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Traits of generalization of problem solution methods exhibited by potential mathematically gifted students when solving problems in a selection process

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Abstract. *Identifying mathematically gifted students is an important objective in mathematics education. To describe skills typical of these students, researchers pose problems in several mathematical domains whose solutions require using different mathematical capacities, such as visualization, generalization, proof, creativity, etc. This paper presents an analysis of the solutions to two problems by 75 students (aged 11 to 14), as part of the selection test for a workshop to stimulate mathematical talent. These problems require the use of the capacity for mathematical generalization of solution methods. We define a set of descriptors of such capacity, use them to analyze students' solutions, and evaluate how well students with high capacity for generalization can be distinguished from average students. The results indicate that the two problems are suitable for identifying potential mathematically gifted students and several descriptors have high discriminatory power to identify students with high or low capacity for generalization.*

Keywords *Mathematical generalization, Generalization of solution methods, Identification of mathematical giftedness, Potential mathematically gifted student, Problem solving, Selection process for workshops*

1 Introduction

Only a few countries have established formal programs to identify and serve mathematically gifted students (MG students from now on). Chamberlin and Chamberlin (2010) reported a significant lack of training and resources available for teachers to identify and support their MG pupils. This gap persists today, so further research is needed in this direction (Leikin, 2021). Leikin (2021) raised the need for research addressing the question "What types of mathematical tasks can best serve for the identification of mathematical giftedness (MG from now on) in earlier ages? In elementary school? In middle school?" (p. 1581). This paper contributes to answer it, since we present the results of a study focused on analyzing the kinds of mathematical generalization (just generalization from now on) made by students to solve two problems and the effectiveness of the problems to discriminate potential MG Students. The problems were part of an admission test for the Spanish mathematical enrichment program ESTALMAT (*EStímulo del TALento MATemático*), aimed at identifying, stimulating, and nurturing potential MG students. The program extends over two years and consists of about 20 3-hour-long workshops per year. The admission test consists of 5 paper and pencil non-routine problems, each with several parts of increasing complexity. The solutions (i.e., the processes of obtaining answers) to each problem are scored by applying agreed marking criteria. The 25 students with the highest total scores are admitted to the program.

Students are considered as mathematically gifted when they quickly grasp new mathematical content and demonstrate characteristics of mathematical reasoning, problem-solving strategies, and creativity that are more complex and elaborated than those of average students of their age or grade (Bicknell, 2008; Krutetskii, 1976; Leikin, 2018). Research results show that problem-solving is a very reliable way of identifying MG students. We present two original problems suitable for identifying students with above-average capacity for generalization. Correctly solving them does not ensure that students are mathematically gifted, but we can affirm that they have potential (Pitta-Pantazi & Leikin, 2018) to be MG students. The 25 students admitted to the ESTALMAT Program can be considered as potential MG students.

Generalization is a main mathematical activity, from young children to mathematicians. To define the hypothesis of his study and choose his research methodology, Krutetskii (1976) "assumed that pupils with different abilities who are capable of learning mathematics are characterized by differences in degree of development of both the ability to *generalize mathematical material* and the ability to *remember generalizations*." (p. 84, original italics). He also added that "Abstractions and generalizations constitute the essence of mathematics" (p. 86).

Researchers have paid attention to MG students' activity of generalization, which is typically assessed by asking students to produce arithmetic, algebraic, or functional generalizations. However, the capacity for generalization is also necessary to recognize mathematical procedures and other types of regularities, such as the generalization of computational strategies specific to certain contexts (Krutetskii, 1976), like the problems we have analyzed.

Mathematical competitions and after-school problem-solving workshops are good contexts for identifying and nurturing MG students (Bicknell, 2008; Leikin, 2021). To correctly solve the problems posed in competitions and workshop admission tests, students must exhibit some above-mentioned specific capacities and skills. This allows researchers to analyze possible differences in the behavior of students with varying levels of success in solving problems. The research we present is situated in this context, since it is based on problems posed to students who took the admission test of the ESTALMAT Program. Analyzing the students' answers to these problems allowed us to reliably identify potential MG students.

The general objective of the research presented in this paper is to identify distinctive characteristics of MG students related to the generalization of methods for solving mathematical problems. To achieve this, we have analyzed the solutions to two problems of the test. This objective is divided into several specific objectives:

1. Identify and categorize different types of generalizations of methods for solving problems and express them as operational descriptors of generalization.

2. Particularize those descriptors to specifically analyze the students' solutions to the problems posed in our experiment.

3. Identify and analyze the evidence of generalization of solution methods in the answers produced by students with different levels of success in solving the problems.

4. Analyze the effectiveness of the problems posed and the descriptors defined to discriminate potential MG students, based on their use of the capacity for generalization of solution methods.

2 Review of literature

Without trying to be exhaustive, we first present a synthetic review of the mathematics education literature on different issues about MG related to identification and attention (2.1). Then we focus on the literature about the topics relevant to our study: the role of problem-solving in MG (2.2), the characterization of traits of generalizations in MG students (2.3), and the identification of MG students in competitions and selection tests (2.4).

2.1 Characteristics of MG students

The identification of MG students is an ever-present research question. Identification is mainly based on mathematical problem-solving, so various kinds of problems are posed to identify specific characteristics of MG students' behavior. Several authors proposed different traits typical of MG students, like flexibility, identification of patterns and relationships, use of efficient problem-solving procedures, inversion of thinking processes, and capacities of generalization, abstraction, and transfer (Krutetskii, 1976; Greenes, 1981; Freiman, 2006). Other researchers have verified the validity of the most important of those characteristics, but the question of what differentiates MG students from average students remains open. Pitta-Pantazi and Christou (2009) argued that this is due to the lack of a commonly agreed definition of MG, of appropriate instruments to identify MG students, and the great diversity of traits of MG.

2.1.1 Mathematical creativity

The construct of creativity is somewhat ambiguous, as proved by the compilation of more than 100 definitions by Treffinger et al. (2002). In mathematics education, there is currently a wide consensus in considering mathematical creativity as integrated by three components: flexibility, originality, and fluency (Leikin & Lev, 2013). Leikin and Sriraman (2022) presented a retrospective analysis of empirical research on mathematical creativity, showing that it can be assessed in quite ways, the multiple-solution tasks (Leikin & Lev, 2013) being one of the most successful, since it allows to assess the three components of creativity. Schoevers et al. (2022) concluded that students with higher levels of creativity perform better in solving geometry problems in general, and particularly in multiple-solution problems.

2.1.2 Visualization

Krutetslii (1976) identified a high capacity of visualization in some MG students, but he did not consider it as a characteristic of MG. Since then, various studies have attempted to shed light on the relationship between visualization and MG. Presmeg (1986) observed that less than 20% of a sample of grade 12 MG students used geometric thinking, which led her to confirm Krutetskii's conclusion. However, more recent research, based on experiments with students in different educational levels, identified visualization skills used by the students and agreed in concluding that visualization is a capacity observed in MG students more than in their average peers (Applebaum, 2017; Mora et al., 2024; Ramírez, 2012).

2.2 Problem-solving

Problem-solving is at the core of MG students' activity, so many publications report on their problem-solving processes. Some focus on analyzing students' use of capacities such as generalization (Amit & Neria, 2008; Gutiérrez et al., 2018; Jablonski & Ludwig, 2022; Krutetskii, 1976), proof (Elgrably & Leikin, 2021; Jablonski & Ludwig, 2022), visualization (Diezmann & Watters, 2002; Lee et al., 2007), or analogy (Lee et al., 2007; Rott, 2013), in contexts like arithmetic (Rott, 2013; Schifter & Russell, 2022), geometry (Elgrably & Leikin, 2021; Lee et al., 2007; Rott, 2013), stochastics (Durak & Tutak, 2019; Jan & Amit, 2012), etc. Most of these authors agree that problem-solving helps to observe students' reasoning associated with some capacities, which are more developed in MG students than in average students.

