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Superreflexive tensor product spaces

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Abstract

The aim of this note is to prove that, given two superreflexive Banach spaces X and Y, then $X\widehat{\otimes}_\pi Y$ is superreflexive if and only if either X or Y is finite-dimensional. In a similar way, we prove that $X\widehat{\otimes}_\varepsilon Y$ is superreflexive if and only if either X or Y is finite-dimensional.

Keywords Superreflexive · Injective tensor product · Projective tensor product · Ultraproducts

Mathematics Subject Classification 46B08 · 46B28 · 46M07

1 Introduction

The study of topological and geometrical properties of Banach spaces has attracted the attention of many researches in functional analysis, the papers [2–4, 8, 10] and the references therein are a good sample of this.

One classical result in tensor product spaces is the well known characterization of the reflexivity in tensor product spaces [9, Theorem 4.21]. Namely, let two reflexive Banach spaces X and Y, one of which has the approximation property. The following are equivalent:

1. $X\widehat{\otimes}_\pi Y$ is reflexive.
2. $(X\widehat{\otimes}_\pi Y)^* = X^*\widehat{\otimes}_\varepsilon Y^*$.
3. Every operator from X to Y^* is compact.
4. $X^*\widehat{\otimes}_\varepsilon Y^*$ is reflexive.

The above characterization yields examples of non-reflexive tensor product spaces even if both factors are reflexive, e.g., $\ell_2\widehat{\otimes}_\pi \ell_2$, which even contains ℓ_1 isometrically

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[9, Example 2.10]. On the other hand, examples of infinite-dimensional Banach spaces X and Y such that $X\widehat{\otimes}_\pi Y$ is reflexive can be given. For instance, given $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} < 1$, then $\ell_p\widehat{\otimes}_\pi \ell_q$ is reflexive in virtue of Pitt theorem [9, Theorem 4.23].

In this note we wonder when a projective tensor product can be superreflexive. As being a condition implying the reflexivity, the above mentioned result [9, Theorem 4.21] will imply that superreflexivity must be an uncommon phenomenon. Going further, a natural question at this point is whether there exists a "non-trivial" superreflexive tensor product space, that is, if there are infinite-dimensional Banach spaces X and Y such that $X\widehat{\otimes}_\pi Y$ is superreflexive. Observe that the answer is no if either X or Y is ℓ_p for $1 < p < \infty$ [3, p. 522].

The main aim of this paper is proving that the answer is no. Indeed, the main theorem of the paper is the following.

Theorem 1.1 *Let X and Y be two superreflexive Banach spaces. The following are equivalent:*

1. *$X\widehat{\otimes}_\pi Y$ is superreflexive.*
2. *Either X or Y is finite dimensional.*

The above result says that the unique possibility for a projective tensor product to be superreflexive is that one of the factors is finite-dimensional, which establishes a big different with the case of reflexivity.

Our methods, which will be focused on studying ultraproducts of injective tensor product, will allow us to derive a similar version for the case of the injective tensor product.

Theorem 1.2 *Let X and Y be two superreflexive Banach spaces. The following are equivalent:*

1. *$X\widehat{\otimes}_\varepsilon Y$ is superreflexive.*
2. *Either X or Y is finite dimensional.*

After a notation section, which is necessary to introduce all the notation about tensor product spaces and ultraproducts, Sect. 3 will be devoted to prove the above mentioned theorems.

2 Notation

For simplicity we will consider real Banach spaces. We denote by B_X and S_X the closed unit ball and the unit sphere, respectively, of the Banach space X. We denote by $L(X, Y)$ the space of all bounded linear operators from X into Y. If $Y = \mathbb{R}$, then $L(X, \mathbb{R})$ is denoted by X^*, the topological dual space of X. A bounded and symmetric subset $A \subseteq B_{X^*}$ is said to be *1-norming* if $\|x\| = \sup_{f\in A} f(x)$ holds for every $x \in X$.

