EXPANDED MODEL FOR ELEMENTARY ALGEBRAIC REASONING LEVELS

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Abstract

The development of algebraic reasoning from the earliest educational levels is an objective that has solid support both from the point of view of research and curricular development. Effectively incorporating algebraic content to enrich mathematical activity in schools requires considering the different degrees of generality of the objects and processes involved in algebraic practices. In this article, we present an expanded version of the model of levels of algebraization proposed within the framework of the Onto-semiotic Approach, establishing sublevels that provide a more microscopic view of the structures involved and the processes of generalization, representation, and analytical calculation at stake. We exemplify the model with mathematical activities that can be approached from primary education, classified according to the different sublevels of algebraization. The use of this expanded model can facilitate the development of didactic-mathematical knowledge of teachers in training on algebraic reasoning and its teaching.

Keywords: Algebraic reasoning, Primary Education, Onto-semiotic Approach, Teacher training.

Contribution to the literature

Godino et al. (2014) proposed a model of elementary algebraic reasoning for primary education that establishes criteria for identifying purely arithmetic mathematical activity and distinguishing it from progressive levels of algebraization. In this paper, we propose an expanded version of that model considering sublevels based on a microscopic view of languages, their treatments, and conversions (Duval, 2017); degrees of generalization and functional reasoning (Blanton et al., 2015; Radford, 2010, 2021; Vergel et al., 2022, 2023); mathematical structures and structural reasoning (Pittalis, 2023; Venkat et al., 2019); and analytical calculation (Filloy et al., 2008; Kaput et al., 2008; Vergel et al., 2022)

1. INTRODUCTION

Interest in the development of algebraic reasoning from the earliest levels of schooling is already a consolidated fact in the community of researchers in mathematics education (Kieran, 2022). Similarly, the need to adopt a broader perspective on the nature of school algebra is accepted (Godino et al., 2014; Malara and Navarra, 2018), understanding that the incorporation of algebraic content at early ages should aim to enrich school mathematical activity and facilitate the transition from arithmetic to algebra (Carraher and Schliemann, 2018).

Algebraic reasoning differs from arithmetic in the form of mathematical practices that emerge from student activities (Hewitt, 2019). These practices involve dealing with indeterminate objects, designating them symbolically, and operating with them analytically (Radford, 2010). However, the presence of unknown quantities represented symbolically is not sufficient to consider an activity as algebraic (Kaput et al., 2008). Symbolic designation is considered algebraic if it is "in the service of expressing generalizations or in the systematic reasoning with symbolized generalizations" (Kaput et al, 2008, p. 49). Practices in which generalization is expressed through other symbol systems (not conventionally algebraic) are considered quasi-algebraic (Kaput et al., 2008).

Supported by theoretical tools of the Onto-semiotic Approach (OSA) to mathematical knowledge and instruction (Godino et al., 2019), Godino et al. (2014) proposed a model of algebraic reasoning for primary education. Their model establishes criteria that allow identification of purely arithmetic mathematical activity (level 0 of algebraization) and distinguishes it from progressive levels of algebraization. In line with the proposals of authors who research in the field of early algebra, two primary levels of proto-algebraic reasoning (levels 1 and 2) are considered to differentiate them from other stable or consolidated forms of algebraic reasoning (level 3). The key idea is to "make explicit the generality" in the field of binary relations (equivalence and order), structures, functions, and modeling of intra- or extra-mathematical situations while operating with that generality. For this purpose, the criteria for delimiting the different levels are based on the type of objects and mathematical processes involved in the mathematical activity, according to the OSA framework: types of representations used, processes of generalization involved, and analytical calculation that is engaged in the corresponding mathematical activity.

The application of the levels of algebraization to the systems of practices allows "to characterize algebraic reasoning (institutional or epistemic perspective of algebra) and algebraic thinking (personal or cognitive perspective of algebra)" (Vergel et al, 2023, p.481), providing criteria to distinguish categories of meanings in the progressive construction of algebraic reasoning/thinking, which is not limited to a single context of application. This characterization offers criteria for curricular and instructional design and explains some conflicts in learning school algebra (Godino et al., 2014). However, given the onto-semiotic complexity of the objects and processes involved in elementary algebraic reasoning, “the boundaries between levels can sometimes be fuzzy” (Godino et al., 2014, p.212), and within each level, it is possible and useful to make distinctions that could lead to the proposal of new sublevels (Godino et al., 2014; Vergel et al., 2022).

The aim of this article is to clarify the distinction between arithmetic, proto-algebraic, and algebraic levels established in the model by Godino et al. (2014), considering sublevels based on a microscopic view of: a) languages, their treatments, and conversions (Duval, 2017); b) the different degrees of generalization and functional reasoning (Blanton et al., 2015; Radford, 2010, 2021; Vergel et al., 2022, 2023); c) the different structures and the structural reasoning involved (Pittalis, 2023; Venkat et al., 2019); and d) the analytical calculation involved (Filloy et al., 2008; Kaput et al., 2008; Vergel et al., 2022).

2. THEORETICAL FRAMEWORK

2.1 Practices, objects, and processes in the analysis of mathematical activity

From the viewpoint of the OSA, mathematical knowledge is constructed through problem-solving activity. The notion of mathematical practice, as an act or expression (verbal, graphic, etc.) carried out by someone to solve a mathematical problem, communicate their solution to others, validate it, or generalize it to new contexts and problems, is the starting point for the analysis of mathematical activity (Godino et al., 2019; Font, et al., 2013).

In systems of mathematical practices, different types of primary mathematical objects participate, that is, entities that can be individualized or separated according to their nature and function: problem situations, languages, concepts, propositions, procedures, and arguments. These objects emerge from systems of mathematical practices through their respective primary processes of problematization, communication, definition, enunciation, algorithmization, and argumentation (Godino et al., 2019). Processes such as modeling, problem solving, or problem creation are understood in the OSA as mega processes, given that they involve several of the former.

The mathematical objects that intervene in mathematical practices and their outcomes, according to the language game in which they participate, can be considered from five dual facets that dialectically complement each other. These facets are considered attributes that, when applied to different primary objects, give rise to the following typology of secondary objects: *ostensive* (material, perceptible) – *non-ostensive* (abstract, ideal, immaterial); *extensive* (particular) – *intensive* (general); *personal* (related to individual subjects) – *institutional* (shared in an institution or community of practices); *significant* (expression, antecedent) – *signified* (content, consequent of a semiotic function[1]); *unitary* (considered globally as a whole) – *systemic* (considered as systems formed by structured components). Furthermore, both primary and secondary objects can be considered from the process-product perspective — that is, an object is an outcome (product) of sequences of practices (process) —, which provides criteria to distinguish types of primary and secondary mathematical processes.

The realization of a mathematical practice mobilizes both the agent (institution or person) who develops it and the environment in which it is carried out (Font et al., 2010). Therefore, the analysis of mathematical activity involves both the description of the sequence of practices and the delimitation of the networks of objects and processes (*ontosemiotic configurations*) that enable such practices.

2.2 Algebraic reasoning from the OSA perspective

From the OSA, Elementary Algebraic Reasoning (EAR) is understood as the system of operative and discursive practices brought into play in solving tasks approachable from primary education in which algebraic objects and processes intervene. The types of algebraic objects considered (Godino et al., 2014) are as follows: a) *binary relations* —of equivalence or order— and their respective properties (reflexive, transitive, and symmetric or antisymmetric); b) *operations and their properties*, performed on elements of various object sets; c) *functions*, their operations and properties; and d) *structures* (semigroup, monoid, group, ring, field, vector space, etc.) and their types and properties.

Binary relations are considered algebraic when used to define new mathematical concepts. The application of properties such as associative, distributive, existence of

[1] In the OSA, a semiotic function is understood as a relation between an antecedent object (expression, signifier) and a consequent object (content, signified) established by a subject (person or institution) according to a criterion or rule of correspondence (Godino et al., 2019).

neutral and inverse elements, allow the application of specific algebraic calculation procedures to solve equations, inequalities, or systems of these, in which objects of indeterminate nature (variables, unknowns, or parameters) are treated analytically, that is, as if they were known numbers (Radford, 2010). Functions are relations or rules that associate the elements of one set with those of another, such that each element of the first set corresponds to one and only one of the second set. They establish and allow generalizing existing relations between quantities that covary together (Blanton and Kaput, 2004). Although the axiomatic study of structures is characteristic of higher algebra, from primary education students begin to familiarize themselves with the configurations of mathematical objects on which operations that comply with specific property systems are defined. That is, a structural treatment in mathematical activity is recognized.

In addition to syntactic calculation, prototypical algebraic processes such as generalization, unitization, and representation are considered. The OSA regards *generalization* in terms of the identification of intensive objects involved in practices. The intensive object does not appear until the subject demonstrates the criterion or rule that is applied to delimit the constituent elements of the set. In addition to the generalization that gives rise to the set, a process of *unitization* can occur, whereby the set becomes something new, different from the elements that constitute it, a unitary entity that can be represented and participate in other practices to give rise to new intensive objects. To do so, the new unitary entity has to be materialized through a gesture, icon, name, or symbol. Thus, a process of *representation* accompanies generalization and unitization. Lastly, symbols detach from the referents they represent to become new objects on which syntactic, analytic, or formal operations are performed (Godino et al., 2014).

In the EAR model proposed by the OSA (Godino et al., 2012, 2014), the algebraic nature of a mathematical practice is linked to the subject performing the activity recognizing the rule that forms the intensive object (generalization), considering the generality as a new unitary entity (unitization), its materialization through any semiotic register (representation), and its subsequent analytical treatment. This way of understanding algebraic reasoning allows algebra to be articulated with the rest of mathematical knowledge: numbers, geometry, measurement, and stochasticity. Indeed, whenever the presence of intensive objects in a mathematical practice is recognized, at any of their levels of generality, a certain degree of algebraic reasoning can be attributed to them, whether the intensive is expressed symbolically or not. Furthermore, it is possible to distinguish the types of configurations of algebraic practices (relational, operational, functional, structural) based on the types of objects and algebraic processes described.

Mathematical practices at level 0 of EAR are those that do not show algebraic characteristics: they involve extensive objects (particulars) or intensive objects of the first degree of generality (natural numbers) expressed through gestural, iconic, natural, or numeric languages. Symbols referring to an unknown value may be involved; however, such a value is obtained because of operations on particular objects. In functional tasks, the relationship is recognized only in particular cases (Godino et al., 2014). At the incipient proto-algebraic level, level 1 of EAR, intensive objects of the second degree participate, whose generality is explicitly recognized through iconic, gestural, natural, or numeric languages. The relations and properties of the operations are applied, and unknown quantities expressed symbolically may be involved; however, operations are not performed using these objects. Equality is used as equivalence. A higher proto-algebraic level, level 2 of EAR, is determined by the use of unknowns or variables expressed with symbolic–literal language to refer to recognized intensives, though tied to spatial and contextual information. In structural practices, equations are of the form $Ax + B = C$ ($A, B, C \in \mathbb{R}$). In functional practices, generality is recognized, but operations are not performed using variables to obtain canonical forms of expression. At the consolidated algebraic level, level 3 of EAR, intensive objects represented in a symbolic–literal manner are generated and operated analytically without referring to contextual information. In structural practices, operations are performed with the unknown to solve equations of type $Ax + B = Cx + D$ ($A, B, C, D \in \mathbb{R}$). Particular functions are involved, and operations are performed using the variables to obtain canonical forms of expression.

3. LITERATURE REVIEW AND SYNTHESIS

In this section, we present a review of the literature on essential aspects of algebraic reasoning that will allow us to have a more structured and analytical view of the levels of EAR.

3.1. Generalization as a Distinctive Feature of Algebraic Reasoning

Despite the diversity of ways to understand generalization, all of them agree that it leads to recognizing a regularity in a set of elements, generating new cases, and obtaining their respective expressions.

> Any effective generalization in the real of numerical and letter symbolism can be regarded from at least two aspects: one must be able to see a similar situation (*where* to apply it), and one must master the generalized type of solution, the generalized scheme of a proof or of and argument (*what* to apply). In either case one must abstract oneself from specific content and single out what is similar, general, and essential in the structures of objects, relationships, or operations (Krutetskii, 1976, p. 237).

Generalization as a process involves: identifying the elements common to all cases, extending the reasoning beyond the range in which it originated, obtaining results broader than those of the particular cases, and providing a direct expression that allows the generation of any term (Ellis, 2007; Radford, 2013).

Given the complexity associated with the process itself, authors such as Krutetskii (1976) and Radford (2010, 2018) believe that it is necessary to distinguish the levels of generalization. For Krutetskii (1976), it is possible to differentiate two levels in an individual's ability to generalize: 1) seeing something general and known to him in what is particular and concrete (subsuming a particular case under a known general object; recognizing and applying a known formula to a particular case) and 2) seeing something general and still unknown to him in what is isolated and particular (deducing a general formula, as a new object, based on different particular cases). On the other hand, in studying the types of numeric-geometric pattern generalization by students, Radford (2010) distinguishes between arithmetic generalization and algebraic generalization. In *arithmetic generalization*, the detected relationship is applied only locally, that is, only to some cases (near or far), without extending to other assumptions or using the observed information and without showing analytical reasoning (Radford, 2010).

Recently, various authors have suggested the need to consider advanced forms of arithmetic generalization very close to the proto-forms of algebraic thinking (Ayala-Altamirano and Molina, 2021; Cooper and Warren, 2011; Radford, 2021; Vergel et al., 2022, 2023). Cooper and Warren (2011) proposed the idea of *quasi-generalization* as an advanced form of arithmetic generalization, as a process by which "students are able to express the generalization in terms of specific numbers" (p. 193). The general rule is perceived as a set of particular relationships that do not acquire a unified form (Cooper and Warren, 2011).

