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ORIGINAL PAPER

First encounter with constructing graphs in the functional thinking approach to school algebra in 3rd and 4th grades

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Abstract

Given the relevance of graphs of functions, we consider their inclusion in primary education from the functional approach to early algebra. The purpose of this article is to shed some light on the students' production and reading of graphs when they solved generalization problems from a functional thinking approach. We aim to explore how 3rd and 4th graders construct graphs associated to functions and what elements they use; and how they read function associated graphs and whether they connect pairs of values to see beyond the data. After four working sessions about functions, we designed and implemented individual interviews to 12 students. Through a qualitative analysis, we highlight that the students can read data in a graph on two different cognitive levels and also construct it from different elements of the graph initially provided. Regarding data reading, we evidence two levels: (a) literal reading of a given element in the graph, and (b) reading beyond the data. The construction of the graph is described with base on the axes, values and labels on the axes, scale of the axes, and construction techniques. We present examples of students' work that evidence that graph construction varied depending on whether it was created from a blank sheet or it was necessary to provide help regarding the axes or the scale of the graph. We describe several techniques used by the students in the representation of data that yield non-canonical representations of a graph and that help glimpse how students are interpreting this representation.

Keywords Primary school · Graphs · Algebraic thinking · Functional thinking · Visual representation

1 Introduction

"Representation and visualization [of concepts and problems] are at the core of understanding in mathematics" (Duval, 1999, p. 3). They are tools that help express objects, concepts or mathematical procedures (Cai, 2005). In particular, representations are considered key for functional thinking because they are used to (a) represent concepts and procedures associated to functions, (b) mediate between subjects, functions and functional relationships, and (c) establish a path to express relationships between variables, which can help connect separate ideas (e.g., Brizuela & Earnest, 2008).

Graphs, tables and diagrams constitute a type of specific representation which Arcavi (2003) calls visual representations. They are products and interpretations of images that allow "unseen" ideas (McCormick et al., 1987) to visually communicate shapes that reinforce the understanding of concepts.The description of the students' work through these representations with algebraic elements helps clarify the role of algebra in primary education.

The students' experiences of representation to give meaning to mathematical ideas have triggered interest in the development of algebraic thinking (e.g., Ayala-Altamirano

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& Molina, 2020) and they establish an open line of research on algebraic thinking (Kieran, 2022).

From a curricular international perspective, US and Canadian curricular documents include algebraic think- ing for primary students (NCTM, 2000; Ontario Public Service, 2020). Algebra standard for Grades 3–5 students requires students to describe and extend patterns and make generalisations about them; and also, to represent

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and analyse functions, verbally and through tables and graphs. Other countries, as Spain, have included algebraic thinking in their curriculum recently, but do not specify contents nor procedures (e.g., Ministerio de Educación y Formación Profesional, 2022).

The students' representations in algebra in primary education has been addressed on various occasions (e.g., Pinto et al., 2022; Radford, 2018). However, visual representations in relation to functional thinking have been dealt with in very few cases at this educative level. Some examples can be found in the use of tables (e.g., Blanton et al., 2015; Torres et al., 2022).

Open questions remain on how primary education children express, reason, and justify algebraic statements through visual representations. "Visualization offers a method of seeing the unseen" (McCormick et al., 1987, p. 3) and they posed certain questions on how visual representations can enhance and broaden children's understanding of algebra. There is evidence that children who justify general statements through visual representations (e.g., Schifter, 2008) can reason more generally on new statements than those children with more primitive reasoning. However, we have not found evidence of how elementary students construct graphs in early algebra context. So our research question is how elementary students construct graphs in functional thinking approach to early algebra.

This paper complements prior research on functional thinking with primary education children, focusing on (Cartesian) graphs as a visual representation, which has not been tackled in previous studies. We decided to focus in the Grade 3 and 4 of primary education, because they still have not previous knowledge with graph representa- tion in math nor other curricular areas in Spain. Conse- quently, due to the students' age, the domain and codomain of the functions were the natural numbers set.

The primary objective of this study is to describe how children in Grade 3 and 4 of primary education construct and read graphs that involve representing functional relationships in contextualized generalization tasks. Graphs allow to evidence how students interpret functions and their elements (particularly, the variables), and the rela- tionships between them, including the generalization of such relationships. To get a wide view of students' understanding of functions, we consider two processes: construction and reading graphs (Ponte, 1984). As graphs are included in the last two years of primary education, we focus on the two previous years, to observe their first encounter from the early algebra approach. The specific research objectives in this study are:

- To explore how children in the 3rd and 4th grades construct graphs associated to functions and, in, what elements of function-related graphs they use.
- To explore how 3rd and 4th graders read function associated graphs and whether they connect pairs of values to see beyond the data.

2 Review of the literature

Recent research challenges so-called limitations in children's capacities to work with funcional topics at primary education, even at pre-school (e.g., Blanton & Kaput, 2004). Nowadays, the topics addressed on functional thinking at primary education have been varied: generalisation (e.g., Ellis, 2007), functional relationships (e.g., Pinto et al., 2022), strategies to solve certain tasks (e.g., Morales et al., 2018), difficulties with certain tasks (e.g., Hidalgo & Cañadas, 2020) or how to foster functional thinking when faced with such difficulties (e.g., Narváez & Cañadas, 2023; Pang & Sunwoo's, 2022) among others.

Representations constitute one of the aspects that have long been of concern to researchers in algebraic thinking and, paticularly in functional thinking. For example, recent studies indicate that the notation of variables is within the reach of children in the early grades of primary education (e.g., Ayala-Altamirano & Molina, 2020; Blanton et al., 2017). Natural language is described as a useful tool for expressing generalizations and consider it a crucial scaffold for the development of more symbolic representations (Radford, 2018; Stephens et al., 2017). Pictorial and manipulative (or concrete materials) representations become useful aiding students in finding the relationships between variables (Moss & McNab 2011). Pinto et al. (2022) evidenced that numerical and natural language were the representations most used by third and fifth graders to express the relationship between variables.

However, there is limited evidence about students' use of visual representations to solve tasks. In a case study, Brizuela et al. (2021) highlighted that tables were used as an organization tool that helped generalize functional relationships. Torres et al. (2022) revealed Grade 2 students' ways of organizing the data in tables enabled to identify the structures in the generalization process. But we do not have found information on the construction, use or reading by primary education children of graphs related to algebraic contents. This is the focus of this paper.

One important antecedent of this study is Ponte (1984), although his research is related Secondary Education. Graphs help visualize how a function behaves: continuity, zeroes, growth intervals, asymptotes, among others, wich are contents of Secondary Education. In our research, we propose to introduce graphs in primary education under the early algebra approach. We follow the Spanish curriculum, which indicates a step-by-step process of modelling using

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mathematical representations to facilitate the understanding and solving everyday problems.

3 Theoretical framework

The aim of early algebra is to foster algebraic thinking since the first years of schooling. Algebraic thinking can be interpreted as an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter-symbolic, but which can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourse of school algebra (Kieran, 1996, p. 275).

Different authors approach algebraic thinking through different focusses (e.g., Carraher et al., 2006; Kaput, 2000). One of them revolves around the notion of function; and is known as functional thinking, which is the subject of this paper.

3.1 Early algebra practices

The framework proposed by Kaput et al. (2008) places four practice elements for algebraic thinking in the classroom: (a) generalizing mathematical relationships and mathematical structures, (b) representation, (c) reasoning with generalized relationships, and (d) justifying generalizations (Blanton et al., 2011). Our main interest lays in how children graph functional elements and relationships and how they reason about them.

Generalization is considered the heart of algebraic thinking (e.g., Mason, 2017). Its representation is necessary to make it visible. Representation in itself helps shape children's understanding of algebraic concepts. Morris (2009) pointed out that the act of representing generalization helps children understand that an action applies to an infinite set of data pairs. This, in turn, can reinforce children's understanding of the expression of generalization. Then, representations of generalizations can be used as important tools with which children can reason and justify algebraic ideas. This will be taken into account when analyzing the work of the students.

3.2 Visual representations

In order to think and reason about mathematical ideas, it is necessary to create an internal representation of such ideas. To be able to communicate these ideas, it is necessary to represent them externally. We consider that cognitive processes such as functional thinking, and manipulating representations, are external given that the communication of these processes requires the use of representations to be external (Hiebert & Carpenter, 1992). Representations are tools that make mathematical concepts and procedures present, and with which individuals address and interact with mathematical knowledge (Rico, 2009). Working with different representations allows people to improve their understanding of the mathematical concept they are representing (Radford, 2000, p. 239).

