot{\psi} &= \Lambda(-\sigma\psi + V\psi - \phi(1-\phi)), \\
\dot{V} &= W, \\
\dot{W} &= (1 + b\psi^2)V + a\psi - b\sigma\psi^2 - b\psi\phi(1-\phi),
\end{aligned} \tag{2.17}$$

where we denote $\frac{d}{d\xi} = \,'$ and $\frac{d}{d\eta} = \dot{}$. The set of critical points of (2.17) for $\Lambda = 0$ is defined by:

$$M_0 := \left\{ (\phi, \psi, V, W) \in \mathbb{R}^4 \mid W = 0, V = \frac{\psi(-a + b\sigma\psi + b\phi(1-\phi))}{(1 + b\psi^2)} \right\} \tag{2.18}$$

Note also that the flow of (2.16) is confined to M_0 when $\Lambda = 0$.

The perturbation theory proposed in [72] provides a manifold M_Λ, which is close to M_0 in a sense that will be specified, and is invariant under the flow associated to (2.17). This manifold lies within a neighborhood of $\Lambda = 0$, specifically $\mathcal{O}(\Lambda)$. This framework allows us to study the problem (2.16) restricted to the manifold M_Λ. To apply this theory, we will use the version established in [91], which has been successfully employed by various authors, see for instance [3, 79] and the references therein.

To apply the theorem, it is essential to verify that M_0 is normally hyperbolic. This requires demonstrating that the Jacobian matrix of the system described by (2.17) at points on M_0 has as many eigenvalues with zero real part as the complementary dimension of the manifold M_0. Specifically, we need to show that exactly two eigenvalues have zero real part. The Jacobian of (2.17) is given by:

$$\begin{bmatrix}
0 & \Lambda & 0 & 0 \\
\Lambda(1-2\phi) & \Lambda(-\sigma + V) & \Lambda\psi & 0 \\
0 & 0 & 0 & 1 \\
-b\psi(1-2\phi) & 2b\psi V + a - 2b\sigma\psi - b\phi(1-\phi) & (1 + b\psi^2) & 0
\end{bmatrix}$$

At $\Lambda = 0$, the eigenvalues of the Jacobian are $\lambda = 0$ with multiplicity two, and $\lambda = \pm\sqrt{1 + b\psi^2}$. Consequently, M_0 is normally hyperbolic.

Let B_R represent the ball of radius R centered at the origin in $\mathbb{R}^2$.

Proposition 2.3.1. [91, Theorem 1]. *If M_0 is a normally hyperbolic manifold, then for any $R > 0$, for any open interval I containing σ, and for any $k \in \mathbb{N}$, there exists a $\Lambda_0 > 0$, depending on R, I, and k, such that for all $\Lambda \in (0, \Lambda_0)$, there exists a manifold M_Λ, given by*

$$\begin{aligned}
M_\Lambda = \Big\{ (\phi, \psi, V, W) \in \mathbb{R}^4 \, & | W = g(\phi, \psi, \Lambda, \sigma), \\
& V = f(\phi, \psi, \Lambda, \sigma), \ (\phi, \psi) \in B_R, \sigma \in I \Big\},
\end{aligned}$$

where f and g are functions in $C^k\left(\overline{B_R} \times \bar{I} \times [0, \Lambda_0]\right)$. This manifold M_Λ is locally invariant under the flow of the system (2.17).

This result implies that for any compact subset of M_0 that includes the critical points of interest, we can identify an invariant manifold for the system (2.17) that lies within an $\mathcal{O}(\Lambda)$ neighborhood of the manifold M_0. Moreover, there exist functions $f(\phi, \psi, \Lambda, \sigma)$ and $g(\phi, \psi, \Lambda, \sigma)$ with any desired level of regularity. This allows us to reduce the order of the equations in our system and focus on studying the system in the form:

$$\begin{aligned} \phi' &= \psi, \\ \psi' &= -\sigma\psi + f(\phi, \psi, \Lambda, \sigma)\psi - \phi(1-\phi), \end{aligned} \tag{2.19}$$

Since the functions f and g are in $C^k\left(\overline{B_R} \times \bar{I} \times [0, \Lambda_0]\right)$, they can be expanded in a Taylor series in Λ as follows:

$$f(\phi, \psi, \Lambda, \sigma) = \sum_{n=0}^{k} f_n(\phi, \psi, \sigma)\Lambda^n + F(\phi, \psi, \Lambda, \sigma)\Lambda^k,$$

where $F(\phi, \psi, \Lambda, \sigma)$ is a continuous function in Λ with $F(\phi, \psi, 0, \sigma) = 0$. A similar expansion applies to the function g.

Observe that, by construction of M_0 (2.18), $f_0 = \frac{\psi(-a+b\sigma\psi+b\phi(1-\phi))}{1+b\psi^2}$ and $g_0 = 0$. The remaining terms can be determined using the fact that the vector field of (2.17) is perpendicular to the normals of M_Λ. Substituting $f = f_0 + \Lambda\bar{f}$, where $\bar{f}$ is the remainder in the Taylor expansion of f, into (2.19), we obtain:

$$\begin{aligned} &\phi' = \psi, \\ &\psi' = \frac{-\sigma\psi - a\psi^2 + \Lambda\bar{f}(\phi, \psi, \Lambda, \sigma)(1+b\psi^2)\psi - \phi(1-\phi)}{1+b\psi^2}, \\ &\phi(-\infty) = 1,\ \phi(+\infty) = 0. \end{aligned} \tag{2.20}$$

Let us now prove the following result, which encapsulates the essence of Theorem 2.1.2.

Proposition 2.3.2. *Let (σ, a, b) be such that there exists a function $H(\phi)$ satisfying the subsolution condition (2.4). Then, there exists a $\bar{\Lambda}$ such that the problem (2.20) has a solution for all $\Lambda \in (0, \bar{\Lambda})$.*

Proof. First, choose $R > 0$ such that the ball B_R encompasses the critical points $(0,0)$, $(1,0)$, and the subsolution $S(\phi)$. We then apply Theorem 2.3.1 for this choice of R. Similarly to Proposition 2.2.1 and Lemma 2.2.1, we need to demonstrate that equation (2.20) has a solution if there exists a function $\bar{S} \in C^1([0,1]) \cap C((0,1))$ that satisfies the following conditions:

$$\begin{aligned} &\bar{S}'(\phi) < \frac{\sigma\bar{S}(\phi) - a\bar{S}(\phi)^2 - \Lambda\bar{f}(\phi, \bar{S}(\phi), \Lambda, \sigma)(1+b\bar{S}^2(\phi))\bar{S}(\phi) - \phi(1-\phi)}{\bar{S}(\phi)(1+b\bar{S}^2(\phi))}, \\ &\bar{S}(0) = \bar{S}(1) = 0, \quad \bar{S}(\phi) > 0. \end{aligned} \tag{2.21}$$

Now, consider $H(\phi)$ as defined in the statement of the Proposition, whose properties we need to establish to complete our argument. By the continuity of the function $\bar{f}$ and applying Theorem 2.3.1, we have $|\bar{f}(\phi, H(\phi), \sigma, \Lambda)| < K$ for $\phi \in (0,1)$ and $\Lambda \in (0, \Lambda_0)$. Substituting H into (2.21), we obtain:

$$H'(\phi) < \frac{\sigma H(\phi) - aH(\phi)^2 - \Lambda \bar{f}(\phi, H(\phi), \Lambda, \sigma)(1 + bH^2(\phi))H(\phi) - \phi(1-\phi)}{H(\phi)(1 + bH^2(\phi))}.$$

To demonstrate that H is a subsolution, it is sufficient to verify that:

$$H'(\phi) < \frac{\sigma H(\phi) - aH(\phi)^2 - \Lambda K(1 + bH^2(\phi))H(\phi) - \phi(1-\phi)}{H(\phi)(1 + bH^2(\phi))},$$

for $\phi \in (0,1)$. Equivalently,

$$H'(\phi) < \frac{\sigma H(\phi) - aH(\phi)^2 - \phi(1-\phi)}{H(\phi)(1 + bH^2(\phi))} - \Lambda K,$$

for $\phi \in (0,1)$. To prove this, it suffices to show that the following difference is uniformly bounded:

$$\frac{\sigma H(\phi) - aH(\phi)^2 - \phi(1-\phi)}{H(\phi)(1 + bH^2(\phi))} - H'(\phi) = \frac{\sigma - \sigma^*}{(1 + bH^2(\phi))}.$$

Since $H(\phi)$ is a solution to (2.3) for σ^*, it follows that $H(\phi)$ is uniformly bounded from below for all $\phi \in (0,1)$. Consequently, there exists a Λ_0 such that the inequality holds for all $\Lambda \in (0, \Lambda_0)$. This guarantees that $H(\phi)$ satisfies the required condition for a subsolution, thus establishing the existence of a solution and concluding the proof of the Proposition. □

A direct consequence of this result, assuming $H(\phi)$ as the solution to (2.3) for $\sigma = \sigma^*(a,b)$, is that it can be shown that for every $\sigma > \sigma^*(a,b)$, there exists a solution to (2.1) for small values of Λ. Proving the Theorem 2.1.2 through this method.

2.4 Non Local Advection Term ($\Lambda > 0$)

In this section, we will investigate the problem (2.1) for general values of $\Lambda > 0$. Unlike the small perturbation of a second-order system considered previously, this scenario involves a fourth-order system. As such, we must employ different techniques to establish the existence of traveling wave solutions. The methods used here are inspired by the work of various authors, including Berestycki, Henderson, Lions, Nadin, Perthame, and Ryzhik, as detailed in [20, 21, 86, 106] and the references therein.

To prove Theorem 2.1.3, we will consider a modified problem that introduces new parameters. Several adjustments to the original problem (2.1) are required for this purpose.

Firstly, we define a truncature function $g(\phi)$ as follows: $g(\phi) = 1$ for $\phi \geq 1$, $g(\phi) = 0$ for $\phi \leq \theta$, $g'(\phi) \geq 0$ for $\phi \in (\theta, 1)$, and $g(\phi) \to 1$ as $\theta \to 0$. Note that $g(\phi)$ is bounded by 1.

Additionally, it is necessary to work within a compact interval $[-\alpha, \alpha]$ and subsequently take the limit as α approaches $+\infty$.

In this way, we arrive at the following problem:

$$\begin{aligned} &\phi'' + \sigma\phi' + g(\phi)u'\phi' + g(\phi)\phi(1-\phi) = 0, \quad \xi \in [-\alpha, \alpha], \\ &\phi(-\alpha) = 1, \quad \phi(+\alpha) = 0, \quad \phi(0) = \theta, \end{aligned} \tag{2.22}$$

where

$$u(\xi) = a(\Gamma * \bar{\phi}) + \frac{b}{2}\left(\Gamma * (\bar{\phi}')^2\right), \tag{2.23}$$

being $\Gamma(\xi) = \frac{1}{2\Lambda}e^{-\frac{|\xi|}{\Lambda}}$, and $\bar{\phi}$ the extension by zero, for $\xi \in (\alpha, \infty)$ and by one, for $\xi \in (-\infty, \alpha)$. Later, V will be recovered as $-u'$ after taking the limit of the parameters.

Let us begin by proving the positivity and monotonicity of the solution ϕ.

Lemma 2.4.1. *Let $\phi : [-\alpha, \alpha] \to \mathbb{R}$ be a solution of (2.22), then $0 \leq \phi \leq 1$ and $\phi' < 0$, for $\xi \in (-\alpha, \alpha)$.*

Proof. Let us prove that $\phi(\xi) \leq 1$ for $\xi \in (-\alpha, \alpha)$. We proceed by reductio ad absurdum. Assume that $\phi(\xi) > 1$ for some ξ. Then, there exists a point $\bar{\xi}$ such that $\phi(\bar{\xi}) > 1$, $\phi'(\bar{\xi}) = 0$, and $\phi''(\bar{\xi}) \leq 0$. However, since ϕ is a solution of the differential equation, we have:

$$\phi''(\bar{\xi}) = -g(\phi)\phi(\bar{\xi})(1 - \phi(\bar{\xi})).$$

Given that $\phi(\bar{\xi}) > 1$ and $g(\phi)$ is positive (as defined previously), the term $\phi(\bar{\xi})(1 - \phi(\bar{\xi}))$ is negative. Consequently,

$$-g(\phi)\phi(\bar{\xi})(1 - \phi(\bar{\xi})) > 0,$$

implying that $\phi''(\bar{\xi}) > 0$. This is a contradiction because we assumed $\phi''(\bar{\xi}) \leq 0$. Therefore, our assumption that $\phi(\xi) > 1$ for some ξ must be false. Hence, $\phi(\xi) \leq 1$ for all $\xi \in (-\alpha, \alpha)$.

Let us analyze the monotonicity of the solution. Let ξ_0 be the first value after $-\alpha$ such that $\phi'(\xi_0) = 0$. The existence of this point is possible by the uniqueness of the initial value problem (I.V.P.) for $\phi'(-\alpha) \neq 0$.

From the differential equation, we have:

$$\phi''(\xi_0) = -g(\phi)\phi(\xi_0)(1 - \phi(\xi_0)) \leq 0.$$

If $\phi(\xi_0) > \theta$, then $g(\phi) > 0$ and thus $\phi''(\xi_0) < 0$, which means $\phi'(\xi)$ was increasing before reaching zero. This would imply that $\phi'(\xi) > 0$ in a neighborhood to the left of ξ_0, which contradicts ξ_0 being the first point where $\phi'(\xi) = 0$.

If $\phi(\xi_0) \leq \theta$, then $g(\phi) = 0$ because of the truncation function, making $\phi''(\xi_0) = 0$. This would imply that ϕ is a constant solution near ξ_0. However, by the uniqueness of the I.V.P., if ϕ were constant, it would contradict the existence of ξ_0 as the first point where $\phi'(\xi) = 0$ since $\phi'(\xi)$ would not change sign.

Therefore, ϕ must be strictly decreasing as it is not possible that $\phi'(\xi) > 0$ for all $\xi \in (-\alpha, \alpha)$.

From this, we deduce that $\phi(\xi) > 0$ for $\xi \in (-\alpha, \alpha)$. □

Remark 2.4.1. *In the following results, α will be considered fixed and will not be explicitly referenced unless it is necessary to indicate dependence on α. We will use the notation $\|\cdot\|_{L^p(-\alpha,\alpha)} := \|\cdot\|_p$ for $1 \leq p \leq \infty$ and $\alpha \in (0, \infty)$.*

Let us start by analyzing the bounds of u as a function of the norm of ϕ and ϕ'.

Lemma 2.4.2. *Let ϕ be a solution of (2.22), then we have*

$$\|u\|_\infty \leq a + \frac{b}{4\Lambda}\|\phi'\|_2^2,$$

$$\|u'\|_\infty \leq \frac{a}{\Lambda} + \frac{b}{4\Lambda^2}\|\phi'\|_2^2.$$

Proof. As a result of the construction of the solution u, and from the expression (2.23), we deduce that:

$$\begin{aligned} u(\xi) &= \int_{-\infty}^{+\infty} a\Gamma(x-y)\bar{\phi}(y) + \frac{b}{2}\Gamma(x-y)\bar{\phi}'^2(y)\,dy \\ &= a\int_{-\infty}^{+\infty} \Gamma(x-y)\bar{\phi}(y) + \frac{b}{2}\int_{-\alpha}^{\alpha} \Gamma(x-y)\phi'^2(y)\,dy \\ &\leq a \max_{y\in\mathbb{R}}\left\{\bar{\phi}(y)\right\}\int_{-\infty}^{+\infty}\Gamma(x-y)\,dy + \frac{b}{2}\max_{y\in\mathbb{R}}\{\Gamma(x-y)\}\int_{-\alpha}^{\alpha}\phi'^2(y)\,dy. \end{aligned}$$

Lemma 2.4.1 states that $\|\bar{\phi}\|_\infty \leq 1$. Combining this with $\|\Gamma\|_\infty \leq \frac{1}{2\Lambda}$, we obtain:

$$u(\xi) \leq a + \frac{b}{4\Lambda}\|\phi'\|_2^2,$$

for all $\xi \in [-\alpha, \alpha]$.

Given that $|\Gamma'(x)| = \frac{1}{\Lambda}\Gamma(x)$ for almost every $x \in \mathbb{R}$, the bound for u' turns out to be

$$\|u'\|_\infty \leq \frac{\|u\|_\infty}{\Lambda}.$$

This establishes the second inequality of the Lemma and completes the proof. ∎

Proposition 2.4.1. *Let ϕ be a solution to (2.22). Then, the following bound*

$$\sigma < 2 + \|u'\|_\infty$$

holds, for $\alpha > \alpha_0 := -\log(\theta)$.

Proof. Let us demonstrate that

$$\sigma + u'(\xi) \leq 2,$$

for some $\xi \in [-\alpha, \alpha]$. By applying this result along with Lemma 2.4.2, we can establish the statement of the Proposition.

The argument will proceed by reductio ad absurdum, assuming that $u'(\xi) > 2 - \sigma$ for all $\xi \in [-\alpha, \alpha]$. Let M be such that

$$\phi(\xi) < Me^{-\xi},\ \xi \in [-\alpha, \alpha]. \tag{2.24}$$

Consider the function

$$\Psi(\xi) = Me^{-\xi} - \phi(\xi),$$

which satisfies

$$-\sigma\Psi' - \Psi'' - u'\Psi' = (\sigma - 1 + \tau u')\Psi - g(\phi)\phi + g(\phi)\phi^2, \tag{2.25}$$

where $\psi(\xi) = Me^{-\xi}$. From the reductio ad absurdum hypothesis, it follows that

$$\sigma - 1 + g(\phi)u' > g(\phi).$$

Thus, the expression (2.25) is strictly positive, indicating that no local minimum exists, as the second derivative at a critical point is negative.

A straightforward approximation argument reveals that if we choose M_0 as the minimum value for which the condition holds, then the function $M_0e^{-\xi} - \phi(\xi)$ achieves its global minimum at either $\xi = -\alpha$ or $\xi = \alpha$. By optimality, this minimum value must be zero, thereby excluding $\xi = \alpha$. Taking $\xi = -\alpha$, we have

$$M_0e^{\alpha} - 1 = 0.$$

From this, we find that $M_0 = e^{-\alpha}$. Additionally, we have

$$M_0e^{-\xi} - \phi(\xi) \geq 0,$$

which leads to a contradiction for $\xi = 0$ if we assume $e^{-\alpha} - \theta < 0$. □

Once we have determined the upper bound for σ, we now proceed to find the lower bound.

Proposition 2.4.2. *Under the hypotheses of Proposition 2.4.1, along with $\frac{b}{2\Lambda^2} < 1$ and $\theta \in (0, 1/3)$, it follows that*

$$(1 - \frac{b}{2\Lambda^2}) \int_{-\alpha}^{\alpha} |\phi'|^2 + \int_{-\alpha}^{\alpha} g(\phi)\phi(1-\phi)^2 \leq 2 + \frac{a}{\Lambda} + \frac{\theta}{\alpha}, \tag{2.26}$$

and

$$\sigma \geq -\frac{6}{5}\frac{\theta}{\alpha}, \tag{2.27}$$

for $\alpha > \alpha_0$.

Remark 2.4.2. *Note that in the previous bounds, there is a term of the form $\frac{\theta}{\alpha}$. This term can be bounded independently of α, specifically:*

$$\frac{\theta}{\alpha} \leq \frac{1}{3\alpha_0},$$

where α_0 was defined in Proposition 2.4.1.

Proof. Consider the differential equation associated with ϕ given by (2.22). Multiplying both sides by $(1 - \phi)$, we obtain:

$$\phi''(1-\phi) + \sigma\phi'(1-\phi) + g(\phi)u'\phi'(1-\phi) + g(\phi)\phi(1-\phi)^2 = 0.$$

Rewriting the terms involving ϕ'' and σ, we obtain the expression

$$[\phi'(1-\phi)]' + \phi'^2 - \frac{\sigma}{2}[(1-\phi)^2]' + g(\phi)u'\phi'(1-\phi) + g(\phi)\phi(1-\phi)^2 = 0.$$

Integrating this over the interval $(-\alpha, \alpha)$

$$\phi'(\alpha) + \int_{-\alpha}^{\alpha} \phi'^2 - \frac{\sigma}{2} + \int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u' + \int_{-\alpha}^{\alpha} g(\phi)\phi(1-\phi)^2 = 0.$$

Rearranging the above expression, we find

$$\int_{-\alpha}^{\alpha} \phi'^2 + \int_{-\alpha}^{\alpha} g(\phi)\phi(1-\phi)^2 = \frac{\sigma}{2} - \phi'(\alpha) - \int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u'. \tag{2.28}$$

Note that the terms on the left-hand side are strictly positive. We now proceed to bound the terms on the right-hand side of the equation.

Using the expression for u in (2.23), we can write u' in (2.28) as the sum of two terms, as follows:

$$-\int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u' = -\int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u_1' - \int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u_2',$$

where $u_1' = a\Gamma * \phi'$ y $u_2' = \frac{b}{2}\Gamma' * (\phi'^2)$. Since $\phi' \leq 0$, it follows that $u_1' \leq 0$, making the term involving u_1' positive. Consequently, we obtain:

$$-\int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u' \leq -\int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u_2'.$$

Using Lemma 2.4.2 and considering the sign of ϕ', we find:

$$-\int_{-\alpha}^{\alpha} g(\phi)(1-\phi)\phi' u' \leq \|u_2'\|_\infty \int_{-\alpha}^{\alpha} -\phi' \leq \|u_2'\|_\infty \leq \frac{b}{4\Lambda^2}\|\phi'\|_2^2,$$

where we have taken into account that $g(\phi) \leq 1$. Therefore, (2.28) can be written as follows:

$$\int_{-\alpha}^{\alpha} \phi'^2 + \int_{-\alpha}^{\alpha} g(\phi)\phi(1-\phi)^2 \leq \frac{\sigma}{2} - \phi'(\alpha) + \frac{b}{4\Lambda^2}\int_{-\alpha}^{\alpha} \phi'^2. \tag{2.29}$$

Let us now estimate the value of $\phi'(\alpha)$. This value can be computed, and its expression is:

$$\phi'(\alpha) = -\theta \frac{\sigma e^{-\sigma\alpha}}{1 - e^{-\sigma\alpha}}.$$

To estimate $\phi'(\alpha)$, we solve the boundary value problem given by $\phi'' + \sigma\phi' = 0$, with $\phi(0) = \theta$ and $\phi(\alpha) = 0$, noting that $g(\phi) = 0$ for $\xi \in [0, \alpha]$. From this, we obtain:

$$|\phi'(\alpha)| \leq \frac{\theta}{\alpha} + |\sigma|\theta.$$

Substituting this estimate into equation (2.29), we get:

$$\left(1 - \frac{b}{4\Lambda^2}\right)\int_{-\alpha}^{\alpha} |\phi'|^2 + \int_{-\alpha}^{\alpha} g(\phi)\phi(1-\phi)^2 \leq \frac{\sigma}{2} + \frac{\theta}{\alpha} + |\sigma|\theta. \tag{2.30}$$

On the other hand, we have:

$$\left(1 - \frac{b}{4\Lambda^2}\right)\int_{-\alpha}^{\alpha} |\phi'|^2 + \int_{-\alpha}^{\alpha} g(\phi)\phi(1-\phi)^2 \geq 0.$$

Therefore, since $\theta \in (0, 1/3)$, it follows that

$$\sigma \geq -\frac{6}{5}\frac{\theta}{\alpha}.$$

Finally, since $\theta \in (0, 1/3)$, and by utilizing Proposition 2.4.1 and Lemma 2.4.2, we obtain:

$$\left(1 - \frac{b}{4\Lambda^2}\right)\int_{-\alpha}^{\alpha} |\phi'|^2 + \int_{-\alpha}^{\alpha} g(\phi)\phi(1-\phi)^2 \leq 2 + \frac{a}{\Lambda} + \frac{\theta}{\alpha} + \frac{b}{4\Lambda^2}\int_{-\alpha}^{\alpha} |\phi'|^2. \quad (2.31)$$

From which we deduce the estimation established in the statement of the Proposition. □

Before proceeding to prove the existence of a solution, let us introduce a preliminary result that will be used to show that $\phi \in C^1(-\alpha, \alpha)$.

Lemma 2.4.3. *Let ϕ and u be solutions of (2.22), then we have*

$$\|\phi'\|_\infty \leq \frac{3(|\sigma| + \|u'\|_\infty)}{2} + \frac{3\Lambda}{4} + \frac{3}{2\Lambda}, \quad (2.32)$$

for $\alpha > \frac{\Lambda}{2}\log(3)$.

Proof. Let $\xi \in [-\alpha, \alpha]$ and consider the expression

$$\int_{-\alpha}^{\xi} \frac{-e^{-\frac{(\xi-y)}{\Lambda}}}{2}\phi''(y)dy + \int_{\xi}^{\alpha} \frac{e^{\frac{(\xi-y)}{\Lambda}}}{2}\phi''(y)dy.$$

Integrating each terms by part, we obtain

$$\int_{-\alpha}^{\xi} \frac{-e^{-\frac{(\xi-y)}{\Lambda}}}{2\Lambda^2}\phi''(y)dy = -\frac{\phi'(\xi)}{2} + \frac{e^{\frac{-(\xi+\alpha)}{\Lambda}}}{2}\phi'(-\alpha) + \int_{-\alpha}^{\xi} \frac{e^{-\frac{(\xi-y)}{\Lambda}}}{2\Lambda}\phi'(y)dy,$$

$$\int_{\xi}^{\alpha} \frac{e^{\frac{(\xi-y)}{\Lambda}}}{2}\phi''(y)dy = -\frac{\phi'(\xi)}{2} + \frac{e^{\frac{(\xi-\alpha)}{\Lambda}}}{2}\phi'(\alpha) + \int_{\xi}^{\alpha} \frac{e^{\frac{(\xi-y)}{\Lambda}}}{2\Lambda}\phi'(y)dy.$$

Combining both expressions and rearranging terms, we can express ϕ' as follows:

$$-\phi'(\xi) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= -\frac{e^{\frac{-(\xi+\alpha)}{\Lambda}}}{2}\phi'(-\alpha) - \frac{e^{\frac{(\xi-\alpha)}{\Lambda}}}{2}\phi'(\alpha), \\
I_2 &= -\int_{-\alpha}^{\xi} \frac{e^{-\frac{(\xi-y)}{\Lambda}}}{2\Lambda}\phi'(y)\,dy - \int_{\xi}^{\alpha} \frac{e^{\frac{(\xi-y)}{\Lambda}}}{2\Lambda}\phi'(y)\,dy, \\
I_3 &= -\int_{-\alpha}^{\xi} \frac{e^{-\frac{(\xi-y)}{\Lambda}}}{2}\phi''(y)\,dy + \int_{\xi}^{\alpha} \frac{e^{\frac{(\xi-y)}{\Lambda}}}{2}\phi''(y)\,dy.
\end{aligned}$$

Let us analyze each term separately. First, we have:

$$I_1 \leq \left(\frac{1}{2} + \frac{e^{\frac{-2\alpha}{\Lambda}}}{2}\right)\|\phi'\|_\infty.$$

Then, I_2 can be bounded as follows

$$I_2 \leq \frac{1}{2\Lambda}\int_{-\alpha}^{\alpha}|\phi'(y)|dy = \frac{1}{2\Lambda},$$

where we have used the identity

$$\int_{-\alpha}^{\alpha}|\phi'(y)|dy = \int_{-\alpha}^{\alpha}-\phi'(y)dy = -\phi(\alpha)+\phi(-\alpha) = 1,$$

since $\phi' \leq 0$.

Finally, since ϕ is a solution to (2.22), I_3 can be expressed as follows:

$$\begin{aligned}&-\int_{-\alpha}^{\xi}\frac{e^{-\frac{(\xi-y)}{\Lambda}}}{2}\left[-(\sigma+g(\phi)u'(y))\phi'(y)-g(\phi)\phi(1-\phi)(y)\right]dy\\&+\int_{\xi}^{\alpha}\frac{e^{\frac{(\xi-y)}{\Lambda}}}{2}\left[-(\sigma+g(\phi)u'(y))\phi'(y)-g(\phi)\phi(1-\phi)(y)\right]dy.\end{aligned}$$

Thus, we find

$$\int_{-\alpha}^{\xi}\frac{e^{-\frac{(\xi-y)}{\Lambda}}}{2}\phi''(y)dy-\int_{\xi}^{\alpha}\frac{e^{\frac{(\xi-y)}{\Lambda}}}{2}\phi''(y)dy \leq \frac{(|\sigma|+\|u'\|_\infty)}{2}+\frac{\Lambda}{4}.$$

Therefore, combining the estimates for I_1, I_2, and I_3, we obtain:

$$-\phi'(\xi) \leq \left(\frac{1}{2}+\frac{e^{\frac{-2\alpha}{\Lambda}}}{2}\right)\|\phi'\|_\infty+\frac{(|\sigma|+\|u'\|_\infty)}{2}+\frac{\Lambda}{4}+\frac{1}{2\Lambda}.$$

Taking norms, we deduce

$$\left(\frac{1}{2}-\frac{e^{\frac{-2\alpha}{\Lambda}}}{2}\right)\|\phi'\|_\infty \leq \frac{(|\sigma|+\|u'\|_\infty)}{2}+\frac{\Lambda}{4}+\frac{1}{2\Lambda}.$$

Then, since $\alpha > \frac{\Lambda}{2}\log(3)$, we conclude the statement of Lemma 2.4.3. □

Once we have established the a priori estimates, we can proceed to prove the existence of a solution to (2.22).

Proposition 2.4.3. *Assume that $\frac{b}{2\Lambda^2} < 1$, $\theta \in (0, \frac{1}{3})$, and $\alpha > \max\left\{\alpha_0, \frac{\Lambda}{2}\log(3)\right\}$. Then, there exists a bounded solution of (2.22) verifying:*

$$\int_{-\alpha}^{\alpha}|\phi'|^2+\int_{-\alpha}^{\alpha}\tau\phi(1-\phi)^2 \leq C_3+C_4\frac{\theta}{\alpha}, \tag{2.33}$$

for

$$-\frac{6}{5}\frac{\theta}{\alpha} \leq \sigma < C_1+C_2\frac{\theta}{\alpha},$$

where $C_i > 0$ for $i \in \{1,2,3,4\}$, are constants independent of θ and α.

Proof. Let us consider the application

$$F_\tau : (\sigma, \phi, u) \to (\theta_\tau, \Phi_\tau, U_\tau)$$

where Φ_τ is the solution of the differential equation:

$$\begin{aligned} &-\Phi_\tau'' - \sigma\Phi_\tau' - g(\phi)\tau u'\Phi_\tau' = \tau g(\phi)\phi(1-\phi), \\ &\Phi_\tau(-\alpha) = 1, \quad \Phi_\tau(+\alpha) = 0. \end{aligned} \tag{2.34}$$

The function $U_\tau = \tau\Gamma * \left(a\bar{\phi} + \frac{b}{2}(\bar{\phi}')^2\right)$, where $\bar{\phi}$ is the extension by constant to all $\mathbb{R}$ of ϕ, satisfies

$$-\Lambda^2 U_\tau'' + U_\tau = \tau a\bar{\phi} + \frac{b}{2}(\bar{\phi}')^2.$$

The value θ_τ is given by the expression:

$$\theta_\tau = \theta - \max_{\xi \geq 0} \phi(\xi) + \sigma.$$

The operator F_τ maps the Banach space $\Omega = \mathbb{R} \times C^1([-\alpha,\alpha]) \times C^1([-\alpha,\alpha])$ onto itself

$$\|(\sigma, \phi, U)\|_\Omega = \max\left\{|\sigma|, \|\phi\|_{C^1([-\alpha,\alpha])}, \|U\|_{C^1([-\alpha,\alpha])}\right\}.$$

.

