

Global-in-time solutions and qualitative properties for the NNLIIF neuron model with synaptic delay

María J. Cáceres ^{*} Pierre Roux[†] Delphine Salort [‡] Ricarda Schneider [§]

June 4, 2019

Abstract

The Nonlinear Noisy Leaky Integrate and Fire (NNLIIF) model is widely used to describe the dynamics of neural networks after a diffusive approximation of the mean-field limit of a stochastic differential equation system. When the total activity of the network has an instantaneous effect on the network, in the average-excitatory case, a blow-up phenomenon occurs. This article is devoted to the theoretical study of the NNLIIF model in the case where a delay in the effect of the total activity on the neurons is added. We first prove global-in-time existence and uniqueness of "strong" solutions, independently of the sign of the connectivity parameter, that is, for both cases: excitatory and inhibitory. Secondly, we prove some qualitative properties of solutions: asymptotic convergence to the stationary state for weak interconnections and a non-existence result for periodic solutions if the connectivity parameter is large enough. The proofs are mainly based on an appropriate change of variables to rewrite the NNLIIF equation as a Stefan-like free boundary problem, constructions of universal super-solutions, the entropy dissipation method and Poincaré's inequality.

Key-words: Leaky integrate and fire models, noise, blow-up, relaxation to steady state, neural networks, delay, global existence, Stefan problem.

AMS Class. No: 35K60, 82C31, 92B20

^{*}Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain. caceresg@ugr.es

[†]Laboratoire de biologie computationnelle et quantitative (LCQB), UMR 7238 CNRS, Université Pierre et Marie Curie, 75205 Paris Cedex 06, France et Laboratoire de Mathématiques d'Orsay (LMO), Université Paris-Sud, Paris-Saclay, Orsay, France. pierre.roux@math.u-psud.fr

[‡]Laboratory of Computational and Quantitative Biology (LCQB), UMR 7238 CNRS, Université Pierre et Marie Curie, 75205 Paris Cedex 06, France. delphine.salort@upcm.fr

[§]Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain. ricarda.schneider87@gmail.com

1 Introduction

Different scientific disciplines study the complex dynamics of neural networks. Over the last decades, mathematicians have been particularly interested in providing specific models to understand the behavior of neurons. One popular approach is to tackle qualitative properties of networks via partial differential equations by deriving mean fields models from stochastic differential equations. Depending on the choice of the intrinsic dynamics of neurons and on the type of synaptic coupling one may obtain different models: Fokker-Planck equations for uncoupled neurons [21], population density models of IF neurons with jumps [17, 15] and Fokker-Planck equations including conductance variables [25, 24], which are mean-field limits of integrate and fire networks assuming Poisson approximations [26, 14, 13]; time elapsed models [22, 23, 20], recently derived as mean-field limits of Hawkes processes [12, 11]; the mean-field McKean-Vlasov equations [19] related to the behaviour of Fitzhugh-Nagumo neurons [18], etc.

In this article we assume that the neurons are described via their membrane potential and that when the membrane potential reaches a critical or *threshold value* V_F , the neurons emit an action potential, also called spike, as a result of depolarization of their membrane, and their voltage values return to a *reset value* V_R ($V_R < V_F$). More precisely, we consider the following PDE model (see [1, 2, 3] for its derivation):

$$\frac{\partial \rho}{\partial t}(v, t) + \frac{\partial}{\partial v} [(-v + \mu(t - D))\rho(v, t)] - a \frac{\partial^2 \rho}{\partial v^2}(v, t) = N(t)\delta(v - V_F), \quad v \leq V_F, \quad (1.1)$$

where the function $\rho(\cdot, t)$ is the probability density of the electric potential of a randomly chosen neuron at time t , $D \geq 0$ is the delay between emission and reception of the spikes (synaptic delay), V_R is the reset potential after firing, and $V_F \in \mathbb{R}$ is the threshold potential. The drift term μ and the firing rate N of the network are given by

$$\mu(t) = \bar{b} + bN(t) \quad \text{with} \quad N(t) = -a \frac{\partial \rho}{\partial v}(V_F, t) \geq 0, \quad (1.2)$$

where $a > 0$ is the diffusion coefficient, the parameter \bar{b} controls the strength of the external stimuli and can be either zero, positive or negative, and b is the *connectivity parameter*. The neurons of the network can be either excitatory or inhibitory. This property is reflected in the NNLI equation through the sign of b : $b > 0$ for average-excitatory networks and $b < 0$ for average-inhibitory ones. The PDE (1.1) is completed with initial and boundary conditions

$$N(t) = N^0(t) \geq 0, \quad \forall t \in [-D, 0], \quad \rho(v, 0) = \rho^0(v) \geq 0, \quad \text{and} \quad \rho(V_F, t) = \rho(-\infty, t) = 0. \quad (1.3)$$

These boundary conditions imply that the following constraint is satisfied $N^0(0) = -a \frac{\partial \rho^0}{\partial v}(V_F)$. In addition, for any strong solution ρ , the total mass is conserved: if ρ^0 is a probability density, then that is also true of ρ at any positive time: $\int_{-\infty}^{V_F} \rho(v, t) dv = \int_{-\infty}^{V_F} \rho^0(v) dv = 1$. The behaviour of the network, and, of course, of the solutions of the NNLIF equation depends strongly on the type of network considered, in terms of the sign of b . In fact, in [4] (among others) it was shown that there are some situations in the case $D = 0$, depending on the initial data and the size of b , where the solutions for the average-excitatory case cannot be global-in-time. The simulations therein suggest that the blow-up phenomenon is reflected in a divergence in finite time of the firing rate. Following these observations, in [9] a criterion for the maximal time of existence for strong solutions was derived (they are called "classical solutions" in this article despite the singularity at point $v = V_R$; we used here the less confusing appellation "strong solutions"). Essentially, it ensures that the solutions exist while the firing rate is finite. In addition, it was obtained that strong solutions for the average-inhibitory case are always global-in-time. In [8], some qualitative properties were proved in the case $D = 0$: uniform bounds in L^∞ for the average-inhibitory case and asymptotic convergence towards steady states for small connectivity parameters.

Nevertheless, all of these previous works analyze a version of the NNLIF equation with no synaptic delay. The synaptic delay D is the short period of time that passes since a nerve impulse is sent from a presynaptic neuron until it finally reaches the postsynaptic neuron. The numerics done in [7] suggest that the synaptic delay in the NNLIF equation prevents the blow-up of the firing rate. Instead, the firing rate is seen to converge to a stationary state, oscillate or increase. In fact, at the microscopic level, it has already been proved that the solutions are global-in-time for the delayed NNLIF model [13]. Moreover, for the population density model of IF neuron with jumps, which arises from the same microscopic approximation as the NNLIF model, it was shown in [17] that the firing rate blows up in finite time in some situations, but in [16] it was proved that this blow-up disappears if a synaptic delay is considered.

In the present paper we deal with the delayed NNLIF equation, which is a modification of the NNLIF model presented in [4] and [9] at the level of the drift term, which includes a synaptic delay $D > 0$. We prove the global-in-time existence of strong solutions for both the average-inhibitory and the average-excitatory cases. Moreover, we analyse the long time behaviour of these solutions for a small connectivity parameter, and we provide a non-existence result of periodic-in-time solutions for a large enough connectivity parameter.

For simplicity of computations, we will suppose that $a = 1$ and $V_F = 0$ during the construction of local-in-time solutions. Nevertheless, these hypotheses are not really a constraint, since we can transform the general

equation into one satisfying the restriction by defining a new density $\bar{\rho}$ as follows:

$$\bar{\rho}(v, t) = \sqrt{a}\rho(\sqrt{a}v + V_F, v). \quad (1.4)$$

For clarity of the following proofs and theorems, we will assume that $\bar{b} = 0$ in the remainder of this paper. We point out that it can also be done without loss of generality using (1.4), since we can pass from the general equation for $\bar{b} \neq 0$ to an equation with $\bar{b} = 0$ by translating the voltage variable v by the factor \bar{b} .

The structure of the paper is as follows: the second and third sections are devoted to the study of the Cauchy problem of Equation (1.1). Taking into account the presence of the delay, we adapt the strategy of [9] in the second section, and rewrite the system (1.1)-(1.2)-(1.3) as a free boundary Stefan-like problem with a nonstandard right hand side consisting of a Dirac Delta source term. We also provide a definition of strong solutions for the new problem, and give some a priori properties for them. The third section is devoted to the proof of local existence and uniqueness of strong solutions for the Stefan-like problem. This is done through a fixed point argument for an integral formulation of the Stefan-like equation. In addition, it is shown how this result can be extended to the Fokker-Planck Equation (1.1). The fourth section contains one of the main results. There we prove the global existence and uniqueness of strong solutions for both the average-inhibitory and the average-excitatory case when $D > 0$. First, we extend to the case at hand the characterization of the maximal time of existence of the solutions in terms of the size of the firing rate, provided in [9] for the case without delay ($D = 0$). Mainly, this ensures that local solutions exist and are unique as long as the firing rate is finite. As in the case $D = 0$, the global existence for the average-inhibitory case is derived, showing that every solution defined until a certain time t_0 can be extended up to a short (but uniform) time ε , since the firing rate up to this additional time $t_0 + \varepsilon$ is uniformly bounded. However, for the average-excitatory case, this uniform bound for the firing rate is not arrived at easily, even with delay (see [7] for numerical results). We overcome this difficulty with a super-solution for the delayed NNLF equation. We do not prove a uniform bound of the firing rate, but we show that for every maximal time of existence the firing rate of the solution is bounded. Therefore, the criterion of maximal time of existence gives us a contraction and consequently we prove global existence of the solution. The fifth section is devoted to studying the long time behaviour of the system (1.1)-(1.2)-(1.3). We show exponentially fast convergence to the steady state of the solutions if the connectivity parameter b is small, extending the results of [4] and [8]. This result is achieved, for both the average-excitatory and the average-inhibitory case, by means of the entropy method, a Poincaré inequality and suitable L^2 estimates of the firing rate. We also study restrictions on the existence of time-periodic solutions, although it is still an

open question whether they exist or not in other situations. Finally, the appendix contains technical tools used extensively in the paper: the Poincaré-like inequality and the entropy equality.

2 The equivalent free boundary Stefan problem

In this section we rewrite Equation (1.1) as a free boundary Stefan problem with a nonstandard right hand side as in [9] was done without delay. With that purpose we perform the three changes of variables presented below. Afterwards, we write the final expression of the equivalent equation, we introduce the notion of strong solution for it and remember some basic a priori properties for this kind of solutions. In all this section, we assume $D > 0$, $a = 1$ and $V_F = 0$, which implies $V_R < 0$.

1. **A first change of variables.** We introduce the following change of variables, which has been widely studied in [10]:

$$y = e^t v, \quad \tau = \frac{1}{2}(e^{2t} - 1). \quad (2.1)$$

Therefore, denoting by $\alpha(\tau) = (\sqrt{2\tau + 1})^{-1}$,

$$t = -\log(\alpha(\tau)), \quad v = y\alpha(\tau), \quad (2.2)$$

and we define

$$w(y, \tau) = \alpha(\tau)\rho(y\alpha(\tau), -\log(\alpha(\tau))). \quad (2.3)$$

Differentiating w with respect to τ , and using that ρ is a solution of (1.1), yields

$$\begin{aligned} \frac{\partial w}{\partial \tau}(y, \tau) &= \alpha'(\tau)\rho(y\alpha(\tau), -\log(\alpha(\tau))) \\ &\quad + y\alpha'(\tau)\alpha(\tau)\frac{\partial \rho}{\partial v}(y\alpha(\tau), -\log(\alpha(\tau))) - \alpha'(\tau)\rho_t(y\alpha(\tau), -\log(\alpha(\tau))) \\ &= -\alpha'(\tau)\frac{\partial^2 \rho}{\partial v^2}(y\alpha(\tau), -\log(\alpha(\tau))) \\ &\quad + \alpha'(\tau)\mu(-\log(\alpha(\tau)) - D)\frac{\partial \rho}{\partial v}(y\alpha(\tau), -\log(\alpha(\tau))) \\ &\quad - \alpha'(\tau)N(-\log(\alpha(\tau)))\delta(y\alpha(\tau) - V_R). \end{aligned} \quad (2.4)$$

Finally, taking into account that $-\alpha'(\tau) = \alpha^3(\tau)$ and

$$\begin{aligned}\frac{\partial w}{\partial y}(y, \tau) &= \alpha^2(\tau) \frac{\partial \rho}{\partial v}(y\alpha(\tau), -\log(\alpha(\tau))), \\ \frac{\partial^2 w}{\partial y^2}(y, \tau) &= \alpha^3(\tau) \frac{\partial^2 \rho}{\partial v^2}(y\alpha(\tau), -\log(\alpha(\tau))),\end{aligned}$$

we obtain

$$\frac{\partial w}{\partial \tau}(y, \tau) = \frac{\partial^2 w}{\partial y^2}(y, \tau) - \alpha(\tau)\mu(t-D) \frac{\partial w}{\partial y}(y, \tau) + M(\tau)\delta\left(y - \frac{V_R}{\alpha(\tau)}\right), \quad (2.5)$$

where $M(\tau) = -\frac{\partial w}{\partial y}(0, \tau) = \alpha^2(\tau)N(t)$, and we use that $\mu(t-D) = \mu(-\log(\alpha(\tau)) - D)$ due to (2.2).