In the context of early algebra, we assume that elementary school students may use different representation while working with functions: (a) spoken natural language, (b) written natural language, (c) pictorial, (d) numerical, (e) algebraic, (f) tabular, (g) graphic systems, and (h) gestures (Carraher et al., 2008; Radford, 2003). Although graphical representation has been given little attention in the field of algebraic thinking in the early years, from statistics education, Friel et al. (2001) noted that "educators have much to learn about the processes involved in reading, analyzing, and interpreting information presented in data graphs and tables" (p. 124).

Our focus is specifically on Cartesian graphs (hereinafter, graphs), as one of the representations usually associated with functions. Graphs are a type of visual representation (Arcavi, 2003). Aside from visual representations as a tool, we are interested in representations in themselves: how they are built and how they are communicated (Selling, 2016).

Following Arcavi (2003), the visualization is.

> [T]he ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

In different countries, as is the case in Spain, the introduction of visual representations of functions is at secondary education. We considered the challenge of working with graphs with children in primary education.

3.2.1 Cartesian graphs

Different types of graphs, share structural characteristics (Kosslyn, 1994), which includes axes, scales, grid lines, reference marks and information on the type of data being related. The simplest graphs are L-shaped, with the X and Y axes. They also have labels to indicate the values or units represented in the axes (Friel et al., 2001). In particular, as Cartesian graphs represent functions on a graph, the X-axis shows the independent variable values, and the Y-axis shows the dependent variable values.

Ponte (1984) considered three processes involved in the understanding of function graphs: (a) reading of graphs, (b) construction of graphs, and (c) interpretation of graphs. They are related, but they can be addressed separately; and we focused on the construction and reading of graphs. The construction of graphs relates identification of the appropriate variables to describe a given situation and grasp its properties as an involved function, or a graph based on given data so the functional relationships are represented. The reading of graphs refers to the understanding and use of the basic principles and conventions of graphs.

Arteaga et al. (2021) established four levels for primary education students to read statistical graphs. We adapt these levels to the functional thinking context as follows:

1. Read the data. When they are asked only to read an element of the graph (e.g., title, value of a variable, etc.).
2. Read within the data. Includes literal reading, to make comparisons between sets of values or conduct calculations based on the values on the graph.
3. Reading beyond the data. Implies a generalization of the values on the graph.
4. Read behind the data. When a critical assessment is required of the contents of the graph.

3.3 Functional thinking

Functional thinking involves generalising relationships between co-varying quantities, representing those relationships in different ways using diverse representations, and reasoning fluently with those representations to interpret and predict the behaviour of functions (Blanton et al., 2011). In mathematics, functions are considered important and have even been proposed as the backbone of the mathematics curriculum (e.g., Freudenthal, 1982; Schwartz, 1990). However, in various countries, including Spain, algebraic thinking has gained ground in primary education in the last decades (Canavarro, 2009; Pincheira et al., 2021; Watanabe, 2008).

Researchers on early algebra consider that functional thinking in the early grades is an important pathway towards algebra (e.g., Carraher & Schliemann, 2007). We assume functional thinking at primary education is "a component of algebraic thinking based on the construction, description, representation and reasoning with and about functions and the elements they are comprised of" (Cañadas & Molina, 2016, p. 210).

Linear functions are first-degree polynomial functions that model situations involving constant change (Ruiz, 2014). Through algebraic symbolism, they can be represented as $f(x) = mx + b$, with m and b real numbers. The x is called the independent variable and the $y/f(x)$ the dependent variable. The set of values that the independent vari- able takes is the domain, while the set of values that the dependent variable can take is the codomain. Due to the prior knowledge of students in Grade 3rd and 4th, and the values of x, y, m and b in this study belong to the natural numbers set.

Then, from a mathematics viewpoint, variables are key elements in functions. Functional thinking focuses on the generalization and expression of a relationship between variables as quantities that vary jointly (Blanton et al., 2011). The practices introduced by Kaput (2008)—described previously— are integrated in all the dimensions of algebraic thinking (e.g., Kaput et al., 2008), particularly in functional thinking. We focus on two of these practices: generalization and representation. Concerning representation, we are particularly interested in graphs. We pay attention to different elements associated to functions such us axes, values of the variables, scales and labels of the axes and different graphs construction techniques.

4 Methods

This is a qualitative, exploratory and descriptive study, with the aim of characterizing the way in which a group of students of 3rd and 4th grades constructed and read graphs during individual and semi-structured interviews.

We designed and implemented four sessions for two groups of students. The research team introduced the tasks to the students and the students completed individual questionnaires with various questions about the tasks that were presented in their usual class. After the sessions, we conducted individual interviews with six students from each one of the groups. The sessions and the interviews were video recorded, and the written work was collected, with a report from the Human Research Ethics Committee of the University of Granada.

Although in this study we focus on the information collected in the interviews, below is a summary of what they worked on during the previous sessions.

4.1 Design and implementation of the work sessions

We developed four working sessions with the whole group (23 from Grades 3 and 27 from Grade 4 (8–10 years old)). Each session lasted 45–60 min, was carried out once a week by three members of the research team, and one of them fulfilled the role of teacher-researcher. Apart from these sessions, students continued to work on the curricular content provided in the textbooks, which did not include algebraic thinking.

In these sessions, we presented generalization contextualized problems involving linear functions, with only number set involved, as our main antecedents did. We introduced the problems through different representations. This promotes students' understanding of the function behavior and help to evidence students' functional thinking (e.g., Blanton et al., 2011; Cañadas et al., 2016).

We followed the inductive reasoning model (Cañadas & Castro, 2007), by asking questions about specific cases and then we guided them towards generalization following such model. We can see the examples of the kind of questions in Table 2.

Each session had three parts. The first part was the problem introduction, when researchers introduced the context and started to work with some particular cases. In the second part, we posed individual questionnaires about the problem introduced. Meanwhile, the researchers observed students' production and selected some examples. In the third part of the session researchers guide a discussion about questionnaires answers and, in some cases, used the examples observed in the previous part. The intention of the discussion was for most students to identify the relationship between the variables and to become familiar with the graphs. We work on correcting some observed errors and on reaffirming and justifying appropriate responses and representations.

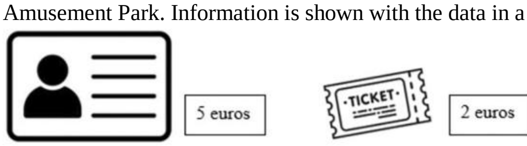
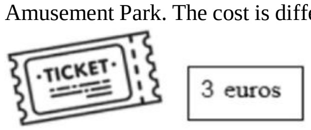
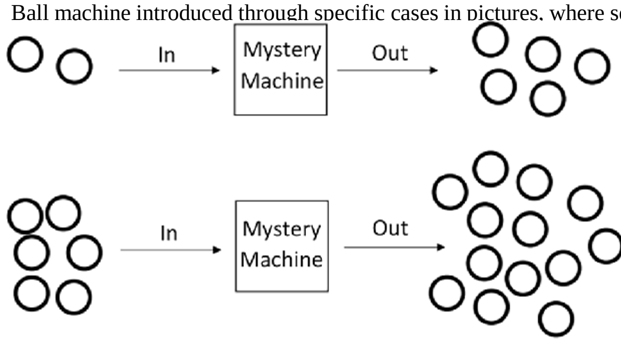
Given that the general goal of the four sessions was to work on different situations involving functions, in each session, we presented different tasks involving various representations—including graphical representation. Table 1 shows a summary of what we did in the four sessions for grades 3 and 4 (see electronic supplementary material for more detail).

In the inclusion of various representations, we followed Selling' progression (2016), going from lesser to greater sophistication. Initially, students worked with only one representation, and then went on to use more than one. Bearing in mind the complexity this might entail for the participants, we introduced tables and graphs last in each session.

Graphical representation was involved from session 1. The students were introduced graphs in the first session, after working on the values in a table (see Fig. 1). We gave them some instructions so that they could understand the meaning of the point, combining reading and construction in the instructions.

By pointing to the X and Y axes, we indicated which numbers correspond to the incoming and outgoing balls. In addition, he pointed on the screen the number 2 on the X-axis (balls going in) and the number 4 on the Y-axis (balls coming out). The researcher also indicated with the finger vertically and horizontally along the grid line to get to the point. Express 'up and to the side' (see Fig. 2). Then ask them to place the rest of the dots for the values on the table.