2.1 Ultrapowers

Given a sequence of Banach spaces $\{X_n : n \in \mathbb{N}\}$ we denote

$$\ell_\infty(\mathbb{N}, X_n) := \left\{ f : \mathbb{N} \longrightarrow \prod_{n\in\mathbb{N}} X_n : f(n) \in X_n\ \forall n \text{ and } \sup_{n\in\mathbb{N}} \|f(n)\| < \infty \right\}.$$

Given a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, consider $c_{0,\mathcal{U}}(\mathbb{N}, X_n) := \{f \in \ell_\infty(\mathbb{N}, X_n) : \lim_{\mathcal{U}} \|f(n)\| = 0\}$. The *ultrapower of* $\{X_n : n \in \mathbb{N}\}$ *with respect to* $\mathcal{U}$ is the Banach space

$$(X_n)_{\mathcal{U}} := \ell_\infty(\mathbb{N}, X_n)/c_{0,\mathcal{U}}(\mathbb{N}, X_n).$$

We will naturally identify a bounded function $f : \mathbb{N} \longrightarrow \prod_{n\in\mathbb{N}} X_n$ with the element $(f(n))_{n\in\mathbb{N}}$. In this way, we denote by $(x_n)_{\mathcal{U}}$ or simply by (x_n), if no confusion is possible, the coset in $(X_n)_{\mathcal{U}}$ given by $(x_n)_{n\in\mathbb{N}} + c_{0,\mathcal{U}}(\mathbb{N}, (X_n))$.

From the definition of the quotient norm, it is not difficult to prove that $\|(x_n)\| = \lim_{\mathcal{U}} \|x_n\|$ holds for every $(x_n) \in (X_n)_{\mathcal{U}}$. If all the spaces X_n are equal to X we will simply write $X_{\mathcal{U}}$.

Observe that, in general, $(X_{\mathcal{U}})^* = (X^*)_{\mathcal{U}}$ if and only if X is superreflexive [7, Theorem 6.4] (see the next subsection for formal definition). However, it follows that $(X^*)_{\mathcal{U}}$ is isometrically a subspace of $(X_{\mathcal{U}})^*$ by the action

$$(x_n^*)(x_n) := \lim_{\mathcal{U}} x_n^*(x_n) \quad (x_n) \in X_{\mathcal{U}}, (x_n^*) \in (X^*)_{\mathcal{U}}.$$

It follows that $S_{(X^*)_{\mathcal{U}}}$ is 1-norming for $X_{\mathcal{U}}$. Indeed, we have the following result, which is more general, whose proof is included for our convenience and for the sake of completeness.

Proposition 2.1 *Let X be a Banach space and $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Let $A \subseteq B_{X^*}$ which is 1-norming for X. Then the set*

$$A_{\mathcal{U}} := \{(f_n) : f_n \in A\ \forall n \in \mathbb{N}\} \subseteq (X^*)_{\mathcal{U}}$$

is a 1-norming set for $X_{\mathcal{U}}$.

Proof Let $(x_n) \in X_{\mathcal{U}}$. Select, for every $n \in \mathbb{N}$, an element $f_n \in A$ such that $f_n(x_n) > \|x_n\| - \frac{1}{n}$. Now $(f_n) \in A_{\mathcal{U}}$, and it is clear that $(f_n)(x_n) = \lim_{\mathcal{U}} f_n(x_n) = \lim_{\mathcal{U}} \|x_n\| = \|(x_n)\|$, as desired. □

2.2 Superreflexive Banach spaces

Given two Banach spaces X and Y, we say that Y is *finitely representable* in X if, for every finite dimensional subspace E of Y and every $\varepsilon > 0$, there exists a finite dimensional subspace F of X and an onto linear mapping $T : E \longrightarrow F$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.

Recall that X is said to be *superreflexive* if every Banach space Y which is finitely representable in X must be reflexive. We refer the reader to [5, Chapter 9] for background.

It is known that a Banach space X is superreflexive if and only if $X_{\mathcal{U}}$ is reflexive for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$ (see the comment after Theorem 1.3.2 in [6]).

Observe also that a Banach space X is superreflexive if and only if X admits an equivalent renorming which is simultaneously uniformly convex and uniformly smooth [5, Theorem 9.18]. Even though we will not enter in the formal definition of uniformly convex and uniformly smooth Banach spaces, observe that a Banach space X is uniformly convex (respectively uniformly smooth) if and only if X^* is uniformly smooth (respectively uniformly convex) [5, Theorem 9.10].

This result allows us to obtain the following consequence from the above mentioned [5, Theorem 9.18]: a Banach space X is superreflexive if and only if X^* is superreflexive.