Similarly, Ayala-Altamirano and Molina (2021) identify between arithmetic and algebraic generalization a form of generalization based on the incipient awareness of the sense of the indeterminate in students' reasoning in generalization processes. In *generalization in action*

> students through their actions seem to perceive the generality, but do not refer to indeterminate quantities analytically. However, the structural sense and the recognition of a structure associated with the problem situation is the evidence that allows inferring that they begin to reason analytically (Ayala-Altamirano and Molina, 2021, p. 215).

In generalization in action, the student recognizes the common element, uses language to describe it, and is capable of employing this generality to determine other cases. However, the relationship is not made explicit at any time and fails to express indeterminate quantities by resorting to particular cases when asked about them (Ayala-Altamirano and Molina, 2021).

Vergel et al. (2022, 2023) used the term sophisticated arithmetic generalization, to differentiate this process from arithmetic generalization. In *sophisticated arithmetic generalization*, a structural view of the relationships is recognized that allows the use of the common characteristic (the generality) to determine any term of the sequence, although the generality is not explicitly expressed analytically (Vergel et al., 2022).

Within algebraic generalization, Radford (2010) distinguishes three types: factual, contextual, and symbolic. *Factual generalization* refers to a generalization of actions in the form of an operational scheme that remains tied to the concrete level of using numerical symbols and to deictic terms and gestures as semiotic means of objectification; the general or the indeterminate remains unnamed, that is, indeterminacy does not reach the level of discourse. The general rule is a formula in action that allows students to successfully address particular cases. In *contextual generalization*, students observe a pattern and can explain it using mathematical elements for any figure within the sequence without it being described by a specific number. Not only are the actions generalized but also the objects resulting from the actions. There is a higher level of abstraction than in factual generalization; therefore, students refer to generality through specific expressions of indeterminacy. Although there are difficulties in appropriately expressing the variables and their relationships, there is a reduction in the semiotic means of objectification. The emerging abstract objects acquire entity through expressions such as "the figure," "the next figure," "for any number," etc. Finally, the expression through alphanumeric language of the general rule is associated with *symbolic generalization*.

3.2. The role of language

Mathematical objects, being ideal, abstract entities, need to be mediated by signs to represent and operate with them. The use of representations is inherent to algebra (Blanton et al., 2015); the system of signs and its own grammatical and syntactic rules allow algebra to express and manipulate generality in an unequivocal and compact manner (Drijvers et al., 2011).

As Radford (2010) proposes, the levels or layers of generality that determine algebraic thinking are closely related to the semiotic representation systems used to express regularity. Although the use of symbolism facilitates the representation and manipulation of essential algebraic objects, it is not the only system for representing algebraic activity. Primary students can use gestural, figural, verbal (oral or written), numeric, tabular, diagrammatic, or graphic languages before or complementary to symbolic language to analytically refer to indeterminate quantities (Arzarello, 2006; Radford, 2011; Torres et al., 2022).

Radford (2018) links a higher degree of sophistication in the expression of algebraic ideas to the idea of *semiotic contraction*, i.e., the concentration of meanings in the fewest number of signs through which generality is expressed. In this sense, diagrammatic, tabular, and graphic languages are attributed an increasing degree of contraction. Diagrammatic language facilitates relating variables and defining a correspondence through sagittal structures or line segments (Blanton, 2008). Diagrams play a decisive role in the development of algebraic reasoning, because they support or make possible the necessary process of particularization of the general rule, allowing conceptual objects to participate in practices from which new objects will emerge. However, more than the construction of a specific diagram, what is relevant in the development of algebraic reasoning is the implicit and hidden mathematical knowledge behind the diagrams used for their representation and manipulation (Giacomone et al., 2022).

Since the appearance of the first mathematical texts, there have been different types of tables: from structures that summarize empirical data to those representing mathematical functions. However, what they have in common is the expression of complex information in a two-dimensional form. "The structure of tables, the transition from a one-dimensional to two-dimensional layout in the location of information, has a far greater significance than might naively be expected" (Campbell-Kelly et al., 2003, p.2). Tables not only facilitate selection, categorization, and calculation with data (Campell-Kelly et al., 2003) but also allow establish and reason with the relationships and properties among the elements (variables), acquiring their use a functional character. Specifically, through tables, pairs of elements of the sets related by the function are organized, changes between variables are identified and described, and both covariation and correspondence relationships are perceived (Blanton, 2008).

Kaput (1993) emphasized the need to study how the first graphical representations can serve as a means of reasoning about quantitative relationships and how they articulate with the symbolic representations of generalizations. For Radford et al. (2008), a graph is a complex mathematical sign that serves to specifically represent certain states of affairs. Graphs, particularly Cartesian graphs, are based on sophisticated syntax and a complex way of conveying relationships between elements that represent aspects of phenomena in the physical world; the Cartesian axes are critical elements of the representation, as are the variables represented on these axes (Johnson, 2022).

Although starting to think algebraically does not require "complete" symbolization, that is, symbolization based on alphanumeric signs, for authors like Blanton et al. (2017) "algebraic reasoning ultimately involves reasoning with perhaps the most ubiquitous cultural artifact of algebra - the conventional symbol system based on variable notation" (p. 182). Alphanumeric symbolism offers multiple possibilities for performing calculations that may be difficult or impossible with other semiotic systems, such as gestural, pictorial, or verbal. It represents an effective language not only for expressing but also for manipulating generality, advancing the analytical nature of algebraic reasoning (Radford, 2018).

3.3 Structures

The term *structure* is used in abstract algebra to designate a set that is closed under one or more composition laws and a set of properties (axioms) that these operations can satisfy. Algebraic structures are classified according to the number and type of operations, as well as the properties that these fulfill: a) with a single internal composition law, such as, semigroups, monoids $(\mathbb{N}, +)$, and groups; b) with two internal composition laws, such as rings $(\mathbb{Z},+,\cdot)$, fields $(\mathbb{Q}, \mathbb{R})$, and lattices; c) with one internal composition law and another external, for example, vector spaces ($\mathbb{R}^2$, $\mathbb{R}^3$); d) with two internal composition laws and one external, the algebras (matrix algebra, function algebra), etc. In these structures, binary relations of preorder (reflexive, transitive), order (antisymmetric preorder), or equivalence (symmetric preorder) can be defined. For example, in every monoid, a preorder associated with its internal operation can be defined, as occurs with the divisibility relation in rings ($\mathbb{Z}$, $\cdot$), or an order in the case of the $\leq$ relation in ($\mathbb{N}$, $+$).

However, within the educational field, there are some ambiguities regarding the meaning of the term structure (Pittalis, 2023; Venkat et al., 2019). It is considered as the network of relations associated with order, addition, and multiplication structures, the identification of general properties that are specified in particular situations as relations between elements, or the recognition of properties within general forms (Pittalis, 2023). In any case, the notion of structure corresponds to the organization of parts of a whole, the entirety of the elements of a regularity, and the relation that exists between those elements (Kieran, 1989). Recognizing the structure in mathematical practices implies seeing an object or expression as a combination of recognizable parts along with connections that place the object as a particular example of a more general type (Hewitt, 2019). It is not enough to recognize a link between two or more objects, but it is necessary to be aware of the use of properties and how they relate to consider that reasoning about the structure is occurring. That is, concluding that 2+5=5+2 is correct by performing calculations and checking the coincidence of the results on both sides of the expression is not evidence of structural reasoning, while it is to use the inherent structure of the operation to assert that the equality is true based on the commutative property of the sum of natural numbers (Hewitt, 2019).

Pittalis (2023) distinguished the sense of structure in arithmetic and algebra. In arithmetic, it implies looking through numbers (recognizing the different ways in which they can be expressed) and performing numerical operations to decompose and recompose, identifying equivalence without calculation, based on their properties. In the case of algebra, this implies seeing relations and properties in an algebraic expression and transforming it accordingly into another equivalent expression. In that continuum that considers the transition from arithmetic to algebra, he introduces the sense of arithmetic-algebraic structure as the ability to:

> see, recognize, conceptualize, utilize, generalize and make aware of structure in a variety of numeric, arithmetic, pattern, and functional situations, in respect to order, additive and

> multiplicative structural relations […] emerges by generalizing actions and processes that make possible the interplay from particular-local relations and properties to general ones, to conceive properties, regularities, and relations as mathematical objects (p. 1869).

These mathematical objects for Pittalis (2023) can be numbers (or classes of them), operations with numbers, patterns (numeric or figural), functions, or variable quantities. A close relationship is observed between the structural sense and the ability to generalize, unify, and reify actions (Sfard, 2020). Thus, Venkat et al. (2019) consider that the difficulty in defining structure is that the term appears intertwined with others such as relations, generality/generalization, and properties that sometimes overlap but are considered different, while at other times they are seen as synonyms. Venkat et al. (2019) link (in the sense of Mason et al., 2009) the mathematical structure to the identification of general properties that manifest in particular situations as relations between elements. They understand properties as implicit behavior and internal relations in a given class of mathematical objects. This leads these authors to consider *emerging structures*, in which relations have a local nature, and differentiate them from *mathematical structures*, supported by general relations. Thus, emerging structures arise in a discourse of particularity (“the even number 6”), while mathematical structures arise in a discourse of generic (“an even number like 6”) /general (“any even number”) relations, applicable within some class of examples (Venkat et al., 2019).

3.4. Analyticity

Reasoning analytically, that is, treating unknown numbers the same as known numbers, is what distinguishes algebra from arithmetic. *Analyticity* is understood in terms of the operational character of indeterminate objects through the application of the properties of operations (commutative, associative, distributive of multiplication with respect to addition, etc.), which involve processes of deduction and generalization (Kaput et al., 2008; Vergel et al., 2022). When deduction is incipient, in the sense that what is generalized is a procedure and not a direct expression (a formula, not necessarily symbolic), so that the relationship is not materialized with unknowns or variables, it is referred to as *proto-analyticity* (Vergel et al., 2022).

Although there is considerable overlap and interaction between the structural and analytical dimensions in algebraic reasoning, analytical reasoning is specifically the type of reasoning that "underpins the transformations and equivalence aspects of equations and equation-solving" (Kieran, 2022, p. 1134). However, not all equations require the same analytical treatment.

Given that to solve an equation of the type $Ax + B = C$ (where *A, B,* and *C* are given particular numbers), students can simply reverse the operations (subtract *B* from *C* and divide by *A*), authors such as Filloy et al. (2008) do not consider equations of the type $Ax + B = C$ as properly algebraic:

> In arithmetic terms, the left side of an equation corresponds to a sequence of operations that are carried out on numbers (whether known or not), and the right side corresponds to the result of having carried out those operations: this is what one might call an arithmetic notion of equality (or of an equation) (p.94)

From the point of view of analytical calculus, there is a leap "FROM working with an unknown on only one side of the equal sign when it is enough to 'undo' the indicated operation" (arithmetic equation, $Ax+ B = C$) "TO dealing with equations where the unknown appears on both sides and therefore has to be operated on" (*algebraic equation*, $Ax+ B = Cx+ D$) (Kilhamm et al., 2019, p.6).

> the arithmetic notion of equality does not apply to an equation such as $Ax \pm B = Cx \pm D$ (where *A, B, C,* and *D* are particular given numbers), and therefore its operational solution involves operations outside the scope of arithmetic, such as operating on the unknown. In order that such operations may acquire sense for the individual and so be brought into use in the process of solving an equation, equations such as those of the form described here (which we will call “non-arithmetic” equations) must in turn be provided with some meaning […] it must be understood that the expressions in both parts of the equality are of the same nature (or structure), and that there is a series of actions that give sense to the equality between them (such as the actions corresponding to the substitution of the numeric value of *x*) (Filloy et al., 2008, p. 94)

For Sfard and Linchevski (1994), while an equation can be solved by treating the literal symbol as an unknown but fixed number, and each side of an equation as a concrete product of operations on this number, in an inequality, the letter plays the role of a variable and the component expressions are functions of this variable. "Unlike the equal sign, the symbol '>' cannot be interpreted as a 'do something' signal" (p. 110). In an inequality, it is necessary to compare the values of the component expressions for different literal symbol values. To solve an inequality of the type $Ax + B < C$, it is not enough to reverse the operations starting from C, to obtain the solution set of values. For example, the sign of A can change the direction of "<" in the sequence when describing the solution set. The inequality $Ax + B < Cx + D$, can be interpreted through the comparison of two linear functions, so that solving it involves finding the set in which the values of the function $f(x) = Ax + B$, are below the values of the function $g(x) = Cx + D$.

3.5. Functions

Functional reasoning is based on "building, describing, and reasoning with and about functions" (Pittalis et al., 2020, p. 632). This includes expressing the relationships between the quantities that vary together and using these expressions to analyze the behavior of a function (Blanton et al., 2011).