Let us consider the ball $B_M = \{(\sigma, \phi, U) \in \Omega \mid \|(\sigma, \phi, U)\|_\Omega < M\}$. We can find a sufficiently large M such that the operator $I - F_\tau$ does not cancel on the boundary of B_M for all $\tau \in [0,1]$. This is equivalent to obtaining a uniform bound for $|\sigma|$, $\|\phi\|_{C^1}$, and $\|U\|_{C^1}$ for any solution of (2.22).

Let us analyze the uniform bounds. By Lemma 2.4.1, we have that $0 \leq \phi \leq 1$. Using Proposition 2.4.2, we obtain that ϕ' is bounded in $L^2(-\alpha,\alpha)$, which implies, thanks to Lemma 2.4.2, that $U \in C^1([-\alpha,\alpha])$.

Furthermore, by Lemma 2.4.1 and Proposition 2.4.2, we have uniform bounds for σ. Lemma 2.4.3 assures that ϕ' is bounded in $C([-\alpha,\alpha])$. This leads us to find the bound M.

In addition, F_τ is absolutely continuous and depends continuously on the parameter τ, due to the $C^1([-\alpha,\alpha])$ bounds obtained.

Therefore, we are able to estimate the Leray-Schauder degree $\deg(I - F_\tau, B_M, 0)$ (see for instance [20, 73, 100]), which is well defined and independent of τ. If we prove $\deg(I - F_0, B_M, 0) \neq 0$, then, due to homotopy invariance, $\deg(I - F_1, B_M, 0) \neq 0$. By the degree property, there exists a fixed point of F_1 in B_M.

Let us calculate $\deg(I - F_0, B_M, 0)$. For $\tau = 0$, the operator F_0 simplifies to:

$$\Phi_{0,\sigma}(\xi) = \frac{e^{-\sigma\xi} - e^{-\sigma\alpha}}{e^{\sigma\alpha} - e^{-\sigma\alpha}}.$$

Therefore, F_0 has the following expression

$$F_0(\sigma, \phi, U) = \left(\sigma + \theta - \frac{1 - e^{-\sigma\alpha}}{e^{\sigma\alpha} - e^{-\sigma\alpha}}, \frac{e^{-\sigma\xi} - e^{-\sigma\alpha}}{e^{\sigma\alpha} - e^{-\sigma\alpha}}, 0\right).$$

Consequently, we have

$$(I - F_0)(\sigma, \phi, U) = \left(\frac{1 - e^{-\sigma\alpha}}{e^{\sigma\alpha} - e^{-\sigma\alpha}} - \theta, \phi - \frac{e^{-\sigma\xi} - e^{-\sigma\alpha}}{e^{\sigma\alpha} - e^{-\sigma\alpha}}, U \right).$$

One can find a $\bar{\sigma}$ that satisfies:

$$\frac{1 - e^{-\bar{\sigma}\alpha}}{e^{\bar{\sigma}\alpha} - e^{-\bar{\sigma}\alpha}} = \theta. \tag{2.35}$$

The degree of F_0 is equivalent to the degree of the function:

$$\mathcal{H}(\sigma, \phi, U) = \left(\theta_0 - \frac{1 - e^{-\sigma\alpha}}{e^{\sigma\alpha} - e^{-\sigma\alpha}}, \phi - \frac{e^{-\bar{\sigma}x} - e^{-\bar{\sigma}\alpha}}{e^{\bar{\sigma}\alpha} - e^{-\bar{\sigma}\alpha}}, U \right) := (H(\sigma), I_\phi - \Phi_{0,\bar{\sigma}}, I_U).$$

Therefore, $\deg(I - F_0, B_M, 0) = \deg(H(\sigma), J, 0)$, where J is an open set containing the unique root of (2.35), by the product of degrees property. This yields $\deg(I - F_0, B_M, 0) = -1$, thereby proving the existence of a fixed point for the operator F_τ. □

Once we have established the existence of a solution in $[-\alpha, \alpha]$, we can take the limit as $\alpha \to \infty$.

Proposition 2.4.4. *If* $\frac{b}{2\Lambda^2} < 1$, *and* $\theta \in (0, \frac{1}{3})$, *then there exists a bounded solution to*

$$\begin{aligned} -\sigma\phi' - \phi'' - g(\phi)u'\phi' &= g(\phi)\phi(1 - \phi), \\ \phi(-\infty) = 1, \quad \phi(+\infty) = 0, &\quad \phi(0) = \theta, \end{aligned} \tag{2.36}$$

where u *is defined as* $u = \Gamma * \left(a\phi + \frac{b}{2}\phi'^2\right)$, *and* ϕ *satisfies*

$$\int_{-\alpha}^{\alpha} \phi'^2 + \int_{-\alpha}^{\alpha} g(\phi)\phi(1 - \phi)^2 \leq C_3, \tag{2.37}$$

for $0 \leq \sigma < C_1$, *being* C_1 *and* C_3 *constants defined in Proposition 2.4.3.*

Proof. Consider an increasing sequence of intervals $\{[-\alpha_n, \alpha_n]\}_n$ with $\alpha_n \to \infty$, and a corresponding sequence $\{\sigma_n, \phi_n, U_n\}_n$, where (σ_n, ϕ_n, U_n) is the solution to (2.22) on each interval $[-\alpha_n, \alpha_n]$. These solutions exist due to Proposition 2.4.3. Moreover, the uniform bounds established for σ_n, ϕ_n, u_n, and their derivatives, which are independent of α_n, ensure that σ_n converges to $\hat{\sigma}$, up to a subsequence. Due to these uniform bounds, $\phi_n \to \hat{\phi}$ and $u_n \to \hat{u}$, up to a subsequence, in the sense of uniform convergence on compact subsets and their derivatives. Consequently, $\hat{\sigma}$, $\hat{\phi}$, and $\hat{u}$ satisfy the differential equation:

$$\begin{aligned} -\hat{\sigma}\hat{\phi}' - \hat{\phi}'' - g(\hat{\phi})\hat{u}'\hat{\phi}' &= g(\hat{\phi})\hat{\phi}(1 - \hat{\phi}), \\ -\Lambda^2\hat{u}'' + \hat{U} &= a\hat{\phi} + \tfrac{b}{2}\hat{\phi}'^2. \end{aligned} \tag{2.38}$$

It only remains to prove that $\hat{\phi}(+\infty) = 0$ and $\hat{\phi}(-\infty) = 1$.

Since $\|\bar{\phi}'\|_2$ is bounded, there exist the values $\hat{\phi}(+\infty)$ and $\hat{\phi}(-\infty)$. By Proposition 2.4.3 we have:

$$\int_{-\infty}^{\infty} g(\phi)\bar{\phi}(1 - \bar{\phi})^2 < +\infty. \tag{2.39}$$

Let us see that $\hat{\phi}(+\infty) = 0$. Note that $\phi_n(\xi) = \theta\frac{e^{-\sigma\xi}-e^{-\sigma\alpha}}{e^{\sigma\alpha}-e^{-\sigma\alpha}}$, for $\xi \geq 0$. Then, we have $\hat{\phi}(\xi) = \theta\frac{e^{-\hat{\sigma}\xi}-e^{-\hat{\sigma}\alpha}}{e^{\hat{\sigma}\alpha}-e^{-\hat{\sigma}\alpha}}$. Therefore, $\bar{\phi}(+\infty) = 0$.

On the other hand, the monotonicity of ϕ_n is inherited by $\hat{\phi}$. Since $\phi_n'(0) < 0$, it follows that $\hat{\phi}(-\infty) > \theta$ and $\hat{\phi}(\infty) = 1$, due to the boundedness of the integral in (2.39). ☐

Finally, to complete the proof of the existence of a solution, it remains to let $\theta \to 0$.

Proposition 2.4.5. *If $\frac{b}{2\Lambda^2} < 1$, then there exist a bounded solution to*

$$\begin{aligned} &-\sigma\phi' - \phi'' - u'\phi' = \phi(1-\phi), \\ &\phi(-\infty) = 1, \quad \phi(+\infty) = 0, \end{aligned} \tag{2.40}$$

where u is defined as $u = \Gamma * \left(a\phi + \frac{b}{2}\phi'^2\right)$, and ϕ satisfies

$$\int_{-\alpha}^{\alpha} |\phi'|^2 + \int_{-\alpha}^{\alpha} \phi(1-\phi)^2 \leq C_3, \tag{2.41}$$

for $0 \leq \sigma < C_1$, where C_1 and C_3 are constants defined in Proposition 2.4.3.

Proof. Let us consider a decreasing sequence $\theta_n \to 0$ and the associated solutions (σ_n, ϕ_n, U_n) to (2.36). Choose a translation ξ such that $\phi_n(0) = \frac{1}{2}$ for all $n \geq 1$. As $\theta \to 0$, and since $g(\phi) \to 1$, the convergence properties follow a proof scheme similar to that used in Proposition 2.4.4. Thus, we obtain that $\sigma_n \to \tilde{\sigma}$, $\phi_n \to \tilde{\phi}$, and $u_n \to \tilde{u}$ up to a subsequence, with uniform convergence on compact sets. Additionally, $\phi_n' \to \tilde{\phi}'$ and $u_n' \to \tilde{u}'$ uniformly on compact sets. Consequently, $\tilde{\sigma}$, $\tilde{\phi}$, and $\tilde{u}$ satisfy the differential equation:

$$\begin{aligned} &-\sigma\phi' - \phi'' - u'\phi' = \phi(1-\phi), \\ &-\Lambda^2 u'' + u = a\phi + \tfrac{b}{2}(\phi')^2, \end{aligned} \tag{2.42}$$

Finally, analogous to the proof of Proposition 2.4.4, it can be shown that $\tilde{\phi}(-\infty) = 1$ and $\tilde{\phi}(+\infty) = 0$. ☐

Theorem 2.1.3 guarantees the existence of traveling waves, and from the estimates derived in the various results, it can be shown that σ lies within the interval:

$$\sigma \in \left[0, 2 + \frac{a}{\Lambda} + \frac{b}{4\Lambda^2}\frac{2+\frac{a}{\Lambda}}{1-\frac{b}{2\Lambda^2}}\right].$$

To demonstrate that the obtained σ is always greater than 2, we use the following reasoning:

Given the bounds on u' provided by the estimates on ϕ and ϕ', and the expression $u = \Gamma * \left(a\phi + \frac{b}{2}(\phi')^2\right)$, we know that u' is bounded. Moreover, it can be shown that $\phi'(\pm\infty) = 0$, $u'(\pm\infty) = 0$, and $u(-\infty) = a$, by repeatedly applying this generalized Lasalle's Invariance Theorem.

Lemma 2.4.4. *(Lasalle's Invariance Theorem, revisited) Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function, and consider the differential equation $x' = f(x)$. Let $V : \mathbb{R}^N \to \mathbb{R}$ be a C^1 function, and suppose $x : [\alpha, +\infty) \to \mathbb{R}^N$ is a positively bounded solution such that $V(x(t)) \to L$, as $t \to +\infty$. Then we have:*

$$\frac{d}{dt}V(x(t)) \to 0.$$

Proof. Let t_n be a sequence such that $x(t_n) \to x_0$. Consider a closed ball $\bar{B}(x_0, r)$ and let $K = \max_{x \in \bar{B}(x_0,r)} |f(x)|$. Choose n_0 such that $x(t_n) \in \bar{B}(x_0, r/2)$ for all $n \geq n_0$. Select $\epsilon \leq \frac{r}{2K}$ such that $x(t) \in B(x_0, r)$ for all $t \in [t_n - \epsilon, t_n + \epsilon]$.

Define $z_n : [-\epsilon, \epsilon] \to \mathbb{R}^N$ by

$$z_n(t) = x(t + t_n).$$

This function z_n represents a uniformly bounded and equicontinuous family of solutions. By the Ascoli-Arzelà theorem, there exists a subsequence $\{z_{n_k}\}$ such that

$$z_{n_k} \to z_0$$

uniformly, where $z_0 : [-\epsilon, \epsilon] \to \mathbb{R}^N$ is a solution of the differential equation with $z_0(0) = x_0$.

Next, let us prove that $V(z(t))$ is constant. We have:

$$V(z(t)) = \lim_{n\to\infty} V(z_n(t)) = \lim_{n\to\infty} V(x(t + t_n)) = \lim_{t\to\infty} V(x(t)) = L.$$

Thus, $V(z(t))$ is constant, and therefore

$$\frac{d}{dt} V(z(t)) = 0.$$

In particular, at $t = 0$,

$$\frac{d}{dt} V(z(0)) = 0 = \frac{d}{dt} V(x_0) = \lim_{n\to\infty} \frac{d}{dt} V(x(t_n)).$$

This completes the proof. □

Given $V(\phi, \phi', u, u') = \phi$, we deduce that $\phi'(\pm\infty) = 0$. Using the expression for u, we get that $u(-\infty) = a$ and $u(+\infty) = 0$. Applying $V(\phi, \phi', u, u') = u$, we also find that $u'(\pm\infty) = 0$.

Now, we can proceed to show that there are no monotonically decreasing traveling wave solutions for $\sigma < 2$.

Lemma 2.4.5. *Given that σ and ϕ are bounded solutions of* (2.40)*, it follows that $\sigma \geq 2$.*

Proof. To prove that $\sigma \geq 2$ by contradiction, assume that $\sigma < 2$. We know that if ϕ is a solution of (2.40), it is monotonically decreasing, satisfies $\phi(+\infty) = 0$, and $\phi'(+\infty) = 0$. Additionally, $u'(+\infty) = 0$, so there exists a sufficiently large $\bar{\xi}$ such that $\|u'(\xi)\| < \epsilon$ for $\xi > \bar{\xi}$.

Consider the differential equation (2.40) and make the following change of variables to polar coordinates:

$$\phi(\xi) = r(\xi)\cos(\omega(\xi)),$$
$$\phi'(\xi) = r(\xi)\sin(\omega(\xi)).$$

By substituting these into the differential equation, we obtain the transformed differential equation in terms of r and ω:

$$r' = -(\sigma + u')r\sin^2(\omega) + r^2\cos^2(w)\sin(w),$$
$$\omega' = -1 - (\sigma + u')\frac{\sin(2\omega)}{2} + r\cos^3(\omega).$$

If we demonstrate that the solution is not monotonically decreasing but instead exhibits infinite oscillations before reaching zero -equivalent to saying that $\omega(\xi)$ remains consistently positive or negative for large values of ξ- we will have reached a contradiction.

We can bound ω' as follows:

$$\omega' = -1 - (\sigma + u')\frac{\sin(2\omega)}{2} + r\cos^3(\omega) \leq -1 + \frac{1}{2}(\sigma + u') + r.$$

Since $r \to 0$ as $\xi \to +\infty$, ω' does not change sign when $\frac{\sigma+u'}{2} + 1 < 0$. This leads to a contradiction, as it implies that the solution would spiral toward zero. $\square$

Chapter 3

Dynamics at the Interface of Cell Populations Governed by Mutual Pressure and Fractional Diffusion

3.1 Introduction and main results

This chapter focuses on the study of the well-posedness of a mathematical model representing the dynamics across the interface between two cell populations. The model consists of a system of coupled differential equations involving a fractional Laplacian, a nonlinear velocity field, and a set of parameters linked to biochemical and biomechanical properties of the system. The equations are given by:

$$\begin{aligned} &\partial_t \phi = -(-\Delta)^{\alpha} \phi + \partial_x u \partial_x \phi + \phi(1-\phi) \\ &-\Lambda^2 \partial_x^2 u + u = a\phi + \frac{b}{2} \left(\partial_x \phi\right)^2, \\ &\phi(0,x) = \phi_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where ϕ represents the cell density, u the velocity, Λ is a mechanical parameter, and $\alpha \in (0,1)$ governs the fractional dissipation term. The term $(-\Delta)^{\alpha}\phi$ represents the fractional Laplacian, defined as follows:

$$(-\Delta)^{\alpha} \phi = C(\alpha)\, \mathrm{P.V.} \int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{|x-y|^{1+2\alpha}} \, \mathrm{d}y.$$

We observe that, since u satisfies the Helmholtz equation, we can express it as

$$u = \Gamma * \left(a\phi + \frac{b}{2} (\partial_x \phi)^2 \right), \tag{3.2}$$

where $\Gamma = \frac{1}{2\Lambda} e^{-\frac{|x|}{\Lambda}}$.

As we have mention in the introduction, the system draws inspiration from active scalar equations, a class of equations where the scalar field ϕ is advected by a velocity that depends on ϕ itself. Active scalar equations have been extensively studied in mathematical fluid dynamics, particularly in connection with fluid models like the incompressible Euler equations and the surface quasi-geostrophic equations. The nonlocality of the velocity, introduced through the elliptic equation, and the fractional dissipation term are key features that influence the system's behavior, similar to other well-known active scalar models.

The nonlinear dependence of the velocity on the scalar field ϕ, which is a distinctive feature of this system, differentiates it from other classical active scalar models, where the velocity typically depends linearly on the scalar.

Overall, the chapter presents a rigorous mathematical framework for understanding the behavior of cell population interfaces influenced by fractional diffusion and nonlinear mechanical interactions, contributing to both the theory of active scalar equations and the study of biological systems.

Regarding the well-posedness of the problem (3.1) with fractional diffusion, we have the following result:

Theorem 3.1.1. *Given an initial condition $\phi_0 \in H^3(\mathbb{R})$, and $\alpha \in (0,1]$, the problem (3.1) admits a unique local solution*

$$\phi \in C\left([0,T^*]; H^3(\mathbb{R})\right) \cap L^2\left([0,T^*], H^{3+\alpha/2}(\mathbb{R})\right).$$

Furthermore, if $\phi_0 \geq 0$, then the solution is global in time and remains uniformly bounded in L^∞.

The proof of this statement is derived using a combination of energy and pointwise methods in the same spirit as in [80]. In that regards, we observe that the fractional Laplacian can also be defined using Fourier transforms:

$$\widehat{(-\Delta)^\alpha}\phi(\xi) = |\xi|^{2\alpha}\hat{\phi}(\xi),$$

and satisfies the following properties:

- Given $\phi(x)$ and $\bar{x} \in \mathbb{R}$ such that $\phi(x) \leq \phi(\bar{x})$ for all $x \in \mathbb{R}$. Then, $-(-\Delta)^\alpha\phi(\bar{x}) \leq 0$. [64] (see also [31])
- Given $\phi(x)$ and $\bar{x} \in \mathbb{R}$ such that $\phi(x) \geq \phi(\bar{x})$ for all $x \in \mathbb{R}$. Then, $-(-\Delta)^\alpha\phi(\bar{x}) \geq 0$.[64] (see also [31])
- Given $\psi \in C^2(\mathbb{R})$, we have that (see Appendix in [81])

$$-\int_{\mathbb{R}} \psi(-\Delta)^\alpha\psi \,\mathrm{d}x = -\int_{\mathbb{R}} \left((-\Delta)^{\alpha/2}\psi\right)^2 \mathrm{d}x \leq 0. \tag{3.3}$$

3.2 Proof of the Main Existence Result

3.2.1 Local well-posedness

We will begin by proving the local existence of a solution to (3.1). To achieve this, we will employ the methods outlined in [98], originally developed by Leray in the 1930s, leading to the following result:

Proposition 3.2.1. *Given an initial condition $\phi_0 \in H^3(\mathbb{R})$, and $\alpha \in (0,1)$, there exists a time $T^* \geq \frac{1}{C(a,b,\Lambda,\alpha)\|\phi_0\|_{H^3}}$, such that the system (3.1) admits a solution $\phi \in C\left([0,T[; C^2(\mathbb{R})\right) \cap C^1\left([0,T^*[; C(\mathbb{R})\right)$.*

To prove this proposition, we begin by considering the following regularized problem:

$$
\begin{aligned}
&\partial_t\phi^\epsilon = J^\epsilon\left[-(-\Delta)^\alpha(J^\epsilon\phi^\epsilon) + \partial_x(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon) + J^\epsilon\phi^\epsilon(1 - J^\epsilon\phi^\epsilon)\right],\\
&u^\epsilon = \Gamma * \left(a\phi^\epsilon + \frac{b}{2}(\partial_x\phi_\epsilon)^2\right),\\
&\phi^\epsilon(0,x) = \phi_0(x),
\end{aligned}
\tag{3.4}
$$

where $J^\epsilon\phi^\epsilon$ is the mollification of the functions $\phi^\epsilon \in L^p(\mathbb{R})$ for $1 \le p < \infty$, defined as

$$
\left(J^\epsilon\phi^\epsilon\right)(x) = \epsilon\int_{\mathbb{R}}\Phi\left(\frac{x-y}{\epsilon}\right)\phi^\epsilon(y)\,\mathrm{d}y, \quad \epsilon > 0,
$$

with Φ being a test function that satisfies

$$
\Phi \in C_c^\infty(\mathbb{R}), \quad \Phi \ge 0, \quad \int_{\mathbb{R}}\Phi = 1.
$$

The mollifiers satisfy certain useful properties outlined in ([98], Lemma 3.5).

We will start by proving the existence of a solution for the regularized problem. (3.4).

Proposition 3.2.2. *Given an initial condition $\phi_0 \in H^3(\mathbb{R})$, there exists a unique solution $\phi^\epsilon \in C^1([0,T_\epsilon], H^3(\mathbb{R}))$ to the regularized differential equation (3.4), for any $\epsilon > 0$.*

Remark 3.2.1. *In the following proofs, we will use C to denote constants that depend on a, b, Λ, and α, and κ_ϵ for constants that depend on ϵ. This notation is adopted to simplify the presentation, as the specific values of these constants are not crucial for the results at this stage.*

Proof. First, we will establish the local-in-time existence of the solution ϕ^ϵ. We will demonstrate that the operator $\bar{F}^\epsilon : H^3 \to H^3$, defined by:

$$
\begin{aligned}
\bar{F}^\epsilon(\phi^\epsilon) &= J^\epsilon F^\epsilon(\phi^\epsilon)\\
&= J^\epsilon\left[-(-\Delta)^\alpha(J^\epsilon\phi^\epsilon) + \partial_x(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon) + J^\epsilon\phi^\epsilon(1 - J^\epsilon\phi^\epsilon)\right],
\end{aligned}
\tag{3.5}
$$

is locally Lipschitz in H^3. We have

$$
\begin{aligned}
\|\bar{F}^\epsilon(\phi_1^\epsilon) - \bar{F}^\epsilon(\phi_2^\epsilon)\|_{H^3} \le& C\|F^\epsilon(\phi_1^\epsilon) - F^\epsilon(\phi_2^\epsilon)\|_{H^3}\\
\le& C\left[\|-(-\Delta)^\alpha J^\epsilon(\phi_1^\epsilon - \phi_2^\epsilon)\|_{H^3}\right.\\
&+ \|\partial_x(J^\epsilon u_1^\epsilon)\partial_x(J^\epsilon\phi_1^\epsilon) - \partial_x(J^\epsilon u_2^\epsilon)\partial_x(J^\epsilon\phi_2^\epsilon)\|_{H^3}\\
&\left.+\|J^\epsilon\phi_1^\epsilon - J^\epsilon\phi_2^\epsilon\|_{H^3} + \|(J^\epsilon\phi_1^\epsilon)^2 - (J^\epsilon\phi_2^\epsilon)^2\|_{H^3}\right]\\
=& I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Let us bound each of the terms. Using the properties of the mollifiers, we have:

$$
\begin{aligned}
I_1 &\le \kappa_\epsilon\|\phi_1 - \phi_2\|_{H^3},\\
I_3 &\le C\|\phi_1 - \phi_2\|_{H^3},\\
I_4 &= \|\left(J^\epsilon\phi_1^\epsilon + J^\epsilon\phi_2^\epsilon\right)\left(J^\epsilon\phi_1^\epsilon - J^\epsilon\phi_2^\epsilon\right)\|_{H^3}\\
&\le \|J^\epsilon(\phi_1^\epsilon + \phi_2^\epsilon)\|_{L^\infty}\|J^\epsilon(\phi_1^\epsilon - \phi_2^\epsilon)\|_{H^3}\\
&\le C\left(\|\phi_1^\epsilon\|_{H^3} + \|\phi_2^\epsilon\|_{H^3}\right)\|J^\epsilon(\phi_1^\epsilon - \phi_2^\epsilon)\|_{H^3}\\
&\le C\|J^\epsilon(\phi_1^\epsilon - \phi_2^\epsilon)\|_{H^3},
\end{aligned}
$$

It remains to estimate the term I_2. To do so, we can add and subtract terms as follows:

$$\begin{aligned}
I_2 &\le \|\partial_x J^\epsilon(u_1^\epsilon - u_2^\epsilon)\partial_x(J^\epsilon\phi_1^\epsilon)\|_{H^3} + \|\partial_x(J^\epsilon u_2^\epsilon)\partial_x J^\epsilon(\phi_1^\epsilon - \phi_2^\epsilon)\|_{H^3} \\
&\le \|\partial_x(J^\epsilon\phi_1^\epsilon)\|_{H^3}\|\partial_x J^\epsilon(u_1^\epsilon - u_2^\epsilon)\|_{L^\infty} + \|\partial_x(J^\epsilon u_2^\epsilon)\|_{L^\infty}\|\partial_x J^\epsilon(\phi_1^\epsilon - \phi_2^\epsilon)\|_{H^3} \\
&\le \kappa_\epsilon\|\phi_1^\epsilon\|_{H^3}\|(u_1^\epsilon - u_2^\epsilon)\|_{L^2} + \kappa_\epsilon\|u_2^\epsilon\|_{L^2}\|\phi_1^\epsilon - \phi_2^\epsilon\|_{H^3}.
\end{aligned}$$

Using the properties of the mollifiers and expressing u^ϵ as the sum of its components, given by the expression (3.2), we obtain:

$$\begin{aligned}
I_2 &\le \kappa_\epsilon\|\phi_1^\epsilon\|_{H^3}\|\Gamma\|_{L^1}\Big(a\|(\phi_1^\epsilon - \phi_2^\epsilon)\|_{L^2} + \frac{b}{2}\|(\partial_x\phi_1^\epsilon)^2 - (\partial_x\phi_2^\epsilon)^2\|_{L^2}\Big) \\
&\quad + \kappa_\epsilon\|u_2^\epsilon\|_{L^2}\|\phi_1^\epsilon - \phi_2^\epsilon\|_{H^3} \\
&\le \kappa_\epsilon\|\phi_1^\epsilon\|_{H^3}\|\Gamma\|_{L^1}\Big(a\|(\phi_1^\epsilon - \phi_2^\epsilon)\|_{L^2} \\
&\quad + \frac{b}{2}\|\partial_x\phi_1^\epsilon + \partial_x\phi_2^\epsilon\|_{L^\infty}\|\partial_x\phi_1^\epsilon - \partial_x\phi_2^\epsilon\|_{L^2}\Big) + \kappa_\epsilon\|u_2^\epsilon\|_{L^2}\|\phi_1^\epsilon - \phi_2^\epsilon\|_{H^3} \\
&\le \kappa_\epsilon\|\phi_1^\epsilon - \phi_2^\epsilon\|_{H^3}.
\end{aligned}$$

Combining all the estimates, we derive:

$$\|\bar{F}^\epsilon(\phi_1^\epsilon) - \bar{F}^\epsilon(\phi_2^\epsilon)\|_{H^3} \le \kappa_\epsilon\|\phi_1^\epsilon - \phi_2^\epsilon\|_{H^3},$$

which establishes that $\bar{F}^\epsilon$ is locally Lipschitz continuous on any open set Ω defined by:

$$\Omega = \Big\{\phi \in H^3 : \|\phi\|_{H^3} \le M\Big\}.$$

Applying the Picard-Lindelöf Theorem in the form of Banach's fixed-point theorem [25], we obtain the existence of a unique solution $\phi^\epsilon \in C^1([0,T_\epsilon], H^3(\mathbb{R}))$ for some $T_\epsilon > 0$. □

We now focus on deriving energy estimates for the approximate problems. The next step is to obtain a uniform bound with respect to ϵ for the entire family of solutions ϕ^ϵ. We state the following result:

Lemma 3.2.1. *Let ϕ^ϵ be the solution to (3.4). Given $\phi_0 \in H^3(\mathbb{R})$, there exists a time T^* such that $\|\phi^\epsilon(t,\cdot)\|_{H^3} \le C$ for $t \in [0,T]$, where $C > 0$ is independent of ϵ and $T \le T^*$.*

Proof. Let us determine the bounds for the L^2 norm of ϕ^ϵ.

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \frac{1}{2}(\phi^\epsilon)^2 \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \Big[-(-\Delta)^\alpha(J^\epsilon\phi^\epsilon) + \partial_x(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon) + J^\epsilon\phi^\epsilon(1 - J^\epsilon\phi^\epsilon)\Big] J^\epsilon\phi \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \Big[- ((-\Delta)^{\alpha/2}(J^\epsilon\phi^\epsilon))^2 + \partial_x(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon)\, J^\epsilon\phi^\epsilon \\
&\qquad\quad + (J^\epsilon\phi^\epsilon)^2(1 - J^\epsilon\phi^\epsilon)\Big]\mathrm{d}x \\
&\quad \le \int_{\mathbb{R}} \partial_x(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon)\, J^\epsilon\phi^\epsilon \,\mathrm{d}x + C\|\phi^\epsilon\|_{L^2}^2(1 + \|\phi^\epsilon\|_{L^\infty}).
\end{aligned}$$

If we analyze the term $\partial_x(J^\epsilon u^\epsilon)$ and integrate by parts, we get

$$\begin{aligned}\int_{\mathbb{R}} &\partial_x(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon)\, J^\epsilon\phi^\epsilon \,\mathrm{d}x\\ &=\frac{-1}{2}\int_{\mathbb{R}}(J^\epsilon\phi^\epsilon)^2\partial_x^2(J^\epsilon u^\epsilon)\,\mathrm{d}x\\ &=\frac{-1}{2}\int_{\mathbb{R}}\frac{J^\epsilon u^\epsilon - aJ^\epsilon\phi^\epsilon - \frac{b}{2}\left(\partial_x(J^\epsilon\phi^\epsilon)\right)^2}{\Lambda^2}(J^\epsilon\phi^\epsilon)^2\,\mathrm{d}x\\ &\leq\frac{1}{2\Lambda^2}\Big(\|J^\epsilon u^\epsilon\|_{L^\infty}\|J^\epsilon\phi^\epsilon\|_{L^2}^2 + a\|J^\epsilon\phi^\epsilon\|_{H^1}^3\\ &\quad+\frac{b}{2}\|\partial_x(J^\epsilon\phi^\epsilon)\|_{L^2}^2\|J^\epsilon\phi^\epsilon\|_{H^1}^2\Big)\,\mathrm{d}x.\end{aligned}$$

Therefore, by combining the estimates and identifying the term with the highest exponent in the norms, we find that

$$\partial_t\int_{\mathbb{R}}\frac{1}{2}(\phi^\epsilon)^2\,\mathrm{d}x \leq C(1+\|\phi^\epsilon\|_{H^3}^4),$$

for some constant $C>0$ that is independent of ϵ.

