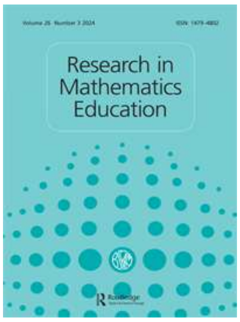

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Research in Mathematics Education

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/rrme20

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RESEARCH IN MATHEMATICS EDUCATION
https://doi.org/10.1080/14794802.2024.2344211

Growing patterns invention by primary education students

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ABSTRACT

This paper addresses the creation of patterns through an open-ended invention task. With a qualitative study, we aimed to characterise the written answers of 76 primary education students in fourth, fifth and sixth grades, identifying the type of pattern they invented. We analysed the representations they used and the structure followed in the sequences they created. In addition, we determined whether there were differences in the invention of patterns among the three school grades in the sample. Results showed that students prefer creating growing patterns using numerical and visual representations. They generally involved a numerical structure more frequently than a geometric one, using an increasing arithmetic progression. They usually formed the pattern by involving a geometric structure, building a figure based on an initial element to which they added more elements to complete a particular figure. In fourth grade, numerical representations were more common than pictorial ones, while in sixth grade, the opposite was true. As age increased, so did the frequency with which students invented a pattern. Results also provided information on the complexity of the types of geometric patterns invented.

ARTICLE HISTORY
Received 1 April 2023
Accepted 13 April 2024

KEYWORDS
Structure; pattern invention; representation

1. Introduction

Several education curricula consider work with algebraic patterns in preschool and primary education as a way to access school algebra (e.g. Ministerio de Educación y Formación Profesional, 2022; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; Ontario Ministry of Education and Training, 2020). There are various reasons for promoting them. Castro-Rodriguez and Castro (2016) pointed out that patterns allow generalisation; they help generate mathematical models and solve problems while establishing the bases to develop algebraic skills. Rittle-Johnson et al. (2017) and Wijns et al. (2021) remarked that patterns provide opportunities to explore regularities and grasp structures, which helps develop numerical thinking.

Working with patterns includes a wide range of skills that can be measured with various tasks and types of patterns. In previous studies, the most common tasks were

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based on activities that show the first terms of a pattern and then ask the resolvers to copy, expand, complete, describe or generalise the pattern (e.g. Fujita & Yamamoto, 2011; Lüken, 2012; Morales et al., 2017; Papic et al., 2011; Rivera, 2013; Wijns et al., 2021). Dörfler (2008) noted that, primarily, this type of task provides strong guidance; the figurative clues and their layout practically exclude trying out other generalisations or continuations; furthermore, they show and describe a previously prescribed general structure, therefore implying there is only one way of continuing the sequence. Accordingly, students solve the tasks by looking for a single, closed answer. He identified the study of "free" generalisation tasks as an open line of research in which generalities can be invented or built.

Overall, literature has highlighted the role of the invention as an opportunity to promote flexible thinking and lead to a deeper understanding of mathematical contents (Baumanns & Rott, 2022). Within the context of the development of algebraic thinking, more studies are focusing on the invention of algebraic problems (e.g. Cai & Hwang, 2020; Cañadas et al., 2018; Fernández-Millán & Molina, 2017) than on the invention of patterns (e.g. Rivera & Rossi-Becker, 2016). Regarding the invention of problems, we have found that this type of task is associated with high cognitive demand. Whoever invents problems must reflect upon the situation's structure more than on the process of solving the problem (Cai et al., 2013). In the case of pattern invention, Rivera and Rossi-Becker (2016) conducted a study with secondary education students (aged 12.5), which concluded that the students could develop well-defined mathematical structures and relate generalisation with visual aspects of the pattern. These researchers indicated that a line of further research would be to work on patterns based on open-ended tasks, making the activity with patterns more interesting and less predictable.

This paper presents a contribution to the open line of research as described by Dörfler (2008) and Rivera and Rossi-Becker (2016). The study analysed responses to an open-ended task focused on inventing patterns in free situations, where students were given no restrictions for inventing. This choice of task was made because it allowed for the examination of students' spontaneous responses and the observation of how they expressed their skills in recognising patterns and mathematical structures. The information gathered from this study will be helpful for teachers seeking to understand better and support the development of structural sense in tasks involving patterns (Lüken, 2012).

2. Objectives of the study

This paper aimed to characterise primary education students' answers when conducting a pattern invention task. The research questions were the following:

- What are the representations that students use when inventing a pattern?
- What are the structures than students use when inventing pattern? Are there any differences in the invented patterns created by school grade?

We are interested in the analysis of representations because they play a cruci al role in the learning of mathematics; on the one hand, they allow us to understand how people know and understand mathematical objects and, on the other hand, they allow us to make them present (Ayala-Altamirano & Molina, 2021). Using multiple representations flexibly promotes a deeper understanding of mathematics (Blanton & Kaput, 2011). Moreover, different representations highlight different aspects of the situation, leading to a more comprehensive understanding of the mathematical concept (Blanton, 2017).

Observing how people assign meanings to mathematical structures and use them is possible through representations. In studying patterns, it is crucial to consider structure as a key aspect. Understanding the relationships between arithmetic or algebraic structures and their geometric dependence on the patterns depicted in a pictorial representation allows us to examine students' comprehension (Outherd & Mitchelmore, 2000).

We highlighted the importance of inventing patterns as a resource to observe how students understand mathematics. From the teachers' perspective, the invention is a way to evaluate students' conception of a particular topic (Cai & Hwang, 2020; Fernández-Millán & Molina, 2017), and it allows students' skills to apply mathematical knowledge (Cañadas et al., 2018). Characterising the type of patterns invented could provide information for teaching patterns by attending to both the representations that have been closest to them and the structures with which they identify pattern creation. Moreover, studying the differences between the school years could help to establish indicators to design tasks longitudinally throughout primary education. Research is needed to determine whether and when experiences with pattern invention work can help strengthen students' ability to successfully tackle grade-level appropriate geometric and numerical pattern tasks (Rivera & Rossi-Becker, 2016). A related concern is to determine the extent to which different types of patterns come naturally to children through pattern invention.

3. Patterns in primary education

This section describes the conceptual framework upon which the study is based and the background leading to characterising patterns in primary education.