2.3 Generalization

Generalization is a central tool in mathematics and a key cognitive process that students must develop (Amit & Neria, 2008). At schools, generalization emerges during the early learning of arithmetic (Baroody & Purpura, 2017). Later, it has a relevant role in the learning of algebra and functions (Amit & Neria, 2008; Radford, 2006; Ramírez et al., 2022). Consequently, researchers have analyzed MG students' learning processes of arithmetic and algebraic thinking.

In his seminal study, Krutetskii (1976) claimed that MG students can make more abstract generalizations than their peers, since they have "the ability for rapid and broad generalization of mathematical objects, relations, and operations" (p. 350). Usual school contexts in which students practice generalization include computing the next terms of numerical sequences (arithmetic generalization), calculating the general term of sequences (algebraic generalization), or formulating abstract functional relationships (functional generalization). Recent research has provided further support to this claim and literature on MG students has shown cases of this form of reasoning (Amit & Neria, 2008; Gutiérrez et al., 2018; Mora et al., 2022; Sheffield, 2009; Singer et al., 2016; Ureña et al., 2022).

When Krutetskii (1976) alluded to the generalization of operations, he meant, in particular, the generalization of *solution methods of problems*. This is a kind of generalization, different from generalizing mathematical facts, objects, or relationships, which has received very little attention in the mathematics education literature. It is also a necessary element of computational thinking (Khine, 2018), which is increasingly present in mathematics curricula from early primary grades. Stacey (1989) noted that, when secondary students identified a generalizable solution method in a part of a geometric pattern problem, they tended to use it in the next parts. Koichu and Kontorovich (2013) and Ramírez and Fernández (2018) posed the well-known billiard problem to students of different levels of mathematical talent, in which the method to calculate the paths of a ball has to be generalized to different sizes of tables. Montejo et al. (2020) posed to potential MG high school students a problem requiring to generalize a strategy to calculate the possible positions of two points on a grid to have the maximum distance between them.

Our study is situated in this approach, since the students had to identify and generalize methods of calculating solutions for several cases of the same type. An original contribution of this paper is that it focuses on students' capacity to generalize solution methods of problems, which requires an approach very different from the generalization of properties or relationships.

2.4 The role of olympiads and enrichment programs in identifying MG students

Several studies have compared the results of identifying MG students by using psychometric tests or mathematical problem-solving. In Spain, Díaz et al. (2008) and Ramírez (2012) compared answers to several psychometric tests and problems posed in the admission test of ESTALMAT. In the international context, Benavides (2008) in Chile, Niederer et al. (2003) in New Zealand, and Al-Hroub (2010) in the UK compared answers to psychometric tests and sets of problems. The findings of these studies, in line with the conclusion raised by Leikin (2018), consistently suggest that solving complex mathematical problems is more effective and reliable to identify MG students than administering other kinds of instruments.

Based on the mentioned results, mathematics education researchers have focused on observing students' behavior in mathematical competitions and other competitive contexts, such as problem-solving tests for admission to enrichment programs. Competitions at different educational levels have a long international tradition (Falk de Losada & Taylor, 2022; Leikin, 2021). Mathematical competitions play an important role in revealing students' capacities that remain hidden in schools (Kahane, 1999), motivating potential MG students and developing their mathematical capacities (Bicknell, 2008). Olympic-style problems are also effective in helping ordinary teachers identify their MG pupils (Miller, 1990). Elgrably and Leikin (2021) compared the abilities of two groups of students in a problem-posing task: candidates or participants in the IMO and mathematics majors who excelled in university mathematics. Their results showed that the olympic students significantly outperformed the majors.

Extracurricular mathematics enrichment activities, like workshops, summer camps, etc., also have a long tradition and are regularly conducted around the world (Kenderov, 2022). To ensure that the participants have above-average mathematical talent, some activities include an admission test, usually based on solving non-routinary problems; e.g., Ramírez et al. (2022) informed on the solutions to a problem asking to formulate a general rule to calculate the area of any of a given kind of figures. They showed the potential of the problem to enable grade 6 students to represent and generalize a quadratic relationship.

3 Theoretical framework

In this paper, we focus on identifying potential MG students by analyzing their solutions to two problems that require the use of the capacity for generalization. We first present the theoretical aspects related to the identification of (potential) MG students and then the generalization of solution methods of problems.

3.1 Identification of mathematical giftedness

Although different authors proposed several definitions of MG, they share the idea that MG students stand out in solving complex problems from their peers in mathematics classes and they evidence high mathematical abilities within their reference school group (Leikin, 2018). Based on the definitions proposed by Diezmann and Watters (2002), Krutetskii (1976), and Leikin (2018), we consider students to be *mathematically gifted* if they stand out above average students of their grade or age in the ease and speed of understanding new mathematical contents, in their ability to perceive and apply complex mathematical structures and procedures, in their effectiveness in correctly solving mathematical problems (even problems that are difficult for most students), and in their ability to generate novel mathematical ideas (relative to their prior mathematical experience). Furthermore, we adhere to the characterization of mathematical problems by Schoenfeld (1985).

For MG to be realized, the students' mathematical abilities have to be developed (Leikin, 2018); then, some researchers (Jablonski & Ludwig, 2022; Pitta-Pantazi & Leikin, 2018) refer to *potential* or *promising* MG students when students' behavior gives initial or partial signs of MG, to distinguish them from those who have already consistently demonstrated their MG. Jablonski and Ludwig (2022) distinguished between "giftedness as a potential achievement and achievement as a visible outcome" (p. 607). This distinction alludes to the belief that MG has an innate component that needs to be nurtured. It also alludes to the fact that MG has to be evaluated for a diversity of contents (arithmetic, geometry, algebra, etc.) and competencies (e.g., generalization, visualization, analogy and transfer, proof, mathematical creativity, etc.) with a variety of activities (e.g., problem-solving and problem-posing). We agree with Leikin (2018) in considering the terms potential and promising as synonymous.

The mathematics education research literature has described numerous characteristics of MG that serve as elements for the identification of MG students, although no specific student exhibits all of them (Freiman, 2006; Krutetskii, 1976). Therefore, to make a valid identification of MG, it is necessary to observe students' production in a variety of contexts adequate to allow students to evidence the use of different characteristics of MG. In practice, when experiments are based only on one context or competency (e.g., asking to solve a set of generalization problems, like in our study), what can be said about successful students is that they are *potential* MG students. Other actions, focusing on different characteristics and competencies, will be necessary to complete the identification and confirm their MG. In our experiment, we focused on observing students' use of generalization in solving two problems, so we can only conclude that the students producing the best solutions are potential MG students.

3.2 Generalization of mathematical problem-solving methods

Various definitions of mathematical generalization can be found in the literature (Amit & Neria, 2008). Krutetskii (1976), apart from analyzing the capacity of generalization of mathematical objects and relations, also observed that, in his experiments, "the ablest pupils … immediately after the first acquaintance with the solution principle using a certain … solution-scheme for a typical problem, this … solution-scheme is applied … to the most diverse variants of examples or problems of the appropriate type" (p. 240). So, the "generalization of the method of solution …, [i.e.,] algorithms for solving the whole class of problems of one type" (p. 259) is a differential trait of MG students: when they solve a problem (or a part of it), they may grasp a general solution method and transfer it to solve other problems (or parts of the problem) of the same type.