Table 1 Summary of the Sessions with the Large Group

Session	Function	Context
1	2x + 5	Amusement Park. Information is shown with the data in a drawing (generic example) 
2	3x	Amusement Park. The cost is different from above, as shown in the following picture (no admission cost) 
3	Comparison of functions (2x + 5 and 3x)	Comparison of Amusement Parks from previous sessions
4	2x + 1	Ball machine introduced through specific cases in pictures, where several cases are seen with a number of ... ays does the same trick and you need 

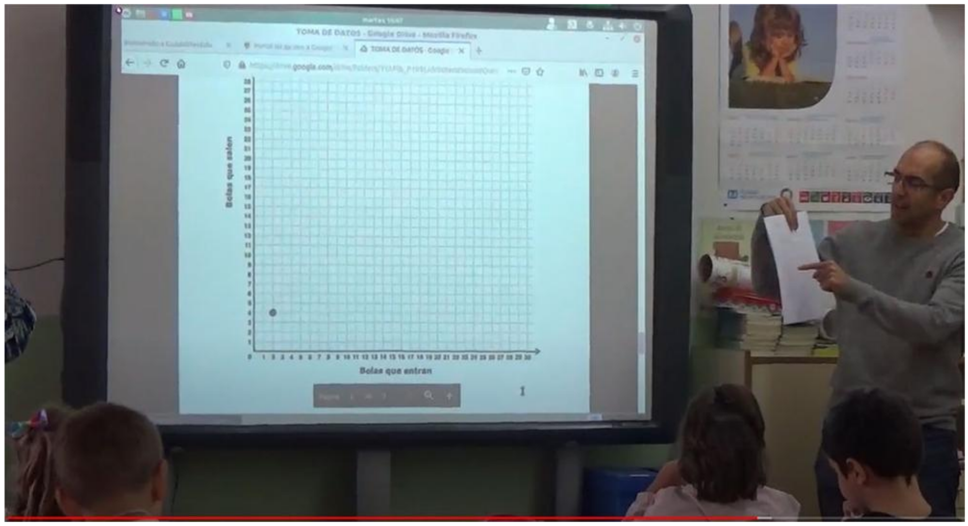


Fig. 1 Introduction to Graphs (First session, 3rd Grade)

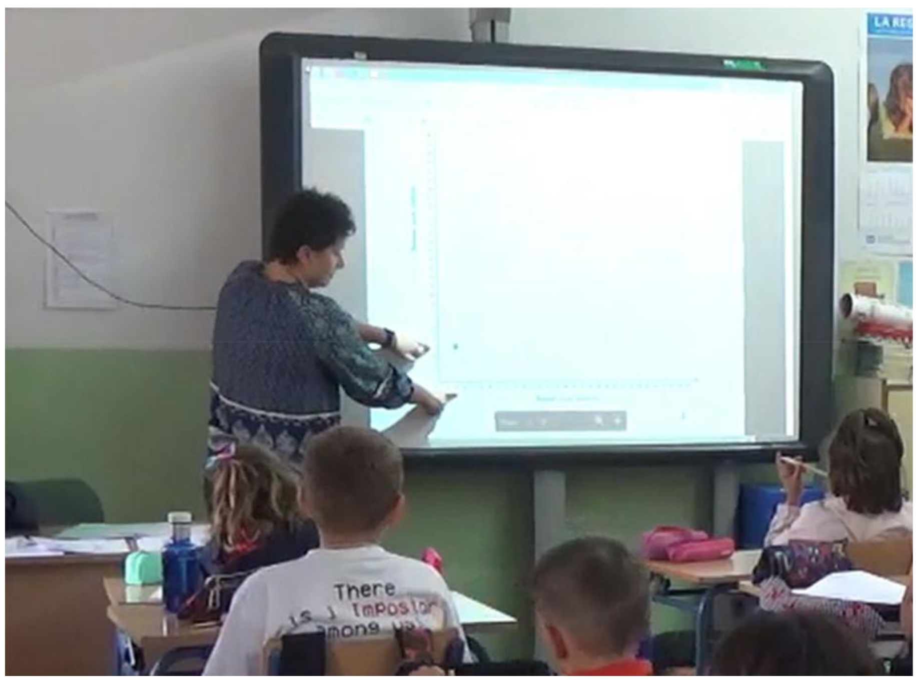


Fig. 2 Indication of the Values Related the axes

There were specific questions related to reading and construction of graphs, as we can observe in Table 2. In sessions 1, 2 and 3 we showed a graph with different dots on which we asked questions related to their reading. We also provided the axes and scale of a graph so they could mark a certain dot related to other cases.

At the end of session 4, we showed only a square grid and asked them to represent the values given on the table in the previous question. Table 2 shows the work that students were asked to do with graphs in each session.

As shown in Table 2, they were asked to construct a graph in the last session. However, we included questions on graphs started with dots that were already represented on them. Then, the questions were related to representing different dots from those represented on the graph. We then asked to compare between two functions represented on the same graph, calling their attention to the point of intersection and then finally proposed a task on the creation of the graph in which they had to involve several of the construction elements (see Table 2).

During the sessions, we guide students to read points on the graph because students were not familiar with this representation. We explained them the meaning of one point in the graph in relation to X and Y axes within the con- text of the variables involved in the problem posed. We are aware that they have the opportunity to observe graphs. The

Table 2 Summary of Work with Graphs in the Large Group

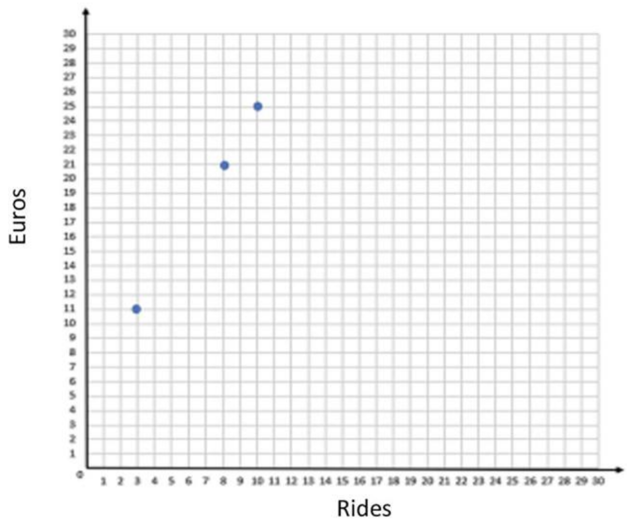
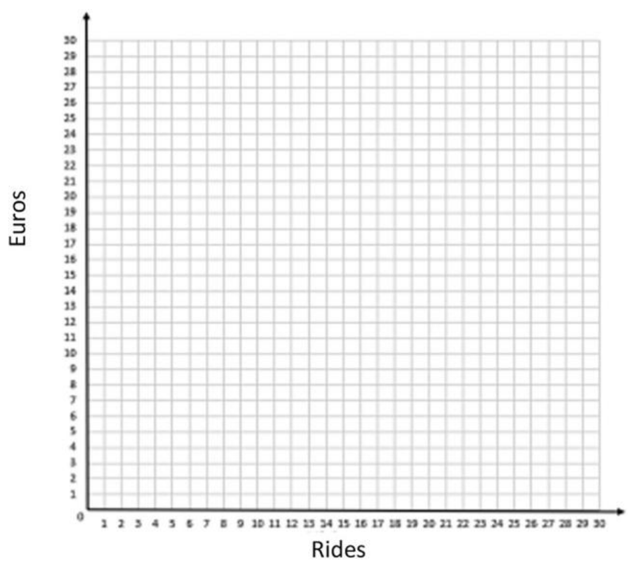
S. no.	Task proposed
1	8. On the graph there are some dots. Each dot is related to the number of rides and the euros paid  8a. If Sofía becomes a member and goes on 3 rides, will she pay 11 euros? 8b. How do you know? 8c. If Sofía becomes a member and goes on 8 rides, will she pay 21 euros? 8d. How do you know? 8e. If Sofía becomes a member and goes on 10 rides, will she pay 20 euros? 8f. How do you know?  9a. If Sofía becomes a member and goes on 4 rides, how much does she have to pay? 9b. Show on the previous graph what Sofía would pay in that case
2	5. On the graph there are some dots. Each dot relates a number of rides and the euros paid by Juan

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Table 2 (continued)