Functional reasoning, a fundamental part of algebraic reasoning, is primarily evident when students establish covariation or correspondence relationships between the variables involved in problem situations. However, from the school perspective, it is appropriate to consider increasing levels of sophistication by differentiating: 1) the function as an assignment of input and output, 2) recursive patterns (particular, general), 3) the function as a covariation relationship, 4) the function as a correspondence relationship, 5) the function as an object (Pittalis et al., 2020). The *input and output* approach refers to the operational and computational aspects of the function, as it does not require awareness of the causal relationship between input and output. The *recursive pattern* is built from the variation of the values of the dependent variable, which is determined on the basis of other determined or previously known values. Thus, it does not consider the variation of the independent variable, and to obtain the value of the latter, it is necessary to calculate all the previous values of the function (Blanton et al., 2011). *Covariation* analyzes how changes in one variable influence the other. A covariation relationship allows us to examine the function in terms of coordinating changes in the values of the dependent and independent variables. A *correspondence relationship* establishes a general rule that describes the relationship between quantities, allowing the analysis and prediction of the function's behavior and knowing a specific value of the dependent variable from the corresponding value of the independent variable, without needing to know other values of the function, offering a holistic view of the functional relationship (Pittalis et al., 2020). Finally, the vision of the *function as an object* is linked to the structural character, it is recognized as part of a family of functions, with its own representations and properties, subject to higher-order operations such as composition, inversion, etc.

The analysis of children's ability to notice relationships between quantities, organize and represent data to make sense of and generalize functional relationships has led various authors to establish levels of sophistication, understood as "reference points" that mark progress in their learning of functions (Stephens et al., 2017). Blanton et al. (2015) identified eight levels of functional reasoning, attending to both the type of relationship recognized (recursive, functional) and the degree of generality (particular or general). Subsequently, Stephens et al. (2017) expanded this model to a total of ten levels of sophistication of functional reasoning, considering in addition to the type of relationship (variational, covariation, correspondence) and its generality, how they express (variables, words) such a relationship.

4. EXPANDED MODEL ELEMENTS OF THE EAR

In the expansion of the EAR model, and based on the literature review, we consider the possibility of establishing different degrees of sophistication in the essential features of algebraic reasoning: generalization, representation, structural reasoning, and functional reasoning.

4.1. Layers of generalization

There are two key ideas that serve as operational criteria to distinguish arithmetic generalization from algebraic or proto-algebraic generalization: deduction and analyticity. For a generalization to be algebraic, there must be a deduction of an expression that allows the value of any term in a sequence to be calculated. On the other hand, the general term of the sequence being expressed in alphanumeric symbolism is not sufficient for the generalization to be the result of algebraic reasoning about the sequence (Radford, 2018), as what differentiates arithmetic reasoning from algebraic is "the analytical manner in which we think when we think algebraically" (p. 9). Therefore, for the generalization to be algebraic, it requires that abduction, that is, the recognition of the common characteristic as something plausible, is used analytically to deduce a formula that necessarily provides the value of any term (Vergel et al., 2022). Thus, we can distinguish three forms of generalization (Figure 1): *pre-algebraic*, when there is no deduction or analytical treatment, and abduction generates a procedure but not a direct expression of generality; *proto-algebraic*, when what is deduced, the common, is expressed through particular instances of the variable and incipient analytical traits are observed (proto-analyticity); and *algebraic*, when the generated formula is used analytically to deduce canonical forms of expression.

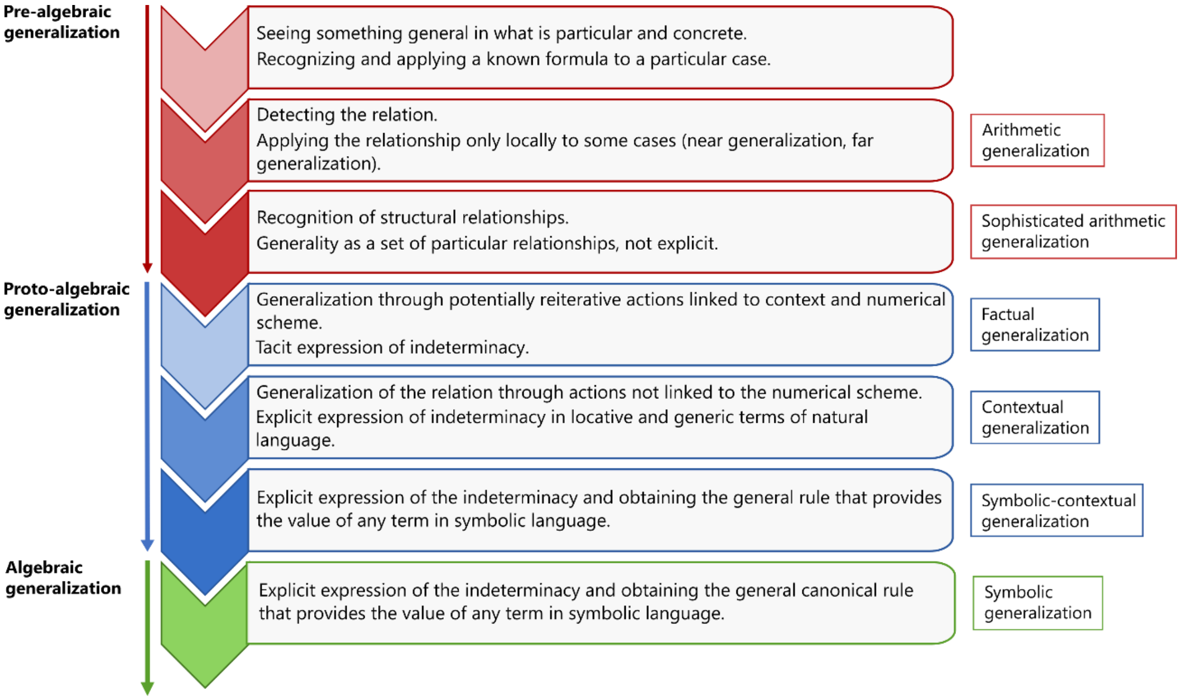


Figure 1. Layers of pre-, proto-, and algebraic generalization.

In this way, the first level identified by Krutetskii (1976) —relating a particular case to a known general object, and recognizing and applying a given formula to a particular case— has a pre-algebraic character, while the second level —deducing a general formula as a new object— would have proto-algebraic or consolidated algebraic traits according to the analytical treatment applied.

Within pre-algebraic generalization, we consider two successive layers: arithmetic generalization (Radford, 2010) and advanced forms of arithmetic generalization, such as sophisticated (Vergel et al., 2022), in-act (Ayala-Altamirano and Molina, 2021), and quasi-generalization (Cooper and Warren, 2011).

We associate factual generalization with an incipient proto-algebraic character because generalization is carried out through actions on numbers and a proto-analytical treatment of indeterminacy (description of procedures or actions that are potentially carried out repetitively) occurs, though this is only expressed tacitly and not explicitly (Radford, 2018). In contextual generalization, although symbolic language has not yet been used, the numerical scheme takes a back seat, identifying and naming generality through locative and generic terms that refer to abstract objects, still conceptually, spatially, and temporally situated (Radford, 2018). Therefore, contextual generalization implies a proto-algebraic character superior to that of factual generalization.

We consider *symbolic-contextual generalization* as proto-algebraic generalization in which, while a general formula is deduced that is expressed in alphanumeric language, such expression retains the contextual dimension of space and time and does not recognize the canonical form of generality. Finally, *algebraic generalization* refers to indeterminate quantities, involves analytical reasoning, and use various forms of representation. That is, what differentiates algebraic generalization from proto-algebraic generalization is that the general rule, the intensive, is recognized as a new unitary entity emerging from the system of practices, materializes symbolically, and becomes a new object upon which actions can be performed. In particular, symbols can be operated with to obtain equivalent or canonical forms of their representation.

4.2. Use of language

The criteria used to establish levels of sophistication in language use are based on both the degree of semiotic contraction of the registers —iconic, natural, diagrammatic, tabular, graphic, or alphanumeric (Radford, 2018)— and the nature of the transformations that different semiotic representations undergo. In this sense, we rely on Duval's (2006) theory of Semiotic Representation Registers (SRR). For this author, the real role of representation systems or languages is not to represent objects but to operate, that is, the transformation of these objects, conditioned by the rules that operate in the semiotic representation register used. He distinguishes two types of possible transformations between representations: treatment and conversion. *Treatment* involves transformation between representations of the same SRR. The decomposition of a number, the decomposition or reorganization of a diagram or geometric figure, or the manipulation of algebraic expressions are examples of treatment in the numeric, figural/graphic, and alphanumeric/algebraic SRRs. The treatments that can be performed vary according to the SRRs and mainly depend on the possible specific semiotic transformations of the selected register. *Conversion* involves transformation between representations of different SRRs without changing the represented object (Duval, 2006). Moving from a figurative representation of a pattern to its numerical representation, translating a statement in natural language into a symbolic expression (for example, from “the square of the difference of two numbers is the product of their sum by their difference" to "$(a-b)^2 = (a+b)(a-b)$") are examples of conversion.

According to Duval (2006), the complexity of converting one representation to another depends on the degree of congruence between the initial and final representations. Two representations are congruent when they meet three conditions: semantic correspondence between the units of meaning; semantic univocity (each unit of meaning in the initial representation corresponds to a single unit of meaning in the final representation); and the same order of presentation of the units of meaning (Duval, 2006). Otherwise, they are considered non-congruent. In a conversion, the represented object does not change, however, the specific lexicon of the representation register must be articulated with the structural properties of the object made ostensive, thus conditioning whether the conversions are congruent or not. This is what determines that the conversion from "the set of points (x, y) on the plane whose ordinate is greater than its abscissa" to "the set of points (x, y) on the plane with $y > x$" is congruent and that it is not the conversion from "the set of points (x, y) on the plane whose coordinates have the same sign" to "the set of points (x, y) on the plane with $xy > 0$" (Duval, 2006).

Based on these ideas, in Figure 2, we establish different degrees of formalization in the use of SRRs.

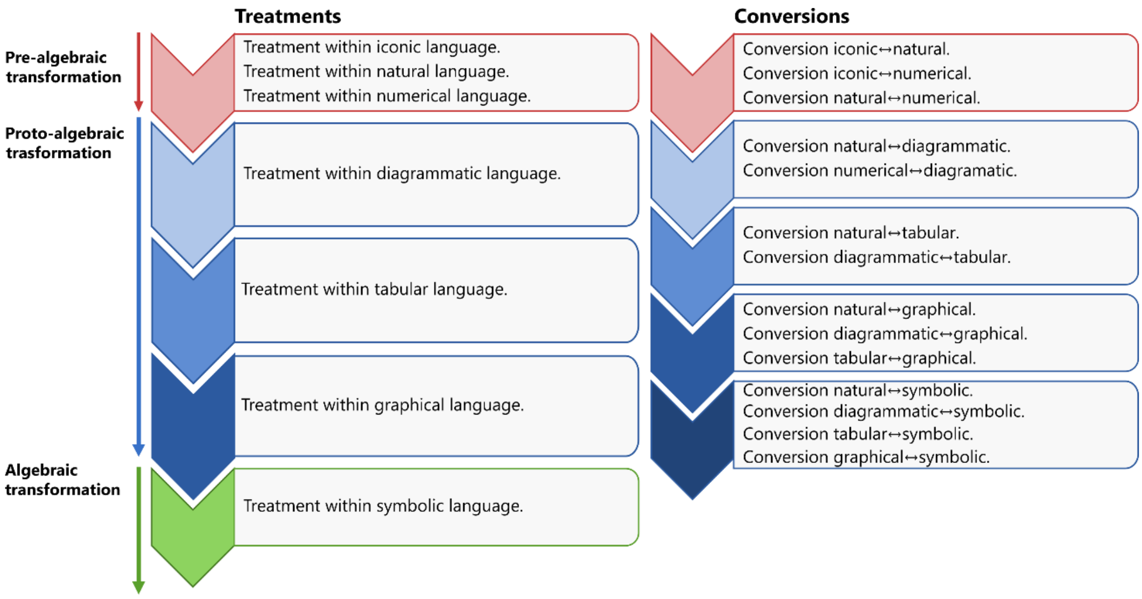


Figure 2. Layers of pre-, proto-, and algebraic SRR transformations.

An increase in the degree of semiotic contraction characterizes the increase in the level of generalization. This contraction of semiotic means implies greater sophistication of the SRRs, which determines the hierarchy shown in Figure 2. In this case, we highlight that the conversion from any SRR to the symbolic constitutes a proto-algebraic type of conversion, as it does not necessarily involve analytical reasoning in this last system of representation, leaving the treatment of symbolic language as the only transformation of fully algebraic nature.

4.3. Degrees of structural reasoning

We consider that an *algebraic structure* is a systemic entity (in the sense of the OSA) determined by the following: a) mathematical objects (for example, natural numbers), b) binary relations among the objects (such as order or equivalence relations), c) operations among the objects (for example, the addition of natural numbers), d) properties of the relations (like the transitivity of order), e) properties of the operations (such as commutativity), and f) correlation between relations and operations (for example, compatibility between order and addition of natural numbers).

In line with Venkat et al. (2019), we understand that structures can have different degrees of generality, as determined in our case by the degree of intension of the involved mathematical objects. This allows us to discuss about degree 1 structures (intensive objects of degree 1, i.e., structures on natural numbers), degree 2 structures (intensive objects of degree 2, i.e., on classes of natural numbers, on rational numbers), etc. Similarly, on the basis of our definition of algebraic structure, it is possible to establish layers of *structural reasoning*, which can be understood as reasoning with and about structures.

We consider *pseudo-structural reasoning* as the use of numbers that exceeds merely arithmetic, in which numerical relations are intended to express generality. This assumes what authors like Kaput et al. (2008) call the *algebraic use of numbers*, that is, reasoning with numerical statements that are analyzed not for calculation purposes, but for their structure in search of a pattern or generalization. Also reasoning with what Fujii (2003) calls *quasi-variables*, i.e., numerical expressions "that indicate an underlying mathematical relationship which remains true whatever the numbers used are" (p. 59).