3.1. Characterisation of patterns

Establishing a precise and agreed definition of a pattern is complicated. However, there is a consensus on two key characteristics: regularity and predictability (McGarvey, 2012; Wijns et al., 2019). A *pattern* is a sequence showing a regular and replicable repetition of objects, numbers, sounds, movements or shapes (Castro-Rodriguez & Castro, 2016; Papic et al., 2011). Regularity, the way elements are arranged and related in a sequence, is termed the *pattern structure* (Mulligan & Mitchelmore, 2009). The structure involved is predictable and entails logical, numerical and spatial relations (Lüken, 2012; Mulligan & Mitchelmore, 2009).

In this study, we focused on *growing patterns*. In this type of pattern, every element that is a part of the sequence is associated with a quantity that increases or decreases systematically according to a set rule, thus expanding or reducing the initial element (Castro-Rodriguez & Castro, 2016; Wijns et al., 2021).

Previous studies (e.g. Pasnak, 2017; Wijns et al., 2021) concluded that growing patterns were more complex than other patterns, such as those of repetition. One reason is that the structure of growing patterns must be inferred by relating two consecutive terms, which requires numerical skills such as counting or arithmetic calculation.

3.2. Growing patterns and their representation

When focusing on how growing patterns are usually represented, three types can be identified: (a) patterns with only numerical representation, (b) patterns with only pictorial representation and (c) patterns with numerical and pictorial representation. The following sections delve into characterising each of these three forms of representation.

3.2.1. Patterns with only numerical representation

Patterns represented only with numbers are related to a more abstract approach. Figure 1 shows a task proposed in a 4th grade textbook (Hale, 2005). It shows only a sequence whose terms are represented only by numbers.

Sometimes, a pattern is accompanied by another variable that can be a numerical representation of the position of each term in a sequence or a different variable in a given context. This type of pattern representation enables the correlation of two data sets and is the initial step in understanding the concept of function. For example, Figure 2 illustrates a task where the pattern representing the amount of money saved (depicted in the second row of the table) needs to be completed, with the first row showing the number of days elapsed.

3.2.2. Patterns with only pictorial representation

Orton et al. (1999) suggest that using pictorial representations instead of number lists can simplify problem-solving by providing a more geometric approach and adding meaning to the task. However, when using visual aids in primary school tasks, it is important to consider whether the pictorial representation is associated with the spatial organisation of elements or not. For instance, in Figure 3, although the sequence is represented pictorially, the objects are not spatially organised, making it impossible to predict the next object in the sequence. On the other hand, Figure 4 displays a pictorial representation with a clear pattern, allowing one to predict the next object in the sequence.

Find the pattern. Use the pattern to find the next three numbers in the sequence. Circle either **growing** or **decreasing** to indicate the type of pattern. Then use words to describe how to find the next number in the pattern.

1. 15 26 37 48 ____ ____ ____ **growing** **decreasing**

Pattern description: ______________________________

Figure 1. Example of a pattern task with numerical representation (Hale, 2005, p. 16).

2. Make a table showing how much money Jeremy will make over the first 15 days if he takes his mother's offer.

# of days	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
total $ earned	2	4	6												

Figure 2. Example of a pattern task with double numerical representation (Hale, 2005, p. 100).

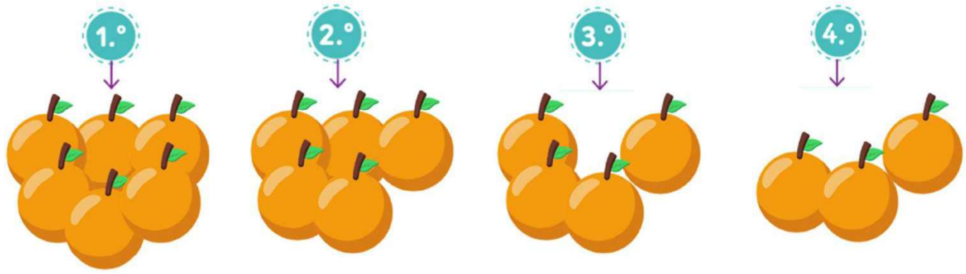


Figure 3. Example of patterns with pictorial representation without spatial organisation and numerical representation of position (Image credit: www.freepik.es).

Arcavi (2003) pointed out that pictorial representations should not only be for illustrative purposes but also be acknowledged as crucial components of reasoning, problem-solving or even proof. According to this author, visualisation favours implementing visual strategies that help students understand the invisible in abstract situations; it can sharpen our understanding or allow us to ask questions that would not be answered by merely working with more abstract situations.

3.2.3. Patterns with numerical and pictorial representation

As we discussed in the previous section, patterns are often represented pictorially. However, when a numerical representation accompanies a pictorial representation, that numerical representation can serve a variety of purposes. For example, it might indicate the number of elements that make up each term, or it might represent some other variable, such as the position of the term in the sequence.

To illustrate this concept, consider Figure 4. This figure shows the number of oranges being picked one at a time, without any spatial organisation. The numerical representation, in this case, indicates the order in which the oranges are being picked.

In Figure 4, the pictorial representation follows the spatial organisation, and numbers are associated with the quantity of elements comprising each term. The patterns with numerical and pictorial representation most frequently found in textbooks or studies are those in which the pictorial representation has a spatial organisation, and the numbers represent the stage in the sequence of each term (e.g. Hale, 2005; Radford, 2010).

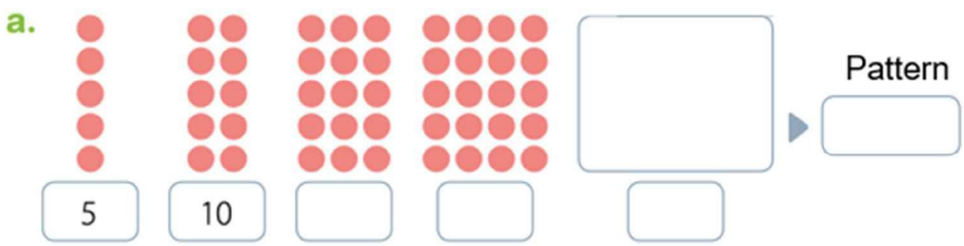


Figure 4. Example of patterns with pictorial representation with spatial organisation and numerical representation of the quantity of elements in each term (Ubilla & Cerda, 2020, p. 157).