Now, let us estimate the bounds of the following seminorm $\|\phi\|_{\dot{H}^3}$.

$$\begin{aligned}\partial_t&\int_{\mathbb{R}}\frac{1}{2}(\partial_x^3\phi^\epsilon)^2\,\mathrm{d}x\\ &=\int_{\mathbb{R}} -(-\Delta)^\alpha(\partial_x^3(J^\epsilon\phi^\epsilon))(\partial_x^3(J^\epsilon\phi^\epsilon)) + (\partial_x^3(J^\epsilon\phi^\epsilon))^2(1-2J^\epsilon\phi^\epsilon)\\ &\quad-4\partial_x(J^\epsilon\phi^\epsilon)\partial_x^2(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)+\partial_x^4(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\\ &\quad+3\partial_x^3(J^\epsilon u^\epsilon)\partial_x^2(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)+3\partial_x^2(J^\epsilon u^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\\ &\quad+\partial_x(J^\epsilon u^\epsilon)\partial_x^4(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\,\mathrm{d}x.\end{aligned}$$

Since we aim to bound the term on the right by the norm $\|\phi^\epsilon\|_{H^3}$ raised to an exponent, we must pay particular attention to terms where fourth-order or higher derivatives might appear, as these terms could complicate the analysis. The remaining terms can be handled more straightforwardly.

Let us examine these problematic terms:

First, consider the fractional Laplacian term. Using (3.3), this term is negative, i.e.,

$$\int_{\mathbb{R}} -(-\Delta)^\alpha(\partial_x^3(J^\epsilon\phi^\epsilon))(\partial_x^3(J^\epsilon\phi^\epsilon))\,\mathrm{d}x \leq 0$$

Therefore, we obtain:

$$\begin{aligned}\partial_t\int_{\mathbb{R}}\frac{1}{2}(\partial_x^3\phi^\epsilon)^2\,\mathrm{d}x \leq&\|\partial_x^3(J^\epsilon\phi^\epsilon)\|_{L^2}^2(1+2\|J^\epsilon\phi^\epsilon\|_\infty)\\ &+4\|\partial_x(J^\epsilon\phi^\epsilon)\|_{L^\infty}\|\partial_x^2(J^\epsilon\phi^\epsilon)\|_{L^2}\|\partial_x^3(J^\epsilon\phi^\epsilon)\|_{L^2}\\ &+\int_{\mathbb{R}}\partial_x^4(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\,\mathrm{d}x\\ &+\int_{\mathbb{R}}3\partial_x^3(J^\epsilon u^\epsilon)\partial_x^2(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\,\mathrm{d}x\\ &+\int_{\mathbb{R}}\Big[3\partial_x^2(J^\epsilon u^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\\ &\qquad+\partial_x(J^\epsilon u^\epsilon)\partial_x^4(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\Big]\mathrm{d}x.\end{aligned}$$

Define

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} \partial_x^4(J^\epsilon u^\epsilon)\partial_x(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\ \mathrm{d}x\\
I_2 &= \int_{\mathbb{R}} 3\partial_x^3(J^\epsilon u^\epsilon)\partial_x^2(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\ \mathrm{d}x\\
I_3 &= \int_{\mathbb{R}} 3\partial_x^2(J^\epsilon u^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon) + \partial_x(J^\epsilon u^\epsilon)\partial_x^4(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\ \mathrm{d}x.
\end{aligned}$$

Let us analyze the terms involving u^ϵ. We know that u^ϵ satisfies the equation (3.4), which implies:

$$-\Lambda^2\partial_x^3 u^\epsilon + \partial_x u^\epsilon = a\partial_x\phi^\epsilon + b\partial_x\phi^\epsilon\partial_x^2\phi^\epsilon, \tag{3.6}$$

$$-\Lambda^2\partial_x^4 u^\epsilon + \frac{1}{\Lambda^2}u^\epsilon = \frac{a}{\Lambda^2}\phi^\epsilon + \frac{b}{2\Lambda^2}(\partial_x\phi^\epsilon)^2 + a\partial_x^2\phi^\epsilon + b((\partial_x^2\phi_\epsilon)^2 + \partial_x\phi^\epsilon\partial_x^3\phi^\epsilon). \tag{3.7}$$

Let us proceed by bounding each term involving u^ϵ. Using equation (3.7) to estimate the term I_1, we obtain:

$$\begin{aligned}
I_1 =& \int_{\mathbb{R}} \frac{1}{\Lambda^4}\left(J^\epsilon u^\epsilon - aJ^\epsilon\phi^\epsilon - \frac{b}{2}J^\epsilon(\partial_x\phi^\epsilon)^2\right)\partial_x(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\ \mathrm{d}x\\
&- \int_{\mathbb{R}} \frac{a}{\Lambda^2}J^\epsilon(\partial_x^2\phi^\epsilon)\partial_x(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\ \mathrm{d}x\\
&- \int_{\mathbb{R}} \frac{b}{\Lambda^2}\left(J^\epsilon(\partial_x^2\phi^\epsilon)^2 + J^\epsilon(\partial_x\phi^\epsilon\partial_x^3\phi^\epsilon)\right)\partial_x(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\ \mathrm{d}x\\
\le& \frac{1}{\Lambda^4}\Big(\|J^\epsilon u^\epsilon\|_{L^\infty} + a\|J^\epsilon\phi^\epsilon\|_{L^\infty}\\
&\qquad + \frac{b}{2}\|J^\epsilon(\partial_x\phi^\epsilon)^2\|_{L^\infty}\Big)\|\partial_x(J^\epsilon\phi^\epsilon)\|_{L^2}\|\partial_x^3(J^\epsilon\phi^\epsilon)\|_{L^2}\\
&+ \frac{a}{\Lambda^2}\|J^\epsilon(\partial_x^2\phi^\epsilon)\|_{L^2}\|\partial_x(J^\epsilon\phi^\epsilon)\|_{L^\infty}\|\partial_x^3(J^\epsilon\phi^\epsilon)\|_{L^2}\\
&+ \frac{b}{\Lambda^2}\Big(\|J^\epsilon(\partial_x^2\phi^\epsilon)^2\|_{L^\infty}\|\partial_x(J^\epsilon\phi^\epsilon)\|_{L^2}\\
&\qquad + \|J^\epsilon(\partial_x\phi^\epsilon\partial_x^3\phi^\epsilon)\|_{L^2}\|\partial_x(J^\epsilon\phi^\epsilon)\|_{L^\infty}\Big)\|\partial_x^3(J^\epsilon\phi^\epsilon)\|_{L^2}\\
\le& C\left(1 + \|J^\epsilon\phi^\epsilon\|_{H^3}^4\right).
\end{aligned}$$

The term I_2 can be bounded as follows, using equation (3.6):

$$\begin{aligned}
I_2 =& \int_{\mathbb{R}} \frac{1}{\Lambda^4}\left(J^\epsilon(\partial_x u^\epsilon) - aJ^\epsilon(\partial_x\phi^\epsilon) - bJ^\epsilon(\partial_x\phi^\epsilon\partial_x^2\phi^\epsilon)\right)\partial_x^2(J^\epsilon\phi^\epsilon)\partial_x^3(J^\epsilon\phi^\epsilon)\mathrm{d}x\\
\le& \frac{1}{\Lambda^4}\Big(\|J^\epsilon(\partial_x u^\epsilon)\|_{L^\infty} + a\|J^\epsilon(\partial_x\phi^\epsilon)\|_{L^\infty}\\
&\qquad + b\|J^\epsilon(\partial_x\phi^\epsilon\partial_x^2\phi^\epsilon)\|_{L^\infty}\Big)\|\partial_x^2(J^\epsilon\phi^\epsilon)\|_{L^2}\|\partial_x^3(J^\epsilon\phi^\epsilon)\|_{L^2}\\
\le& C\left(1 + \|J^\epsilon\phi^\epsilon\|_{H^3}^4\right).
\end{aligned}$$

Finally, let us compute I_3. By integrating the term on the right by parts, we obtain:

$$\begin{aligned}
I_3 =& \int_{\mathbb{R}} \frac{5}{2}\partial_x^2(J^\epsilon u^\epsilon)\left[\partial_x^3(J^\epsilon\phi^\epsilon)\right]^2 \, \mathrm{d}x \\
=& \int_{\mathbb{R}} \frac{5}{2\Lambda^2}\left(J^\epsilon u^\epsilon - aJ^\epsilon\phi^\epsilon - \frac{b}{2}J^\epsilon\left(\partial_x\phi^\epsilon\right)^2\right)\left[\partial_x^3(J^\epsilon\phi^\epsilon)\right]^2 \, \mathrm{d}x \\
\leq& \frac{5}{2\Lambda^2}\left(\|J^\epsilon u^\epsilon\|_{L^\infty} + a\|J^\epsilon\phi^\epsilon\|_{L^\infty} + \frac{b}{2}\|J^\epsilon\left(\partial_x\phi^\epsilon\right)\|_{L^\infty}^2\right)\|\partial_x^3(J^\epsilon\phi^\epsilon)\|_{L^2}^2 \\
\leq& C\left(1+\|J^\epsilon\phi^\epsilon\|_{H^3}^4\right).
\end{aligned}$$

Therefore, by combining all the bounds, we conclude that

$$\partial_t \int_{\mathbb{R}} \frac{1}{2}(\partial_x^3\phi^\epsilon)^2 \, \mathrm{d}x \leq C\left(1+\|J^\epsilon\phi^\epsilon\|_{H^3}^4\right) \leq \bar{C}\left(1+\|\phi^\epsilon\|_{H^3}^4\right).$$

Define $E(\phi) = (1+\|\phi\|_{L^2}^2 + \|\partial_x^3\phi\|_{L^2}^2)$. Using the obtained bounds, we have:

$$\frac{1}{2}\partial_t E(\phi^\epsilon) \leq CE(\phi^\epsilon)^2,$$

from which we deduce that:

$$E(\phi^\epsilon(t)) \leq \frac{1}{\frac{1}{E(\phi_0)} - 2Ct}.$$

Consequently, $\|\phi^\epsilon\|_{H^3}$ will be uniformly bounded for all $\epsilon > 0$ and for all $t \in (0,T)$, where $T < T^*$ and $T^* \geq \frac{1}{2\bar{C}E(\phi_0)}$.

Thus, we have shown that $\phi^\epsilon \in L^\infty([0,T], H^3(\mathbb{R}))$. Additionally, since the problem is parabolic, we gain some extra regularity and find that $\phi^\epsilon \in L^2([0,T], H^{3+\alpha/2}(\mathbb{R}))$, with $T < T^*$. To see this, we can repeat the computation for the bound on $\|\partial_x^3\phi^\epsilon\|_{L^2}^2$ and use

$$\begin{aligned}
\int_{\mathbb{R}} -(-\Delta)^\alpha(\partial_x^3(J^\epsilon\phi^\epsilon))(\partial_x^3(J^\epsilon\phi^\epsilon)) \, \mathrm{d}x =& \int_{\mathbb{R}} -\left((-\Delta)^{\alpha/2}(\partial_x^3(J^\epsilon\phi^\epsilon))\right)^2 \, \mathrm{d}x \\
=& -\|(-\Delta)^{\alpha/2}(\partial_x^3(J^\epsilon\phi^\epsilon))\|_{L^2}^2
\end{aligned}$$

we obtain

$$\partial_t \int_{\mathbb{R}} \frac{1}{2}(\partial_x^3\phi^\epsilon)^2 \, \mathrm{d}x + \|(-\Delta)^{\alpha/2}(\partial_x^3(J^\epsilon\phi^\epsilon))\|_{L^2}^2 \leq \bar{C}\left(1+\|\phi^\epsilon\|_{H^3}^4\right).$$

From this, we deduce that $\phi^\epsilon \in L^2([0,T], H^{3+\alpha/2}(\mathbb{R}))$, with $T < T^*$. □

So far, we have identified a sequence of approximate problems whose solutions share a common lifespan and exhibit the desired regularity. To establish the existence of a local solution to (3.1), we need to pass to the limit in this sequence. To achieve this, we are going to proof the sequence of solutions is Cauchy in the space $\phi^\epsilon \in C([0,T^*], H^1)$.

Considering the difference between two terms in the sequence, let us bound $\|\phi^\epsilon - \phi^{\epsilon'}\|_{H^1}$.

First of all, we will start with the $\|\phi^\epsilon - \phi^{\epsilon'}\|_{L^2}^2$

$$
\begin{aligned}
\frac{1}{2}\partial_t\|\phi^\epsilon - \phi^{\epsilon'}\|_{L^2}^2 = & \frac{1}{2}\int_{\mathbb{R}} \partial_t(\phi^\epsilon - \phi^{\epsilon'})(\phi^\epsilon - \phi^{\epsilon'}) \\
\leq & \int_{\mathbb{R}} -(-\Delta)^\alpha (J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'})(J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}) \\
& + \int_{\mathbb{R}} [\partial_x J^\epsilon u^\epsilon \partial_x J^\epsilon \phi^\epsilon - \partial_x J^{\epsilon'} u^{\epsilon'} \partial_x J^{\epsilon'}\phi^{\epsilon'}](J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}) \\
& + \int_{\mathbb{R}} [J^\epsilon\phi^\epsilon(1 - J^\epsilon\phi^\epsilon) - J^{\epsilon'}\phi^{\epsilon'}(1 - J^{\epsilon'}\phi^{\epsilon'})](J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}) \\
\leq & \int_{\mathbb{R}} \partial_x J^\epsilon u^\epsilon [\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}](J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}) \\
& + \int_{\mathbb{R}} \partial_x J^{\epsilon'}\phi^{\epsilon'} [\partial_x J^\epsilon u^\epsilon - \partial_x J^{\epsilon'} u^{\epsilon'}](J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}) \\
& + \int_{\mathbb{R}} [1 - (J^\epsilon\phi^\epsilon + J^{\epsilon'}\phi^{\epsilon'})](J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'})^2 \\
= & J_1 + J_2 + J_3.
\end{aligned}
$$

Let us examine each term. For the term J_1, integrating by parts yields:

$$
J_1 = -\int_{\mathbb{R}} \partial_x^2 J^\epsilon u^\epsilon (J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'})^2 \leq C\|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{L^2}^2.
$$

Using the expression for u^ϵ given by equation (3.4), the term I_2 can be expressed as:

$$
\begin{aligned}
J_2 = & a\int_{\mathbb{R}} \partial_x J^{\epsilon'}\phi^{\epsilon'} [\Gamma * (J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'})](J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}) \\
& + \frac{b}{2}\int_{\mathbb{R}} \partial_x J^{\epsilon'}\phi^{\epsilon'} \left[\Gamma * \left((\partial_x J^\epsilon\phi^\epsilon)^2 - (\partial_x J^{\epsilon'}\phi^{\epsilon'})^2\right)\right](J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}) \\
\leq & a\|\partial_x J^{\epsilon'}\phi^{\epsilon'}\|_{L^\infty}\|\Gamma\|_{L^1}\|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{L^2}^2 \\
& + \frac{b}{2}\|\partial_x J^{\epsilon'}\phi^{\epsilon'}\|_{L^\infty}\|\Gamma\|_{L^1}\|\partial_x J^\epsilon\phi^\epsilon \\
& + \partial_x J^{\epsilon'}\phi^{\epsilon'}\|_{L^\infty}\|\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}\|_{L^2}\|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{L^2} \\
\leq & C(\|\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}\|_{L^2}^2 + \|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{L^2}^2).
\end{aligned}
$$

The bound for the term J_3 is straightforward, and thus we obtain:

$$
\frac{1}{2}\partial_t\|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{L^2}^2 \leq C\|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{H^1}^2.
$$

Now, let us estimate $\|\partial_x\phi^\epsilon - \partial_x\phi^{\epsilon'}\|_{L^2}^2$.

$$
\begin{aligned}
\frac{1}{2}&\partial_t\|\partial_x\phi^\epsilon - \partial_x\phi^{\epsilon'}\|_{L^2}^2 \\
&=\frac{1}{2}\int_{\mathbb{R}} \partial_t(\partial_x\phi^\epsilon - \partial_x\phi^{\epsilon'})(\partial_x\phi^\epsilon - \partial_x\phi^{\epsilon'}) \\
&\leq \int_{\mathbb{R}} -(-\Delta)^\alpha(\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'})(\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&\quad + \int_{\mathbb{R}} [\partial_x^2 J^\epsilon u^\epsilon \partial_x J^\epsilon \phi^\epsilon - \partial_x^2 J^{\epsilon'} u^{\epsilon'} \partial_x J^{\epsilon'}\phi^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&\quad + \int_{\mathbb{R}} [\partial_x J^\epsilon u^\epsilon \partial_x^2 J^\epsilon \phi^\epsilon - \partial_x J^{\epsilon'} u^{\epsilon'} \partial_2^2 J^{\epsilon'}\phi^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&\quad + \int_{\mathbb{R}} [\partial_x J^\epsilon \phi^\epsilon(1-2J^\epsilon\phi^\epsilon) - \partial_x J^{\epsilon'}\phi^{\epsilon'}(1-2J^{\epsilon'}\phi^{\epsilon'})](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&=I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Now, let us analyze each term. Clearly, using (3.3), we have $I_1 \leq 0$. Next, we examine the term I_2 by adding and subtracting terms as needed.

$$
\begin{aligned}
I_2 = &\int_{\mathbb{R}} \partial_x^2 J^\epsilon u^\epsilon[\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&+ \int_{\mathbb{R}} \partial_x J^{\epsilon'}\phi^{\epsilon'}[\partial_x^2 J^\epsilon u^\epsilon - \partial_x^2 J^{\epsilon'} u^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}).
\end{aligned}
$$

The first term can be easily bounded, but bounding the second term requires using the differential equation satisfied by u^ϵ.

$$
\begin{aligned}
\int_{\mathbb{R}} &\partial_x J^{\epsilon'}\phi^{\epsilon'}[\partial_x^2 J^\epsilon u^\epsilon - \partial_x^2 J^{\epsilon'} u^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&=\frac{1}{\Lambda^2}\int_{\mathbb{R}} \partial_x J^{\epsilon'}\phi^{\epsilon'}[J^\epsilon u^\epsilon - J^{\epsilon'} u^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&\quad - \frac{a}{\Lambda^2}\int_{\mathbb{R}} \partial_x J^{\epsilon'}\phi^{\epsilon'}[J^\epsilon \phi^\epsilon - J^{\epsilon'} \phi^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&\quad - \frac{b}{2\Lambda^2}\int_{\mathbb{R}} \partial_x J^{\epsilon'}\phi^{\epsilon'}[(\partial_x J^\epsilon \phi^\epsilon)^2 - (\partial_x J^{\epsilon'} \phi^{\epsilon'})^2](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}).
\end{aligned}
$$

From this, using similar estimates as those for J_2, we deduce that

$$
I_2 \leq C\|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{H^1}^2.
$$

For the term I_3, by adding and subtracting terms, we find

$$
\begin{aligned}
I_3 = &\int_{\mathbb{R}} \partial_x J^\epsilon u^\epsilon[\partial_x^2 J^\epsilon\phi^\epsilon - \partial_x^2 J^{\epsilon'}\phi^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
&+ \int_{\mathbb{R}} \partial_x^2 J^{\epsilon'}\phi^{\epsilon'}[\partial_x J^\epsilon u^\epsilon - \partial_x J^{\epsilon'} u^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}).
\end{aligned}
$$

Integrating by parts the first term, we have

$$
\begin{aligned}
I_3 = &-\frac{1}{2}\int_{\mathbb{R}} \partial_x^2 J^\epsilon u^\epsilon(\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'})^2 \\
&+ \int_{\mathbb{R}} \partial_x^2 J^{\epsilon'}\phi^{\epsilon'}[\partial_x J^\epsilon u^\epsilon - \partial_x J^{\epsilon'} u^{\epsilon'}](\partial_x J^\epsilon\phi^\epsilon - \partial_x J^{\epsilon'}\phi^{\epsilon'}) \\
\leq &C\|J^\epsilon\phi^\epsilon - J^{\epsilon'}\phi^{\epsilon'}\|_{H^1}^2.
\end{aligned}
$$

where we have applied estimates similar to those used for J_2. Finally, to analyze I_4, it is clear that

$$I_4 \leq C\|J^\epsilon \phi^\epsilon - J^{\epsilon'} \phi^{\epsilon'}\|_{H^1}^2.$$

Therefore, we obtain

$$\partial_t \|\phi^\epsilon - \phi^{\epsilon'}\|_{H^1}^2 \leq C\|J^\epsilon \phi^\epsilon - J^{\epsilon'} \phi^{\epsilon'}\|_{H^1}^2.$$

By adding and subtracting terms we get

$$\partial_t \|\phi^\epsilon - \phi^{\epsilon'}\|_{H^1}^2 \leq C\|J^\epsilon \phi^\epsilon - \phi^\epsilon\|_{H^1}^2 + \|\phi^\epsilon - \phi^{\epsilon'}\|_{H^1}^2 + \|J^{\epsilon'} \phi^{\epsilon'} - \phi^{\epsilon'}\|_{H^1}^2.$$

And utilizing the following property of mollifiers:

$$\|J_\epsilon u - u\|_{L^2} \leq C\epsilon \|u\|_{H^1}.$$

We prove that the sequence is Cauchy. Since the sequence of approximated solutions is Cauchy in space $C([0,T^*], H^1).$, we can interpolate to show that there exists a limit.

$$\phi \in C([0,T^*], H^{3-}).$$

Furthermore, by applying the standard argument for parabolic regularity gains (see [98]), we can additionally establish that

$$\phi \in C([0,T^*], H^3) \cap L^2(0,T^*, H^{3+\alpha}).$$

To establish uniqueness, we will use a proof by contradiction. For this, we consider two solutions of (3.1) and attempt to estimate bounds on the norm $\|\phi_1 - \phi_2\|_{H^1}$. The methodology to be employed will follow the same steps as the estimates used to demonstrate that the sequence is Cauchy, and in a similar way, we will arrive at the following inequality:

$$\partial_t \|\phi_1 - \phi_2\|_{H^1}^2 \leq C\|\phi_1 - \phi_2\|_{H^1}^2.$$

Applying Gronwall's inequality, we get

$$\|\phi_1 - \phi_2\|_{H^1}^2 \leq \|\phi_1(0,x) - \phi_2(0,x)\|_{H^1}^2 e^{Ct}.$$

From this expression, we can deduce the uniqueness of the solution to (3.1).

3.2.2 Proof of global existence

Once we have established the local-in-time existence of a classical solution for (3.1), we can proceed to prove that this solution is global in time.

Proposition 3.2.3. *Assume that $0 \leq \phi_0 \in H^3$ is the initial data for (3.1). Then, the solution ϕ remains non-negative and satisfies*

$$\sup_{t\leq T} \|\phi(t,\cdot)\|_{L^\infty} \leq \max\{1, \|\phi_0\|_\infty\},$$

and

$$\sup_{t\leq T} \|\partial_x \phi(t,\cdot)\|_{L^\infty} \leq Ce^T$$

for all $t \in [0,T]$.

Proof. First, let us prove that the solution remains non-negative. Since the initial data $\phi_0(x) \geq 0$ for $x \in \mathbb{R}$, at points $\bar{x}$ where $\phi_0(\bar{x}) = 0$, we have $\partial_x \phi_0(\bar{x}) = 0$ because $\phi_0 \in H^3(\mathbb{R})$. Moreover, $-(-\Delta)^\alpha \phi_0(\bar{x}) \geq 0$ due to the positivity of ϕ_0. Thus, we have

$$\partial_t \phi(t, \bar{x})\Big|_{t=0} = -(-\Delta)^\alpha \phi(0, \bar{x}) + \partial_x u(0, \bar{x}) \partial_x \phi_0(\bar{x}) + \phi_0(\bar{x})(1 - \phi_0(\bar{x})) \geq 0.$$

This implies that there exists a t_0 such that $\phi(t, x) \geq 0$ for $t \in [0, t_0)$.

To show that the solution remains positive for all $t \in [0, T]$, assume, for contradiction, that there exists a pair $(\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}$ such that $\phi(\bar{t}, \bar{x}) = 0$. At this point, we have $\partial_x \phi(\bar{t}, \bar{x}) = 0$ and $-(-\Delta)^\alpha \phi(\bar{t}, \bar{x}) \geq 0$. Thus, $\partial_t \phi(\bar{t}, \bar{x}) \geq 0$ follows, which implies that $\phi(t, x) > 0$ for all $t \geq \bar{t}$ in any neighborhood of $\bar{x}$. Therefore, the solution ϕ remains positive for all $t \in [0, T]$.

Next, we need to check if $\|\phi(t, \cdot)\|_{L^\infty(\mathbb{R})}$ is uniformly bounded in time. Define $\phi(t, x_t) = \max_{x \in \mathbb{R}} |\phi(t, x)|$ and analyze how this maximum evolves over time. By applying the Rademacher Theorem, we can show that $\phi(t, x_t)$ is differentiable with respect to time (see [64]). Therefore, we can calculate:

$$\partial_t \phi(t, x_t) = -(-\Delta)^\alpha \phi(t, x_t) + \partial_x u(t, x_t) \partial_x \phi(t, x_t) + \phi(t, x_t)(1 - \phi(t, x_t)).$$

Because of the properties of the fractional Laplacian, we have

$$-(-\Delta)^\alpha \phi(t, x_t) \leq 0,$$

thus

$$\partial_t \phi(t, x_t) \leq \phi(t, x_t)(1 - \phi(t, x_t)).$$

From this, we deduce that the maximum will be bounded by a logistic function, leading to the result that $\|\phi\|_{L^\infty} \leq \max\{1, \|\phi_0\|_\infty\}$ for all $t \in [0, T]$.

Next, we analyze the behavior of $\|\partial_x \phi(t, \cdot)\|_{L^\infty(\mathbb{R})}$ for $t \in [0, T]$. To do this, we will use $(-\Delta)^\alpha (\partial_x \phi)$. Since $\phi \in C([0, T], H^3) \cap L^2([0, T], H^{3+\alpha/2}(\mathbb{R}))$, and by Sobolev embeddings, $\phi(t, \cdot) \in C^{2+1/2+\alpha/2}(\mathbb{R})$ for almost every $t \in [0, T]$. Therefore, $(-\Delta)^\alpha (\partial_x \phi)$ is continuous in x for almost every $t \in [0, T]$.

Using the previous procedure, we have:

$$\partial_t \partial_x \phi = -(-\Delta)^\alpha (\partial_x \phi) + \frac{u - a\phi - \frac{b}{2}(\partial_x \phi)^2}{\Lambda^2} \partial_x \phi + \partial_x u \partial_x^2 \phi + \partial_x \phi (1 - 2\phi).$$

Define $M(t) = \max_{x \in \mathbb{R}} \partial_x \phi(t, x) \geq 0$ and $m(t) = \min_{x \in \mathbb{R}} \partial_x \phi(t, x) \leq 0$. We aim to determine $\partial_t \|\partial_x \phi(t, \cdot)\|_{L^\infty}$. Suppose $\|\partial_x \phi(t, \cdot)\|_{L^\infty} = M(t)$. Then, $M(t)$ satisfies the following integral expression:

$$M(t) \leq M(0) + \int_0^T \left[\frac{u - a\phi - \frac{b}{2} M^2}{\Lambda^2} M + M \right] ds.$$

Performing the same computation for $m(t)$ and noting that m is negative, we get:

$$(-m)(t) \leq (-m)(0) + \int_0^T \left[\frac{u - a\phi - \frac{b}{2} m^2}{\Lambda^2} (-m) + (-m) \right] ds.$$

Observe that both expressions have the same structure, allowing us to use the same bounds for u and ϕ to approximate them. Hence, by noting that $\|u\|_{L^\infty} \leq a\|\phi\|_{L^\infty} + \frac{b}{2}\|\partial_x \phi\|^2_{L^\infty}$, we obtain:

$$M(t) \leq M(0) + \int_0^T \frac{\Lambda^2 + 2a\|\phi\|_{L^\infty}}{\Lambda^2} M \, ds,$$

$$(-m)(t) \leq (-m)(0) + \int_0^T \frac{\Lambda^2 + 2a\|\phi\|_{L^\infty}}{\Lambda^2} (-m) \, ds,$$

from which the uniform bound on the derivative follows. $\square$

Chapter 4

Singular patterns in Keller Segel-type models

4.1 Introduction

The investigation of collective behavior in biological systems, such as defense strategies, invasion patterns, and resilience, has become a prominent field of research that necessitates a multidisciplinary approach. In addition to the inherent value of studying species like birds, fish, and ants, these models also offer insights into microscopic processes such as cell morphogenesis and tumor development. Recent advancements in microscopy and antibody concentration techniques have made it possible to directly model collective behavior at both cellular and subcellular levels. This has revitalized interest in the chemotaxis models originally developed by Keller and Segel (KS), which describe the movement of biological entities in response to chemical gradients. Over time, these models have been extended and adapted to new contexts, leading to more versatile applications. While the classical KS model involves a coupled system of reaction-diffusion equations to represent cell dynamics and chemoattractant concentration, contemporary approaches introduce more flexible variants, including hyperbolic and parabolic limits, to achieve better alignment with experimental observations.

In this Chapter, we study two variants of the flux-saturated Keller-Segel (FSKS) model, which could offer a better representation of biological phenomena, particularly in terms of saturation of velocities and cell movement limits.

$$\begin{aligned}&\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - a\,U\,\frac{\partial Q}{\partial x}\right)\\&\mathcal{T}(Q) = U,\end{aligned} \tag{4.1}$$

where $\mathcal{T} = \mathcal{T}(Q)$ is one of the following linear differential operators

$$\mathcal{T}(Q) = \frac{\partial^2 Q}{\partial t \partial x} - \nu \frac{\partial^2 Q}{\partial x^2}, \tag{4.2}$$

or

$$\mathcal{T}(Q) = \tau \frac{\partial Q}{\partial t} + \alpha \frac{\partial Q}{\partial x} - \nu \frac{\partial^2 Q}{\partial x^2}. \tag{4.3}$$

The parameters $\alpha \geq 0$ and $\nu \geq 0$ stands for the transport and diffusion coefficients respectively. The flux function $\Phi = \Phi(s)$ is a bounded, regular, increasing and odd function. The value $c > 0$ is defined as

$$c = \lim_{s \to \infty} \Phi(s),$$

and it is finite. Also $\Phi \in C^1(\mathbb{R})$ in order to have uniquenees of the initial value poblems. The value $\mu = \Phi'(0)$ is the kinematic viscosity for small velocities and near $u_x = 0$ the flow means

$$U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) \sim \mu U^{m-1} U_x,$$

being $m \geq 1$ a parameter that measures the porosity of the medium.

The FSKS model combines the porous medium operator with flux-saturation mechanisms to address some limitations of the classical KS model, such as the difficulty to capture compact-support soliton-type solutions often observed in experimental settings. These solutions, characterized by sharp interfaces, are especially relevant in modeling collective cell invasion and propagation patterns, which cannot be reproduced with linear diffusion terms in traditional models.

One of the central objectives of this work is the analysis of soliton-type patterns with compact support in the FSKS model. These patterns, which resemble biological pulses or wavefronts, emerge in situations where traditional KS models fail. Different flux-saturated operators, including relativistic and hyperbolic tangent forms, are explored to capture these solutions. We analyze cases where transport and diffusion terms are combined, as well as when the time evolution of the chemoattractant is negligible, resulting in a self-generated potential by the cell density.

In Figure 1 we provide various configurations of some of the results that we obtain here, coming from the analysis of the nonlinear variants of the KS models that we will study throughout the Chapter.