Based on Krutetskii (1976), Polya (1957), and Radford (2006), we consider that the *capacity for generalization* is a complex process consisting of knowing how to perform two types of mathematical activities: i) recognizing a particular property in some cases and inducing a general property valid for all elements of the set to which the cases belong, and ii) recognizing characteristics of a particular mathematical problem-solving method performed one or several times, inducing an abstract method sharing those characteristics, and being able to apply it in any other related but distinct problem. To generalize a solution method, students must first familiarize with a problem's solution, to realize the essence of the idea under the solution method; this should lead them to be aware that such method can be adapted to use it in other similar problems (Maj, 2011). The problems we have analyzed are of the second type, since, to solve them, students have to identify, in the first questions of the problem, new solution methods, generalize them, and apply them to solve the following questions of the problem.

Very few authors have studied the generalization of solution methods and, as far as we know, there is no detailed characterization of this kind of generalization. Then, we have defined a set of *descriptors* of i) the kinds of solution methods that students can perform when solving a sequence of related questions in a problem or a sequence of related problems and ii) the levels of generalization of those methods:

DG0. Non-codable answers: Blank answers or solutions that provide no useful information.

DG1.Types of solution methods

DG1.1. Solutions that do not meet some requirement of the problem: Solutions based on methods using operations, numbers, figures, etc. that are incompatible with the data or conditions in the problem.

DG1.2. Trial and error: Solutions based on methods using random data or other elements of the problem.

DG1.3. Recursive: Solutions based on recursive-style methods.

DG1.4. Partially optimal: Solutions based on an optimal method poorly applied by the student.

DG1.5. Optimal: Solutions based on an optimal method correctly applied by the student.

DG2. Levels of generalization (of optimal solution methods)

DG2.1. Uninitiated generalization: Solutions showing that the student has neither abstracted nor generalized an optimal solution method.

DG2.2. Partial generalization: Solutions showing that the student has started to abstract and generalize an optimal solution method by noticing some regularities.

DG2.3. Complete generalization: Solutions showing that the student has abstracted and generalized an optimal solution method.

These descriptors are partially based on other well-established descriptors of generalization and partially emergent. We first used literature on early algebra to define the descriptors: descriptors DG1 are based on the strategies to calculate near and far terms of pattern problems: trial and error, recursive, functional (Stacey, 1989; Lannin, 2005); descriptors DG2 are based on the levels of generalization by Radford (2006): DG2.2 relates with arithmetic generalization and DG2.3 with algebraic factual and contextual generalizations. After using the first definitions of the descriptors to analyze a sample of students' answers, we improved them to adapt them more accurately to the specific characteristics of the solution methods used in problems P3 and P4, resulting in the above definitions. In Section 4, we present the particularizations of these descriptors to the characteristics of the specific methods that students should generalize to solve the problems used in our study.

4 Research methodology

This study is qualitative, descriptive, and exploratory. Its general objective is to identify distinctive characteristics of MG students' styles of generalization of solution methods of mathematical problems.

4.1 The experiment

The population of this research comprised the 312 students in grades 6 (primary school), 7, and 8 (secondary school) who took the admission test for the ESTALMAT Program. The 25 students who obtained the highest total score on the test were admitted to participate in the workshop. We discarded the students who scored 0 points in problems P3 and P4, resulting in a reduced population of 289 students. Then we selected a convenience sample of 75 students, based on their total scores on the admission test: the top 25 scores (the admitted students), the middle 25 scores, and the bottom 25 scores. In this way, we can compare the outcomes of the most successful students (which we consider as potential MG students) with the less successful parts of the population. The data analyzed are their written solutions to P3 and P4. To maintain anonymity, each student was assigned a number.

The admission test consisted of five original and varied problems, dealing with visual thinking, logical thinking, intuition, creativity, generalization, ability to organize ideas, etc., and diverse mathematical contents. Each problem has several parts with increasing complexity and only requires mathematical content knowledge typical of ordinary students in grade 6.

To select the problems of the test for this study, we analyzed the mathematical capacities necessary to solve each problem. We selected problems P3 and P4 because they are the only ones requiring generalization of solution methods. We present the statement of each problem and then analyze its characteristics related to the solution methods to be generalized.

Problem P3

Three fairies live in a palace with many floors, numbered 1, 2, 3, 4, 5, ... There are two magic wands on each floor, one red and one blue. To move between floors, the fairies must touch a wand.

When a fairy touches the red wand, she can go 10 floors up or 10 floors down; e.g., if a fairy is on floor 37 and touches the red wand, she can go to floor 47 or 27.

When a fairy touches the blue wand, she can go up to another floor that is three times the floor she is on plus one; e.g., if the fairy is on floor 5, she can go up to floor 16 (16 = 3×5+1). The fairy can also move in the opposite direction; for instance, if she is on floor 13, she can go down to floor 4, because 13=3×4+1.

3a) The Forest fairy lives on floor 1. Could she go to floor 13? Could she go to floor 40? What about floor 93? What about floor 57? If she can go to any of these floors, explain which wands she touched and in which order. If you think she cannot go to some of these floors, explain why she cannot.

3b) Could you state a property that all floor numbers that the Forest fairy can reach have in common?

3c) The Moon fairy lives on floor 2. Describe how she can go to floor 57.

3d) The Water fairy lives on floor 18. Could she use the two wands to go to floor 5?

3e) Can two or the three fairies meet on any floor? If you think they can, tell us on which floor, which fairies meet there, and how they would go there. If you think they cannot, give a reason.

Two methods must be generalized to solve this problem. Students have to recognize, based on their answer to part 3a, that a fairy can only reach the floors whose number has specific unit digits: if the Forest fairy touches the blue wand, she can go from floor 1 to 4 (1×3+1); if she uses the blue wand three more times, she can go to floors 13 (4×3+1), 40 (13×3+1), and 121 (40×3+1). To complete the solution, students have to recognize and generalize the property that, if the fairy continues using the blue wand, she can only reach floors ending in 4, 3, 0, or 1. In part 3e, students are expected to calculate the floors accessible by each fairy and realize that they cannot meet.

Students also have to generalize the optimal method to find the shortest number of steps to go to any reachable floor: repeatedly touch the blue wand until the fairy reaches a floor whose number has the same unit digit as the target floor; then, touch the red wand as many times as needed to go to the target floor.

Problem P4

We want to tile a rectangular wall measuring 3m by 7m using only square tiles, which may be different, with integer side lengths and using the minimum possible number of tiles. The minimum number [Figure 1] *is 5 tiles: two tiles with sides of 3m and three tiles with sides of 1m. The design shown in the figure is a solution; there are other designs, but we consider them equal, since they use the same tiles, although arranged differently.*

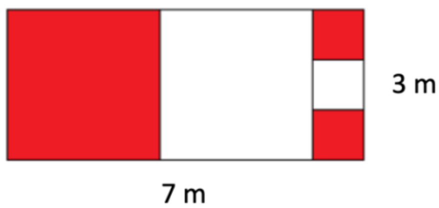


Figure 1.

4a) To tile a rectangular wall of 8m by 5m [Figure 2] *using the same criterion as above, how many square tiles do we need, and what are their sizes? Draw a design using those tiles.*

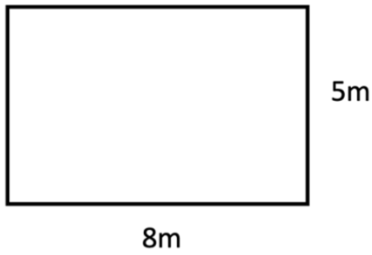


Figure 2.

4b) If we only have square tiles with sides of 2m, 4m, and 6m, and we use the same criterion as above, calculate the number of square tiles required, along with their sizes, to tile a 22m by 6m wall [Figure 3]*? Draw a design using those tiles.*

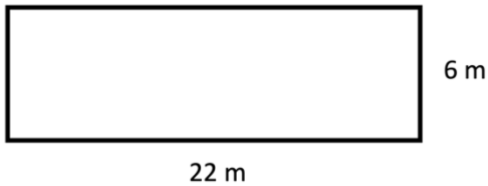


Figure 3.