Structural reasoning is *incipient* if the properties of operations and relations are applied locally, whereas it is considered *partial* when the properties of operations and relations are applied generally but the correlation between operations and binary relations is not involved. *Structural reasoning* is *complete* when the properties of the operations and binary relations are generally applied and correlated.

4.4. Degrees of functional reasoning

Considering the works of Blanton et al. (2015) and Stephens et al. (2017), to determine the layers of functional reasoning (Figure 3), we focus on the degree of intension (particular, general) and type (recursive pattern, covariation, correspondence) of the functional relationship. Moreover, we consider the contextual or not nature of the rule and the analytical treatment.

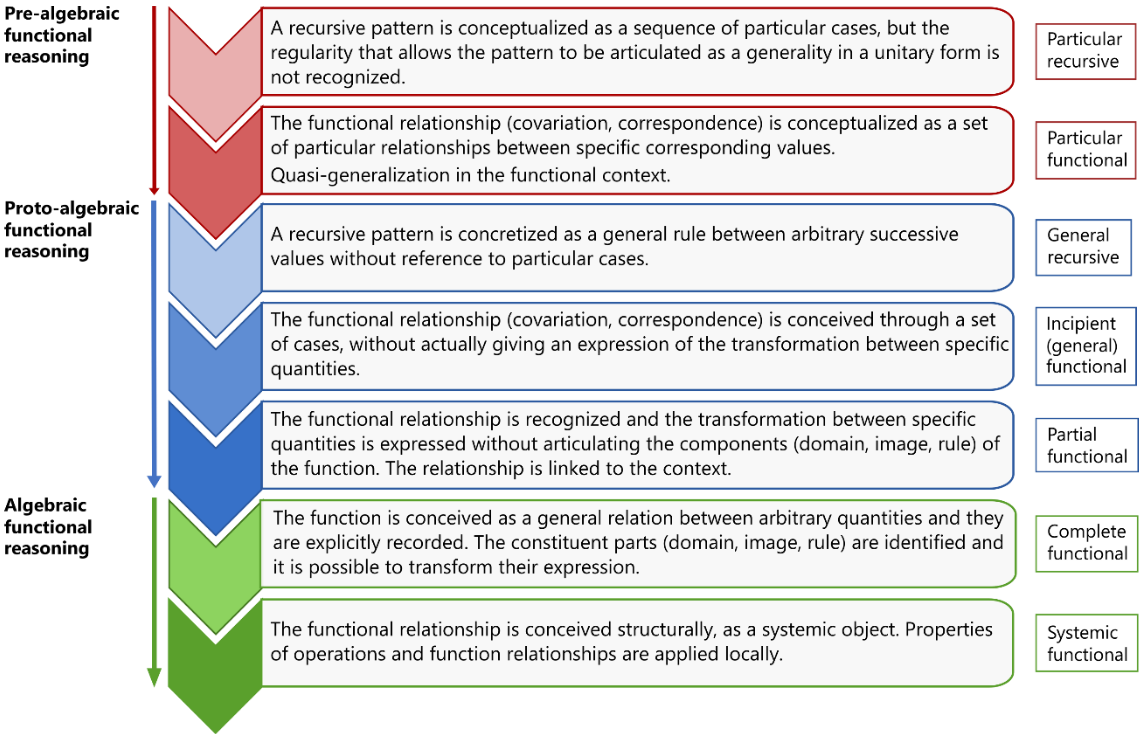


Figure 3. Layers of functional reasoning

We consider as pre-algebraic functional reasoning when the relationship is conceived as a set of particular instances. In *particular recursive reasoning*, a recursive pattern is conceptualized as a sequence of particular cases, without recognizing the regularity that defines the pattern as a generality in unitary form (Blanton et al., 2015). In *particular functional reasoning*, the functional relationship of covariation or correspondence, is concretized as a set of particular relations between specific corresponding values. It assumes a quasi-generalization (Cooper and Warren, 2011) in the functional context.

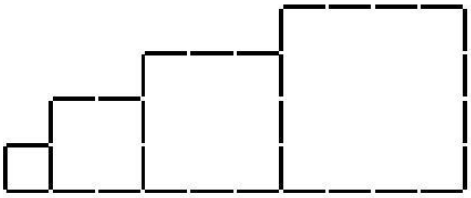
Proto-algebraic functional reasoning is that in which the functional relationship is recognized in general, but neither the structure of the function is recognized nor the expression of the relationship is transformed. In *general recursive reasoning*, a recursive pattern is made explicit as a general rule between arbitrary successive values without reference to particular cases (Blanton et al., 2015). Although it is possible to identify features of contextual generalization (Radford, 2018) in practices in which a general recursive pattern is recognized, the function is not involved as a covariation or correspondence relationship. *Incipient (general) functional reasoning* leads to qualitative recognition of the general relationship (covariation or correspondence) or its recognition through a set of cases, but without giving an expression of the transformation between the specific quantities (primitive general-functional reasoning of Blanton et al., 2015; basic-functional of Stephens et al., 2017). In *partial functional reasoning,* the functional relationship is identified and the transformation between specific quantities is expressed, but domain, image, and correspondence rule are not articulated. There is no syntactic calculation involved in the rule's treatment. At this level, we consider features of the emerging general-functional and condensed general-functional reasoning of Blanton et al. (2015) and Stephens et al. (2017) when the function and its constituent parts retain a contextual dimension.

Finally, algebraic functional reasoning involves considering the function as a structure. In *complete (general) functional reasoning*, the general relationship of correspondence between arbitrary quantities is recognized and explicitly recorded in symbolic form (non-contextual condensed general-functional). The limits of the generality of the relationship are perceived, i.e. in which set of values the relationship is maintained and when it ceases to be so (object-function for Blanton et al., 2015). Once the correspondence rule is constructed, it is possible to decompose its constituent parts, differentiating the variables from the operations in the context of the problem and transforming it to obtain equivalent expressions. In *systemic functional reasoning*, the function acts as an object, with its own properties and multiple representations, and the properties of operations and function relations are recognized and applied locally (incipient structural reasoning with real functions of a real variable).

5. EXPANDED MODEL FOR THE EAR

In this section, we expand the EAR model developed by Aké et al. (2013) and Godino et al. (2014) by distinguishing two sublevels of algebraic reasoning within each of the four levels proposed by these authors. Considering the intrinsic complexity of the elements of algebraic reasoning detailed in the previous section (generalization, representation, structural reasoning, and functional reasoning), we describe the characteristics of the mathematical practices that allow the definition of different sublevels within the arithmetic (pre-algebraic), proto-algebraic, and algebraic levels of the previous model. We illustrate the descriptions of the different sublevels through tasks that motivate functional-type practices, in which patterns and functions are involved (Problems 1 and 4), and structural-type practices, involving the properties of operations, equations, and inequalities (Problems 2 and 3). These are described as follows:

Problem 1. Ana entertains herself by making increasingly larger squares (linked to each other) with sticks as shown in the image. When she has nine squares, she wonders: how many sticks will I need to make the tenth square?



Problem 2. There are six seats between chairs and stools. Chairs have four legs and stools have three. In total, there are 20 legs. How many chairs and stools are there? (Godino et al., 2014)

Problem 3. Juan has 6 more pencils than Ana. Antonio has 8 more pencils than Juan. If Ana and Juan have more pencils than Antonio, how many pencils could Ana have?

Problem 4. Laura has been given a toy worm made of pieces. The worm with the head and one piece measures 9 cm. The worm with the head and two pieces measures 13 cm.

a) How many pieces did the worm have when it measured 25 cm? Explain your answer.
b) Write in mathematical form the length of the worm when, in addition to the head, a given number of pieces are added. Explain your answer.

First, we present a general description of each EAR level adapted from the proposal by Godino et al. (2014). Next, we describe the criteria that define the sublevels, focusing on the elements (generalization, representation, structural and functional reasoning) and degrees of sophistication described in Section 4. We exemplify the application of the new criteria to analyze the algebraic activity involved in solving these problems. When possible, we analyze practices developed by students; otherwise, we analyze solutions anticipated by the authors.

5.1 Pre-algebraic activity. Level 0 of the EAR

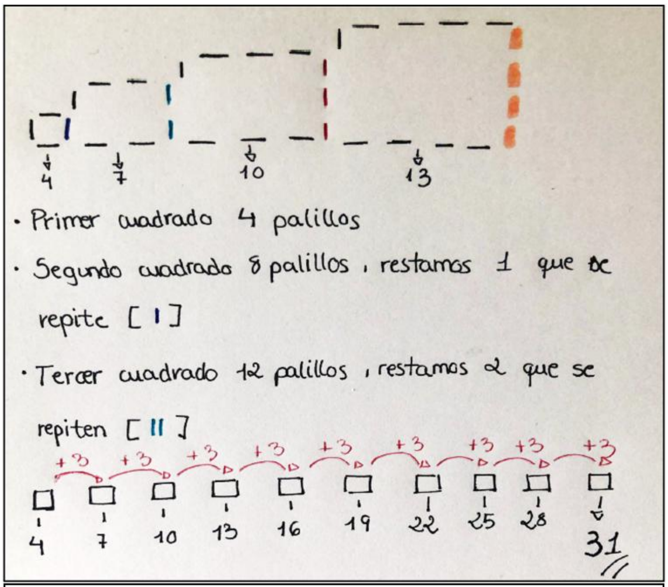
At the EAR level 0, practices involve and operate with extensive objects or first-degree of generality intensive objects expressed through gestural, iconic, natural, or numeric languages. Equality is used only in its operational sense. In functional practices or patterns, the relationship is recognized only in specific cases (Godino et al., 2014).

5.1.1. Level 0.I. Arithmetic activity

We propose the following criterion:

> Practices at level 0.I of EAR are those in which an arithmetic generalization occurs or a particular recursive reasoning is involved.

Figure 4 includes a level 0.I solution to Problem 1.

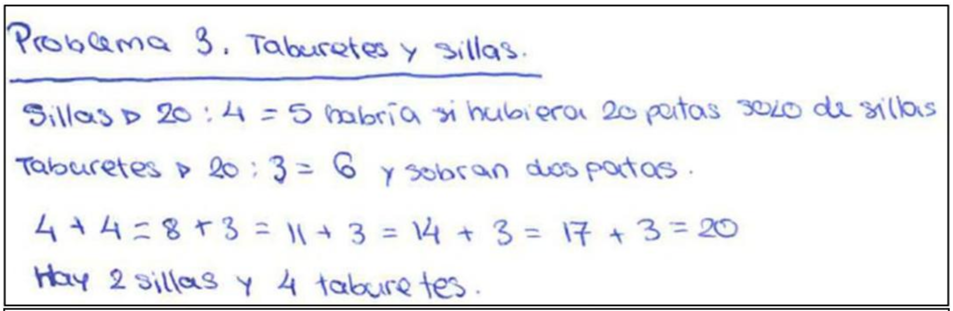


Translation:

First square 4 sticks
Second square 8 sticks, we subtract 1 that repeats [|]
Third square 12 sticks, we subtract 2 that repeat [||]

Figure 4. Solution of level 0.I to Problem 1.

In the mathematical practices shown in Figure 4, a treatment in the iconic/figural register occurs leading to the decomposition of the figure into each square that composes it. The student detects the relationship (+3) and applies it locally only to some cases (arithmetic generalization). A recursive pattern is conceptualized (to calculate an element you need the value corresponding to the previous element) on particular elements, without explicitly stating the general rule in a unitary form, that is, particular recursive reasoning. Therefore, it is a mathematical activity of level 0.I.


Problema 3. Taburetes y sillas.
Sillas ▸ 20 : 4 = 5 habría si hubiera 20 patas solo de sillas
Taburetes ▸ 20 : 3 = 6 y sobran dos patas.
4 + 4 = 8 + 3 = 11 + 3 = 14 + 3 = 17 + 3 = 20
Hay 2 sillas y 4 taburetes.

Translation:

Problem 3. Stools and chairs.
Chairs ‣ 20 : 4 = 5 there would be if there were 20 legs only of chairs
Stools ‣ 20 : 3 = 6 and two legs are left over.
4 + 4 = 8 + 3 = 11 + 3 = 14 + = 17 + 3 = 20
There are 2 chairs and 4 stools.

Figure 5. Solution of level 0.I to Problem 2.

Figure 5 includes one student's solution to Problem 2. The student uses a trial-and-error strategy to solve the problem, operating with first-degree intensive objects (natural numbers) using natural and numeric language. There is no structural treatment, and equality has only an operational sense. Therefore the mathematical activity is of level 0.I.

5.1.2. Level 0. II. Sophisticated arithmetic activity

We consider the following rule for assigning this level:

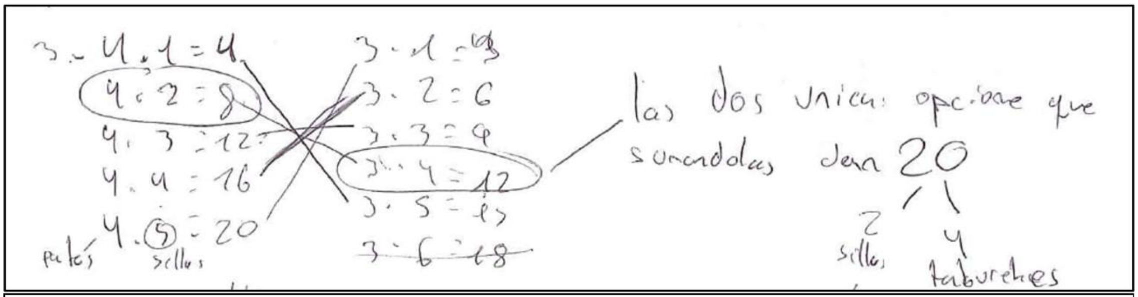
> EAR level 0.II practices are those in which sophisticated arithmetic generalization occurs. From a structural viewpoint, pseudo-structural reasoning is involved. In functional practices, particular functional reasoning (covariation or correspondence) is observed.