2.3 Tensor product spaces

The projective tensor product of X and Y, denoted by $X\widehat{\otimes}_\pi Y$, is the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_\pi := \inf\left\{\sum_{n=1}^k \|x_n\|\|y_n\| : z = \sum_{n=1}^k x_n \otimes y_n\right\},$$

where the infimum is taken over all such representations of z. The reason for taking completion is that $X \otimes Y$ endowed with the projective norm is complete if and only if either X or Y is finite dimensional (see [9, P.43, Exercises 2.4 and 2.5]).

It is well known that $\|x \otimes y\|_\pi = \|x\|\|y\|$ for every $x \in X$, $y \in Y$, and that the closed unit ball of $X\widehat{\otimes}_\pi Y$ is the closed convex hull of the set $B_X \otimes B_Y = \{x \otimes y : x \in B_X, y \in B_Y\}$.

Observe that the action of an operator $G\colon X \longrightarrow Y^*$ as a linear functional on $X\widehat{\otimes}_\pi Y$ is given by

$$G\left(\sum_{n=1}^k x_n \otimes y_n\right) = \sum_{n=1}^k G(x_n)(y_n),$$

for every $\sum_{n=1}^k x_n \otimes y_n \in X \otimes Y$. This action establishes a linear isometry from $L(X, Y^*)$ onto $(X\widehat{\otimes}_\pi Y)^*$ (see e.g., [9, Theorem 2.9]). All along this paper we will use the isometric identification $(X\widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ without any explicit mention.

Recall that given two Banach spaces X and Y, the *injective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\varepsilon Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\|_\varepsilon := \sup\left\{\sum_{i=1}^n |x^*(x_i)y^*(y_i)| : x^* \in S_{X^*}, y^* \in S_{Y^*}\right\},$$

where $u = \sum_{i=1}^{n} x_i \otimes y_i$ (see [9, Chapter 3] for background). Observe that, from the very definition, the set $S_{X^*} \otimes S_{Y^*} := \{x^* \otimes y^* : x^* \in S_{X^*}, y^* \in S_{Y^*}\} \subseteq B_{(X\widehat{\otimes}_\varepsilon Y)^*}$ is a 1-norming subset for $X\widehat{\otimes}_\varepsilon Y$.

3 Proof of the results

Observe that, given two Banach spaces X and Y, then X and Y can be seen as subspaces of both $X\widehat{\otimes}_\pi Y$ and $X\widehat{\otimes}_\varepsilon Y$. Consequently, to $X\widehat{\otimes}_\pi Y$ or $X\widehat{\otimes}_\varepsilon Y$ be superreflexive then both X and Y must be superreflexive.

Before exhibiting the proof of Theorems 1.1 and 1.2 we will need the following lemma.

Lemma 3.1 *Let X and Y be two superreflexive Banach spaces and let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Then the mapping $\phi : X_{\mathcal{U}}\widehat{\otimes}_\varepsilon Y_{\mathcal{U}} \longrightarrow (X\widehat{\otimes}_\varepsilon Y)_{\mathcal{U}}$ defined by*

$$\phi\left(\sum_{i=1}^{p}(x_n^i) \otimes (y_n^i)\right) := \left(\sum_{i=1}^{p} x_n^i \otimes y_n^i\right)$$

defines a linear isometry.

Proof The linearity is immediate, so let us prove that it is an isometry. Let $z \in X_{\mathcal{U}}\widehat{\otimes}_\varepsilon Y_{\mathcal{U}}$ with $z \neq 0$. We can assume up to a density argument that $z = \sum_{i=1}^{p}(x_n^i) \otimes (y_n^i)$ for certain $(x_n^i) \in X_{\mathcal{U}}$ and $(y_n^i) \in Y_{\mathcal{U}}$. Let $\varepsilon > 0$. By the definition of the injective norm we can find $f \in S_{(X_{\mathcal{U}})^*}$ and $g \in S_{(Y_{\mathcal{U}})^*}$ such that

$$\|z\| - \varepsilon < (f \otimes g)(z).$$

Since X and Y are superreflexive then $(X_{\mathcal{U}})^* = (X^*)_{\mathcal{U}}$ and, similarly, $(Y_{\mathcal{U}})^* = (Y^*)_{\mathcal{U}}$. Consequently, $f = (f_n) \in S_{(X^*)_{\mathcal{U}}}$ and $g := (g_n) \in S_{(Y^*)_{\mathcal{U}}}$. Now we have