4c) Now the wall is a square with sides of 9m. We only have square tiles with sides of 1m, 2m, 4m, 5m, and 7m. We can tile it in two different ways using the minimum number of tiles. Find them and draw each tiling [in Figure 4].

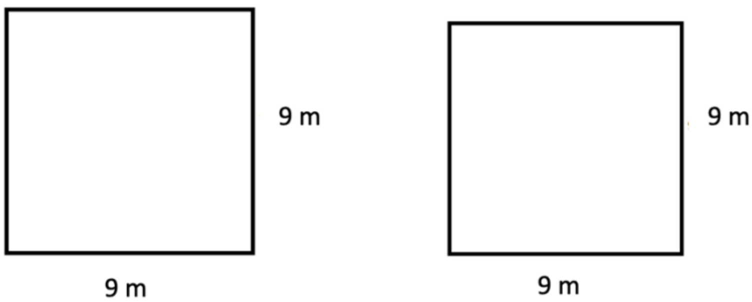


Figure 4.

4d) We can use square tiles with integer side lengths and decimal side lengths with .5 as the only decimal: 0.5m, 1m, 1.5m, 2m, 2.5m, ...,15.5m, 16m, and 16.5m. We have a wall measuring 371.25m^2 and want to tile it by using as the largest tile only one square tile with a side of 16.5m. What is the minimum set of tiles that allows us to do it? Justify your answer.

Students have to identify and generalize, from the example given, the optimal tiling method: first, place the largest possible tile as many times as possible; then, fill in the gap using the largest possible tile as many times as possible; repeat the procedure with smaller tiles until the whole rectangle is tiled. For instance, to tile the rectangle in Figure 3, we place three 6m tiles, then one 4m tile, and finally two 2m tiles. Part 4d does not specify a shape for the wall, but students, based on the context of the previous parts of the problem, assumed that the wall is a rectangle.

4.2 Methodology of analysis of data

We first conducted a pilot evaluation of a small sample of six students with different levels of performance on the test. The four authors independently analyzed their solutions and, based on the initial definitions of the descriptors of generalization, determined which descriptors were evidenced in each problem. We triangulated our assignments of descriptors, discussed the differences, and agreed on new definitions of the descriptors. We then analyzed the solutions of the six students again and repeated the triangulation. Finally, we produced the general definitions of the descriptors presented in 3.2.

The next step was to produce initial versions of particularized descriptors for P3 and P4. By applying the same triangulation methodology, we obtained particularized versions of the definitions of the descriptors for P3 and P4 (Tables 1 and 2).

The descriptors of generalization, and the particularizations to P3 and P4, represent processes of reasoning that can be observed in students' solutions and allow us to efficiently and reliably determine whether the students evidenced or not their capacity for generalization of solution methods and in which ways it is evidenced. In section 5 we show examples of solutions evidencing the most useful descriptors.

Problem P3 requires generalizing two solution methods: how to use the wands and which floors can be reached by each fairy. For this reason, we have defined some pairs of descriptors for the same degree of generalization. Note that DG1.3 and DG1.4 do not apply to P3, so Table 1 only includes the descriptors valid for P3.

Table 1. Descriptors of generalization used to analyze the solutions to P3.

Code	Description
	DG1.Types of solution methods
DG1.1 Solutions that do not meet some requirement of the problem	Calculations (made several times) of movement from one floor to another that do not match the wand definitions
DG1.2 Trial and error	Calculations (made several times) using randomly chosen wands
DG1.5 Optimal	Calculations (made several times) with optimally chosen wands for going to any reachable floor
	DG2. Levels of generalization
DG2.1a Uninitiated generalization of the use of wands	They do not use the optimal wand selection method because they make, at least in part, random selection of wands. They have not abstracted that they must use blue wand to reach a floor with the same unit digit as the target floor
DG2.1b Uninitiated generalization of accessible floors	Their solutions do not mention the unit digits of floors, except the specific floors they have reached in their calculations, or provide other kind of explanation. They have not abstracted that only floors with certain unit digits can be reached
DG2.2 Partial generalization of accessible floors	Their solutions consist of one of these arguments (or a combination of them), based on the cases they have calculated: - Each fairy can only reach floors with certain unit digits (but they do not mention all digits for each fairy) - Some fairies can only reach floors with certain unit digits (but they do not mention all fairies)
DG2.3a Complete generalization of the use of wands	They apply systematically (though not necessarily from 3a) the optimal method for choosing wands

Code	Description
DG2.3b Complete generalization of accessible floors	They explain that each fairy can only reach floors ending in certain unit digits and name all digits of each fairy

Table 2. Descriptors of generalization used to analyze the solutions to P4.

Code	Description
DG1.Types of solution methods	
DG1.1 Tilings that do not meet some requirement of the problem	They draw an apparently correct tiling, but i) they use non-square tiles, or ii) the dimensions of the tiles they write are larger (smaller) than the correct ones, so the tiles actually overlap (leave gaps) in the rectangle
DG1.2 Trial and error	They cover the rectangle with tiles chosen randomly (at least the first ones) or chosen without considering the rule of using the minimum number of tiles
DG1.3 Recursive	They make incorrect repetitive tilings, because they use an excessive number of (usually small) tiles of the same size
DG1.4 Partially optimal	In some rectangles, they use the optimal strategy correctly, but in others they try to apply it but make incorrect tilings
DG1.5 Optimal	Tilings made by selecting the optimal tiles to cover the rectangles
DG2. Levels of generalization	
DG2.1 Uninitiated generalization of the tiling method	They have not abstracted the optimal tiling method, since they cover the rectangles with inadequate tiles
DG2.2 Partial generalization of the tiling method	They have not fully abstracted and generalized the optimal tiling method, since they start using adequate tile (the largest possible), but then continue with inappropriate tiles
DG2.3 Complete generalization of the tiling method	They systematically apply the optimal tiling method (although not necessarily in 4a)

4.2.1 Qualitative analysis

For each problem, and based on Tables 1 and 2, we recorded the descriptors of generalization evidenced in the students' solutions in summary tables (Figure 5), to record the students evidencing each descriptor.

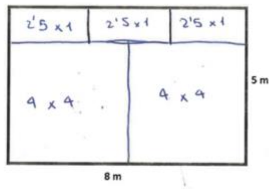
Code	Definition	Examples	Comments	Students
DG1.1. Tilings that do not meet some requirements of the problem	They draw an apparently correct tiling, but i) they use non-square tiles or ii) the dimensions they write are larger (smaller) than the correct ones, so that the tiles actually overlap (leave gaps) in the rectangle.	**111 – a** 	The student makes a tiling that does not match the dimensions of the rectangle. In this answer, we observe in the upper part that the length of the rectangle is divided into three pieces of 2.5m which would add up to 7.5m.	203, 612, 619, 9260, 9264, 330, 547, 178, 350, 554, 9272, 344, 511,543, 111, 505, 560, 157, 122, 364, 586, 534, 9254, 121, 133, 146, 147, 169, 206, 356, 357, 360, 559, 571, 603

Figure 5. Fragment of the table to classify the answers to P4.

The result of this qualitative analysis is a classification of the solutions to each problem according to the descriptors evidenced. Section 5 includes examples of solutions evidencing the most relevant descriptors.

4.2.2 Quantitative analysis

To analyze the effectiveness of the problems and descriptors in discriminating potential MG students with high capacity for generalization of solution methods, we need to identify differences between students' behaviors. A way to do it is to classify the answers to each problem and identify which descriptors were evidenced by the students with higher scores more (or less) often than by those with lower scores. As defined in the literature on educational research methods (e.g., Allen & Yen, 1979), discrimination is a tool that analyzes the characteristics of an item relative to the test as a whole of which it is a part (hence, if the test were changed, the discrimination of the item could be different). Calculating the discrimination of the problems and the descriptors of generalization allows us to decide whether they are adequate to identify specific students' traits of reasoning that differentiate high- and low-scoring students and, consequently, differentiate between potential MG and average students.