> Below, one possible level 0.II solution to Problem I is shown:

> Square 2 is obtained from 1 by adding 3. Similarly, square 3 is obtained from 2 by adding 3. Counting in the drawing, I observe that for square 3, I need 10 sticks.

The number of sticks I will need for element 10 will be the result of the sum: 10+3+3+3+3+3+3+3= 31 sticks.

In this solution, the relationship between particular values is conceptualized: the number of sticks needed for element 10 can be determined by adding 3 to the number of sticks needed for element 3 seven times. However, this relationship is not expressed as the multiplication of 3 by the difference between both values, never managing to explicitly express that generality analytically (Vergel et al., 2022).



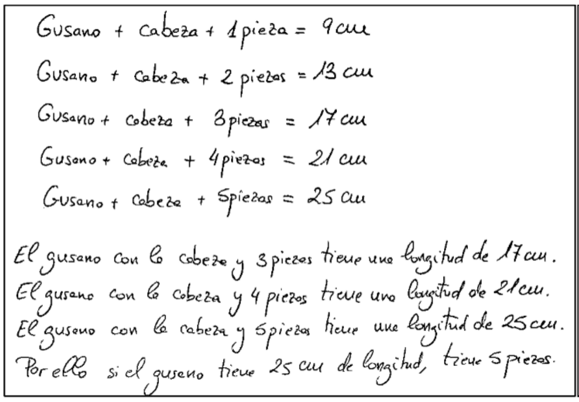
Translation:

The only two options that add up to 20

2 chairs 4 stools

Figure 6. Solution of level 0.II to Problem 2.

Figure 6 includes one student's solution to Problem 2. Through their practices, the student seems to recognize the structure associated with the situation (pseudo-structural use of numerical relationships): the first five multiples of 4 and the first five multiples of 3 are obtained. Possible combinations corresponding to 6 seats are connected with lines (if there is 1 chair there would be 5 stools, if there are 2 chairs there would be 3 stools, etc.). The solution is obtained by selecting the only possible combination whose sum is 20. The activity developed corresponds to a level 0.II.



Translation:

Worm + head + 1 piece = 9 cm
Worm + head + 2 pieces = 13 cm
Worm + head + 3 pieces = 17 cm
Worm + head + 4 pieces = 21 cm
Worm + head + 5 pieces = 25 cm
The worm with the head and 3 pieces has a length of 17 cm.
The worm with the head and 4 pieces has a length of 21 cm.
The worm with the head and 5 pieces has a length of 25 cm.
Therefore, if the worm is 25 cm long, it has 5 pieces.

Figure 7. Solution of level 0.II to Problem 4 (a).

Figure 7 shows one student's solution to the first part of Problem 4. The student conceptualizes the functional relationship between the number of pieces and the length of the worm as a set of particular relationships between the corresponding specific values (particular functional reasoning). In this way, the student determines which number of pieces corresponds to 25 cm in length. The mathematical activity in this case is level 0.II.

5.2. Primary proto-algebraic activity. Level 1 of EAR

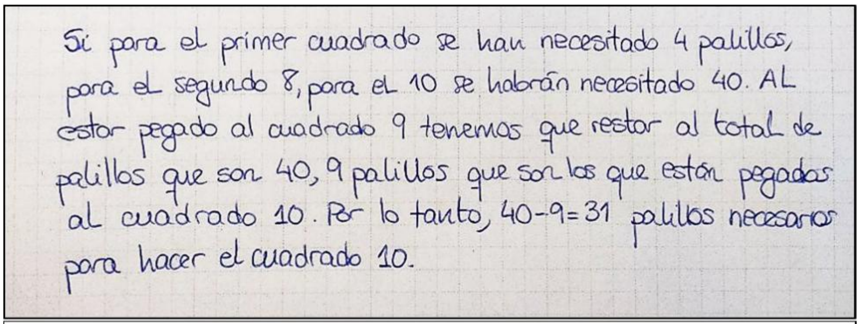
At level 1 of EAR, second-degree intensive objects participate (classes of first-degree intensive objects), whose generality is explicitly recognized through natural, iconic, numeric, or diagrammatic languages. Equality is used as equivalence (Godino et al., 2014). The types of structures, generalizations, and transformations between the registers involved allow differentiating the following sublevels.

5.2.1. Level 1.I. Incipient proto-algebraic

We propose the following criterion to assign the incipient proto-algebraic level:

> Conversions between natural and diagrammatic languages and treatment in diagrammatic language, are characteristics of level 1.I. From a structural perspective, incipient structural reasoning with first-degree structures is involved. Practices that imply factual generalization are also typical of this first proto-algebraic level.

Figure 8 includes one student's solution to Problem 1, in which the mathematical activity is considered to be proto-algebraic of level 1.I.

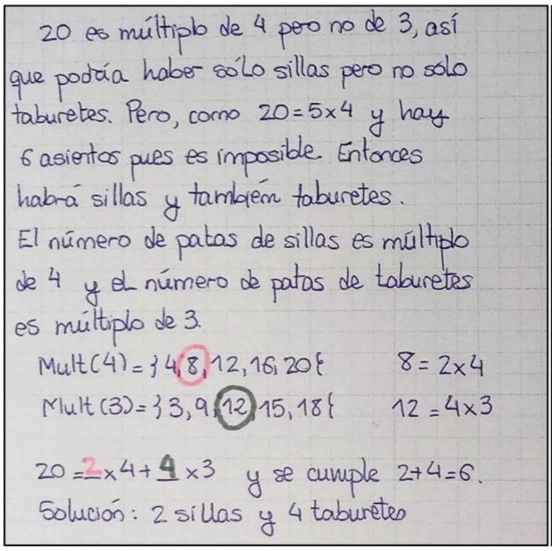

Si para el primer cuadrado se han necesitado 4 palillos, para el segundo 8, para el 10 se habrán necesitado 40. AL estar pegado al cuadrado 9 tenemos que restar al total de palillos que son 40, 9 palillos que son los que están pegados al cuadrado 10. Por lo tanto, 40-9=31 palillos necesarios para hacer el cuadrado 10.

Translation:

If for the first square 4 sticks were needed, for the second square 8, for the 10th square 40 sticks will have been needed. Since the square is attached to square 9, we have to subtract from the total number of sticks, which are 40, 9 sticks that are attached to the square 10. Therefore, 40-9=31 sticks to make the square 10.

Figure 8. Solution of level 1.I to Problem 1. Factual generalization.

In this solution, the indeterminate is not explicitly stated; rather, it is expressed through concrete actions. A factual type of generalization (Radford, 2018) occurs: which is generalized is the operational scheme that determines the number of sticks (multiplying by 4 the number corresponding to the square's position and subtracting the previous position which is the number of sticks shared with the previous figure), linked to the concrete level of specific numerical values. The practice corresponds to a level 1.I of EAR.



Translation:
20 is a multiple of 4 but not of 3, so there could be only chairs but not only stools. But, as 20=5×4 and there are 6 seats, it is impossible. So there will be chairs and also stools. The number of legs of chairs is a multiple of 4 and the number of legs of stools is a multiple of 3.
Mult(4)={4,8,12,16,20}
Mult(3)={3,9,12,15,18}
20=2×4+4×3 and the following is true 2+4=6
Solution: 2 chairs and 4 stools

Figure 9. Solution of level 1.I to Problem 2. Incipient structural reasoning with first-degree structure.

In the solution to Problem 2 included in Figure 9, incipient structural reasoning with first-degree structures is observed, as relations and properties of operations with first-degree intensive objects are explicitly stated and locally applied, specifically, the divisibility relation and its properties with natural numbers. Therefore, the mathematical activity corresponds to a level 1.I of EAR.

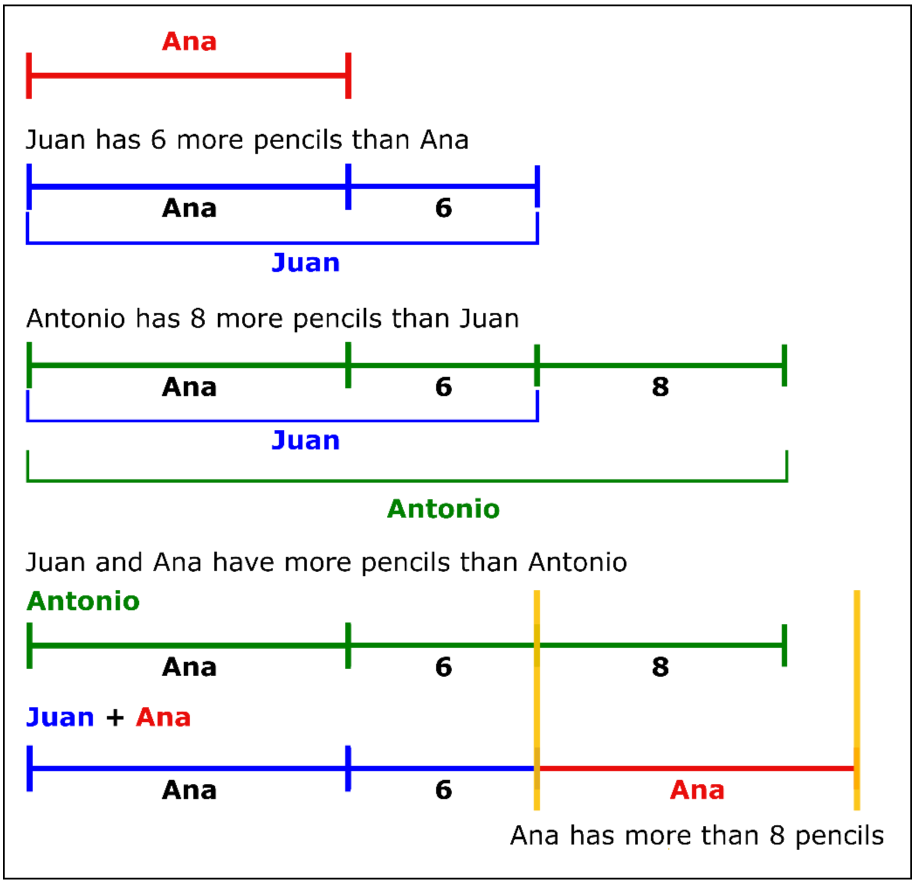


Figure 10. Solution of level 1.I to Problem 3. Treatment in diagrammatic language.

Figure 10 includes a solution to Problem 3 using the diagrammatic register. The segments represent the quantities of pencils of Ana, Juan (starting from Ana's) and Antonio (starting from Juan's). There are congruent conversions between the natural and the diagrammatic, as well as a treatment in the diagrammatic language to determine the relationships between the lengths of these segments. Solving the problem using this strategy may require several attempts to construct segments that fit the condition "Juan and Ana have more pencils than Antonio". The mathematical activity corresponds to level 1.I of EAR.

5.2.2. Level 1.II. Emergent proto-algebraic

We propose the following criterion:

> Conversions between natural and tabular languages are characteristic of sublevel 1.II, although the latter is used only as a record of particular values. Symbolic language can be used to express unknown quantities; however, it is not operated with. From a structural viewpoint, partial or complete structural reasoning of

degree 1 is brought into play. From a functional perspective, general recursive reasoning is involved.

The application of the properties of the order relation in the monoid of natural numbers and its relation with the addition operation can be used to solve Problem 3.

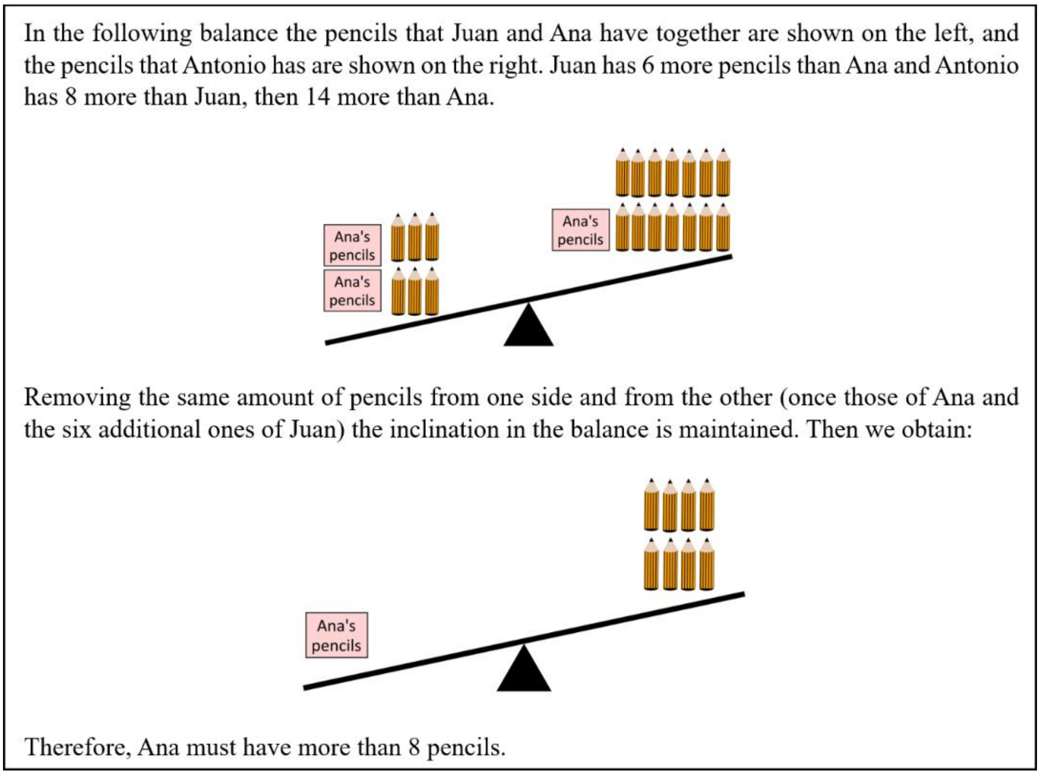
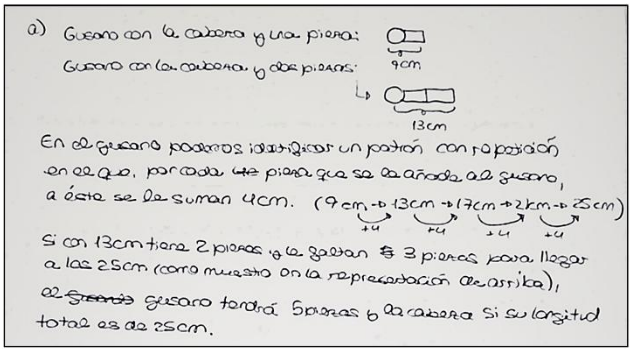


Figure 11. Solution of level 1.II to Problem 3. Complete structural reasoning of degree 1.