Therefore, w satisfies the equation

$$\begin{cases} \frac{\partial w}{\partial \tau}(y, \tau) = \frac{\partial^2 w}{\partial y^2}(y, \tau) - \alpha(\tau)\mu(t-D) \frac{\partial w}{\partial y}(y, \tau) + M(\tau)\delta_{\frac{V_R}{\alpha(\tau)}}(y) & y \in]-\infty, 0], \quad \tau \in \mathbb{R}_+, \\ M(\tau) = -\frac{\partial w}{\partial y}(0, \tau) & \tau \in \mathbb{R}_+, \\ w(-\infty, \tau) = w(0, \tau) = 0 & \tau \in \mathbb{R}_+, \\ w(y, 0) = w_I(y) & y \in (-\infty, 0]. \end{cases}$$

2. A second change of variables. With the change of variable (denoting $t_\omega = -\log(\alpha(\omega))$),

$$x = y - \int_0^\tau \mu(t_\omega - D) \alpha(\omega) d\omega = y - \int_0^\tau \mu(-\log(\alpha(\omega)) - D) \alpha(\omega) d\omega, \quad (2.6)$$

and defining $s(\tau) := -\int_0^\tau \mu(-\log(\alpha(\omega)) - D) \alpha(\omega) d\omega$, for $\tau \in \mathbb{R}_+$, the function u given by $u(x, \tau) = w(y, \tau)$ satisfies

$$\begin{cases} \frac{\partial u}{\partial \tau}(x, \tau) = \frac{\partial^2 u}{\partial x^2}(x, \tau) + M(\tau)\delta_{s(\tau) + \frac{V_R}{\alpha(\tau)}}(x) & x \in]-\infty, s(\tau)], \tau \in \mathbb{R}_+, \\ M(\tau) = -\frac{\partial u}{\partial x}(s(\tau), \tau) & \tau \in \mathbb{R}_+, \\ u(-\infty, \tau) = u(s(\tau), \tau) = 0 & \tau \in \mathbb{R}_+, \\ u(x, 0) = u_I(x) & x \in (-\infty, 0]. \end{cases} \quad (2.7)$$

3. A third change of variables. In the system (2.7) the boundary $s(\tau)$ depends on $\mu(t-D) = \bar{b} + bN(t-D)$, and it, in turn, on M . Therefore, we have to remove the explicit t dependency in the term $N(t-D)$. In the case $D = 0$ (see [9]) it is easy since $\mu(t-D) = \bar{b} + bN(t-D) = \bar{b} + bM(\tau)\alpha^{-2}(\tau)$.

However, if $D > 0$ the relation is more involved, because $t = \frac{1}{2} \log(2\tau + 1)$ and $\tau = \frac{1}{2} (e^{2t} - 1)$, and thus, if we consider the time $t - D$, there is a related τ_D : $\tau_D = \frac{1}{2} (e^{2(t-D)} - 1)$, which makes complicated the handling of D . We overcome this difficulty considering $\bar{D} = (1 - e^{-2D}) > 0$. With this choice $\tau_D = \tau (1 - \bar{D}) - \frac{\bar{D}}{2}$ and we obtain:

$$N(t - D) = \alpha^{-2}(\tau_D) M(\tau_D) = \alpha^{-2} \left((1 - \bar{D})\tau - \frac{1}{2}\bar{D} \right) M \left((1 - \bar{D})\tau - \frac{1}{2}\bar{D} \right), \quad (2.8)$$

which depends only on τ . Thus, the initial synaptic delay D is translated into the delay \bar{D} , which is scaled between 0 and 1, being $\bar{D} = 0$ if $D = 0$ and $\bar{D} = 1$ if $D = \infty$. In this way, using (2.8) we can rewrite $s(\tau)$ in terms of $M(\tau)$, avoiding the dependence in t

$$\begin{aligned} s(\tau) &= -\bar{b}(\sqrt{2\tau + 1} - 1) - b \int_0^\tau N(t_\omega - D) \alpha(\omega) d\omega \\ &= -\bar{b}(\sqrt{2\tau + 1} - 1) - b \int_0^\tau \alpha^{-2} \left((1 - \bar{D})\omega - \frac{1}{2}\bar{D} \right) M \left((1 - \bar{D})\omega - \frac{1}{2}\bar{D} \right) \alpha(\omega) d\omega. \end{aligned} \quad (2.9)$$

The change of variables $z = (1 - \bar{D})\omega - \frac{1}{2}\bar{D}$ yields

$$s(\tau) = -\bar{b}(\sqrt{2\tau + 1} - 1) - \frac{b}{\sqrt{1 - \bar{D}}} \int_{-\frac{1}{2}\bar{D}}^{(1 - \bar{D})\tau - \frac{1}{2}\bar{D}} M(z) \alpha^{-1}(z) dz. \quad (2.10)$$

Taking into account that $M^0(\tau) = \alpha^2(\tau) N^0(\frac{1}{2} \log(2\tau + 1))$, finally leads to the following equivalent Stefan-like equation

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial \tau}(x, \tau) = \frac{\partial^2 u}{\partial x^2}(x, \tau) + M(\tau) \delta_{s(\tau) + \frac{v_R}{\alpha(\tau)}}(x), & x < s(\tau), \tau > 0, \\ s(\tau) = -\bar{b}(\sqrt{2\tau + 1} - 1) - \frac{b}{\sqrt{1 - \bar{D}}} \int_{-\frac{1}{2}\bar{D}}^{(1 - \bar{D})\tau - \frac{1}{2}\bar{D}} M(z) \alpha^{-1}(z) dz, & \tau > 0, \\ M(\tau) = -\frac{\partial u}{\partial x}(s(\tau), \tau), & \tau > 0, \\ M(\tau) = M^0(\tau) > 0, & \tau \in (-\frac{\bar{D}}{2}, 0], \\ u(-\infty, \tau) = u(s(\tau), \tau) = 0, & \tau > 0, \\ u(x, 0) = u_I(x), & x < 0. \end{array} \right. \quad (2.11)$$

where $\bar{D} \in [0, 1]$ and $\alpha(\tau) = \frac{1}{\sqrt{2\tau + 1}}$. Let us remark that this problem is well defined since $\alpha(\tau) \in \mathbb{R}^+$, $\forall \tau > -\frac{1}{2}$. Also, we note that if $\bar{D} = 0$, the system (2.11) reduces to the system studied in [9].

We conclude this section with the notion of strong solutions for this kind of system and with some a priori properties, that will be useful for the rest of the computations of the present work.

Definition 2.1 (Strong solutions for the Stefan-like problem) *Let $u^0(x)$ be a non-negative $C^0((-\infty, 0]) \cap C^1((-\infty, V_R) \cup (V_R, 0]) \cap L^1((-\infty, 0))$ function such that $u^0(0) = 0$. Suppose that $\frac{du^0}{dx}$ vanish at $-\infty$ and admits finite left and right limits at V_R . We say that u is a strong solution of (2.11) (equivalently (2.7)) with initial datum u^0 on the interval $J = [0, T)$ or $J = [0, T]$, for a given $T > 0$ if:*

1. $M(\tau)$ is a continuous function for all $\tau \in J$.
2. u is continuous in the region $\{(x, \tau) : -\infty < x \leq s(\tau), \tau \in J\}$ and for all $\tau \in J$, $u(\tau) \in L^1((-\infty, s(\tau)))$.
3. If we denote $s_1(\tau) := s(\tau) + \frac{V_R}{\alpha(\tau)}$, then $\partial_{xx}u$ and $\partial_\tau u$ are continuous in the region $\{(x, \tau) : x \in (-\infty, s(\tau)) \setminus \{s_1(\tau)\}, \tau \in J \setminus \{0\}\}$.
4. $\partial_x u(s_1(\tau)^-, \tau)$, $\partial_x u(s_1(\tau)^+, \tau)$, $\partial_x u(s(\tau)^-, t)$ are well defined.
5. $\partial_x u$ vanishes at $-\infty$.
6. Equations (2.11) (equivalently (2.7)) are satisfied in the sense of distributions in the region $\{(x, \tau) : -\infty < x \leq s(\tau), \tau \in J\}$ and pointwise (in the classical sense of a function's derivative) in $\{(x, \tau) : x \in (-\infty, s(\tau)) \setminus \{s_1(\tau)\}, \tau \in J \setminus \{0\}\}$.

By "satisfied in the sense of distributions", we mean that the time-dependent distribution associated to u (by canonical injection of $L^1_{loc}(\Omega)$ into $D'(\Omega)$ for any open set Ω) satisfies the equality in the space of distributions (the time derivative of u has to be taken in the sense of distributions too). This definition includes the notion of solution given in [9] for the case without transmission delay ($D = \bar{D} = 0$). We call these solutions "strong" because they are C^1 in time and C^2 in space regularity away from the singularity begotten by the delta Dirac mass in the equation. As we can see in the following lemma, the solution u has a jump of size $M(\tau)$ at the point $x = s_1(\tau)$; it is another – equivalent – way of defining "satisfied in the sense of distributions" for this particular case, as one can prove by standard – tedious – computations with suitable test functions.

We gather in the following lemma some a priori properties of these solutions.

Lemma 2.2 (A priori properties) *Let u be a solution to (2.11) (equivalently (2.7)) in the sense of the previous definition. Then:*

1. The mass is conserved: $\int_{-\infty}^{s(\tau)} u(x, \tau) dx = \int_{-\infty}^0 u^0(x) dx, \forall t > 0$.
2. The flux across the moving point $s_1(\tau) := s(\tau) + \frac{V_R}{\alpha(\tau)}$ is exactly the strength of the source term:

$$M(\tau) := -\partial_x u(s(\tau), \tau) = \partial_x u(s_1(\tau)^-, \tau) - \partial_x u(s_1(\tau)^+, \tau).$$

3. If $\bar{b} \leq 0$ and $b < 0$ (respectively, $\bar{b} \geq 0$ and $b > 0$), the free boundary $s(\tau)$ is a monotone increasing (respectively, decreasing) function of time.

Proof. The proof of properties 1. and 2. is exactly the same as in [9][Lemma 2.3], because it does not take into account the expression of $s(t)$. Property 3. is obvious with the form of s .

3 Local existence and uniqueness

In this section we introduce an implicit integral equation for M . Then, thanks to the form of that equation, it will be possible to solve it for local time using a fixed point argument. In addition, since we will be able to prove that the fixed point function is a contraction, we will also get the local uniqueness of M . This is useful, since once M is known, (2.11) (equivalently (2.7)) decouples, and u can be calculated easily by a Duhamel formula.

The inclusion of the transmission delay, $D > 0$, in the model produces that the NNLIF equation (1.1) becomes linear for $t < D$, since the drift and the diffusion terms depend on the initial condition, instead of the firing rate $N(t)$. This fact is translated to the system (2.11) (equivalently (2.7)) when $\tau \leq \frac{1}{2}(e^{2D} - 1) = \frac{\bar{D}}{2(1-\bar{D})}$, which means $t \leq D$ in the original time variable. Thus, the boundary is free with a delay but it is constrained on any short period of time, since on every time interval of size less than $\frac{1}{2}(e^{2D} - 1) = \frac{\bar{D}}{2(1-\bar{D})}$, systems (2.7) and (2.11) are equivalent to a linear system with non-standard right-hand side. To be more precise, we can solve them by studying the following equivalent linear system:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial \tau}(x, \tau) = \frac{\partial^2 u}{\partial x^2}(x, \tau) + M(\tau) \delta_{s(\tau) + \frac{v_R}{\alpha(\tau)}}(x) & x \in]-\infty, s(\tau)], \tau \in \mathbb{R}_+, \\ s(\tau) = - \int_0^\tau I(\omega) d\omega & \tau \in \mathbb{R}_+, \\ u(-\infty, \tau) = u(s(\tau), \tau) = 0 & \tau \in \mathbb{R}_+, \\ M(\tau) = - \frac{\partial u}{\partial x}(s(\tau), \tau) & \tau \in \mathbb{R}_+, \\ u(x, 0) = u_I(x), & x < 0. \end{array} \right. \quad (3.1)$$

where $I \in \mathcal{C}^0([0, +\infty))$ is an abstract input function. In our case, the system (2.7) (equivalently (2.11)) can be written as System (3.1) with $I(\omega) = \mu(-\log(\alpha(\omega)) - D) \alpha(\omega)$, on every time interval of size less than $\frac{1}{2}(e^{2D} - 1) = \frac{\bar{D}}{2(1-\bar{D})}$, since I does not depend on u on this time, but on the values on the previous time intervals. Therefore, System (2.7) (equivalently (2.11)) can be considered as a linear system on every time interval of size less than $\frac{1}{2}(e^{2D} - 1) = \frac{\bar{D}}{2(1-\bar{D})}$.

The notion of solution of definition 2.1 and the *a priori* properties still apply for equation (3.1) if we assume I such that the free boundary $s(\tau)$ is a monotone function on time.

Denoting $G(x, \tau, \xi, \eta) = \frac{1}{\sqrt{4\pi(\tau - \eta)}} e^{-\frac{(x - \xi)^2}{4(\tau - \eta)}}$, we prove the following theorems.