We have adopted the *discrimination index* (*d*), as characterized by Allen and Yen (1979): it evaluates "whether a person who does well on the test as a whole (that is, a person who presumably is high on the trait being measured) is more likely to get the particular item correct than a person who does poorly on the test as a whole … [and] whether an item discriminates between those examinees who do well and those who do poorly on the test as a whole" (p. 120). In other words, *d* assesses whether a particular test item performs as well as the test as a whole and identifies the best students; in a well-designed test, the best items are those in which students with the highest and lowest scores obtain the same score on the item and the overall test (Allen & Yen, 1979). Then, *d* is suitable to differentiate between high and low performers according to a given criterion, the capacity of generalization in our case: the discrimination index of a problem (P3 or P4) or descriptor assesses its efficacy in discriminating the students with the highest capacity for generalization of solution methods and, hence, potential MG students.

To calculate the discrimination index *d* of an item under analysis, "examinees [are] ordered on the basis of their total test scores" (Allen & Yen, 1979, p. 122) and then their item scores are compared. To do it, two parts of the sample are selected, comprising the subjects with higher and lower total scores in the whole test. The best option for our study is to select groups consisting of the top third (highest performers) and bottom third (lowest performers) of the sample. The central third of students is not used. The discrimination index *d* for each problem P3 and P4 is calculated as follows:

$d = P_t – P_b$, where

P_t is the mean score on the problem of the students in the top third

P_b is the mean score on the problem of the students in the bottom third

The index *d* evaluates "the degree to which responses to one item are related to responses to the other items in the test" (p. 120). The total number of items on the test does not influence the value of *d*, since it is only based on the arithmetic mean of the students' scores on the item analyzed.

As *d* is mostly used to analyze dichotomous items, it is assumed that it varies between –1 and +1. The discriminatory power of an item increases as *d* moves away from 0: if $d = 0$, there was the same number of correct answers by subjects in both thirds, so the item does not discriminate at all. If $d = +1$ ($d = –1$), all subjects in the top (bottom) third and no subjects in the bottom (top) third answered correctly the item, so the item has perfect discrimination. We will show that values of *d* close to –1 are meaningful in our study. The discrimination is (Ebel & Frisbie, 1965):

Very high	when	$0{,}40 \leq \lvert d \rvert \leq 1$
High	when	$0{,}30 \leq \lvert d \rvert < 0{,}40$
Low	when	$0{,}20 \leq \lvert d \rvert < 0{,}30$
Very low	when	$0 \leq \lvert d \rvert < 0{,}20$

The steps to calculate *d* for P3 or P4 are: i) order the 75 students in the sample according to their total scores in the admission test (range 0–50 points). ii) Identify the 25 students with the highest total scores (top third) and the 25 students with the lowest total scores (bottom third). iii) Calculate the values of P_t and P_b for the problem; the scores of P3 and P4 (range 0–10 points) are non-dichotomous, so it is necessary to standardize $P_t – P_b$ to the range [–1, +1] by dividing it by 10. And iv) calculate $d = (P_t – P_b)/10$ and interpret its values for each problem.

We also use *d* to analyze the discriminatory power of the descriptors (Tables 1 and 2), to identify solution methods that are more typical of students with a higher capacity for generalization of solution methods. The steps to calculate *d* for the descriptors are similar to those described above, with some differences: a) the meanings of the descriptors are particular for each problem so, now, the unit of analysis is not the test, but each problem; b) consequently, the students are ordered, and the top and bottom thirds are made, based on the problem scores; c) Pt – Pb is calculated for each descriptor in each problem; d) the descriptors are dichotomous items (1 = the descriptor is evident in a solution, 0 = it is not evident), so $P_t – P_b$ is in the range [–1, 1] and it is not necessary to standardize it, so $d = P_t – P_b$.

If *d* is negative for a descriptor, the descriptor has been more evidenced in solutions by students in the bottom third; hence, it may be associated with the use of inefficient or incorrect solution methods. The values of *d* allow us to get conclusions about the characteristics of each problem and descriptor allowing us to discriminate the students with higher capacity for generalization of solution methods and being potential MG students.

5 Analysis of data and results

This section presents the analysis of the data obtained after calculating *d* for the problems and the descriptors of generalization evidenced in the solutions. We analyze the power of discrimination of i) P3 and P4 and ii) the descriptors of generalization evidenced in the solutions to P3 and P4. We present all data but analyze only the descriptors that provide relevant information.

5.1 Analysis of the discriminatory power of the problems

The discrimination index of P3 and P4 allows us to determine if the problems are high discriminators within the test, i.e., if they allow us to identify potential MG students. Table 3 shows that the power of discrimination of both problems is very high, indicating that they differentiate very well the students with the best capacity for generalization of solution methods.

Table 3. Indices d for P3 and P4

	Problems	
	P3	P4
Mean problem score of students in the bottom third	1,70	0,63
Mean problem score of students in the top third	6.26	5,86
Index *d*	0.46	0.52

5.2 Analysis of the discriminatory power of the descriptors in P3

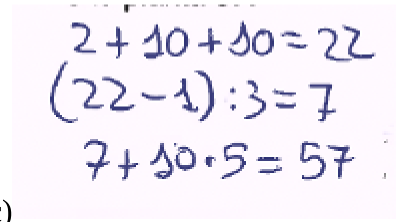
Table 4 shows the values of *d* for each descriptor. Below, we analyze the descriptors of generalization whose values of *d* show high or very high discrimination in P3.

Table 4. Indices d for the descriptors of generalization in P3

	Descriptors							
	DG 1.1	DG 1.2	DG 1.5	DG 2.1a	DG 2.1b	DG 2.2	DG 2.3a	DG 2.3b
Number of evidences by students in the bottom third	6	15	0	17	16	0	1	0
Number of evidences by students in the top third	0	6	19	8	5	10	15	11
Index *d*	–0,24	–0,36	0,76	–0,36	–0,44	0,40	0,56	0,44

Two descriptors related to the methods of obtaining the solutions (DG1.x) provide interesting information:

The index *d* of DG1.2 reveals that P3 makes a high *negative* discrimination of students with poor capacity of generalization of solving methods, since the students in the bottom third tended to use trial and error (Figure 6) more often than those in the top third. Then, DG1.2 helps to discriminate students who are not potential MG students.

3c) 

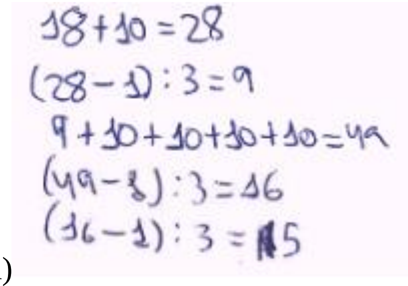
3d) 

Figure 6. Student 203’s solutions to parts 3c and 3d

Descriptor DG1.5 has very high *positive* discrimination of students who identified and used the optimal method to calculate the shortest way to arrive at the desired floor (Figure 7). The discrimination is positive because students in the top third used this method much more often than students in the bottom third. Thus, the systematic use of this optimal method discriminates very well potential MG students showing a capacity to identify regularities in the problem-solving processes.