$$\begin{aligned}
\|z\| - \varepsilon < (f_n) \otimes (g_n)(z) &= \sum_{i=1}^{p}(f_n)(x_n^i)(g_n)(y_n^i) \\
&= \sum_{i=1}^{p} \lim_{n,\mathcal{U}} f_n(x_n^i) \lim_{n,\mathcal{U}} g_n(y_n^i) \\
&= \sum_{i=1}^{p} \lim_{n,\mathcal{U}} f_n(x_n^i) g_n(y_n^i) \\
&= \lim_{n,\mathcal{U}} \sum_{i=1}^{n} f_n(x_n^i) g_n(y_n^i)
\end{aligned}$$

Now observe that we can consider $(f_n\otimes g_n)\in S_{(X\widehat{\otimes}_\varepsilon Y)_{\mathcal{U}}^*}$. Evaluating the above element at $\phi(z)$ we get

$$\begin{aligned}(f_n\otimes g_n)(\phi(z)) &= (f_n\otimes g_n)\left(\sum_{i=1}^p x_n^i\otimes y_n^i\right)\\ &=\lim_{n,\mathcal{U}}(f_n\otimes g_n)\left(\sum_{i=1}^p x_n^i\otimes y_n^i\right)\\ &=\lim_{n,\mathcal{U}}\sum_{i=1}^p f_n(x_n^i)g_n(y_n^i)\end{aligned}$$

The above proves that

$$\begin{aligned}\|z\|-\varepsilon<(f_n)\otimes(g_n)(z) &= (f_n\otimes g_n)(\phi(z))\\ &\leq \|(f_n\otimes g_n)\|_{((X\widehat{\otimes}_\varepsilon Y)_{\mathcal{U}})^*}\|\phi(z)\|_{(X\widehat{\otimes}_\varepsilon Y)_{\mathcal{U}}}.\end{aligned}$$

Since $\|(f_n\otimes g_n)\|_{((X\widehat{\otimes}_\varepsilon Y)_{\mathcal{U}})^*}=\lim_{n,\mathcal{U}}\|f_n\otimes g_n\|_{(X\widehat{\otimes}_\varepsilon Y)^*}=1$ we infer $\|z\|-\varepsilon\leq \|\phi(z)\|$. The arbitrariness of $\varepsilon>0$ implies $\|z\|\leq\|\phi(z)\|$.

To prove that $\|\phi(z)\|\leq\|z\|$ let $\varepsilon>0$. Since $S_{X^*}\otimes S_{Y^*}$ is 1-norming for $X\widehat{\otimes}_\varepsilon Y$ we infer that $(S_{X^*}\otimes S_{Y^*})_{\mathcal{U}}$ is 1-norming for $(X\widehat{\otimes}_\varepsilon Y)_{\mathcal{U}}$ in virtue of Proposition 2.1. Consequently, we can find two sequences $(h_n)\subseteq S_{X^*}$ and $(j_n)\subseteq S_{Y^*}$ such that

$$\|\phi(z)\|-\varepsilon<(h_n\otimes j_n)(\phi(z)).$$

Recalling the definition of $\phi(z)$ we get

$$\begin{aligned}(h_n\otimes j_n)(\phi(z)) &= (h_n\otimes j_n)\left(\sum_{i=1}^p x_n^i\otimes y_n^i\right)\\ &=\lim_{\mathcal{U}}(h_n\otimes j_n)\left(\sum_{i=1}^p x_n^i\otimes y_n^i\right)\\ &=\lim_{\mathcal{U}}\sum_{i=1}^p h_n(x_n^i)j_n(y_n^i)\\ &=\sum_{i=1}^p\lim_{\mathcal{U}} h_n(x_n^i)j_n(y_n^i)\end{aligned}$$

in virtue of the linearity of the limit through $\mathcal{U}$.