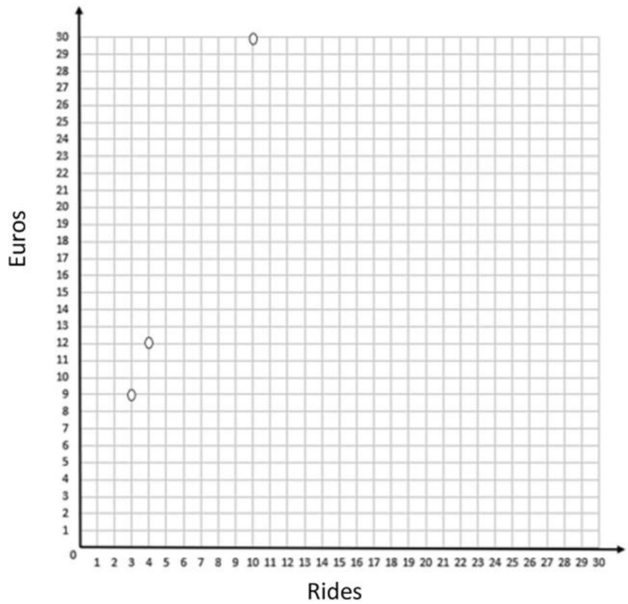
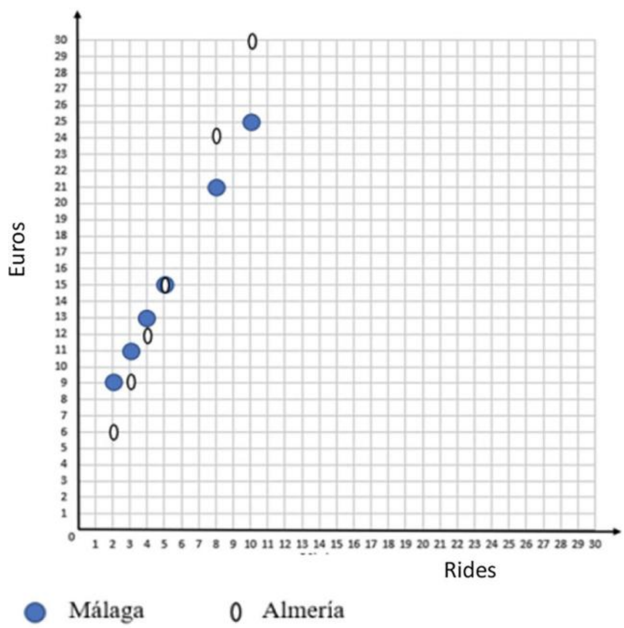
S. no.	Task proposed
	 5a. Look at the graph on the left. If Juan goes on 3 rides; will he pay 9 euros? How do you know? 5b. Look at the graph on the left. If Juan goes on 10 rides; will he pay 30 euros? How do you know? 5c. If Juan goes on 4 rides, will he pay 16 euros? How do you know? 5d. If Juan goes on 5 rides, how much will he pay? 5e. Mark on the graph when Juan goes on 5 rides
3	5. On the following graph there is information on some of the rides taken. Look at the graph and answer:  5a. Which amusement park is for the black dots? 5b. Which amusement park is more expensive if you go on 4 rides? 5c. Which amusement park is more expensive if you go on 5 rides? 5d. Which amusement park is more expensive if you go on 10 rides? 5e. Which amusement park do you think is more expensive if you go on 100 rides?

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Table 2 (continued)

S. no.	Task proposed
4	3 c Represent some numbers from the previous questions on a graph.

Note: S = Session

description associated with the "construction of graphs" involves students' individual performances in drawing the axes, situating the coordinates including scaling the involved quantities, labels, etc. Also, as we used similar tasks in the sessions and in the interview so the prompts indeed can involve some kind of guidance and, as a consequence, some kind of students' learning.

4.2 Participants

Two groups participated in the sessions, 26 from Grades 3 and 27 from Grade 4 (8–10 years old) in Spain. Concerning students' prior knowledge directly related to our research interests, they had never worked on contents related to functions before in their regular classes, previously to their participation in our research. They had solved some tasks related to numerical and geometrical increasing patterns the previous grade, their generalization and their representation, as part of the same project encompassing this study. The students had not worked with tables or graphs before this experience.

We used the sessions to introduce students to functional thinking through different representation, particularly graphs. Once the sessions finished, we selected 12, 6 from Grade 3, and 6 from Grade 4 to participate of individual interviews. We will here concentrate on those participat- ing in the interviews for two main reasons. First, they knew what a graph was in the context of functional thinking and had some experience with them from the sessions. Second, because the interviews allowed us to have more detailed information about the work they developed on graphs than in the sessions with the whole group.

We selected these students for the interviews based on their performance during the sessions and the advice or suggestion of each group's teacher. Performance was considered in terms of the generalization shown: (a) generalization of some kind from the beginning (some used symbolic representation, but this was not the main criterion) (b) did not generalize at first, but they did at the end of the sessions (in any representation), and (c) did not generalize at first, and continued having difficulties to generalize at the end or they were not able to. In addition, we needed students with a willingness to talk during the interview.

4.3 Interviews

We conducted individual, semi-structured interviews with the 12 students selected. Two members of the research team —who are also authors of this paper— were present. Each interview lasted around 30 min, and time was not a limitation for them.

The purpose of the interviews was to delve into the construction and use students made of visual representations.

We used the context of the ball machines—inspired by the idea of Dienes' (1971) "functions machines", which had also been used during the sessions with the whole group. First, we presented the situation involving the function 3x + 1. We showed students pictures representing a ball machine with different specific cases (see Fig. 3). We also had other cases

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of functions considered less complex because there are no constant or because they are more familiar with the notion of double ($y = 3 \times$ and $y = 2x + 2$).

The context of the function machine was difficult for some students, when the constant was not identified. Other contexts such as the amusement park where admission had a fixed price and every ride had a price, did distinguish these elements and were easier for most of the students during the sessions. We moved on to amusements park (see Fig. 4) taks context if we observed difficulties with the first one.

Following the same method as in the sessions, we guided them towards generalization by working with specific cases and various representations. In all cases, we presented the task with table and graph representations.

4.3.1 Worksheets for Interviews

We took three different worksheets to the interviews. Initially we gave them a squared piece of paper with nothing (Fig. 5, sheet 1). In the second one, we added the Cartesian axes (Fig. 3, sheet 2) and in the third one, the axes with some natural numbers represented (Fig. 3, sheet 3). We tried to use only the first sheet and would only offer the second or third one when students would not do anything with the previous one.

4.3.2 Interview protocol

We designed an interview protocol, and below is a fragment of it related to graphical representation.

Procedure: give children a blank sheet. Introduce graphs after having worked with several specific cases of a function. Then, ask them to write the information down on a piece of squared paper (sheet 1).

When starting on sheet 1, give them the possibility of doing different representations. If they do nothing, ask them to make a graph and wait. If they do nothing, provide them sheet 2. If they still do nothing, give them sheet 3.

This allows us to address several stages of graphic construction:

1. They build the graph axes.
2. They trace the number line on each axis.
3. They represent the numbers on scale.
4. They add titles to the axes.
5. They place the data correctly.

Similarly, we consider their reasoning using the graph to express the functional relationship.

6. They read, communicate or reason using the graph. How they relate the values? Are they able to generalize based on the graphical representation given by looking at the dot trend? This can be seen during and after the construction. Here we can ask: Can you add another different dot? How does the graph continue? We ask about another dot not represented and whether they think it would belong to the graph or not.
7. They communicate or reason using the graph.

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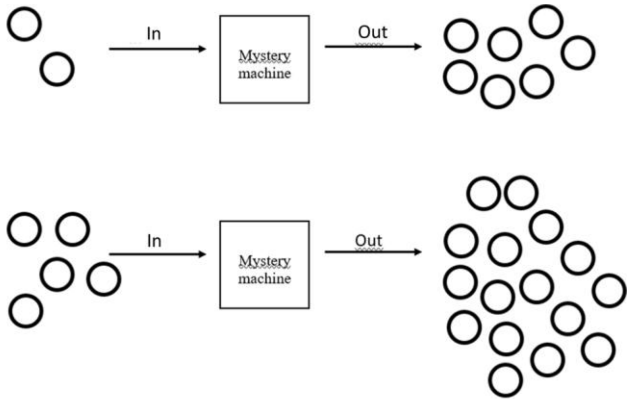

Fig. 3 Specific cases of the ball machine for the interviews (3x + 1)

Fig. 4 Generic amusement park example

4.4 Data analysis

We analyzed the recordings and students' written work from the interviews. We transcribed the former and scanned the latter. We performed a qualitative analysis focused on the following elements.

Concerning the construction of graphs, we followed the elements of Cartesian graphs described in the theoretical framework (Friel et al., 2001; Kosslyn, 1994), considering:

- Axes (L-shaped graph).
- Values on the axes.
- Scale of the axes.
- Labels on the axes.
- Values on the graph.
- Technique used.

We used these elements to describe students' work, as indicators in order to observe student's construction of graph.

To analyze students' readings of the graphs, we adapted the categories of Arteaga et al. (2021) and considering the type of questions: (a) read the data, (b) read within the data, and (c) read beyond the data.

In the results, we used an alphanumeric code beginning with 3 or 4 for the grade. Next, an S for "student", and a digit to identify them in the order in which we collected data.

5 Results and interpretation

This section provides the results and their interpretation related to the two study objectives.

5.1 Construction of graphs

Before asking students to make a graph, they had already worked with specific cases (see Fig. 3 and the interview protocol). This work was previously done in numerical representation, and they had represented data on a table (sometimes with help) and verbally with the general rule or even algebraic symbolism. Students were asked to trans- fer the information from the table or another representa- tion to the graph, but they were also asked to represent new specific cases on the graph or even the general trend.