The balance model allows the recognition of the complete numerical relationship expressed in the inequality as it undergoes transformations (Linchevski and Herscovics, 1996). In this solution, a complete structural reasoning with intensives of degree 1 is observed: the cancellation of identical terms on both sides of the inequality requires the articulation of the order relation and the properties of operations with natural numbers. Therefore, it is a practice of level 1.II.



Translation:

Worm with the head and one piece: 9 cm
Worm with the head and two pieces: 13 cm
In the worm we can identify a repeating pattern in which, for each piece added to the worm, 4 cm are added to it.
If with 13cm it has 2 pieces and 3 pieces missing to reach 25 cm (as shown in the representation above), the worm will have 5 pieces and the head if its total length is 25 cm.

Figure 12. Solution of level 1.II to Problem 4 (a). General recursive reasoning.

To determine the number of pieces the worm has in the first part of Problem 4, the student whose solution is included in Figure 12 determines the relation "for each piece added to the worm, 4 cm are added to it". We observe that the student, despite noting the covariation between the quantities compared, does not use it to determine the number of pieces the worm has with a length of 25 cm, but completes the series by adding these 4 cm, until the required number of pieces is determined. Therefore, the solution falls on the recognition of a general recursive relation, which is characteristic of level I.II.

5.3. Level 2 of EAR. Advanced proto-algebraic activity

Level 2 of EAR is defined by the use of unknowns, generalized numbers, or variables that, although they may be expressed symbolically, remain tied to the context. Conversions between different SRRs occur. In functional practices, the function appears as a covariation or correspondence relationship. Generality is recognized, but variables are not operated with to obtain canonical forms of expression. In structural practices, arithmetic equations and inequalities are solved without operating with the unknown (Godino et al., 2014).

5.3.1. Level 2.I. Consolidated proto-algebraic

The consolidated proto-algebraic level is characterized as follows:

> Treatment in tabular language is characteristic of level 2.I. The tabular register is used as an icon of a structure of relationships, to identify and describe changes between variables, both in covariation and correspondence. Incipient structural reasoning with second-degree structures intervenes. In structural practices, equations of the form $Ax + B = C$ ($A, B, C \in \mathbb{R}$) are solved. This level includes practices that involve contextual generalization or incipient (general) functional reasoning.

Below, a possible solution to Problem 1 is included, in which the mathematical activity is considered to be at level 2.I.

> To calculate the number of sticks needed to build a square in a given position, I always have to multiply the position of the square by 4 and subtract the position of the previous figure. For the square in position 10, I will need 4×10-9=31 sticks.

In this solution, a general rule ("multiply the position of the square by 4 and subtract the position of the previous figure") is obtained that allows determining the number of sticks to build a square without being restricted to numerical representation. It is expressed in generic terms but linked to the context (contextual generalization) (Radford, 2018). This is, as a result, an activity of level 2.I of EAR.

Taking into account that the number of seats is 6 and that each chair provides 4 legs and each stool 3, a table can be created with all possible cases.

×4, ×3

Number of seats	1	2	3	4	5	6
Legs of chairs	4	8	12	16	20	24
Legs of stools	3	6	9	12	15	18

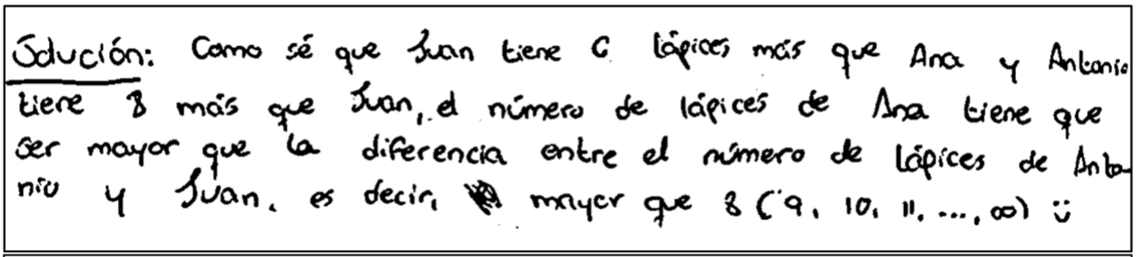
We look for which combination of chair legs and stool legs add up to 20, so that the number of chairs and stools add up to 6.

Then, there must be 2 chairs and 4 stools, since there are 6 seats in total (2+4=6) and the total number of legs is 20 (8+12=20).

Number of seats	1	2	3	4	5	6
Legs of chairs	4	8	12	16	20	24
Legs of stools	3	6	9	12	15	18

Figure 13. Solution of level 2.I to Problem 2. Structural use of tabular register.

In the solution proposed in Figure 13 to Problem 2, there is a conversion from natural language to tabular language. The table acts as a structure of relations, and its treatment allows obtaining knowledge about characteristics of the variables and how they covary. This is an activity of level 2.I.



Solución: Como sé que Juan tiene 6 lápices más que Ana y Antonio tiene 8 más que Juan, el número de lápices de Ana tiene que ser mayor que la diferencia entre el número de lápices de Antonio y Juan, es decir, mayor que 8 (9, 10, 11, ..., ∞)

Translation:

Since I know that Juan has 6 more pencils than Ana and Antonio has 8 more than Juan, the number of pencils of Ana has to be greater than the difference between the number of pencils of Antonio and Juan, that is, greater than 8 (9, 10, 11, ..., ∞).

Figure 14. Solution of level 2.I to Problem 3. Incipient functional reasoning.

In Figure 14, the student recognizes a generalized relationship between the number of pencils of Ana, Antonio, and Juan, but does not express it through mathematical transformation between generic quantities. In other words, a general correspondence is recognized in a primitive form (Blanton et al., 2015), showing incipient functional reasoning. Mathematical activity is associated with EAR level II.

At level 2.I, local relationships and the properties of operations on second-grade intensive objects are applied. In particular, mathematical practices in which the properties of ratios and proportions are locally used (for example, order or equivalence relationships between fractions) are associated with this level. Thus, in Problem 4, recognizing the direct proportionality relation between the number of pieces added after the worm's head and their total length, to establish and solve the equation $4x=20$, where x is the number of pieces added, and 20 (=25-5) is the length of the worm without considering the head, is an activity of level 2.I.

5.3.2. Level 2.II. Sophisticated proto-algebraic

We propose the following criterion for assigning the sophisticated proto-algebraic level:

> Level 2.II includes treatment in graphical language. Partial or complete structural reasoning of degree two is involved. Inequalities of the form $Ax + B < C$ or $Ax + B > C$ (A, B, $C \in \mathbb{R}$) are solved. This level also includes practices that involve symbolic-contextual generalization or partial functional reasoning.

The analytical calculation involved in solving inequalities entails a greater ontosemiotic complexity than that required in solving equations, as the intervening and emerging objects from these systems of practices (variables) are of a higher level than those involved in solving equations (unknowns) (Aké et al., 2013).

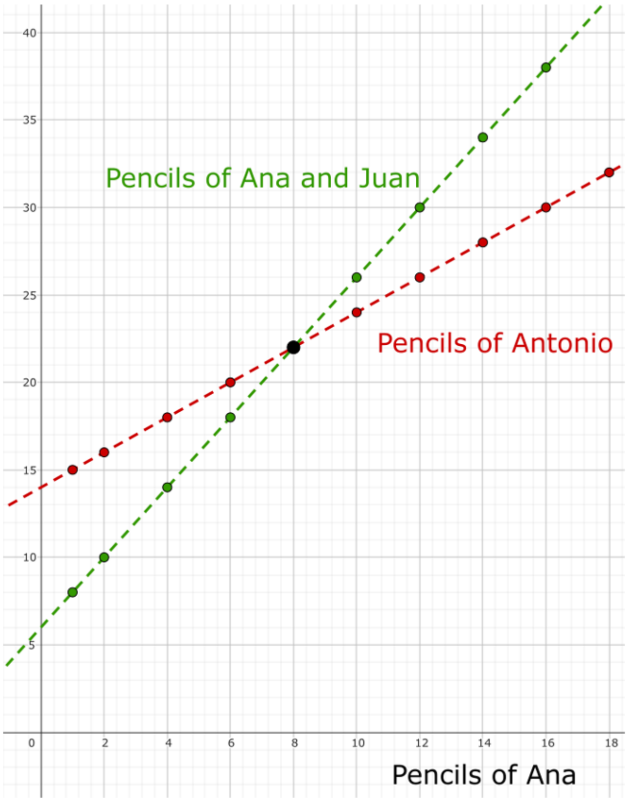
A solution to Problem 1 of level 2.II would be as follows:

> To calculate the number of sticks I need to build a square in position n, I determine the number of sticks that constitute the square, that is, $4n$ and subtract the number of sticks corresponding to one side of the previous square, that is, $n-1$. Therefore, we have $4n-(n-1)$. For the square in position 10, I will need $4 \times 10-(10-1)=31$ sticks.

In the previous solution, a general formula that determines the number of sticks needed to construct the square at a given position *n* is deduced. This rule is expressed in symbolic language but retains the spatial-contextual dimension and does not operate with literal symbol to obtain a canonical form of generality. This type of generalization, which we have named symbolic-contextual, is characteristic of level 2.II of EAR.

In Figure 15, a possible solution to Problem 3 is included, which involves the conversion from numeric to graphic languages and the treatment in this last register to interpret the graphs of two affine functions.

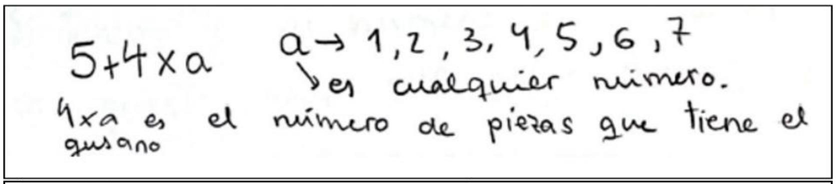
We graphically represent the number of pencils that Juan and Ana have together (in green) and the number of pencils that Antonio has (in red) as a function of the number of pencils that Ana has.



We observe that, if Ana has less than 8 pencils, then Ana and Juan have less than Antonio (the green points are below the red ones), and just when Ana has 8 pencils they have the same (marked in black on the graph). If Ana has 8 pencils, then Antonio will have the same as Ana and Juan. If she has more, Ana and Juan will have more than Antonio (the green dots are above the red ones).

Figure 15. Solution of level 2.II to Problem 3. Graphical conversion and treatment.

The student must choose which variable (in this case, Ana's pencils) and what values will be represented on the x-axis. Then, the functions (number of pencils of Ana and Juan, number of pencils of Antonio), whose domain and range take discrete values, are represented. Deciding what number of pencils Ana can have so that together with Juan's they add up to more than Antonio's, involves interpreting the graph of the functions, and the meaning of one being above the other in a certain range of the variable (treatment in the graphic register). The mathematical activity is considered to be of level 2.II.



Translation:

a is any number.

$4xa$ is the number of pieces the worm has.

Figure 16. Solution of level 2.II to Problem 4 (b). Partial functional reasoning.

Figure 16 shows the response of a student to the second part of Problem 4. The student correctly expresses the rule symbolically, but does not correctly articulate it with the meaning of the variables. The independent variable a is "any number," referring to the number of pieces of the worm (natural number). Then they indicate that $4\times a$ is the "number of pieces the worm has" and not the length. Therefore, this is a case of partial functional reasoning, and the algebraic activity carried out by the student is of level 2.II.

5.4. Level 3 of EAR. Algebraic activity

In level 3 of EAR, intensive objects represented in symbolic-literal form are generated, which participate in analytical calculations without referring to contextual information (treatment in symbolic language). Unknowns, generalized numbers, and specific functions are involved, and variables are operated with to obtain canonical forms of expression (Godino et al., 2014).

5.4.1. Level 3.I. Incipient algebraic

We propose the following criterion for assigning Level 3.I:

> In structural practices of level 3.I, the unknown is operated with to solve equations of the type $Ax + B = Cx + D$ ($A, B, C, D \in \mathbb{R}$). Practices that involve symbolic generalization and those in which complete functional reasoning appears are also considered at this level.