In the introduction of patterns with numerical and pictorial representation, it is important to consider that students do not spontaneously relate these two aspects. Radford (2011) showed in a longitudinal study that second grade students first focused on the numerical aspects and ignored the spatial organisation in the visual representation of each term. Considering both aspects, they reproduced and continued the pattern with the teacher's support. Then, from third to fourth grade, students considered numerical and spatial aspects from the beginning.

3.3. Growing patterns and their structure

As pointed out earlier, the structure of *growing patterns* is related to a rule that systematically increases or decreases a pattern. If we focus on the numerical aspects involved in the pattern, then our interest will lie in its numerical structure. If we focus on the spatial aspects of the pattern, our interest will lie in the geometric structure.

3.3.1. Numerical structure of growing patterns

There are different types of growing patterns when focusing on the relationship between two consecutive terms, for example, (a) *arithmetic progression*, whose difference between two consecutive terms will always be the same; (b) *geometric progression,* where the ratio between a term and the previous one is always the same; (c) *Fibonacci progression* where each term (starting from the third one) is found by adding the two previous terms (or applying another mathematical operation to the two previous terms); and (d) *quadratic sequence* where the rule to find its elements follows a quadratic expression.

In all the types of growing patterns mentioned above, there are two types of structures: (a) recursive structure, which allows predicting the next term of the sequence, and (b) functional structure, which allows predicting any element in the sequence. The recursive structure of the sequence shown in Figure 4 is to add five at a time, while its functional structure is 5n, that is, multiplying the position of the term by five.

3.3.2. Geometric structure of growing patterns

We can establish two sub-categories of patterns with pictorial representation when noting how individual elements are arranged in a two-dimensional space. From this perspective, first, we refer to patterns arranged not according to a geometric structure and, further to Arcavi's ideas (2003), this type of representation can be illustrative and accessory (see Figure 3).

Second, we refer to the patterns following a structure based on the layout of each element called *geometric growing patterns* (Warren & Cooper, 2008; Wijns et al., 2019). In this case, the structure given with a pictorial representation can help students notice the relations due to their concrete and visual nature (Wilkie, 2022). Rivera and Rossi-Becker (2016) found that representations involving more than one stage in a single image imply a much higher level of difficulty (see Figure 5A) compared to those representing stages independently (see Figure 5B).

Visualising and identifying structures to help students make abstractions and generalisations has been key to developing their algebraic reasoning (Hunter & Miller, 2022). According to Warren and Cooper (2008), geometric growing patterns should be included in classrooms because (i) they are a visual representation of numerical patterns, (ii) they

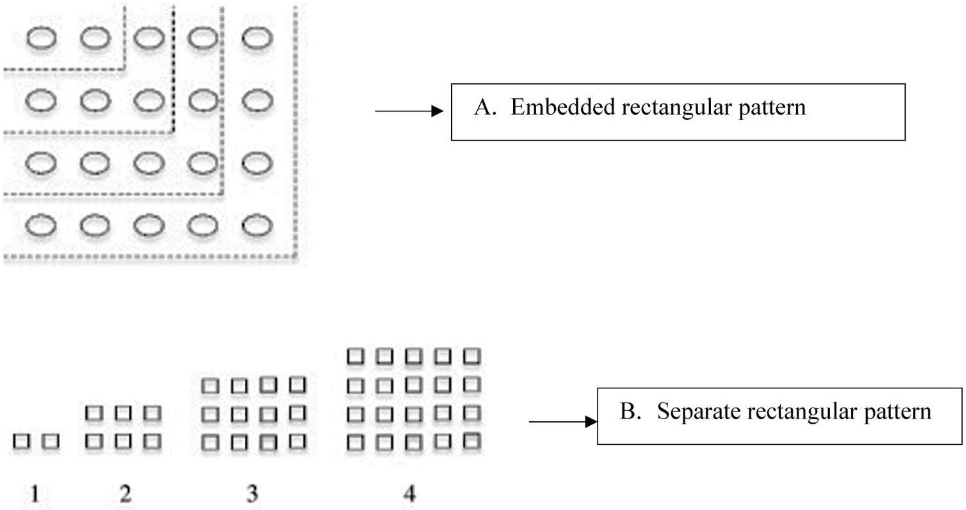


Figure 5. Example of a geometric growing pattern (Rivera & Rossi-Becker, 2016).

can be used as an informal introduction to the concept of variable, and (iii) they can be used to generate equivalent expressions. These studies reached these conclusions by presenting students with pictorial representation patterns and analysing the strategies to perceive their structures.

3.4. Pattern invention tasks

Literature has highlighted the role of the invention as an opportunity to promote flexible thinking and lead to a deeper understanding of mathematical contents (Baumanns & Rott, 2022). However, it is essential to consider that there is not only one type of invention task, based on the ideas of Stoyanova (2000), who referred to problem-solving, distinguishing three categories of invention tasks: (a) free situations, (b) semi-structured situations, and (c) structured situations. In the first one, students have no restrictions to invent, while in the semi-structured ones, they are asked to consider patterns based on some experience or in contexts expressed with pictures or texts. Finally, structured situations are those which reformulate the patterns given or change their condition.

As mentioned above, in research on algebraic development, there are more studies on the invention of algebraic problems related to an equation (e.g. Cai & Hwang, 2020; Cañadas et al., 2018; Fernández-Millán & Molina, 2017). One of the few studies focusing on the invention of patterns was conducted by Rivera and Rossi-Becker (2016). They offered secondary education students (aged 12.5) semi-structured activities called semi-free patterning. In this type of task, the stages in a pattern are ambiguous, given that they show two terms of the pattern and the sequence can be continued in different ways. Therefore, there is not only one answer. Its objective is to study the process of building and justifying structures. They concluded that open tasks encourage students to develop well-defined mathematical structures. Furthermore, they noted that students related visual structures (interpreted as whole or partial configurations) with formulas generalising the numerical structure of the pattern. We aimed to broaden the findings of the work conducted by Rivera and Rossi-Becker (2016) by analysing the answers of fourth, fifth, and sixth-grade primary education students (aged 10–12), who were presented with a free invention task without being shown the terms of the pattern.