Theorem 3.1 *If the function u is solution of (3.1) in the sense of definition (2.1), then the continuous function M satisfies*

$$\begin{aligned} M(\tau) = & -2 \int_{-\infty}^0 G(s(\tau), \tau, \xi, 0) \frac{du^0}{dx}(\xi) d\xi \\ & + 2 \int_0^\tau M(\eta) \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta), \eta) d\eta \\ & - 2 \int_0^\tau M(\eta) \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) d\eta \quad (3.2) \end{aligned}$$

Proof. The proof is exactly the same as in [9]. Therefore, we only sketch it. The main idea is to use the following Green's identity: $\frac{\partial}{\partial \xi} \left(G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right) - \frac{\partial}{\partial \tau} (Gu) = 0$. The result is obtained by integration of this identity in the regions $\{\xi \in (-\infty, s(\eta) + V_R \alpha(\eta)^{-1}), \eta \in (0, \tau)\}$ and $\{\xi \in (s(\tau) + V_R \alpha(\tau)^{-1}, s(\tau)), \eta \in (0, \tau)\}$ and addition of the results. Computations are then done on the different terms to express them in function of M , G , $\partial_x G$ and u^0 under integrals. \square

Theorem 3.2 *Let $u^0(x)$ be a non-negative $C^0((-\infty, 0]) \cap C^1((-\infty, V_R) \cup (V_R, 0]) \cap L^1((-\infty, 0))$ function such that $u^0(0) = 0$, and suppose that $\frac{du^0}{dx}$ vanishes at $-\infty$ and admits finite left and right limits at V_R . Then, there exists $T \in \mathbb{R}_+$ and a unique function $M \in \mathcal{C}^0([0, T])$ that satisfies (3.2) on $[0, T]$.*

Proof. Let $\sigma \in \mathbb{R}_+$, $m := 1 + 2 \sup_{x \in (-\infty, V_R) \cup (V_R, 0]} \left| \frac{du^0}{dx}(x) \right|$ and consider the space

$$C_{\sigma, m} = \{M \in \mathcal{C}^0([0, \sigma]) \mid \|M\|_\infty := \sup_{\tau \in [0, \sigma]} |M(\tau)| \leq m\}.$$

We define in this space the functional

$$\begin{aligned} \mathcal{T}(M)(\tau) := & -2 \int_{-\infty}^0 G(s(\tau), \tau, \xi, 0) \frac{du^0}{dx}(\xi) d\xi + 2 \int_0^\tau M(\eta) \frac{\partial G}{\partial x}(s(\tau), \tau, s(\eta), \eta) d\eta \\ & - 2 \int_0^\tau M(\eta) \frac{\partial G}{\partial x}(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) d\eta. \end{aligned} \quad (3.3)$$

The proof of the theorem is obtained if we show that this functional has a unique fixed point. To do so, we start proving that for σ small enough $\mathcal{T} : C_{\sigma, m} \rightarrow C_{\sigma, m}$.

On the one hand,

$$\begin{aligned} \left\| 2 \int_{-\infty}^0 G(s(\cdot), \cdot, \xi, 0) \frac{du^0}{dx}(\xi) d\xi \right\|_{\infty} &= 2 \sup_{x \in]-\infty, V_R[\cup]V_R, 0]} \left| \frac{du^0}{dx}(x) \right| \left\| \int_{-\infty}^0 G(s(\cdot), \cdot, \xi, 0) d\xi \right\|_{\infty} \\ &= 2 \sup_{x \in]-\infty, V_R[\cup]V_R, 0]} \left| \frac{du^0}{dx}(x) \right|. \end{aligned}$$

On the other hand, the positive valued applications

$$\phi_1 : \tau \mapsto 2m \int_0^{\tau} \left| \frac{\partial G}{\partial x}(s(\tau), \tau, s(\eta), \eta) \right| d\eta,$$

and

$$\phi_2 : \tau \mapsto 2m \int_0^{\tau} \left| \frac{\partial G}{\partial x}(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) \right| d\eta,$$

are continuous on $(0, \sigma]$. A direct computation gives

$$\left| \frac{\partial G}{\partial x}(s(\tau), \tau, s(\eta), \eta) \right| = \frac{1}{2\sqrt{4\pi}} \frac{|s(\tau) - s(\eta)|}{|\tau - \eta|^{\frac{3}{2}}} e^{-\frac{(s(\tau) - s(\eta))^2}{4(\tau - \eta)}}$$

Since the function I is bounded on every compact set (because it is continuous), we can denote $I_{\sigma} = \sup_{\omega \in [0, \sigma]} |I(\omega)|$,

and we have $|s(\tau) - s(\eta)| = \left| \int_{\eta}^{\tau} I(\omega) d\omega \right| \leq I_{\sigma} |\tau - \eta|$. Substituting in the previous expression and putting back the integral, we get $\phi_1(\tau) \leq \frac{mI_{\sigma}}{\sqrt{4\pi}} \int_0^{\tau} \frac{1}{\sqrt{\tau - \eta}} d\eta = \frac{mI_{\sigma}}{\sqrt{4\pi}} \sqrt{\tau}$. We directly get $\lim_{\tau \rightarrow 0} \phi_1(\tau) = 0$ and ϕ_1 is continuous on $[0, \sigma]$, with $\phi_1(0) = 0$. Similarly, we have

$$|s(\tau) - s(\eta) - \frac{V_R}{\alpha(\tau)}| \geq |V_R| - I_{\sigma} |\tau - \eta| \geq |V_R| - I_{\sigma} \tau,$$

and using the inequality $ze^{-z^2} \leq e^{-z^2/2}$, we can write

$$\phi_2(\tau) = \frac{1}{2\sqrt{4\pi}} \int_0^{\tau} \frac{|s(\tau) - s(\eta) - \frac{V_R}{\alpha(\eta)}|}{|\tau - \eta|^{\frac{3}{2}}} e^{-\frac{(s(\tau) - s(\eta) - \frac{V_R}{\alpha(\eta)})^2}{4(\tau - \eta)}} d\eta,$$

and then

$$\phi_2(\tau) \leq \frac{1}{\sqrt{4\pi}} \int_0^{\tau} \frac{1}{\tau - \eta} e^{-\frac{(s(\tau) - s(\eta) - \frac{V_R}{\alpha(\tau)})^2}{8(\tau - \eta)}} d\eta \leq \frac{1}{\sqrt{4\pi}} \int_0^{\tau} \frac{1}{\tau - \eta} e^{-\frac{(|V_R| - I_{\sigma}\tau)^2}{8(\tau - \eta)}} d\eta.$$

When τ is small enough, $|V_R| - I_\sigma \tau \geq \frac{1}{2}|V_R|$ and we have, making the change of variable $z = \frac{|V_R|}{2\sqrt{8(\tau-\eta)}}$,

$$\begin{aligned}\phi_2(\tau) &\leq \frac{1}{\sqrt{4\pi}} \int_0^\tau \frac{1}{\tau-\eta} e^{-\frac{(|V_R| - I_\sigma \tau)^2}{8(\tau-\eta)}} d\eta \leq \frac{1}{\sqrt{4\pi}} \int_0^\tau \frac{1}{\tau-\eta} e^{-\frac{V_R^2}{32(\tau-\eta)}} d\eta \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{|V_R|}{2\sqrt{8\tau}}}^{+\infty} \frac{1}{z} e^{-z^2} dz.\end{aligned}$$

we also get $\lim_{\tau \rightarrow 0} \phi_2(\tau) = 0$ and ϕ_2 is continuous on $[0, \sigma]$, with $\phi_2(0) = 0$.

Thus, there exists an interval $[0, \zeta] \subset [0, \sigma]$ such that

$$\sup_{\tau \in [0, \zeta]} |\phi_1(\tau)| < \frac{1}{2} \quad \text{and} \quad \sup_{\tau \in [0, \zeta]} |\phi_2(\tau)| < \frac{1}{2}.$$

Hence, we have, for this $\zeta \in \mathbb{R}_+$ and for all $M \in C_{\zeta, m}$,

$$\left\| 2 \int_0^\tau M(\eta) \frac{\partial G}{\partial x}(s(\cdot), \cdot, s(\eta), \eta) d\eta \right\|_\infty \leq \sup_{\tau \in [0, \zeta]} 2m \int_0^\tau \left| \frac{\partial G}{\partial x}(s(\tau), \tau, s(\eta), \eta) \right| d\eta < \frac{1}{2},$$

and also

$$\left\| 2 \int_0^\tau M(\eta) \frac{\partial G}{\partial x}(s(\cdot), \cdot, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) d\eta \right\|_\infty < \frac{1}{2}.$$

Collecting the previous bounds, we obtain, for σ small enough and depending only on m and I , thus depending only on $\frac{du^0}{dx}$ and g , that the map \mathcal{T} is defined as a function from $C_{\sigma, m}$ into $C_{\sigma, m}$.

Finally, we prove that for this choice of σ the functional \mathcal{T} is a contraction. We have for all $M, N \in C_{\sigma, m}$, for all $\tau \in [0, \sigma]$,

$$\begin{aligned}|\mathcal{T}(M)(\tau) - \mathcal{T}(N)(\tau)| &\leq \frac{1}{m} \|M - N\|_\infty \left(2m \int_0^\tau \left| \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta), \eta) \right| d\eta \right. \\ &\quad \left. + 2m \int_0^\tau \left| \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) d\eta \right| \right) \\ &\leq \frac{1}{m} \left(\sup_{\tau \in [0, \sigma]} |\phi_1(\tau)| + \sup_{\tau \in [0, \sigma]} |\phi_2(\tau)| \right) \|M - N\|_\infty. \quad (3.4)\end{aligned}$$

As $m \geq 1$, we have

$$\frac{1}{m} \left(\sup_{\tau \in [0, \sigma]} |\phi_1(\tau)| + 2 \sup_{\tau \in [0, \sigma]} |\phi_2(\tau)| \right) < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m} \leq 1.$$

Thus \mathcal{T} is a contraction on the complete metric space $C_{\sigma, m}$. It admits a unique fixed point M in that space.

□

Theorem 3.3 (Local existence of the linear problem) *Let $u^0(x)$ be a non-negative function in $C^0((-\infty, 0]) \cap C^1((-\infty, V_R) \cup (V_R, 0]) \cap L^1((-\infty, 0))$, such that $u^0(0) = 0$ and suppose that $\frac{du^0}{dx}$ vanishes at $-\infty$ and admits finite left and right limits at V_R . Then, there exists a unique strong solution u for the problem (3.1).*

Proof. The proof is omitted, since it is performed as in [9] [Corollary 3.3]. Let us only point out that, once M is known the equation for u decouples, and u can be calculated via Duhamel's formula

$$u(x, \tau) = \int_{-\infty}^0 G(x, \tau, \xi, 0) u^0(\xi) d\xi - \int_0^\tau M(\eta) G(x, \tau, s(\eta), \eta) d\eta + \int_0^\tau M(\eta) G(x, \tau, s_1(\eta), \eta) d\eta,$$

where $s_1(\tau) = s(\tau) + \frac{V_R}{\alpha(\tau)}$. This formula provides the smoothness and decay of u to prove that this mild solution (or solution in the integral sense) is a strong solution in the sense of Definition 2.1. \square

Theorem 3.4 (Local existence of the non-linear Stefan-problem) *Let $u^0(x)$ be a non-negative function in $C^0((-\infty, 0]) \cap C^1((-\infty, V_R) \cup (V_R, 0]) \cap L^1((-\infty, 0))$, such that $u^0(0) = 0$ and suppose that $\frac{du^0}{dx}$ vanishes at $-\infty$ and admits finite left and right limits at V_R . Then, there exists a unique strong solution u for the problem (2.7) (equivalently (2.11)) in the sense of definition 2.1.*

Proof. The system (2.7) (equivalently (2.11)) can be written as System (3.1) considering the function $I(\omega) = \mu(-\log(\alpha(\omega)) - D)\alpha(\omega)$, on every time interval of size less than $\frac{1}{2}(e^{2D} - 1) = \frac{\bar{D}}{2(1-\bar{D})}$. Then, for time $\tau \in [0, \frac{1}{2}(e^{2D} - 1)]$ (equivalently $\tau \in [0, \frac{\bar{D}}{2(1-\bar{D})}]$), there is a unique local solution defined on $[0, T_1]$ for System (2.7) (equivalently (2.11)), which is the solution for I associated with the initial datum M^0 .

If $T_1 = \frac{1}{2}(e^{2D} - 1)$, then we use $u(\cdot, T_1)$ and the values of M in the interval $[0, \frac{1}{2}(e^{2D} - 1)]$ as initial values for System (3.1) in order to have a solution on $[0, T_2]$, $T_2 \leq e^{2D} - 1$.

We repeat this procedure until we find the maximal time of existence for the solution of (2.7) (equivalently (2.11)). \square

The existence and uniqueness proved in Theorem 3.4 can be translated into our initial system (1.1)-(1.2)-(1.3) recovering ρ and N by undoing the changes of variables (2.1) and (2.6).