Below, we show a solution to Problem 1, where the mathematical activity is considered to be of level 3.I.

> To calculate the number of sticks needed to construct a square in position n, I determine the number of sticks necessary to form the complete square,

i.e. $4n$, and subtract the number of sticks corresponding to one side (shared) of the previous squares, i.e. $n-1$. Thus, we obtain $4n-(n-1)=4n-n+1=3n+1$. For the square at position 10, I will need $3\times 10-1=31$ sticks.

In this solution to Problem 1, the general correspondence relationship is recognized in symbolic language, operating with literal symbols to obtain the canonical form of representation. Therefore, the activity is of level 3.I.

Next, we show a possible solution to Problem 2 of level 3.I.

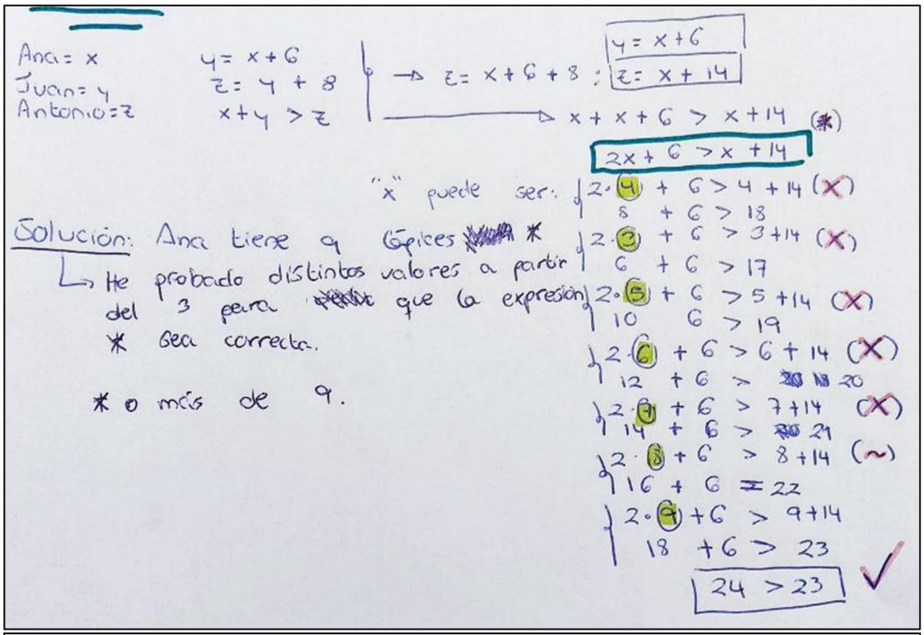
Each chair has 4 legs and each stool has 3. If we call S the number of chairs and T the number of stools, we can express the total number of legs as

$$4S+3T=20.$$

Moreover, since there are 6 seats, $T+S=6$, or equivalently, $T=6-S$. Substituting in the expression for the total number of legs gives $S+3(6-S)=20$, from where $S+18=20$, then $S=2$. If there are two chairs, then there are 4 stools.

In the above mathematical practices, a system of two equations with two unknowns, the number of chairs and the number of stools, (conversion from natural to symbolic language) is posed, and a substitution technique is used to solve it (treatment in the symbolic register). Therefore, the activity is of level 3.I.

Figure 17 shows one level 3.I solution to Problem 3 given by a secondary education student.



Translation:

Solution: Ana has 9 pencils. I have tried different values starting from 3 to make the expression
* correct.

* or more than 9

Figure 17. Solution of level 3.I to Problem 3.

There is a conversion between the natural and symbolic registers. The student assigns a literal symbol to the number of pencils of each child (Ana's, *x*; Juan's, *y*; Antonio's, *z*) and symbolically establishes the relationship between them (for example, "Juan has 6 more pencils than Ana" goes to "$y = x + 6$"). Although the student comes to pose the inequality, they do not operate with the unknown to solve it, but they do so to canonically express the total number of pencils that Ana and Juan have ($x + x + 6$ as $2x + 6$), showing that they detach the literal symbol from the context (treatment in symbolic language). Solving the inequality by operating with the unknown is characteristic of a consolidated algebraic level, as described below.

5.4.2. Level 3.II. Consolidated algebraic

We propose the following criterion for assigning Level 3.II:

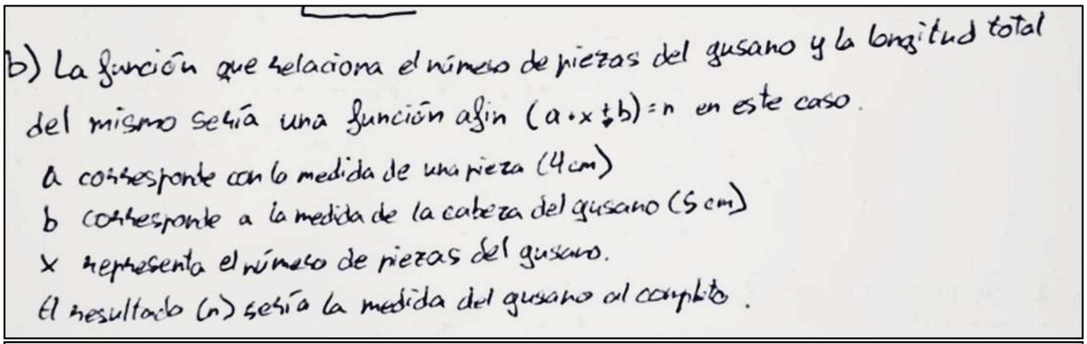
> Structural practices in which the unknown is operated with to solve inequalities of type $Ax + B < Cx + D$ ($A, B, C, D \in \mathbb{R}$) are considered to be of level 3.I. From a functional perspective, this level involves systemic functional reasoning.

The student whose solution to Problem 3 is included in Figure 13 could have solved the inequality $2x + 6 > x + 14$:

$$2x + 6 > x + 14 \Rightarrow 2x - x > 14 - 6 \Rightarrow x > 8,$$

to conclude that Ana should have more than 8 pencils. In this case, the mathematical activity is of level 3.II.

The first functions that students name in secondary education are linear and affine functions. Recognizing a function as part of one of these families and locally applying its properties (systemic functional reasoning) is associated with EAR level 3.II. Thus, at the beginning of secondary education, a solution to the second part of Problem 4 such as the one included in Figure 18, can be expected.



Translation:

The function relating the number of pieces of the worm and the total length of the worm would be an affine function $(ax + b) = n$ in this case.
a corresponds to the size of one piece (4cm).
b corresponds to the size of the head of the worm (5cm).
x represents the number of pieces of the worm.
The result (n) would be the size of the complete worm.

Figure 18. Solution of level 3.II to Problem 4 (b). Systemic functional reasoning.

Once the student recognizes the affine function that relates the number of pieces worm can have and its total length it is expressed as

$$l(n) = 4n + 5$$

they can determine the inverse function of the previous one $n(l) = \frac{l-5}{4}$, which allows obtaining the number of pieces knowing the length of the worm. Therefore, the student could know the number of pieces the worm had when it was 25 cm long and solve the first part of Problem 4:

$$n(25) = \frac{25 - 5}{4} = 5 \text{ pieces.}$$

The treatment of the function as an object, which can be inverted, is typical of EAR level 3.II.

5. SYNTHESIS AND IMPLICATIONS

To ensure adequate teaching of algebra in primary education, it is necessary to develop tools that can be incorporated into teacher training programs that guide the analysis and design of tasks that promote algebraic reasoning in schoolchildren (Hohensee, 2017; Zapatera and Quevedo, 2021). This involves training future teachers to: a) encourage the search for regularities and properties and generalization through patterns; b) recognize functional relationships; c) develop relational thinking; d) familiarize students with the idea of unknowns and variables; e) use multiple representations (diagrams, tables, graphs) to help students develop ways of algebraic thinking and their consideration in modeling and problem solving; and f) advance in the use of symbolic language as a way of thinking (Carraher and Schlieman, 2018; Malara and Navarra, 2018; Warren, et al., 2016).

Generalization is at the heart of algebraic reasoning, and involves recognizing a regularity, generating new cases, and obtaining the expression of the general rule (Radford, 2010). However, as we have shown, there is not a single form of generalization, but different layers closely related to the scope of the rule, the semiotic representation registers used to elaborate the expression of the regularity, and its analyticity. On the one hand, what is generalized in most of the early algebraic activity are the structural aspects of the relationships and numerical patterns (Kieran et al., 2017); before generalizing, it is necessary to examine the structure of the mathematical situation. Teachers should not tacitly assume that prior work with arithmetic allows the recognition of underlying structures as entities (Radford, 2011). This reflection leads us to consider different types of structures, attending to the degree of intention of the objects on which properties, operations, and relationships are built, and to consider different degrees of structural reasoning according to how many of these elements of the structure are involved and how they are used.

On the other hand, it is not possible to separate the activity of generalization from the representations used to characterize and reason with the generalized relationships (Blanton et al., 2015; Radford, 2018). To foster algebraic reasoning in primary education, teachers have to develop their expertise on how to help students transition their representations (Malara and Navarra, 2018), knowing the role of pre- and proto-algebraic representations. Primary education students are capable of using non-conventional personal representations, such as gestures or words, to express indeterminate quantities (Radford, 2011). Furthermore, through progressive instruction, students can learn to use literal symbols or other representations such as tables or graphs (Blanton, et al., 2017; Carraher and Schliemann, 2018). Both diagrammatic and tabular registers facilitate the development of strategies that allow progression from more informal strategies (intuitive, trial and error, or arithmetic) towards more formal (algebraic) strategies, especially in the early modeling of problems involving equations. Models such as the balance model can offer the opportunity to semantically and syntactically establish a basis for solving linear equations (Filloy and Rojano, 1989). Graphical representation of functions by hand allows students to give meaning to the symbols in algebraic formulas, recognize their structure, and reason with and about the formulas (Kop et al., 2020a, 2020b). Since each of these representation systems has its own signs and rules, which condition the degree of generality of the mathematical practices they support, it has been possible to establish layers in the use of language, distinguishing transformations of pre, proto, and algebraic character.

Ultimately, algebraic reasoning involves the use of alphanumeric symbols (Blanton et al., 2017). Alphanumeric symbolism constitutes a precise and condensed semiotic system allowing to efficiently perform computations which could be difficult to perform within other semiotic system of representation (Radford, 2018). As a consequence, and based on previous work (Godino et al., 2014; Radford, 2010), the levels of our model are characterized by a progressive increase in the degree of semiotic contraction. However, a premature introduction to symbolic language can lead to an early development of formalization in which the symbols “become semantically empty” (Zeljić, 2015, p. 432). Promoting the development of pre- and proto-algebraic representations as a preliminary step to the introduction of the symbolic language can help to give meaning to alphanumeric symbols. In this sense, the layer stratification we propose in our work can serve as a guide of a progressive instructional process towards meaningful representation systems with higher level of semiotic contraction.

Moreover, representation systems are of great importance in the manipulation of functional relationships, where generalization and the study of the structure play an essential role. Given the importance of functional reasoning in the development of elementary algebra, we consider it necessary to pay attention to the various degrees of sophistication, not as "the shifts observed [or observable changes] in student thinking" (Stephens et al., 2017, p. 150), but in terms of the degree of intention and the nature of the functional relationship, its representation, and analytical treatment. Thus, we have distinguished layers of pre-algebraic functional reasoning when the relationship is conceived as a set of particular cases, proto-algebraic when the functional relationship is identified in a general manner, but its structure is not recognized nor its expression is transformed, and algebraic functional reasoning, when the function is conceived in a structural manner. In each of these layers, some degrees established by Blanton et al. (2015) and Stephens et al. (2017) are recognized.

In this paper, we started from the model of EAR levels by Godino et al. (2014), with the intention of clarifying the characteristics and limits between the arithmetic, proto-algebraic, and algebraic levels established in that framework. In the Annex, we include a synthesis of the characteristic features of the expanded model of EAR levels that we have developed in this study. This new expanded model of EAR can be used in teacher training as a guide for the analysis of algebraic reasoning manifested by students when solving mathematical tasks, or in the a priori analysis of anticipated solutions. It offers a detailed perspective of the progression in algebraic competence in the mathematical activity carried out by students. The sublevels we propose are determined on the basis of the following aspects: (i) the representations, their treatments and conversions; (ii) the different degrees of generalization and functional reasoning; (iii) mathematical structures and structural reasoning, and (iv) the analytical calculation involved. Although we consider our proposal to be sufficiently detailed and operational, it is clear that it does not exhaust all possibilities. For example, it is possible to compare the degree of sophistication of different types of conversions within each layer, both by the registers involved and by the congruence of these transformations. In addition, one could distinguish whether structural reasoning is partial or complete (use of the properties of operations and relations in a general and correlated manner).

We consider that this new expanded model of EAR, together with the illustrative examples of the different sublevels, will facilitate the work already started in teacher training for the development of algebraic reasoning in primary education (Aké et al, 2013; Burgos and Godino, 2018, 2022). These works show that despite the complexity of achieving such competence, it favors the analysis of implicated meanings and potential difficulties in the solution and creation of algebraic tasks.

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References

Aké, L. Godino, J. D., Gonzato, M. & Wilhelmi, M. R. (2013). Proto-algebraic levels of mathematical thinking. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 1-8). IGPME.

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa. Special Issue on Semiotics, Culture, and Mathematical Thinking, 9*(1), 267-300

Ayala-Altamirano, C. & Molina, M. (2021). El proceso de generalización y la generalización en acto. Un estudio de casos. *PNA*, *15*(3), 211-241.