4. Method

We present a qualitative, exploratory and descriptive study to characterise the answers of a group of students when inventing patterns.

4.1. Participants

In total, we worked with a sample of 76 students from three groups of different levels of primary education in a school in Granada (Spain): 25 fourth grade students (aged nine–10), 25 fifth grade students (aged 10–11) and 27 sixth grade students (aged 11–12). Regarding the students' prior knowledge directly related to our study objective, we found that they had not worked on algebraic contents (patterns, generalisation, among others) in their mathematics lessons, nor had they worked on invention tasks (open tasks) nor used different representations to express relations.

This research conforms to the ethical guidelines, including compliance with the legal requirements of the country of study (Spain). The subjects participating in this study have given their informed consent to allow transcriptions. In the case of children, parents have signed permission for the sessions and application of questionnaires.

4.2. Design and implementation of the work sessions

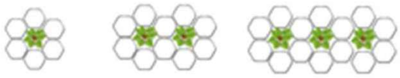
Four lesson sessions were conducted for each group, lasting between 45 and 60 min. The sessions were conducted once a week, and a research team member fulfilled the role of teacher-investigator. The sessions were video recorded.

Each session started with an introduction of the context to be worked on, which was delved into through questionnaires which the students completed individually. Given that the general objective of the four sessions was to work on different situations involving growing patterns, at each one, we presented different types of tasks: copy, expand, complete, describe and generalise. Only the last session included pattern-inventing tasks. Below is the information related to the contexts and patterns involved in the tasks before the invention task was analysed for each school grade: fourth, fifth, and sixth.

4.2.1. Description of the questionnaires

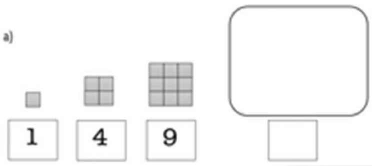
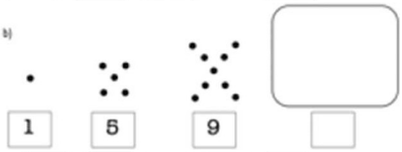
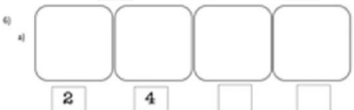
In the three grades, we began by presenting situations related to everyday contexts, such as forming playgroups in the case of the 4th grade (sessions 1 and 2), the design of a green area in the school in 5th and 6th grade (sessions 1 and 2), the layout of tables and chairs, for all grades (session 3). The patterns of the first three sessions followed an arithmetic progression. Table 1 shows the name of each context and the structures involved. Following the ideas of Wijns et al. (2019), the questionnaires of the first three sessions presented tasks with varying degrees of difficulty in the following order: copy, expand, extrapolate, recognise and generalise.

Table 1. Contexts of the sessions and questionnaires given.

Context	Numerical structure	Geometric structure	Session 1	Session 2	Session 3
Soccer match	+5 y = 5n		4th	4th	
Tiles around a flowerpot	+4 y = 4n + 2		5th 6th	5th 6th	
Guests seated around a table	+3 y = 3n + 2				4th 5th 6th

In Session 4 for all grades, we administered a questionnaire with a mathematical context focusing on the task had numerical and pictorial representations, specifically geometric growing patterns. We tried to present different mathematical structures and geometric representations. Although in the previous sessions, we only presented patterns which followed an arithmetic progression, in the fourth session, we included a pattern with a quadratic progression and another with a Fibonacci progression, as some students referred to these during the classroom discussion. Table 2 describes the tasks presented.

Table 2. Questionnaire questions Session 4.

	Task proposed	Numerical structure	Spatial structure	Task objective
1		Quadratic progression $y = n^2$	Yes	Expand and generalise
2		Arithmetic progression +4 y = 4n − 3	Yes	Expand and generalise
3		Arithmetic progression +3 y = 3n + 1	No	Expand and generalise
4		Fibonacci progression	No	Expand and generalise
5			Yes	Invent (Semi-free) Expand and generalise
6			No	Invent (Semi-free) Expand and generalise
7	Invent a sequence following any pattern you wish.			Invent (free)

4.2.2. Description of the pattern invention open task

This study analysed the pattern invention open task, the seventh task shown in Table 2. This task elicited the free invention of a pattern without including any supporting representation. We thus looked at the skill to invent. Although the task was open and could provide various answers, only one was requested. Neither did we indicate which method to follow, so it was also an open-method task (Yeo, 2017).

4.3. Data analysis

We qualitatively analysed the written answers of all students in the pattern invention task following the process shown in Figure 6. The categories come from a deductive-inductive process derived from literature and our interpretation of the data.

First, we focused on the representations used, following our frame of reference; they could be (a) only numerical, (b) only pictorial, or (c) numerical and pictorial. In the case of numerical representations, we also identified the role of the numbers involved: (a) represented the terms of the sequence; (b) represented the position of each term within the sequence; (c) represented another variable.

We then analysed the answers, identifying whether they involved a structure or not. To identify a numerical structure, we analysed each term of the sequences represented both numerically and pictorially and noted whether there was an underlying rule to describe the systematic increase or decrease of terms. Students sometimes explained the rules they followed, and this information complemented our analyses. If no explicit or implicit rule was detected to explain the increase or decrease of terms, we considered no numerical structure was identified. Therefore, no numerical pattern could be observed.

The geometric structure was found in those answers represented pictorially. In this case, we analysed how the elements were arranged in the two-dimensional space. If they were only decorative or illustrative pictures, these representations were classified as a sequence without a geometric structure. Therefore, no geometric growing pattern had been invented.

In the next phase, the sequences that did follow a structure were compared and grouped again. Three categories arose from this comparison based on the construction of the visual or geometric part. This is described in Table 3.

We considered that a student had created a pattern when, upon being given a representation, whether numerical, pictorial or numerical and pictorial, a numerical regularity or spatial arrangement of the elements could be inferred from it that allowed continuing the sequence. We considered we had a pattern when there was a representation with a geometric or numerical structure. A student may have conducted a representation without an implicit structure, and in this situation, we considered the student had not managed to create a pattern. This study looked at the classification of patterns which were evidenced.