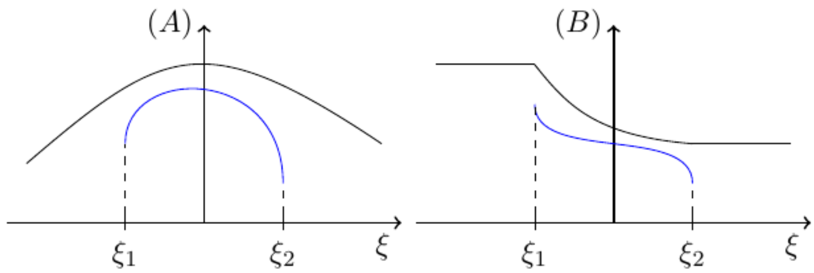


Figure 4.1: Pattern prototypes with compact support associated with flux–saturated operators. Pattern prototypes with compact support associated with saturated-flux operators. In blue we represent the cell concentration and in black the profile of the chemoattractant..

The Chapter is structured as follows. Section 4.2 is devoted to defining the solution concept and the soliton-type geometric structure of the solutions we seek. In this sense, we define the block-type solution, which will be the object of study in this paper. Section 4.3 deals with the case in which the chemoattractant gradient is transported without diffusion, proving that any maximal solution of the associated dynamic system is a block-type solution. Section 4.4 deals with the case where transport and diffusion terms are combined in the chemoattractant. Conditions are given for the existence and non-existence of block-type solutions. In the case of diffusion without transport of the chemoattractant, a complete analysis of all types of traveling waves is carried out (particularly those cases in which there are block-type solutions), classifying all types of solutions according to the system parameters.

4.2 Block solitons moving at a constant speed.

Let us consider the general system of cell evolution U together with the chemoattractant Q

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right) \\ \tau_1 \frac{\partial Q}{\partial t} &+ \tau_2 \frac{\partial}{\partial t}\left(\frac{\partial Q}{\partial x}\right) + \alpha \frac{\partial Q}{\partial x} - \nu \frac{\partial^2 Q}{\partial x^2} = U. \end{aligned} \tag{4.4}$$

In this model, the coefficient $\nu \geq 0$ is the viscosity of the chemoattractant. Additionally, we assume that $a > 0$, $\alpha > 0$ and $m \geq 1$. The parameters τ_1 and τ_2 are taken greater than or equal to zero, in fact in the models considered τ_1 and τ_2 take values $\{0, 1\}$.

We are going to study the existence of biological blocks that move at a constant speed. Mathematically, the concept of block solutions is associated to that of traveling waves type solutions of the previous problem (4.4) for cell dynamics whose mass is concentrated in a bounded region. If $\sigma > 0$ denotes a speed of propagation, we look for solutions of the kind $U(t,x) = u(x - \sigma t)$, $Q(t,x) = q(x - \sigma t)$, where u, $q : \mathbb{R} \to \mathbb{R}$ are scalar functions and u has the mass concentrated in a compact interval. Formally, the resulting system for u, q verifies

$$\begin{aligned} -\sigma u'(\xi) &= \left(u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(z)}\right) - au(\xi)q'(\xi)\right)', \\ (\alpha - \sigma\tau_1)q'(\xi) &- (\sigma\tau_2 + \nu)q''(\xi) = u(\xi), \end{aligned} \tag{4.5}$$

where $\xi := x - \sigma t$.

A first question to consider is the concept of solution for (4.4). The appropriate framework for our analysis is that of solutions of bounded variation. However, the theory of existence in the context of bounded variation solutions for KS-type systems is not sufficiently fully established, and this is not the aim of our paper. To avoid entering the theory of bounded variation functions of several variables, we are going to focus on our study on the equation (4.5) directly.

The cell structure that gives rise to the u component is going to be assumed to be much larger and heavier than the molecular structure of the chemoattractant given by the q component, therefore a singularization of the component

u can be expected. This appreciation is supported by the presence of a flux–saturated as the basis of the movement of u in the first equation On the other hand, we expect a milder behavior of the chemoattractant q. The formation of discontinuities in q is not expected if

$$\sigma\tau_2 + \nu > 0. \tag{4.6}$$

Even in the degenerate case $\tau_2 = 0 = \nu$ the existence of fronts is not apparent because in q, even in that case, we have a linear transport equation. These reasons make us assume that q is of class 1 in $\mathbb{R}$ while u is only going to be a bounded variation function. We are in a position to consider distributional solutions imposing that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - au(\xi)q'(\xi) + \sigma u(\xi)\right)\psi'(\xi)d\xi = 0,\\
&(\alpha - \sigma\tau_1)\int_{\mathbb{R}} q(\xi)\psi'(\xi)d\xi - (\sigma\tau_2 + \nu)\int_{\mathbb{R}} q'(\xi)\psi'(\xi)d\xi = -\int_{\mathbb{R}} u(\xi)\psi(\xi)d\xi.
\end{aligned} \tag{4.7}$$

holds, for each $\psi \in \mathcal{D}(\mathbb{R})$. In this expression the role of $u'(\xi)$ has to be clarified. When a function $u \in BV(\mathbb{R})$ its derivative in the sense of the distributions Du decomposes as an absolutely continuous part $u'(\xi)$ and a singular part $D_s u$ that is orthogonal to the Lebesgue measure. The singular part of the measure is not easy to absorb, see [4]. The set $\mathcal{S} = \mathrm{supp}\{D_s u\}$ is called the set of singularities of u. It is common in this type of operators that $\mathcal{S}$ is a finite set and also $u \in C^1(\mathbb{R}\backslash\mathcal{S})$.

A block structure is going to be requested on u. This concept of block solution materializes in the existence of a compact interval $[\xi_1, \xi_2]$, not reduced to a point, such that $u(\xi) > 0$, for a.e. $\xi \in [\xi_1, \xi_2]$, and $u(\xi) = 0$, otherwise. For q no restrictions will be imposed on its support. One last assumption is that the singularities of u have been formed by the saturation of the cell flux. If $\bar\xi \in \mathcal{S}$ is a singular point, then the lateral limit values are always defined. The point $\bar\xi$ is said of saturation to the left if

$$\lim_{\xi\to\bar\xi} u'(\xi) = -\infty, \qquad \lim_{\xi\to\bar\xi^-} u(\xi) \geq \lim_{\xi\to\bar\xi^+} u(\xi),$$

while it will be saturation to the right if

$$\lim_{\xi\to\bar\xi} u'(\xi) = +\infty, \qquad \lim_{\xi\to\bar\xi^-} u(\xi) \leq \lim_{\xi\to\bar\xi^+} u(\xi).$$

If $\bar\xi$ is a boundary point of the support, then the saturation condition is only one-sided, that is, if $\bar\xi = \xi_1$, then $u(\xi) = 0$, for $\xi < \xi_1$, which means

$$\lim_{\xi\to\xi_1^+} u'(\xi) = \infty,$$

and a symmetric condition on ξ_2.

Lemma 4.2.1. *Assume that (4.6) holds, then there are no saturation points inside the support.*

Proof. It follows from (4.7) that the function $u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - au(\xi)q'(\xi) + \sigma u(\xi)$ has zero weak derivative. By Stampacchia's Lemma $u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - au(\xi)q'(\xi)+\sigma u(\xi) = K$, for some constant K. We can assume that this constant is going to be zero when considering ξ outside the support of u. Therefore, we have

$$u^{m-1}(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - aq'(\xi) + \sigma = 0, \qquad \text{a.e. } \xi \in [\xi_1, \xi_2]. \tag{4.8}$$

Whence, both values

$$\lim_{\xi \to \bar{\xi}^{\pm}} u(\xi) = u_{\pm},$$

verify

$$u_{\pm}^{m-1}c - aq'(\bar{\xi}) + \sigma = 0,$$

where it has been used that we have a saturation on the left. Then, both one–sided limits are equal. That implies the continuity of u in $\bar{\xi}$. Using the regularity and the (4.6) condition in the second equation of (4.7) we get that q'' is defined in $\bar{\xi}$. In particular the function

$$\xi \to cu^{m-1}(\xi) - q'(\xi) + \sigma,$$

has infinite derivative at $\bar{\xi}$, and it is an increasing function in a neighborhood of that point. This would give us ξ values such that $\xi \to cu^{m-1}(\xi) - q'(\xi) + \sigma < 0$, which is contradictory to (4.8) since $\Phi(s) < c$, for all $s \in \mathbb{R}$. $\square$

In conclusion, assuming (4.6) we can define a block–type solution as follows:

Definition 4.2.1. Given an interval $[\xi_1, \xi_2]$, we will say that a pair of functions

$$u \in C^0[\xi_1, \xi_2] \cap C^1(\xi_1, \xi_2) \text{ and } q \in C^0[\xi_1, \xi_2] \cap C^2(\xi_1, \xi_2)$$

constitute a block-type solution as long as

- $u(\xi) > 0$, for each $\xi \in [\xi_1, \xi_2]$ and both verify

$$\begin{aligned} &u^{m-1}(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - aq'(\xi) + \sigma = 0 \\ &(\alpha - \sigma\tau_1)q'(\xi) - (\sigma\tau_2 + \nu)q''(\xi) = u(\xi). \end{aligned} \tag{4.9}$$

- The singular points are $\mathcal{S} = \{\xi_1, \xi_2\}$, and both are lateral saturation points for u, that is

$$\lim_{\xi \to \xi_1^+} u'(\xi) = \infty, \qquad \lim_{\xi \to \xi_2^-} u'(\xi) = -\infty.$$

Under these conditions we will discuss throughout the paper the existence of a cell block moving at speed σ, and this will be obtained by extending by zero the cell density u outside the interval $[\xi_1, \xi_2]$, q extends to a function from class 1 to $\mathbb{R}$ through two straight lines.

Taking $g = \Phi^{-1}$, in the sense of the composition of applications, then $g : (-c, c) \to \mathbb{R}$ is a C^1 function defined as:

$$g(y) = s \longleftrightarrow y = \Phi(s).$$

If, in addition, we define $r(\xi) = g'(\xi)$, we get:

$$u' = u\, g\left(\frac{ar - \sigma}{u^{m-1}}\right).$$

Finally, we can rebuild from (4.9) the following system:

$$\begin{aligned} u' &= u\, g\left(\frac{ar - \sigma}{u^{m-1}}\right), \\ r' &= \frac{(\alpha - \sigma\delta_1)r - u}{\sigma\delta_2 + \nu}. \end{aligned} \tag{4.10}$$

Note that under the hypothesis on g in the previous section, this system is defined only for values of (u, r) belonging to the domain Γ defined by

$$\Gamma := \left\{(u, r) : u > 0, |(ar - \sigma)u^{1-m}| < c\right\}. \tag{4.11}$$

such that $\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where

$$\Gamma_0 = \left\{(u, r) \in \Gamma : u > 0,\ r = \frac{\sigma}{a}\right\}, \quad \Gamma_- = \left\{(u, r) \in \Gamma : r > \frac{\sigma}{a}\right\}$$

and

$$\Gamma_+ = \left\{(u, r) \in \Gamma : r < \frac{\sigma}{a}\right\}.$$

We denote by $\gamma = \partial\Gamma = \gamma_+ \cup \gamma_- \cup \{(0, \frac{\sigma}{a})\}$, where

$$\gamma_\pm := \left\{(u, r) \in (0, \infty) \times \mathbb{R} : \frac{ar - \sigma}{u^{m-1}} = \mp c\right\}. \tag{4.12}$$

A block-type solution corresponds to a maximal solution of (4.10) such that

$$\begin{aligned} \lim_{\xi \to \bar{\xi}^\pm} u(\xi) &= u_\pm, \\ \lim_{\xi \to \bar{\xi}^\pm} r(\xi) &= r_\pm, \end{aligned} \tag{4.13}$$

where (ξ_-, ξ_+) is the maximum interval of definition and $(u_\pm, r_\pm) \in \gamma_\pm$.

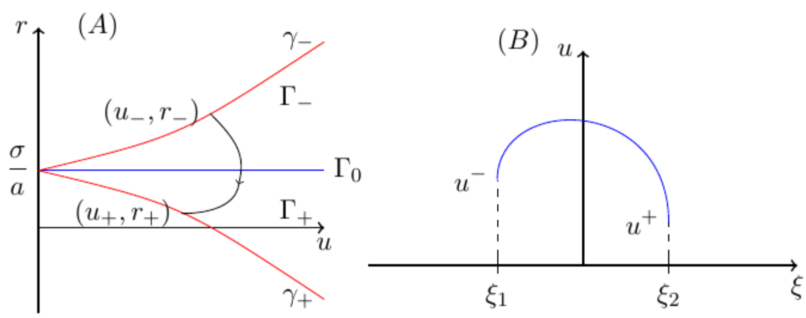


Figure 4.2: (A) Representation of the field of tangent vectors of (4.10) and (B) representation of a block type solution.

Proposition 4.2.1. *Under these conditions, the spatial support of the solution is necessarily bounded.*

Proof. Let us prove it by reductio ad absurdum. Assume that $\xi^+ = +\infty$. Then, there exists a sequence $\xi_n \to +\infty$ such that

$$u'(\xi_n) \to 0.$$

This implies

$$\frac{ar(\xi_n) - \sigma}{u^{m-1}(\xi_n)} \to 0.$$

Since u^+ is bounded, this implies that $r(\xi^n) \to \frac{\sigma}{a}$, but this is not possible since $r^+ \in \ \gamma^+$. Therefore $\xi^+ < +\infty$. The reasoning for ξ^- is analogous. $\square$

4.3 Transport in the gradient of Q

This section is devoted to analyze the case where the gradient of the chemoattractant $(\partial_x Q)$ is solution of the non-homogeneous linear transport equation:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right) \\ \frac{\partial}{\partial t}\left(\frac{\partial Q}{\partial x}\right) &- \nu \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x}\right) = U. \end{aligned} \tag{4.14}$$

In this model we asume that $\nu > 0$ so we are in a non-degenerate situation (remember that $a > 0$). The case $m = 1$ can be analyzed following the guidelines of the case $m > 1$, therefore we also assume that $m > 1$. As we will see, this model can be seen as a particular case of the one analyzed in the next section where the values would have another expression, but we have considered studying this case first for clarity in the exposition.

Hence, we will focus on study the existence of block solutions, defined in the previus section, which correspond with the search of orbits of the diferential equation

$$\begin{aligned} u' &= u\, g\left(\frac{ar - \sigma}{u^{m-1}}\right) \\ r' &= -\frac{u}{\sigma + \nu}. \end{aligned} \tag{4.15}$$

This orbits connect γ_- with γ_+, where $\gamma_\pm$ were defined in (4.12). Therefore, for any inicial condition in Γ defined in (4.11), we will be able to find a block solution, see Figure 4.3.

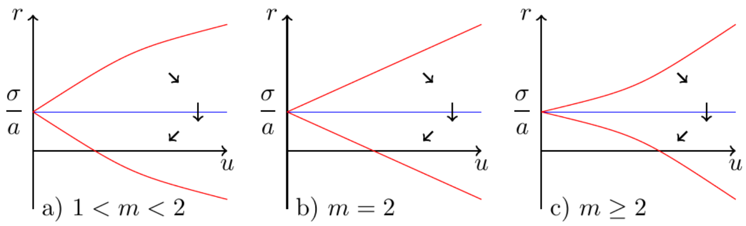


Figure 4.3: Representation of the tangent vector field associated to (4.15) .

Theorem 4.3.1. *Every maximal solution of (4.15) is a block solution.*

A key ingredient to prove this Theorem are the following results, in which we will show that there exist an orbit connecting γ^- with γ^+, for every initial condition in the line $s = \frac{\sigma}{a}$.

Proposition 4.3.1. *Consider the initial conditions $r(0) = \frac{\sigma}{a}$ and $u(0) = u_0 > 0$ associated to (4.15). Then, there exists a block solution (u, r) corresponding to these initial data.*

Proof. Let us prove that the maximal solution of the previous problem gives rise to a block solution, by seeing that it connect a point in γ^- with a point in γ^+, according to the definition of the previous section.

We can check that $u(\xi)$ has uni-modal shape with a unique maximum at $\xi = 0$, and $r(\xi)$ is a strictly decreasing function. Therefore, we can prove the existence of the finite limits

$$\lim_{\xi \to \xi_\pm} (u(\xi), r(\xi)) =: (u_\pm, r_\pm). \tag{4.16}$$

Due to the decrease of r we obtain $r_+ < \frac{\sigma}{a} < r_-$, and taking limits in the inequality

$$\left| \frac{ar - \sigma}{u^{m-1}} \right| < c,$$

we deduce that $u_\pm$ are necesarily strictly positive.

Let us see that

$$r_+ = \frac{\sigma - c u_+^{m-1}}{a}.$$

If $r_+ \neq \frac{\sigma - c u_+^{m-1}}{a}$, then (u^+, r^+) is not in the boundary of Γ, and by a prolongation argument we get $\xi^+ = +\infty$. Therefore (u^+, r^+) will be a critical point. However, there are no critical points in the problem, so $\xi^+ < +\infty$. Using again a prolongability argument we obtain that (u^+, r^+) is in the boundary of Γ.

Also, we can prove that

$$r_- = \frac{\sigma + c u_-^{m-1}}{a},$$

by using similar arguments. □

Once demonstrated the existence of solutions for initial conditions in the vertical isocline, we will proceed to prove that for every initial condition in Γ, the associated solutions always reach the vertical isocline.

Lemma 4.3.1. *Every maximal solution of (4.15) intersect the curve Γ_0.*

Proof. Let $u, r : (\xi^-, \xi^+) \to \mathbb{R}$ a solution of (4.15), and assume that for some value $\xi_0 \in (\xi^-, \xi^+)$ we have $(u_0, r_0) := (u(\xi_0), r(\xi_0)) \in \Gamma$. Let us prove that if $r_0 > \frac{\sigma}{a}$, then there exits a $\xi_1 \in (\xi^-, \xi^+)$, such that $r(\xi_1) = \frac{\sigma}{a}$. Similarly, if $r_0 < \frac{\sigma}{a}$, we can find a value ξ_2 such that $r(\xi_2) = \frac{\sigma}{a}$, which will conclude the proof.

Assume that $r_0 > \frac{\sigma}{a}$ and $r(\xi) > \frac{\sigma}{a}$, for all $\xi \in (\xi^-, \xi^+)$ and hence, $u'(\xi) > 0$ and there exist the $\lim_{\xi \to \xi^+} u(\xi)$. If this limit is finite, then $\xi^+ = +\infty$ and we will have a critical point, but this is not possible. Therefore, we have

$$\lim_{\xi \to \xi^+} u(\xi) = +\infty. \tag{4.17}$$

Since $u(\xi) \geq u_0$ and $r(\xi) \leq r_0$, then we obtain:

$$g\left(\frac{ar(\xi)-\sigma}{u^{m-1}(\xi)}\right) \leq g\left(\frac{ar_0-\sigma}{u_0^{m-1}}\right),$$

for $\xi \in (\xi_0, \xi^+)$. Using that $r \to \frac{ar-\sigma}{u^{m-1}}$ is increasing in r, then we deduce that $u \to \frac{ar-\sigma}{u^{m-1}}$ is decreasing in u and $y \to g(y)$ is derecreasing. Defining M as the value obtained above, we have that

$$u'(\xi) \leq Mu(\xi).$$

Then, combining the Gronwall Lemma together with (4.17) we deduce $\xi_+ = +\infty$. However, this can not be possible, because $r'(\xi) = -\frac{1}{\sigma+\mu}u(\xi)$ and $u(\xi) \geq u_0$, for $\xi \in (\xi_0, \xi^+)$, and we obtain:

$$r'(\xi) \leq -\frac{1}{\sigma+\mu}u_0 < 0, \ \forall \xi \in (\xi_0, \xi^+),$$

which is a contradiction. □

Therefore, thanks to these two results and the fact that the solutions are invariant under time translation, the proof of Proposition 4.3.1 follows.

4.4 Transport and diffusion in the chemoattractant

In this section we consider that the chemoattractant concentration is solution of a linear transport-diffusion equation.

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right), \\ \frac{\partial Q}{\partial t} &+ \alpha\frac{\partial Q}{\partial x} - \nu\frac{\partial^2 Q}{\partial x^2} = U, \end{aligned} \tag{4.18}$$

where α is the transport speed coefficient of the chemoattractant density and ν stands for its diffusion coefficient. Our goal is to study the existence of orbits that connect γ^- with γ^+, defined in (4.12), of the differential equation:

$$\begin{aligned} u' &= u\, g\left(\frac{ar-\sigma}{u^{m-1}}\right), \\ r' &= \frac{1}{\nu}\left((\alpha-\sigma)r - u\right). \end{aligned} \tag{4.19}$$

The analysis of the existence of such orbits will be studied in terms of the parameter m and the relation between α and σ.

Remark 4.4.1. *If $\sigma = \alpha$, the differential equation (4.19) is similar to (4.15). Therefore, we will have existence of block solutions for $\sigma > 0$, thanks to Theorema 4.3.1.*

The main result describing the existence of block solution in this context is the following one.

Theorem 4.4.1. *Block solutions exist if one of the following condition hold true*

- *If* $1 < m < 2$ *and* $\sigma > 0$.
- *If* $m \geq 2$, $\sigma > 0$ *and* $\alpha \leq \alpha^*$, *for* $\alpha^* > 0$, *where*

$$\alpha^* = \left(\frac{a}{m-1}\right)^{\frac{m-1}{2m-3}} \frac{1}{c^{3-2m}}(m-2)^{\frac{m-2}{2m-3}}. \tag{4.20}$$

- *If* $m \geq 2$, $\sigma > \sigma^*(\alpha) > 0$ *and* $\alpha > \alpha^* > 0$, *where*

$$\sigma^*(\alpha) = \alpha - \frac{a}{m-1} c^{\frac{1}{1-m}} \left(\frac{m-2}{\alpha}\right)^{\frac{m-2}{m-1}}. \tag{4.21}$$

Remark 4.4.2. *In the case* $m = 2$, *we have that* $\sigma^*(\alpha) = \alpha - \frac{a}{c}$, *which is the limit when* $m \to 2$ *of* (4.21), *and therefore solving* $\sigma^*(\alpha) = 0$, *we obtain the value* $\alpha^* = \frac{a}{c}$.

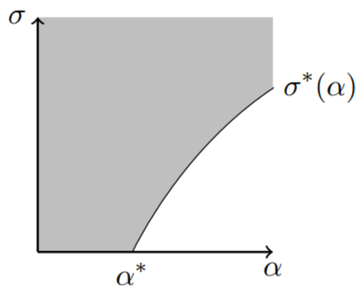


Figure 4.4: Representation of the region of existence obtained in the Theorem 4.4.1 for $m \geq 2$.

To carry out the proof of Theorem 4.4.1, we are going to introduce a series of previous results.

Let us denote by η the horizontal isocline, whose equation is $r = \frac{1}{\alpha - \sigma} u$ and represents the points $(u, r) \in \Gamma$, where $r' = 0$. Firstly, we will analyze the case in which η has positive slope, and we will focus on finding some initial values $(\bar{u}, \bar{r})$ from which we can construct the solution.

Lemma 4.4.1. *Let* $1 < m \leq 2$. *If* η *has positive slope and cuts* γ^-, *then there exist* $(\bar{u}, \bar{r}) \in \Gamma_-$ *such as:*

$$r' < 0, \ u' > 0, \ \forall (u, r) \in B,$$

where $B := B_{(\bar{u},\bar{r})} = \{(u, r) \in \Gamma_- : \ u \leq \bar{u}, \quad r \geq \bar{r}\}$.

Proof. Since $1 < m \leq 2$, then γ^- is the graph of a concave function or a straight line. It is easy to see that an increasing line below $u = 0$ will intersect γ^- at a single point. u^*, see Figure 4.5. Therefore, it allow us to find a curved triangular region, denoted by B, just build taking as vertex of B any points $(\bar{u}, \bar{r}) \in \Gamma^-$ such that $\bar{u} > u^*$ and $\bar{r} > \gamma_-(u^*)$. $\square$

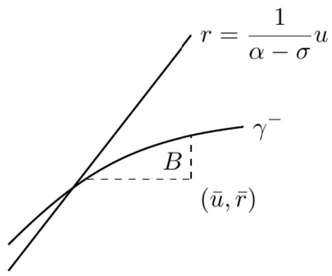


Figure 4.5: Scheme of the proof of the Lemma 4.4.1

The points $(\bar{u}, \bar{r})$, defined in Lemma 4.4.1, will allow us to construct the desired orbits of (4.19) when these are taken as initial data.

Proposition 4.4.1. *If η has positive slope and cuts γ^-, then there exists a block solution of (4.19).*

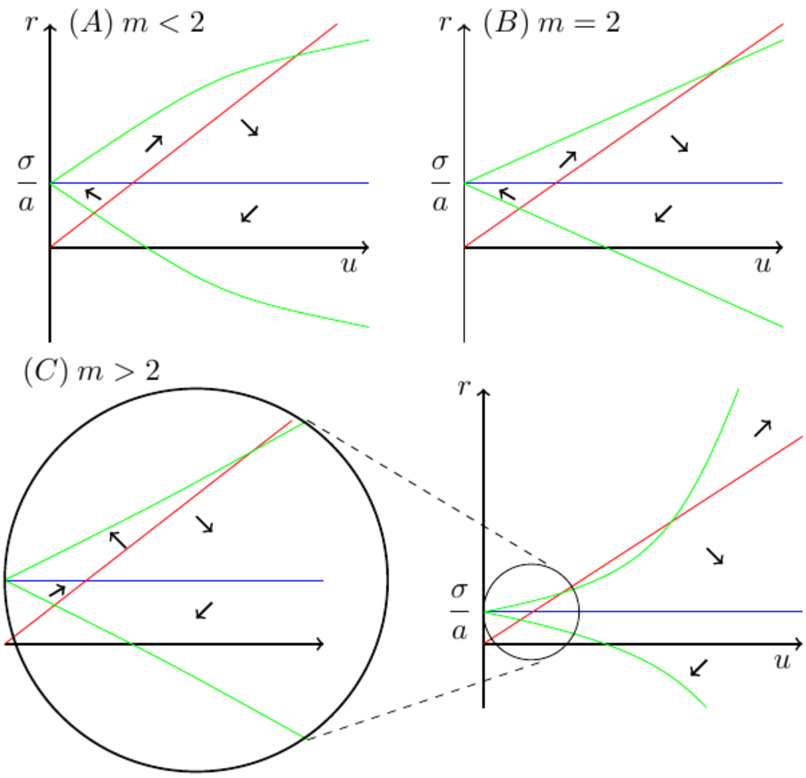


Figure 4.6: Representation of the phase diagram as a function of m, when η intersects γ^+.

Proof. In the case $1 < m \leq 2$, let us take the points $(\bar{u}, \bar{r})$, previously defined in the Lemma 4.4.1, as the initial condition of the problem (4.19). The solution of the initial value problem $(\bar{u}, \bar{r})$ remains in B as long as it is defined. Bendixson's theorem assures us that this solution has to touch γ^-, as $\xi \to \xi^-$, since there is no equilibrium points in the set B. Moreover, this solution will always intersect the line $r = \frac{\sigma}{a}$, for some $\xi_1 \in (\xi^-, \xi^+)$. This is because $r' < 0$ and $u' > 0$, for

$r > \frac{\sigma}{a}$. An argument similar to that used in Lemma 4.3.1 allow us to prove the existence of a value ξ_1 at which the solution intersects the straight line $r = \frac{\sigma}{a}$.

We cannot know how the solutions (u_1, r_1), connecting γ^- with $r = \frac{\sigma}{a}$, will behave once they go through the line $r = \frac{\sigma}{a}$.

Let us define $u^* = u_1(\xi_1)$. Observe that the solution of (4.19), with initial condition $u(0) = \tilde{u}$, $r(0) = \frac{\sigma}{a}$, will touch γ^- when $\xi \to \xi^-$, for any value $\tilde{u} \geq u^*$. This is due to the fact that the orbits of the autonomous systems can not intersect.

Therefore, to finish the proof it remains to find a value $\tilde{u}$ such that the solution of the initial value problem $(\tilde{u}, \frac{\sigma}{a})$ reaches γ^+.

Let $(\hat{u}, \hat{r})$ be the intersection of the straight line η with the curve γ^+. Then, the solution of the initial value problem $(\hat{r}, u^*)$ touches γ^+ as $\xi \to \xi^+$, due to the fact that $r' < 0$ for $r \leq \hat{r}$. In addition, this solution will intersect the straight line $r = \frac{\sigma}{a}$ at a point $(\tilde{u}, \frac{\sigma}{a})$, by a symmetric argument to the one made in Lemma 4.3.1.

In the case $m > 2$, the proof is carried out in a similar way. Indeed, if η intersects γ^-, this intersection can be made at two points or at one tangent point. In both cases, we can define

$$\underline{u} = \max \left\{ u \in (0, +\infty) : \frac{1}{\alpha - \sigma} u = \gamma^-(u) \right\}.$$

Therefore, the set

$$A = \{(u, r) \in \Gamma_- : u > \underline{u}, \quad r \geq \eta(u)\}$$

is negatively invariant, since $u' > 0$ and $r' \geq 0$, for all $(u, r) \in A$. Therefore, following the same ideas as in the previous case, we can show that the solution connects γ^- to $r = \frac{\sigma}{a}$, reaching the line $r = \frac{\sigma}{a}$, for some $\xi_1 \in (\xi^-, \xi^+)$. So, as in the previous case, we are able to find an initial condition $(\tilde{u}, \frac{\sigma}{a})$ whose solution connects γ^- with γ^+. □

Let us now study the case in wich η has negative slope.

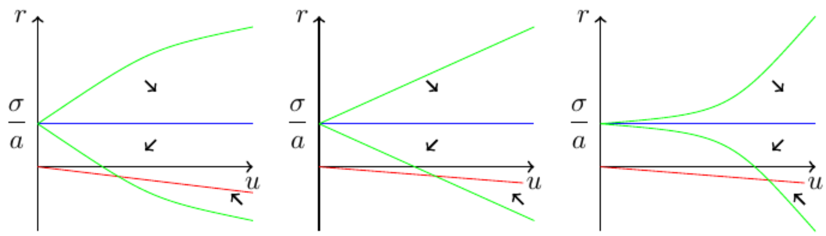
Proposition 4.4.2. *If η has negative slope, then there exists a block solution of* (4.19).

Proof. The directions of the vector field show that the solution with initial data $(\hat{u}, \frac{\sigma}{a})$ will touch γ^- and γ^+, for any point $(\hat{u}, \frac{\sigma}{a})$ in the straight line $r = \frac{\sigma}{a}$ which is to the left of the intersection point of the straight line η with γ^+. The proof argument is similar to the one made in the proof of Proposition 4.2.1. □

These results inform us under which conditions we can find block solutions. To finish the proof of the Theorem 4.4.1, it is necessary to see what relationships between the parameters allow us to obtain these solutions.

Proof. (Proof of Theorem 4.4.1). The equation of the line η is $\frac{1}{\alpha - \sigma} u$. If $\sigma > \alpha$, η has negative slope, and by Proposition 4.4.2 we have existence of a block solution. If $\sigma = \alpha$, we have also existence of solution by arguing as in Remark 4.4.1.

On the other hand, if $\sigma < \alpha$, we have to study the relative position between γ^- and η. Proposition 4.4.1 establishes the existence of solution when η has



(A) $1 < m < 2$ (B) $m = 2$ (C) $m > 2$

Figure 4.7: Representation of the phase diagram as a function of m, when η is decreasing.

positive slope. Therefore, we have to analyze the possibilities of intersection between η and γ^-.

If $1 < m < 2$, η will always intersect γ^-, when $0 < \sigma < \alpha$.