Blanton, M. (2008). *Algebra and the elementary classroom: Transforming thinking, transforming practice*. Heinemann.

Blanton, M. Brizuela, B., Gardiner, A., Sawrey, K. & Newman-Owens, A. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics, 95*(2), 181-202.

Blanton, M., Brizuela, B. M., Murphy, A., Sawrey, K. & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds´s thinking about generalizing functional relationship. *Journal for Research in Mathematics Education*, *46*(5), 511-559.

Blanton, M. & Kaput, J. (2004). Elementary grades students' capacity for functional thinking. In M. Johnsen & A. Berit (Eds.), *Proceedings of the 28thinternational group of the psychology of mathematics education* (Vol. 2, pp. 135–142). Bergen University College.

Blanton, M. L., Levi, L., Crites, T. & Dougherty, B. J. (2011). *Developing essential understanding of algebraic thinking for teaching mathematics in grades 3-5*. NCTM.

Burgos, M. & Godino, J.D. (2018). Recognizing algebrization levels in an inverse proportionality task by prospective secondary school mathematics teachers. In L. Gómez Chova, A. López Martínez & I. Candel Torres (Eds.), *Proceedings of EDULEARN18 Conference* (pp. 2483–2491), IATED Academy.

Burgos, M. & Godino, J.D. (2022). Assessing the Epistemic Analysis Competence of Prospective Primary School Teachers on Proportionality Tasks. *International Journal of Science and Mathematics Education*, *20*, 367-389.

Campbell-Kelly, M., Croarken, M., Flood, R. & Robson, E. (2003). The history of Mathematical Tables. From Summer to Spreadsheets; Oxford University Press. USA. http://uruk-warka.dk/mathematics/ER7%20etal-tables.pdf

Carraher, D. & Schliemann, A. (2018). Cultivating early algebraic thinking. In C. Kieran (Ed.), *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds: The Global Evolution of an Emerging Field of Research and Practice* (pp. 107-138), Springer.

Cooper, T. J. & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 187–214). Springer.

Drijvers, P., Goddijn, A. & Kindt, M. (2011). Algebra education: exploring topics and themes. In P. Drijvers (Eds.), *Secondary algebra education* (pp. 5-26). Sense Publishers.

Duval, R. (2006). Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational Studies in Mathematics, 61*, 103–131. https://doi.org/10.1007/s10649-006-0400-z

Duval, R. (2017). *Understanding the mathematical way of thinking – The registers of semiotic representations*. Springer.

Ellis, A. B. (2007). A taxonomy for categorizing generalizations. *The Journal of the Learning Sciences, 16*(2), 221-262.

Filloy, E., Puig, L. & Rojano, T. (2008). *Educational algebra. A theoretical and empirical approach.* Springer.

Filloy, E. & Rojano, T. (1989). Solving equations: the transition from arithmetic to algebra. *For the learning of Mathematics, 9*(2), 19-25.

Font, V., Godino, J. D. & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*, 97-124.

Font, V., Planas, N. & Godino, J. D. (2010). Modelo para el análisis didáctico en educación matemática. *Infancia y Aprendizaje, 33*(2), p. 89-105.

Fujii, T. (2003). Probing Students' Understanding of Variables through Cognitive Conflict Problems: Is the Concept of a Variable So Difficult for Students to Understand? In N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (pp. 49–65). PME.

Giacomone, B., Godino, J.D., Blanco, T.F. & Wilhelmi, M. R. (2022). Onto-semiotic Analysis of Diagrammatic Reasoning. *International Journal of Science and Mathematics Education, 21,* 1495–1520. https://doi.org/10.1007/s10763-022-10316-z

Godino, J. D., Aké, L., Gonzato, M. & Wilhelmi, M. R. (2014). Niveles de algebrización de la actividad matemática escolar. Implicaciones para la formación de maestros. *Enseñanza de las Ciencia*s, *32*(1), 199-219.

Godino, J. D. Batanero, C. & Font, V. (2019). The onto-semiotic approach: implications for the prescriptive character of didactic. *For the Learning of Mathematics, 39* (1), 37-42.

Godino, J. D., Castro, W., Aké, L. & Wilhelmi, M. R. (2012). Naturaleza del razonamiento algebraico elemental. *BOLEMA, 26* (42B), 483-511.

Hewitt, D. (2019). “Never carry out any arithmetic”: the importance of structure in developing algebraic thinking. In U.T. Jankvist, M. Van den Heuvel-Panhuizen & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 558-565), Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Hohensee, C. (2017). Preparing elementary prospective teachers to teach early algebra. *Journal of Mathematics Teacher Education, 20,* 231-257.

Johnson, H. L. (2022) Task design for graphs: rethink multiple representations with variation theory, *Mathematical Thinking and Learning, 24* (2), 91-98, DOI: 10.1080/10986065.2020.1824056

Kaput, J. (1993). The urgent need for proleptic research in the representation of quantitative relationships. In T. A. Romberg, T. P. Carpenter, & E. Fennema (Eds.), *Integrating research on the graphical representation of functions* (pp. 279–312). Lawrence Erlbaum Associates.

Kaput, J.J., Blanton, M., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D.W. Carraher, & M.L. Blanton (Eds.), *Algebra in the early grades* (pp. 19–55). Routledge.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner y C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (Vol. 4, pp. 33-56). NCTM.

Kieran, C. (2022) The multi-dimensionality of early algebraic thinking: background, overarching dimensions, and new directions. *ZDM Mathematics Education, 54*, 1131–1150. https://doi.org/10.1007/s11858-022-01435-6.

Kieran, C., Pang, J. S., Ng, S. F., Schifter, D. & Steinweg, A. S. (2017). Topic Study Group 10: Teaching and learning of early algebra. In G. Kaiser (Ed.), *The Proceedings of the 13th International Congress on Mathematical Education*. Springer.

Kilhamn, C., Röj-Lindberg, AS. & Björkqvist, O. (2019). School Algebra. In Kilhamn, C. & Säljö, R. (Eds.), *Encountering Algebra* (pp. 1-11). Springer. https://doi.org/10.1007/978-3-030-17577-1_1

Kop, P.M., Janssen, F.J., Drijvers, P.H. & van Driel, J.H. (2020a). The relation between graphing formulas by hand and students' symbol sense. *Educational Studies in Mathematics, 105*, 137–161. https://doi.org/10.1007/s10649-020-09970-3

Kop, P. M., Janssen, F. J., Drijvers, P. H. & van Driel, J. H. (2020b). Promoting insight into algebraic formulas through graphing by hand. *Mathematical Thinking and Learning, 25*(2), 125-144. https://doi.org/10.1080/10986065.2020.1765078

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. University of Chicago Press.

Linchevski, L. & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics*, *30*, 39–65. doi:https://doi.org/10.1007/BF00163752.

Malara, N. A. & Navarra G., (2018). New words and concepts for early algebra teaching: sharing with teachers epistemological issues in early algebra to develop students' early algebraic thinking. In C. Kieran (Ed.). *Teaching and Learning Algebraic Thinking with 5- to 12-year-olds. The Global Evolution of an Emerging Field of Research and Practic*e (pp. 27-50). Springer.

Mason, J., Stephens, M. & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21*(2), 10-32.

Pittalis, M. (2023). Young Students' Arithmetic-Algebraic Structure Sense: an Empirical Model and Profiles of Students. *International Journal of Sciences and Mathematics Education, 21*, 1865–1887. https://doi.org/10.1007/s10763-022-10333-y

Pittalis, M., Pitta-Pantazi, D. & Christou, C. (2020). Young students' functional thinking modes: The relation between recursive patterning, covariational thinking, and correspondence relations. *Journal for Research in Mathematics Education, 51*(5), 631–674. https://doi.org/10.5951/jresematheduc-2020-0164

Radford, L., Miranda, I. & Guzmán, J. (2008). Relative motion, graphs and the heteroglossic transformation of meanings: a semiotic analysis. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), *Proceedings of the Joint 32nd Conference of the International Group for the Psychology of Mathematics Education and the 30th North American Chapter* (vol. 4, pp. 161-168). The Ohio State University.

Radford, L. (2010). Layers of generality and types of generalization in pattern activities. *PNA*, *4*(2), 37-62.

Radford, L. (2011). Grade 2 Students´ Non-Symbolic Algebraic Thinking. In J. Cai y E. Knuth (Eds.), *Early algebraization. A global dialogue from multiple perspectives* (pp. 303-322). Springer-Verlag.

Radford, L. (2013). Three key concepts of the theory of objectification: knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, *2*(1), 7-44.

Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 3-25). Springer. https://doi.org/10.1007/978-3-319-68351-5_1

Radford, L. (2021). O ensino-aprendizagem da ágebra na teoria da objetivação. In V. Moretti & L. Radford (Eds.), *Pensamento algébrico nos anos iniciais: Diálogos e complementaridades entre a teoriada objetivação e a teoria histórico-cultural* (pp. 171-195). Livraria da Física.

Sfard, A. & Linchevski, L. (1994). The gains and the pitfalls of reification – the case of algebra. *Educational Studies in Mathematics*, *26*, 191–228

Sfard, A. (2020). Commognition. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_100031

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M. L., Knuth, E. & Gardiner, A. M. (2017). A learning progression for elementary students' functional thinking a learning progression for elementary students' functional. *Mathematical Thinking and Learning*, *19*(3), 143–166. https://doi.org/10.1080/10986065.2017.1328636

Torres, M.D., Brizuela, B.M., Cañadas, M.C. & Moreno, A. (2022) Introducing Tables to Second-Grade Elementary Students in an Algebraic Thinking Context. *Mathematics, 10* (1), 56. https://doi.org/10.3390/math10010056

Venkat, H., Askew, M., Watson, A. & Mason, J. (2019). Architecture of mathematical structure. *For the Learning of Mathematics, 39*(1), 13–17.

Vergel, R., Godino, J. D., Font, V. & Pantano, O. L. (2023). Comparing the views of the theory of objectification and the onto-semiotic approach on the school algebra nature and learning. *Mathematics Education Research Journal, 35,* 475–496 https://doi.org/10.1007/s13394-021-00400-y.

Vergel, R., Radford, L. & Rojas, P.J. (2022). Zona conceptual de formas de pensamiento aritmético "sofisticado" y proto-formas de pensamiento algebraico: una contribución a la noción de zona de emergencia del pensamiento algebraico. *Bolema, 36*(74), 1174-1192.

Warren, E., Trigueros, M. & Ursini, S. (2016). Research on the learning ad teaching of algebra. In Á. Gutierrez, G.C. Leder & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education* (pp. 73-108). Sense Publishers.

Zapatera, A. & Quevedo, E. (2021). The Initial Algebraic Knowledge of Preservice Teachers. *Mathematics, 9,* 2117.

Zeljić, M. (2015). Modelling the relationships between quantities: Meaning in literal expressions. *Eurasia Journal of Mathematics, Science and Technology Education, 11*(2), 431-442. https://doi.org/10.12973/eurasia.2015.1362a

Annex

Description of the characteristic features of the expanded EAR model. The objects corresponding to a particular EAR level could appear at higher levels.

Level of EAR	**Objects**		**Processes**		
	Degree of intension	**Languages**	**Generalization Functional reasoning**	**Transformations between registers**	**Analytical calculus Structural reasoning**
Arithmetic					
0.I	• Extensive objects. • First-degree of generality intensive objects.	• Gestual. • Iconic. • Natural. • Numeric.	• Arithmetic generalization. • Particular recursive reasoning.	• Treatments within iconic, natural, and numeric languages.	• Operations with extensive and first-degree intensive objects.

				• Conversions between iconic, natural, and numeric languages.	• Operational use of the equality.
0.II	• Quasi-variables.		• Sophisticated arithmetic generalization. • Particular functional reasoning.		• Pseudo-structural reasoning.
Primary proto-algebraic					
1.I	• Second-degree of generality intensive objects. • Emergent structures of first-degree.	• Diagrammatic.	• Factual generalization.	• Treatment within diagrammatic language. • Conversions between natural and diagrammatic languages.	• Incipient structural reasoning with structures of first-degree. • Equality used as equivalence
1.II	• Complete structures of first-degree.	• Tabular as a record of particular values • Symbolic to represent unknown quantities without operating with them.	• General recursive reasoning.	• Treatment within tabular language as a record of particular values. • Treatment within the symbolic language as a receptor or unknown quantity. • Conversion between natural and tabular languages.	• Partial or complete structural reasoning with structures of first-degree.
Advanced proto-algebraic					
2.I	• Unknowns. • Generalized numbers. • Variables. • Emergent structures of second-degree.	• Tabular. • Graphic. • Symbolic.	• Contextual generalization. • Incipient (general) functional reasoning.	• Treatment within tabular language to identify and describe changes between variables. • Conversion between the different languages.	• Incipient structural reasoning with structures of second-degree. • Resolution of equations of the type: $Ax + B = C$
2.II	• Complete structures of second-degree.		• Symbolic-contextual generalization. • Partial functional reasoning.	• Treatment within graphic language.	• Partial or complete structural reasoning with structures of second-degree. • Resolution of inequations of the type: $Ax + B < C$
Algebraic					

3.I	• Particular functions and their properties.	• Natural. • Numeric. • Diagrammatic. • Tabular. • Graphic. • Symbolic.	• Symbolic generalization. • Complete functional reasoning.	• Treatment within symbolic language.	• Resolution of equations of the type: $Ax + B = Cx + D$
3.II	• Functions, their properties and operations.		• Systemic functional reasoning.		• Resolution of inequations of the type: $Ax + B < Cx + D$