Corollary 3.5 *Let ρ^0 be a non-negative $C^0((-\infty, V_F]) \cap C^1((-\infty, V_R) \cup (V_R, V_F]) \cap L^1((-\infty, V_F))$ function such that $\rho^0(V_F) = 0$ and $\frac{d\rho^0}{dv}$ decays at $-\infty$ and admits finite left and right limits at V_R . Then there exists a maximal $T^* \in (0, +\infty]$ and there exists a unique strong solution to the problem (1.1)-(1.2)-(1.3) with $D > 0$ on the time interval $[0, T^*)$.*

Remark 3.6 *Using Duhamel's formula for u and going back to ρ , we can see that as long as ρ^0 is fast decaying*

at $-\infty$ (for any polynomial function f , the quantities $f(v)\rho(v)$ and $f(v)\frac{d}{dv}\rho^0(v)$ go to 0 as v goes to $-\infty$), then $\rho(\cdot, t)$ is fast decaying at $-\infty$ too for every positive t . This property will be implicitly used in other sections.

Remark 3.7 *The same proof gives strong solutions for the coupled excitatory-inhibitory system with positive delay studied in [7]: for a short time, the two equations decouple.*

4 Global existence of solutions for the delayed model

In this section we derive the main result of the work: the global existence of solutions for the delayed model (1.1). The result is obtained directly for the average-inhibitory case (as in the case without transmission delay [9]), while for the average-excitatory case, it has to be derived through some of the properties of super-solutions.

4.1 A criterion for the maximal time of existence

The key step to obtain the main results of the paper is a criterion for the maximal time of existence of solutions, summarized in Theorem 4.2. It ensures that solutions exists while the firing rate is finite. With that purpose, first we show an auxiliary proposition, Proposition 4.1, which provides the tool to prove Theorem 4.2. Then using Theorem 4.2 we derive Proposition 4.4 which will allows to obtain the global existence of solutions for the inhibitory case. Their proofs are all omitted or sketched, since they are the same as in [9][Proposition 4.1, Theorem 4.2, Proposition 4.3], because they are all consequences of the local existence result of Theorem 3.4.

Proposition 4.1 *Suppose that the hypotheses of Theorem 3.4 hold and that u is a solution to (2.7) (equivalently (2.11)) in the time interval $[0, T]$. Assume in addition, that*

$$U_0 := \sup_{x \in (-\infty, s(t_0 - \varepsilon)]} |\partial_x u(x, t_0 - \varepsilon)| < \infty \quad \text{and that} \quad M^* = \sup_{t \in (t_0 - \varepsilon, t_0)} M(t) < \infty,$$

for some $0 < \varepsilon < t_0 \leq T$. Then, $\sup\{|\partial_x u(x, t)| : x \in (-\infty, s(t)], t \in [t_0 - \varepsilon, t_0)\} < \infty$, with a bound depending only on the quantities M^ and U_0 .*

Using this proposition we obtain the same criterion as in the case without synaptic delay [9]:

Theorem 4.2 *Suppose that the hypotheses of Theorem 3.4 hold. Then the solution u can be extended up to a maximal time $0 < \bar{T} \leq \infty$ given by*

$$\bar{T} = \sup\{t \in (0, +\infty] : M(t) < \infty\}.$$

In terms of the original system (1.1)-(1.2)-(1.3), we have the following maximal time of existence result:

Theorem 4.3 (Maximal time of existence) *Let ρ^0 be a non-negative function in $C^0((-\infty, V_F]) \cap C^1((-\infty, V_R) \cup (V_R, V_F]) \cap L^1((-\infty, V_F))$ such that $\rho^0(V_F) = 0$ and $\frac{d\rho^0}{dv}$ decays at $-\infty$ and admits finite left and right limits at V_R . Then there exists a unique strong solution to the problem (1.1)-(1.2)-(1.3) with $D \geq 0$ on the time interval $[0, T^*)$ where $T^* > 0$ can be characterized by*

$$T^* = \sup\{t > 0 : N(t) < \infty\}.$$

Proof. The case $D = 0$ is proved in the article [9]. In the case $D > 0$, we use theorem 4.2 and get the result directly. \square

Using Theorem 4.2 we derive the key result for the global existence in the inhibitory case:

Proposition 4.4 *Suppose that the hypotheses of Theorem 3.4 hold and that u is a solution to (2.11) (equivalently (2.7)) in the time interval $[0, t_0]$ for $b \leq 0$. Then there exists $\varepsilon > 0$ small enough and independent of t_0 , such that, if*

$$\bar{U} := \sup_{x \in (-\infty, s(t_0 - \varepsilon)]} |\partial_x u(x, t_0 - \varepsilon)| < \infty \quad (4.1)$$

then $\sup_{t_0 - \varepsilon < t < t_0} M(t) < \infty$, for $0 < \varepsilon < t_0$.

The proof follows as in [9][Proposition 4.3]. The non-positivity of the connectivity parameter b is crucial, because, in that case, $s(t)$ is a monotone non-decreasing function, which implies a uniform bound for M .

Finally, combining Theorem 4.2 with the previous result we obtain the global existence and uniqueness of strong solutions for the inhibitory case with synaptic delay for equivalent systems (2.7) and (2.11).

Proposition 4.5 *Suppose that the hypotheses of Theorem 3.4 hold and that $b \leq 0$. Then there exists a unique global-in-time strong solution u for system (2.11) (equivalently (2.7)) in the sense of Definition 2.1 with initial datum u^0 . In addition, if both b and \bar{b} are negative, $s(t)$ is a monotone increasing function.*

This proposition, translated to the initial delayed Fokker-Planck equation (1.1) provides the global existence for the inhibitory case, as follows:

Theorem 4.6 (Global existence - inhibitory case) *Let ρ^0 be a non-negative function in $C^0((-\infty, V_F]) \cap C^1((-\infty, V_R) \cup (V_R, V_F]) \cap L^1((-\infty, V_F))$, such that $\rho^0(V_F) = 0$ and $\partial_v \rho^0$ admits finite left and right limits at V_R . Suppose that $\partial_v \rho^0$ decay at $-\infty$, then there exists a unique strong solution to the problem (1.1)-(1.2)-(1.3) with $b \leq 0$ and $D \geq 0$ on the time interval $[0, T^*)$ with $T^* = +\infty$.*

4.2 Super-solutions and control over the firing rate

We are not able to obtain the global existence of solutions for the average-excitatory case as it is done before for the average-inhibitory. The difficulty is the extension of the proposition 4.4 for the case $b > 0$, which implies a uniform bound for M is the same in the average-excitatory case. Thus we have to proceed with a different strategy, by means of a super-solution, to prove that the firing rate of any local solution cannot diverge in finite time. Then, applying the criterion of Theorem 4.3 the result is reached. We start introducing the notion of super-solution.

Definition 4.7 *Let $T \in \mathbb{R}_+$, $D \geq 0$ and $\bar{b} = 0$, $(\bar{\rho}, \bar{N})$ is said to be a (strong) super-solution to (1.1)-(1.2)-(1.3) on $(-\infty, V_F] \times [0, T]$ if for all $t \in [0, T]$ we have $\bar{\rho}(V_F, t) = 0$ and*

$$\partial_t \bar{\rho} + \partial_v [(-v + b\bar{N}(t-D))\bar{\rho}] - a\partial_{vv}\bar{\rho} \geq \delta_{v=V_R}\bar{N}(t), \quad \bar{N}(t) = -a\partial_v \bar{\rho}(V_F, t), \quad (4.2)$$

on $(-\infty, V_F] \times [0, T]$ in the distributional sense and on $((-\infty, V_F] \setminus V_R) \times [0, T]$ in the classical sense, with arbitrary values for \bar{N} on $[-D, 0)$.

We choose $\bar{b} = 0$ just for convenience as we can do it without loss of generality (as said in the introduction). Notice that for a solution in $C^{2,1}((-\infty, V_R) \cup (V_R, V_F] \times [0, T]) \cap C^0((-\infty, V_R] \times [0, T])$, the condition reduces to satisfy the property in the classical sense in $(-\infty, V_R) \cup (V_R, V_F] \times [0, T]$ and having a decreasing jump discontinuity for the derivative on V_R of size at least \bar{N}/a .

Notice also that when $T < D$, $\bar{N}(t-D)$ is an arbitrary initial datum, and thus if we find such a super-solution $\bar{\rho}$, then for every constant $\alpha > 0$, the function $\alpha\bar{\rho}$ is also a super-solution.

We start proving the following comparison property between strong solutions and super-solutions of (1.1)-(1.2)-(1.3).

Theorem 4.8 *Let $D > 0$, $0 < T < D$ and $\bar{b} = 0$. Let (ρ, N) be a strong solution of (1.1)-(1.2)-(1.3) on $(-\infty, V_F] \times [0, T]$ for the initial condition (ρ^0, N^0) and let $(\bar{\rho}, \bar{N})$ be a strong super-solution of (1.1)-(1.2)-(1.3) on $(-\infty, V_F] \times [0, T]$. Assume that*

$$\forall v \in (-\infty, V_F], \quad \bar{\rho}(v, 0) \geq \rho^0(v) \quad \text{and} \quad \forall t \in [-D, 0), \quad \bar{N}(t) = N^0(t).$$

Then,

$$\forall (v, t) \in (-\infty, V_F] \times [0, T], \quad \bar{\rho}(v, t) \geq \rho(v, t) \quad \text{and} \quad \forall t \in [0, T], \quad \bar{N}(t) \geq N(t).$$

Proof. First, we prove that if $\bar{\rho}(v, t) \geq \rho(v, t)$ then $\bar{N}(t) \geq N(t)$. Due to the Dirichlet boundary condition for ρ and the definition of super-solution we have $\rho(V_F, t) = \bar{\rho}(V_F, t) = 0$ on $[0, T]$. Thus, as long as $\bar{\rho}(v, t) \geq \rho(v, t)$ holds, we have

$$-a \frac{\bar{\rho}(V_F, t) - \bar{\rho}(v, t)}{V_F - v} \geq -a \frac{\rho(V_F, t) - \rho(v, t)}{V_F - v}.$$

And taking the limit $v \rightarrow V_F$ we get $\bar{N}(t) \geq N(t)$.

Then, denoting $w = \bar{\rho} - \rho$, we have for all $(v, t) \in (-\infty, V_F] \times [0, T]$,

$$\partial_t w + \partial_v(-vw) + b\bar{N}(t - D)\partial_v \bar{\rho} - bN(t - D)\partial_v \rho - a\partial_{vv}w \geq \delta_{v=V_R}(\bar{N}(t) - N(t)).$$

As we assume $T < D$ we have by hypothesis $\bar{N}(t - D) = N^0(t - D)$ for all $t \in [0, T]$. Thus, as long as $w \geq 0$ holds, since $\bar{N}(t) \geq N(t)$,

$$\partial_t w + \partial_v[(-v + bN^0(t))w] - a\partial_{vv}w \geq 0.$$

As $w(\cdot, 0) \geq 0$, by a standard maximum principle theorem, we have $\forall t \in [0, T]$, $w(\cdot, t) \geq 0$, and we conclude the proof. \square

Now, for fixed continuous (thus bounded) N^0 and the choice $\bar{N}(t) = N^0(t)$ in $[-D, 0]$, we look for a super-solution on $[0, D]$ of the form

$$\bar{\rho}(v, t) = e^{\xi t} f(v), \tag{4.3}$$

where ξ is large enough and f is a carefully selected function, such that, the function $\bar{\rho}$ satisfies (4.2), which means

$$(\xi - 1)f + (-v + bN^0(t))f' - af'' \geq \delta_{v=V_R}\hat{N}(t), \quad \hat{N}(t) = -af'(V_F). \tag{4.4}$$

We show that f defined as follow

$$f : (-\infty, V_F] \rightarrow \mathbb{R}_+ \\ v \mapsto \begin{cases} 1 & \text{on } (-\infty, V_R] \\ e^{V_R - v}\psi(v) + \frac{1}{\delta}(1 - \psi(v))(1 - e^{\delta(v - V_F)}) & \text{on } (V_R, V_F] \end{cases}$$

verifies (4.4). To complete the definition of f we explain which are ψ and δ :

1. For $\varepsilon > 0$ small enough, such that $\frac{V_F + V_R}{2} + \varepsilon < V_F$, we consider $\psi \in C_b^\infty(\mathbb{R})$ satisfying $0 \leq \psi \leq 1$ and

$$\psi \equiv 1 \text{ on } \left(-\infty, \frac{V_F + V_R}{2}\right) \text{ and } \psi \equiv 0 \text{ on } \left(\frac{V_F + V_R}{2} + \varepsilon, +\infty\right).$$

2. For $B > 0$, such that $|-v + bN^0(t)| \leq B, \forall t \in [-D, 0], \forall v \in (V_R, V_F)$, we take $\delta > 0$ such that $a\delta - B \geq 0$.

Notice that f being the sum of two continuous non-negative functions that never vanish at the same point, we have

$$\inf_{v \in (V_R, \frac{V_F + V_R}{2} + \varepsilon)} f(v) > 0.$$

With these choices, $\bar{\rho}(v, t)$ is a super-solution on $[0, D]$ for ξ large enough, because:

- On $(-\infty, V_R)$, $\bar{\rho}$ is independent of v , thus the definition is satisfied if and only if $\xi \geq 1$.
- Around the V_R point the inequality (4.4) has to hold in the sense of distribution, that is in our case

$$f'(V_R^+) - f'(V_R^-) \leq f'(V_F)$$

This inequality is satisfied because $f'(V_R^-) = 0$, $f'(V_R^+) = -1$ and $f'(V_F) = -1$.