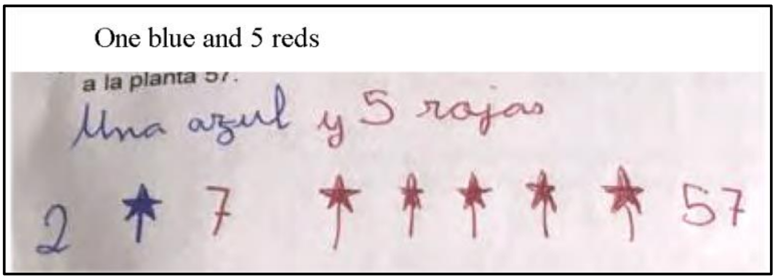
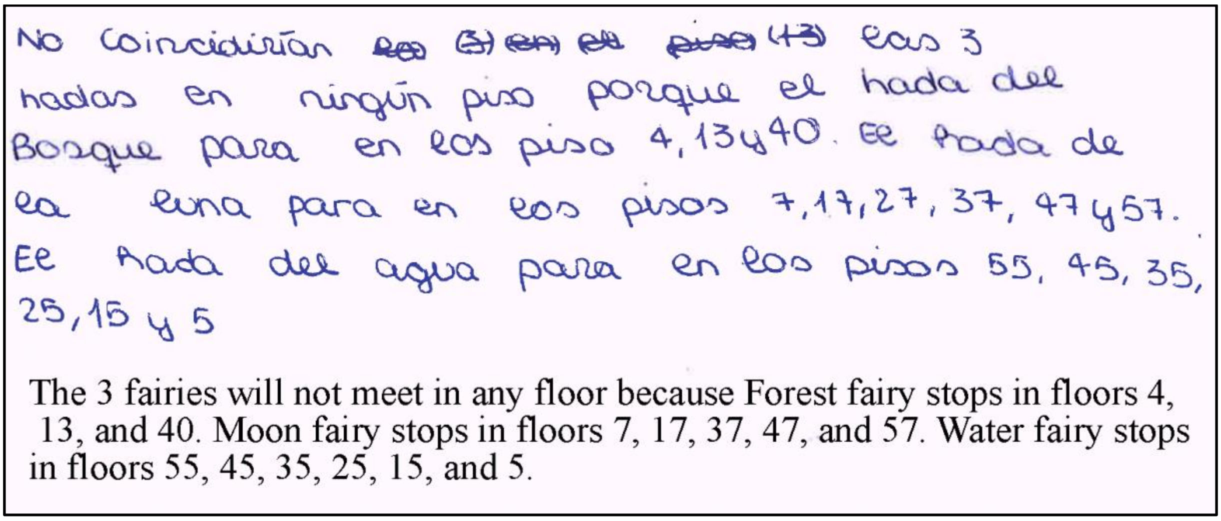


Figure 7. Student 340’s solution to part 3c

Regarding the levels of generalization evidenced by students in P3 (DG2), all descriptors provide interesting results:

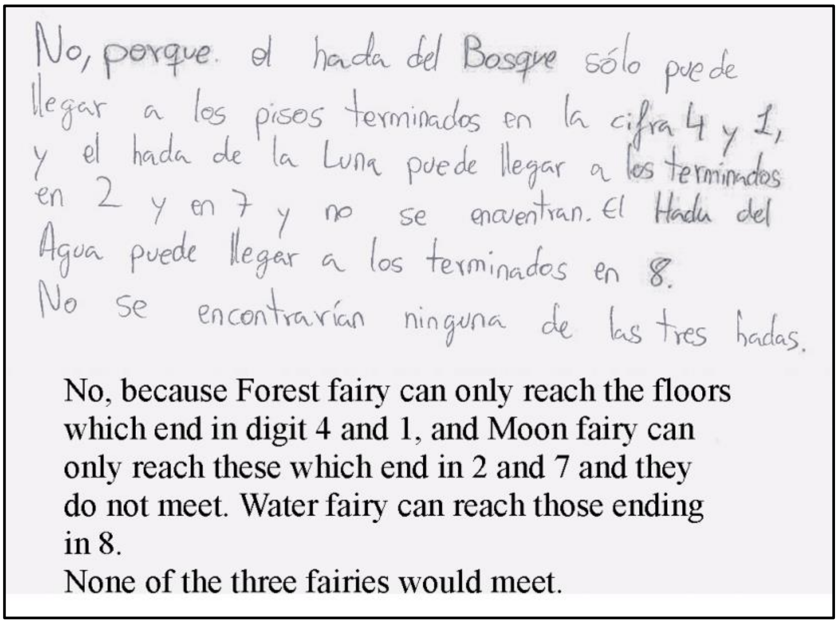
Descriptors DG2.1a and DG2.1b produce high and very high negative discrimination, respectively: DG2.1a was evidenced when students did not have an idea of how to find answers, so they randomly used the wands and, in most cases, failed to identify the optimal method for using them (Figure 6). DG2.1b was evidenced by students who did not note that only floors with specific unit digits are available to each fairy (Figure 8). These behaviors are typical of students who are unable to identify hidden regularities in the processes of solving the parts of the problem. Then, these descriptors are useful to identify students with low capacity for generalization.


No coincidirán ~~los (3) (en) el piso~~ (43) las 3 hadas en ningún piso porque el hada del Bosque para en los piso 4, 13 y 40. El hada de la luna para en los pisos 7, 17, 27, 37, 47 y 57. El hada del agua para en los pisos 55, 45, 35, 25, 15 y 5

The 3 fairies will not meet in any floor because Forest fairy stops in floors 4, 13, and 40. Moon fairy stops in floors 7, 17, 37, 47, and 57. Water fairy stops in floors 55, 45, 35, 25, 15, and 5.

Figure 8. Student 111's solution to part 3e

DG2.2 produces a very high discrimination (*d* = 0.40) of students who progressed in recognizing some regularity in the accessible floors but did not achieve complete generalization. The student in Figure 9 identified some unit digits for each fairy but not all of them. The discrimination is positive because most students evidencing partial generalization had higher problem scores, while no student with lower scores did it. Hence, DG2.2 discriminates very well students with a high capacity for generalization.

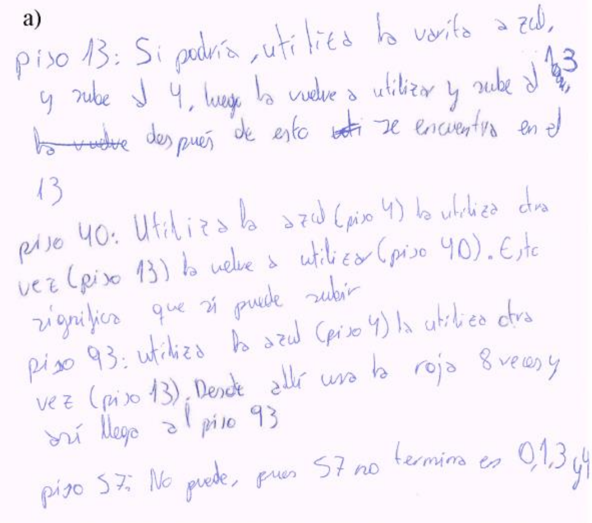

No, porque el hada del Bosque sólo puede llegar a los pisos terminados en la cifra 4 y 1, y el hada de la Luna puede llegar a los terminados en 2 y en 7 y no se encuentran. El Hada del Agua puede llegar a los terminados en 8.
No se encontrarían ninguna de las tres hadas.

No, because Forest fairy can only reach the floors which end in digit 4 and 1, and Moon fairy can only reach these which end in 2 and 7 and they do not meet. Water fairy can reach those ending in 8.
None of the three fairies would meet.