5.1.1 Axes

Eight out of the twelve students interviewed (three from Grade 3 and five from Grade 4) built the graph without guidance to start tracing the axes. The other four students had to be given sheet 2 or 3 (see Fig. 5).

Below is a description of the characteristics concern- ing the construction of axes. Four Grade 3 students (3S1, 3S2, 3S3 and 3S6) and two from Grade 4 (4S2 and 4S6) exchanged the axes of the dependent and independent vari- ables, considering that the vertical axis was the depend- ent variable and the horizontal one was the independent variable.

For example, 4S6 traced the axes perpendicularly, assigned labels to them and correctly indicated their val- ues. Figure 6 shows he switched the axes, placing the val- ues of the balls coming out (independent variable) on the *x*-axis, and the balls going in (dependent variable) on the *y*-axis. This did not alter the graph reading.

We can observe in Fig. 6 that 4S6 represented dots out of the range of the numbers that he assigned to the axis. In the following fragment of the interview, we evidenced that the student perceived the regularity between consecutive dots and used it to construct the graph even for new data.

(1) I (interviewer): How are you placing the dots?
(2) 4S6: It would be for the 4 on the 1, the 7 on the 2? (He points with the pencil at the horizontal and vertical positions of the dots).
(3) I: OK, and if we continue putting dots, where on the graph should we put them?
(4) 4S6: Well, by looking at what is happening here (4S6 draws the dots taking into account the same distance

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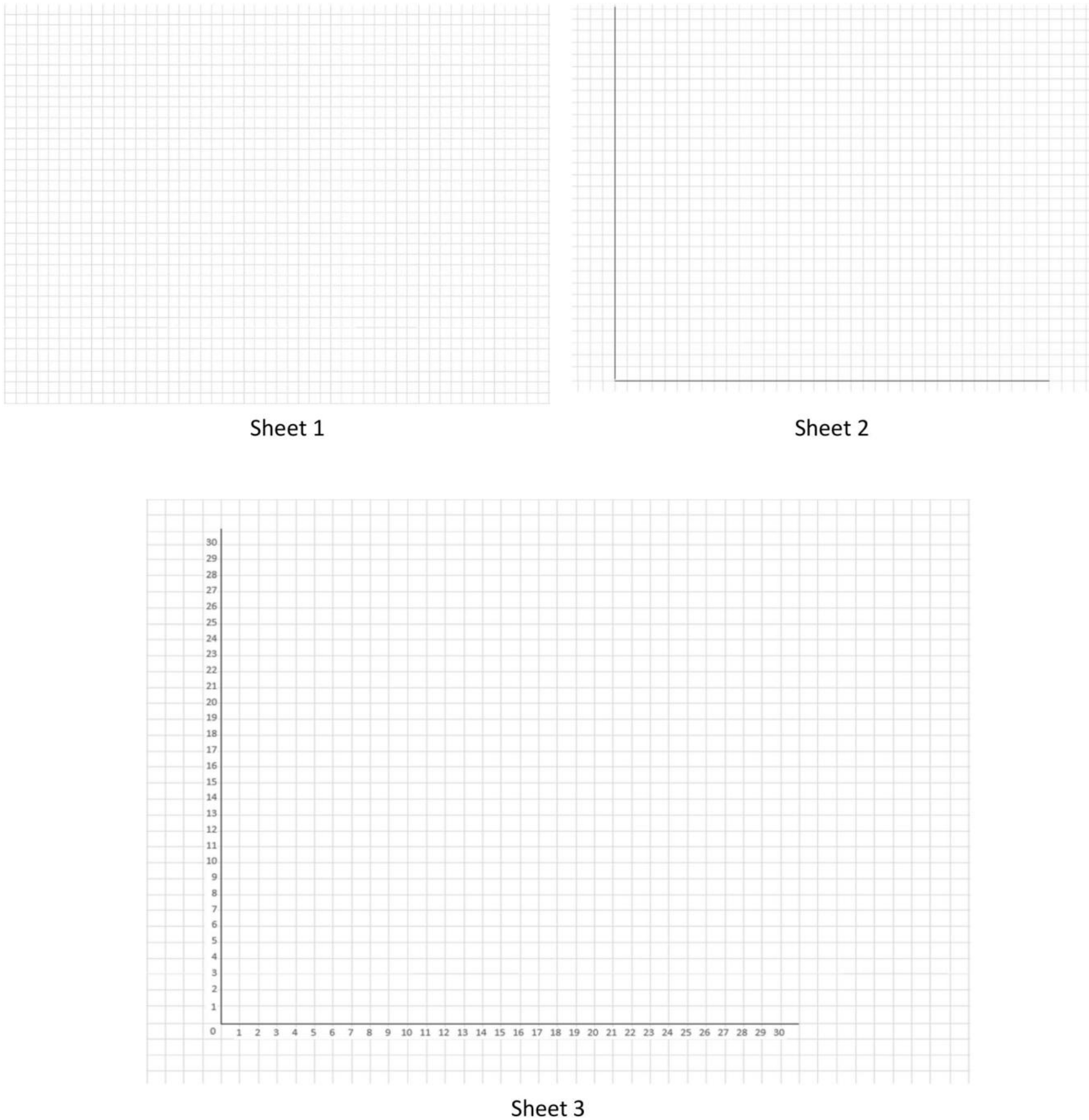

Fig. 5 Sheets Used during the Interviews

from one to the other until he reaches the end of the sheet, see Fig. 6).

3S3 traced the axes perpendicularly, respecting the scale, and placed the intersection of the axes as the ori- gin of reference and represented the natural numbers in the correct order on both axes. As shown in Fig. 7, this student, as well as switching the axes, inverted their posi- tion, by placing the x-axis on the top. This did not alter

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the reading of the graph or the interpretation of the data since the position of the axes is due to convention.

In the case of 3S2 (see Fig. 8), we found that, aside from switching the axes (she didn't have any labels assigned but during the interview she referred to the val- ues of both variables), she noted the values inside the Cartesian axes.

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5.1.2 Values on the axes

The origin of the axes was something difficult for three Grade 3 students and two Grade 4 students. That was the case with the graph by 4S5 (see Fig. 9). This student put 1 as the first value on both axes. However, on the *X*-axis he placed it at the beginning of the axis, while on the *Y*-axis he did not.

Two students —one from Grade 3 (3S2) and another one from Grade 4 (4S2)— did not initiate the two axes at 0. For example, 3S2 (see Fig. 7) started both axes at 1. In the case of 4S2, the first values for each variable is 1 but, also, the start of the axes is not shared (see Fig. 10).

4S4 considered different starts for both axes. He explicitly expressed that the x-axis began at 0, while the y-axis had its starting dot at 1 (see Fig. 11).

Other students, however, represented the axes and organized the data according to the natural order of the numbers on the line and with a common point for both exes. Yet similarly, they did not represent all the values of the variable, but rather limited themselves to the values that they were asked about and, at times, without considering a scale. For example, 3S5 limited the size of the axes to the largest value of the data in the Table 22 (see Fig. 12). He asked the researcher for help to place the maximum value on the sheet limit, saying, "I don't know how to do it. I want to stop at 22" (22 was the maximum value of the ordinate in the data used).

5.1.3 Scale of the axes

Below we describe the type of scale considered in the axes. Three Grade 3 students identified the axes with number lines (representing the natural numbers in order and respecting constant distances between the units, or referring to these distances). In Grade 4, four out of the six students did so.

For Grade 3, we present the case of 3S4. He transferred the data from the table to the graph maintaining both the values and the order in which they appeared on the table (see Fig. 13). He identified the Y axis with a number line but with a line in X axis, with no correct order of numbers and different intervals between them. We also found that 3S4 placed some values of the axis inside the graph.

Although on the graph by 3S4 the axes were properly traced, he did not relate the *X* axe with number line model. he marked the value of *X* (e.g., 4) and then sought to represent the equivalent of y (13). For the next pair of values, he represented the 3 after the 4 but to represent the 10 he did consider that it should be lower than the 13. This is how he represented all the pairs. While he did not take into account the order of the numbers on the x-axis, he did consider it on the y-axis. Therefore, it evidences that for this student, the *X*-axe are not representations of variables taking values on a numerical set.