- On $(V_R, \frac{V_F + V_R}{2} + \varepsilon)$, we choose ξ such that

$$(\xi - 1) \inf_{v \in (V_R, \frac{V_F + V_R}{2} + \varepsilon)} f(v) \geq \sup_{v \in (V_R, \frac{V_F + V_R}{2} + \varepsilon)} (B|f'(v)| + a|f''(v)|),$$

which is possible because $\inf_{v \in (V_R, \frac{V_F + V_R}{2} + \varepsilon)} f(v) > 0$. Then the super-solution inequality (4.4) holds.

- On $(\frac{V_F + V_R}{2} + \varepsilon, V_F)$, the desired inequality (4.4) holds since

$$(-v + bN^0(t))f' - af'' = e^{\delta(V_F - v)} [a\delta - (-v + bN^0(t))] \geq e^{\delta(V_F - v)} [a\delta - B] \geq 0.$$

Given this super-solution on $[0, D]$ for any fixed continuous $N^0(t)$, we can prove global existence for local solutions.

Theorem 4.9 (Global existence - excitatory and inhibitory cases) *Let ρ^0 be a non-negative function in $C^0((-\infty, V_F]) \cap C^1((-\infty, V_R) \cup (V_R, V_F]) \cap L^1((-\infty, V_F))$, such that $\rho^0(V_F) = 0$ and $\frac{d\rho^0}{dv}$ decays at $-\infty$ and admits finite left and right limits at V_R ; let $N^0 \in C^0([-D, 0])$ with $D > 0$. Let (ρ, N) be the corresponding strong solution of (1.1)-(1.2)-(1.3) on the maximal interval of existence $[0, T^*)$. Then $T^* = +\infty$.*

Proof. Assume the maximal time of existence T^* is finite, this means, using Theorem 4.3, that the firing rate N diverges when $t \rightarrow T^*$. We prove that this is a contradiction with the fact that $\bar{\rho}$ given by (4.3) is a super-solution.

As the solution was showed previously to be unique, we assume without loss of generality that $T^* = \frac{D}{2} < D$ by using the new initial conditions

$$\tilde{\rho}^0(v) = \rho(v, T^* - \frac{D}{2}) \quad \forall v \in (-\infty, V_F] \quad \text{and} \quad \tilde{N}^0(\tilde{t}) = N\left(T^* - \frac{D}{2} + \tilde{t}\right), \quad \tilde{t} \in [-D, 0).$$

As $\tilde{\rho}^0$ is continuous and vanish at V_F and $-\infty$, it belongs to $L^\infty((-\infty, V_F])$ and therefore there exists $\alpha \in \mathbb{R}_+^*$ such that the super-solution $\bar{\rho}$ we constructed satisfies $\alpha \bar{\rho}(v, 0) \geq \tilde{\rho}^0(v)$, for all $v \in (-\infty, V_F]$, where we use the fact that $\bar{\rho}$ never vanish on $(-\infty, V_F)$. Then, by Theorem 4.8, we have

$$N\left(T^* - \frac{D}{2} + \tilde{t}\right) = \tilde{N}(\tilde{t}) \leq \bar{N}(\tilde{t}) = ae^{\xi \tilde{t}} \quad \forall \tilde{t} \in \left[0, \frac{D}{2}\right).$$

Thus, $N(t) \leq ae^{\xi(t-T^*+\frac{D}{2})}$ for all $t \in [T^* - \frac{D}{2}, T^*)$. Therefore, by continuity, there is no divergence of the firing rate N when $t \rightarrow T^*$, and thus by Theorem 4.3 we reach a contradiction. \square

Remark 4.10 *As we remark for the local existence, the result about global existence works for the coupled excitatory-inhibitory system with positive delay studied in [7].*

5 Qualitative properties and long time behaviour of solutions

The aim of this part is to extend the results about long time behaviour obtained in [4] and [8] for the case without delay to the case where a synaptic delay is considered. All the results can be extended, assuming that the delay is small enough with respect to the other parameters of the model, in particular in comparison with the connectivity b .

In all this section, we use two standard technical results fully stated in Appendix A (entropy method and Poincaré's inequality). We also recall here that a steady state of the system (1.1)-(1.2)-(1.3) is defined as a solution of the problem

$$\begin{aligned} \frac{\partial}{\partial v} [(-v + bN_\infty)\rho_\infty] - a \frac{\partial^2 \rho_\infty}{\partial v^2} &= \delta_{v=V_R} N_\infty, \\ N_\infty &= -a \frac{\partial \rho_\infty}{\partial v}(V_F), \quad \rho_\infty(V_F) = 0, \quad \rho_\infty(-\infty) = 0, \\ \rho_\infty(v) &\geq 0, \quad \int_{-\infty}^{V_F} \rho_\infty(v) dv = 1. \end{aligned}$$

Such steady states are exactly the same for the case without transmission delay ($D = 0$) and for the case with

synaptic delay ($D > 0$). They are studied in [4] and a form is given:

$$\rho_\infty(v) = \frac{N_\infty}{a} e^{-\frac{(v-bN_\infty)^2}{2a}} \int_{\max(v, V_R)}^{V_F} e^{\frac{(w-bN_\infty)^2}{2a}} dw.$$

We point out that this formula is not explicit, since N_∞ depends on ρ_∞ . However, using that ρ_∞ is a probability density we can obtain an implicit equation for N_∞ [4].

Moreover, it is proved in [4] that for small enough positive b or every negative b , there is a unique steady state (ρ_∞, N_∞) . For $b > 0$ large enough, there are no steady states.

Using the results of [4], it is also possible to directly deduce for small enough values of b (*via* the continuous dependence in b and the value at 0) that

$$\lim_{b \rightarrow 0} N_\infty^b > 0, \quad (5.1)$$

where we denote by N_∞^b the stationary firing rate for the system (1.1)-(1.2)-(1.3) with connectivity parameter b .

5.1 Uniform estimates on the firing rate

We separate the average-inhibitory and the average-excitatory cases. Indeed, in the average-inhibitory case ($b < 0$), all the results are uniform with respect to the initial data, which is not the case when $b > 0$ (average-excitatory case), in accordance with the blow-up structure of Equation (1.1) in the case without delay. In the previous section we proved the global existence of the solution when synaptic delay is considered. However, in the proofs, the structure of Equation (1.1), in the average-excitatory case, leads to results which strongly depends on the initial data.

The following theorem, which is similar to Theorem 3.1 of [8], provides some L^2 control over the firing rates, and thus, supplies the main tool to prove the long time behaviour result of Theorem 5.3.

Theorem 5.1 *Let $b_1 > 0$ a connectivity parameter such that there exists a stationary state of System (1.1)-(1.2)-(1.3) and let $\rho_\infty^{b_1}$ be the corresponding stationary state. Let V_M with $V_F > V_M > V_R$ and define $S(b_1, V_M) := \int_{V_M}^{V_F} \frac{(\rho_\infty^0)^2}{\rho_\infty^{b_1}} dv$, with an initial datum chosen such that $S(b_1, V_M) < +\infty$. Then:*

i) For all $b \leq 0$ there exists a constant C independent of $S(b_1, V_M)$ and there exists a time $T > 0$ depending

only on V_M and $S(b_1, V_M)$, such that for all intervals $J \subset (T, +\infty)$:

$$\int_J N(t)^2 dt \leq C(1 + |J|). \quad (5.2)$$

ii) Assume $b > 0$ is small enough depending on $S(b_1, V_M)$, the delay D and V_M , then there exists a constant C such that for all $t \geq 0$:

$$\int_0^t N(t)^2 dt \leq C(1 + t). \quad (5.3)$$

Remark 5.2 Let us mention that the condition on the delay only appear in the average-excitatory case in Theorem 5.1.

Proof. The main idea of the proof is to use entropy type inequalities and to compare the solution of Equation (1.1) with a kind of super-solution which is a stationary state of Equation (1.1) with a strictly positive connectivity parameter in a similar manner as in [8] [Theorem 3.1], with the difficulty that now there is a synaptic delay in the equation.

Let us assume V_R positive without loss of generality, by translating the voltage variable, and introduce some notations and a useful function. We set the function γ as in [8] and defined by

$$\forall v \in (-\infty, V_F], \quad \gamma(v) = 1_{v > \alpha} e^{\frac{-1}{\beta - (V_F - v)^2}},$$

where $\beta = (V_F - \alpha)^2$ and $V_F > \alpha > V_R > 0$. Given a steady state $(\rho_\infty^{b_1}, N_\infty^{b_1})$ associated to the parameter b_1 and a solution (ρ, N) associated to the parameter b and the initial condition (ρ^0, N^0) , we denote in the following

$$H(v, t) = \frac{\rho(v, t)}{\rho_\infty^{b_1}(v)} \quad \text{and} \quad W(v, t) = \rho_\infty^{b_1}(v) (H(v, t))^2 = \frac{\rho(v, t)^2}{\rho_\infty^{b_1}(v)}$$

Let us first prove Theorem 5.1 in the inhibitory case ($b \leq 0$). Let $I(t) = \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv$. Following the computations made in [8][Section 3.1.1], where the fact that the support of $\gamma \subset (\alpha, V_F]$ with $V_F > \alpha > V_R > 0$ was used, we find that, for all α close enough to V_F , there exist $C_1, C_2, C_3 \in \mathbb{R}_+^*$ independent of b such that

$$\frac{d}{dt} I(t) \leq -C_1 N(t)^2 + C_2 + (bN(t - D) - C_3) I(t). \quad (5.4)$$

We have, due to $I(0) < +\infty$ (because $S(b_1, V_M) < +\infty$) and $b \leq 0$,

$$\frac{d}{dt} I(t) \leq C_2 - C_3 I(t).$$

Thus, there exists a time $T > 0$ depending on $I(0)$, such that, for $t > T$: $I(t) \leq 2\frac{C_2}{C_3}$. This conclude the proof of Theorem 5.1 i) by integrating the differential inequality (5.4) on every interval $J \subset [T, +\infty[$.

Let us now assume that $b > 0$. Let us again consider $I(t) = \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv$. Following again the computations made in [8][Section 3.1.2], we obtain that there exist C_1, C_2, C_3, C_4 and $C_5 = \frac{\gamma(V_F)}{N_\infty^{b_1}}$ depending only on γ and $\rho_\infty^{b_1}$ such that for all $t > 0$,

$$\frac{d}{dt} I(t) \leq \left(-C_4 + C_2 b^2 N(t-D)^2 \right) I(t) + C_1 b^2 N(t-D)^2 - C_5 N(t)^2 + C_3. \quad (5.5)$$

Let $\bar{M} = \max(I(0), \frac{C_3}{C_4})$. As $I(0)$ is strictly bounded by $2\bar{M}$, by continuity of I , there exists a closed interval $[0, \eta]$ on which I is bounded by $2\bar{M}$. Thus, let $[0, \bar{B}]$ ($[0, \bar{B})$ if $\bar{B} = +\infty$) the largest closed interval on which I is bounded by $2\bar{M}$. Assume \bar{B} is finite. On this interval $[0, \bar{B}]$, we have

$$\frac{d}{dt} I(t) \leq -C_4 I(t) + \left(C_1 + 2C_2 \bar{M} \right) b^2 N(t-D)^2 - C_5 N(t)^2 + C_3.$$

Let $C_6 = C_1 + 2C_2 \bar{M}$. As $C_3 - C_4 \bar{M} \leq 0$, we have

$$\frac{d}{dt} (I(t) - \bar{M}) \leq -C_4 (I(t) - \bar{M}) + C_6 b^2 N(t-D)^2 - C_5 N(t)^2.$$

Gronwall's Lemma implies that, for all $t \in [0, \eta]$,

$$I(t) - \bar{M} \leq e^{-C_4 t} (I(0) - \bar{M}) + e^{-C_4 t} \int_0^t \left(C_6 b^2 N(s-D)^2 - C_5 N(s)^2 \right) e^{C_4 s} ds. \quad (5.6)$$

Therefore, assuming $b^2 \leq \frac{C_5}{C_6 e^{C_4 D}}$, we can write

$$\begin{aligned} \int_0^t \left(C_6 b^2 N(s-D)^2 - C_5 N(s)^2 \right) e^{C_4 s} ds &= C_6 b^2 \int_0^t N(s-D)^2 e^{C_4 s} ds - C_5 \int_0^t N(s)^2 e^{C_4 s} ds \\ &\leq C_6 b^2 \int_{-D}^0 N(s)^2 e^{C_4 s + C_4 D} ds \\ &= C_6 e^{C_4 D} b^2 \int_{-D}^0 N(s)^2 e^{C_4 s} ds \\ &\leq C_6 e^{C_4 D} b^2 \int_{-D}^0 N^0(s)^2 ds. \end{aligned}$$

Hence, $I(t) \leq \bar{M} + e^{-C_4 t} C_6 e^{C_4 D} b^2 \int_{-D}^0 N^0(s)^2 ds$. For b small enough,

$$\bar{M} + e^{-C_4 t} C_6 e^{C_4 D} b^2 \int_{-D}^0 N^0(s)^2 ds < 2\bar{M},$$

which implies $I(\bar{B}) < 2\bar{M}$ and by continuity of I there exists $\varepsilon \in \mathbb{R}_+^*$ such that I is stricly bounded by $2\bar{M}$ on $[\bar{B}, \bar{B} + \varepsilon]$. This is a contradiction to the maximality of \bar{B} . Thus, $\bar{B} = +\infty$ and $I(t) \leq 2\bar{M}$, for all $t \in \mathbb{R}_+$.