If $m = 2$, the slope of η must be greater than the slope of γ^-, which is a line in this case. This is fulfilled when $\frac{c}{a} < \frac{1}{\alpha-\sigma}$, i.e.,

$$\sigma > \alpha - \frac{a}{c} \tag{4.22}$$

Finally, for $m > 2$, the intersection points are given by the roots of the equation:

$$\frac{cu^{m-1}}{a} + \frac{\sigma}{a} - \frac{u}{\alpha - \sigma} = 0.$$

The existence of roots of this equation is equivalent to prove that the minimum takes negative values, this holds true for

$$\sigma > \alpha - \frac{a}{m-1} c^{\frac{1}{1-m}} \left(\frac{m-2}{\alpha}\right)^{\frac{m-2}{m-1}} = f(\alpha).$$

Note that if $m = 2$ this equation coincides with (4.22), as $m \to 2$.

It can be seen that $f(\alpha) \leq 0$, if $\alpha \leq \alpha^*$, where

$$\alpha^* = \left(\frac{a}{m-1}\right)^{\frac{m-1}{2m-3}} \frac{1}{c^{3-2m}} (m-2)^{\frac{m-2}{2m-3}}.$$

If $\alpha > \alpha^*$, then the expresion $\sigma > f(\alpha)$ is equivalent to $\sigma > \sigma^*(\alpha)$ which is given by (4.21) □

4.4.1 Non-existence of block solution.

Once we have analyzed the existence of solution in the previous section, let us see under what conditions we can prove the non-existence of block solutions.

For this purpose, we will consider the function $\theta : (-c, c) \to \mathbb{R}$ defined as:

$$\theta(y) = \frac{(\alpha - \sigma)y}{\nu} - (m-1)yg(y).$$

Observe that the function satisfies that $\theta(-c) = \theta(c) = -\infty$, therefore there exists the value $\theta_0 = \max_{y \in (-c,c)} \theta(y)$.

On the other hand, let us consider the function $\omega : (0, +\infty) \to \mathbb{R}$ defined as:

$$\omega(u) = \frac{au - \sigma(\alpha - \sigma)}{\nu u^{m-1}}.$$

Arguing as before, let us consider the value $\omega_0 = \max_{u \in (0,+\infty)} \omega(u)$, whose expression is:

$$\omega_0 = \frac{1}{v} \left(\frac{\sigma(\alpha - \sigma)}{m-2} \right)^{2-m} \left(\frac{a}{(m-1)} \right)^{m-1}.$$

With these two constants we obtain the following non-existence result.

Theorem 4.4.2. *If $\theta_0 > \omega_0$, then there is no block-type solution.*

Proof. Let us take $y(\xi) = \frac{ar(\xi)-\sigma}{u^{m-1}(\xi)}$, which satisfies the differential equation

$$\begin{aligned} u' &= ug(y), \\ y' &= \frac{\frac{a}{v}\left[(\alpha - \sigma)\left[\frac{u^{m-1}y+\sigma}{a}\right] - u\right] - (m-1)u^{m-1}yg(y)}{u^{m-1}}. \end{aligned} \tag{4.23}$$

Observe that a block-type solution is now a connection between $y = c$ and $y = -c$. Taking $\bar{y}$ such that $\theta(\bar{y}) > \max_{u \in (0,+\infty)} \omega(u)$ then the expression of the second equation of (4.23) provides $y' > 0$, for all $u \in (0, +\infty)$. Therefore, it would not be possible to connect $y = c$ with $y = -c$. □

Remark 4.4.3. *For example, in the Wilson operator $g(u)$ is defined by $g(u) = \frac{1}{\mu}\frac{u}{1-\frac{|u|}{c}}$. Then, we can calculate explicitly the values of θ_0 and ω_0. Those values are:*

$$\begin{aligned} \theta_0 &= c\left(\sqrt{\left(\frac{\alpha - \sigma}{\nu} + (m-1)\frac{c}{\mu}\right)} - \sqrt{(m-1)\frac{c}{\mu}}\right)^2, \\ \omega_0 &= \frac{1}{\nu}\left(\frac{\sigma(\alpha-\sigma)}{m-2}\right)^{2-m}\left(\frac{a}{m-1}\right)^{m-1}. \end{aligned}$$

Combining and approximating them we obtain the following inequality

$$\sigma^{\frac{m}{2}-1}(\alpha - \sigma)^{\frac{m}{2}} \geq \Theta\left(\sqrt{\left(\frac{\alpha}{\nu} + (m-1)\frac{c}{\mu}\right)} + \sqrt{(m-1)\frac{c}{\mu}}\right), \tag{4.24}$$

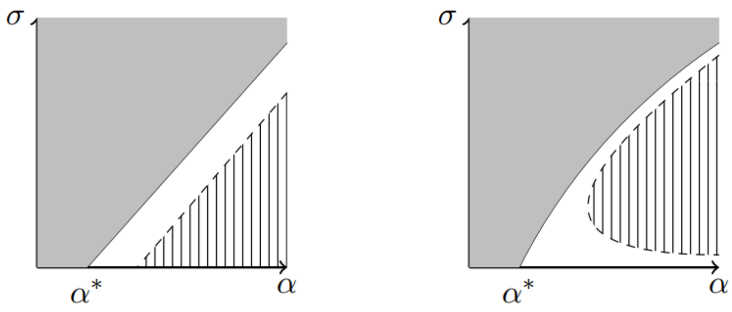
where

$$\Theta = \sqrt{\frac{\nu}{c}}\left(\frac{a}{m-1}\right)^{\frac{m-1}{2}}\frac{1}{(m-2)^{\frac{2-m}{2}}}.$$

Using Theorem 4.4.1 and Theorem 4.4.2 we can establish the region of existence and non-existence of solution for a given parametric configuration. (See Figure 4.8)

Observe that the left-hand side of the inequality 4.24 has uni-modal shape and, therefore, we obtain an interval of σ-values for which there is no solution, with all parameters fixed. We can see this behavior in Figure 4.8. Note that in the limit case $m = 2$, the region of non-existence is bounded by a straight line.

Moreover, it has been numerically observed, it is possible to find a block-type solution under certain parameter settings for $m > 2$ in the region between the non-existence zone and $\sigma = 0$.



(A) $m = 2$. (B) $m = 3$.

Figure 4.8: Representation of the region of existence (gray region) and non-existence (pattern region) of solution in Wilson's model, for $c = \mu = a = \nu = 1$.

4.4.2 Case $m = 1$

In the previous sections we have always considered the case $m > 1$, for which we have shown the existence and non-existence of solution under certain configurations of the parameters. But what happens in the case $m = 1$?

Taking $m = 1$, the system can be written as follows

$$\begin{aligned} u' &= ug\left(ar - \sigma\right), \\ r' &= \frac{1}{\nu}((\alpha - \sigma)r - u). \end{aligned} \tag{4.25}$$

Here, γ^+ and γ^- are horizontal straight lines and it is posible that the point $(u, r) = (0, 0)$ belongs to Γ. If we remove the $\sigma = \alpha$ and $\sigma = c$ cases, an isocline analysis reveals the situations given by Figure 4.9. In this case, due to the shape of the curves γ^+ and γ^-, we have more difficulties in finding block solutions of the equation, i.e., solutions that touch these curves. This fact is due to the behavior of the function $g(u)$ as $u \to \pm c$. In fact, we can show that we may not find a block solution, for a certain behavior of the function g, as the following result shows.

Proposition 4.4.3. *Assume that*

$$\frac{1}{g(u)} = \mathcal{O}(c - u), \text{ as } u \to c,$$

then there is no block-type solution.

Remark 4.4.4. *A type of flux-saturated function that satisfies this condition is the Wilson operator.*

Proof. We are going to use a reductio ad absurdum argument to prove the Lemma. Suppose that there is a block solution, then there would be a connection between γ^+ and γ^-. This means we can find a solution branch $(\bar{u}, \bar{r})$ in the interval $(\xi^-, \xi^- + \epsilon]$ that starts over points of γ^-.

On the other hand, we can consider the problem

$$r'(u) = \frac{1}{\nu}\frac{(\alpha - \sigma)r - u}{u\, g(ar - \sigma)}, \quad r(u_0) = r_0, \tag{4.26}$$

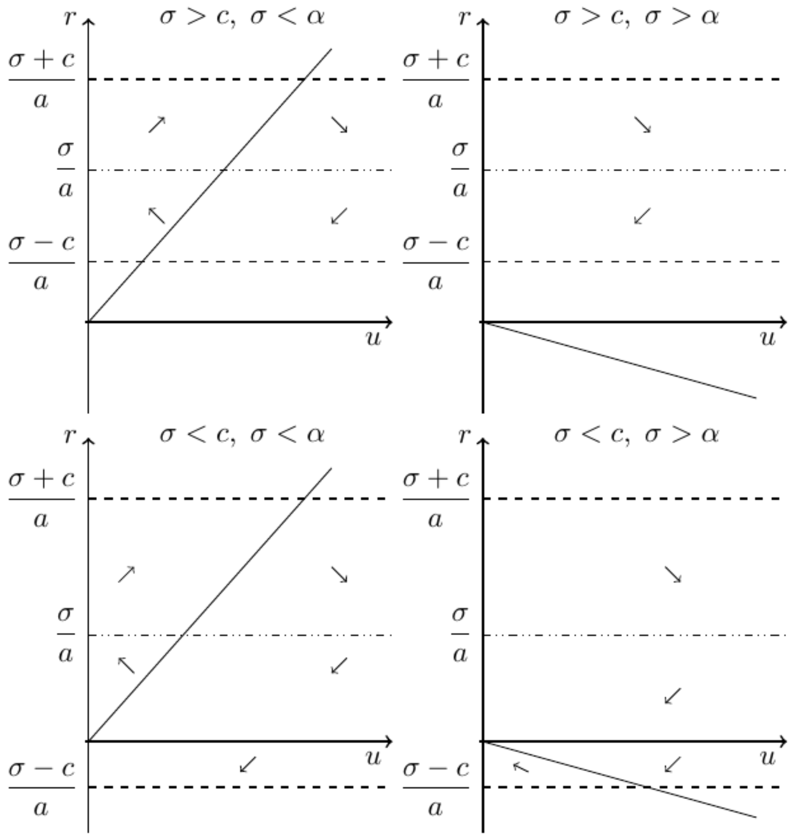


Figure 4.9: Representation of the phase diagram of (4.25), as a function of σ,m and α.

where $(u_0, r_0) \in \gamma^-$. The Picard-Lindelof theorem can be applied to (4.26) extended by zero, prove uniqueness of solution that takes the form $r(u) = \frac{\sigma+c}{a}$.

However, we had assumed that there was a solution branch $(\bar{u}, \bar{r})$ that in the form $\bar{r}(\bar{u})$ would be the solution of the problem (4.26), which is not possible due to the previous uniqueness argument. $\square$

On the other hand, we can also establish conditions for the existence of solution.

Proposition 4.4.4. *Assume that*

$$\frac{1}{g(u)} = \mathcal{O}((c-u)^{1/p}), \ \text{as } u \to c,$$

then there exists a block–type solution.

Remark 4.4.5. *We can find some flux–saturated functions satisfying these conditions on g, such as the hyperbolic tangent operator, or the Larson operator to which it is associated $g(u) = \frac{1}{\mu}\frac{u}{\sqrt[p]{1-\left(\frac{|u|}{c}\right)^p}}$.*

The non-uniqueness of the problem (4.26), under the hypothesis of Proposition 4.4.4, will allow us to prove the existence of a solution for this model. To do this, we will consider the following result.

Lemma 4.4.2. *Consider the equation*

$$x' = x^{\frac{1}{p}} a(t,x),\ x > 0,\ p > 1, \tag{4.27}$$

where the function a admits a continuous extension in a neighborhood of $(0,0)$ and $a(0,0) > 0$. Then, the initial value problem (4.27) with $x(0) = 0$ has a solution $x(t) > 0$ in a neighborhood of the right–hand side of $t = 0$.

Proof. Let us make the change of variable $y = x^{\frac{p-1}{p}}$. We have the following initial value problem:

$$y' = \frac{p-1}{p} a\left(t, y^{\frac{p}{p-1}}\right),\ y(0) = 0.$$

By using the Picard theorem, the problem has a solution $y(t)$ with $y'(0) = \frac{p-1}{p} a(0,0) > 0$. Therefore, we have that $y(t) > 0$ on positive values in the neighborhood of $t = 0$.

□

Remark 4.4.6. *If $a(0,0) < 0$, then the neighborhood is on the left–hand side of $t = 0$.*

We can now proceed to prove Proposition 4.4.4.

Proof. We will use the same constructive scheme developed in the proof of Proposition 4.4.1.

First, we will consider $\tilde{u}$ sufficiently large, such that $r' < 0$ and u' has an uni-modal shape, under the conditions $u > \tilde{u}$ and $r \in (\frac{\sigma+c}{a}, \frac{\sigma-c}{a})$.

Using Lemma 4.4.2 and system (4.26), we are be able to launch solutions from the curves γ^+ and γ^-. Taking the initial conditions such that $u_0 > \tilde{u}$, these solutions will always touch the curve $r = \frac{\sigma}{a}$ (see the proof of Lemma 4.3.1).

Let us denote by (u^-, r^-) one of the solution launched from γ^- and (u^+, r^+) one of the solution launched from γ^+. Those solutions touch the curve $r = \frac{\sigma}{a}$ at some value u_1^- and u_1^+, respectively. Asume $u_1^- < u_1^+$, the opposite case can be treated similarly. Then, we have that the solution (u^+, r^+), once it crosses the line $r = \frac{\sigma}{a}$, it will always touch the curve γ', since $u' > 0$, and it cannot touch the orbit of the solution (u^-, r^-). Therefore, the solution (u^+, r^+) connects γ^- with γ^+. □

4.4.3 Diffusion without transport

Formally, letting $\alpha = 0$ in the model (4.18) leads to the following system:

$$\begin{aligned} &\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right), \\ &\frac{\partial Q}{\partial t} - \nu\frac{\partial^2 Q}{\partial x^2} = U, \end{aligned} \tag{4.28}$$

corresponding to a process where diffusion of the chemoattractant dominates the dynamics. As we discussed in the previous section, using the jump condition we can derive a differential system for the description of entropy solutions. In this case, the equations can be obtained by letting $\alpha = 0$ in equation (4.19)

$$\begin{aligned} u' &= u\, g\left(\frac{ar-\sigma}{u^{m-1}}\right), \\ r' &= -\frac{1}{\nu}\left(u+\sigma r\right). \end{aligned} \tag{4.29}$$

Therefore, we can analyze the existence of block–type solutions of this problem as a particular case of (4.19). From Theorem 4.4.1 we have the following result.

Corollary 4.4.1. *There is a block–type solution of* (4.29)*, for all* $\sigma > 0$*.*

4.4.4 Transport without diffusion in Q

In this last section we consider that the chemoattractant concentration is solution of a linear transport-diffusion equation, which corresponds to the case $\tau_1 = 1$ and $\tau_2 = 0 = \nu$ in equation (4.4), namely

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left(U^m \Phi\left(U^{-1}\frac{\partial U}{\partial x}\right) - aU\frac{\partial Q}{\partial x}\right), \\ \frac{\partial Q}{\partial t} &+ \alpha\frac{\partial Q}{\partial x} = U. \end{aligned} \tag{4.30}$$

With a similar argument to the one in Section 2, we would obtain (4.7). However, in this case, the expression obtained for q' does not need to be regular since (4.6) is not satisfied. However, if $\sigma \neq \alpha$ we can expect that $q \in H^1_{loc}(\mathbb{R})$ and

$$(\alpha-\sigma)q'(\xi) = u(\xi), \quad \text{a.e. } \xi \in \mathbb{R}.$$

Using this in the first expression of (4.7), as in section 2, we obtain the existence of a value K such as

$$u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right) - \frac{a}{\alpha-\sigma}u^2(\xi) + \sigma u(\xi) = K, \quad \text{a.e. } \xi \in \mathbb{R},$$

where u' is the Radon–Nikodym derivative. Since $u^m(\xi)\Phi\left(\frac{u'(\xi)}{u(\xi)}\right)$ is assumed to be 0 if $u(\xi) = 0$, it follows that if u has compact support, then outside this support $u = 0$, and thus $K = 0$. Therefore, solutions of (4.30) will satisfy the equation

$$\begin{aligned} u' &= ug\left(\frac{\frac{a}{\alpha-\sigma}u-\sigma}{u^{m-1}}\right), \\ q' &= \frac{u}{\alpha-\sigma}, \end{aligned} \tag{4.31}$$

at the points of its support.

Remark 4.4.7. *Since condition* (4.6) *is not verified, there is no clear description of the block–type solutions. Therefore, we will describe the maximal branches solutions of* (4.33) *in order to describe a possible connection between them.*

Because of the large casuistry, the description of the chemoattractant will not be discussed here. In this section, it will be assumed that $\sigma \neq \alpha$*, since the solution* $u = cte$ *is obtained for* $\sigma = \alpha$*.*

Setting

$$H(u) = \frac{\frac{a}{\alpha-\sigma}u - \sigma}{u^{m-1}} \tag{4.32}$$

the first equation of (4.31) can be written as

$$u' = u\, g\left(H(u)\right), \tag{4.33}$$

and the expression of the quimioatracctant follows after integrate the second equation of (4.31).

Since the function g is only defined in $(-c, c)$, it is important to know the values of u for which $H(u) \in (-c, c)$. We will distinguish several cases based on the values of α, σ, m.

Case 1. If $\sigma > \alpha$, H is always negative because $H(0) = -\infty$ and there is no root of H.

Case 1.1. If $m > 2$, H is increasing and $H(\infty) = 0$.

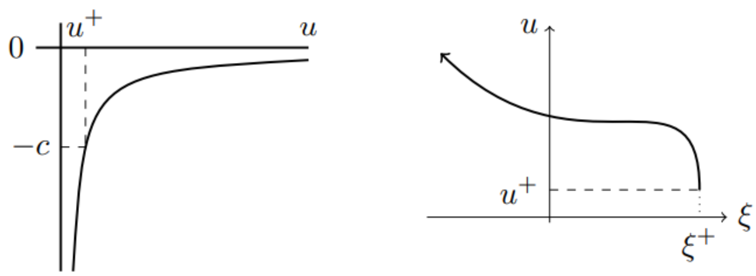


Figure 4.10: Behavior of H in Case 1.1 and the profile of the solution.

Solutions live in the interval (u^+, ∞) and are decreasing. It can be extended to $-\infty$, and $u(-\infty) = +\infty$. For a finite value $u(\xi^+) = u^+$ and $u'(\xi^+) = -\infty$. This can be observed in Figure 4.10.

Case 1.2. When $m = 2$, H is again increasing, negative but $H(+\infty) = -\dfrac{a}{\sigma - \alpha} < 0$. This case opens two possibilities see Figure 4.11.

1.2.① The first alternative is $\sigma > \alpha + \frac{a}{c}$, which is similar to the Case 1.1, see Figure 4.11.

1.2.② The second case corresponds with $\sigma \leq \alpha + \frac{a}{c}$. In this situation there are no solutions because there are no points in which the differential equation is defined.

Case 1.3. In the case $1 < m < 2$ the function H satisfies $H(\infty) = -\infty$ and there is only a critical point at u^* with a maximum value H^*, which are given by

$$H^* := H(u^*) < 0, \qquad u^* = \frac{\sigma(\alpha - \sigma)(1 - m)}{a(2 - m)}. \tag{4.34}$$

According to the relative position of H^* with respect to c, we can distinguish the following cases, see Figure 4.12:

1.3.① If $H^* \leq -c$, there are no solutions since there are no points for which the differential equation is defined.

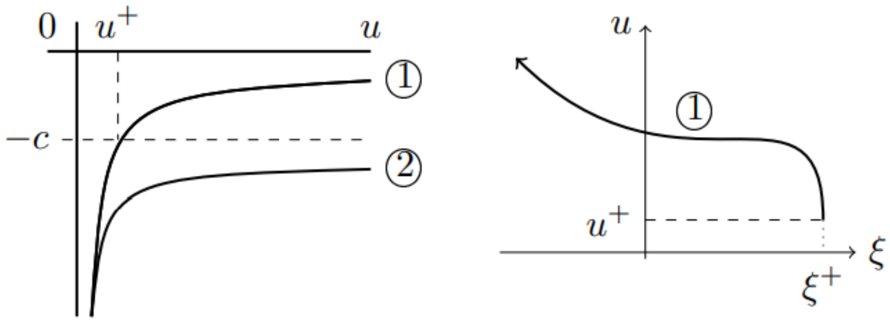


Figure 4.11: Behavior of H in Case 1.2 and the profile of the solution. Note that for ② there is no solution, because the differential equation is not defined

1.3.② If $-c < H^*$, the differential equation is only defined for $u \in (u^+, u^-)$. Therefore, the solutiones are defined in a bounded interval (ξ_-, ξ_+), where $u(\xi^-) = u^-$, $u(\xi^+) = u^+$, and $u'(\xi^-) = u'(\xi^+) = -\infty$.

,

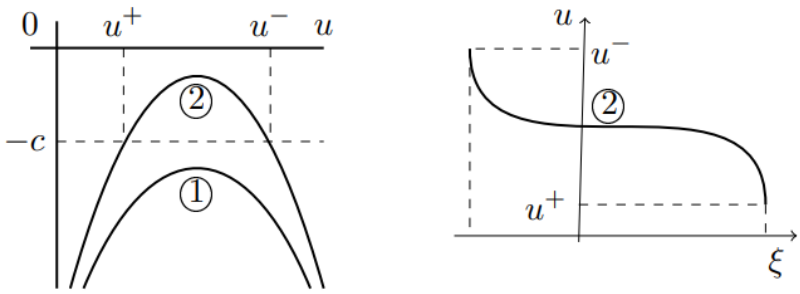


Figure 4.12: Behavior of H in Case 1.3 and the profile of the solution. Note that for ① there is no solution, because the differential equation is not defined.

Case 2. If $0 < \sigma < \alpha$, then $H(0) = -\infty$, but H changes sign in $\hat{u}$, which is given by

$$\hat{u} = \frac{\sigma(\alpha - \sigma)}{a}. \tag{4.35}$$

Case 2.1. In the case $1 < m < 2$, H has no critical points and $H(+\infty) = +\infty$. Therefore, there are two types of traveling waves, one increasing and one decreasing that are represented in Figures 4.13.

Case 2.2. If $m = 2$ and $0 < \alpha < \sigma$, H has no critical points, but there is a finite asymptotic value $H(+\infty) = \dfrac{a}{\alpha - \sigma} > 0$. The position of this asymptotic value with respect to c gives us three different situations:

2.2.① If $\dfrac{a}{\alpha - \sigma} > c$. This is a scenario similar to Case 2.1, see the graph in Figure 4.14.

2.2.② If $\dfrac{a}{\alpha - \sigma} = c$. The modification in this case is the non-existence of a point r_+ that cuts the graph of H to the c level (see Figure 4.14), and now the

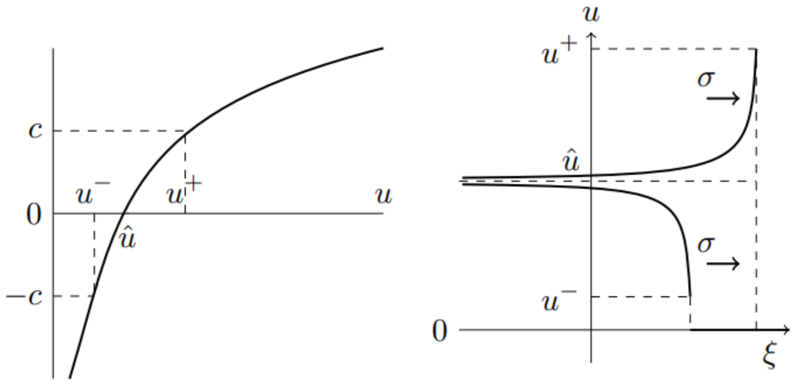


Figure 4.13: Behavior of H in the case $1 < m < 2$ and $0 < \sigma < \alpha$. Notice that both traveling waves are continued by zero.

solutions end at infinity. They can reach infinity in finite or infinite time depending on the properties of Φ.

2.2.③ If $0 < \dfrac{a}{\alpha - \sigma} < c$. This is a situation similar to the previous one, but infinite is reached in infinite time, see Figure 4.14.

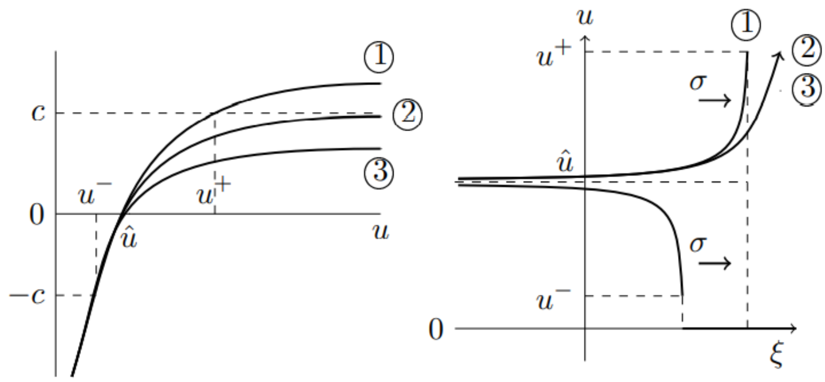


Figure 4.14: Behavior of H in the case $m = 2$ and $0 < \alpha < \sigma$. In this case the traveling waves that have a finite height are continued by zero. The traveling wave of type ② and ③ are not bounded, and, therefore, is conditioned by a more general theory of the initial value problem.

Case 2.3. In the case $m > 2$, $H(+\infty) = 0$, and the function H reaches a maximum level $H^* > 0$. Then, it is necessary to compare this number with c and we can define three different scenarios and the values of σ for which the different traveling waves are defined, see Figure 4.15.

Conclusion and summary

The idea of this last section is to determine under which conditions we can find block–type solutions. To do this we have analyzed all the different solution profiles satisfying equation (4.31).

We have basically found two types of profiles that we can denominate, according to their character, as increasing or decreasing profiles. Both are

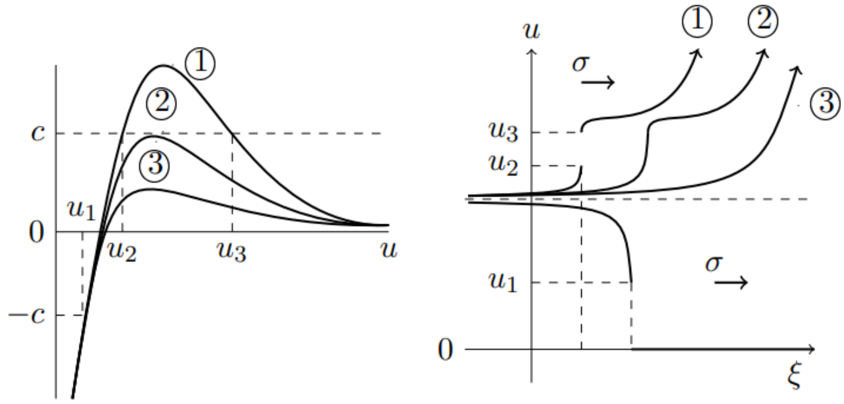


Figure 4.15: Behavior of H in the case $m > 2$ and $0 < \alpha < \sigma$.

separated by the level u^*, defined in (4.35), which defines the point of possible sign change of the H function.

However, the only type of compact support solution we have found has a decreasing profile, see Figure 4.16. This compact support solution exist for $\sigma > \alpha$, $1 < m < 2$ and $H^* > -c$. The last condition can be expressed, after several standard calculations, as:

$$\frac{\sigma^{(2-m)}}{(\sigma-\alpha)^{m-1}} > \frac{c}{2-m}\left(\frac{m-1}{a(2-m)}\right)^{m-1}.$$

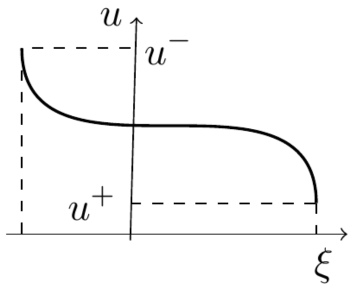


Figure 4.16: Representation of the only block solution of (4.30)

Chapter 5

Mean-Field Limits for Stochastic Interacting Particles on Digraph Measures

5.1 Introduction

This Chapter investigates a physical system composed of N particles that interact through a network, represented by a graph G. Each vertex corresponds to a particle, while edges depict pairwise interactions. The system's dynamics are governed by stochastic differential equations (SDEs), which define the evolution of particle states $X_i(t) \in \mathbb{R}^d$. The equations are expressed as:

$$\begin{aligned}
&\mathrm{d}X_i(t) = f(X_i(t))\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N} A_{ij}^N g(X_i, X_j)\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N} \hat{A}_{ij}^N h(X_i, X_j)\mathrm{d}B_t^i, \\
&X_i(0) = X_i^0, \quad i = 1, \ldots, N. \qquad (5.1)
\end{aligned}$$

where $A_{i,j}^N$ is the adjacency matrix for deterministic interactions , and $\hat{A}_{ij}^N$ accounts for their influence on noise term. A_{ij}^N and $\hat{A}_{ij}^N$ are not necessarily identical. The functions f, g and h are Lipschitz and bounded. The noise term contains a sequence $\{B_t^i\}$, for $1 \leq i \leq N$, of independent and identically distributed Brownian motions in $\mathbb{X} \subseteq \mathbb{R}^d$. The initial conditions $\{X_i^0\}_{i=1,..,N}$ form a sample of independently and identically distributed random variables with a probability distribution $\mu^0 \in \mathcal{P}(\mathbb{X})$. $X_i(t)$ represents, at the microscopic level, the trajectory of particle i.

The study focuses on deriving the mean-field limit for these interacting particle systems while considering suitable graph limits, particularly in cases of sparse and intermediate density graphs, utilizing a representation called digraph measure (DGM).

Previous work on mean-field limits has treated different configurations of adjacency matrices, with approaches varying from fully exchangeable to non-exchangeable particles. Classical mean-field limits have employed techniques involving graphons, digraph measures, or graph operators, consistently demonstrating convergence to a Vlasov equation.

In this Chapter, we concentrate on stochastic interactions and derive the corresponding Vlasov-Fokker-Planck equation. The aim is to establish a measure $\mu(t,x)$ representing the typical evolution of particles over time and space, ultimately converging to a weak solution of the Vlasov-Fokker-Planck equation. The empirical measure, defined as:

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_i(t)}, \text{ for } t\in[0,T],$$

is critical for representing the law of the entire system.

To establish the mean-field limit, we will employ the coupling method techniques [50, 51]. Additionally, we will utilize the approaches introduced by Sznitman [118] and incorporate the methods used in [12, 22]. This comprehensive framework will allow us to investigate the mean-field limit of the particle system described by (5.1).

To apply this framework, we need to introduce the following system of independent processes.

$$\begin{aligned} X_u(t) =& X_u(0) + \int_0^t f(X_u(s))\,\mathrm{d}s + \int_0^t \int_I \int_{\mathbb{X}} g(X_u(s),y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\ &+ \int_0^t \int_I \int_{\mathbb{X}} h(X_u(s),y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B_s^u, \end{aligned} \tag{5.2}$$

where $\mu_{u,t} = \mathcal{L}(X(t)|U=u)$, and U is a uniform random variable on I. The initial conditions $X_u(0) = X_u^0$, for $u\in I$, form a sample of independently and identically distributed random variables with a probability distribution $\bar{\mu}^0\in\mathcal{P}(\mathbb{X})$. And the finite positive Borel measures $\eta,\hat{\eta}\in\mathcal{B}(I,\mathcal{M}_+(I))$ are the fiber measures associated to the adjacency matrices of the graphs $A_{i,j}^N$ and $\hat{A}_{i,j}^N$.