The first author coded students' answers as a first step. Next, the second and third authors verified the categories assigned to each answer. To ensure the inter-reliability of the coding, after the first author's coding, we submitted the coding to a calibration that included joint sessions for coding and discussion of disagreements.

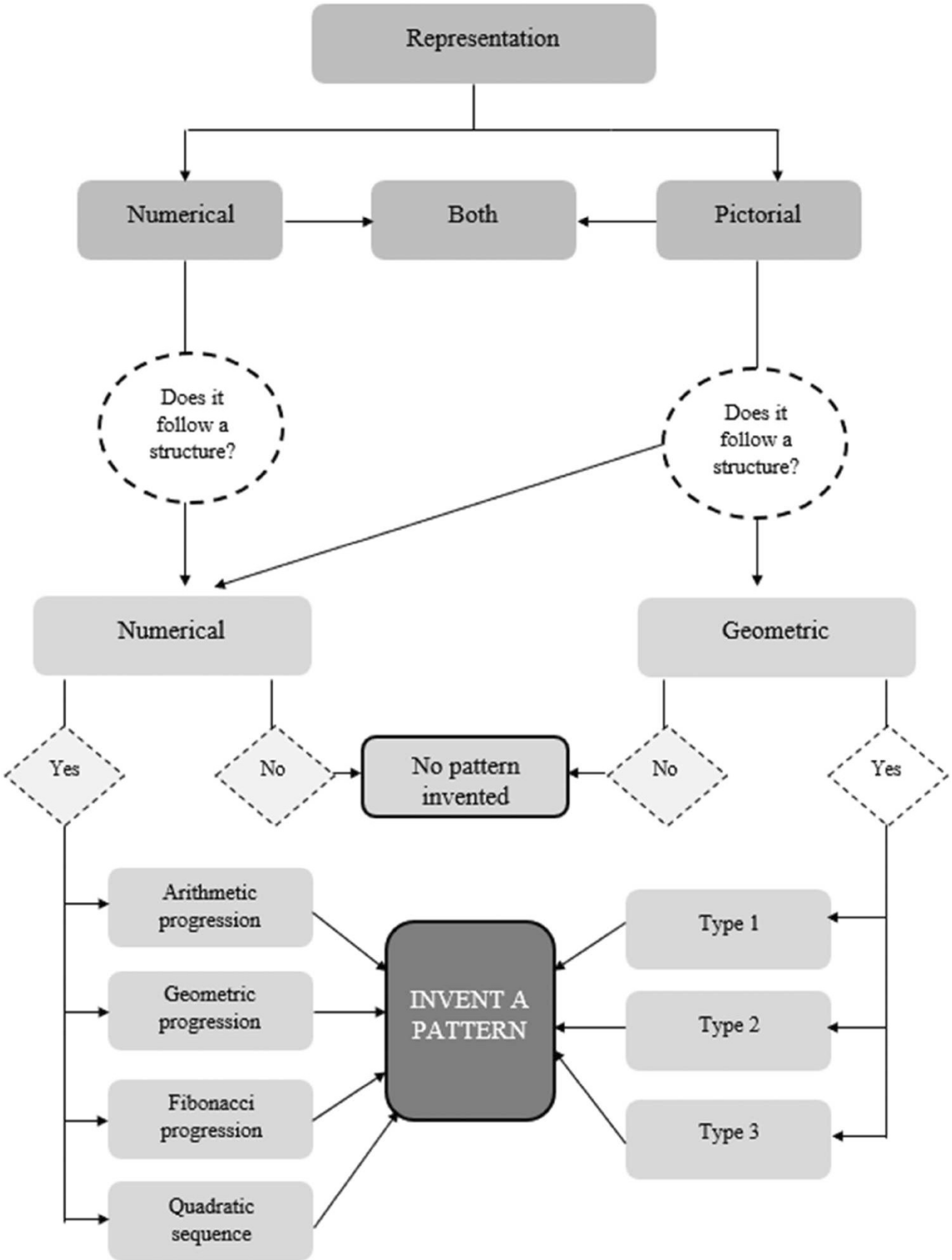
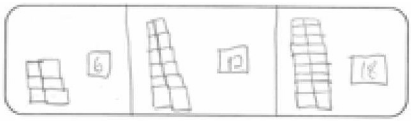
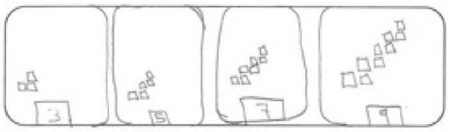
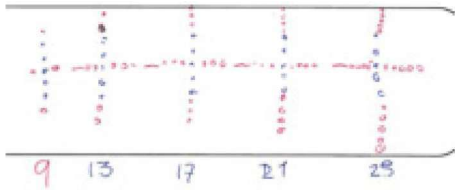


Figure 6. Analysis process.

5. Results

These results were obtained from analysing the pattern invention task conducted by 4th, 5th and 6th-grade students in primary education regarding the representations used and the structure involved. The task proposed did not ask for a particular type of pattern. Therefore, students could propose repetition or growing patterns or create no pattern.

Table 3. Categories for geometric structures.

Construction of a geometric structure	Description	Example
Type 1	Starting with an initial element which is repeated and to which the same is added	
Type 2	Starting with an initial element to which other elements, different from the original, are added, thus completing a certain figure.	
Type 3	The initial element is reproduced by increasing it in all "to scale" dimensions (2D expansion). Similar figures are formed.	

5.1. Representations used by students

When inventing a pattern, in general, we found that the students in the three grades more frequently used numerical and pictorial representation together. The joint use of the pictorial and numerical representation prevailed more in the 5th grade (19 out of 25 students, 76%) than in the 4th grade (11 out of 25 students, 44%) or the 6th grade (15 out of 26 students, 58%). This is shown in the chart in Table 4.

In the three primary education grades, we found that the numerical representation, together with the pictorial one, helped present the quantity of the elements arranged in the pattern. However, there were four cases, in the three school grades, which four cases in the three school grades used numerical representation to identify a position and not the quantity of elements. We consider it of relevance because by managing to identify the position of the elements and the elements, it is evidenced that the child has identified two sets of data that he/she relates simultaneously. This can be related to the transition to functional thinking (Torres et al., 2023). This is an open line within the field of algebraic thinking that would be worth further investigation.