Hence, integrating (5.5) between 0 and t we obtain

$$-I(0) \leq (C_1 + 2C_2 \bar{M}) b^2 \int_{-D}^{t-D} N(s)^2 ds - C_5 \int_0^t N(s)^2 ds + C_3 t,$$

that gives, for b small enough,

$$\frac{C_5}{2} \int_0^t N(s)^2 ds \leq I(0) + (C_1 + 2C_2 \bar{M}) b^2 \int_{-D}^0 N^0(s)^2 ds + C_3 t,$$

which finishes the proof of Theorem 5.1. \square

5.2 Convergence towards steady states for small connectivities

Now, we can prove our main result of this section, that ensures that even with delay, the solutions converge exponentially fast to the steady state, when the connectivity parameter is small, and with an appropriate initial condition. We do not include the linear case ($b = 0$), which was proved in [4] with better hypotheses and better results.

Theorem 5.3 *Let b_1 such that there exists a steady state $(\rho_\infty^{b_1}, N_\infty^{b_1})$ for (1.1)-(1.2)-(1.3) with $\bar{b} = 0$. Let $V_M \in (V_R, V_F)$, (ρ^0, N^0) an initial condition and $S(b_1, V_M) := \int_{V_M}^{V_F} \frac{\rho^0(v)^2}{\rho_\infty^{b_1}(v)} dv$. If there exists C_0 such that $\rho^0(\cdot) \leq C_0 \rho_\infty^{b_1}(\cdot)$ (consequently $S(b_1, V_M) < +\infty$), then:*

- *If $b > 0$, there exists $\eta \in \mathbb{R}_+^*$ depending only on $\rho^0, N^0, S(b_1, V_M), D$ and V_M such that if $0 < b < \eta$, then there exist $A_0, \mu, \gamma \in \mathbb{R}_+^*$ depending only on (ρ^0, N^0) such that*

$$\begin{aligned} \int_{-\infty}^{V_F} \rho_\infty \left(\frac{\rho - \rho_\infty}{\rho_\infty} \right)^2 (v, t) dv &\leq A_0 e^{-\mu t} \left[\int_{-\infty}^{V_F} \rho_\infty \left(\frac{\rho^0 - \rho_\infty}{\rho_\infty} \right)^2 (v) dv \right. \\ &\quad \left. + \frac{8b^2}{a} \int_{-D}^0 (N^0(s) - N_\infty)^2 e^{\gamma(s+D) - \int_{-D}^0 (N^0(u) - N_\infty)^2 du} ds \right]. \end{aligned}$$

- *If $b < 0$, there exists $\eta \in \mathbb{R}_+^*$, independent of $S(b_1, V_M)$ such that if $-\eta < b < 0$, there exist $T, \mu, \gamma \in \mathbb{R}_+^*$*

depending only on the initial condition, and $A_0 \in \mathbb{R}_+^*$ depending only on the values of N on $[0, T]$, such that for all $t \in (T, +\infty)$,

$$\begin{aligned} \int_{-\infty}^{V_F} \rho_{\infty} \left(\frac{\rho - \rho_{\infty}}{\rho_{\infty}} \right)^2 (v, t) dv &\leq A_0 e^{-\mu t} \left[\int_{-\infty}^{V_F} \rho_{\infty} \left(\frac{\rho^0 - \rho_{\infty}}{\rho_{\infty}} \right)^2 (v) dv \right. \\ &\quad \left. + \frac{8b^2}{a} \int_{-D}^0 (N^0(s) - N_{\infty})^2 e^{\gamma(s+D) - \int_{-D}^0 (N(u) - N_{\infty})^2 du} ds \right]. \end{aligned}$$

Proof. Let $G : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a convex function of class C^2 . As $\rho^0 \leq C_0 \rho_{\infty}^{b_1}$ and because the rate of decay for ρ_{∞}^b is asymptotically the same for every b at $-\infty$, we have for every b such that a stationary state exists another constant \tilde{C}_0 such that $\rho^0(\cdot) \leq \tilde{C}_0 \rho_{\infty}^b(\cdot)$. Then, the following entropy equality holds, applying theorem A.2.

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{V_F} \rho_{\infty} G \left(\frac{\rho}{\rho_{\infty}} \right) dv &= -a \int_{-\infty}^{V_F} \rho_{\infty} \left[\frac{\partial}{\partial v} \left(\frac{\rho}{\rho_{\infty}} \right) \right]^2 G'' \left(\frac{\rho}{\rho_{\infty}} \right) dv \\ &\quad - N_{\infty} \left[G \left(\frac{N}{N_{\infty}} \right) - G \left(\frac{\rho}{\rho_{\infty}} \right) - \left(\frac{N}{N_{\infty}} - \frac{\rho}{\rho_{\infty}} \right) G' \left(\frac{\rho}{\rho_{\infty}} \right) \right] \Big|_{v=V_R} \\ &\quad + b(N(t-D) - N_{\infty}) \int_{-\infty}^{V_F} \frac{\partial \rho_{\infty}}{\partial v} \left[G \left(\frac{\rho}{\rho_{\infty}} \right) - \frac{\rho}{\rho_{\infty}} G' \left(\frac{\rho}{\rho_{\infty}} \right) \right] dv. \end{aligned}$$

Now, denote $q(v, t) = \frac{\rho(v, t)}{\rho_{\infty}(v)}$ and $F(t) = \frac{N(t)}{N_{\infty}}$. We choose $G(x) = (x-1)^2$. Hence, for all $\varepsilon \in (0, \frac{1}{2})$, using the inequality $(a+b)^2 \geq \varepsilon(a^2 - 2b^2)$, yields

$$G(F(t)) - G(q(V_R, t)) - \left(F(t) - q(V_R, t) \right) G'(q(V_R, t)) = (F(t) - q(V_R, t))^2 \geq \varepsilon(F(t) - 1)^2 - 2\varepsilon(q(V_R, t) - 1)^2.$$

Then, according to the expression of ρ_{∞} , the Poincaré-like inequality (see Appendix A, Theorem A.1) and the Sobolev injection of $H^1(I)$ in $L^{\infty}(I)$ for a sufficiently small neighborhood I of V_R , there exists $C \in \mathbb{R}_+^*$ such that

$$|q(V_R, t) - 1|^2 \leq C \int_{-\infty}^{V_F} \rho_{\infty}(v) \left(\frac{\partial q}{\partial v} \right)^2 (v, t) dv.$$

Thus, for ε satisfying $2CN_{\infty}\varepsilon \leq \frac{a}{2}$, we have

$$\begin{aligned} &-N_{\infty} \left[G \left(\frac{N}{N_{\infty}} \right) - G \left(\frac{\rho}{\rho_{\infty}} \right) - \left(\frac{N}{N_{\infty}} - \frac{\rho}{\rho_{\infty}} \right) G' \left(\frac{\rho}{\rho_{\infty}} \right) \right] \Big|_{v=V_R} \\ &\leq -N_{\infty}\varepsilon G \left(\frac{N(t)}{N_{\infty}} \right) + \frac{a}{2} \int_{-\infty}^{V_F} \rho_{\infty} \left(\frac{\partial q}{\partial v} \right)^2 (v, t) dv. \end{aligned} \tag{5.7}$$

On the other hand, as $G(x) - xG'(x) = 1 - x^2$, we have by integration by parts,

$$b(N(t-D) - N_\infty) \int_{-\infty}^{V_F} \frac{\partial \rho_\infty}{\partial v} [G(q) - qG'(q)] dv = 2b(N(t-D) - N_\infty) \int_{-\infty}^{V_F} \rho_\infty q \frac{\partial q}{\partial v} dv.$$

Using the inequality $cd \leq \bar{\varepsilon}c^2 + \frac{1}{\bar{\varepsilon}}d^2$ with $\bar{\varepsilon} = \frac{a}{2}$, $c = \sqrt{\rho_\infty} \frac{\partial q}{\partial v}$ and $d = 2b(N(t-D) - N_\infty)\sqrt{\rho_\infty}q$, we obtain

$$\begin{aligned} & b(N(t-D) - N_\infty) \int_{-\infty}^{V_F} \frac{\partial \rho_\infty}{\partial v} [G(q) - qG'(q)] dv \\ & \leq \frac{a}{2} \int_{-\infty}^{V_F} \rho_\infty \left(\frac{\partial q}{\partial v} \right)^2 dv + \frac{8(b(N(t-D) - N_\infty))^2}{a} \int_{-\infty}^{V_F} \rho_\infty q^2 dv \\ & \leq \frac{a}{2} \int_{-\infty}^{V_F} \rho_\infty \left(\frac{\partial q}{\partial v} \right)^2 dv + \frac{8(b(N(t-D) - N_\infty))^2}{a} \int_{-\infty}^{V_F} \rho_\infty (q-1)^2 dv \\ & \quad + \frac{8(b(N(t-D) - N_\infty))^2}{a}. \end{aligned} \tag{5.8}$$

Collecting the previous bounds (5.7) and (5.8), we have

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{V_F} \rho_\infty(v) G(q(v, t)) dv & \leq -N_\infty \varepsilon G\left(\frac{N(t)}{N_\infty}\right) + \frac{8(b(N(t-D) - N_\infty))^2}{a} \int_{-\infty}^{V_F} \rho_\infty(v) G(q(v, t)) dv \\ & \quad - a \int_{-\infty}^{V_F} \rho_\infty \left(\frac{\partial q}{\partial v} \right)^2 dv + \frac{8(b(N(t-D) - N_\infty))^2}{a}. \end{aligned}$$

Using Poincaré's inequality (Theorem A.1) and the specific form of G we chose, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{V_F} \rho_\infty(v) G(q(v, t)) dv & \leq -\frac{\varepsilon}{N_\infty} (N(t) - N_\infty)^2 + \frac{8b^2}{a} (N(t-D) - N_\infty)^2 \\ & \quad + \left(-\nu a + \frac{8(b(N(t-D) - N_\infty))^2}{a} \right) \int_{-\infty}^{V_F} \rho_\infty(v) G(q(v, t)) dv, \end{aligned}$$

where ν is the Poincaré constant of Theorem A.1. Theorem A.1 holds for $b \in [-\eta, \eta]$, for some small enough $\eta > 0$; henceforth, we assume this is the case.

Then, denoting

$$E(t) = \int_{-\infty}^{V_F} \rho_\infty(v) G(q(v, t)) dv,$$

and $\phi(t) = -\nu at + \frac{8b^2}{a} \int_0^t (N(s-D) - N_\infty)^2 ds$, and using Gronwall's lemma, the previous inequality becomes

$$\begin{aligned}
E(t) &\leq e^{\phi(t)} E(0) + \frac{8b^2}{a} e^{\phi(t)} \int_0^t (N(s-D) - N_\infty)^2 e^{-\phi(s)} ds \\
&\quad - \frac{\varepsilon}{N_\infty} e^{\phi(t)} \int_0^t (N(s) - N_\infty)^2 e^{-\phi(s)} ds \\
&\leq e^{\phi(t)} E(0) + \frac{8b^2}{a} e^{\phi(t)} \int_{-D}^{t-D} (N(s) - N_\infty)^2 e^{-\phi(s+D)} ds \\
&\quad - \frac{\varepsilon}{N_\infty} e^{\phi(t)} \int_0^t (N(s) - N_\infty)^2 e^{-\phi(s)} ds.
\end{aligned}$$

But, for $s > 0$,

$$-\phi(s+D) = \nu a s + \nu a D - \frac{8b^2}{a} \int_0^{s+D} (N(u-D) - N_\infty)^2 du \leq -\phi(s) + \nu a D.$$

Using (5.1) and the fact that in Theorem A.1 the Poincaré's constant ν is independent of b , we can choose b such that

$$\frac{8b^2 e^{\nu a D}}{a} \leq \frac{\varepsilon}{N_\infty^b}.$$

With this choice, we can write

$$\begin{aligned}
E(t) &\leq e^{\phi(t)} E(0) + \frac{8b^2}{a} e^{\phi(t)} \int_{-D}^0 (N(s) - N_\infty)^2 e^{-\phi(s+D)} ds \\
&\quad - \frac{\varepsilon}{N_\infty} e^{\phi(t)} \int_{t-D}^t (N(s) - N_\infty)^2 e^{-\phi(s)} ds \\
&\leq e^{\phi(t)} \left(E(0) + \frac{8b^2}{a} \int_{-D}^0 (N^0(s) - N_\infty)^2 e^{-\phi(s+D)} ds \right).
\end{aligned}$$

- If $b > 0$: By Theorem 5.1, in the average-excitatory case, there exist constants $\beta_1, \beta_2 \in \mathbb{R}_+$ such that

$$\phi(t) \leq -\nu a t + \frac{8b^2}{a} (\beta_1 + \beta_2 t).$$

Thus, for b small enough ($\beta_1(b)$ and $\beta_2(b)$ do not grow when b goes to 0), $e^{\phi(t)} \leq A_0 e^{-\mu t}$, with A_0, μ positive real numbers.