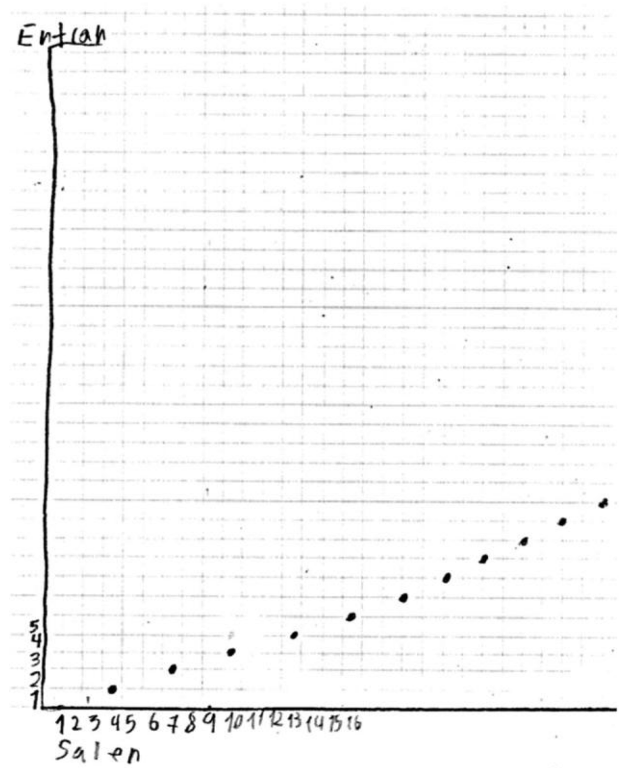

Fig. 6 Graph by 4S6. Entran = In; Salen = Out

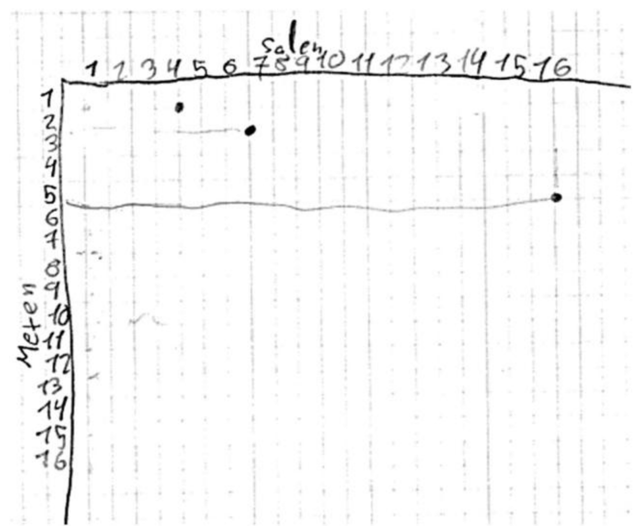

Fig. 7 Graph by 3S3. Meten = In; Salen = Out

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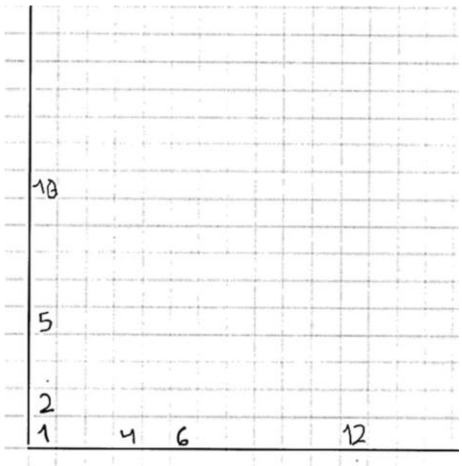


Fig. 8 Graph by 3S2

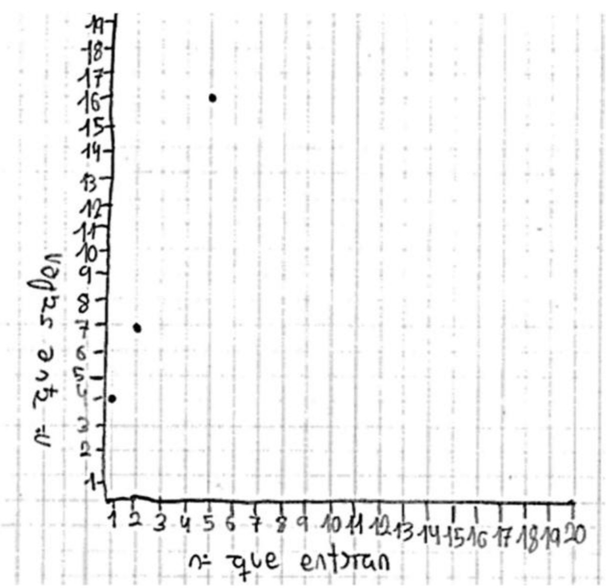


Fig. 9 Graph by 4S5. No que entran = N entering; No que salen = N. that come out (No que entran = N entering; No que salen = N. that come out)

5.1.4 Labels on the axes

Three students in Grade 3 labeled axes and all the students from Grade 4 did so. The labels showed that they were all aware that they had to summarize what the information shown in each axis meant, but each one expressed it in varying degrees of detail. Some students labeled it with the letters “E” and “S”, summarizing the Spanish for “In” and

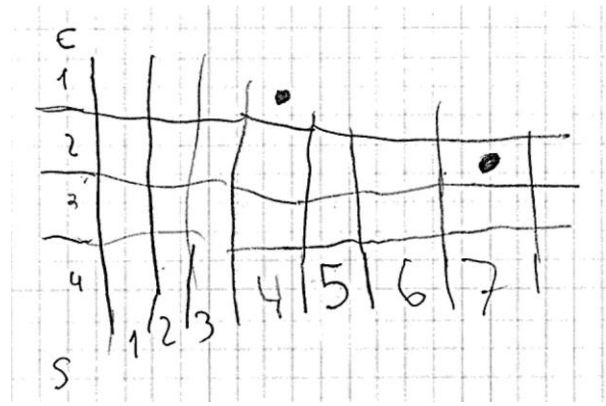
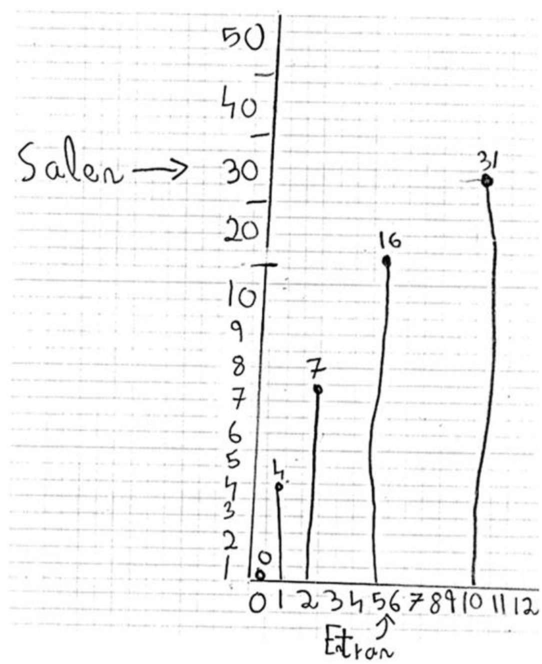


Fig. 10 Graph by 4S2



“Out” (see Fig. 10). Most of them wrote “In” and “Out”, as shown, for example in Figs. 11 and 12. Others

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Fig. 11 Graph proposed by 4S4. Salen = Out; Entran = In

were more specific and noted, “Number…”, or “Quantity…”, as shown in Fig. 9.

5.1.5 Construction techniques

We found that the students represented the variable values with different techniques. We present here some examples. 4S2 built a graph which, although the axes only represent the data available, is useful to represent the value pairs (see Fig. 10). We noted that although the technique may be useful to construct the graph, the independent variable values were inverted. This student represents the number

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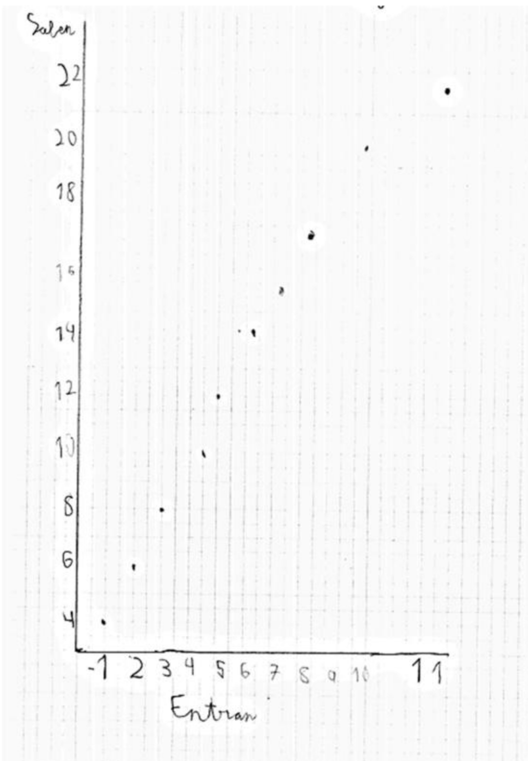


Fig. 12 Graph by 3S5. Salen = In; Entran = Out

of balls coming out with “S” and the ones going in with “E”, each one on its corresponding axis.