The goal is to demonstrate convergence of the empirical measure to a probability measure $\mu_{u,t}$ satisfying the Vlasov-Fokker-Planck equation given by:

$$\begin{aligned} &\partial_t\bar{\mu}_t + \partial_x\left(\bar{\mu}_t f(x) + \int_I \mu_{u,t}\int_I\int_{\mathbb{X}} g(x,y)\ \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}u\right) \\ &+\frac{1}{2}\partial_x^2\left(\int_I \mu_{u,t}\left[\int_I\int_{\mathbb{X}} h(x,y)\ \mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right]^2\ \mathrm{d}u\right) = 0. \end{aligned} \tag{5.3}$$

This will be proved in Theorems 5.3.1 and 5.3.2, where in each result, we will handle the limit measure $\mu_{u,t}$ differently. However, in both cases, we will obtain the same Vlasov-Fokker-Plank equation.

The Chapter is organized as follows. In Section 5.2, we introduce various measure and probability spaces, along with the associated metrics, that will serve as the foundation for establishing the convergence of the empirical measure and of DGMs. Section 5.3 states the key results and interpretations of the work, along with the necessary assumptions for proving these results. The proof of Theorem 5.3.1 is presented in Section 5.4, where we focus on studying the independent process (5.2). The existence and uniqueness of the solution to this problem will be explored in the same section. In Section 5.5, we employ coupling methods to illustrate the convergence in law of the empirical measure towards the solution of (5.2), thereby proving Theorem 5.3.1. Section 5.6 is dedicated to proving Theorem 5.3.2 using techniques similar to those employed in the preceding sections. Finally, Section 5.7 covers examples of Vlasov-Fokker-Plank equations for some DGMs.

5.2 Metric Measure Spaces and Digraph Measures

We are going to introduce various spaces and notation that we will use. We are concerned with the dynamics of particles in a finite time interval, denoted as $[0, T]$, where we fix $T > 0$ as a parameter. Our analysis takes place within the framework of a filtered probability space $(\Omega, \mathcal{F}, \{F_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{F_t\}$ denotes a filtration that complies with standard conditions. The particle trajectories $X(t)$ take values in $\mathbb{X} \subseteq \mathbb{R}^d$.

First of all, let us determine the measurement space over which we are going to define our digraphs. Let I be a complete metric space, and consider the space $\mathcal{B}(I, \mathcal{M}_+(I))$, which is the space of bounded measurable functions from I to $\mathcal{M}_+(I)$, where $\mathcal{M}_+(I)$ is the set of all finite Borel positive measures on I. Also, consider $\mathcal{C}(I, \mathcal{M}_+(I))$ as the space of continuous functions from I to $\mathcal{M}_+(I)$. Let us define $\mathcal{B}(\mathbb{X})$ the space of bounded measurable functions from $\mathbb{X}$ to $\mathbb{R}^d$ and

$$L_f := \sup_{x,y\in\mathbb{X},x\neq y} \frac{|f(x) - f(y)|}{|x - y|},$$

the Lipschitz constant of $f \in \mathcal{C}(\mathbb{X})$. Let $\mathcal{BL}(\mathbb{X})$ be the space of bounded Lipschitz continuous functions and

$$\mathcal{BL}_1(\mathbb{X}) = \{f \in \mathcal{BL}(\mathbb{X}) : \mathcal{BL}(f) := \|f\|_\infty + L_f \leq 1\}.$$

Let $\eta^x, \nu^x \in \mathcal{M}_+(I)$, then we can define the bounded Lipschitz distance as:

$$d_{BL}(\eta^x, \nu^x) := \sup_{f\in\mathcal{BL}_1(\mathbb{X})} \int_{\mathbb{X}} f(y) \ \left(\eta^x(\mathrm{d}y) - \nu^x(\mathrm{d}y)\right).$$

Hence, given $\eta, \nu \in \mathcal{B}(I, \mathcal{M}_+(I))$, define the uniform bounded Lipschitz metric:

$$d_\infty(\eta, \nu) := \sup_{x\in\mathbb{X}} d_{BL}(\eta^x, \nu^x).$$

and the bounded Lipschitz norm (on the space of all finite signed Borel measures):

$$\|\eta\| := \sup_{x\in\mathbb{X}} \sup_{f\in\mathcal{BL}_1(\mathbb{X})} \int_{\mathbb{X}} f(y) \ \eta^x(\mathrm{d}y).$$

We have the following Proposition [95, Proposition 2.6]:

Proposition 5.2.1. *Let I be a complete separable metric space. Assume I is compact. Then $(\mathcal{B}(I, \mathcal{M}_+(I)), d_\infty)$ and $(\mathcal{C}(I, \mathcal{M}_+(I)), d_\infty)$ are complete metric spaces.*

In our problem, we will consider $I = [0, 1]$. Since we are working with DGMs, and in our problem we have a graph represented by a matrix, we need to approximate this graph by a DGM. Consider a partition of the interval

$I = [0,1]$ as given by $I_i^N =]\frac{i-1}{N}, \frac{i}{N}]$, for $1 < i \leq N$, and $I_1^N = [0, \frac{1}{N}]$. Let us build the digraph measures associated with the matrices $A_{i,j}^N$ and $\hat{A}_{i,j}^N$ as follows

$$\begin{aligned} \eta_{A^N}^u(v) &:= \sum_{i=1}^N \mathbf{1}_{I_i^N}(u) \sum_{j=1}^N \frac{A_{i,j}^N}{N} \delta_{\frac{j}{N}}(v), \\ \eta_{\hat{A}^N}^u(v) &:= \sum_{i=1}^N \mathbf{1}_{I_i^N}(u) \sum_{j=1}^N \frac{\hat{A}_{i,j}^N}{N} \delta_{\frac{j}{N}}(v), \end{aligned} \tag{5.4}$$

where $\mathbf{1}_{\frac{j}{N}}(x)$ is the indicator function and $\frac{j}{N}$ serves as the representative of the set I_j^N and . We recall that the sequence of graphs $\{A^N\}_{N\geq 1}$ converges to the digraph measure η if and only if $d_\infty(\eta_{A^N}, \eta) \to 0$, as $N \to \infty$. Similarly, we aim for $\hat{A}^N$ to converge to the digraph measure $\hat{\eta}$. Observe that $\eta_{A^N} \in \mathcal{B}(I, \mathcal{M}_+(I)) \bigcap \mathcal{C}(I, \mathcal{M}_+(I))$.

In order to analyze the collection of probability laws, let us consider the following space of probability measures

$$\mathcal{N} := \{\nu \in \mathcal{P}(I \times \mathcal{C}([0,T], \mathbb{X})) : \pi_1 \circ \nu = \lambda\},$$

where λ denotes the Lebesgue measure on I and π_1 is the projection map associated to the first coordinate. To compare probability measures we make use of the following Wasserstein-2 metric:

$$W_2(\mu,\nu) := \left(\inf\left\{\mathbb{E}|X - \tilde{X}|^2 : \mathcal{L}(X) = \mu, \mathcal{L}(\tilde{X}) = \nu\right\}\right)^{\frac{1}{2}}, \ \mu,\nu \in \mathcal{P}(\mathbb{R}^d).$$

As the probability measures also depend on time, we can define the following Wasserstein-2 metric:

$$\begin{gathered} W_{2,t}(\mu,\nu) := \left(\inf\left\{\mathbb{E}\|X - \tilde{X}\|_{*,t}^2 : \mathcal{L}(X) = \mu, \mathcal{L}(\tilde{X}) = \nu\right\}\right)^{\frac{1}{2}}, \\ t \in [0,T], \quad \mu,\nu \in \mathcal{P}(\mathcal{C}([0,T],\mathbb{X})), \end{gathered}$$

where $\|X\|_{*,t} := \sup_{0\leq s\leq t} |X_s|$ for $X \in \mathcal{C}([0,T],\mathbb{X})$, $t \in [0,T]$. Additionally, as we are also working with digraph measures, we must also measure distances taking into account the graph's heterogeneity variable. For this purpose, we have the following two measures:

$$W_{2,t}^{\mathcal{N},2}(\mu,\nu) := \left(\int_I [W_{2,t}(\mu_u,\nu_u)]^2 \ \mathrm{d}u\right)^{\frac{1}{2}}, \ t \in [0,T], \ \mu,\nu \in \mathcal{N},$$

$$W_{2,t}^{\mathcal{N},\infty}(\mu,\nu) := \sup_{u\in I} W_{2,t}(\mu_u,\nu_u), \ t \in [0,T], \ \mu,\nu \in \mathcal{N}.$$

Since $I = [0,1]$, the following inequality holds:

$$W_{2,t}^{\mathcal{N},2}(\mu,\nu) \leq W_{2,t}^{\mathcal{N},\infty}(\mu,\nu).$$

We will work in the space $\mathcal{N}$ which, equipped with the distance $W_{2,t}^{\mathcal{N},2}(\mu,\nu)$, is a complete metric space.

Remark 5.2.1. *Notice that the Wasserstein metric can be compared with the bounded Lipschitz metric, denoted as* $d_{\mathcal{BL}}$*, because the following inequality holds:*

$$W_{2,t}(\mu_u, \nu_u) \geq \sup_{f \in \mathcal{BL}_1} \left| \int_{\mathbb{R}^d} f(x)\ \mu_{u,t}(\mathrm{d}x) - \int_{\mathbb{R}^d} f(x)\ \nu_{u,t}(\mathrm{d}x) \right|,\ \mu, \nu \in \mathcal{N},$$

i.e. $d_{\mathcal{BL}}(\mu_{u,t}, \nu_{u,t}) \leq W_{2,t}(\mu_u, \nu_u)$ *(see [76]).*

5.3 Main Results

Before stating our results, we introduce our main assumptions.

Assumptions (H) Let us assume that f, h and g are bounded and Lipschitz continuous functions. We denote $B_f = \sup_{x \in \mathbb{X}} |f(x)|$, $B_h = \sup_{(x,y) \in \mathbb{X}^2} |h(x,y)|$ and $B_g = \sup_{(x,y) \in \mathbb{X}^2} |g(x,y)|$, and $|f(x) - f(y)| \leq L_f |x - y|$ holds for $x, y \in \mathbb{X}$, $|h(x_1, y_1) - h(x_2, y_2)| \leq L_h(|x_1 - x_2| + |y_1 - y_2|)$ and $|g(x_1, y_1) - g(x_2, y_2)| \leq L_g(|x_1 - x_2| + |y_1 - y_2|)$ for $(x_i, y_i) \in \mathbb{X}^2$, where L_f, L_h and L_g are the respective Lipschitz constants.

We remark that the Lipschitz assumptions are in many systems not a severe restriction, e.g., if the system is globally dissipative, one may use cut-off arguments to make sufficiently smooth nonlinearities Lipschitz in all parts of phase space relevant for the long-term dynamics. Of course, there are situations, where Lipschitz assumptions fail, e.g., for singular interaction terms but we are not going to pursue this direction here.

Let $[0, T]$ be the time domain for the system dynamics for some $T > 0$. Consider a filtered probability space $(\Omega, \mathcal{F}, \{F_t\}_{t \in [0,T]}, \mathbb{P})$, where $\{F_t\}$ denotes a filtration that complies with standard conditions. The particle trajectories $X(t)$ take values in $\mathbb{X}$. The random variable U is given on a compact Polish probability space $(I, \mathfrak{B}(I), \mu_I)$, which we select as $I = [0, 1]$ with the usual Lebesgue measure here but point out that more general choices might turn out to be useful as well. Indeed, recall that I will be space to track the node labels upon passing to the mean-field limit, so choosing a more geometric space for $(I, \mathfrak{B}(I), \mu_I)$ could track distances on the graph but here we work measure-theoretically so the Borel-Maraham theorem guarantees that no important measure-theoretic information is lost by considering $I = [0, 1]$. Regarding the assumptions about the graph, we assume that $\sup_{1 \leq i \leq N} \sum_{j=1}^{N} (A_{i,j}^N)^2 = \mathcal{O}(N)$ and $\sup_{1 \leq i \leq N} \sum_{j=1}^{N} (\hat{A}_{i,j}^N)^2 = \mathcal{O}(N)$ as $N \to \infty$.

Finally, we need to impose conditions on the initial data. We assume that the initial data $X_i(0)$, for $1 \leq i \leq N$, are independent and identically distributed, and we have finite second moments initially $\mathbb{E}|X_i(0)^2| < +\infty$, for all $i \in \{1, 2, \ldots, N\}$. Moreover, assume that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}|X_i^0 - X_{\frac{i}{N}}^0|^2 = 0,$$

for any random variable X_i^0 and $X_{\frac{i}{N}}^0$, for $i = 1, \ldots, N$, where $\mu^0 = \mathcal{L}(X_i^0)$ and $\bar{\mu}^0 = \mathcal{L}(X_{\frac{i}{N}}^0)$, for $i = 1, \ldots, N$.

After establishing the key assumptions of our problem (5.1), we proceed to present our main theorem, where we establish the convergence in law of the empirical measure towards a probability measure that satisfies the Vlasov-Fokker-Planck equation. The proof of this theorem will be provided in Section 5.5.

Theorem 5.3.1. *Under the assumptions (H), let us consider sequences of graphs* $\{A^N_{i,j}\}_{N\geq 1}$ *and* $\{\hat{A}^N_{i,j}\}_{N\geq 1}$*. Assume that there exists* η *and* $\hat{\eta}$ *(DGM) such that* η_{A^N} *converges to* η *and* $\eta_{\hat{A}^N}$ *converges to* $\hat{\eta}$*. In this case, the empirical measure* μ_N *converges in probability to a measure* $\bar{\mu} \in \mathcal{P}(C([0,T],\mathbb{X}))$*, where* $\bar{\mu}$ *represents the law of the solution* X *of equation (5.2). And* $\bar{\mu}$ *solves the Vlasov-Fokker-Plank equation in the weak sense*

$$\begin{aligned}\partial_t\bar{\mu}_t + \partial_x\left(\bar{\mu}_t f(x) + \int_I \mu_{u,t}\int_I\int_{\mathbb{X}} g(x,y)\ \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}u\right) = \\ + \frac{1}{2}\partial_x^2\left(\int_I \mu_{u,t}\left[\int_I\int_{\mathbb{X}} h(x,y)\ \mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right]^2\ \mathrm{d}u\right).\end{aligned} \tag{5.5}$$

Note that when studying this problem, we have assumed that the heterogeneity variable of the graph is a random variable given by a uniform distribution. We can modify this hypothesis and obtain better properties regarding the law $\mu_{u,t}$.

As we did before, let us consider the following system of independent processes:

$$\begin{aligned}X_u(t) =& X_u(0) + \int_0^t f(X_u(s))\ \mathrm{d}s + \int_0^t\int_I\int_{\mathbb{X}} g(X_u(s),y)\ \mu_{v,s}(\mathrm{d}y)\eta_u(\mathrm{d}v)\mathrm{d}s \\ &+ \int_0^t\int_I\int_{\mathbb{X}} h(X_u(s),y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B^u_s,\end{aligned} \tag{5.6}$$

where $\mu_{u,t} = \mathcal{L}(X_u(t))$, for $u \in I$. Since we do not take u as a random variable, we need to modify the way we understand the law of $X_u(t)$ for each u and impose more properties on this law. To do this, we will modify our space, and instead of working in $\mathcal{N}$, we will work in the following space:

$$\begin{aligned}\tilde{\mathcal{N}} := \Big\{\nu = (\mu_u, u\in I) \in \left(\mathcal{P}(\mathcal{C}([0,T],\mathbb{X}))\right)^I : u \to \mu_u \text{ is measurable}, \\ \sup_{u\in I}\int_{\mathcal{C}_d}\|x\|^2_{*,T}\ \mu_u(\mathrm{d}x) < \infty\Big\}.\end{aligned}$$

This space equipped with the distance $W^{\mathcal{N},\infty}_{2,t}$ is a complete metric space.

In addition to the assumptions (H), we must establish one more set of assumptions:

Assumptions ($\tilde{H}$) Let us assume that the law of probability $\bar{\mu}_0$ of the initial condition is measurable with respect to the heterogeneity variable of the graph. In other words, the mapping $u \in I \to \bar{\mu}^0_u \in P(\mathbb{X})$ is measurable. Moreover, assume that $W_2\left(\bar{\mu}^0_{u_1},\bar{\mu}^0_{u_2}\right) \leq \alpha\,|u_1 - u_2|$, for $u_1, u_2 \in I$ and $\alpha \in \mathbb{R}_+$.

With the additional assumptions about our system, we obtain the following result:

Theorem 5.3.2. *Under the assumptions (H) and $(\tilde{H})$, let us consider a sequence of graphs $\{A^N_{i,j}\}_{N\geq 1}$ and $\{\hat{A}^N_{i,j}\}_{N\geq 1}$. Assume that there exists η and $\hat{\eta}$ (DGM) such that η_{A^N} converges to η and $\eta_{\hat{A}^N}$ converges to $\hat{\eta}$. In this case, the empirical measure μ_N converges in probability in $\mathcal{P}(C([0,T],\mathbb{X}))$ to the measure $\bar{\mu}$. And $\bar{\mu}$ solves the Vlasov-Fokker-Plank equation in the weak sense*

$$\begin{aligned}\partial_t\bar{\mu}_t + \partial_x\left(\bar{\mu}_t f(x) + \int_I \mu_{u,t}\int_I\int_{\mathbb{X}} g(x,y)\ \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}u\right) = \\ +\frac{1}{2}\partial_x^2\left(\int_I \mu_{u,t}\left[\int_I\int_{\mathbb{X}} h(x,y)\ \mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right]^2\ \mathrm{d}u\right).\end{aligned} \tag{5.7}$$

Observing that in both theorems, we obtain similar results regarding the probability $\bar{\mu}$ that is, in both, we demonstrate that our empirical measure converges to it in probability and satisfies the Vlasov-Fokker-Planck equation. However, in each case, the law represents a different concept.

In the first case, $\mu_{u,t}(x)$ belongs to the space $\mathcal{P}(I\times C([0,t],\mathbb{X}))$, so we are considering that our probability measure determines the probability of finding a particle at position x with heterogeneity variable u, for each time instant $t\in[0,T]$. In contrast, in the second case, for each fixed $u\in I$, the probability measure $\mu_{u,t}(x)$ determines the probability of finding the particle at position x.

These different ways of understanding each probability measure lead us to observe $\bar{\mu}$ in different ways: In the first case, it is understood as a marginal as we integrate over all possible values that u can take, giving us $\bar{\mu}$ as the probability law of X. Meanwhile, in the second case, since we have a probability law for each u, by integrating with respect to u, we are measuring the average probability of finding particles at position X.

5.4 Existence of Solutions

First of all, we are going to prove the existence and uniqueness of solution of a solution to (5.2).

Proposition 5.4.1. *Under the assumptions (H), for every random variable U on I there exists a unique solution to (5.2).*

Proof. To prove the existence of solution for the equation (5.2), we will consider an operator defined on the space $\mathcal{N}$, and look for a fixed point of this operator. That is, let us consider the mapping $\mu\in\mathcal{N}\mapsto\mathcal{F}(\mu)\in\mathcal{N}$, where $\mathcal{F}(\mu)$ is the law associated to the solution of the equation

$$\begin{aligned}X^\mu_u(t) =& X^\mu_u(0) + \int_0^t f(X^\mu_u(s))\ \mathrm{d}s + \int_0^t\int_I\int_{\mathbb{X}} g(X^\mu_u(s),y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\ &+\int_0^t\int_I\int_{\mathbb{X}} h(X^\mu_u(s),y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B^u_s.\end{aligned} \tag{5.8}$$

Note that if we have a fixed point, then $\mathcal{F}(\mu)=\mathcal{L}(X^\mu)=\mu$, so we would prove the existence of solution. First of all, we must verify that the operator is well defined and that for each $\mu \in \mathcal{N}$ we have existence of solution of the problem. Let us first prove that the operator is well defined. To do this, let us take $\mu \in \mathcal{N}$, and $X_u^0(t) = X_u(0)$, $\forall t \in [0,T]$ and $u \in I$. Consider the following recurrence equation

$$\begin{aligned} X_u^n(t) = X_u^{n-1}(0) &+ \int_0^t f(X_u^{n-1}(s))\mathrm{d}s \\ &+ \int_0^t \int_I \int_{\mathbb{X}} g(X_u^{n-1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\ &+ \int_0^t \int_I \int_{\mathbb{X}} h(X_u^{n-1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B_s^u, \end{aligned} \tag{5.9}$$

where $X_u^k(0) = X_u^0(0)$, for all $k \geq 1$. Let us prove that $\{X_u^n\}_{n\geq 0}$ is Cauchy. Since $X_u^{n+1} - X_u^n$ is a martingale, we can use Burkholder-Davis-Gundy's inequality ([71]) and obtain

$$\mathbb{E}\|X_u^{n+1} - X_u^n\|_{*,t}^2 \leq K_{BDG}\mathbb{E}[X_u^{n+1} - X_u^n]_t. \tag{5.10}$$

In this case the $K_{BDG} = 4$. Estimating the right-hand term we have

$$\begin{aligned} \mathbb{E}|X_u^{n+1}(t) - X_u^n(t)|^2 &\leq 3\mathbb{E}\int_0^t |f(X_u^n(s)) - f(X_u^{n-1}(s))|^2\ \mathrm{d}s \\ &+ 3\mathbb{E}\int_0^t \left|\int_I \int_{\mathbb{X}} g(X_u^n(s), y)\mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v) - g(X_u^{n-1}(s), y)\mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\right|^2 \mathrm{d}s \\ &+ 3\mathbb{E}\Big|\int_0^t \int_I \int_{\mathbb{X}} \Big(h(X_u^n(s), y)\mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v) \\ &\qquad - h(X_u^{n-1}(s), y)\mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\Big)\mathrm{d}B_s^u\Big|^2. \end{aligned}$$

Using properties of stochastic integrals [105], i.e., by Itô isometry, we can rewrite the last integral term to convert it to a deterministic integral and obtain

$$\begin{aligned} \mathbb{E}|X_u^{n+1}(t) - X_u^n(t)|^2 &\leq 3\mathbb{E}\int_0^t |f(X_u^n(s)) - f(X_u^{n-1}(s))|^2\ \mathrm{d}s \\ &+ 3\mathbb{E}\int_0^t \left|\int_I \int_{\mathbb{X}} g(X_u^n(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v) - g(X_u^{n-1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\right|^2 \mathrm{d}s \\ &+ 3\mathbb{E}\int_0^t \left|\int_I \int_{\mathbb{X}} h(X_u^n(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v) - h(X_u^{n-1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right|^2 \mathrm{d}s. \end{aligned}$$

Using that the functions g, h and f are Lipschitz, we get

$$\begin{aligned} \mathbb{E}|X_u^{n+1}(t) - X_u^n(t)|^2 \leq 3\mathbb{E}&\int_0^t L_f^2|X_u^n(s) - X_u^{n-1}(s)|^2\ \mathrm{d}s \\ &+ 3\mathbb{E}\int_0^t \left(\int_I \int_{\mathbb{X}} L_g|X_u^{n-1}(s) - X_u^n(s)|\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\right)^2 \mathrm{d}s \\ &+ 3\mathbb{E}\int_0^t \left(\int_I \int_{\mathbb{X}} L_h|X_u^{n-1}(s) - X_u^n(s)|\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right)^2 \mathrm{d}s. \end{aligned}$$

Therefore, we get

$$\mathbb{E}|X_u^{n+1}(t) - X_u^n(t)|^2 \leq 3\left(L_f^2 + L_g^2\|\eta\|^2 + L_h^2\|\hat{\eta}\|^2\right)\mathbb{E}\int_0^t |X_u^n(s) - X_u^{n-1}(s)|^2\ \mathrm{d}s.$$

By taking the supremum in s of the difference between X^n and X^{n-1}, in the right-hand side integral, it follows

$$\mathbb{E}|X_u^{n+1}(t) - X_u^n(t)|^2 \leq 3\left(L_f^2 + L_g^2\|\eta\|^2 + L_h^2\|\hat{\eta}\|^2\right)\mathbb{E}\int_0^t \|X_u^n - X_u^{n-1}\|_{*,s}^2 \,\mathrm{d}s.$$

By using (5.10), we arrive at the expression:

$$\mathbb{E}\|X_u^{n+1} - X_u^n\|_{*,t}^2 \leq 12\left(L_f^2 + L_g^2\|\eta\|^2 + L_h^2\|\hat{\eta}\|^2\right)\int_0^t \mathbb{E}\|X_u^n - X_u^{n-1}\|_{*,s}^2 \,\mathrm{d}s.$$

Defining $C = 12\left(L_f^2 + L_g^2\|\eta\|^2 + L_h^2\|\hat{\eta}\|^2\right)$ and iterating this, we get that:

$$\mathbb{E}\|X_u^{n+1} - X_u^n\|_{*,t}^2 \leq C^n\frac{T^n}{n!}\mathbb{E}\|X_u^1 - X_u^0\|_{*,s}^2,$$

where $\mathbb{E}\|X_u^1 - X_u^0\|_{*,s}^2$ is bounded, due to the assumptions on the initial data and the fact that the functions f,g and h are bounded. It follows that the sequence $\{X_u^n\}_{n\geq 0}$ is Cauchy and converges uniformly in probability at $u \in I$ to X_u^μ, which satisfies the equation (5.8). Therefore, we have that $\mathcal{F}$ is a well-defined map from $\mathcal{N}$ to $\mathcal{N}$. It remains to be seen that the application $\mathcal{F}$ has a fixed point in this space. To do this, let us consider $\mu, \nu \in \mathcal{N}$, with X_u^μ and X_u^ν their respective solutions to (5.8), and let us estimate $\mathbb{E}|X_u^\mu - X_u^\nu|^2$ as follows

$$\begin{aligned}
\mathbb{E}|X_u^\mu - X_u^\nu|^2 &\leq 3\mathbb{E}\int_0^t |f(X_u^\mu(s)) - f(X_u^\nu(s))|^2 \,\mathrm{d}s \\
&+ 3\mathbb{E}\int_0^t \int_I \left|\int_{\mathbb{X}} g(X_u^\mu(s), y)\mu_{v,s}(\mathrm{d}y) - g(X_u^\nu(s), y)\nu_{v,s}(\mathrm{d}y)\right|^2 \eta^u(\mathrm{d}v)\mathrm{d}s \\
&+ 3\mathbb{E}\int_0^t \int_I \left|\int_{\mathbb{X}} h(X_u^\mu(s), y)\mu_{v,s}(\mathrm{d}y) - h(X_u^\nu(s), y)\nu_{v,s}(\mathrm{d}y)\right|^2 \hat{\eta}^u(\mathrm{d}v)\mathrm{d}s,
\end{aligned}$$

where we have used again Itô's isometry. By adding and subtracting in the second and third term,$g(X_u^\mu(s), y)\nu_{v,s}(\mathrm{d}y)$ and $h(X_u^\mu(s), y)\nu_{v,s}(\mathrm{d}y)$, respectively. We can bound the above expression by the sum of the following terms.

$$\begin{aligned}
\mathbb{E}|X_u^\mu - X_u^\nu|^2 \leq 3\mathbb{E}\int_0^t &|f(X_u^\mu(s)) - f(X_u^\nu(s))|^2 \,\mathrm{d}s \\
&+ 6\mathbb{E}\int_0^t \int_I \left|\int_{\mathbb{X}} g(X_u^\mu(s), y)\ (\mu_{v,s}(\mathrm{d}y) - \nu_{v,s}(\mathrm{d}y))\right|^2 \eta^u(\mathrm{d}v)\mathrm{d}s \\
&+ 6\mathbb{E}\int_0^t \int_I \left|\int_{\mathbb{X}} (g(X_u^\mu(s), y) - g(X_u^\nu(s), y))\ \nu_{v,s}(\mathrm{d}y)\right|^2 \eta^u(\mathrm{d}v)\mathrm{d}s \\
&+ 6\mathbb{E}\int_0^t \int_I \left|\int_{\mathbb{X}} h(X_u^\mu(s), y)(\mu_{v,s}(\mathrm{d}y) - \nu_{v,s}(\mathrm{d}y))\right|^2 \hat{\eta}^u(\mathrm{d}v)\mathrm{d}s \\
&+ 6\mathbb{E}\int_0^t \int_I \left|\int_{\mathbb{X}} (h(X_u^\mu(s), y) - h(X_u^\nu(s), y))\nu_{v,s}(\mathrm{d}y)\right|^2 \hat{\eta}^u(\mathrm{d}v)\mathrm{d}s.
\end{aligned}$$

Using the Lipschitz condition of the functions f, g and h, and the Remark 5.2.1, we have

$$\begin{aligned}
\mathbb{E}|X_u^\mu - X_u^\nu|^2 \leq (3L_f^2 + 6(L_g^2\|\eta\| + L_h^2\|\hat{\eta}\|))\mathbb{E}\int_0^t &|X_u^\mu(s) - X_u^\nu(s)|^2 \,\mathrm{d}s \\
+ 6B_g^2\int_0^t \int_I &|W_{2,s}(\mu_{v,s}, \nu_{v,s})|^2 \ \eta^u(\mathrm{d}v)\mathrm{d}s \\
+ 6B_h^2\int_0^t \int_I &|W_{2,s}(\mu_{v,s}, \nu_{v,s})|^2 \ \hat{\eta}^u(\mathrm{d}v)\mathrm{d}s.
\end{aligned}$$

Taking the supremum for $v \in I$ in the 2-Wasserstein metric, we arrive at the following expression:

$$\mathbb{E}|X_u^\mu - X_u^\nu|^2 \leq (3L_f^2 + 6(L_g^2\|\eta\| + L_h^2\|\hat{\eta}\|))\mathbb{E}\int_0^t |X_u^\mu(s) - X_u^\nu(s)|^2 \, \mathrm{d}s \\ + 6(B_g^2\|\eta\| + B_h^2\|\hat{\eta}\|)\int_0^t |W_{2,s}^{\mathcal{N},\infty}(\mu_s, \nu_s)|^2 \, \mathrm{d}s.$$

It then, follows from Grönwall's inequality that

$$\mathbb{E}|X_u^\mu - X_u^\nu|^2 \leq 6(B_g^2\|\eta\| + B_h^2\|\hat{\eta}\|)e^{(3L_f^2+6(L_g^2\|\eta\|+L_h^2\|\hat{\eta}\|))T}\int_0^t |W_{2,s}^{\mathcal{N},\infty}(\mu, \nu)|^2 \, \mathrm{d}s.$$

By this inequality and using the Burkholder-Davis-Gundy (BDG) inequality again, we get that

$$\mathbb{E}\|X_u^\mu - X_u^\nu\|_{*,t}^2 \leq 24(B_g^2\|\eta\| + B_h^2\|\hat{\eta}\|)e^{(3L_f^2+6(L_g^2\|\eta\|+L_h^2\|\hat{\eta}\|))T}\int_0^t |W_{2,s}^{\mathcal{N},\infty}(\mu, \nu)|^2 \, \mathrm{d}s.$$

This means that

$$|W_{2,t}^{\mathcal{N},\infty}(\mathcal{F}(\mu), \mathcal{F}(\nu))|^2 \leq C\int_0^t |W_{2,s}^{\mathcal{N},\infty}(\mu, \nu)|^2 \, \mathrm{d}s, \tag{5.11}$$

where $C = 24(B_g^2\|\eta\| + B_h^2\|\hat{\eta}\|)e^{(3L_f^2+6(L_g^2\|\eta\|+L_h^2\|\hat{\eta}\|))T}$. This expression gives the path-wise uniqueness of the solution, and also allow us to prove the existence of solution. For this purpose we will build an iterative process, as follows. Consider $\nu = (\mathcal{L}(Z_u), u \in I)$, where $Z_u(t) = X_u(0)$ for all $u \in I$ and $t \in [0, T]$. Iterating this, and using (5.11), we get

$$W_{2,T}^{\mathcal{N},2}(\mathcal{F}^{n+1}(\nu), \mathcal{F}^n(\nu)) \leq \frac{C^nT^n}{n!}|W_{2,T}^{\mathcal{N},\infty}(\mathcal{F}(\nu), \nu)|.$$

It follows that the sequence $\mathcal{F}^n(\nu)$ is Cauchy for n large enough, where we have used that $W_{2,T}^{\mathcal{N},\infty}(\mathcal{F}(\nu), \nu) < \infty$, due to the assumptions on the initial data and the fact that the functions f, g, and h are bounded. This sequence will have a limit since $\mathcal{N}$ is a complete metric space, and hence there exists $\mu = (\mathcal{L}(X_u), u \in I) \in \mathcal{N}$ solution of (5.2), for the initial data $X_u(0)$. □

Lemma 5.4.1. *Let $\bar{\mu}$ be the law associated to the solution X of (5.2), with $\mu_{u,t} = \mathcal{L}(X(s)|U = u)$ being the conditional probability with respect to the random variable U. Then $\mu_{u,t}$ is a weak solution to the following Vlasov-Fokker-Planck equation:*

$$\partial_t\mu_{u,t} + \partial_x\left(\mu_{u,t}f(x) + \mu_{u,t}\int_I\int_{\mathbb{X}} g(x,y) \; \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\right) = \\ + \frac{1}{2}\partial_x^2\left(\mu_{u,t}\left[\int_I\int_{\mathbb{X}} h(x,y) \; \mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right]^2\right). \tag{5.12}$$

Proof. Consider a test function $\phi \in C_c^\infty(\mathbb{X})$, and let us compute the time derivative of $\phi(X_u(t))$, where u is a sample of the random variable U. To do that, let us make use of Itô's formula. We have

$$\begin{aligned}\mathrm{d}\phi = \Bigg(&\partial_x\phi f(X_u(t)) + \partial_x\phi \int_I\int_{\mathbb{X}} g(X_u(t),y)\ \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\\ &+\frac{1}{2}\left[\int_I\int_{\mathbb{X}} h(x,y)\mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right]^2 \partial_x^2\phi\Bigg)\,\mathrm{d}t\\ &+\partial_x\phi\int_I\int_{\mathbb{X}} h(x,y)\mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B_t^u.\end{aligned} \tag{5.13}$$

If we calculate the expectation of this expression and integrate by parts, we obtain the weak formulation (5.12). □

With the objective of demonstrating the convergence of the solution, it is crucial to investigate, how the solution depends on the digraph measure η. Specifically, we want to establish suitable continuity properties. Such continuity proofs will lay the foundation for later proving the convergence to a mean-field limit.