- If $b \leq 0$: By Theorem 5.1, in the inhibitory case, there exists $T \in \mathbb{R}_+$ and there exist constants $\beta_1, \beta_2 \in \mathbb{R}_+$ such that $\phi(t) \leq -\nu a t + \frac{8b^2}{a} (\beta_1 + \beta_2 (t - T))$, for $t \geq T$. Thus, for b small enough (these β_1 and β_2 don't depend on b) and t large enough, $e^{\phi(t)} \leq A_0 e^{-\mu t}$, with A_0, μ positive real numbers and A_0 depending on

the values of N between 0 and T . \square

5.3 Non-existence of periodic solutions for a large connectivity parameter

Numerical results for the NNLIIF models have been presented in [7]. On one hand, we observe how the blow-up is avoided if we include a synaptic delay. For this case, we have a small value of b combined with a concentrated initial condition, which produces the blow-up of the solution without delay [4]. For this value of b there is a unique steady state [4], and the solution tends to it, after avoiding the blow-up due to the delay, which is consistent with Theorem 5.3.

On the other hand, also it is known that a blow-up situation happens for large value of b [4]. If we include the delay, the solutions avoid the blow-up, but they do not tend to an equilibrium, since for large values of b there is no steady state [4]. Numerically, no periodic solutions are observed and the firing rates seem to grow slowly all the time with limit $+\infty$, but without diverging in finite time [7].

Here we clarify a bit the situation by proving analytically that it is impossible for periodic solutions to exist when b is greater than $V_F - V_R$ in the case $V_F \leq 0$ and $\bar{b} = 0$.

Theorem 5.4 *If $b > V_F - V_R$, $\bar{b} = 0$ and $V_F \leq 0$, then for any $D \geq 0$ there are no strong periodic solutions to equation (1.1)-(1.2)-(1.3) such that $\int_{-\infty}^{V_F} |v| \rho^0(v) dv < +\infty$.*

Proof. Assume there exists a T -periodic solution (ρ, N) of equation

$$\partial_t \rho + \partial_v [(-v + bN(t - D))\rho] - a\partial_{vv}\rho = \delta_{V_R}(v)N(t)$$

such that $v\rho^0 \in L^1((-\infty, V_F))$. Then, as we said before, $v \mapsto v\rho(v, t) \in L^1((-\infty, V_F))$ for all time since the decay assumption propagates. Denoting $\Phi(v) := \frac{1}{T} \int_0^T \rho(v, t) dt$, we have $\int_{-\infty}^{V_F} \Phi(v) dv = 1$ and, Φ satisfies

$$\partial_v \left(-v\Phi + b\frac{1}{T} \int_0^T N(t - D)\rho(v, t) dt \right) - a\partial_{vv}\Phi = \delta_{v=V_R}\bar{N}, \quad (5.9)$$

where $\bar{N} := \frac{1}{T} \int_0^T N(t) dt = \frac{1}{T} \int_0^T N(t - D) dt$. Now, we multiply equation (5.9) by v and we integrate:

$$\int_{-\infty}^{V_F} v\Phi dv - b\bar{N} + (V_F - V_R)\bar{N} = 0.$$

Hence, $\int_{-\infty}^{V_F} v\Phi dv = (b - (V_F - V_R))\bar{N}$. As $V_F \leq 0$, the integral is negative, leading to $(b - (V_F - V_R)) < 0$, which is a contradiction. \square

Corollary 5.5 *If $b > V_F - V_R$, $V_F \leq 0$ and $\bar{b} = 0$, there is no steady state to equation (1.1)-(1.2)-(1.3).*

Remark 5.6 *This corollary upgrade a result of [4] that classifies the number of steady states. Indeed, the theorem 3.1 of [4] tell us (in the case $\bar{b} = 0$) that there is no steady state when*

$$b > \max \left(2(V_F - V_R) , 2V_F \int_0^{+\infty} e^{-\frac{s^2}{2}} \frac{e^{\frac{sV_F}{\sqrt{a_0}}} - e^{\frac{sV_R}{\sqrt{a_0}}}}{s} ds \right),$$

and that there are at least two steady states when $b > V_F - V_R$ and $0 < 2a_0b < (V_F - V_R)^2V_R$. The case $b > V_F - V_R$ and $V_F \leq 0$ was not covered by the result.

6 Conclusions and open problems

This paper focuses on the NNLIF system with a synaptic delay. For the average-excitatory case without transmission delay, solutions can blow-up in finite time [4] due to the divergence of the firing rate [9]. Nevertheless, at microscopic level, it has been proved that if the synaptic delay is taken into account, solutions are always global-in-time [13]. Moreover, for delayed NNLIF system numerical results show that the blow-up phenomenon is avoided [7]. In this work we prove the global-in-time existence for the delayed NNLIF model, confirming the previous numerical observations. The techniques developed in [9], for the case without synaptic delay, are not enough for the average-excitatory delayed system, since it is not possible to find a uniform bound for the firing rate. We reach the proof by combining these techniques with the construction of super-solutions [8], which provide a control of the firing rate. Moreover, we show qualitative properties of the solutions: a priori estimates on the firing rate, exponential convergence of the solutions to the steady state when the connectivity parameter is small enough, and some obstructions to the existence of time-periodic solutions in order to have insights in their potential domain of existence in terms of the model parameters.

In conclusion, we complete the mathematical analysis of the NNLIF system proving global-in-time existence for the delayed NNLIF model and making progress in the study of its long time behaviour. In this way we contribute to a better understanding of the NNLIF model, with a rich variety of phenomena, able to reproduce biological facts. In the light of these and earlier results we can conclude that the blow-up phenomenon, observed when the synaptic delay is neglected, appears due to this simplification. However, when the transmission delay is included in the model, solutions tend to the unique steady state when the connectivity parameter b is small enough. For b large enough so that no steady states exist, we also prove that there are no periodic solutions and it seems that the firing rate increases in time, as it was observed numerically in [7]. The methods developed in

this article could also be used to prove similar results for the NNLIF model with delay and refractory state for both one [5] and two populations [6, 7].

Several questions regarding the NNLIF model remain open: what happens with the solutions of the non-delayed NNLIF model after a blow-up phenomenon, the convergence towards the stationary state for average-inhibitory case (when entropy methods break), the existence and stability of periodic solutions and the analysis of possible multistability phenomena.

A Technical results from the literature

For completeness we include in this appendix two theorems often used in the study of NNLIF models: a Poincaré-like inequality and an entropy equality.

Theorem A.1 (Poincaré-like inequality) *There exist $\eta > 0$ and $\nu > 0$ depending on η such that for all $b \in [-\eta, \eta]$, for any measurable function h satisfying $\int_{-\infty}^{V_F} \rho_\infty(v)h(v)dv = 1$, the following inequality holds: $\nu \int_{-\infty}^{V_F} \rho_\infty(v)(h(v) - 1)^2 dv \leq \int_{-\infty}^{V_F} \rho_\infty(v) \left[\frac{\partial h}{\partial v} \right]^2 (v) dv$, where ρ_∞ is the steady state associated with b .*

The proof of this result was done in [4] for $b = 0$, and it is easily extended for small enough b , because it only uses the behaviour of ρ_∞ around V_F and $-\infty$, which does not depend on b , as it was explained in [8]. That behaviour does not change asymptotically when b goes from 0 to a small value, and therefore ν is independent of b for small enough b .

In the rest of the appendix we prove the following entropy equality, where we assume $\bar{b} = 0$, without loss of generality.

Theorem A.2 (Entropy equality) *For all convex function $G : \mathbb{R}_+ \rightarrow \mathbb{R}$ of class C^2 and any initial condition (ρ^0, N^0) satisfying the hypotheses of Theorem 4.9, if there exists $C_0 > 0$ such that $\rho^0(v) \leq C_0 \rho_\infty(v)$, for all $v \in (-\infty, V_F]$, then for all $t > 0$, there exists $C(t)$ such that for all $v \in (-\infty, V_F]$, $\rho(v, t) \leq C(t) \rho_\infty(v)$ and the corresponding strong solution ρ of (1.1)-(1.2)-(1.3) satisfies*

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{V_F} \rho_\infty G\left(\frac{\rho}{\rho_\infty}\right) dv &= -a \int_{-\infty}^{V_F} \rho_\infty \left[\frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) \right]^2 G''\left(\frac{\rho}{\rho_\infty}\right) dv \\ &\quad - N_\infty \left[G\left(\frac{N}{N_\infty}\right) - G\left(\frac{\rho}{\rho_\infty}\right) - \left(\frac{N}{N_\infty} - \frac{\rho}{\rho_\infty}\right) G'\left(\frac{\rho}{\rho_\infty}\right) \right] \Big|_{v=V_R} \\ &\quad + b(N(t-D) - N_\infty) \int_{-\infty}^{V_F} \frac{\partial \rho_\infty}{\partial v} \left[G\left(\frac{\rho}{\rho_\infty}\right) - \frac{\rho}{\rho_\infty} G'\left(\frac{\rho}{\rho_\infty}\right) \right] dv. \quad (\text{A.1}) \end{aligned}$$

Proof. According to Section 3, for $a = 1$, $V_F = 0$ and $t < D$, if we denote

$$\mathcal{G}(x, \tau, \xi, \eta) = \frac{1}{\sqrt{4\pi(\tau - \eta)}} e^{-\frac{(x - \xi)^2}{4(\tau - \eta)}}$$

the heat kernel, the solution ρ satisfies

$$\begin{aligned} \rho(v, t) = & e^{2t} \int_{-\infty}^0 \mathcal{G} \left(e^t v - \int_0^{\frac{1}{2}(e^{2t}-1)} e^{-s} \mu(s - D) ds, \frac{1}{2}(e^{2t} - 1), \xi, 0 \right) \rho^0(e^t \xi) d\xi \\ & - e^t \int_0^{\frac{1}{2}(e^{2t}-1)} M(\eta) \mathcal{G} \left(e^t v - \int_0^{\frac{1}{2}(e^{2t}-1)} e^{-s} \mu(s - D) ds, \frac{1}{2}(e^{2t} - 1), s(\eta), \eta \right) d\eta \\ & + e^t \int_0^{\frac{1}{2}(e^{2t}-1)} M(\eta) \mathcal{G} \left(e^t v - \int_0^{\frac{1}{2}(e^{2t}-1)} e^{-s} \mu(s - D) ds, \frac{1}{2}(e^{2t} - 1), s_1(\eta), \eta \right) d\eta, \quad (\text{A.2}) \end{aligned}$$

where M only depends on N^0 as long as $t < D$.

Due to the forms of \mathcal{G} and ρ_∞ , the last two terms decrease at least as fast as ρ_∞ at $-\infty$. The first term satisfies

$$\begin{aligned} & e^{2t} \int_{-\infty}^0 \mathcal{G} \left(e^t v - \int_0^{\frac{1}{2}(e^{2t}-1)} e^{-s} \mu(s - D) ds, \frac{1}{2}(e^{2t} - 1), \xi, 0 \right) \rho^0(e^t \xi) d\xi \\ & \leq e^{2t} C_0 \int_{-\infty}^0 \mathcal{G} \left(e^t v - \int_0^{\frac{1}{2}(e^{2t}-1)} e^{-s} \mu(s - D) ds, \frac{1}{2}(e^{2t} - 1), \xi, 0 \right) \rho_\infty(e^t \xi) d\xi, \\ & = e^t C_0 \int_{-\infty}^0 \mathcal{G} \left(e^t v - \int_0^{\frac{1}{2}(e^{2t}-1)} e^{-s} \mu(s - D) ds, \frac{1}{2}(e^{2t} - 1), e^{-t} \xi, 0 \right) \rho_\infty(\xi) d\xi. \quad (\text{A.3}) \end{aligned}$$

Hence, there exists $C(t)$ such that $\rho(v, t) \leq C(t) \rho_\infty(v)$, $\forall v \in (-\infty, V_F]$. As we said previously, it is then true for all a, V_F, t using changes of variables and successive time intervals of length less than D .