As for the nuances related to the grade differences, we noted independent differences in the choice of numerical or pictorial representation (see Figure 7). We noted that among 4th-grade students, there was a most abundant of patterns with numerical representation, but as students advanced in grade, this changed, and there were more pictorially represented patterns. Thus, we found that in the 4th grade, the numerical representations were the most frequent (eight students versus pictorial with six students). In the 5th grade, the quantity of numerical representations versus pictorial ones was the

Table 4. Representations involved in the creation of a pattern.

Representation	4th grade	5th grade	6th grade
Pictorial	6	3	7
Numerical	8	3	4
Pictorial and numerical	11	19	15

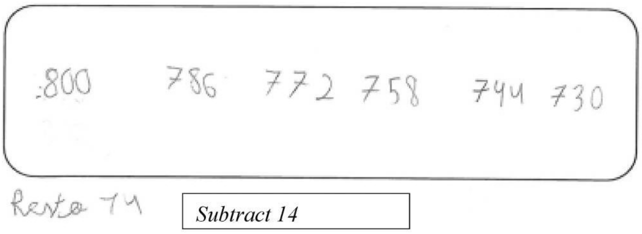


Figure 7. Numerical structure in a decreasing arithmetic progression (5th grade).

same (3 students), and in the 6th grade, they preferred the pictorial representation (7 students) over the numerical one (4 students).

The representations were formed in a primarily abstract or mathematical context save for a few exceptions, four cases in the 5th grade and one in the 6th, where students used real contexts such as a bunch of grapes, the petals on a flower, or tiles on the floor, to arrange the elements of their pattern.

5.2. Structures used by students

When characterising a pattern, the replicability and predictability of the sequence is important (McGarvey, 2012; Wijns et al., 2019), that is where structure plays a key role. We considered there was no evidence of having created a pattern when students formed a representation without a numerical or geometric structure.

From the sample, 76% of the students created a growing pattern (58 out of 76 students). The remaining students (11 in 4th grade, 4 in 5th and 3 in 6th) showed no structure in their proposed sequence. In all grades, there was a most abundant of patterns that followed only numerical structure: in 4th grade, 64% of the answers (9 out of 14 students); in 5th grade, 62% of the answers (13 out of 21 students) and in 6th grade, 57% of the answers (13 out of 23 students).

The presence of patterns in the students' answers increased as their school level rose. This is shown in the Table 5. When analysing the results grade by grade, in the 4th grade, 56% of the students created a pattern (14 out of 25 students); in the 5th grade, 84% of the students (21 out of 25) and in the 6th grade, 88% of the students (23 out of 26).

In all grades, there were most abundant patterns that followed an only numerical structure: in 4th grade, 64% of the answers (9 out of 14 students); in 5th grade, 62% of the answers (13 out of 21 students) and in 6th grade, 57% of the answers (13 out of 23 students). The highest percentage of patterns following a geometric structure was found in the 6th grade (43%) compared to the lower grades. In the 4th and 5th grades, we found a similar percentage of answers showing a pattern (36% and 38%, respectively).

Below are details of the results obtained for the numerical and geometric structures followed by the students.

Table 5. Structures evidenced in the patterns.

Structure	4th grade	5th grade	6th grade
Without structure	11	4	3
Geometric	5	8	10
Numerical	9	13	13

5.2.1. Numerical structures followed

The type of numerical structure chosen for the invention task was mostly a growing arithmetic progression structure in the three primary education grades, 4th, 5th and 6th. The numerical structure characterising the arithmetic progression was +3 for the 4th grade (9 students), +2 (5 students) y +3 and +8 (4 students) in the 5th grade, and +2 (6 students), +1 (5 students) y +5 (4 students) in the 6th grade.

Only six students (1 in 4th grade and 5 in 5th) chose a decreasing arithmetic progression. An example can be seen in Figure 7, whose recursive structure is subtracting 14, as the pupil explains.

Out of all the students who created a pattern, only 2 proposed a geometric progression: one in the 5th grade and another in the 6th. Figure 8 shows a 6th-grade student's answer explaining that the rule was to multiply by two every time.

In the Table 6 reflects students' choices based on the type of structure involved. We found no other type of patterns appearing spontaneously in the previous sessions, such as Fibonacci progressions or quadratic sequences.

5.2.2. Geometric structures followed

A geometric structure exists in a pattern when the visual arrangement of the elements involved in the construction of the pattern generates regularities, and its continuity is predictable. At other times, the arrangement of the visual or pictorial elements shows no regularities, and the sequence cannot be continued singly. Figure 9 is an example of a pattern whose recursive numerical structure is to add two, but its pictorial representation does not follow a geometric structure.

Twenty-three students proposed patterns following a geometric structure. The variety of geometric structures in the growing patterns provided a more detailed characterisation than counting elements or arithmetic calculations (Pasnak, 2017; Wijns et al., 2021). Students applied several strategies to create a geometric structure. For example, see Figure 10. Students completed rows with dots, creating an increasing/growing pattern (+3). While both followed the same numerical structure, the geometric structure was different. The student in the 4th grade added a row with three dots (Figure 13a). While the student in the 6th grade added a dot at each end of the figure (Figure 13b).

This is how the categories described in Table 3 emerged from comparing these patterns. Figure 13a shows the type 1 construction by six students (2 in each grade). This construction consisted of starting with an initial element (three dots) and repeating in the following term to which the same starting initial element is added.

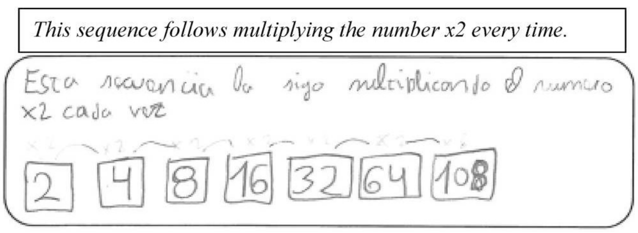


Figure 8. Numerical structure in geometric growing (6th grade).

Table 6. Types of numerical structures evidenced.