Although they had sheet 1, 4S3 highlighted the auxil- iary lines, as shown on Fig. 14. She also plotted the space as 4S2 did. 4S3 ordered the numbers on the axes correctly. We found a difference on where she placed the numbers in each axis. While on the *y*-axis the numbers appear to indi- cate a surface area, on the x-axis they appear to indicate a line (except for 1). But when it came to representing the dot, they did so as the intersection of the lines.

Others such as 4S4, drew a bar graph to help their con- struction and reading (Fig. 11). The student placed a bar from the indeterminate variable value, placed the ordinate value above the bar and finally put a dot. However, these students had not yet worked with them because in Spain they are introduced in 5th and 6th grades.

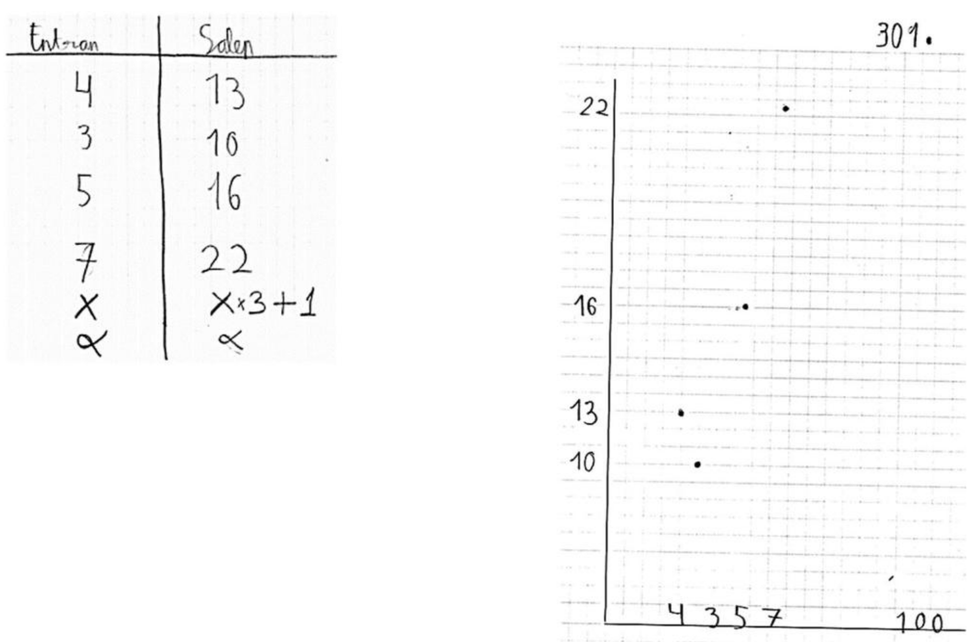
5.2 Reading of graphs

This is the literal reading of the elements in the graphs or the reading that went beyond what was on the graph.

5.2.1 Read the data

Some students found difficulties constructing graphs, but did read them correctly. They described what the dots meant taking into account the numerical values and the

Fig. 13 Table and graph by 3S4. Salen = Out; Entran = In



(a)

(b)

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graph axes. This was the case of 3S1 and 3S2. Below is a fragment of the interview with 3S1:

I: This dot, what is it? (Indicating the dot (2,7))
3S1: Well, two and seven" (Indicating with finger from the dot, the two on the x-axis and the seven on the y-axis).

This fragment of the interview evidences a direct reading of a point represented in the graph.

5.2.2 Reading beyond the data

Three students' (3S4, 3S5 y 4S6) read beyond the information shown on the graph. They perceived the rate of change to continue the graph beyond the data on the table.

When 4S6 finished representing the data on the graph, there were more dots than the ones on the table and the researcher asked him to explain what he had done. 4S6 replied, "It would be the 4 of 1, the 7 of 2… (pause). Wait, I've made a mistake. I put 11 and it should be 10 because three times three is nine plus one is ten". The moment the student realized there was a dot which did not comply with what he expected according to the functional relationship identified, there was a pause. The student tried to fit this dot into his hypothesis of the generalization rule (Torres et al., 2021). The researcher encouraged 4S6 to continue representing dots following his conjecture, and we can see the graph he built in Fig. 6. He prepared a graph conjecturing the trend of the functional relationship (see lines (1)–(4), below Fig. 6). This moment evidences also that the student had internalized the idea that the variable values belonged to a numerical set and that the relationship between domain and codomain comprised the functional relationship.

The interviewer asked 3S5 (see Fig. 12): "Can you tell me where we can place the dot corresponding to the entrance of 20 balls?" 3S5 replied while pointing to an area on the desk space, "It doesn't fit, we would need another page above it to be able to draw it". 3S5 took a piece of paper and placed it above the one with their graph, indicating he identified the trend followed by the given function.

3S4 expressed the trend with a gesture indicating the position which the remaining dots would have and also alluded to their going, "From here upwards along the diagonal" (see Fig. 15), referring to its similarity with the layout of the dots.

In contrast, other students located the dots taking into account only one variable. In the case of 3S1, who placed the dot (100, 301) increasing only the value of the independent variable, as shown in Fig. 16 (highlighting the tracing of the axis on the paper and that the student covered with her arm).

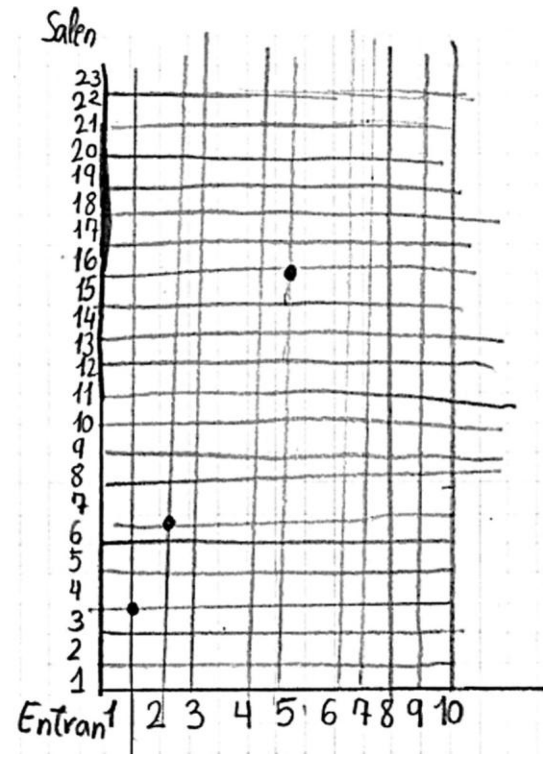

Fig. 14 Construction of a graph using auxiliary lines (4S3). Salen = Out/ Entran = In

3S1 evidenced the trend of the function $y = 3 \times$ by changing the function during the interview, asking them to read a graph already completed. This student evidenced the trend from another graph representing the relationship $y = 3x$. This is shown in Fig. 17.

3S1 did not identified the trend for the function $y = 3x + 1$ but she did with $y = 3x$. This fact could show that functions which involve multiplicative structure and an a independent term means greater difficulties for this students than those that involve only multiplicative structure with tables and graphs representations, as with other representations (e.g., Cañadas et al., 2016).

Once we gave student 3S2 sheet 3, with the axes and the values of the axes, she represented the variable values correctly. When asked whether it would make sense to have other dots in different areas inside the Cartesian axes (areas marked in red in Fig. 18, which the interviewer pointed out with her finger), she said it would. This proves that the student does not consider each value of the independent variable to correspond to a unique value of the dependent variable and that she does not identify the trend of the graph of that function.