Figure 9. Student 594's solution to part 3e

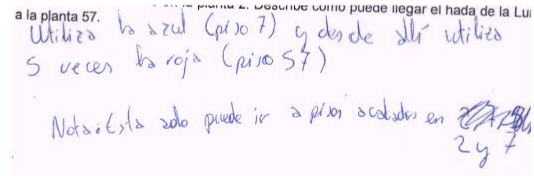
Descriptors DG2.3a and DG2.3b make a very high discrimination of students who succeeded in making complete generalizations. Their discrimination is positive because a majority of the students in the top third generalized the three solution methods, while only one student in the bottom third evidenced DG2.3a. Figure 10 presents a solution showing that the student had grasped an optimal procedure of using the wands, since, in all the questions, she used the optimal procedure to get the answer. Figure 11 presents a solution evidencing DG2.3b, since the student had identified the relationships between floors and wands, so was able to calculate all the floors available to each fairy.

a)



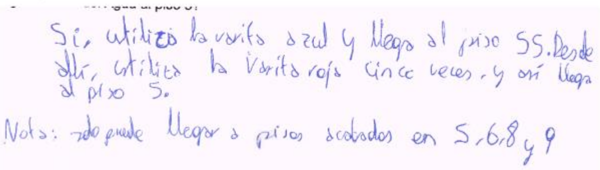
floor 13: She could, she uses the blue wand, and goes up to [floor] 4, then she uses it again and goes up to 13, then she is in [floor] 13
floor 40: She uses the blue (floor 4) she uses it once more (floor 13) she uses it again (floor 40). This means that she can go up.
floor 93: she uses the blue (floor 4) she uses it once more (floor 13). From there she uses the red [wand] 8 times and so she arrives to floor 93.
floor 57: She cannot, since 57 does not ends in 0, 1, 2, 3 and 4.

c)



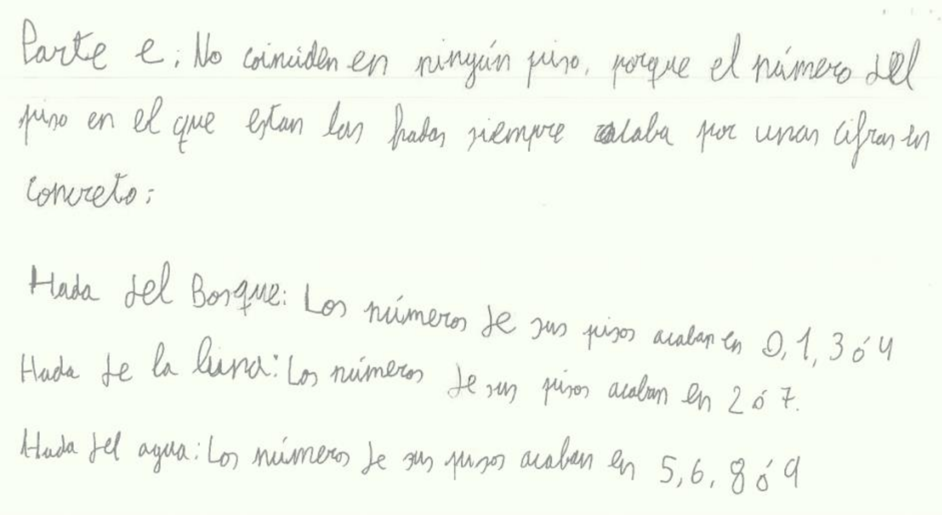
She uses the blue (floor 7) and from there she uses five times the red (floor 57).
Note: This [fairy] can only reach floors ending in 2 and 7

d)



Yes, she uses the blue wand and goes to floor 55. From there, she uses the red wand five times, and so she reaches floor 5.
Note: [she] can only reach floors ending in 5, 6, 8, and 9

Figure 10. Student 9260's solutions to parts 3a, 3c, and 3d



Parte e: No coinciden en ningún piso, porque el número del piso en el que estan las hadas siempre acaba por unas cifras en concreto:

Hada del Bosque: Los números de sus pisos acaban en 0, 1, 3 ó 4
Hada de la luna: Los números de sus pisos acaban en 2 ó 7.
Hada del agua: Los números de sus pisos acaban en 5, 6, 8 ó 9

Part e. They do not meet in any floor, because the number of the floor where the fairies are always ends in some specific digits:
Forest fairy: The numbers of her floor end in 0, 1, 3 or 4
Moon fairy: The numbers of her floor end in 2 or 7
Water Fairy: The numbers of her floor end in 5, 6, 8 or 9

Figure 11. Student 185's solution to part 3e

The values of *d* in Table 4 show relationships between descriptors of DG1 and DG2 that are qualitatively consistent with the quality of the responses associated with those descriptors: when students used poor solving strategies (not meeting the requirements of the problem or trial and error), then they could not find a solving strategy allowing them to correctly generalize the use of wands (DG2.1a) or the access to floors (DG2.1b). Conversely, when students used correct solution strategies (DG1.5), then they were able to convert them into generalized strategies (DG2.2, DG2.3a,b). Quantitatively, these relationships are clear in Table 4, with values of *d* under -0.20 for DG1.1, DG1.2, and DG2.1a,b and over 0.40 for DG1.5, DG2.2, and DG2.3a,b.

As a final synthesis of the analysis of P3, we argue that this problem is very adequate to obtain a variety of solutions evidencing the different descriptors of each type of generalization and allowing high or very high discriminatory power of potential MG students with high capacity to generalize solution methods. Concerning the descriptors, some have proved to be useful in identifying potential MG students (DG1.5, DG2.2, and DG2.3) while others are useful in identifying students who do not show MG characteristics (DG1.2 and DG2.1). Therefore, in ordinary groups of students, P3 helps identify both students with low and high MG.

5.3 Analysis of the discriminatory power of the descriptors in P4

Table 5 shows the values of *d* for each descriptor. Below we analyze the descriptors whose values of *d* show high or very high discrimination for P4. The discussion made in section 5.2 about the meanings of negative and positive values of *d* is also valid here.

Table 5. Indices d for the descriptors of generalization in P4

	Descriptors							
	DG 1.1	DG 1.2	DG 1.3	DG 1.4	DG 1.5	DG 2.1	DG 2.2	DG 2.3
Number of evidences by students in the bottom third	18	4	4	1	0	7	1	0
Number of evidences by students in the top third	10	3	7	6	14	2	6	15
Index *d*	–0,32	-0,04	0,12	0,20	0,56	–0,20	0,20	0,60

Descriptor DG1.1 offers interesting information about solutions based on inefficient or incorrect methods. It made high *negative* discrimination of students with a low capacity for generalization of the solution method, who used non-square tiles (Figure 12-1) or squares with incorrect dimensions (Figure 12-2). The discrimination is negative because these solutions are mostly used by students in the bottom third.

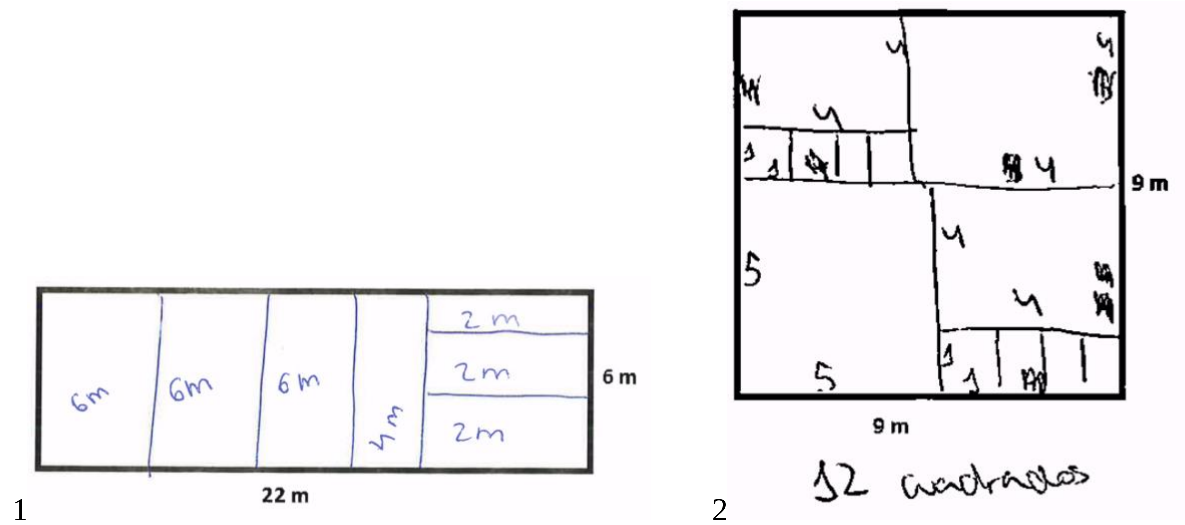


Figure 12. Solutions by students 121 and 203 to parts 4b and 4c, respectively.