Structure Numerical	4th grade	5th grade	6th grade
Growing arithmetic	7	6	1
Decreasing arithmetics	1	5	0
Growing geometric	1	2	1

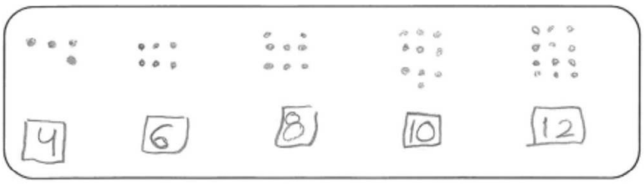


Figure 9. Pattern without a geometric structure (4th grade).

Another sort of construction, type 2, is shown in Figure 11. This spatial construction consists of starting with an initial element (three squares) and gradually adding elements, not the initial one, to complete a certain figure. In this case, it appears to form a snake. The patterns with type 2 geometric structures are the most common in all grades (3 in the 4th, 5 in the 5th and 5 in the 6th) based upon the completion of a figure.

The last type of spatial construction amounts to type 3. This construction consists of reproducing an initial element (the four dots forming a square) and increasing it in all dimensions "to scale" (2D expansion). An example of this type is shown in Figure 12. This structure appeared in the productions of 5 students (2 in 5th grade and 3 in 6th grade).

Considering the construction of the geometric structure, we came up with the following chart (Table 7) according to the data analysed. The patterns students in all school levels preferred to create were those of type 2, followed by type 1 and then type 3. As for the differences among grades, there were none in type 1. Type 3 patterns have been manifested in the upper grades. This may be because this type of construction requires a higher cognitive demand when working with the two dimensions of scale and magnification.

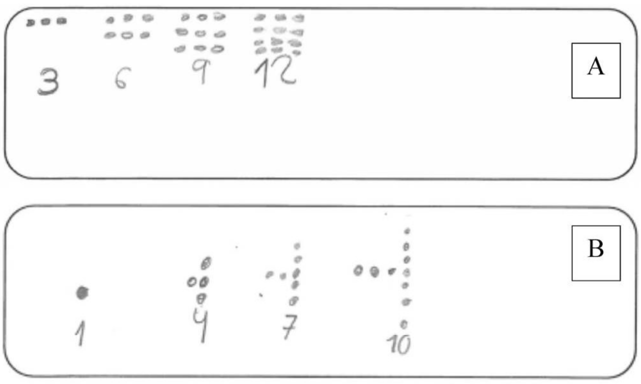


Figure 10. Patterns with a geometric structure (4th and 6th grades, respectively).

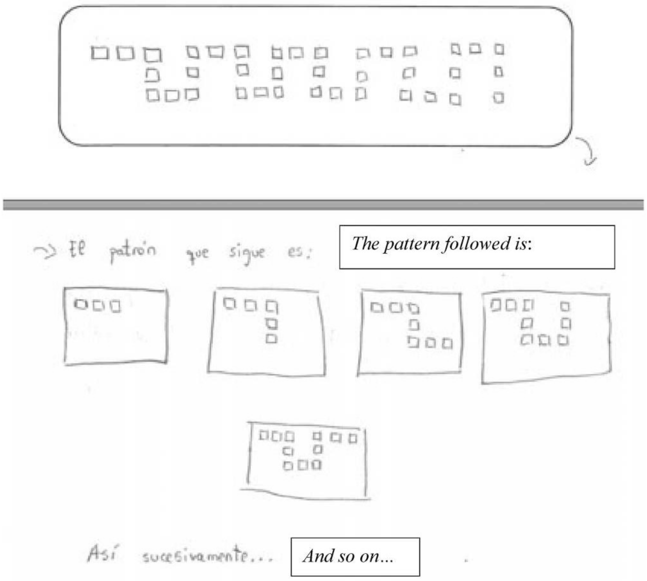


Figure 11. Pattern with type 2 geometric structure.

5.3. Relationship between the representation and the structures used by students

When a pattern was pictorially and numerically represented, it was impossible to clearly establish whether the geometric structure prevailed somehow over the numerical one or vice versa. We found examples of pictorial representations, which were auxiliary or derivative, that appeared to be used by students to represent the quantity of elements they already had predetermined with a numerical structure (for example, Figure 13). In this case, the pictorial representation is subordinated to the numerical structure implied in the pattern and does not imply a spatial arrangement responding to a geometric structure.

On the other hand, we also found auxiliary or derivative numerical representations that appeared to have been developed after considering the spatial structure of the pattern (Figure 14). The associated numerical structure (difference three arithmetic

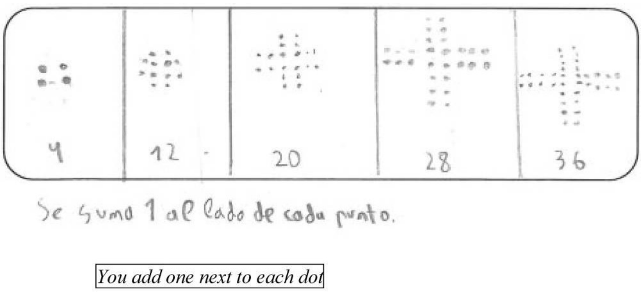


Figure 12. Pattern with type 3 geometric structure.

Table 7. Types of geometric structures evidenced.

Structure Geometric	4th grade	5th grade	6th grade
Type 1	2	2	2
Type 2	3	5	5
Type 3	0	2	3

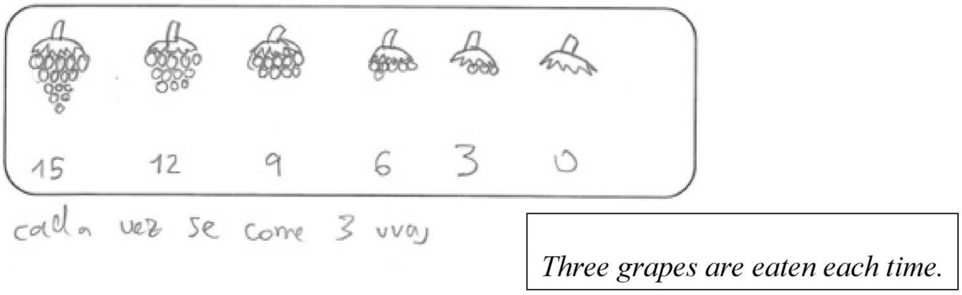


Figure 13. Example of pattern with only numerical structure.

progression) appeared due to the construction of a figure considering a concrete form, giving the pictorial representation and the geometric structure a significant role in the pattern.