Proposition 5.4.2. *Given $\hat{\eta}_1,\hat{\eta}_2 \in \mathcal{BC}(I,\mathcal{M}_+(I))$, let μ_1 and μ_2 be the laws of the solutions of (5.2) for the DGMs $\hat{\eta}_1$ and $\hat{\eta}_2$. Then,*

$$\left[W_{2,s}^{\mathcal{N},2}(\mu^1,\mu^2)\right]^2 \le \hat{C}_1\hat{C}_2\left[d_\infty(\hat{\eta}_1,\hat{\eta}_2)\right]^2 e^{\hat{C}_1 T}, \tag{5.14}$$

where $\hat{C}_1 = 3\left[2B_g^2\|\eta\|^2+3B_h^2\|\hat{\eta}_1\|^2\right]e^{\left(3L_f^2+3\left[2L_g^2\|\eta\|^2+3L_h^2\|\hat{\eta}_1\|^2\right]\right)T}$ and $\hat{C}_2 = \frac{3B_h^2T}{\left[2B_g^2\|\eta\|^2+3B_h^2\|\hat{\eta}_1\|^2\right]}$.

Given $\eta_1,\eta_2 \in \mathcal{BC}(I,\mathcal{M}_+(I))$, let μ_1 and μ_2 be the laws of the solutions of (5.2) for the DGMs η_1 and η_2. Then,

$$\left[W_{2,s}^{\mathcal{N},2}(\mu^1,\mu^2)\right]^2 \le C_1C_2\left[d_\infty(\eta_1,\eta_2)\right]^2 e^{C_1 T}, \tag{5.15}$$

where $C_1 = 3\left[3B_g^2\|\eta_1\|^2+2B_h^2\|\hat{\eta}\|^2\right]e^{\left(3L_f^2+3\left[3L_g^2\|\eta_1\|^2+2L_h^2\|\hat{\eta}\|^2\right]\right)T}$ and $C_2 = \frac{3B_g^2T}{\left[3B_g^2\|\eta_1\|^2+2B_h^2\|\hat{\eta}\|^2\right]}$.

Proof. We will proceed to prove only the first statement of the proposition, as the proof idea is the same for both cases. Consider two different DGMs $\hat{\eta}_1$ and $\hat{\eta}_2 \in \mathcal{B}(I,\mathcal{M}_+(I))\bigcap\mathcal{C}(I,\mathcal{M}_+(I))$ with the respective solutions they induce denoted by X^1 and X^2. We have

$$\begin{aligned}&\mathbb{E}|X_u^1(t)-X_u^2(t)|^2\\ &\le 3\mathbb{E}\int_0^t |f(X_u^1(s))-f(X_u^2(s))|^2\ \mathrm{d}s\\ &+3\mathbb{E}\int_0^t\left|\int_I\int_{\mathbb{X}} g(X_u^1(s),y)\ \mu_{v,s}^1(\mathrm{d}y)\eta^u(\mathrm{d}v)-g(X_u^2(s),y)\ \mu_{v,s}^2(\mathrm{d}y)\eta^u(\mathrm{d}v)\right|^2\mathrm{d}s\\ &+3\mathbb{E}\Bigg|\int_0^t\int_I\int_{\mathbb{X}}\Bigg(h(X_u^1(s),y)\ \mu_{v,s}^1(\mathrm{d}y)\hat{\eta}_1^u(\mathrm{d}v)\\ &\qquad\qquad -h(X_u^2(s),y)\ \mu_{v,s}^2(\mathrm{d}y)\hat{\eta}_2^u(\mathrm{d}v)\Bigg)\,\mathrm{d}B_s^u\Bigg|^2.\end{aligned}$$

Let us start with the term related to Brownian motion. We can rewrite this term again using standard properties of stochastic integrals. Furthermore, by adding and subtracting terms, we get

$$
\begin{aligned}
&3\mathbb{E}\left|\int_0^t \int_I \int_{\mathbb{X}} \Big(h(X_u^1(s),y)\ \mu_{v,s}^1(\mathrm{d}y)\hat{\eta}_1^u(\mathrm{d}v) - h(X_u^2(s),y)\ \mu_{v,s}^2(\mathrm{d}y)\hat{\eta}_2^u(\mathrm{d}v)\Big)\ \mathrm{d}B_s^u\right|^2 \\
&\leq +9\mathbb{E}\int_0^t \left|\int_I \int_{\mathbb{X}} \Big[h(X_u^1(s),y) - h(X_u^2(s),y)\Big]\ \mu_{v,s}^1(\mathrm{d}y)\hat{\eta}_1^u(\mathrm{d}v)\right|^2 \mathrm{d}s \\
&\quad + 9\mathbb{E}\int_0^t \left|\int_I \int_{\mathbb{X}} h(X_u^2(s),y)\ \Big[\mu_{v,s}^1(\mathrm{d}y) - \mu_{v,s}^2(\mathrm{d}y)\Big]\ \hat{\eta}_1^u(\mathrm{d}v)\right|^2 \mathrm{d}s \\
&\quad + 9\mathbb{E}\int_0^t \left|\int_I \int_{\mathbb{X}} h(X_u^2(s),y)\ \mu_{v,s}^2(\mathrm{d}y)\left[\hat{\eta}_1^u(\mathrm{d}v) - \hat{\eta}_2^u(\mathrm{d}v)\right]\right|^2 \mathrm{d}s.
\end{aligned}
$$

Using the properties of h, and just like we did in the previous proof, we can bound these terms as follow

$$
\begin{aligned}
&3\mathbb{E}\left|\int_0^t \int_I \int_{\mathbb{X}} \Big(h(X_u^1(s),y)\mu_{v,s}^1(\mathrm{d}y)\hat{\eta}_1^u(\mathrm{d}v) - h(X_u^2(s),y)\mu_{v,s}^2(\mathrm{d}y)\hat{\eta}_2^u(\mathrm{d}v)\Big)\ \mathrm{d}B_s^u\right|^2 \\
&\leq +9L_h^2\|\hat{\eta}_1\|^2\mathbb{E}\int_0^t \left|X_u^1(s) - X_u^2(s)\right|^2 \mathrm{d}s \\
&\quad + 9B_h^2\|\hat{\eta}_1\|^2 \int_0^t \left|W_{2,s}^{\mathcal{N},\infty}(\mu^1,\mu^2)\right|^2 \mathrm{d}s \\
&\quad + 9B_h^2 T\left[d_\infty(\hat{\eta}_1,\hat{\eta}_2)\right]^2.
\end{aligned}
$$

If we go back to the original inequality and bound the remainder terms using the same methodology as we did for the Brownian term, we obtain the following expression

$$
\begin{aligned}
\mathbb{E}|X_u^1(t) - X_u^2(t)|^2 \leq{}& \Big(3L_f^2 + 3\left[2L_g^2\|\eta\|^2 + 3L_h^2\|\hat{\eta}_1\|^2\right]\Big)\,\mathbb{E}\int_0^t |X_u^1 - X_u^2|_{*,s}^2\,\mathrm{d}s \\
&+ 3\left[2B_g^2\|\eta\|^2 + 3B_h^2\|\hat{\eta}_1\|^2\right]\int_0^t \left[W_{2,s}^{\mathcal{N},\infty}(\mu^1,\mu^2)\right]^2 \mathrm{d}s \\
&+ 9B_h^2 T\left[d_\infty(\hat{\eta}_1,\hat{\eta}_2)\right]^2.
\end{aligned}
$$

Applying the BDG inequality and Grönwall's inequality we get

$$
\mathbb{E}|X_u^1 - X_u^2|_{*,t}^2 \leq C_1\left(\int_0^t \left[W_{2,s}^{\mathcal{N},\infty}(\mu^1,\mu^2)\right]^2 \mathrm{d}s + C_2\left[d_\infty(\hat{\eta}_1,\hat{\eta}_2)\right]^2\right),
$$

where $C_1 = 3\left[2B_g^2\|\eta\|^2 + 3B_h^2\|\hat{\eta}_1\|^2\right]e^{\left(3L_f^2+3\left[2L_g^2\|\eta\|^2+3L_h^2\|\hat{\eta}_1\|^2\right]\right)T}$ and $C_2 = \frac{3B_h^2 T}{\left[2B_g^2\|\eta\|^2+3B_h^2\|\hat{\eta}_1\|^2\right]}$. By definition of $W_{2,t}^{\mathcal{N},\infty}(\mu^1,\mu^2)$, we have the following:

$$
\left[W_{2,s}^{\mathcal{N},\infty}(\mu^1,\mu^2)\right]^2 \leq C_1\left(\int_0^t \left[W_{2,s}^{\mathcal{N},\infty}(\mu^1,\mu^2)\right]^2 \mathrm{d}s + C_2\left[d_\infty(\hat{\eta}_1,\hat{\eta}_2)\right]^2\right).
$$

Applying Grönwall's inequality, we find

$$
\left[W_{2,s}^{\mathcal{N},\infty}(\mu^1,\mu^2)\right]^2 \leq C_1C_2\left[d_\infty(\hat{\eta}_1,\hat{\eta}_2)\right]^2 e^{C_1T}.
$$

And thus, we obtain the desired inequality

$$
\left[W_{2,s}^{\mathcal{N},2}(\mu^1,\mu^2)\right]^2 \leq C_1C_2\left[d_\infty(\hat{\eta}_1,\hat{\eta}_2)\right]^2 e^{C_1T}.
$$

This finishes the proof. □

5.5 Proof of Theorem 5.3.1

Once the existence of the solution of the independent problem has been proven and the continuity of the solution with respect to the DGM has been shown, the next step is to determine how close the solutions of the problem (5.1) are to the solutions of the independent problem (5.2). To compare trajectories of solutions between equation (5.1) and equation (5.2), we will associate the particles X_i with the particles $X_{u(i)}$. To achieve this, we will consider a partition of the interval I defined by I_i^N, as introduced previously. For each interval, we will select a representative $u(i) = \frac{i}{N}$. Consequently, we will compare the trajectories of X_i^N and $X_{\frac{i}{N}}$. To proceed with this matter, it is important to consider that each one is described by a distinct stochastic process. However, to identify both particles as similar, we make the assumption that for our choice of the random variable $u = \frac{i}{N}$, both particles exhibit the same Brownian motions, i.e., $B_t^i = B_t^{\frac{i}{N}}$. This strategy of aligning the trajectories based on u enables us to evaluate the similarity between the two particles, despite their different stochastic processes.

Proposition 5.5.1. *Under the hypotheses of Theorem 5.3.1, the following inequality*

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\|X_i^N(t) - X_{\frac{i}{N}}\|_{*,t}^2 \leq C\frac{16}{N}\sum_{i=1}^{N}\mathbb{E}|X_i^0 - X_{\frac{i}{N}}^0|^2$$
$$+ C\left(\frac{2}{N}(B_g^2 + B_h^2)\mathcal{O}(1) + B_g^2\left[d_\infty\left(\eta_{A^N}, \eta\right)\right]^2 + B_h^2\left[d_\infty\left(\eta_{\hat{A}^N}, \hat{\eta}\right)\right]^2\right),$$

holds, where $C = 48Te^{(16L_f^2+96(L_g^2+L_h^2)\mathcal{O}(1))T}$.

Proof. Fix $t \in [0, T]$. Let us compare X_i^N and $X_{\frac{i}{N}}$, solutions of (5.1) and (5.2), respectively.

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\|X_i^N(t) - X_{\frac{i}{N}}\|_{*,t}^2 \leq \frac{16}{N}\sum_{i=1}^{N}\mathbb{E}|X_i^0 - X_{\frac{i}{N}}^0|^2$$
$$+ \frac{16}{N}\sum_{i=1}^{N}\mathbb{E}\int_0^t |f(X_i^N(s)) - f(X_{\frac{i}{N}}(s))|^2\,\mathrm{d}s$$
$$+ \frac{16}{N}\sum_{i=1}^{N}\mathbb{E}\int_0^t \left|\frac{1}{N}\sum_{i=1}^{N}A_{i,j}^N g(X_i^N(s), X_j^N(s))\right.$$
$$\left. - \int_I\int_{\mathbb{X}} g(X_{\frac{i}{N}}(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^{\frac{i}{N}}(\mathrm{d}v)\right|^2 \mathrm{d}s$$
$$+ \frac{16}{N}\sum_{i=1}^{N}\mathbb{E}\int_0^t \left|\frac{1}{N}\sum_{j=1}^{N}\hat{A}_{ij}^N h(X_i^N(s), X_j^N(s))\mathrm{d}s\right.$$
$$\left. - \int_I\int_{\mathbb{X}} h(X_{\frac{i}{N}}(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^{\frac{i}{N}}(\mathrm{d}v)\right|^2 \mathrm{d}s.$$

Hence, we get that $\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\|X_i^N(t) - X_{\frac{i}{N}}\|_{*,t}^2 \leq \mathrm{E}_0 + \mathrm{E}_f + \mathrm{E}_g + \mathrm{E}_h$, so let us estimate each term. Notice that, by the construction of the solution, we have assumed that $B_t^i = B^{\frac{i}{N}}$. Let us start with E_f. We can bound the term depending on the function f in a similar way to what we have been doing throughout this work. Using the fact that the function f is Lipschitz,

$$\mathrm{E}_f \leq L_f^2 \frac{16}{N} \sum_{i=1}^{N} \int_0^t \mathbb{E}|X_i^N(s) - X_{\frac{i}{N}}(s)|^2 \, \mathrm{d}s.$$

To estimate E_g, we will add and subtract different similar terms to find

$$\begin{aligned}
\mathrm{E}_g \leq \frac{48}{N}\Bigg(& \int_0^t \sum_{i=1}^{N} \mathbb{E}\left| \frac{1}{N} \sum_{j=1}^{N} A_{i,j}^N (g(X_i^N(s), X_j^N(s)) - g(X_{\frac{i}{N}}(s), X_{\frac{j}{N}}(s))) \right|^2 \mathrm{d}s \\
& + \int_0^t \sum_{i=1}^{N} \mathbb{E}\Bigg| \frac{1}{N} \sum_{j=1}^{N} A_{i,j}^N \Big(g(X_{\frac{i}{N}}(s), X_{\frac{j}{N}}(s)) \\
& \qquad\qquad - \int_{\mathbb{X}} g(X_{\frac{i}{N}}(s), y) \; \mu_{\frac{j}{N},s}(\mathrm{d}y) \Big) \Bigg|^2 \mathrm{d}s \\
& + \int_0^t \sum_{i=1}^{N} \mathbb{E}\left| \int_I \int_{\mathbb{X}} g(X_{\frac{i}{N}}(s), y) \; \mu_{v,s}(\mathrm{d}y) \left(\eta_{A^N}^{\frac{i}{N}}(\mathrm{d}v) - \eta^{\frac{i}{N}}(\mathrm{d}v) \right) \right|^2 \mathrm{d}s \Bigg),
\end{aligned}$$

where $\eta_{A^N}^{\frac{i}{N}}$ is given by (5.4). We can denote the previous estimate as $\mathrm{E}_g \leq \mathrm{E}_g^1 + \mathrm{E}_g^2 + \mathrm{E}_g^3$. Let us bound each term separately. Using the Lipschitz property of g, the term E_g^1 can be bounded as follows:

$$\begin{aligned}
\mathrm{E}_g^1 & \leq L_g^2 \frac{48}{N^3} \int_0^t \sum_{i=1}^{N} \Bigg[\sum_{j=1}^{N} (A_{i,j}^N)^2 \cdot \mathbb{E}\Bigg[\sum_{j=1}^{N} \Big(|X_i^N(s) - X_{\frac{i}{N}}(s)| \\
& \qquad\qquad + |X_j^N(s) - X_{\frac{j}{N}}(s)| \Big)^2 \Bigg] \Bigg] \mathrm{d}s \\
& \leq L_g^2 \frac{96}{N^3} \int_0^t \sum_{i=1}^{N} \Bigg[\sum_{j=1}^{N} (A_{i,j}^N)^2 \cdot \sum_{j=1}^{N} \Big(\mathbb{E}|X_i^N(s) - X_{\frac{i}{N}}(s)|^2 \\
& \qquad\qquad + \mathbb{E}|X_j^N(s) - X_{\frac{j}{N}}(s)|^2 \Big) \Bigg] \mathrm{d}s \\
& \leq \frac{96}{N} L_g^2 \mathcal{O}(1) \int_0^t \sum_{i=1}^{N} \mathbb{E}|X_i^N(s) - X_{\frac{i}{N}}(s)|^2 \mathrm{d}s,
\end{aligned}$$

where we have used that $\sum_{j=1}^{N} (A_{i,j}^N)^2 = \mathcal{O}(N)$, for all $1 \leq i \leq N$. Similarly, the term E_g^3 can be bounded by using the fact that g is a Lipschitz and bounded function:

$$\mathrm{E}_g^3 \leq 48 T B_g^2 \left[d_\infty(\eta_{A^N}, \eta) \right]^2. \tag{5.16}$$

Finally, we only need to estimate E_g^2. Note that this term can be rewritten by expanding the square as follows:

$$\begin{aligned}
\mathrm{E}_g^2 = \frac{48}{N^3} \int_0^t \sum_{i,j,k=1}^{N} \mathbb{E}\Bigg[& \left(A_{i,j}^N g(X_{\frac{i}{N}}(s), X_{\frac{j}{N}}(s)) - \int_{\mathbb{X}} g(X_{\frac{i}{N}}(s), y) A_{i,j}^N \; \mu_{\frac{j}{N},s}(\mathrm{d}y) \right) \\
& \left(A_{i,k}^N g(X_{\frac{i}{N}}(s), X_{\frac{k}{N}}(s)) - \int_{\mathbb{X}} g(X_{\frac{i}{N}}(s), y) A_{i,k}^N \; \mu_{\frac{k}{N},s}(\mathrm{d}y) \right) \Bigg] \mathrm{d}s.
\end{aligned} \tag{5.17}$$

We observe that, due to the independence of the $X_{\frac{i}{N}}$ as constructed in (5.2), all terms will be 0 except when $k = j$ or $k = i$. Therefore, by making use of the boundedness of g we obtain

$$\mathrm{E}_g^2 \leq \frac{96}{N} B_g^2 \int_0^t \sum_{i=1}^{N} \frac{1}{N^2} \sum_{j=1}^{N} (A_{i,j}^N)^2.$$

Using that $\sum_{j=1}^{N} (A_{i,j}^N)^2 = \mathcal{O}(N)$, for all $1 \leq i \leq N$, we have

$$\mathrm{E}_g^2 \leq \frac{96}{N} B_g^2 T \mathcal{O}(1).$$

The bounds for the term E_h are calculated in the same way as the previous term. Putting all the estimates together, we obtain:

$$\begin{aligned}
&\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \| X_i^N(t) - X_{\frac{i}{N}} \|_{*,t}^2 \\
&\qquad \leq \left(16L_f^2 + 96(L_g^2 + L_h^2)\mathcal{O}(1)\right) \int_0^t \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} |X_i^N(s) - X_{\frac{i}{N}}(s)|^2 \, \mathrm{d}s \\
&\qquad + \frac{96}{N}(B_g^2 + B_h^2) T \mathcal{O}(1) + \frac{16}{N} \sum_{i=1}^{N} \mathbb{E} |X_i^0 - X_{\frac{i}{N}}^0|^2 \\
&\qquad + 48T \left(B_g^2 \left[d_\infty \left(\eta_{A^N}, \eta \right) \right]^2 + B_h^2 \left[d_\infty \left(\eta_{\hat{A}^N}, \hat{\eta} \right) \right]^2 \right).
\end{aligned}$$

Grönwall's inequality leads to

$$\begin{aligned}
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \| X_i^N(t) - X_{\frac{i}{N}} \|_{*,t}^2 &\leq C \frac{16}{N} \sum_{i=1}^{N} \mathbb{E} |X_i^0 - X_{\frac{i}{N}}^0|^2 \\
&+ C \left(\frac{2}{N}(B_g^2 + B_h^2)\mathcal{O}(1) + B_g^2 \left[d_\infty \left(\eta_{A^N}, \eta \right) \right]^2 + B_h^2 \left[d_\infty \left(\eta_{\hat{A}^N}, \hat{\eta} \right) \right]^2 \right),
\end{aligned}$$

where $C = 48T e^{(16L_f^2 + 96(L_g^2 + L_h^2)\mathcal{O}(1))T}$. □

To prove Theorem 5.3.1, we have to demonstrate that the empirical measure $\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^N}$ converges in probability to $\bar{\mu} = \int_I \mu_u du$.

Proof. (of Theorem 5.3.1) The convergence in probability, follows from the convergence for both random variables: U and the one generated by the Brownian motion. To do this, it is sufficient to establish that:

$$\lim_{N \to +\infty} E_u \times \mathbb{E} \left| \int_{\mathbb{X}} f(y) \; \bar{\mu}_N(\mathrm{d}y) - \int_{\mathbb{X}} f(y) \; \bar{\mu}(\mathrm{d}y) \right|^2 = 0, \tag{5.18}$$

for every bounded and Lipschitz function f, where E_u denotes the expectation related to the random variable U. We know that $\mu_{i/N,t} = \mathcal{L}(X(t)|U = i/N)$, is the probability measure $\bar{\mu}_t$ conditioned on $U = i/N$. Therefore we obtain

$$\begin{aligned}
&\lim_{N \to +\infty} E_u \times \mathbb{E} \left| \int_{\mathbb{X}} f(y) \; \bar{\mu}_N(\mathrm{d}y) - \int_{\mathbb{X}} f(y) \; \bar{\mu}(\mathrm{d}y) \right|^2 \\
&\quad \leq \lim_{N \to +\infty} \frac{1}{N} \sum_{i=1}^{N} E_u \times \mathbb{E} \left| f(X_i) - f\left(X_{\frac{i}{N}}\right) \right|^2.
\end{aligned}$$

Taking into account the Lipschitz property of f and using Proposition 5.5.1, we can conclude

$$\begin{aligned}
&\lim_{N\to+\infty} E_u \times \mathbb{E}\left|\int_{\mathbb{X}} f(y)\ \bar{\mu}_N(\mathrm{d}y) - \int_{\mathbb{X}} f(y)\ \bar{\mu}(\mathrm{d}y)\right|^2 \\
&\leq \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} E_u \times \mathbb{E}\left|X_i - X_{\frac{i}{N}}\right|^2_{*,t} = 0.
\end{aligned}$$

This finishes the proof of Theorem 5.3.1. □

5.6 Proof of Theorem 5.3.2

Finally, let us prove Theorem 5.3.2. Unlike before, since our probability measure will now take actual input values $u \in I$, instead of considering u as a random variable, we will work in the space $\tilde{\mathcal{N}}$, in which we must prove the measurability with respect to u. The overall proof idea of Theorem 5.3.2 will follow very similar ideas as before, yet certain details that need to be modified in some proofs due to working with measurability in the variable u. Therefore, in this section, we will not go into detail in all the proofs but will only introduce those steps that are different from the previous ones. Let us begin by demonstrating the existence and uniqueness of the solution to (5.6).

Proposition 5.6.1. *Under de assumptions H and $\tilde{H}$, there exist a unique solution X_u, for $u \in I$, to (5.6), where $\mu_t = (\mathcal{L}(X_u) : u \in I) \in \tilde{\mathcal{N}}$. Moreover, $\mu_{u,t}$ is a weak solution of the Vlasov-Fokker-Plank equation:*

$$\begin{aligned}
\partial_t\mu_{u,t} + \partial_x\left(\mu_{u,t}f(x) + \mu_{u,t}\int_I\int_{\mathbb{X}} g(x,y)\ \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\right) = \\
+\frac{1}{2}\partial_x^2\left(\mu_{u,t}\left[\int_I\int_{\mathbb{X}} h(x,y)\ \mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right]^2\right), \forall u \in I.
\end{aligned} \tag{5.19}$$

Proof. The proof will be carried out in the same way as we did in Proposition 5.4.1. We will consider an operator $\tilde{\mathcal{F}}$ defined in the space $\mathcal{L}$ and seek a fixed point. To do this, let us consider the mapping $\mu \in \tilde{\mathcal{N}} \mapsto \tilde{\mathcal{F}}(\mu) \in \tilde{\mathcal{N}}$, where $\tilde{\mathcal{F}}(\mu)$ is the law associated with the solution of the equation:

$$\begin{aligned}
X_u^\mu(t) =& X_u^\mu(0) + \int_0^t f(X_u^\mu(s))\ \mathrm{d}s + \int_0^t\int_I\int_{\mathbb{X}} g(X_u^\mu(s),y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\
&+ \int_0^t\int_I\int_{\mathbb{X}} h(X_u^\mu(s),y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}v_s^u.
\end{aligned} \tag{5.20}$$

Note that if we have a fixed point then $\tilde{\mathcal{F}}(\mu) = \mathcal{L}(X^\mu) = \mu$, so this would prove the existence of solution. First, we must prove that the operator is well-defined, meaning that there exists a solution for (5.20), for every $\mu \in \tilde{\mathcal{N}}$, and furthermore, the solutions are in $\tilde{\mathcal{N}}$. To do this, let us take $\mu \in \mathcal{N}$, and let $X_u^0(t) = X_u(0)$, $\forall t \in [0,T]$ and $u \in I$. Consider the following recurrence equation:

$$
\begin{aligned}
X_u^n(t) = X_u^{n-1}(0) &+ \int_0^t f(X_u^{n-1}(s))\ \mathrm{d}s \\
&+ \int_0^t \int_I \int_{\mathbb{X}} g(X_u^{n-1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\
&+ \int_0^t \int_I \int_{\mathbb{X}} h(X_u^{n-1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B_s^u,
\end{aligned} \tag{5.21}
$$

where $X_u^k(0) = X_u^0(0)$, for all $k \geq 1$. Notice that if we prove that the sequence $\{X_u^n\}_{n\geq 1}$ is Cauchy, and moreover, for each $n \in \mathbb{N}$, $\mathcal{L}(X_u^n : u \in I) \in \tilde{\mathcal{N}}$, then we will have that the sequence has a limit, and this limit belongs to $\tilde{\mathcal{N}}$ since this space is complete with the Wasserstein distance $W_{2,t}^{\tilde{\mathcal{N}},\infty}$.