On the other hand, if we see $(h_n) \otimes (j_n) \in (X_{\mathcal{U}} \widehat{\otimes}_\varepsilon Y_{\mathcal{U}})^*$ we get

$$\begin{aligned}(h_n) \otimes (j_n)(z) &= (h_n) \otimes (j_n) \left(\sum_{i=1}^{p} (x_n^i) \otimes (y_n^i) \right) \\ &= \sum_{i=1}^{p} (h_n)(x_n^i)(j_n)(y_n^i) \\ &= \sum_{i=1}^{p} \lim_{\mathcal{U}} h_n(x_n^i) j_n(y_n^i)\end{aligned}$$

With all the above we get

$$\|\phi(z)\| - \varepsilon < (h_n \otimes j_n)(\phi(z)) = (h_n) \otimes (j_n)(z) \leq \|(h_n) \otimes (j_n)\|_{(X_{\mathcal{U}} \widehat{\otimes}_\varepsilon Y_{\mathcal{U}})^*} \|z\|.$$

Now observe that since $h_n \in S_{X^*}$ it follows that $\|(h_n)\|_{X^*_{\mathcal{U}}} = \lim_{\mathcal{U}} \|h_n\| = 1$. Analogously we get $\|(j_n)\|_{Y^*_{\mathcal{U}}} = 1$. Hence $\|(h_n) \otimes (j_n)\|_{(X_{\mathcal{U}} \widehat{\otimes}_\varepsilon Y_{\mathcal{U}})^*} = 1$ and we get

$$\|\phi(z)\| - \varepsilon \leq \|z\|.$$

The arbitrariness of $\varepsilon > 0$ implies $\|\phi(z)\| = \|z\|$ and the lemma is finished. □

Now we can provide the proof of Theorem 1.1.

Proof of Theorem 1.1 (2)⇒(1). If the dimension of X is N, then X is isomorphic to ℓ_1^N. Consequently, $X \widehat{\otimes}_\pi Y$ is isomorphic to $\ell_1^N \widehat{\otimes}_\pi Y = \ell_1^N(Y)$ [9, Example 2.6], which is superreflexive since Y is superreflexive.

(1)⇒(2). Assume that both X and Y are infinite dimensional. Take any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, and let us prove that $(X \widehat{\otimes}_\pi Y)_{\mathcal{U}}$ is not reflexive. This is equivalent to proving that its dual $Z := (X \widehat{\otimes}_\pi Y)^*_{\mathcal{U}}$ is not reflexive. Observe that $(L(X, Y^*))_{\mathcal{U}} = ((X \widehat{\otimes}_\pi Y)^*)_{\mathcal{U}}$ is isometrically a subspace of Z. Consequently, $(X^* \widehat{\otimes}_\varepsilon Y^*)_{\mathcal{U}}$ is an isometric subspace of Z. By Lemma 3.1 we infer that Z contains an isometric copy of $(X^*)_{\mathcal{U}} \widehat{\otimes}_\varepsilon (Y^*)_{\mathcal{U}}$.

Let us prove that Z contains an isometric copy of $\ell_2 \widehat{\otimes}_\varepsilon \ell_2$. Indeed, since X^* is infinite dimensional then ℓ_2 is finitely representable in X^* by Dvoretzky theorem (c.f. e.g., [1, Theorem 12.3.6]). By [1, Proposition 11.1.12] we get that ℓ_2 is an isometric subspace of $X^*_{\mathcal{U}}$. Similarly ℓ_2 is an isometric subspace of $Y^*_{\mathcal{U}}$. Consequently, $\ell_2 \widehat{\otimes}_\varepsilon \ell_2$ is isometrically a subspace of $X^*_{\mathcal{U}} \widehat{\otimes}_\varepsilon Y^*_{\mathcal{U}}$ since the injective tensor product respects subspaces. Consequently, $\ell_2 \widehat{\otimes}_\varepsilon \ell_2$ is isometrically a subspace of Z.

This implies that Z is not reflexive since $\ell_2 \widehat{\otimes}_\varepsilon \ell_2$ is not reflexive (c.f. e.g., [9, Theorem 4.21]). □

A similar proof to the above one yields also the proof of Theorem 1.2.

Theorem 1.2 (1)⇒(2) follows the same ideas than the corresponding implication in Theorem 1.1.

To prove that (2)$\Rightarrow$(1) assume that X is finite-dimensional. Then X is isomorphic to ℓ_∞^N. This implies that $X\widehat{\otimes}_\varepsilon Y$ is isomorphic to $\ell_\infty^N\widehat{\otimes}_\varepsilon Y = \ell_\infty^N(Y)$ [9, Section 3.2], from where the superreflexivity of $X\widehat{\otimes}_\varepsilon Y$ follows. □

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