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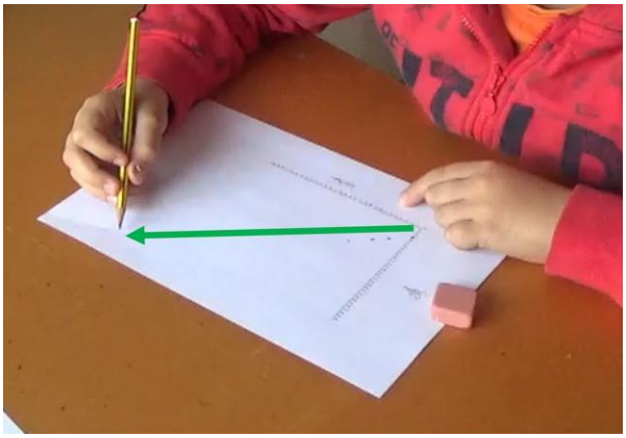


Fig. 15 Evidence of the trend indicated by 3S4

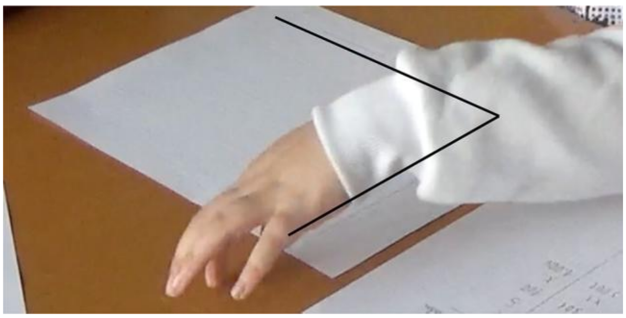
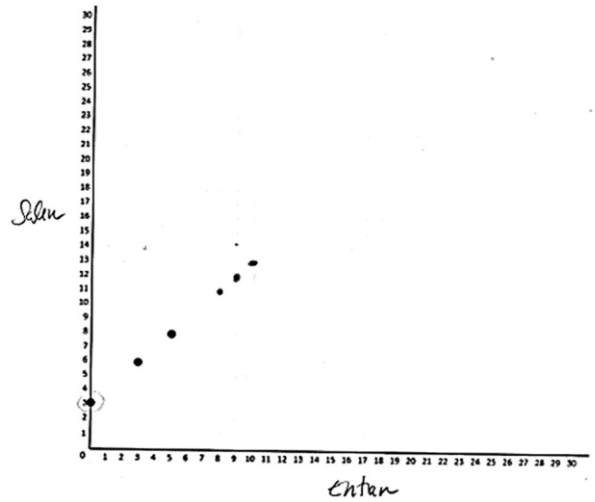


Fig. 16 Location of the dot (100, 301) according to 3S1

6 Discussion

Although most of the students plotted the axes of the graph by themselves from some of the data represented in the table, there were 4 students who did not do so. This is a skill related to graphs in general, not only related to functional thinking. However, not knowing how to represent basic elements such as axes limits their access to this representation of functional relationships.

In the sessions and in the interviews we followed the same order in the introduction of representations, moving from pictorial to numerical, to tabular and finally to graphs. During the sessions, we found some difficulties in the students regarding graphs and we tried to solve them during the debate. However, difficulties in the interviews and described in this paper were different. This may be because the work in the interviews was more autonomous. This may also be due to the fact that different students were involved in the discussion of the sessions, we could not identify some difficulties or solved them with their peers, which was not possible during the interviews.



It was not possible to find evidence about reading beyond the data during the sesssions, although we suspected that

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Fig. 17 Graph y = 3× shown to 3S1. Salen = Out; Entran = In

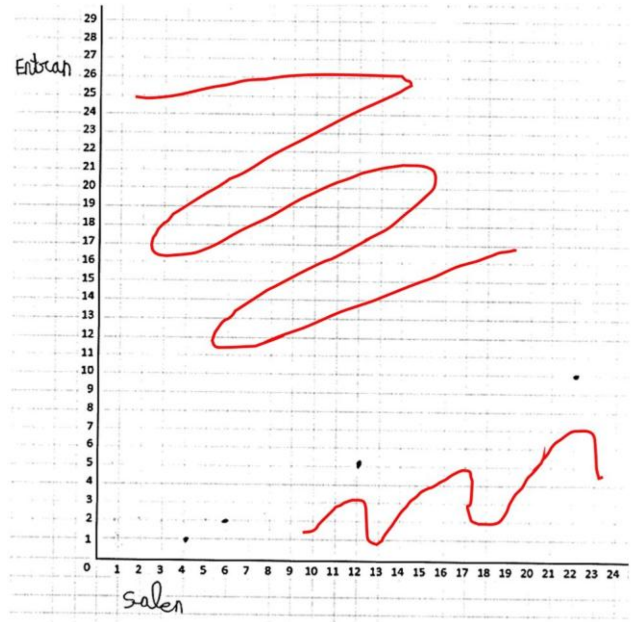

Fig. 18 Possible areas on the graph to locate dots by 3S2. Salen = Out/ Entran = In

some did. During the interviews we did identify those students who were able to do so, sometimes by gesturing or drawing on the graph.

Through the different examples presented, we found that considering the graph appropriate or not depends on the conventions followed by the mathematical community. These too should be made explicit when graphs are introduced. For example, student 3S3 did not use two of these conventions related to the position of the axes, but it could be another way of representing data.

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Concerning the strategy used in the construction, we observed the inclusion vertical lines following the researcher's instruction in the sessions to get to the point. They also draw the grid with horizontal and vertical lines, probably guided by the work in the sessions in relation to 'up and to the side'.

7 Conclusions

This study constitutes a contribution in the field concerning graphs in the context of early algebra. We had found no prior studies addressing this representation related to functions at these ages, in spite of its known advantages given it is a usual visual representation when studying algebra and functions.

Martí et al. (2010) pointed out that the level of understanding of a simple graph is very superficial at the end of primary education and beginning of Secondary and that it barely advances in the following years. Therefore, the identification of different levels of reading and construction of graphs could be a starting point to design specific tasks that help advance the understanding of graphs.

We focused on the construction and reading of graphs by students. This provides information on the graph characteristics which students have focused on and those which are not so evident for them, or which have not caught their attention. The results presented here provide grounds for analyzing how they use this representation to better understand the mathematical idea underlying the problem.

Concerning the first specific research objective, about the construction of graphs associated to functions, we evidence a series of conventions and implicit characteristics which should be taken into account in the teaching of graphs. Although the graph examples help everyone to have an idea of what a Cartesian graph is like, there is information that is usually not given, assuming the students will learn it.

The authors of this paper, as researchers and teachers of mathematics education, have become aware that there is much more work to be done in this area than expected based on our teaching experience and the literature review conducted.

From the perspective of the construction of graphs, we began with mathematical elements considered in prior studies and identified some of the indicators of the same which have allowed us to advance in our research objectives. We identified the axes, scales, values and labels of the axes; the location of the values on a graph and the construction technique of the graph, as elements which help describe students' work. These elements have also helped us describe some of the difficulties students have when building graphs. Notably, location of the axes, relative position of the two axes, start of the scales for the two axes, recognition of a

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number line on each one of the axes, representation according to the order of the natural numbers on each one of the axes, the common start of both axes, where the values of the axes are placed (inside or outside the graph), where the value starts on each axis, whether they have a maximum value, are some of these elements. How to identify the space where each dot is represented was also a turning point in the students' work.

With regard to the second specific research objective, which refers to the reading of graphs associated to functions involving natural numbers, we have noted two levels of understanding. A first level which requires extracting data directly present in the graph (direct or literal reading), and a second level (reading beyond the data) which requires an overall interpretation of the graph trends related to the generalization of that being represented. The generalisation offered from a graphical representation goes through the evidence of a tendency of dots obtained from specific reading and construction guidelines and procedures. At these levels, we observed how students elaborate strategies to place the dots forgetting the context of the task. With graphic representation, they move on to a more abstract context, in which the meaning of the initial task can sometimes be forgotten.

When reading the graph, they all did so without any difficulty, although not all showed they had understood the idea that each value on the *x* variable corresponds to a single value on the *y* variable or that the graph has a trend. These elements not only help us describe the students' work, but they should be taken into account when graphs are taught to children of these ages.

Students have presented some difficulties that can be due to the fact that graphical representations have not been included in ordinary classroom by the age of these students in Spain. We propose graphical representation as an interesting and complementary way of working functional thinking at primary education. It would help students to visualize relationships and support them in the generalization proccess.

We noted some limitations in our study. The number of students and the number of tasks were reduced. It would be good to work with more students with different profiles, and also to include tasks with differente characteristics to explore ways for students to expand their functional thinking involving graphs. This work could contribute to complement a hypothetical learning progression (Stephens et al., 2017) to incorporate graph representation in functional thinking.

Students had some difficulties with graph construction, but some of them did read them correctly. This could be due to the fact that the students had not been previously taught or practiced with graphs in a planed way considering different elements. This is significant information regarding the elements to be taken into account when working with graphs in primary education classrooms. We should also analyze

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the teaching of elements directly related to graphs, namely the following:

- *Axes*. Where they are located and how. Where the independent and the dependent variables go.
- *Data on the axes*. Order of the numbers, scale of the two axes and the presence of 0.
- Area of space where the value pairs should be represented (intersection of lines).
- *Labels on the axes*. Meaning of the values on each axis.

The understanding of these elements rather than the procedural undertaking (where to put what), may be a precondition to access a deeper level of interpretation of graphs.

The use of graphical representation in tasks involving functions at primary education can help students to visualise the behaviour of functions and, with them, promote knowledge of this mathematical notion. It is proposed to do so starting with the representation or reading of points (particular cases) on the graph, until reaching the trend of the function in that representation. Furthermore, the handling of this representation can be useful for other mathematical and scientific fields, where graphs are common at higher educational levels.

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