6. Discussion and conclusion

In our study, we focused on characterising how students responded when given tasks that involved inventing patterns. Such tasks are considered to have a high cognitive demand, but they are also an effective way to learn about a student's understanding and skills in mathematics (Cai & Hwang, 2020; Cañadas et al., 2018). Our study emphasizes the importance of pattern invention as a tool for assessing a student's mathematical comprehension. We examined the spontaneous responses of students to understand how they represented growing patterns and identified the structures involved. Additionally, we found some differences in pattern construction based on the grade level of the students in our sample; these differences will be discussed transversely in the next two sections. One of the study's contributions is to describe the process followed when identifying patterns and provide evidence on the importance of focusing attention both on the representation and the numerical and geometric structure.

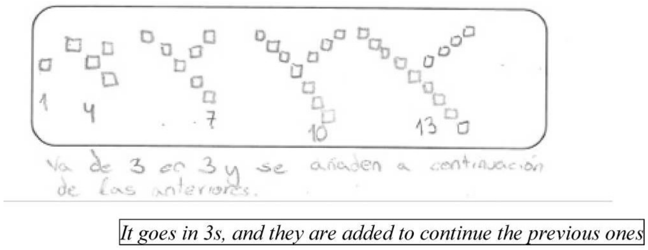


Figure 14. The pattern in which the geometric structure prevails over the numerical one.

6.1. Representations used in the invented patterns

We are interested in the analysis of representations because they allow us to understand how people know and understand mathematical objects and, on the other hand, they allow us to make them present to others and to oneself (Rico, 2009). In this study in the representations of patterns, we found that students mostly used pictorial and numerical representations together. These were formed in a primarily abstract or mathematical context, save for a few exceptions where students used real contexts. This can be explained by considering the tasks immediately before the analysis task, which was all focused on a mathematical context and represented both pictorially and numerically. Previous research has indicated that the types of problems students are capable of inventing can be influenced by the nature of the tasks they have previously been exposed to (Torres et al., 2023). This demonstrates that the invention and the representation used are based on previously solved cases, emphasizing the importance of presenting varied representations in classroom interventions.

In all grades, a preference has been found for the use of pictorial and numerical representation together. When focusing on answers which involved only numerical or only pictorial representations, it was interesting to note that in the lowest grade (4th), they tended to focus first on the numerical (Radford, 2011) while the pictorial prevailed in the highest grade. These cases could be due to the fact that students are traditionally trained much more on tasks with a numerical rather than a visual component.

The importance of the representations provided by the students in this type of tasks plays a key role in the interpretation of the structure. The evidence of the structure in the work with patterns is positioned as key to be able to understand the reasoning of students in the search for regularities. This topic will be further discussed in the following section.

6.2. Structure implied in the invented patterns

The key characteristics of identifying algebraic patterns are their regularity and predictability. Therefore, the structure is key to distinguishing between what is or is not a pattern. In the 4th grade, we found that 11 out of 25 students had not invented a pattern, as neither a numerical nor a geometric structure was involved in the representation. In the following grades, the percentage of students who did not invent a pattern was lower: in the 5th grade, 4 out of 25 students and in the 6th grade, 3 out of 26 students. The difficulty observed in the 4th grade could be because the cognitive demand for invention is high in younger students. Cai et al. (2013) pointed out in the context of invention of problems that this type of task is highly demanding as it requires reflecting upon the structure of the problems. This could explain our results and broaden the outcome of this research to the context of the invention of patterns.

Comparing the structures of the patterns created in all grades, we found that the prevailing structure was the numerical one. However, the use of geometric structures increased with the grades (5, 9 and 10 students, respectively, in the 4th, 5th and 6th grades). This finding arising from the context of the invention of patterns is consistent with the study by Radford (2011), showing how students, over the years, consider both the numerical and the spatial aspects when solving tasks that require completing or continuing patterns. Managing to invent a pattern with two different types of structures evidences cognitive flexibility in students. This highlights the role of the invention as an opportunity to promote flexible thinking and lead to a deep understanding of mathematical contents (Baumanns & Rott, 2022).

A contribution to this study is the characterisation of the three types of geometric structures we analysed and their apparent degree of complexity, with patterns with a type 3 structure being the most complex and used, preferably in the 6th grade. These findings complement previous research, such as that by Rivera and Rossi-Becker (2016), who described the difficulty of visual representations when stating that those which involve more than one stage in a single image are the most complex. In our case, no differences in the complexity of type 1 geometric patterns were found by always adding the same initial element. However, it was more complex according to age by adding elements to complete certain figures, expanding the initial one.

These results provide indicators to teach patterns in the highest grades of Primary Education. On the one hand, they show the potential of students to work simultaneously with pictorial and numerical representations (Orton et al., 1999) and to identify less complex numerical and geometric structures. On the other hand, the differences observed among the grades could recommend implementing teaching strategies favouring the use of different numerical patterns from arithmetic progressions and the use in the lower grades of geometric structures of the various types to familiarise students with a greater wealth of patterns and establish various strategies to identify them (Pasnak, 2017; Wijns et al., 2021).

In this regard, an open line of research would be exploring the existence of auxiliary representations when a numerical or geometric structure related to the figure aspects of a pattern prevails (Rivera & Rossi-Becker, 2016). The numerical structure could appear as a result of the geometric construction. That is, the pictorial representation and the geometric structure would take on a significant role in the invention of a pattern. This could be helpful for knowledge focused on visual skills and also to understand algebraic thinking regarding the relationships established between geometric and numeric structures. Moreover, the teaching implications of this study lie upon the invention of problems. This type of tasks do not only provide students the opportunity of demonstrating their knowledge and what they can do with it (Baumanns & Rott, 2022), but they also allow teachers to observe patterns in students' learning and mathematical thinking.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work has been developed within the projects with references Agencia Estatal de Investigación [grant number PID2020-113601GB-I00] financed by MCIN/AEI/10.13039/501100011033.

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