To conclude we prove the entropy equality A.1. First, we calculate $\frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right)$:

$$\frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) = \frac{1}{\rho_\infty} \frac{\partial \rho}{\partial v} - \frac{\rho}{\rho_\infty^2} \frac{\partial \rho_\infty}{\partial v}$$

and thus

$$\frac{\partial \rho}{\partial v} = \rho_\infty \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) + \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v},$$

and

$$\frac{\partial^2 \rho}{\partial v^2} = \rho_\infty \frac{\partial^2}{\partial v^2} \left(\frac{\rho}{\rho_\infty} \right) + 2 \frac{\partial \rho_\infty}{\partial v} \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) + \frac{\rho}{\rho_\infty} \frac{\partial^2 \rho_\infty}{\partial v^2}.$$

These expressions allow us to do the following computations

$$\begin{aligned}
\frac{\partial}{\partial t} \left[\rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right] &= \frac{\partial \rho}{\partial t} G' \left(\frac{\rho}{\rho_\infty} \right) \\
&= \left(\frac{\partial}{\partial v} \left[(v - bN(t - D)) \rho \right] + a \frac{\partial^2 \rho}{\partial v^2} + \delta_{V_R} N \right) G' \left(\frac{\rho}{\rho_\infty} \right) \\
&= \left(v \rho_\infty \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) + v \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} + \rho \right. \\
&\quad \left. - bN(t - D) \rho_\infty \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) - bN(t - D) \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} + \delta_{V_R} N \right. \\
&\quad \left. + a \rho_\infty \frac{\partial^2}{\partial v^2} \left(\frac{\rho}{\rho_\infty} \right) + 2a \frac{\partial \rho_\infty}{\partial v} \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) + a \frac{\rho}{\rho_\infty} \frac{\partial^2 \rho_\infty}{\partial v^2} \right) G' \left(\frac{\rho}{\rho_\infty} \right) \\
&= \left(\left(v \rho_\infty + 2a \frac{\partial \rho_\infty}{\partial v} \right) \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) + a \rho_\infty \frac{\partial^2}{\partial v^2} \left(\frac{\rho}{\rho_\infty} \right) \right. \\
&\quad \left. + \frac{\rho}{\rho_\infty} \left[v \frac{\partial \rho_\infty}{\partial v} + \rho_\infty + a \frac{\partial^2 \rho_\infty}{\partial v^2} \right] - bN(t - D) \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} \right. \\
&\quad \left. - bN(t - D) \rho_\infty \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) + \delta_{V_R} N \right) G' \left(\frac{\rho}{\rho_\infty} \right).
\end{aligned}$$

As ρ_∞ is a steady state, we have

$$\begin{aligned}
\frac{\rho}{\rho_\infty} \left[v \frac{\partial \rho_\infty}{\partial v} + \rho_\infty + a \frac{\partial^2 \rho_\infty}{\partial v^2} \right] - bN(t - D) \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} \\
= \frac{\rho}{\rho_\infty} \left[bN_\infty \frac{\partial \rho_\infty}{\partial v} - \delta_{V_R} N_\infty \right] - bN(t - D) \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} \\
= b(N_\infty - N(t - D)) \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} - \delta_{V_R} N_\infty \frac{\rho}{\rho_\infty}, \quad (\text{A.4})
\end{aligned}$$

and combining the two previous computations

$$\begin{aligned}
\frac{\partial}{\partial t} \left[\rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right] &= \left(\left(v \rho_\infty + 2a \frac{\partial \rho_\infty}{\partial v} \right) \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) + a \rho_\infty \frac{\partial^2}{\partial v^2} \left(\frac{\rho}{\rho_\infty} \right) \right. \\
&\quad \left. + \delta_{V_R} N_\infty \left(\frac{N}{N_\infty} - \frac{\rho}{\rho_\infty} \right) + b(N_\infty - N(t - D)) \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} \right. \\
&\quad \left. - bN(t - D) \rho_\infty \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) \right) G' \left(\frac{\rho}{\rho_\infty} \right).
\end{aligned}$$

On the other hand, since $\frac{\partial}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right) = \frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) G' \left(\frac{\rho}{\rho_\infty} \right)$ and

$$\frac{\partial^2}{\partial v^2} G \left(\frac{\rho}{\rho_\infty} \right) = \left[\frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) \right]^2 G'' \left(\frac{\rho}{\rho_\infty} \right) + \frac{\partial^2}{\partial v^2} \left(\frac{\rho}{\rho_\infty} \right) G' \left(\frac{\rho}{\rho_\infty} \right),$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left[\rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right] &= \left(v \rho_\infty + 2a \frac{\partial \rho_\infty}{\partial v} \right) \frac{\partial}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right) + a \rho_\infty \frac{\partial^2}{\partial v^2} G \left(\frac{\rho}{\rho_\infty} \right) \\ &\quad - a \rho_\infty \left[\frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) \right]^2 G'' \left(\frac{\rho}{\rho_\infty} \right) + \delta_{V_R} N_\infty \left(\frac{N}{N_\infty} - \frac{\rho}{\rho_\infty} \right) G' \left(\frac{\rho}{\rho_\infty} \right) \\ &\quad + b(N_\infty - N(t - D)) \frac{\partial \rho_\infty}{\partial v} \frac{\rho}{\rho_\infty} G' \left(\frac{\rho}{\rho_\infty} \right) - bN(t - D) \rho_\infty \frac{\partial}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right). \end{aligned}$$

Then, using the equation for ρ_∞ , we write

$$\begin{aligned} \frac{\partial}{\partial t} \left[\rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right] &= \frac{\partial}{\partial v} \left(v \rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right) + a \frac{\partial^2}{\partial v^2} \left(\rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right) \\ &\quad + \delta_{V_R} N_\infty \left[\left(\frac{N}{N_\infty} - \frac{\rho}{\rho_\infty} \right) G' \left(\frac{\rho}{\rho_\infty} \right) + G \left(\frac{\rho}{\rho_\infty} \right) \right] \\ &\quad - a \rho_\infty \left[\frac{\partial}{\partial v} \left(\frac{\rho}{\rho_\infty} \right) \right]^2 G'' \left(\frac{\rho}{\rho_\infty} \right) - bN(t - D) \rho_\infty \frac{\partial}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right) \\ &\quad - bN_\infty \frac{\partial \rho_\infty}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right) + b(N_\infty - N(t - D)) \frac{\partial \rho_\infty}{\partial v} \frac{\rho}{\rho_\infty} G' \left(\frac{\rho}{\rho_\infty} \right). \end{aligned}$$

Finally, as $\rho(V_F, t) = \rho_\infty(V_F) = 0$ and $\frac{\partial \rho_\infty}{\partial v}(V_F) = -aN_\infty \neq 0$, we can apply l'Hôpital's rule :

$$\forall t \in \mathbb{R}_+, \quad \lim_{v \rightarrow V_F} \frac{p(v, t)}{\rho_\infty(v)} = \lim_{v \rightarrow V_F} \frac{\frac{\partial p}{\partial v}(v, t)}{\frac{\partial \rho_\infty}{\partial v}(v)} = \frac{N(t)}{N_\infty},$$

and integrating we obtain $\int_{-\infty}^{V_F} \frac{\partial}{\partial v} \left(v \rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right) dv = \left[v \rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right]_{-\infty}^{V_F} = 0$ and

$$\begin{aligned} \int_{-\infty}^{V_F} a \frac{\partial^2}{\partial v^2} \left(\rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right) dv &= \left[a \frac{\partial}{\partial v} \left(\rho_\infty G \left(\frac{\rho}{\rho_\infty} \right) \right) \right]_{-\infty}^{V_F} \\ &= \left[a \left(\frac{\partial \rho}{\partial v} - \frac{\rho}{\rho_\infty} \frac{\partial \rho_\infty}{\partial v} \right) G' \left(\frac{\rho}{\rho_\infty} \right) + a \frac{\partial \rho_\infty}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right) \right]_{-\infty}^{V_F} \\ &= \left(-N + \frac{N}{N_\infty} N_\infty \right) G' \left(\frac{N}{N_\infty} \right) - N_\infty G \left(\frac{N}{N_\infty} \right) \\ &= -N_\infty G \left(\frac{N}{N_\infty} \right), \end{aligned}$$

where every integral is defined and finite thanks to the inequality $\rho(v, t) \leq C(t) \rho_\infty(v)$.

Eventually, we have, by integration by parts and using boundary conditions,

$$\int_{-\infty}^{V_F} bN(t - D) \rho_\infty \frac{\partial}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right) dv = -bN(t - D) \int_{-\infty}^{V_F} \frac{\partial \rho_\infty}{\partial v} G \left(\frac{\rho}{\rho_\infty} \right) dv$$

and the result comes from this integral and the previous computations. \square

References

- [1] R. BRETTE AND W. GERSTNER, *Adaptive exponential integrate-and-fire model as an effective description of neural activity*, Journal of neurophysiology, 94 (2005), pp. 3637–3642.
- [2] N. BRUNEL, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking networks*, J. Comp. Neurosci., 8 (2000), pp. 183–208.
- [3] N. BRUNEL AND V. HAKIM, *Fast global oscillations in networks of integrate-and-fire neurons with long firing rates*, Neural Computation, 11 (1999), pp. 1621–1671.
- [4] M. J. CÁCERES, J. A. CARRILLO, AND B. PERTHAME, *Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states*, Journal of Mathematical Neuroscience, 1-7 (2011).
- [5] M. J. CÁCERES AND B. PERTHAME, *Beyond blow-up in excitatory integrate and fire neuronal networks: refractory period and spontaneous activity*, Journal of theoretical Biology, 350 (2014), pp. 81–89.
- [6] M. J. CÁCERES AND R. SCHNEIDER, *Blow-up, steady states and long time behaviour of excitatory-inhibitory nonlinear neuron models*, Kinetic and Related Models, 10 (2017), pp. 587–612.
- [7] M. J. CÁCERES AND R. SCHNEIDER, *Analysis and numerical solver for excitatory-inhibitory networks with delay and refractory periods*, ESAIM: Mathematical Modelling and Numerical Analysis, 52 (2018), pp. 1733–1761.
- [8] J. CARRILLO, B. PERTHAME, D. SALORT, AND D. SMETS, *Qualitative properties of solutions for the noisy integrate & fire model in computational neuroscience*, Nonlinearity, 25 (2015), pp. 3365–3388.
- [9] J. A. CARRILLO, M. D. M. GONZÁLEZ, M. P. GUALDANI, AND M. E. SCHONBEK, *Classical solutions for a nonlinear fokker-planck equation arising in computational neuroscience*, Comm. in Partial Differential Equations, 38 (2013), pp. 385–409.
- [10] J. A. CARRILLO AND G. TOSCANI, *Asymptotic L^1 -decay of solutions of the porous medium equation to self-similarity*, Indiana University Mathematics Journal, (2000), pp. 113–142.
- [11] J. CHEVALLIER, *Mean-field limit of generalized hawkes processes*, Stochastic Processes and their Applications, 127 (2017), pp. 3870–3912.

- [12] J. CHEVALLIER, M. J. CÁ CERES, M. DOUMIC, AND P. REYNAUD-BOURET, *Microscopic approach of a time elapsed neural model*, Mathematical Models and Methods in Applied Sciences, 25 (2015), pp. 2669–2719.
- [13] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ, *Particle systems with a singular mean-field self-excitation. Application to neuronal networks*, Stochastic Processes and their Applications, 125 (2015), pp. 2451–2492.
- [14] F. DELARUE, J. INGLIS, S. RUBENTHALER, E. TANRÉ, ET AL., *Global solvability of a networked integrate-and-fire model of mckean-vlasov type*, The Annals of Applied Probability, 25 (2015), pp. 2096–2133.
- [15] G. DUMONT AND P. GABRIEL, *The mean-field equation of a leaky integrate-and-fire neural network: measure solutions and steady states*, arXiv preprint arXiv:1710.05596, (2017).
- [16] G. DUMONT AND J. HENRY, *Population density models of integrate-and-fire neurons with jumps: well-posedness*, J. Math. Biol., 67 (2013), pp. 453–81.
- [17] ———, *Synchronization of an excitatory integrate-and-fire neural network*, Bull. Math. Biol., 75 (2013), pp. 629–648.
- [18] R. FITZHUGH, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical journal, 1 (1961), pp. 445–466.
- [19] S. MISCHLER, C. QUININAO, AND J. TOUBOUL, *On a kinetic Fitzhugh–Nagumo model of neuronal network*, Communications in Mathematical Physics, 342 (2016), pp. 1001–1042.
- [20] S. MISCHLER AND Q. WENG, *Relaxation in time elapsed neuron network models in the weak connectivity regime*, ACAP, (2018), pp. 1–30.
- [21] K. NEWHALL, G. KOVAČIČ, P. KRAMER, D. ZHOU, A. V. RANGAN, AND D. CAI, *Dynamics of current-based, poisson driven, integrate-and-fire neuronal networks*, Comm. in Math. Sci., 8 (2010), pp. 541–600.
- [22] K. PAKDAMAN, B. PERTHAME, AND D. SALORT, *Dynamics of a structured neuron population*, Nonlinearity, 23 (2010), pp. 55–75.
- [23] K. PAKDAMAN, B. PERTHAME, AND D. SALORT, *Relaxation and self-sustained oscillations in the time elapsed neuron network model*, SIAM Journal on Applied Mathematics, 73 (2013), pp. 1260–1279.

- [24] B. PERTHAME AND D. SALORT, *On a voltage-conductance kinetic system for integrate & fire neural networks*, Kinet. Relat. Models, 6 (2013), pp. 841–864.
- [25] A. V. RANGAN, G. KOVAČIČ, AND D. CAI, *Kinetic theory for neuronal networks with fast and slow excitatory conductances driven by the same spike train*, Physical Review E, 77 (2008), pp. 1–13.
- [26] A. RENART, N. BRUNEL, AND X.-J. WANG, *Mean-field theory of irregularly spiking neuronal populations and working memory in recurrent cortical networks*, in Computational Neuroscience: A comprehensive approach, J. Feng, ed., Chapman & Hall/CRC Mathematical Biology and Medicine Series, 2004.