The proof that $\{X_u^n\}_{n\geq 0}$ is Cauchy, is identical to the one carried out in Proposition 5.4.1, therefore, we will not develop it. Let us focus on proving that for each $n \in \mathbb{N}$, the mapping $u \in I \mapsto \mathcal{L}(X_u^n, B^u)$ is measurable. We will prove it by induction. By construction and assumptions $\tilde{H}$, measurability holds for $n = 0$. Suppose it holds for $n = 0, \ldots, N-1$. Let us now establish that measurability holds for $n = N$. To show this, it is sufficient to show that it is measurable for any collection of times, that is,

$$
I \ni u \mapsto \mathcal{L}\left(X_u^N(t_1), B^u(t_1), \ldots, X_u^N(t_m), B^u(t_m)\right) \in \mathcal{P}\left(\left(\mathbb{R}^d \times \mathbb{R}^d\right)^m\right).
$$

is measurable for all $0 \leq t_1 \leq \cdots \leq t_m \leq T$ and $m \in \mathbb{N}$. It further suffices to demonstrate that

$$
I \ni u \mapsto \mathbb{E}\left[\prod_{i=1}^m \left(\alpha_i\left(X_u^N(t_i)\right)\beta_i\left(B^u(t_i)\right)\right)\right] \in \mathbb{R},
$$

is measurable, for all $0 \leq t_1 \leq \cdots \leq t_m \leq T, m \in \mathbb{N}$ and bounded and continuous functions $\{\alpha_i, \alpha_i : i = 1, \ldots, m\}$ on $\mathbb{R}^d$. Let us now establish that $X_u^N(t)$ is measurable. To achieve this, consider $X_u^{N,\delta}(t)$ as a solution to the following auxiliary process

$$
\begin{aligned}
X_u^{N,\delta}(t) =& X_u^{N-1}(0) + \int_0^t f(X_u^{N-1}\left(\left\lfloor\frac{s}{\delta}\right\rfloor\delta\right))\ \mathrm{d}s \\
&+ \int_0^t \int_I \int_{\mathbb{X}} g\left(X_u^{N-1}\left(\left\lfloor\frac{s}{\delta}\right\rfloor\delta\right), y\right)\ \mu_{v,\lfloor\frac{s}{\delta}\rfloor\delta}(dy)\eta^u(\mathrm{d}v)\mathrm{d}s \\
&+ \int_0^t \int_I \int_{\mathbb{X}} h\left(X_u^{N-1}\left(\left\lfloor\frac{s}{\delta}\right\rfloor\delta\right), x\right)\ \mu_{v,\lfloor\frac{s}{\delta}\rfloor\delta}(dx)\hat{\eta}^u(\mathrm{d}v)dB_s^u,
\end{aligned}
$$

where $\delta \in (0,1)$. Note that, due to the construction of the auxiliary process, for each $\delta > 0$, we can express $X_u^{N,\delta}(t)$ as a finite sum of terms that depend on

$$
\left\{X_u^{N-1}(0), X_u^{N-1}(\delta), \ldots, X_u^{N-1}\left(\left\lfloor\frac{t}{\delta}\right\rfloor\delta\right)\right\} \quad \text{and} \quad \left\{B_0^u, B_\delta^u, \ldots, B^u_{\left(\lfloor\frac{t}{\delta}\rfloor\delta\right)}\right\}.
$$

Furthermore, $X_u^{N,\delta}(t)$ converges to $X_u^k(t)$ in probability as $\delta \to 0$, for each $u \in I$. So it suffices to prove that

$$
I \ni u \mapsto \mathbb{E}\left[\prod_{i=1}^m \left(\alpha_i\left(X_u^{k,\delta}(t_i)\right)\beta_i\left(B_{t_i}^u\right)\right)\right] \in \mathbb{R},
$$

is measurable, for all $0 \leq t_1 \leq \cdots \leq t_m \leq T, m \in \mathbb{N}$ and bounded and continuous functions $\{\alpha_i, \beta_i : i = 1, \ldots, m\}$ on $\mathbb{R}^d$. Fix $t \in [0, \bar{T}]$. Since the measurability holds for $N-1$, it further suffices to show that

$$X_u^{N,\delta}(t) = \gamma\left(u, X_u^{N-1}, B^u\right),$$

for some measurable function $\gamma : I \times \mathcal{C}([0,T]) \times \mathcal{C}([0,T]) \to \mathbb{R}$. Using that $X_u^{N,\delta}(t)$ is a finite sum depending on $\left\{X_u^{N-1}(0), X_u^{N-1}(\delta), \ldots, X_u^{N-1}\left(\lfloor \frac{t}{\delta} \rfloor \delta\right)\right\}$ and $\left\{B_0^u, B_\delta^u, \ldots, B^u_{\left(\lfloor \frac{t}{\delta} \rfloor \delta\right)}\right\}$ continuously, we have that $\gamma(u, \cdot, \cdot)$ is continuous on $\mathcal{C}([0,T]) \times \mathcal{C}([0,T])$, for each $u \in I$, and that $h(\cdot, x, w)$ is measurable on I for each $(x, w) \in \mathcal{C}([0,T]) \times \mathcal{C}([0,T])$. Therefore, γ is measurable and this proves the mesurability for $n = N$. In summary, we have thus proven that the operator $\tilde{\mathcal{F}}$ is well-defined. Now, we need to analyze the existence of a fixed point for the operator. However, this is nothing more than proving that given two laws μ and ν in $\tilde{\mathcal{N}}$, it holds that:

$$|W_{2,t}^{\tilde{\mathcal{N}},\infty}(\mathcal{F}(\mu), \mathcal{F}(\nu))|^2 \leq C \int_0^t |W_{2,s}^{\tilde{\mathcal{N}},\infty}(\mu, \nu)|^2 \, \mathrm{d}s, \tag{5.22}$$

for a certain constant $C > 0$. The proof of this inequality is similar to the one carried out in Proposition 5.4.1, so we omit the details. The last inequality gives the path-wise uniqueness of the solution, and also allow us to prove the existence of solution. For this purpose we will build an iterative process, as follows. Consider $\nu = (\mathcal{L}(Z_u) : u \in I)$, where $Z_u(t) = X_u(0)$ for all $u \in I$ and $t \in [0,T]$. Iterating this, and using (5.11), we get:

$$W_{2,T}^{\tilde{\mathcal{N}},\infty}(\tilde{\mathcal{F}}^{n+1}(\nu), \tilde{\mathcal{F}}^n(\nu)) \leq \frac{C^n T^n}{n!} |W_{2,T}^{\tilde{\mathcal{N}},\infty}(\tilde{\mathcal{F}}(\nu), \nu)|.$$

It follows that the sequence $\left\{\tilde{\mathcal{F}}^n(\nu)\right\}_n$ is Cauchy for n large enough, where we have used that $W_{2,T}^{\tilde{\mathcal{N}},\infty}(\tilde{\mathcal{F}}(\nu), \nu) < \infty$, due to the assumptions on the initial data and the fact that the functions f, g, and h are bounded. This sequence will have a limit since $\tilde{\mathcal{N}}$ is a complete metric space, and hence there exists $\mu = (\mathcal{L}(X_u), u \in I) \in \tilde{\mathcal{N}}$ solution of (5.2), for the initial data $X_u(0)$. □

Once the existence and uniqueness of the solution to (5.6) have been established, the next step is to study how these solutions depend on the graph. In other words, what properties the solutions have with respect to the DGMs: η and $\hat{\eta}$, and with respect to the graph's heterogeneity variable u. For this purpose, we have the following result.

Proposition 5.6.2. *Given $\hat{\eta}_1, \hat{\eta}_2 \in \mathcal{BC}(I, \mathcal{M}_+(I))$, let μ_1 and μ_2 be the laws of the solutions of (5.6) for the DGMs $\hat{\eta}_1$ and $\hat{\eta}_2$. Then,*

$$\left[W_{2,s}^{\tilde{\mathcal{N}},\infty}(\mu^1, \mu^2)\right]^2 \leq \hat{C}_1 \hat{C}_2 \left[d_\infty(\hat{\eta}_1, \hat{\eta}_2)\right]^2 e^{\hat{C}_1 T}, \tag{5.23}$$

where $\hat{C}_1 = 3\left[2B_g^2\|\eta\|^2 + 3B_h^2\|\hat{\eta}_1\|^2\right] e^{\left(3L_f^2 + 3\left[2L_g^2\|\eta\|^2 + 3L_h^2\|\hat{\eta}_1\|^2\right]\right)T}$ and $\hat{C}_2 = \frac{3B_h^2 T}{\left[2B_g^2\|\eta\|^2 + 3B_h^2\|\hat{\eta}_1\|^2\right]}$.

Given $\eta_1, \eta_2 \in \mathcal{BC}(I, \mathcal{M}_+(I))$, let μ_1 and μ_2 be the laws of the solutions of (5.6) for the DGMs η_1 and η_2. Then,

$$\left[W_{2,s}^{\tilde{N},\infty}(\mu^1, \mu^2)\right]^2 \leq C_1 C_2 \left[d_\infty(\eta_1, \eta_2)\right]^2 e^{C_1 T}, \tag{5.24}$$

where $C_1 = 3\left[3B_g^2\|\eta_1\|^2 + 2B_h^2\|\hat{\eta}\|^2\right] e^{\left(3L_f^2 + 3\left[3L_g^2\|\eta_1\|^2 + 2L_h^2\|\hat{\eta}\|^2\right]\right)T}$ *and* $C_2 = \frac{3B_g^2 T}{\left[3B_g^2\|\eta_1\|^2 + 2B_h^2\|\hat{\eta}\|^2\right]}$.

Given $u_1, u_2 \in I$, let μ_u be the law of the solution of (5.6). Then,

$$\begin{aligned}\left[W_{2,T}(\mu_{u_1}, \mu_{u_2})\right]^2 \leq C \Big(&\left[W_2(\bar{\mu}_{u_1}^0, \bar{\mu}_{u_2}^0)\right]^2 \\ &+ 2T\left[B_g^2 d_{BL}^2(\eta^{u_1}, \eta^{u_2}) + B_h^2 d_{BL}^2(\hat{\eta}^{u_1}, \hat{\eta}^{u_2})\right]\Big),\end{aligned}$$

where $C = 16e^{16\left(L_f^2 + 2L_g^2|\eta|^2 + 2L_h^2|\hat{\eta}|^2\right)T}$.

Proof. The proof of the first two results is analogous to the one carried out in Proposition 5.4.2, so we will not go into detail. Just note that since the structure of the two equations, (5.2) and (5.6), is analogous, the bounds will be the same thanks to the relationship between the measures $W^{\cdot,2}$ and $W^{\cdot,\infty}$ and the fact that the set over which u takes variables is the interval $I = [0, 1]$.

Let $u_1, u_2 \in I$, and let μ be the law of the solution to (5.6). We know that for each $u \in I$, X_u satisfies the equation:

$$\begin{aligned}X^u(t) =& X^u(0) + \int_0^t f(X^u(s))\ \mathrm{d}s + \int_0^t \int_I \int_{\mathbb{X}} g(X^u(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\ &+ \int_0^t \int_I \int_{\mathbb{X}} h(X^u(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B_s^u.\end{aligned} \tag{5.25}$$

To remove the dependence of B_s^u with respect to u, we will employ the coupling method (see [50]). This method allows us to relate our stochastic process with the following system, as in the limit both systems satisfy the same dynamics:

$$\begin{aligned}X^u(t) =& X^u(0) + \int_0^t f(X^u(s))\ \mathrm{d}s + \int_0^t \int_I \int_{\mathbb{X}} g(X^u(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}s \\ &+ \int_0^t \int_I \int_{\mathbb{X}} h(X^u(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\mathrm{d}B_s.\end{aligned} \tag{5.26}$$

Therefore, since we are interested in analyzing what happens in the limit, it will be enough to prove the property for the second system. Let us proceed to bound $\mathbb{E}|X_{u_1}(t) - X_{u_2}(t)|^2$:

$$\begin{aligned}&\mathbb{E}|X_{u_1}(t) - X_{u_2}(t)|^2 \leq 4\mathbb{E}|X_{u_1}(0) - X_{u_2}(0)|^2 \\ &+ 4\mathbb{E}\int_0^t |f(X_{u_1}(s)) - f(X_{u_2}(s))|^2\ \mathrm{d}s \\ &+ 4\mathbb{E}\int_0^t \Big|\int_I \int_{\mathbb{X}} g(X^{u_1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^{u_1}(\mathrm{d}v) - g(X^{u_2}(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^{u_2}(\mathrm{d}v)\Big|^2 \mathrm{d}s \\ &+ 4\mathbb{E}\int_0^t \Big|\int_I \int_{\mathbb{X}} h(X^{u_1}(s), y)\ \mu_{v,s}(\mathrm{d}y)\eta^{u_1}(\mathrm{d}v) - h(X^{u_2}(s), y)\ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^{u_2}(\mathrm{d}v)\Big|^2 \mathrm{d}s.\end{aligned}$$

We have used the Hölder inequality and the properties of stochastic integrals. By adding and subtracting terms in the last two integrals, we obtain the following:

$$
\begin{aligned}
\mathbb{E}|X_{u_1}(t)-X_{u_2}(t)|^2 \le\ & 4\mathbb{E}|X_{u_1}(0)-X_{u_2}(0)|^2 \\
&+ 4\mathbb{E}\int_0^t |f(X_{u_1}(s)) - f(X_{u_2}(s))|^2\ \mathrm{d}s \\
&+ 8\mathbb{E}\int_0^t \left|\int_I\int_{\mathbb{X}} (g(X^{u_1}(s),y) - g(X^{u_2}(s),y))\ \ \mu_{v,s}(\mathrm{d}y)\eta^{u_1}(\mathrm{d}v)\right|^2 \mathrm{d}s \\
&+ 8\mathbb{E}\int_0^t \left|\int_I\int_{\mathbb{X}} g(X^{u_1}(s),y)\ \mu_{v,s}(\mathrm{d}y)\,(\eta^{u_1}(\mathrm{d}v) - \eta^{u_2}(\mathrm{d}v))\right|^2 \mathrm{d}s \\
&+ 8\mathbb{E}\int_0^t \left|\int_I\int_{\mathbb{X}} (h(X^{u_1}(s),y) - h(X^{u_2}(s),y))\ \ \mu_{v,s}(\mathrm{d}y)\hat{\eta}^{u_1}(\mathrm{d}v)\right|^2 \mathrm{d}s \\
&+ 8\mathbb{E}\int_0^t \left|\int_I\int_{\mathbb{X}} h(X^{u_1}(s),y)\ \mu_{v,s}(\mathrm{d}y)\,(\eta^{u_1}(\mathrm{d}v) - \hat{\eta}^{u_2}(\mathrm{d}v))\right|^2 \mathrm{d}s.
\end{aligned}
$$

Using the definition of bounded Lipschitz distance for the DGM and leveraging the Lipschitz and boundedness properties of the functions f, h, and g, we obtain the following estimate:

$$
\begin{aligned}
\mathbb{E}|X_{u_1}(t)-X_{u_2}(t)|^2 \le\ & 4\mathbb{E}|X_{u_1}(0)-X_{u_2}(0)|^2 \\
&+ 4\left(L_f^2 + 2L_g^2\|\eta\|^2 + 2L_h^2\|\hat{\eta}\|^2\right)\mathbb{E}\int_0^t |X_{u_1}(s)-X_{u_2}(s)|^2\ \mathrm{d}s \\
&+ 8T\left[B_g^2 d_{BL}^2(\eta^{u_1},\eta^{u_2}) + B_h^2 d_{BL}^2(\hat{\eta}^{u_1},\hat{\eta}^{u_2})\right].
\end{aligned}
$$

Taking the supremum in s of the difference $|X_{u_1}(s) - X_{u_2}(s)|$, and using the BDG inequality, we get:

$$
\begin{aligned}
\mathbb{E}\|X_{u_1}-X_{u_2}\|_{*,t}^2 \le\ & 16\mathbb{E}|X_{u_1}(0)-X_{u_2}(0)|^2 \\
&+ 16\left(L_f^2 + 2L_g^2\|\eta\|^2 + 2L_h^2\|\hat{\eta}\|^2\right)\mathbb{E}\int_0^t \|X_{u_1}-X_{u_2}\|_{*,s}^2\ \mathrm{d}s \\
&+ 32T\left[B_g^2 d_{BL}^2(\eta^{u_1},\eta^{u_2}) + B_h^2 d_{BL}^2(\hat{\eta}^{u_1},\hat{\eta}^{u_2})\right].
\end{aligned}
$$

Using the Grönwall's inequality yields

$$
\begin{aligned}
\mathbb{E}\|X_{u_1}-X_{u_2}\|_{*,t}^2 \le C\Big(&\mathbb{E}|X_{u_1}(0)-X_{u_2}(0)|^2 \\
&+2T\left[B_g^2 d_{BL}^2(\eta^{u_1},\eta^{u_2}) + B_h^2 d_{BL}^2(\hat{\eta}^{u_1},\hat{\eta}^{u_2})\right]\Big),
\end{aligned}
$$

where $C = 16e^{16\left(L_f^2+2L_g^2|\eta|^2+2L_h^2|\hat{\eta}|^2\right)T}$. By the definition of the Wasserstein distance, we have:

$$
\begin{aligned}
[W_{2,T}(\mu_{u_1},\mu_{u_2})]^2 \le C\Big(&\mathbb{E}|X_{u_1}(0)-X_{u_2}(0)|^2 \\
&+2T\left[B_g^2 d_{BL}^2(\eta^{u_1},\eta^{u_2}) + B_h^2 d_{BL}^2(\hat{\eta}^{u_1},\hat{\eta}^{u_2})\right]\Big).
\end{aligned}
$$

As this inequality holds for any random variable $X_u(0)$, if we take the infimum, we get:

$$
\begin{aligned}
[W_{2,T}(\mu_{u_1},\mu_{u_2})]^2 \le C\Big(&\left[W_2(\bar{\mu}_{u_1}^0,\bar{\mu}_{u_2}^0)\right]^2 \\
&+2T\left[B_g^2 d_{BL}^2(\eta^{u_1},\eta^{u_2}) + B_h^2 d_{BL}^2(\hat{\eta}^{u_1},\hat{\eta}^{u_2})\right]\Big).
\end{aligned}
$$

Thus, we obtain the desired inequality and conclude the proof. □

After establishing the existence of solution for (5.6) and studying the properties of it, let us proceed to determine how close the solutions of (5.1) are to those of (5.6). To accomplish this, as we did in the previous section, we will compare the trajectories of solutions between equation (5.1) and equation (5.2), associating the particles X_i with the particles $X_u(i)$. To facilitate this comparison, we will consider a partition of the interval I defined by I_i^N, as introduced earlier. Within each interval, we will select a representative $u(i) = \frac{i}{N}$. Consequently, we will compare the trajectories of X_i^N and $X_{\frac{i}{N}}$. It is important to point out that each one is described by a distinct stochastic process. However, to identify both particles as similar, we make the assumption that for our choice of the random variable $u = \frac{i}{N}$, both particles exhibit the same Brownian motions, i.e., $B_t^i = B_t^{\frac{i}{N}}$.

Proposition 5.6.3. *Under the hypotheses of Theorem 5.3.2, the following inequality*

$$\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\|X_i^N(t) - X_{\frac{i}{N}}\|_{*,t}^2 \leq C\frac{16}{N}\sum_{i=1}^{N}\mathbb{E}|X_i^0 - X_{\frac{i}{N}}^0|^2$$
$$+ C\left(\frac{2}{N}(B_g^2 + B_h^2)\mathcal{O}(1) + B_g^2\left[d_\infty\left(\eta_{A^N},\eta\right)\right]^2 + B_h^2\left[d_\infty\left(\eta_{\hat{A}^N},\hat{\eta}\right)\right]^2\right),$$

holds, where $C = 48Te^{(16L_f^2+96(L_g^2+L_h^2)\mathcal{O}(1))T}$.

Proof. Since the proof of this result is identical to the one carried out in the previous section, we will not go into detail about the proof. □

We can now proceed to prove Theorem 5.3.2.

Proof. We want to prove that the empirical measure μ^N converges in probability to the measure $\bar{\mu}$. To do this, it is sufficient to establish that:

$$\lim_{N\to+\infty}\mathbb{E}\left|\int_{\mathbb{X}} k(y)\ \mu_N(\mathrm{d}y) - \int_{\mathbb{X}} k(y)\ \bar{\mu}(\mathrm{d}y)\right|^2 = 0,$$

for every bounded and Lipschitz function k. To show that this tends to zero, let us add and subtract some term:

$$\mathbb{E}\Big|\int_{\mathbb{X}} k(y)\ \mu^N(\mathrm{d}y) - \int_{\mathbb{X}} k(y)\ \bar{\mu}(\mathrm{d}y)\Big|^2 \tag{5.27}$$
$$\leq \mathbb{E}\left|\int_{\mathbb{X}} k(y)\ \mu^N(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{X}} k(y)\delta_{X_{\frac{i}{N}}}(y)\right|^2$$
$$+ \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\left(\int_{\mathbb{X}} k(y)\ \delta_{X_{\frac{i}{N}}}(\mathrm{d}y) - \int_{\mathbb{X}} k(y)\ \mu_{\frac{i}{N}}(\mathrm{d}y)\right)\right|^2$$
$$+ \mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{X}} k(y)\ \mu_{\frac{i}{N}}(\mathrm{d}y) - \int_{\mathbb{X}} k(y)\ \bar{\mu}(\mathrm{d}y)\right|^2. \tag{5.28}$$

Let us see that each term converges to zero. If we expand the first term, we get:

$$\mathbb{E}\left|\int_{\mathbb{X}} k(y)\ \mu^N(\mathrm{d}y) - \frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{X}} k(y)\ \delta_{X_{\frac{i}{N}}}(y)\right|^2 \leq \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}|k(X_i) - k(X_{\frac{i}{N}})|^2.$$

Using k is Lipschitz and by Proposition 5.6.3, we deduce that it tends to zero. Let us proceed with the second term of (5.28).

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\left(\int_{\mathbb{X}} k(y)\ \delta_{X_{\frac{i}{N}}}(\mathrm{d}y) - \int_{\mathbb{X}} k(y)\ \mu_{\frac{i}{N}}(\mathrm{d}y)\right)\right|^2 \leq \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left|k(X_{\frac{i}{N}}) - \mathbb{E}|k(X_{\frac{i}{N}})|\right|^2.$$

Using that k is bounded, we prove that this term converges to zero. Finally, for the last term of (5.28), we can write it as:

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{X}} k(y)\ \mu_{\frac{i}{N}}(\mathrm{d}y) - \int_I\int_{\mathbb{X}} k(y)\ \mu_u(\mathrm{d}y)du\right|^2.$$

Using Proposition 5.6.2, which establishes the continuity of $\hat{\mu}_u$ with respect to u, allows us to demonstrate that this term tends to zero. Consequently, $\mu^N \to \bar{\mu}$ in probability. □

5.7 Examples

As demonstrated earlier, under certain conditions, the empirical measure of the stochastic process (5.1) converges to the measure $\bar{\mu}(t,x)$ that satisfies the differential equation:

$$\begin{aligned}\partial_t\bar{\mu}_t + \partial_x\left(\bar{\mu}_t f(x) + \int_I \mu_{u,t}\int_I\int_{\mathbb{X}} g(x,y)\ \mu_{v,t}(\mathrm{d}y)\eta^u(\mathrm{d}v)\mathrm{d}u\right) = \\ + \frac{1}{2}\partial_x^2\left(\int_I \mu_{u,t}\left[\int_I\int_{\mathbb{X}} h(x,y)\ \mu_{v,t}(\mathrm{d}y)\hat{\eta}^u(\mathrm{d}v)\right]^2 \mathrm{d}u\right).\end{aligned} \tag{5.29}$$

Now, let us very briefly explore different examples of stochastic processes and how this theorem would apply to these processes, along with the corresponding limiting probability equation.

Consider first the case where the Brownian term is only multiplied by a constant, and the graph connects all nodes with each other, i.e., the adjacency matrix $A_{ij} = 1$ for $i = 1, .., N$. We obtain the following stochastic process:

$$\mathrm{d}X_i(t) = f(X_i(t))\mathrm{d}t + \frac{1}{N}\sum_{j=1}^{N} g(X_i, X_j)\mathrm{d}t + \sigma\mathrm{d}B_t^i. \tag{5.30}$$

In this case, the limit differential equation is:

$$\partial_t \bar{\mu}_t + \partial_x \left(\bar{\mu}_t f(x) + \bar{\mu}_t \int_{\mathbb{X}} g(x,y) \ \mu_t(\mathrm{d}y) \right) = \frac{\sigma^2}{2} \partial_x^2 \bar{\mu}_t. \tag{5.31}$$

Now, assume that the limiting DGMs η and $\hat{\eta}$ have sufficiently good properties, and η, $\hat{\eta} \in \mathcal{B}(I, \mathcal{M}_{+,abs}(I)) \bigcap \mathcal{C}(I, \mathcal{M}_{+,abs}(I))$, where $\mathcal{M}_{+,abs}(I)$ is the set of all finite Borel positive measures on I absolutely continuous with respect to the Lebesgue measure m. In this case, denoting W and $\hat{W}$ as the respective Radon-Nikodym derivatives of the previous DGMs, the system can be seen as a representation through graphons. In this space, the adjacency matrices are represented through graphons, and the limiting equation has the form

$$\begin{aligned} &\partial_t \bar{\mu}_t + \partial_x \left(\bar{\mu}_t f(x) + \int_I \mu_{u,t} \int_I \int_{\mathbb{X}} W(u,v) g(x,y) \mu_{v,t}(\mathrm{d}y) \mathrm{d}v \mathrm{d}u \right) = \\ &\quad + \frac{1}{2} \partial_x^2 \left(\int_I \mu_{u,t} \left[\int_I \int_{\mathbb{X}} \hat{W}(u,v) h(x,y) \mu_{v,t}(\mathrm{d}y) \mathrm{d}v \right]^2 du \right). \end{aligned} \tag{5.32}$$

This differential equation is consistent with the results obtained in other works [12, 22] for graphons. Yet, our result is a clear generalization as there are many DGMs that have no graphon representation as we do not assume any absolute continuity of the graph limit. For example, we can apply our result to certain sparse graphs, such as certain ring networks, where the limiting DGM would be $2\delta_u(v)$. The resulting differential equation is then:

$$\begin{aligned} &\partial_t \bar{\mu}_t + 2\partial_x \left(\bar{\mu}_t f(x) + \int_I \mu_{u,t} \int_{\mathbb{X}} g(x,y) \ \mu_{u,t}(\mathrm{d}y) \mathrm{d}u \right) = \\ &\quad + \partial_x^2 \left(\int_I \mu_{u,t} \left[\int_{\mathbb{X}} h(x,y) \ \mu_{u,t}(\mathrm{d}y) \right]^2 \mathrm{d}u \right). \end{aligned} \tag{5.33}$$

In particular, all the DGM examples discussed for the deterministic particle system case in [95] can be carried over now to the stochastic setting, which provides also a clear indication that our approach to mean-field limits via DGMs is quite robust.

Part III

Conclusion and perspective

Chapter 6

Conclusion and perspective

In this chapter, we present some ongoing projects and future works that have emerged throughout the development of this thesis. We will focus on the main projects that currently hold our interest.

The thesis is centered around two main areas of study: pattern formation in biological models, which arises from biomechanical interactions (Chapters 2 and 3) or biochemical interactions (Chapter 4), and the mean-field limit for stochastic interacting particles on digraph measures (Chapter 5).

Pattern Formation in Biological Models

- In Chapter 2, we addressed the model introduced by J-F Joanny et al. [113], which is characterized by establishing a novel relationship between pressure and velocity:

  $$\begin{aligned} &\partial_t \phi + V \partial_x \phi = \partial_x^2 \phi + \phi(1-\phi), \\ &\Lambda^2 \partial_x^2 V - V = 2\Lambda V_0 \partial_x \left(\phi + \beta(\partial_x \phi)^2\right), \end{aligned} \tag{6.1}$$

  We focused on studying the existence of traveling waves in this model, demonstrating that for small Λ, the range of wave speeds for which traveling waves exist has a well-defined structure. This result stems from the combination of Fenichel's techniques with those of dynamical systems.

  On the other hand, the existence of traveling waves with this type of velocity-pressure relation, when $\Lambda > 0$ and $\beta = 0$, has been studied by various authors [84, 106]. In our case, we extended this result to $\beta > 0$, showing that for given Λ, β, and V_0, satisfying $\frac{V_0\beta}{\Lambda} < 1$, there exists at least one σ for which traveling waves exist.

  The remaining questions in this case concern understanding the structure of the set of σ for which traveling waves exist. Specifically, whether this set is open or closed, connected or disconnected, or even discrete. Furthermore, it is worth investigating whether the estimate $\frac{V_0\beta}{\Lambda} < 1$ can be improved, as it depends on the estimation techniques and the methodology employed to prove the existence of solutions.

- In Chapter 3, as mentioned in the Introduction, we focus on studying a modification of the model by Joanny et al. and proving the existence of solutions for this model. This section is part of an ongoing collaboration with R. Granero, whose second part aims to demonstrate pattern formation in the model, particularly investigating whether traveling waves can be found that satisfy the equation:

$$
\begin{aligned}
&-(-\partial_{xx})^{\alpha}\phi + \sigma\phi' + u'\phi' + \phi(1-\phi) = 0 \\
&-\Lambda^2 u'' + u = a\phi + \frac{b}{2}\left(\phi'\right)^2, \\
&\phi(-\infty) = 1,\ \phi(\infty) = 0, \quad x \in \mathbb{R},
\end{aligned}
\tag{6.2}
$$

  Several authors have studied the existence of traveling waves for models of the form:

$$
\partial_t \phi = -(-\Delta)^{\alpha}\phi + f(\phi),
$$

  where $f(\phi)$ can represent a Fisher term (Cabré and Roquejoffre [34, 33]), a bistable term [59, 83], or a combustion term (Roquejoffre et al. [103]). Our objective is to prove the existence of solutions to (6.2), as, unlike the models studied, our approach couples the velocity equation as a consequence of considering pressure. In the case of a solution's existence, we aim to determine how biological parameters influence potential constraints on σ and α.

- Apart from the pressure model presented by Joanny [113], there is another model of interest that also studies biomechanical interactions by examining the internal pressure that a population exerts on itself as it grows, which acts as a possible regulatory mechanism in tissue growth. This model was introduced by Shraiman [116] and is given by the following system:

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= \phi\beta\left(1 - \frac{\phi A}{K}\right) - \alpha\phi^2 + \frac{\mu}{\kappa+\mu}\phi(\alpha\phi - \langle\alpha\phi\rangle), \\
\frac{\partial A}{\partial t} &= A\left[\alpha\phi - \frac{\mu}{\kappa+\mu}(\alpha\phi - \langle\alpha\phi\rangle)\right],
\end{aligned}
\tag{6.3}
$$

  where A represents the area of the cell tissue, and $\langle\phi\rangle(t,x) = \int_{-\epsilon}^{\epsilon}\phi(t,x-y)k(y)dy$, with k being a regularizing function with compact support on $(-\epsilon,\epsilon)$. Here, ϵ is the zone of influence of each cell with its surroundings. The term $(\phi - \langle\phi\rangle)$ reflects the regulation of the cell based on its interaction with its environment, and this term arises from the treatment of the internal pressure.

  In Shraiman's model, cell migration is not considered. This aspect was added and analyzed numerically by B. Blanco [24], leading to the following model:

$$
\begin{aligned}
\frac{\partial \phi}{\partial t} &= \nu\nabla\cdot(J(\phi)) + \phi G(\phi) \\
G(\phi) &= \beta\left(1 - \frac{\phi A}{K}\right) - \alpha\phi + \frac{\mu}{\kappa+\mu}(\alpha\phi - \langle\alpha\phi\rangle) \\
\frac{\partial A}{\partial t} &= A\left[\alpha\phi - \frac{\mu}{\kappa+\mu}(\alpha\phi - \langle\alpha\phi\rangle)\right]
\end{aligned}
\tag{6.4}
$$

where J represents the flux. In her paper, Blanco analyzed the flux given by:

$$J(\phi) = \left(\frac{\phi}{\sqrt{\phi^2 + \left(\frac{\nu}{c}\right)^2 |\nabla \phi|^2}} \nabla(\phi^m) \right)$$

Our goal is to study what happens, from a mathematical perspective, with this model for different fluxes. Specifically, we aim to determine the existence of solutions and whether some form of traveling waves can be found. Finally, regarding the study of pattern formation in biological models, in Chapter 4, we addressed various Keller-Segel models with saturated fluxes, which examined interactions driven by biochemical agents. We demonstrated the existence of soliton-type solutions, which commonly appear in scientific experiments when modeling these aspects. The next objective is to study what happens when we combine pressure models, such as (6.4) and (6.1), with biochemical interactions, which arise when cells interact with their surrounding environment, thereby leading to multiscale models. One possible model of this kind could be the following:

$$\begin{aligned} \partial_t \phi + V \partial_x \phi &= \partial_x^2 \phi + \phi(1-\phi) + F(\phi, C), \\ \Lambda^2 \partial_x^2 V - V &= 2\Lambda V_0 \partial_x \left(\phi + \beta(\partial_x \phi)^2 \right), \\ \partial_t C &= G(\phi, C), \end{aligned} \tag{6.5}$$

where C is the concentration of the chemical agent.

Mean-field limit for stochastics interacting particles Lastly, in Chapter 5, we specifically investigate particle systems modeled by stochastic differential equations (SDEs), where the mean-field limit converges to a Vlasov-Fokker-Planck-type equation. Starting from conventional approaches in stochastic analysis, we explore the network connectivity between particles using digraph measures (DGMs). DGMs offer a tool to capture sparse, intermediate, and dense network/graph interactions in the mean-field, thereby extending beyond more classical approaches such as graphons or extended graphons [89].

A possible line of research is to explore what would happen in the case where the interactions are not Lipschitzian but instead present some type of singularity. For example, consider the recent study conducted by Jabin, Bresch and Duerinckx [28], in which they introduce a new approach to justify mean-field limits for first- and second-order particle systems with singular interactions.

Part IV

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