DG1.5 highlights correct solutions based on the optimal tiling method, evidenced only by students in the top third who consistently showed (Figure 13) that had understood and generalized ways to use the minimum possible number of tiles and always used the optimal tiling method. Then, DG1.5 discriminates very well students with a high capacity for generalization of solution methods.

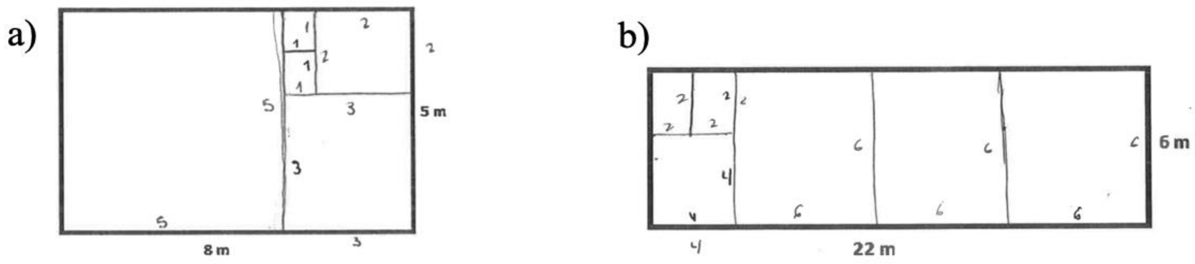


Figure 13. Student 301's solutions to parts 4a and 4b.

Descriptor DG2.3 made very high discrimination, as it was evidenced by most students in the top third but none in the bottom third. The solution in Figure 14 evidenced DG2.3.

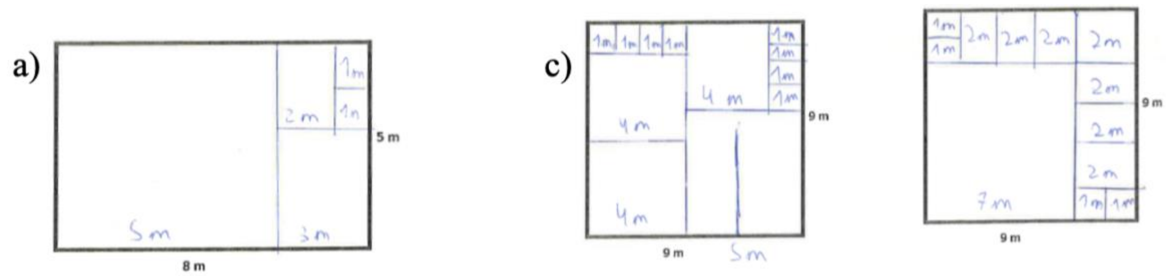


Figure 14. Student 9273's solutions to parts 4a and 4c

As for P3, the values of *d* in Table 5 show qualitatively consistent relationships between descriptors of DG1 and DG2: when students used poor tiling strategies (DG1.1, DG1.2, DG1.3), they failed to find a correct generalized method for tiling the surfaces (DG2.1). Conversely, when students used a correct tiling strategy (DG1.5), they succeeded in converting it into a generalized strategy (DG2.2, DG2.3). Quantitatively, as there is a diversity of poor solution strategies in P4, the values of *d* are close to 0. There are very clear relationships, DG1.1 – DG2.1, DG1.4 – DG2.2, and DG1.5 – DG2.3, but we cannot affirm that there is a relationship of DG1.2 and DG1.3 with DG2.2 and DG2.3, since their values of *d* are low or very low.

As a final synthesis of the analysis of P4, this problem is adequate to let students evidence their different capacities of generalization, from average students to potential MG students. The descriptors DG1.5 and DG2.3 are useful in identifying potential MG students, while DG1.2 is useful in identifying non-MG students.

6 Conclusions

Identifying potential MG students requires observing their problem-solving abilities across various mathematical topics and their use of a range of processes of reasoning. We have presented novel insights into this research line by analyzing solutions by 11 to 14 year-old students to two problems requiring them to identify specific solution methods, abstract and generalize them, and apply them in different ways to answer related questions. These problems are original, since they are not within the usual contexts of arithmetic, algebraic, or functional generalizations, but they ask students to think about procedural aspects of the solutions. A contribution of our study is to raise the importance of analyzing students' capacity to generalize solution methods of mathematical problems, in line with Krutetskii (1976).

Our research has four objectives. To achieve the first one, we have presented a set of original descriptors for the capacity to generalize solution methods, that characterize different ways and levels of using, abstracting, and generalizing solution methods. The proposed operative characterization of the processes of generalization, necessary to solve this kind of problem, is an original contribution to the research on students' capacity for generalization.

To accomplish the second objective, we have used the descriptors to analyze the solutions to two problems posed in the admission test to ESTALMAT. We have particularized the descriptors into specific operational descriptors for each problem (Tables 1 and 2), ensuring a reliable analysis of students' solutions.

Regarding the third objective, the qualitative analysis of the students' solutions has shown evidence of different strategies of generalizing solution methods that align with the descriptors defined. This study offers an interpretation of generalization in contexts different from the usual ones and provides examples of various forms of generalization of solution methods of problems.

As for the fourth objective, we have conducted a quantitative analysis of the descriptors evidenced in the solutions, by calculating their discrimination index *d*. We have identified the descriptors making high or very high discrimination of the solutions by students with higher or lower problem scores. These results lead us to argue that:

- Our problems are suitable for identifying specific forms of generalization of solution methods typical of MG students.

- Some descriptors are associated with ways of reasoning of MG students (DG1.5, DG2.2, DG2.3) or students with low mathematical capacity (DG1.1, DG1.2, DG2.1).
- Both problems require to generalize solution methods, but their contexts are quite different. This results in a diversity of solution methods, with some descriptors discriminating high-scoring students in both problems (DG1.5 and DG2.3) or only in a problem (DG2.2 for P3). We argue that this is a consequence of the richness of the problems and the diversity of their mathematical contexts.
- The results for each problem show consistency between the values of *d* for sub-sets of descriptors of DG1 and DG2. This allows us to empirically confirm the theoretical validity of the descriptors we have presented to characterize the generalization of mathematical problem-solving methods and their usefulness to identify potential MG students.

Possible limitations of our study are i) the length of the problem statements and the students' lack of familiarity with such type of problems. This may have caused some students with low reading comprehension to have difficulty understanding the mathematical contents involved and solution methods, preventing them from providing correct solutions. ii) Our results cannot be generalized, since they are based on a moderate-sized sample and solutions to two specific problems. The values of *d* obtained for P3 and P4 in our experiment cannot be extended to other sets of problems, that might have different difficulties; however, the values of *d* for the descriptors are independent of the test, since they are based only on students' scores in those problems.

The theoretical framework and the research methodology used in our study seem to be suitable for future research experiments analyzing students' processes of generalization of any kind of solution methods and problems. Then, this study paves the way for future research into the identification of potential MG students' capacity for generalization of problem solution methods. Several directions of research can be envisaged, like the possible relationship between previous problem-solving experience, the scores obtained, and the discrimination power of the problems and descriptors.

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